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CONTINUUM MODELING OF
TWO-PHASE FLOWS

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1. INTRODUCTION

Continuum modeling of two-phase flows can essentially be achieved in two ways. The first approach, the so-called continuum theory of mixtures, ignores the details of the flow occurring on the microscopic level, while the second one is the result of some averaging procedure. Although they both lead, as expected, to the same set of basic equations, they differ strongly in their spirit when closure equations have to be found. In the present report, we have attempted to give a brief critical review of both approaches, to compare them and to discuss some of the major difficulties which arise. It is shown that the application of the continuum theory of mixtures is, in most cases, questionable and that the only appropriate way of finding closure equations, besides correlating experimental results, consists in a careful investigation of the microscopic flow pattern associated with an adequate averaging technique.

2. FUNDAMENTAL ASSUMPTIONS

In order to bring out what we think are the really essential difficulties, and to keep the presentation comparatively simple, without jeopardizing its generality, we make the following assumptions:

- (a) We consider only two-phase systems consisting of two distinct localized phases but include two cases. Each of the two phases may consist of the same pure substance or, alternatively, each may consist of a different pure substance.
- (b) Both phases are supposed to be chemically inert.
- (c) Polar and electromagnetic effects are neglected.
- (d) All interfaces between the phases are devoid of any physical property such as mass, surface tension, surface energy, etc.

3. DESCRIPTION ON A MICROSCALE

The geometrical structure of a given two-phase medium is completely determined by the specification of the union Σ of all interfaces separating both phases at any given time t . In what follows, we shall call a micro-level description, a level of description whose length and time scales (ℓ_1, t_1) are much larger than the largest of the length and time scales associated with discrete processes taking place on a molecular or micro-structural level in either phase, and at the same time much smaller than the smallest length and time-scales which characterize the space-time heterogeneity of the system. At that level of description, the standard conservation equations of continuum physics hold in each phase separately and are all of the familiar form:

$$\frac{\partial}{\partial t} \left(\rho^\alpha f^\alpha \right) + \nabla \cdot \left(\rho^\alpha f^\alpha \mathbf{v}^\alpha + \mathbf{j}^\alpha \right) = \rho^\alpha r^\alpha \quad ; \quad \alpha = 1, 2 \quad , \quad (1)$$

where $\rho^\alpha, \underline{v}^\alpha$ stand respectively for the density and the velocity of phase α . The specific value f^α (per unit mass of α) of any conserved extensive quantity F may be found in table 1, along with its diffusive flux $\underline{j}_\alpha$ and specific rate of production r_α . The associated jump conditions on the interface Σ are given by:

$$\sum_{\alpha=1}^2 \dot{f}^\alpha = 0, \quad (2)$$

where $\dot{f}^\alpha$ is defined as:

$$\dot{f}^\alpha = \rho^\alpha f^\alpha (\underline{v}^\Sigma - \underline{v}^\alpha) \cdot \underline{n}^\alpha - \underline{j}^\alpha \cdot \underline{n}^\alpha. \quad (3)$$

Here, $\underline{n}^\alpha$ is the outward unit normal to the generally time-dependent surface (Σ) , while $\underline{v}^\Sigma$ is the velocity of a geometrical point on that surface.

Table 1

F	f^α	$\underline{j}^\alpha$	r^α	$\dot{f}^\alpha$
Mass	1	0	0	$-\rho^\alpha (\underline{v}^\alpha - \underline{v}^\Sigma) \cdot \underline{n}^\alpha = \dot{m}^\alpha$
Momentum	$\underline{v}^\alpha$	$-\underline{\sigma}^\alpha$	$\underline{F}^\alpha$	$\dot{m}^\alpha \underline{v}^\alpha + \underline{\sigma}^\alpha \cdot \underline{n}^\alpha = \dot{\underline{p}}^\alpha$
Total Energy	$e^\alpha = u^\alpha + \frac{1}{2}(\underline{v}^\alpha)^2$	$\underline{q}^\alpha - \underline{\sigma}^\alpha \cdot \underline{v}^\alpha$	$\underline{F}^\alpha \cdot \underline{v}^\alpha$	$\dot{m}^\alpha e^\alpha - (\underline{q}^\alpha - \underline{\sigma}^\alpha \cdot \underline{v}^\alpha) \cdot \underline{n}^\alpha = \dot{E}^\alpha$

In table 1, the symbols $\underline{\sigma}^\alpha$, $\underline{q}^\alpha$, e^α , u^α and $\underline{F}^\alpha$ denote, as usual, the Cauchy stress-tensor, the heat-flux, the specific total energy, the specific internal energy and the external force per unit mass of α .

The above microlevel conservation equations and jump conditions have to be supplemented by microlevel equations of state, and constitutive laws describing the behavior of each separate phase and also the transfer pro-

cesses which occur at the interfaces. We assume here that their form can be obtained by using the classical procedure of thermodynamics of irreversible processes. This last step essentially consists in postulating the existence of a so-called accompanying equilibrium state [1] for each subsystem of dimension ℓ_1 , whose specific entropy s^α may be expressed by an equation of state as a function of its internal energy, its deformation parameters and possibly various internal variables accounting for dissipative relaxation mechanisms taking place on the atomic, molecular level, or on some microstructural level larger than λ_1 , but smaller than ℓ_1 . As a result, the equation governing the behavior of s^α is of the form given by equation (1) if the following identifications are made:

$$f^\alpha \equiv s^\alpha \quad ; \quad j^\alpha \equiv q^\alpha/T^\alpha \quad ; \quad \rho^\alpha r^\alpha \equiv \dot{\theta}^\alpha \quad , \quad (4)$$

where T^α is the temperature and $\dot{\theta}^\alpha$ is the volumetric rate of entropy production in the bulk of phase α . On the other hand, defining $\dot{f}^\alpha$ as

$$\dot{f}^\alpha \equiv \dot{s}^\alpha = \dot{m}^\alpha s^\alpha - \frac{q^\alpha}{T^\alpha} \cdot \underline{n}^\alpha \quad (5)$$

we see that the rate of entropy production per unit area on Σ is given by:

$$\dot{s} = \sum_{\alpha=1}^2 \dot{s}^\alpha = \sum_{\alpha=1}^2 (\dot{m}^\alpha s^\alpha - q^\alpha/T^\alpha \cdot \underline{n}^\alpha) \quad . \quad (6)$$

Moreover it is known from the second principle of thermodynamics that $\dot{\theta}^\alpha$ and $\dot{s}$, which both exhibit the familiar bilinear form in terms of force and fluxes, satisfy separately the following inequalities:

$$\dot{\theta}^\alpha \geq 0, \quad \alpha = 1, 2, \quad (7)$$

and

$$\dot{s} \geq 0. \quad (7a)$$

4. MACROLEVEL DESCRIPTION

Solving the above microlevel equations with the associated boundary conditions is a formidable task, since there are two deformable materials, occupying distinct regions of physical space and separated from each other by a large number of definite interfaces. Intuition, however, suggests that if the observer accepts a coarser-grained, or macrolevel, description of the medium, i.e. on a length and time scale much larger than (ℓ_1, t_1) , the latter may be viewed as a mixture of two overlapping equivalent continua, each of them undergoing its own process. When this point of view is adopted, the interfaces are smeared out across the equivalent continua and the real thermal and mechanical interactions across the interfaces become average volumetrically distributed interactions. There has been a number of papers devoted to this topic in recent years, among which two different approaches may be distinguished:

(a) The macrolevel approach considered as a phenomenological continuum theory of mixtures.

(b) The macrolevel approach regarded as the result of a detailed averaging procedure.

Although they both use the macrolevel description, they differ substantially in their spirit.

In the first one, which has been reviewed in [2], the starting point is the system of macrolevel conservation equations which is known, a priori, to have the usual conservation-law form derived for the so-called two-fluid theory developed for fluids whose mixing takes place on a molecular level (classical mixture theory) [3-4]. Here it is assumed that the real two-phase

medium may be replaced by a mixture of two equivalent coexisting phases to which appropriate thermodynamic extensive and intensive properties may be ascribed, namely density, momentum, internal energy, temperature, etc. A thermodynamic foundation for the behavior of the system is provided using either the methods of classical or rational thermodynamics, but no attempt is made at deriving the macrolevel equations from the microlevel ones.

In the second approach, which has also been extensively studied either in connection with two-phase fluid flows [5-6-7-8], or with the theory of porous media [9-10-11], the macrolevel equations are derived from the microlevel ones by averaging procedures. It is useful to note here that the purpose of the averaging procedure is not to yield a new unexpected set of equations that could not be guessed a priori but to establish a clear link between the quantities appearing on the macrolevel and those already introduced on the microscale. As a matter of fact, it is obvious that the averaged conservation equations will also have a form identical to that of the two-fluid theory of classical mixtures.

Although the first approach looks much more compact and elegant, it has a number of drawbacks:

(a) Except in very special cases, the constitutive assumptions that can be inferred from it have not proved very successful.

(b) The physical meaning of such thermodynamic properties as the internal energy and the temperature of the overlapping continua is not clear.

(c) There is a great deal of controversy about the form which should be given to the dissipation inequality ([2], [12]), the latter being postulated on the basis of questionable macrolevel arguments instead of being merely derived by averaging.

The second approach, on the other hand, although it requires tedious calculations, avoids difficulties (b) and (c) and also gives the exact expression of macrolevel diffusive fluxes in terms of microlevel quantities, which might prove useful when tackling the problem of closing the set of averaged equations.

From now on, we shall concentrate on the latter approach and try to make its connection with the first approach clear, because various authors do not use the same terminology and, when they do, there remain ambiguities of meaning. We shall quote the main results and omit the details of the derivation which may be found elsewhere ([5-8], [13]).

5. MICROLEVEL EQUATIONS AS PARTIAL DIFFERENTIAL EQUATIONS FOR DISTRIBUTIONS

In order to avoid some of the conceptual difficulties raised and partly discussed or resolved in some of the above-mentioned papers, we shall consistently use the appropriate mathematical tool for multiphase media, i.e. distribution theory [14-15], following MARLE [9] who first applied it in a simpler case.

In equation (1), ρ^α , f^α , $\underline{v}^\alpha$, $\underline{j}^\alpha$, r^α may be considered as smooth functions of $\underline{x}$ and t , only in the domain occupied by each phase α . Let them be analytically continued in the other phase by some arbitrary function of $\underline{x}$ and t , making them smooth functions, which are defined everywhere in R^4 and locally integrable. Now, instead of considering (Σ) as a surface moving with time in the physical space, it proves convenient here to think of the interface as a manifold (Σ) in a four-dimensional space $(\underline{x}, t)$. Where such a point of view is taken, the interface Σ becomes a fixed hypersurface in R^4 . Accordingly, the characteristic function χ^α of each phase, defined by:

$$\begin{cases} \chi^\alpha(\xi) = 1 & \text{if event } \xi \text{ occurs in } \alpha, \\ \chi^\alpha(\xi) = 0 & \text{if event } \xi \text{ does not occur in } \alpha, \end{cases} \quad (9)$$

where:

$$\xi = (t, x), \quad (10)$$

i.e.

$$\xi_0 = t; \xi_i = x_i, \quad i = 1, 2, 3, \quad (11)$$

may be considered a distribution in R^4 , and the same is true of its products by the analytically continued functions f^α , ρ^α , v^α , In view of the classical properties of distributions, equation (1) may be written in the form [13]:

$$\frac{\partial}{\partial t} (\chi^\alpha \rho^\alpha f^\alpha) + \nabla \cdot [(\rho^\alpha f^\alpha v^\alpha + j^\alpha) \chi^\alpha] - \rho^\alpha r^\alpha \chi^\alpha + \dot{f}^\alpha \Delta^\Sigma; \alpha = 1, 2, \quad (12)$$

where:

$$\Delta^\Sigma = \frac{\delta^\Sigma}{[1 + (v^\Sigma \cdot n^\alpha)^2]^{1/2}} \quad (13)$$

denotes a surface distribution defined in R^4 , whose support is Σ . This last equation may be thought of as being a microlevel balance equation for phase α , valid everywhere and at all times, as long as it is considered in the sense of distributions. Equation (12) is the basic equation which will be used to derive the fundamental averaged conservation equations needed either in the two-fluid approach or in the diffusion approach to be defined later.

6. SPACE-TIME AVERAGES. MACROLEVEL DESCRIPTION.

The most natural way of introducing statistical considerations consists in taking ensemble averages, i.e., over an ensemble of "macroscopically

identical" systems. From that point of view, any microlevel physical quantity is considered as a particular replica of a random function, and the corresponding macrolevel quantity is just its mathematical expectation. Although ensemble averaging exhibits a number of advantages, we shall only deal here with space-time averages because:

- (a) It is not obvious that a field variable can in all cases be considered a particular sample of a random function,
- (b) Experimentalists, for practical reasons, take space-time averages of one particular replica of the random function,
- (c) The link between ensemble averages and space-time averages is not clear unless the medium under consideration is space-time statistically stationary and ergodic.

Quite generally, the space-time average of any microlevel quantity $a(\xi)$, whether it is smooth or not, is defined by the following convolution product:

$$\langle a(\xi) \rangle = a(\xi) * w(\xi) , \quad (15)$$

where $w(\xi)$ is some normalized even weighting function of compact support, such that:

$$\int_{R^4} w(\xi) d\xi = 1. \quad (16)$$

The great majority of authors dealing with averages in multiphase media use volume or time, or time-volume averages, which correspond in fact to the obvious choice of a space-time window for the weighting function $w(\xi)$. This is not sufficient, however, to ensure the smoothness of the average $\langle a(\xi) \rangle$, which gives rise to mathematical problems of an entirely artificial nature, associated with an inappropriate choice for the weighting function

[5-6-8]. In order to avoid such difficulties, we shall require that $w(\xi)$ be of class C^∞ , in which case the averaged quantities are infinitely smooth functions of ξ [14-15]. Moreover, the choice of a C^∞ weighting function is just as physical as that of a window since it is undoubtedly closer to the transfer function of a real instrument. A typical example of such a weighting function is provided by the familiar hat-shaped test-functions introduced in the theory of distributions, i.e.,

$$w(\xi) = \begin{cases} \frac{1}{K} \cdot \exp - \frac{1}{r^2 - \xi^2} & \text{if } |\xi| < r, \\ 0 & \text{if } |\xi| > r. \end{cases} \quad (17)$$

where:

$$K = \int \exp - \frac{1}{r^2 - \xi^2} d\xi.$$

According to the type of experiment, one may have to consider time averages or space averages, or both. In order to account for all situations at the same time, we only study here the general case of combined space-time averages, all the more since any measurement, however local and instantaneous, corresponds precisely to a space-time average taken over some volume and some time period. The only physically relevant question here is whether the width of the time (or space) window should be made very small in order to obtain "instantaneous" (or "local") quantities, or, on the contrary, much larger. The proper choice of the width of the "pseudo-window" $w(\xi)$, i.e., of its support, should be conducted with great detail in each specific case. Quite generally, unless only gross overall data are required, the width of the window should be small enough compared to some length or time (L , T) scale on which the average $a(\xi)$ varies significantly in order to be able to talk about a point, and large compared to the largest length and time scale characterizing the heterogeneity of the material (λ_3 , θ_3) in order to smooth

out the erratic fluctuations of the averages. For this reason, the width (ℓ_2, t_2) of the "pseudo-window", which is a quantitative space-time measure of the new macrolevel of description, should satisfy the following inequality:

$$\begin{aligned} \lambda_1 &<< \ell_1 << \lambda_2 < \lambda_3 << \ell_2 << L \\ \theta_1 &<< t_1 << \theta_2 < \theta_3 << t_2 << T. \end{aligned} \quad (18)$$

Moreover, when the above conditions are met, the apparent arbitrariness of the weighting function will be immaterial because the corresponding averaged quantities will be practically insensitive to the particular choice made for the weighting function $w(\xi)$. A good discussion of these problems may be found in [9].

7. SPACE-TIME AVERAGED FIELD EQUATIONS

It follows from equations (12) and (15), and the application of the rule for the calculation of the derivative of a convolution product that the general form of any space-time averaged field equation is:

$$\frac{\partial}{\partial t} \langle \chi^\alpha_\rho f^\alpha \rangle + \nabla \cdot \langle \chi^\alpha (\rho^\alpha f^\alpha \underline{v}^\alpha + \underline{j}^\alpha) \rangle = \langle \rho^\alpha r^\alpha \chi^\alpha \rangle + \langle \dot{f}^\alpha \Delta^\Sigma \rangle, \quad \alpha = 1, 2 \quad (19)$$

As noted earlier, functions such as $\chi^\alpha_\rho f^\alpha w(\xi)$, $\dot{f}^\alpha \Delta^\Sigma w(\xi)$, will be of class C^∞ , therefore all the averaged quantities appearing in equation (19) have a well-defined meaning and a smooth behavior at each point $\underline{x}$ and at any instant t . As a consequence, the macrolevel equation (19) will describe on the macroscale (ℓ_2, t_2) , for each value of α , the evolution of an equivalent fictitious phase α occupying the entire space. We have, therefore, mapped the originally "discontinuous" physical space onto a new one in which two fictitious phases having averaged equivalent properties coexist at each point $\underline{x}$, whatever the value of t .

7.1 TWO-FLUID THEORY

Equation (19) corresponds to the so-called two-fluid theory. Beside the phase averages previously defined, it is customary to introduce intrinsic (or conditional averages) denoted by the symbol $\bar{F}^\alpha$ such that:

$$\bar{F}^\alpha = \frac{\langle \chi^\alpha F^\alpha \rangle}{\langle \chi^\alpha \rangle} = \frac{\langle \chi^\alpha F^\alpha \rangle}{\phi^\alpha}, \quad (20)$$

where

$$\phi^\alpha = \langle \chi^\alpha \rangle, \quad (21)$$

and also:

$$\bar{\bar{F}}^\alpha = \frac{\langle \rho^\alpha F^\alpha \chi^\alpha \rangle}{\langle \rho^\alpha \chi^\alpha \rangle} = \frac{\overline{\rho^\alpha F^\alpha}}{\overline{\rho^\alpha}}, \quad (22)$$

because, with each extensive microlevel quantity, it is quite natural to associate a physically meaningful mass-weighted average. Using the above definitions and decomposing f^α into its average $\bar{f}^\alpha$ and its associated space-time fluctuation f'^α , i.e.

$$f^\alpha = \bar{f}^\alpha + f'^\alpha, \quad (23)$$

it is straightforward to show that substitution for $f^\alpha, j^\alpha, r^\alpha$ from table 1 into equation (19) yields a set of conservation equations which are, as expected, (see Section (4)) formally identical to the two-fluid theory equations derived in the classical theory of fluid mixtures [16] [3]. These are:

$$\frac{\partial \rho^{*\alpha}}{\partial t} + \nabla \cdot (\rho^{*\alpha} \bar{\mathbf{V}}^\alpha) = \langle \dot{m}^\alpha \Delta^\Sigma \rangle, \quad (24)$$

$$\frac{\partial}{\partial t} (\rho^{*\alpha} \bar{\mathbf{V}}^\alpha) + \nabla \cdot (\rho^{*\alpha} \bar{\mathbf{V}}^\alpha \otimes \bar{\mathbf{V}}^\alpha - \underline{\underline{\sigma}}^{*\alpha}) = \rho^{*\alpha} \bar{\mathbf{F}}^\alpha + \langle \dot{p}^\alpha \Delta^\Sigma \rangle, \quad (25)$$

$$\frac{\partial}{\partial t} (\rho^{*\alpha} \bar{\mathbf{e}}^\alpha) + \nabla \cdot (\rho^{*\alpha} \bar{\mathbf{e}}^\alpha \bar{\mathbf{V}}^\alpha + \underline{\underline{q}}^{*\alpha} - \underline{\underline{\sigma}}^{*\alpha} \cdot \bar{\mathbf{V}}^\alpha) = \rho^{*\alpha} \bar{\mathbf{F}}^\alpha \cdot \bar{\mathbf{V}}^\alpha + \langle \dot{E}^\alpha \Delta^\Sigma \rangle, \quad (26)$$

where the partial densities $\rho^{*\alpha}$, stress-tensors $\underline{\underline{\sigma}}^{*\alpha}$ and heat fluxes $\underline{\underline{q}}^{*\alpha}$

are defined by:

$$\rho^{*\alpha} = \phi \frac{\alpha}{\rho}, \quad (27)$$

$$\underline{\sigma}^{*\alpha} = \phi \left(\underline{\sigma}^{\alpha} - \overline{\rho^{\alpha} \underline{v}'^{\alpha} \otimes \underline{v}'^{\alpha}} \right) \quad (28)$$

$$\underline{q}^{*\alpha} = \phi \left[\underline{q}^{\alpha} - \underline{\sigma}^{\alpha} \cdot \underline{v}'^{\alpha} + \overline{\rho^{\alpha} e'^{\alpha} \underline{v}'^{\alpha}} - \overline{\rho^{\alpha} (\underline{v}'^{\alpha} \otimes \underline{v}'^{\alpha}) \cdot \underline{v}^{\alpha}} \right]. \quad (29)$$

In view of the fact that the moment of momentum equation implies that $\underline{\sigma}^{\alpha}$ is symmetric, the set of differential equations (24), (25) and (26) should, in principle, provide the instantaneous and local values of the fundamental thermodynamic variables describing the medium, i.e. $\rho^{*\alpha} = \overline{\rho^{\alpha}}$, $\underline{e}^{*\alpha} = \overline{\underline{e}^{\alpha}}$, $\underline{v}^{*\alpha} = \overline{\underline{v}^{\alpha}}$, ($\alpha = 1, 2$), on the condition that they be supplemented with equations of state for each fictitious phase α , if they exist, constitutive laws for the partial stress tensors and heat fluxes as well as for the now volumetrically distributed interfacial interaction terms $\langle \underline{m}^{\alpha} \Delta \Sigma \rangle$, $\langle \underline{p}^{\alpha} \Delta \Sigma \rangle$, $\langle \underline{E}^{\alpha} \Delta \Sigma \rangle$, and boundary conditions. Moreover, the dissipation inequality expressing the second principle of thermodynamics and placing a constraint on the behavior of the medium under consideration should be formulated.

Before even trying to discuss constitutive theories for such media, it is worthwhile noting that when we go over from the usually intractable microlevel description to the macrolevel two-fluid equations, a very large amount of information is lost in the process of averaging, even though no essential physical simplification has been made up till now. But this does not jeopardize the usefulness of such a description all the more since the

mathematical difficulty of the problem greatly decreases. Although this is the only completely formal macrolevel approach to the problem, it turns out, however, that the two-fluid theory is still extremely complex since it involves 10 coupled scalar partial differential equations with the scalar unknowns $\rho^{*\alpha}$, $\bar{e}^\alpha$, $\bar{v}^\alpha$ ($\alpha = 1, 2$) and the need for 28 scalar closure equations for $q^{*\alpha}$, $q^{*\alpha}$, $\langle \dot{m}^\alpha \Delta^\Sigma \rangle$, $\langle \dot{p}^\alpha \Delta^\Sigma \rangle$, $\langle \dot{E}^\alpha \Delta^\Sigma \rangle$. This is why, just like in the classical theory of mixtures, it is worthwhile introducing the so-called diffusion theory, where the two coexisting phases are viewed as one single medium considered as a whole, in which diffusion occurs [17].

When this is done, more information is lost, in particular concerning details of the individual behavior of each phase and it is expected that the theory will only hold when some physical conditions are satisfied by both the medium and the process which is taking place (see section 9.1).

7.2 DIFFUSION THEORY

Whether it is applicable or not (see section 9.1), the corresponding equations may be derived very easily from (24), (25) and (26) by mere addition over the phase index α , just retaining one of the two partial mass conservation equations.

Defining, as is done in the classical theory of mixtures, the average density,

$$\rho^m = \sum_{\alpha=1}^2 \rho^{*\alpha} \quad (30)$$

the average specific density of any extensive quantity F by

$$\rho^m f^m = \sum_{\alpha=1}^2 \rho^{*\alpha} f^\alpha, \quad (31)$$

the overall stress-tensor and heat flux of the mixture by

$$\underline{\sigma}^m = \sum_{\alpha=1}^2 [\underline{\sigma}^{*\alpha} - \rho^{*\alpha}(\underline{\bar{V}}^\alpha - \underline{V}^m) \otimes (\underline{\bar{V}}^\alpha - \underline{V}^m)], \quad (32)$$

and

$$\underline{q}^m = \sum_{\alpha=1}^2 \{ \underline{q}^{*\alpha} - \underline{\sigma}^{*\alpha}(\underline{\bar{V}}^\alpha - \underline{V}^m) + \rho^{*\alpha} \underline{e}^\alpha (\underline{\bar{V}}^\alpha - \underline{V}^m) - \underline{V}^m [\rho^{*\alpha}(\underline{\bar{V}}^\alpha - \underline{V}^m) \otimes (\underline{\bar{V}}^\alpha - \underline{V}^m)] \}, \quad (33)$$

and the mass diffusion flux:

$$\underline{J}^\alpha = \rho^{*\alpha}(\underline{\bar{V}}^\alpha - \underline{V}^m), \quad (34)$$

we obtain finally the following familiar equations of classical diffusion theory, which, except for the last one, are identical to those corresponding to a single fluid

$$\frac{\partial \rho^m}{\partial t} + \nabla \cdot (\rho^m \underline{V}^m) = 0, \quad (35)$$

$$\frac{\partial}{\partial t} (\rho^m \underline{V}^m) + \nabla \cdot (\rho^m \underline{V}^m \otimes \underline{V}^m - \underline{\sigma}^m) = \rho^m \underline{F}^m, \quad (36)$$

$$\frac{\partial}{\partial t} (\rho^m \underline{e}^m) + \nabla \cdot (\rho^m \underline{e}^m \underline{V}^m - \underline{\sigma}^m \cdot \underline{V}^m + \underline{q}^m) = \rho^m \underline{F}^m \cdot \underline{V}^m + \sum_{\alpha=1}^2 \underline{\bar{F}}^\alpha \cdot \underline{J}^\alpha, \quad (37)$$

$$\frac{\partial \rho^{*\alpha}}{\partial t} + \nabla \cdot (\rho^{*\alpha} \underline{V}^m + \underline{J}^\alpha) = \langle \dot{m}^\alpha \Delta^\Sigma \rangle, \quad \alpha = 1 \text{ or } 2. \quad (38)$$

Equations (35), (36), (37), (38) constitute a set of 6 scalar equations which can, in principle, be solved for the six fundamental thermodynamic variables describing the mixture, namely, $\rho^m, \underline{e}^m, \underline{V}^m$ and $\rho^{*\alpha}$, on the condition that thirteen scalar closure equations are found for $\underline{\sigma}^m, \underline{q}^m, \underline{J}^\alpha$ and $\langle \dot{m}^\alpha \Delta^\Sigma \rangle$, $\alpha = 1 \text{ or } 2$.

7.3 GENERAL REQUIREMENTS IMPOSED ON THE MACROLEVEL CLOSURE EQUATIONS FOR TWO-PHASE MEDIA

At this point, if we deliberately ignore the fact that an experimentalist does not measure such quantities as e^m or $\bar{e}^\alpha$ and usually prefers to determine intensive quantities such as pressure and temperature, whatever their meaning in this context, the whole foregoing derivation may be considered as absolutely general and physically rigorous. The only requirement, from a purely numerical point of view, is to formulate the appropriate closure equations, as stated above. This may be achieved in many different ways, for example, by correlating a large number of experimental data. In order to be more specific, let us restrict the discussion to cases where the diffusion theory is preferable. Whatever method is chosen to obtain the closure equations, they will all be given by space-time functionals expressing the unknown quantities $\underline{\sigma}^m$, $\underline{q}^m$, $\underline{J}^\alpha$ and $\langle \dot{m}^\alpha \Delta^\Sigma \rangle$ in terms of the basic independent variables, ρ^m , e^m , $\underline{v}^m$, $\rho^{*\alpha}$, i.e.

$$\begin{pmatrix} \underline{\sigma}^m \\ \underline{q}^m \\ \underline{J}^\alpha \\ \langle \dot{m}^\alpha \Delta^\Sigma \rangle \end{pmatrix} = \mathcal{F}_{\substack{x' \in D \\ -\infty < t' \leq t}} [\rho^m(\underline{x}', t'), e^m(\underline{x}', t'), \underline{v}^m(\underline{x}', t'), \rho^{*\alpha}(\underline{x}', t')] \quad (39)$$

which should satisfy a few important conditions:

- (a) They should be as simple as possible in order to avoid mathematical difficulties.
- (b) They should be compatible with the principles of galilean invariance and of material symmetry. The latter, for example, requires that the macrolevel constitutive laws be isotropic, if the material is isotropic on the macrolevel. These principles place strong restrictions on the form of functionals $\mathcal{F}$.

(c) $\mathcal{F}$ should not violate the second law of thermodynamics.

This last question brings us back to the problem of the thermodynamic description of two-phase flows on the macrolevel.

8. MACROLEVEL THERMODYNAMICS OF TWO-PHASE FLOWS

Obviously, one would like to be able to define on the macrolevel, as is done on the microlevel, a physically meaningful internal energy, to write an equation of state as well as a Gibbs equation for each continuum and, finally, to formulate the proper dissipation inequalities resulting from the second law of thermodynamics.

8.1 INTERNAL ENERGY. INTERNAL ENERGY EQUATION

An equation of state for each continuum can only be written if its specific internal energy is introduced. From the very definition of e^α , we have:

$$\overline{e^\alpha} = \overline{u^\alpha} + \frac{1}{2} \overline{(v'^\alpha)^2} + \frac{1}{2} \overline{(v^\alpha)^2}. \quad (40)$$

Therefore, if we want to consider, as is usually done in the continuum theory of mixtures, the total energy as the sum of the specific kinetic energy associated with the velocity $\overline{v^\alpha}$ of phase α and of the thermodynamic internal energy, the latter should obviously be written as:

$$u^{*\alpha} = \overline{u^\alpha} + \frac{1}{2} \overline{(v'^\alpha)^2}, \quad (41)$$

which means that a "turbulent" kinetic energy term has to be included in this apparent internal energy $\bar{u}^{\alpha*}$, besides the term which is usually expected, i.e. $\bar{u}^{\alpha}$. This term seems to have been systematically overlooked in papers devoted to continuum theory of mixtures [2][12], although it is the only interpretation of the average internal energy which makes the resulting internal energy equation formally identical to that of the two-fluid theory of classical mixtures, i.e.

$$\frac{\partial}{\partial t} (\rho^{\alpha*} \bar{u}^{\alpha*}) + \nabla \cdot (\rho^{\alpha*} \bar{u}^{\alpha*} \bar{V}^{\alpha} + \bar{q}^{\alpha*}) = \bar{\sigma}^{\alpha*} : \nabla \bar{V}^{\alpha} + \left[\frac{(\bar{V}^{\alpha})^2}{2} \langle \dot{m}^{\alpha} \Delta^{\Sigma} \rangle - \bar{V}^{\alpha} \langle \dot{p}^{\alpha} \Delta^{\Sigma} \rangle + \langle \dot{E}^{\alpha} \Delta^{\Sigma} \rangle \right]. \quad (42)$$

A similar problem arises in conjunction with the diffusion theory where:

$$\rho^m e^m = \sum_{\alpha=1}^2 \rho^{\alpha*} \bar{e}^{\alpha} = \frac{1}{2} \rho^m (\bar{V}^m)^2 + \sum_{\alpha=1}^2 \rho^{\alpha*} \left[\bar{u}^{\alpha*} + \frac{1}{2} (\bar{V}^{\alpha} - \bar{V}^m)^2 \right]. \quad (43)$$

Here again the formal similarity of the internal energy equation with that derived in the diffusion theory of classical mixtures requires that it be defined by:

$$\rho^m u^m = \sum_{\alpha=1}^2 \rho^{\alpha*} \left[\bar{u}^{\alpha*} + \frac{1}{2} (\bar{V}^{\alpha} - \bar{V}^m)^2 \right], \quad (44)$$

although there is no obvious physical justification for this choice.

Accordingly, the equation governing the behavior of u^m may be written:

$$\frac{\partial}{\partial t} (\rho^m u^m) + \nabla \cdot (\rho^m u^m \bar{V}^m + \bar{q}^m) = \bar{\sigma}^m : \nabla \bar{V}^m + \sum_{\alpha=1}^2 \bar{F}^{\alpha} \cdot \bar{J}^{\alpha} \quad (45)$$

Of course, the difficulties first mentioned disappear if the "turbulent" kinetic energy and the diffusion kinetic energy are negligible with respect to the other contributions as is the case in single phase compressible turbulent flows or in strongly interacting mixtures (see section 9.1).

8.2 EXISTENCE OF AN EQUATION OF STATE AND OF A GIBBS EQUATION FOR EACH CONTINUUM CONSIDERED SEPARATELY OR FOR THE MIXTURE AS A WHOLE

In complete thermodynamic equilibrium, we would have:

$$u^{\alpha*} = u^{\alpha}; \quad \bar{s}^{\alpha} = s^{\alpha}; \quad \bar{\rho}^{\alpha} = \rho^{\alpha}. \quad (46)$$

Accordingly, a very simple fundamental equation of state:

$$\bar{s}^{\alpha} = \bar{s}^{\alpha}(u^{\alpha*}, \bar{\rho}^{\alpha}), \quad (47)$$

identical to that of the original microlevel phase α , would hold for the equivalent continuum α . It is very unlikely, however, that equation (47) may still rigorously be applied even locally and instantaneously during any real process, due to interfacial transfers and statistical fluctuations. Whether a fundamental equation of state of the form (47) with $u^{\alpha*}$ and $\bar{\rho}^{\alpha}$ as independent variables, as postulated by workers in the field of fluid two-phase flows, or with $\bar{u}^{\alpha}$ and $\rho^{*\alpha}$, as assumed by specialists of continuum theory of mixtures, makes sense at least approximately, depends on the specific problem which is investigated, on the accuracy which is needed, and on the width of the space-time window. A direct consequence of this observation is that the existence of some simple Gibbs equation for each continuum becomes questionable. Indeed it has been shown both for fluid two-phase flows [6] and also in the much simpler case of single-

phase compressible-gas turbulent flows [18] that such an equation does not hold. Accordingly such quantities as $\bar{p}^\alpha$ and $\bar{T}^\alpha$ which are the conditional averages of the microlevel thermodynamic pressures and temperatures cannot be considered in all generality as macrolevel thermodynamic intensive parameters conjugate of some average deformation variable.

8.3 DISSIPATION INEQUALITIES

In view of equations (4)-(8), (19) and (23), the equation governing the evolution of the average entropy of each phase may be written as:

$$\frac{\partial}{\partial t} (\rho^{\alpha} \bar{s}^{\alpha}) + \nabla \cdot \{ \rho^{\alpha} \bar{s}^{\alpha} \underline{V}^{\alpha} + \phi^{\alpha} (\underline{q}^{\alpha} / T^{\alpha}) + \rho^{\alpha} \overline{s'^{\alpha} \underline{V}'^{\alpha}} \} = \phi^{\alpha} \bar{\dot{\theta}}^{\alpha} + \langle \dot{s}^{\alpha} \Delta^{\Sigma} \rangle \quad \alpha = 1, 2, \quad (48)$$

which yields the following macrolevel dissipation inequality for each continuum α :

$$\frac{\partial}{\partial t} (\rho^{\alpha} \bar{s}^{\alpha}) + \nabla \cdot \{ \rho^{\alpha} \bar{s}^{\alpha} \underline{V}^{\alpha} + \phi^{\alpha} (\underline{q}^{\alpha} / T^{\alpha}) + \rho^{\alpha} \overline{s'^{\alpha} \underline{V}'^{\alpha}} \} - \phi^{\alpha} \bar{\dot{\theta}}^{\alpha} - \langle \dot{s}^{\alpha} \Delta^{\Sigma} \rangle \geq 0, \quad \alpha = 1, 2, \quad (49)$$

with

$$\sum_{\alpha=1}^2 \langle \dot{s}^{\alpha} \Delta^{\Sigma} \rangle \geq 0. \quad (50)$$

For the mixture as a whole, summation of the above equations over the phase index α gives

$$\begin{aligned} \frac{\partial}{\partial t} (\rho^m \bar{s}^m) + \nabla \cdot (\rho^m \bar{s}^m \underline{V}^m + \sum_{\alpha=1}^2 [\phi^{\alpha} (\underline{q}^{\alpha} / T^{\alpha}) + \\ + \overline{s^{\alpha} J^{\alpha}} + \rho^{\alpha} \overline{s'^{\alpha} \underline{V}'^{\alpha}}]) = \sum_{\alpha=1}^2 \phi^{\alpha} \bar{\dot{\theta}}^{\alpha} + \langle \dot{s} \Delta^{\Sigma} \rangle, \end{aligned} \quad (51)$$

or

$$\frac{\partial}{\partial t} (\rho^m s^m) + \nabla \cdot \left\{ \rho^m s^m \mathbf{V}^m + \sum_{\alpha=1}^2 [\phi^\alpha (\mathbf{q}^\alpha / T^\alpha) + \bar{s}^\alpha \mathbf{J}^\alpha + \rho^{*\alpha} \bar{s}^\alpha \mathbf{V}^\alpha] \right\} \geq 0. \quad (52)$$

A number of useful conclusions may be drawn from the above results:

(a) There are indeed several dissipation inequalities which should be satisfied in the two-fluid theory ([2], [12]) and not just one for the whole mixture as is assumed most of the time by specialists of the continuum theory of mixtures.

(b) The dissipation inequalities are not identical to the ones usually postulated, due to interfacial transfers and statistical fluctuations.

(c) They are not identical with the ones which may be derived from the averaged conservation equations (24) (25) (26), and some average fundamental equation of the form (47), which is not surprising in view of the conclusions of the foregoing section.

(d) The rate of entropy production terms does not appear as a mere bilinear form from which thermodynamic flows and forces can be easily identified.

9. THE CLOSURE PROBLEM

For the sake of simplicity and clarity, we shall address ourselves exclusively to cases where both phases are fluids. As has already been emphasized in section 4, two different approaches to the problem of closure can be taken, one which deliberately ignores what happens on the microlevel (continuum theory of mixtures), and one which calculates the different statistical contributions to the macrolevel quantities. The latter may be

conveniently called after HEWITT "flow-pattern modeling". But before any attempt is made at closing the set of conservation equations, listed previously, one should decide which of the two theories (two fluid or diffusion theory) is the most appropriate for a particular application.

9.1 CHOICE OF THE PROPER THEORY-TERMINOLOGY

The particular choice to be made depends on:

- (a) The applicability of each of them to the physical situation under consideration.
 - (b) The amount of information which is needed in practice as well as the required degree of precision.
 - (c) The number of closure equations which are necessary and, accordingly, the mathematical complexity of the resulting equations.
- For practical purposes, both (b) and (c) suggest that the diffusion theory be selected. Indeed, for obvious reasons, it is the diffusion theory equations, or rather simplified versions of them, which are favored most of the time. But, as indicated in (a), its physical relevance should be discussed. As a matter of fact, it is known from the classical theory of fluid mixtures [3-4] that the diffusion theory, although it applies to the great majority of cases and is very accurate, requires that the two fluids be in thermal equilibrium and that the diffusion velocities be sufficiently small for the kinetic energy of diffusion to be negligible [3-4]. Here at the macrolevel, where two coexisting equivalent phases are considered as a result of the averaging process, it is not quite clear that these two

requirements are always met. Whether the average temperatures $\bar{T}^\alpha$ of the two fictitious continua may be assumed equal and their velocities $\bar{V}^\alpha$ to be close to one another should be examined in each specific case. This will depend on the width of the space-time windows, on the external actions imposed on the medium and on the intensity of the momentum and energy transfer between the two original phases on the microlevel, which, in particular, implies that the time scales characterizing such interactions be small compared to the macrolevel time scale T associated with the phenomenon under consideration. Such a condition is not likely to be satisfied when high-frequency instabilities exist or high-frequency waves are propagated. Local average thermal equilibrium ($\bar{T}^1 = \bar{T}^2$) is certainly achieved when the process undergone by two different substances is isothermal. Also, when two phases of the same pure substance flow across a pipe, the preceding condition is probably satisfied with a good approximation, if the classical thermodynamic phase-equilibrium conditions hold on the microlevel and if area-averaged quantities are introduced. On the other hand, such a theory is expected to break down when, for example, a hot fluid flows inside a cold, adiabatic porous solid matrix. In order to avoid ambiguities, it is useful at this point to stress that the so-called U.V.U.T (unequal velocities-unequal temperatures), E.V.U.T. (equal velocities - unequal temperatures), U.V.E.T. (unequal velocities - equal temperatures), E.V.E.T. (equal velocities - equal temperatures) models, as described by HEWITT [19], either coincide or are simplified versions of the two-fluid (U.V.U.T.) diffusion (U.V.,E.T.) theories.

9.2 CLOSURE EQUATIONS DERIVED FROM THE CONTINUUM THEORY OF MIXTURES

The continuum theory of mixtures, originally treated by classical thermodynamics [17], and meant for classical fluid mixtures, where diffusion occurs on the molecular level, has also been intensively studied and extended in recent years by a number of authors in the context of rational thermodynamics (see Atkin's review [2]). Both methods claim to treat very general cases and to provide a systematic procedure for the derivation of so-called constitutive laws.

The first one, i.e. classical thermodynamics of irreversible processes, whose validity and physical basis are unquestionable on the microlevel (ℓ_1, t_1) , might well be misleading on the macroscale (ℓ_2, t_2) . As is well known, one of the crucial assumptions is that of the existence of a fundamental equation of state and accordingly of a Gibbs equation, which, in the case of the diffusion theory, would express the average entropy s^m as a function of $\rho^m, u^m, \rho^{*\alpha}$, and possibly of some appropriate internal variables accounting for dissipation mechanisms. Moreover, the resulting entropy equation should exhibit the familiar bilinear form in terms of fluxes and forces. As seen in section 8, the completion of such a program is, in the majority of cases, highly unlikely, unless drastic simplifying assumptions [20] are made which would ensure that some kind of local thermodynamic equilibrium prevails on the macrolevel: in particular, the contribution of the statistical fluctuations should all be negligible or the medium should be space-time statistically stationary. Although this method is successful in the derivation of Darcy's

law for the isothermal creeping flow of a fluid inside a rigid porous matrix, it breaks down even when time-averaged single-phase flows are considered: as a matter of fact, although it is usually assumed that an average equation of state $\bar{p} = f(\bar{\rho}, \bar{T})$ holds with a good enough approximation for compressible-gas turbulent flows, it is known that satisfactory closure equations for the Reynolds stress tensor cannot be derived with the aid of classical thermodynamics of irreversible processes.

The procedure of rational thermodynamics, on the other hand, is sufficiently arbitrary and general to account for nearly any kind of situation. But it hinges heavily on a dissipation inequality (or dissipation inequalities), which in general case is questionable, and also requires the proper choice of independent arguments for the functional constitutive laws to be made a priori.

From the foregoing discussion it is obvious that we should take a more pragmatic attitude and give up any hope, except perhaps under exceptional circumstances, of discovering a systematic method to provide general forms of closure equations valid for any type of fluid two-phase flows.

9.3 CLOSURE EQUATIONS DERIVED FROM FLOW PATTERN MODELING

Here the closure laws are inspired by an inspection of the actual patterns (bubbly, annular, etc.) of two-phase flows and a careful study of their microlevel physics. In the diffusion theory, for example, it is useful to remember that all unknown quantities are, generally speaking, the sum of three different contributions (see section 7.2). The first one is just the average of corresponding microlevel quantities for which constitutive laws are known. The second one is a diffusion term associated with the difference

between the barycentric velocities of both equivalent continua. The third one is a Reynolds-type "turbulent" contribution.

It is, of course, impossible to indicate any general guideline: each specific case has to be analyzed separately. It is essential, however, to try:

(a) to evaluate the orders of magnitude of each term and see which are negligible

(b) to find out appropriate closure relations for the remaining terms either by correlating experimental results obtained with flows of a given class - in particular exhibiting the same type of flow pattern - or by studying analytically highly simplified physical models of the flow.

This is the motivation for elaborate experiments that have been made recently in two-phase flows in order to investigate not just the overall behavior of the flow but also its detailed behavior.

10. TWO-PHASE FLOW IN A FIXED, RIGID ADIABATIC, IMPERMEABLE AND CHEMICALLY INERT PIPE

For the purpose of illustrating the foregoing considerations, it is useful to give a derivation of the equations which govern two-phase flow in a pipe. The following notation will be used:

Ox_3 : axis of the pipe

S : wall of the pipe

$A(x_3)$: area of the cross-section of the pipe

I : interface between the two phases

S^α : portion of the wall which is in contact with medium α .

$$\frac{\partial}{\partial t} (\phi^{\alpha} \bar{\rho}^{\alpha} \bar{f}^{\alpha}) + \frac{\partial}{\partial x_3} \left\{ \left[\phi^{\alpha} \bar{\rho}^{\alpha} \bar{f}^{\alpha} \bar{V}^{\alpha} + \phi^{\alpha} (\bar{j}^{\alpha} + \bar{j}_T^{\alpha}) \right] \cdot \underline{e}_3 \right\} = \phi^{\alpha} \bar{\rho}^{\alpha} \bar{r}^{\alpha} + \langle \dot{f}^{\alpha} \Delta^I \rangle + \langle \dot{f}^{\alpha} \Delta^S \rangle, \quad (56)$$

where $\underline{e}_3$ stands for the unit vector of the axis Ox_3 . In the case of a single-phase flow, they reduce to

$$\frac{\partial}{\partial t} (\bar{\rho} \bar{f}) + \frac{\partial}{\partial x_3} \left\{ \left[\bar{\rho} \bar{f} \bar{V} + \bar{j} + \bar{j}_T \right] \cdot \underline{e}_3 \right\} = \bar{\rho} \bar{r} + \langle \dot{f} \Delta^S \rangle. \quad (57)$$

The width of the time window should be chosen according to the prescriptions given earlier. On the other hand, the width of the x_3 -window [support of $w''(x_3)$] can be made as small as required, the limiting case of "mathematical" area averaging being recovered when

$$w''(x_3) = \delta(x_3).$$

Using Table 1, it is now quite straightforward to derive the required conservation equations. Although it is by no means necessary, we shall assume from now on that the wall is fixed, rigid, adiabatic, impermeable and chemically inert - in which case

$$\left. \begin{aligned} \Delta S^{\alpha} &= \delta S^{\alpha} \\ \dot{m}^{\alpha} &= 0 \\ \dot{E}^{\alpha} &= 0 \\ \underline{p}^{\alpha} &= \underline{\sigma}^{\alpha} \cdot \underline{n}^{\alpha} \end{aligned} \right\} \text{ on the wall}$$

Two-Fluid Model ($\alpha = 1, 2$)

Mass Conservation:

$$\frac{\partial}{\partial t} (\phi^{\alpha} A \bar{\rho}^{\alpha}) + \frac{\partial}{\partial x_3} (\phi^{\alpha} A \bar{\rho}^{\alpha} \bar{V}_3^{\alpha}) = \dot{m}^{\alpha} \Delta^I * w. \quad (58)$$

Momentum Conservation:

$$\begin{aligned} \frac{\partial}{\partial t} (\phi^{\alpha} A \bar{\rho}^{\alpha} \bar{V}_i^{\alpha}) + \frac{\partial}{\partial x_3} \left[\phi^{\alpha} A (\bar{\rho}^{\alpha} \bar{V}_i^{\alpha} \bar{V}_3^{\alpha} - \bar{\sigma}_{i3}^{\alpha} - \sigma_{T_{i3}}^{\alpha}) \right] &= \phi^{\alpha} \bar{\rho}^{\alpha} A F_i^{\alpha} + \\ &+ \dot{p}^{\alpha} \Delta^I * w + \underline{\sigma}^{\alpha} \cdot \underline{n}^{\alpha} \delta S^{\alpha} * w. \end{aligned} \quad (59)$$

Total energy or enthalpy:

$$\frac{\partial}{\partial t} (\phi^\alpha A \bar{\rho}^\alpha \bar{e}^\alpha) + \frac{\partial}{\partial x_3} \left[\phi^\alpha A (\bar{\rho}^\alpha \bar{e}^\alpha \bar{V}_3^\alpha + \bar{q}_3^\alpha - \bar{\sigma}_{3j}^\alpha \bar{V}_j^\alpha + \bar{q}_T^\alpha) \right] = \phi^\alpha A \bar{\rho}^\alpha \bar{F}^\alpha \bar{V}^\alpha + \bar{E}^\alpha \Delta I_{*w} \quad (60)$$

Defining the specific total enthalpy of fluid α and $\underline{Q}_T^\alpha$ by

$$H^\alpha = e^\alpha + p^\alpha / \rho^\alpha, \quad \underline{Q}_T^\alpha = -\bar{\tau}^\alpha \underline{V}'^\alpha + \bar{\rho}^\alpha \overline{H'^\alpha \underline{V}'^\alpha}, \quad (61)$$

we may rewrite the above equation in the form

$$\frac{\partial}{\partial t} (\phi^\alpha A \bar{\rho}^\alpha \bar{H}^\alpha) + \frac{\partial}{\partial x_3} \left[\phi^\alpha A (\bar{\rho}^\alpha \bar{H}^\alpha \bar{V}_3^\alpha + \bar{q}_3^\alpha - \bar{\tau}_{3j}^\alpha \bar{V}_j^\alpha + \bar{Q}_T^\alpha) \right] - \frac{\partial}{\partial t} (\phi^\alpha A \bar{p}^\alpha) = \phi^\alpha A \bar{\rho}^\alpha \bar{F}^\alpha \bar{V}^\alpha + \bar{E}^\alpha \Delta I_{*w} \quad (62)$$

Diffusion Model (m - referred to barycentric velocity)

Mass Conservation:

$$\frac{\partial}{\partial t} (A \rho^m) + \frac{\partial}{\partial x_3} (\rho^m A V_3^m) = 0 \quad (63)$$

Partial-Mass Conservation:

$$\frac{\partial}{\partial t} (A \phi^{\alpha-\alpha} \rho^{\alpha-\alpha}) + \frac{\partial}{\partial x_3} \left[A (\phi^{\alpha-\alpha} \rho^{\alpha-\alpha} V_3^m + J_3^\alpha) \right] = \dot{m}^\alpha \Delta I_{*w} \quad (64)$$

Momentum Conservation:

$$\frac{\partial}{\partial t} (A \rho^m V_i^m) + \frac{\partial}{\partial x_3} \left[A \left(\rho^m V_i^m V_3^m - \sum_{\alpha=1}^2 \phi^\alpha \left\{ \bar{\sigma}_{i3}^\alpha + \sigma_{Ti3}^\alpha + \sigma_{Di3}^\alpha \right\} \right) \right] = \sum_{\alpha=1}^2 A \phi^{\alpha-\alpha} F_i^\alpha + \sum_{\alpha=1}^2 \left(\underline{\underline{\sigma}}^\alpha \cdot \underline{\underline{n}}^\alpha \delta S^\alpha \right)_{*w} \quad (65)$$

where:

$$\underline{\underline{\sigma}}_T^\alpha = -\bar{\rho}^\alpha \underline{\underline{V}}'^\alpha \otimes \underline{\underline{V}}'^\alpha$$

and

$$\underline{\sigma}_D^\alpha = - \bar{\rho}^\alpha \bar{V}^\alpha \otimes (\bar{V}^\alpha - \underline{V}^m) .$$

Note that

$$\sum_{\alpha=1}^2 \phi^\alpha \underline{\sigma}_D^\alpha = - \sum_{\alpha=1}^2 \bar{\rho}^\alpha (\bar{V}^\alpha - \underline{V}^m) \otimes (\bar{V}^\alpha - \underline{V}^m) .$$

Energy Conservation:

$$\begin{aligned} \frac{\partial}{\partial t} (A \rho^m e^m) + \frac{\partial}{\partial x_3} \left\{ A \left[\rho^m e^m v_3^m + \sum_{\alpha=1}^2 \phi^\alpha \left[\bar{q}_3^\alpha - \bar{\sigma}_{3j}^\alpha \bar{V}_j^\alpha + q_{T3}^\alpha + \right. \right. \right. \\ \left. \left. \left. + \bar{\rho}^\alpha \bar{e}^\alpha (\bar{V}_3^\alpha - v_3^m) \right] \right] \right\} = \rho^m \underline{F}^m \cdot \underline{V}^m . \end{aligned} \quad (66)$$

Pipe-Flow Approximation

Up to this point no really restrictive simplification has been made. It is useful, however, to remember that the above, averaged equations make sense only when we deal with a real pipe whose length L is much larger than some characteristic transverse dimension. In other words, if

$$\frac{L}{A} \frac{dA}{dx_3} \ll 1 ,$$

we may safely assume that

$$\bar{V}_1^\alpha, \bar{V}_2^\alpha \ll \bar{V}_3^\alpha; v_1^m, v_2^m \ll v_3^m .$$

With these assumptions, the averaged equations may be further simplified: in particular we may omit the equations of momentum in directions x_1 and x_2 . In the important case of the diffusion model, one obtains the following set of equations:

$$\frac{\partial}{\partial t} (A \rho^m) + \frac{\partial}{\partial x_3} (\rho^m A V_3^m) = 0 ; \quad (67)$$

$$\frac{\partial}{\partial t} (A \phi_{\rho}^{\alpha-\alpha}) + \frac{\partial}{\partial x_3} [A \phi_{\rho}^{\alpha-\alpha} V_3^m + A \phi_{\rho}^{\alpha-\alpha} (\overline{V_3^\alpha} - V_3^m)] = \dot{M}^\alpha ; \quad (68)$$

$$\begin{aligned} \frac{\partial}{\partial t} (A \rho^m V_3^m) + \frac{\partial}{\partial x_3} [A (\rho^m V_3^m + \sum_{\alpha=1}^2 \phi_{\rho}^{\alpha-\alpha}) - \sum_{\alpha=1}^2 A \phi_{\tau}^{\alpha-\alpha} \\ + \sum_{\alpha=1}^2 A \phi_{\rho}^{\alpha-\alpha} (\overline{V_3'^\alpha})^2 + A \sum_{\alpha=1}^2 \phi_{\rho}^{\alpha-\alpha} (\overline{V_3^\alpha} - V_3^m)^2] = -\tau_w C - \rho^m A g. \end{aligned} \quad (69)$$

$$\begin{aligned} \frac{\partial}{\partial t} \{A \rho^m [h^m + \frac{1}{2} (V_3^m)^2]\} + \frac{\partial}{\partial x_3} \{A [h^m + \frac{1}{2} (V_3^m)^2 + g x_3] \rho^m V_3^m \\ + A \sum_{\alpha=1}^2 \phi_{\rho}^{\alpha-\alpha} (\overline{q_3^\alpha} - \overline{\tau_{33}^\alpha} \overline{V_3^\alpha}) + A \sum_{\alpha=1}^2 \phi_{\rho}^{\alpha-\alpha} [-(\overline{\tau_{33}^\alpha} \cdot \overline{V_3'^\alpha})_3 + \overline{\rho^{\alpha H} V_3'^\alpha}] \\ + A \sum_{\alpha=1}^2 \phi_{\rho}^{\alpha-\alpha} (\overline{V_3^\alpha} - V_3^m)\} = 0, \end{aligned} \quad (70)$$

where

$$\rho^m h^m = \sum_{\alpha=1}^2 \phi_{\rho}^{\alpha-\alpha} + \frac{1}{2} \sum_{\alpha=1}^2 \phi_{\rho}^{\alpha-\alpha} (\overline{V_3'^\alpha})^2 + \frac{1}{2} \sum_{\alpha=1}^2 \phi_{\rho}^{\alpha-\alpha} (\overline{V_3^\alpha} - V_3^m)^2. \quad (71)$$

Assuming further that.

- (a) the mean flow is time independent
- (b) the viscous stress tensor, the heat flux and the turbulent terms are negligible, and disregarding higher-order terms in $(\overline{V_3^\alpha} - V_3^m)$, which is consistent with the use of the diffusion model where the slip velocity is expected to be small, we obtain

$$\frac{\partial}{\partial x_3} (\rho^m A V_3^m) = 0 ; \quad (72)$$

$$\frac{\partial}{\partial x_3} [A(\phi^{\alpha-\alpha} \rho^{\alpha} V_3^m) + A\phi^{\alpha-\alpha} \bar{\rho}^{\alpha} (V_3^{\alpha} - V_3^m)] = \dot{M}^{\alpha} ; \quad (73)$$

$$\frac{\partial}{\partial x_3} \{A[\rho_m (V_3^m)^2 + p^m]\} = -\rho^m A g - \tau_w C ; \quad (74)$$

$$\frac{\partial}{\partial x_3} \{A\rho_m V_3^m [h^m + \frac{1}{2} (V_3^m)^2 + g x_3]\} + A \sum_{\alpha=1}^2 \phi^{\alpha-\alpha} \bar{h}^{\alpha} (V_3^{\alpha} - V_3^m) = 0. \quad (75)$$

It is worthwhile noting that the last term of the last equation may also be written:

$$A \sum_{\alpha=1}^2 \phi^{\alpha-\alpha} \bar{h}^{\alpha} (V_3^{\alpha} - V_3^m) = \frac{\phi^1 \phi^2 \bar{\rho}^1 \bar{\rho}^2}{\rho_m} (\bar{h}^1 - \bar{h}^2) (\bar{V}_3^1 - \bar{V}_3^2) \quad (76)$$

where $\bar{h}^1 - \bar{h}^2$ is related to the latent heat of vaporization in the case of a liquid-vapor flow.

If the so-called homogeneous model is considered for a liquid-vapor flow, one introduces even more drastic simplifications since it is assumed that there is no slip, no pressure difference between the phases, no departure from thermodynamic equilibrium in a given cross-section. Therefore

$$\rho^{\alpha} = \bar{\rho}^{\alpha} = \rho_{\text{sat.}}^{\alpha} (p^m) , \quad (77)$$

$$h^{\alpha} = \bar{h}^{\alpha} = h_{\text{sat.}}^{\alpha} (p^m) , \quad (78)$$

which, in turn, means that ρ^m and h^m are known functions of p^m and ϕ^{α} , or

$$\rho^m = \rho^m(p^m, \phi^{\alpha}) , \quad (79)$$

$$h^m = h^m(p^m, \phi^{\alpha}) . \quad (80)$$

Along with

$$\frac{\partial}{\partial x_3} (\rho^m A V_3^m) = 0, \quad (81)$$

$$\frac{\partial}{\partial x_3} \{A[\rho^m (V_3^m)^2 + p^m]\} = -\rho^m A g - \tau_w C, \quad (82)$$

$$\frac{\partial}{\partial x_3} \{A \rho^m V_3^m [h^m + \frac{1}{2} (V_3^m)^2 + g x_3]\} = 0, \quad (83)$$

they constitute a system of 5 scalar equations for the five unknowns p^m , ρ^m , h^m , V_3^m and ϕ^α , if τ_w is known, i.e., if a closure equation is found for τ_w as a functional of h^m , ϕ^α , V_3^m . If it is required, the volumetric rate of mass of vapor production may be simply calculated from the partial mass conservation equation

$$\dot{M}^\alpha = \frac{\partial}{\partial x_3} [A(\phi^\alpha \rho^\alpha V_3^\alpha)]. \quad (84)$$

In the case of the so-called generalized homogeneous model for a liquid vapor flow, the no-slip condition is relaxed. The resulting conservation equations are equations (72), (73), (74) and (75), while thermal equilibrium still requires that (79) and (80) be satisfied. From equations (72), (74), (75), (79) and (80), the unknowns $p^m, \rho^m, h^m, V_3^m, \phi^\alpha$ may be determined if both τ_w and the slip velocity:

$$S = \bar{V}_3^\alpha - V_3^m \quad (85)$$

are specified in terms of h^m, ϕ and V_3^m by closure relations. Just like in the previous case, equation (73) merely gives the value of $\dot{M}^\alpha$.

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