FIRST ORDER PERTURBATION EFFECTS IN IRON-DOMINATED TWO-DIMENSIONAL SYMMETRICAL MULTIPOLES*

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Abstract

The effects of several perturbations are investigated. They are: modification of the shape of a pole, error excitation, displacement, and rotation of a pole. The effects are described in terms of changes of multipole coefficients. General relationships between some of these coefficients are described, and formulae are derived that allow their calculation for a model 2N-pole magnet. Numerical values of these coefficients are given for a quadrupole, sextupole, and octupole.

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1. Introduction

Symmetrical multipole magnets are very important elements in particle accelerators and are becoming of increasing importance for particle microscopes. Both applications require very accurate field distributions, and it is therefore necessary to be able to estimate the effects of fabrication and assembly tolerances. Effects of assembly errors are of particular interest since they introduce asymmetries that cause harmonics which are not present in the symmetric magnet. These harmonics are usually more harmful than the undesired harmonics that are present even in a well designed symmetric multipole magnet. Magnets used for the above mentioned purposes belong usually to one of the following two categories:

1. Conventional iron magnets, where the field distribution is dominated by the iron configuration, while the location of the conductors is only of minor importance.

2. Magnets where the field configuration is dominated by the configuration of the, usually superconducting, conductors, while the iron (usually only a shield) is only of minor importance.

This discussion deals only with the first type of magnet, the other will be the subject of a separate paper.

To make the subject tractable for a general investigation, three basic approximations are made throughout this paper:

a) Only two-dimensional magnets are discussed; this means that the results are applicable only to magnets that are long compared to their aperture.

b) The effects of perturbations are discussed only to first order in the perturbation parameters; this means in particular that linear superposition is assumed to be valid when the effects of more than one perturbation are discussed.

c) It is assumed that the unperturbed magnet with $2\cdot N$ poles is symmetric, i.e., that the geometry of the magnet does not change upon rotation of the whole magnet by the angle π/N.

The effects of perturbations will be expressed in terms of generation or changes of multipole coefficients, which are usually the quantities of interest for beam dynamics calculations. However it should be pointed out that the techniques used for calculation of the multipole coefficients of the perturbed model magnet can also be employed for calculation of field changes.

Two basic types of considerations are made in this paper:

α) General properties of some multipole coefficients produced by perturbations, and general relations between some of the coefficients describing effects of perturbations.

β) Numeric evaluations of the effects of the basic types of perturbations for a model magnet. These calculations imply, of course, some further approximations which will be discussed later.

The following types of perturbations will be discussed:

1) Error excitation of a pole
2) Linear displacement of a pole
3) Rotation of a whole pole about the center of the magnet
4) Poleface modification
5) Conductor-related perturbations.

The results of the model calculations for the first four types of perturbations are represented in tables 1-3 and figs 4,6-11, and their use is summarized in sec. 4.5. It should be noted that although throughout this paper magnetic multipoles are discussed, most of the results apply equally to electrostatic multipoles.

2. Basic Formulae, Normalization, and Notation

The coordinate system is chosen such that the fields are in the x-y plane of a Cartesian coordinate system with the point x = y = 0 coinciding with the center of the unperturbed magnet. The field components H_x, H_y in the iron- and conductor-free region of the magnet can be derived from a scalar potential V or a vector potential which needs to have only a component A in the direction perpendicular to the x-y plane. The field components are obtained from the potentials through:

$$H_x = -\partial V/\partial x = \partial A/\partial y \quad , \tag{1a}$$

$$H_y = -\partial V/\partial y = -\partial A/\partial x \quad . \tag{1b}$$

Introducing the complex quantities $z = x + iy = r\,e^{i\phi}$, $F(z) = A + iV$ and $H = H_x + iH_y$, and indicating the complex conjugate of a quantity by an asterisk, the field components can be obtained from the complex potential F through[1])

$$H^* = i \quad dF(z)/dz \quad . \tag{2a}$$

It will also be useful to express the fields in terms of the radial and azimuthal field components H_r and H_ϕ, and the complex quantity $\mathcal{H}(r,\phi) = H_r + i\,H_\phi$. Since H_r and H_ϕ are the field components in a Cartesian coordinate system that is rotated by ϕ with respect to the x-y system, $\mathcal{H}$ is related to H through

$$\mathcal{H} = H \cdot e^{-i\phi} \quad . \tag{2b}$$

It should be noted that H^* is a function of the complex variable z, but $\mathcal{H}$ (or $\mathcal{H}^*$) cannot be expressed as a function of z only. F(z) can be expanded into a Taylor series about the origin,

$$F(z) = \sum_{n=0}^{\infty} C_n z^n \quad , \tag{3a}$$

and the complex expansion coefficients C_n are called the multipole coefficients. Since for a symmetric 2N-pole magnet it must hold

$$\mathcal{H}(r,\phi + \pi/N) = -\mathcal{H}(r,\phi) \quad ,$$

one obtains with eqs. (2) and (3a) for a symmetric 2N-pole magnet:

$$C_n \cdot e^{in\pi/N} = -C_n, \; n = 1, 2, \ldots$$

Assuming without loss of generality that $C_0 = 0$, all C_n are therefore zero unless $n = N(2m+1)$, $m = 0, 1, 2, \ldots$, giving

$$F(z) = \sum_{m=0}^{\infty} C_{N(2m+1)} \cdot z^{N(2m+1)} \quad . \tag{3b}$$

The term $C_N z^N$ in this series will be called the fundamental harmonic, its odd multiples the allowed harmonics and the terms that cannot appear because of symmetry will be called forbidden harmonics.

Assuming further that each pole of the unperturbed 2N-pole magnet has a symmetry axis and that the vertex of one pole is on the x axis, the following must be true:

$$\mathcal{H}(r,\phi) = \mathcal{H}^*(r,-\phi) \quad .$$

Using eqs. (2) and (3b) it follows that all coefficients in (3b) are imaginary. Introducing the real quantity $d_n = C_n/i$, eq. (3b) gives for the unperturbed magnet:

$$F_0(z) = i \sum_{m=0}^{\infty} d_{N(2m+1)} \cdot z^{N(2m+1)} \quad . \tag{3c}$$

Throughout this paper, all distances are normalized so that the distance from the center of the unperturbed magnet to the vertices of the poles are unity, and the unperturbed excitation is normalized so that

$$d_N \equiv 1 \quad . \tag{3d}$$

The most obvious physical significance of the multipole coefficients C_n in eq. (3a) is obtained by comparing the absolute value $|H_n|$ of the contribution of the term $C_n z^n$ to the field to the absolute value $|H_N|$ of the contribution of the term $C_N z^N$ to the field. From eq. (2a) follows

$$|H_n|/|H_N| = n\cdot|C_n|r^{N-n}\big/N|C_N| \quad .$$

With the normalizations introduced above, one then obtains at the aperture radius of the magnet

$$|H_n|/|H_N| = |C_n|\cdot n/N \quad . \tag{3e}$$

The difference between quantities describing the perturbed and the unperturbed magnet will be indicated by Δ, and the type of perturbation that causes the effect will be indicated either as subscript or in parentheses unless a special symbol is used.

It will prove to be useful later to use the function $D(a)$ which is defined in this paper as:

$$D(a) \begin{cases} = 1 \text{ when } a \text{ equals zero or a positive or negative integer} \\ = 0 \text{ for all other values of } a \end{cases}$$

3. Model Independent Relations between Multipole Coefficients

3.1. GENERAL RELATIONSHIPS

For the consideration of all basic perturbation effects it is assumed that the unperturbed magnet is described by eq. (3c), i.e., all poles are equally excited (except for the alternating sign), each pole has a symmetry axis, the symmetry axis of one pole coincides with the x-axis, and the geometry of the magnet reproduces itself upon rotation by π/N. The basic perturbations listed in sec. 1 are considered to be associated with the pole whose vertex lies on the (positive) x-axis. Describing the effects of a particular perturbation associated with that pole by $\Delta C_n(0)$, the effect of that same perturbation applied to a pole whose symmetry axis is rotated by α is described by $\Delta C_n(\alpha)$ and is obtained as follows: describing the respective effects by $\Delta\mathcal{H}_0(r,\phi)$ and $\Delta\mathcal{H}_\alpha(r,\phi)$, assuming that both poles are excited with the same polarity, and referring to figs. 1(a) and 1(b), drawn for a poleface perturbation, it must hold:

$$\Delta\mathcal{H}_\alpha(r,\phi) = \Delta\mathcal{H}_0(r,\phi-\alpha) \quad .$$

From eqs. (2) and (3a) follows then

$$\Delta C_n(\alpha) = \Delta C_n(0) \cdot e^{-in\alpha} \quad . \qquad (4)$$

The right side of eq. (4) has to be multiplied by -1 when the rotated pole has an excitation opposite to the reference pole and the perturbation is of type 2, 3, or 4 of the basic perturbations listed in sec. 1.

Most perturbations associated with the reference pole are not symmetric with respect to the x-axis. Referring to figs. 1(a) and 1(c) and using the same procedure as above, one obtains from

$$\Delta\mathcal{H}_{1c}(r,\phi) = (\Delta\mathcal{H}_{1a}(r,-\phi))^* :$$

$$\Delta C_n(1c) = -(\Delta C_n(1a))^* \quad . \tag{5}$$

3.2. ERROR EXCITATION OF THE REFERENCE POLE

Error excitation of a pole can have several causes, for instance leakage current in the coils, shorted coil turns, unequal saturation characteristics of the iron in the pole base, or cracks in the iron.

The magnitude of the error excitation will be described by the additional excitation ε, which can also be considered as the change of the scalar potential of the pole. Since the absolute value of the unperturbed potentials is, according to eqs. (3c) and (3d), for all practical purposes one, $|\varepsilon|$ can also be considered as the absolute value of the relative change of excitation. The effect of error excitation of the reference pole is described by $\Delta C_n(x)$. Since this perturbation has no asymmetry with respect to the x-axis, it follows from eq. (5) that $\Delta C_n(x)$ must be purely imaginary. $\Delta C_n(x)$ will therefore be represented by

$$\Delta C_n(x) = i\varepsilon \cdot j_n \quad . \tag{6}$$

To find which of the j_n are negligibly small, two cases of superposition of error excitation are considered.

a) The excitation of all poles are alternatingly changed by $\pm\varepsilon$, i.e., the relative excitation of the whole magnet is changed by ε. Neglecting saturation effects, the resulting change in the complex potential is, according to eq. (3c),

$$\Delta F = \varepsilon \cdot i \sum_{m=0}^{\infty} d_{N(2m+1)} \cdot z^{N(2m+1)} \quad . \tag{7}$$

On the other hand, according to eq. (4), the change ΔC_n of the multipole coefficients as a consequence of this superposition of perturbations becomes:

$$\Delta C_n = i\varepsilon \cdot j_n \cdot \sum_{m=0}^{2N-1} e^{-in\cdot m\cdot \pi/N} \cdot (-1)^m$$

$$\Delta C_n = i\varepsilon \cdot j_n \cdot \sum_{m=0}^{2N-1} e^{-i\pi m(n/N+1)} \tag{8a}$$

If every term in the sum in eq. (8a) equals one, the sum equals 2N. If the individual terms in the sum are not one, and one employs the explicit formula for this geometrical sum, it becomes evident that the sum becomes zero, so that eq. (8a) can be written:

$$\Delta C_n = i\varepsilon \cdot 2N \cdot j_n \, D\left(\frac{1}{2}\left(\frac{n}{N} + 1\right)\right) \tag{8b}$$

From eqs. (7) and (8b) then follows:

$$j_{N(2m+1)} = d_{N(2m+1)}/2N \tag{8c}$$

This means that the allowed harmonics are generated by error excitation of a pole only to the extent of their presence in the unperturbed magnet. Since the allowed harmonics are different in each individual magnet, but should be very small in any well designed magnet, the generation of allowed harmonics by error excitation can therefore in general be neglected and will not be discussed any further.

b) The excitation of all poles is changed by $+\varepsilon$, resulting in a change of the scalar potential of <u>each</u> pole by $+\varepsilon$. The resulting change of the magnetic fields is of course zero unless the cause of the excitation change is a change of current in conductors that are both very close to the magnet aperture and located very asymmetrically in the space between adjacent poles. Since even in that case the effect will be very small and different in each individual magnet, this possibility will not be discussed further. Superposition of these perturbations and describing their total effect by ΔC_n gives with the same procedure as above:

$$\Delta C_n = i \cdot \varepsilon\, j_n \sum_{m=0}^{2N-1} e^{-inm\pi/N} = i \cdot \varepsilon \cdot 2N \cdot j_n \cdot D(n/2N) \quad .$$

Since for this case all $\Delta C_n = 0$ (except for the inconsequential case $n = 0$) it follows that

$$j_{N\cdot 2m} = 0 \quad ; \quad m = 1, 2, \ldots . \tag{9}$$

From eqs. (8c) and (9) follows then that for all practical purposes, error excitation does not generate any harmonics that are multiples of the fundamental harmonic.

3.3. LINEAR DISPLACEMENT OF THE REFERENCE POLE

It is convenient to introduce separate coefficients describing the effects of radial displacement ($\Delta C_n(rd)$) of the reference pole by ε ($\varepsilon > 0$ representing displacement of the reference pole in the x direction), and the effects of azimuthal displacement ($\Delta C_n(ad)$) of the reference pole by ε ($\varepsilon > 0$ representing displacement of the reference pole in the y direction). Since the perturbation caused by the radial displacement of the reference pole is symmetric with respect to the x axis, the reasoning that led to eq. (5) demands that $\Delta C_n(rd) = -(\Delta C_n(rd))^*$, requiring that $\Delta C_n(rd)$ is imaginary. It is therefore convenient to introduce

$$\Delta C_n(rd) = i \cdot \varepsilon \cdot b_n \quad . \tag{10a}$$

Introducing for the moment $\Delta C_n(ad,\varepsilon)$ to also indicate the displacement, similar reasoning requires that

$$\Delta C_n(ad,\varepsilon) = -(\Delta C_n(ad,-\varepsilon))^* \quad .$$

Since $\Delta C_n(ad,\varepsilon)$ is in this first order perturbation theory proportional to ε, it follows then that $\Delta C_n(ad)$ must be real, making the following notation convenient:

$$\Delta C_n(ad) = \varepsilon \cdot a_n \quad . \tag{10b}$$

If the reference pole is displaced by ε in the direction $e^{i\gamma}$ (fig. 2(a)), the resulting change in multipole coefficients is given by:

$$\Delta C_n(2a) = \varepsilon \cdot (a_n \cdot \sin \gamma + i \cdot b_n \cdot \cos \gamma) \tag{11a}$$

Displacing a pole, whose symmetry axis is in the direction α, in the direction $e^{i\gamma}$ (fig. 2(b)) gives, with eqs. (11a) and (4):

$$\Delta C_n(2b) = \varepsilon(a_n \sin (\gamma-\alpha) + i b_n \cos (\gamma-\alpha))\, e^{-in\alpha} \quad . \tag{11b}$$

This can also be expressed as

$$\Delta C_n(2b) = 0.5 \cdot i \cdot \varepsilon \left[(b_n - a_n)\, e^{i(\gamma-\alpha)} + (b_n + a_n)\, e^{-i(\gamma-\alpha)}\right] e^{-in\alpha} \quad . \tag{11c}$$

To obtain general relations between some of the a_n and b_n, all poles are now displaced by ε in the direction $e^{i\gamma}$ Then α for the m'th pole is $m \cdot \pi/N$ and one obtains for the resulting change $\Delta C_n(\text{all})$:

$$\Delta C_n(\text{all}) = 0.5 \cdot i \cdot \varepsilon \sum_{m=0}^{2N-1} \left[(b_n - a_n)\, e^{i(\gamma - m\pi/N)} + (b_n + a_n)\, e^{-i(\gamma - m\pi/N)}\right] e^{-im\pi(1+n/N)}$$

$$\Delta C_n(\text{all}) = i \cdot N \cdot \varepsilon \left\{(b_n - a_n)\, e^{i\gamma}\, D\left(\frac{1}{2}\left(\frac{n+1}{N} + 1\right)\right) + (b_n + a_n)\, e^{-i\gamma}\, D\left(\frac{1}{2}\left(\frac{n-1}{N} + 1\right)\right)\right\} \tag{12a}$$

Since this superposition of perturbations is, on the other hand, equivalent to a displacement of the whole magnet by ε in the direction $e^{i\gamma}$, the resulting change in the complex potential is, with eq. (3c), given by:

$$\Delta F = F_0(z - \varepsilon \cdot e^{i\gamma}) - F_0(z) \quad ,$$

giving to first approximation in ε:

$$\Delta F = -i \cdot N \cdot \varepsilon \cdot e^{i\gamma} \cdot \sum_{m=0}^{\infty} (2m+1)\, d_{N(2m+1)} \cdot z^{N(2m+1)-1} \qquad (12b)$$

Comparison of eq. (12a) with eq. (12b) yields:

$$a_{N(2m+1)+1} + b_{N(2m+1)+1} = 0 \qquad (13a)$$

$$a_{N(2m+1)-1} - b_{N(2m+1)-1} = (2m+1) \cdot d_{N(2m+1)} \qquad (13b)$$

3.4. ROTATION OF THE REFERENCE POLE ABOUT THE CENTER OF THE MAGNET

Rotating the reference pole by the angle ε (in radians, $\varepsilon > 0$ represents rotation in the mathematically positive sense), describing the resulting effects by $\Delta C_n(r)$, and using the same reasoning that led to the introduction of a_n (eq. (10b)) leads to the result that $\Delta C_n(r)$ must be real. $\Delta C_n(r)$ will therefore be represented by

$$\Delta C_n(r) = \varepsilon \cdot \rho_n \quad . \qquad (14)$$

Rotating all poles by ε and describing the resulting effect with ΔC_n gives, using again eq. (4):

$$\Delta C_n = \varepsilon \rho_n \sum_{m=0}^{2N-1} e^{-m\pi(n/N + 1)} = \varepsilon \cdot 2N \cdot \rho_n \cdot D\left(\frac{1}{2}\left(1 + \frac{n}{N}\right)\right) \qquad (15a)$$

Since this combination of perturbations represents a rotation of the whole magnet, one obtains from eq. (3c) for the resulting change ΔF of the complex potential to first order in ε:

$$\Delta F = F_0(z\, e^{-i\varepsilon}) - F_0(z) = -i\varepsilon z\, F_0'(z) = \varepsilon \cdot N \sum_{m=0}^{\infty} (2m+1) \cdot d_{N(2m+1)} \cdot z^{N(2m+1)} \qquad (15b)$$

Comparison of eqs. (15a) and (15b) gives:

$$\rho_{N(2m+1)} = (m+1/2) \cdot d_{N(2m+1)} \; ; \; m = 0, 1, \ldots \qquad (15c)$$

As in the case of error excitation of the reference pole, allowed harmonics are generated through rotation of a pole only to the extent that they are present in the magnet and can therefore in general be neglected.

3.5. POLEFACE MODIFICATION

Although it is impossible to make general statements about the negligibility of some coefficients produced by poleface modifications, the following consideration can be of interest.

When fabricating the poles of a magnet it is often possible to insure that all poles are identical by either using the same die to punch the laminations or by machining all poles in the same process on a milling machine. While the poles are then guaranteed to be identical, they are not necessarily symmetric, and the question arises how the magnet should be assembled. Using an octupole magnet as an example, one could be tempted to assemble it as schematically indicated in fig. 3(b) to insure that the lines lying symmetrically between the poles remain lines with zero scalar potential. It is clear that an assembly according to fig. 3(a) will probably be preferable since it leaves the geometry invariant under a rotation of π/N, but it is of interest what the quantitative difference in harmonic content of the two magnets would be.

Considering the asymmetry as a perturbation of an originally symmetric pole, and describing the resulting harmonics for the reference pole in the normal position of the reference magnet and reference pole (as in fig. 1(a)) by $\Delta C_n(0)$, then the harmonics caused by the asymmetry for the magnet in fig. 3(a) become, with the use of eq. (4):

$$\Delta C_n(3a) = e^{-in\pi/2N} \cdot \Delta C_n(0) \sum_{m=0}^{2N-1} e^{-im\pi(1+n/N)}$$

$$\Delta C_n(3a) = e^{-in\pi/2N} \cdot \Delta C_n(0) \cdot 2N \cdot D\left(\frac{1}{2}\left(1 + \frac{n}{N}\right)\right) \tag{16a}$$

Going to fig. 3(b) and considering first the effects of the perturbations of the pair of poles next to the +x-axis, and then adding up the effects of the perturbations on all other pairs of poles, one obtains with eqs. (4) and (5) for the assembly indicated in fig. 3(b):

$$\Delta C_n(3b) = (\Delta C_n(0) \cdot e^{-in\pi/2N} + \Delta C_n(0)^* \, e^{in\pi/2N}) \cdot \sum_{m=0}^{N-1} e^{-imn2\pi/N}$$

$$\Delta C_n(3b) = 2N \cdot \mathrm{Re}(\Delta C_n(0) \cdot e^{-in\pi/2N}) \cdot D(n/N) \tag{16b}$$

In this case, all multiples of N are generated and they become

$$\Delta C_{N\cdot m}(3b) = 2N \cdot \mathrm{Re}((-i)^m \cdot \Delta C_{N\cdot m}(0)) \tag{16c}$$

Comparing eq. (16a) with eqs. (16b) and (16c) leads to the following conclusion: Although the amplitude of the allowed harmonics generated by the asymmetry in an assembly as in fig. 3(b) can at most equal the amplitude of the allowed harmonics generated in an assembly as in fig. 3(a), the latter is preferable since the former generates all multiples of N. This is true particularly since ΔC_{2N} can be very damaging for the beam dynamics.

4. Evaluation of Perturbation Effects for a Model Magnet

It is clear that model calculations cannot describe effects resulting from detailed characteristics of specific magnets, such as amplitudes of allowed harmonics, saturation effects, details of pole contours or coil configuration. The model should also have the property that the effects of perturbations can be calculated analytically. Consequently the results of these calculations can serve only as a guide to describe the effects of perturbations in a real magnet. The applicability can, however, be judged by taking into account some of the above mentioned general relationships (for instance eqs. (13)), and by mentally going through the exercise of correcting the procedure below to take particular properties of a real magnet into account. The result will in most cases be that the results of the analytical evaluations are very accurate when the effects are of practical significance.

4.1. CHOICE OF MODEL AND DERIVATION OF BASIC FORMULAE AND PROCEDURES

The effects of perturbations will be calculated for a reference pole located as in fig. 1(a). The unperturbed 2N-pole magnet model consists of identical poles at scalar potentials ±1 giving an unperturbed complex potential

$$F_0(z) = iz^N \quad . \tag{17a}$$

From this follows for the equation describing the reference pole

$$\text{Im } F_0(z) = \text{Re}(z^N) = r^N \cos N\phi = 1 \tag{17b}$$

The reference pole can therefore also be described by the complex parameter representation

$$z^N = e^{iN\phi}/\cos N\phi = 1 + i \cdot \tan N\phi \quad , \tag{17c}$$

$$z(\phi) = e^{i\phi}/(\cos N\phi)^{1/N} \quad . \tag{17d}$$

The significance of $N\phi$ is best seen with eq. (17c). This equation expresses that in the z^N plane the reference pole is a straight line just as the pole of an ideal dipole magnet. In this geometry $N\phi$ is the angle between the straight lines connecting the origin with the vertex of the pole and with the point under consideration on the pole. From this follows that for the reference pole, the region $|N\phi| < \pi/3$ is of primary interest.

With the exception of the discussion in sec. 4.6, the perturbations of the reference pole will be represented by the equivalent changes of the scalar potential on the unperturbed surface of the reference pole. To evaluate the effect of the perturbations of the scalar potential, the solution to the Dirichlet problem[2]) in a unit circle will be used: If the scalar potential V is known as a function of angle ψ on the circumference of a unit circle that is centered with respect to the origin of a complex W plane, the complex potential inside the unit circle is given by

$$F(W) = F(0) + \frac{i}{\pi} \cdot \int_{-\pi}^{\pi} \frac{W\, e^{-i\psi}}{1-W\, e^{-i\psi}} \cdot V(\psi) \cdot d\psi \quad . \tag{18}$$

To find the function that maps analytically the inside of the magnet onto the inside of the unit circle, one can make use of the fact that the complex potential $F(W(z))$ equals iz^N when the poles have, alternatingly, scalar potentials ± 1. Because of the symmetry of the problem one can assume without loss of generality that the points

$$z_m = \lim_{r\to\infty} (r\, e^{i\pi(m+0.5)/N}) \quad ; \quad m = 0, 1, \ldots 2N-1$$

map onto the points

$$W_m = e^{i\pi(m+0.5)/N} \quad ; \quad m = 0, 1 \ldots 2N-1 \quad .$$

Applying eq. (18) to the unperturbed magnet gives therefore

$$F_0(W(z)) = iz^N = \frac{i}{\pi} \int_{-\pi/2N}^{\pi/2N} \sum_{m=0}^{2N-1} \cdot \frac{W\,e^{-i(\psi+m\pi/N)}}{1 - W\,e^{-i(\psi+m\pi/N)}} \quad (-1)^m \cdot d\psi$$

Expanding the denominator in the sum in this equation into a geometric series, performing the sum over m, and representing the remaining terms in the sum by the closed expression for a geometrical sum gives

$$iz^N = \frac{2i}{\pi} \cdot \int_{-\pi/2N}^{\pi/2N} \frac{N \cdot (W\,e^{-i\psi})^N}{1 - (W\,e^{-i\psi})^{2N}}\, d\psi \quad .$$

Performing the integration gives

$$z^N = \frac{2}{i\pi} \cdot \ell n \left(\frac{1+iW^N}{1-iW^N}\right) \quad . \qquad (19a)$$

Solving for W^N yields

$$W^N = \tan\left(\frac{\pi}{4} z^N\right) \quad . \qquad (19b)$$

Describing a point on the poleface with ϕ through eq. (17c) and its map with $e^{i\psi}$, the relation between ϕ and ψ can be found from eqs. (19) and can be expressed in the following forms:

$$N\phi = \arctan \left\{\frac{2}{\pi}\, \ell n \left[\tan\left(\frac{N\psi}{2} + \frac{\pi}{4}\right)\right]\right\}, \qquad (20a)$$

$$N\psi = 2 \arctan \left(e^{\pi/2} \tan N\phi\right) - \pi/2 \qquad (20b)$$

$$N\psi = 2 \arctan\left\{\tanh\left[\frac{\pi}{4} \tan (N\phi)\right]\right\}. \qquad (20c)$$

The derivative $d\psi/d\phi$ will also be of interest and is easily obtained from eq. (20c):

$$d\psi/d\phi = \frac{\pi/2}{\cosh\left[\frac{\pi}{2} \tan N\phi\right] \cdot \cos^2 N\phi} \quad . \qquad (20d)$$

Graphs of $N\psi$ and $d\psi/d\phi$ vs $N\phi$ are given in fig. 4. It will also be necessary to expand W^m into a Taylor series in z. This is easily done by expressing W^m as

$$W^m = z^m(\pi/4)^{m/N} \cdot \left(\frac{\tan \pi/4 \ z^N}{\pi/4 \ z^N}\right)^{m/N} \quad ,$$

expanding the content of the last parenthesis in $(\pi z^N/4)^2$, and then applying the binomial theorem. This gives, with $k = m/N$, and using the abbreviation $s = z(\pi/4)^{1/N}$:

$$W^m = s^m\left(1+s^{2N} \frac{k}{3} + s^{4N} \frac{k}{18}(k+\frac{7}{5}) + s^{6N} \frac{k}{81} \left(\frac{k^2}{2} + k \frac{21}{10} + \frac{62}{35}\right) + \ldots\right) \quad (21a)$$

The first omitted term in this series is proportional to s^{8N+m}.

The procedure for evaluation of the effect of most perturbations will be as follows: The perturbation will first be expressed in terms of an equivalent change of the scalar potential on the surface of the unperturbed reference pole. This perturbation potential ΔV is originally known as a function of location on the reference pole, and through eq. (20c) is also known as a function of ψ. Using this in eq. (18) and expanding in $W e^{-i\psi}$ gives for the perturbation ΔF of the complex potential, when the unimportant term $\Delta F(0)$ is neglected:

$$\Delta F = \sum_{m=1}^{\infty} W^m \int_{-\pi}^{\pi} i \cdot e^{-im\psi} \Delta V(\psi) \cdot d\psi/\pi \quad . \tag{22a}$$

Abbreviating the integrals in this expression by

$$I_m = \int_{-\pi}^{\pi} i \; e^{-im\psi} \Delta V(\psi) \cdot d\psi/\pi \quad , \tag{22b}$$

and expressing W^m by

$$W^m = \sum_{n=m}^{\infty} z^n \cdot K_{n,m} \quad , \tag{21b}$$

with the $K_{n,m}$ obtained from eq. (21a), gives

$$\Delta F = \sum_{n,m} z^n \cdot K_{n,m} \cdot I_m \quad . \tag{22c}$$

Using the properties of $K_{n,m}$ and introducing $m' =$ largest integer $< n/2N$, one can therefore write for the change ΔC_n of the multipole coefficients:

$$\Delta C_n = \sum_{m=0}^{m'} K_{n,n-2Nm} \cdot I_{n-2Nm} \quad . \tag{22d}$$

The problem of evaluating the effects of perturbations is thus reduced to finding the $\Delta V(\psi)$, evaluating the integrals in eq. (22b) and performing the summation in eq. (22d).

Figure 5 shows the original and displaced or modified position of an element of the poleface. Using the ordinary vector representation for the displacement ($\vec{\Delta z}$) and the unperturbed field ($\vec{H}$), the change of the scalar potential at the unperturbed position of the element is to first order in $|\vec{\Delta z}|$ given by $\Delta V = \vec{\Delta z} \cdot \vec{H}$. Using now the complex representation of the vector $\vec{\Delta z}$ and $\vec{H}$ gives:

$$\Delta V = \mathrm{Re}(\Delta z \cdot H^*) \quad .$$

Using eqs. (2a) and (17a) then yields:

$$\Delta V = -N \cdot \mathrm{Re}(\Delta z \cdot z^{N-1}) \quad . \tag{23}$$

4.2. ERROR EXCITATION OF THE REFERENCE POLE

Increasing the scalar potential of the reference pole by ε gives

$$\Delta V = \begin{cases} \varepsilon & \text{for } |\psi| \leq \pi/2N \\ 0 & \text{otherwise} \end{cases}$$

It follows therefore

$$I_m = i\varepsilon/\pi \int_{-\pi/2N}^{\pi/2N} e^{-im\psi}\, d\psi = i \cdot \varepsilon \cdot 2/\pi \quad \frac{\sin\,(m\pi/2N)}{m} \tag{24}$$

4.3. LINEAR DISPLACEMENT OF THE REFERENCE POLE

For azimuthal displacement by ε, $\Delta z = i \cdot \varepsilon$ and one obtains from eqs. (23) and (17c)

$$\Delta V(ad) = \varepsilon N \sin((N-1)\cdot\phi)/(\cos N\phi)^{1-1/N} \quad . \tag{25a}$$

Substituting this expression into eq. (22b), using the fact that $\Delta V(ad)$ is an odd function of ψ, and introducing ϕ as the integration variable gives

$$I_m(ad) = \frac{2\varepsilon N}{\pi} \int_0^{\pi/2N} \sin m\psi \cdot \frac{\sin(N-1)\psi}{(\cos N\phi)^{1-1/N}} \; \frac{d\psi}{d\phi} \; d\phi \quad . \tag{26a}$$

Expressing $d\psi/d\phi$ through eq. (20d), and introducing $\alpha = N\phi$, $\beta = N\psi$ gives then

$$I_m(ad) = \varepsilon \cdot \int_0^{\pi/2} \sin\,(m/N\;\beta) \cdot \frac{\sin(1-1/N)\alpha}{(\cos\,\alpha)^{3-1/N}\;\cosh(\pi/2\;\tan\alpha)} \cdot d\alpha \tag{27a}$$

β is obtained from α by replacing in eq. (20c) $N\phi$ by α and $N\psi$ by β.

For radial displacement by ε one obtains in similar fashion the following equations

$$\Delta V(rd) = -\varepsilon N \cdot \cos(N-1)\phi/(\cos N\phi)^{1-1/N} \tag{25b}$$

$$I_m(rd) = -\,i\,\frac{2\varepsilon N}{\pi} \cdot \int_0^{\pi/2N} \cos m\psi \; \frac{\cos(N-1)\phi}{(\cos\;N\phi)^{1-1/N}} \cdot \frac{d\psi}{d\phi} \cdot d\phi \tag{26b}$$

$$I_m(rd) = -\,i\varepsilon \int_0^{\pi/2} \cos(m/N\;\beta) \cdot \frac{\cos(1-1/N)\alpha}{(\cos\;\alpha)^{3-1/N}\cdot\cosh(\pi/2\;\tan\;\alpha)} \; d\alpha \tag{27b}$$

4.4. ROTATION OF THE REFERENCE POLE ABOUT THE CENTER OF THE MODEL MAGNET

For a rotation about the origin by ε radians in the mathematically positive direction, $\Delta z = i\cdot\varepsilon z$ and eqs. (23), (17c), and (20a) give:

$$\Delta V(r) = \varepsilon N \tan N\phi = \frac{2N\varepsilon}{\pi} \ln\left[\tan\left(\frac{N\psi}{2} + \frac{\pi}{4}\right)\right] \quad . \tag{28}$$

Substituting this into eq. (22b), using the fact that $\Delta V(r)$ is an odd function of ψ, and introducing $\beta = N\psi$ as new integration variable yields, with $k = m/N$:

$$I_m(r) = \varepsilon \cdot \frac{4}{\pi^2} \int_0^{\pi/2} \sin(k\beta) \cdot \ln\left[\tan\left(\frac{\beta}{2} + \frac{\pi}{4}\right)\right] \cdot d\beta \quad . \tag{29a}$$

Integrating this by parts to eliminate the integrable singularity at the upper limit then gives

$$I_m(r) = \varepsilon \cdot \frac{4}{k\pi^2} \cdot \int_0^{\pi/2} \frac{\cos(k\beta) - \cos(k\cdot\pi/2)}{\cos\beta} \cdot d\beta \quad . \tag{29b}$$

According to eq. (15c), for the model magnet $\rho_{N(2m+1)} = 0$ for m = 1, 2,... Without presenting the mathematical details here, it can also be shown that

$$\rho_{Nm} = \begin{cases} 0.25 & \text{for } m = 2 \\ 0 & \text{for } m > 2 \end{cases} \quad . \tag{30}$$

The procedure to obtain this result for the model magnet is as follows: One considers the case where the poles are alternatingly rotated by $\pm\varepsilon$, giving information about all ρ_{2Nm}. Although one can calculate the resulting effects with the methods used in this paper, it is in this particular case easier to use another method: With the above mentioned combination of rotations, the asymptotes of the unperturbed polefaces remain V = 0 lines. Applying the transformation z^N to the area that is bound by the rotated reference pole and its unperturbed asymptotes leads to a geometry from which eq. (30) follows immediately for even multiples of N, while its validity for odd multiples of N folows from eq. (15c).

4.5. DISCUSSION, USE, AND APPLICATION OF TABLES 1-3

Although the analytical approach to evaluate the I_m could have been carried further in some cases, it was decided not to do so because it seemed unavoidable to use a computer to prepare the tables, if only to avoid human error. To calculate the tables, the expressions for the I_m (eqs. (24), (27a), (27b), (29b)) were numerically evaluated and used together with eqs. (21a) and (22d) to obtain the values for the ΔC_n. None of the general relationships and properties of the coefficients (eqs. (8c), (9), (13), (15c), (30)) were utilized in these evaluations to make it possible to check the correctness of the computer program. After verification that all numbers that should be zero were sufficiently small (order 10^{-12}) the printout was modified to give zeroes where they belong, improving the legibility of the tables. Because of space limitations and of the practical significance of $|H_n|/|H_N|$ (eq. (3e)), it was decided to reproduce for this publication only the quantities $n \cdot \Delta C_n/N$ for N = 2, 3, 4.

To make a judgement about the applicability of the tables to particular magnets, eqs. (8c) and (15c) can be valuable, but the best criterion is probably eq. (13b). Considering a quadrupole (N = 2) and multiplying both sides of eq. (13b) by $[N(2m+1)-1]/N = 2m + 1/2$, the right side of that equation would be well below 10^{-2} for $m = 1$ if the magnet is reasonably well designed, indicating that the linear displacement coefficients for $n = 5$ are likely to be quite accurate. For $m = 2$ and a well designed magnet, $22.5\ d_{10}$ will still be somewhat below 10^{-2}, i.e., eq. (13b) will be more significantly violated, indicating that the linear displacement coefficients for $n = 9$ are not very accurate, and for $n = 13$ matters are even worse. The tables for larger N show similar tendencies, leading to the conclusion that the tables have to be used with caution above $n = 5N$. They are carried farther only to show the general trend of the results of the model calculations.

To facilitate the use of the tables, it is again emphasized that they give the effects of perturbations of the reference pole, which is located as indicated in fig. 1(a). Equation (4) allows application to any other pole, and care should be taken to take the sign of excitation properly into account. Equations (11) can simplify the evaluation of the effects of displacements of poles.

It is worth noting that the excitation error tables are valuable not only for the primary intended purpose. They can, for instance, be used to plan correction by asymmetric excitation of measured and otherwise uncorrectable field errors. Another possibility is their use to study the feasibility of weak multipurpose correction magnets, for instance a 12-pole magnet that provides, with two independent sets of coils, both a skew quadrupole field and a sextupole field.

A combination of errors that can easily occur is the radial displacement of all poles by ε. In this case it is obvious that only the fundamental and the allowed harmonics are perturbed and the effect is described by

$$\Delta C_{N(2m+1)} = i\varepsilon \cdot 2N \cdot b_{N(2m+1)} \; ; \; m = 0, 1, \ldots \qquad (31)$$

Many multipole magnets consist of two subassemblies and both displacement and rotation errors can occur when the two halves are assembled. One can, for instance, easily imagine that the upper half of a magnet as shown in fig. 3(b) is rotated by δ with respect to the lower half, with the rotation axis being at z_0, usually someplace outside the magnet. This is equivalent to the combination of a pure rotation of the upper half by δ about the origin of the magnet, and a displacement of the upper half by $-z_0 \cdot i\delta$. To simplify the treatment of these problems they are symmetrized by applying 1/2 of the total error with opposite sign to each half of the magnet.

Rotating every pole in the upper half of a magnet as in fig. 3(b) by $0.5\,\varepsilon$, and assuming the pole directly above the x-axis in the first quadrant to be on scalar potential 1, one obtains with eqs. (4) and (14):

$$\Delta C_n(\text{upper half}) = 0.5\,\varepsilon \cdot \rho_n \sum_{m=0}^{N-1} e^{-in\alpha_m} (-1)^m ; \; \alpha_m = \frac{\pi}{2N} + m \cdot \frac{\pi}{N} \quad .$$

Since every pole of the lower half of the magnet has an excitation opposite to that of the symmetrically located pole in the upper half, but the rotation is in the opposite direction, the contribution of the lower half is identical to that of the upper half, except every α_m has to be replaced by $-\alpha_m$. One therefore obtains for the case where every pole in the upper half is rotated by $0.5\,\varepsilon$ and every pole in the lower half rotated by $-0.5\,\varepsilon$:

$$\Delta C_n = \varepsilon \cdot \rho_n \cdot \mathrm{Re}\left(e^{-i\pi n/2N} \sum_{m=0}^{N-1} e^{-i\pi m(1+n/N)} \right) \quad .$$

If every term in the sum equals 1, n/N has to be an odd integer giving $\pm i$ for the factor in front of the sum, yielding

$$\Delta C_n = 0 \quad \text{for } n/N = \text{odd integer} \quad . \tag{32a}$$

Excluding the case n/N = odd integer and using the closed expression for the sum gives:

$$\Delta C_n = \varepsilon \cdot \rho_n \cdot \mathrm{Re}\, e^{-i\pi n/2N} \cdot \frac{1 - e^{-i\pi(N+n)}}{1 + e^{-i\pi n/N}}$$

$$\Delta C_n = \varepsilon \cdot \rho_n \cdot \frac{1 - (-1)^{n+N}}{2\cos(\pi n/2N)}$$

From this follows

$$\Delta C_n = \begin{cases} \varepsilon\, \rho_n/\cos(\pi n/2N) & \text{for } n + N = \text{odd integer} \\ 0 & \text{for } n + N = \text{even integer} \end{cases} \tag{32b}$$

Equation (32b) clearly includes eq. (32a), so that eq. (32b) describes all ΔC_n. Applying the same method to the case where every pole in the upper half of the magnet is displaced by $0.5 \cdot \varepsilon\, e^{i\gamma}$, and every pole in the lower half by $-0.5\, \varepsilon\, e^{i\gamma}$, gives with eq. (11c):

$$\Delta C_n = \begin{cases} i \cdot 0.5 \cdot \varepsilon \left(e^{i\gamma} \dfrac{b_n - a_n}{\cos(\frac{\pi}{2}\frac{n+1}{N})} + e^{-i\gamma} \dfrac{b_n + a_n}{\cos(\frac{\pi}{2}\frac{n-1}{N})} \right) & \text{for } n+N = \text{even integer} \\ 0 & \text{for } n+N = \text{odd integer} \end{cases} \tag{33a}$$

With $\varepsilon \cdot \cos\gamma = \Delta x$, $\varepsilon \cdot \sin\gamma = \Delta y$ and $.5 \cdot \left(\dfrac{b_n + a_n}{\cos(\frac{\pi}{2}\frac{n-1}{N})} \pm \dfrac{b_n - a_n}{\cos(\frac{\pi}{2}\frac{n+1}{N})} \right) = G_{\substack{x\\y}}$,

this becomes

$$\Delta C_n \begin{cases} = \Delta y \cdot G_y + i\Delta x \cdot G_x & \text{for } n+N = \text{even integer} \\ = 0 & \text{for } n+N = \text{odd integer} \end{cases} \tag{33b}$$

It should be noted that for this particular type of assembly error, effects caused by displacement cannot be partly compensated by rotation errors since they produce different harmonics, i.e., if one type produces only even harmonics, the other produces only odd harmonics. The effects can be sizeable. For instance, using the numerical values from the tables in the general eqs. (32b) and (33a) gives for $N = 3$, $n = 4$, 5 and $\gamma = \pi/2$: $4/3\ \Delta C_4 = -\varepsilon \cdot 1.28$; $5/3 \cdot \Delta C_5 = -\varepsilon \cdot 0.51$. Tables that include $b_n \pm a_n$ and the quantities entering eqs. (32b) and (33b) can be obtained from the author upon request.

4.6. POLEFACE MODIFICATION

The purpose of discussing poleface modifications quantitatively for the model magnet is the desirability to get a better understanding of their effects and to obtain some quantitative information that can give guidance for "shimming" of a magnet (in the design phase or after it has been built) as well as obtaining some quantitative information allowing to set machining tolerances for the polefaces. In contrast to the preceding sections, dipole magnets are included in the discussion for obvious reasons.

The basic procedures to obtain the effect of a modification of a poleface of the model magnet is the same as for the previous evaluations: The modification of the reference pole is described by the equivalent change $\Delta V(\psi)$ of the scalar potential and ΔV is then used in eq. (22a). To simplify the evaluation it is assumed that the "bump" on the poleface is so localized for all m of interest that $e^{-im\psi}$ can be written outside the integral in eq. (22a). For this to be true, the bump width $|\Delta z|$ should be small enough that the equivalent $|\Delta\psi|$ is small compared to $\pi/2m$. One therefore obtains:

$$|\Delta\psi| = |\frac{d\psi}{dz}| \cdot |\Delta z| << \pi/2m$$

Using eqs. (17d) and (20d) one obtains

$$|\frac{dz}{d\psi}| = 2 \cosh [\frac{\pi}{2} \tan N\phi] \cdot (\cos N\phi)^{1-1/N}/\pi \quad , \tag{34}$$

giving as condition for $|\Delta z|$

$$|\Delta z| << \cosh (\frac{\pi}{2} \tan N\phi) \cdot (\cos N\phi)^{1-1/N}/m \quad . \tag{35}$$

The right side of this inequality is plotted in fig. 6 for m = 1. Since it is often more convenient to describe the location of a point on the poleface by r, fig. 7 shows the relationships between $N\phi$ and r. If inequality (35) is not satisfied, a bump can of course be broken up into several parts that are treated separately.

If a bump on the reference pole has the height h (h > 0 corresponding to addition of iron), then ΔV becomes, with eqs. (2a) and (17a)

$$\Delta V = h \cdot |H| = Nh/(\cos N\phi)^{1-1/N} \quad . \tag{36}$$

Introducing S for the common factor $\Delta V \cdot \Delta\psi/\pi$ in (eq. (22a)) gives

$$S = \Delta V \cdot \Delta\psi/\pi = \Delta V \cdot |\Delta z| \cdot |\frac{d\psi}{dz}|/\pi$$

Using eqs. (34) and (36) in this expression and introducing the bump area $a = h\,|\Delta z|$ ($a > 0$ corresponding to addition of iron) yields:

$$S = a \cdot \frac{N/2}{\cosh\,(\pi/2\,\tan N\phi)\cdot(\cos N\phi)^{2-2/N}} \tag{37}$$

Figure 8 shows a plot of $2S/aN$ vs $N\phi$, and it indicates how the effect of pole-face modifications diminishes when they are far removed from the aperture. With this expression, ΔF becomes

$$\Delta F = i \cdot S \cdot \sum_{m=1}^{\infty} W^m\, e^{-im\psi} \tag{38}$$

The resulting multipole coefficients are obtained from eq. (22d), with $I_m = iS \cdot e^{-im\psi}$. Since there are too many multipole coefficients of interest for too many N, they are not easily plotted, but are of course easily obtainable with the help of a computer. However the interesting case where every pole of the magnet has the same perturbation is easily represented graphically. In that case, ΔF becomes

$$\Delta F = i \cdot 2NS \cdot \sum_{m=0}^{\infty} W^{N(2m+1)} \cdot e^{-iN(2m+1)\psi} \quad . \tag{39}$$

With eq. (21a) this can be written:

$$\Delta F = i \cdot 2NS \cdot \sum_{m=0}^{\infty} z^{N(2m+1)} \cdot L_{2m+1} \tag{40a}$$

$$L_1 = \frac{\pi}{4} \cdot e^{-iN\psi} \tag{40b}$$

$$L_3 = \left(\frac{\pi}{4}\right)^3 \left(e^{-i3N\psi} + \frac{1}{3}\, e^{-iN\psi}\right) \tag{40c}$$

$$L_5 = \left(\frac{\pi}{4}\right)^5 \left(e^{-i5N\psi} + e^{-i3N\psi} + \frac{2}{15}\, e^{-iN\psi}\right) \tag{40d}$$

$$L_7 = \left(\frac{\pi}{4}\right)^7 \left(e^{-i7N\psi} + \frac{5}{3}\, e^{-i5N\psi} + \frac{11}{15}\, e^{-i3N\psi} + \frac{17}{315}\, e^{-iN\psi}\right) \quad . \tag{40e}$$

In figs. 9-11, the real and imaginary parts of $3L_3$, $5L_5$. and $7L_7$ are plotted vs $N\phi$, and together with the plot for $2S/aN$ the multipole coefficients are easily obtained. If every pole has two identical bumps located symmetrically with respect to its symmetry axis, eq. (40a) has to be replaced by

$$\Delta F = i \cdot 4NS \cdot \sum_{m=0}^{\infty} z^{N(2m+1)} \cdot \mathrm{Re}(L_{2m+1}) \tag{40f}$$

It is noteworthy that in the latter case the zero crossings of Re(L) have a practical significance: By locating bumps at these zero crossings for shimming purposes, ΔC_{3N} and ΔC_{5N} can be controlled independently, which will obviously simplify the process. The values of $N\phi$ for the zero crossings are: 22.75° for $\mathrm{Re}(L_3)$, 15° and 45° for $\mathrm{Re}(L_5)$, and 11.25°, 33.75°, and 56.25° for $\mathrm{Re}(L_7)$.

4.7. PERTURBATIONS RESULTING FROM CONDUCTORS

Conductor-related effects are discussed only for completeness' sake since they have usually only weak effects in iron dominated magnets. Therefore without giving any details only the procedures are briefly outlined that are practical when dealing with conductors.

If one considers the two dimensional fields produced by a round conductor with axisymmetric current distribution, it is easy to show that even in the presence of saturating iron, the fields outside the conductor depend only on the net current, and not on its distribution. For computational purposes it is therefore permissible to replace such a conductor by a current filament. For the magnet model used here, it is impossible to consider the effects of only one filament, since the "air"-region is completely surrounded by iron. It is therefore imperative to have at least a fictitious current return, which is chosen here to be at the origin. If a current filament with unity current is located at $W_0(z_0)$, and the return current goes through the origin, it is easy to verify that the complex potential is

$$F(W) = -\frac{1}{2\pi} \cdot \ln \frac{(1-W/W_0)(1-W\,W_0^*)}{W} \quad . \tag{41a}$$

If every current element is represented in this manner, the singularity at $W = 0$ must disappear since by necessity the sum of all currents must be zero. Knowing this fact, one can therefore study the effects stemming from current filaments by using the fictitious complex potential

$$F(W) = -\frac{1}{2\pi} \ln\left((1-W/W_0)(1-W\,W_0^*)\right) \tag{41b}$$

Effects stemming from small displacements of a conductor are easily obtained by calculating from eq. (41b)

$$\Delta F = \Delta W_0 \cdot \partial F/\partial W_0 + \Delta W_0^* \cdot \partial F/\partial W_0^* \quad , \tag{42}$$

expanding in W and using eq. (22d).

It should be pointed out, however, that for the effects of perturbations that are identical in every space between poles, another procedure is often more convenient. Applying the transformation z^N to a sector containing two poles leads to dipole geometry which sometimes is easier for evaluation, particularly if the poles are mapped onto a straight line with a Schwarz-Christoffel transformation.

Just to indicate how strongly the effects of conductors vanish as soon as they are not in the immediate vicinity of the magnet aperture, it is worth noting that the undesired multipole coefficients decay at least like $e^{-\pi r^N}$.

Concluding Remarks

Since it is the author's experience that the following is sometimes either forgotten or not completely understood, a word of caution against misuse of the multipole coefficients: The multipole coefficients are the coefficients of a Taylor expansion in z. The convergence radius of this expansion equals the distance from the origin to the next singularity (which for most of the cases discussed here is at the magnet aperture) and the expansion will give nonsense if used outside its convergence radius. This argument does not apply to the unperturbed model magnet since it was intentionally chosen to give exactly iz^N for the complex potential everywhere.

Although effects of perturbations on multipole coefficients are usually of primary interest, the field perturbations themselves might sometimes be of concern. With eqs. (2a) and (18) one obtains for the model magnet

$$H^* = -\frac{W'}{\pi} \int_{-\pi}^{\pi} \frac{e^{i\psi} \cdot \Delta V(\psi)}{(e^{i\psi} - W)^2} \cdot d\psi \quad . \tag{43}$$

The integral can be numerically evaluated for every specific $W(z)$ and it is worth noting that this evaluation is trivial for the case of poleface modifications under the assumptions made in sec. 4.6, giving in that case an explicit analytical expression for H^*.

This paper dealt exclusively with symmetric magnets. Consideration of general multipole magnets where not all pole-vertices have the same distance from the origin have been excluded despite their practical importance because the parameter space becomes so large that the presentation of numerical results becomes difficult. But it should be pointed out that the procedure used for the model calculations is also applicable to these more general magnets; the formulae get more complicated and require more extensive use of computers, but are basically still quite manageable. Work on quadrupoles and sextupoles with a specific type of asymmetry is in progress. It is also noteworthy that the transformation mapping the poles onto a circle holds also for the case where the total number of poles $2N$ is an odd number, as long as the poles are described by eq. (17b). This type of magnet is sometimes used to generate rotating fields and the analysis carried through here can be applied to them also.

Acknowledgment

The author wants to thank Mr. R. Yourd for writing the computer programs that were used to obtain the tables and most of the figures.

References

1) These relations can be found in books on electromagnetism or functions of a complex variable; for instance: Fuchs, Shabat, Functions of a Complex Variable and Some of Their Applications, (Addison-Wesley Publishing Co., Reading, Mass., 1961) pp. 139-141

2) ibid., pp. 209-218

Figure Captions

Fig. 1(a). Normal position of reference pole.

Fig. 1(b). Rotated reference pole.

Fig. 1(c). Reference pole with antisymmetric perturbation.

Fig. 2(a). Displacement of reference pole.

Fig. 2(b). Displacement of rotated reference pole.

Fig. 3(a). Assembly of octupole with invariance of geometry against rotation by $\pi/8$.

Fig. 3(b). Assembly of octupole with invariance of $V = 0$ lines.

Fig. 4. Relation between $N\psi$, $d\psi/d\phi$ and $N\phi$.

Fig. 5. Displacement of a poleface element.

Fig. 6. Relation between allowable modification width and $N\phi$ for reference pole.

Fig. 7. Relation between r and $N\phi$ of reference pole.

Fig. 8. Relation between common poleface perturbation factor and $N\phi$.

Fig. 9. Relation between $3L_3$ and $N\phi$.

Fig. 10. Relation between $5L_5$ and $N\phi$.

Fig. 11. Relation between $7L_7$ and $N\phi$.

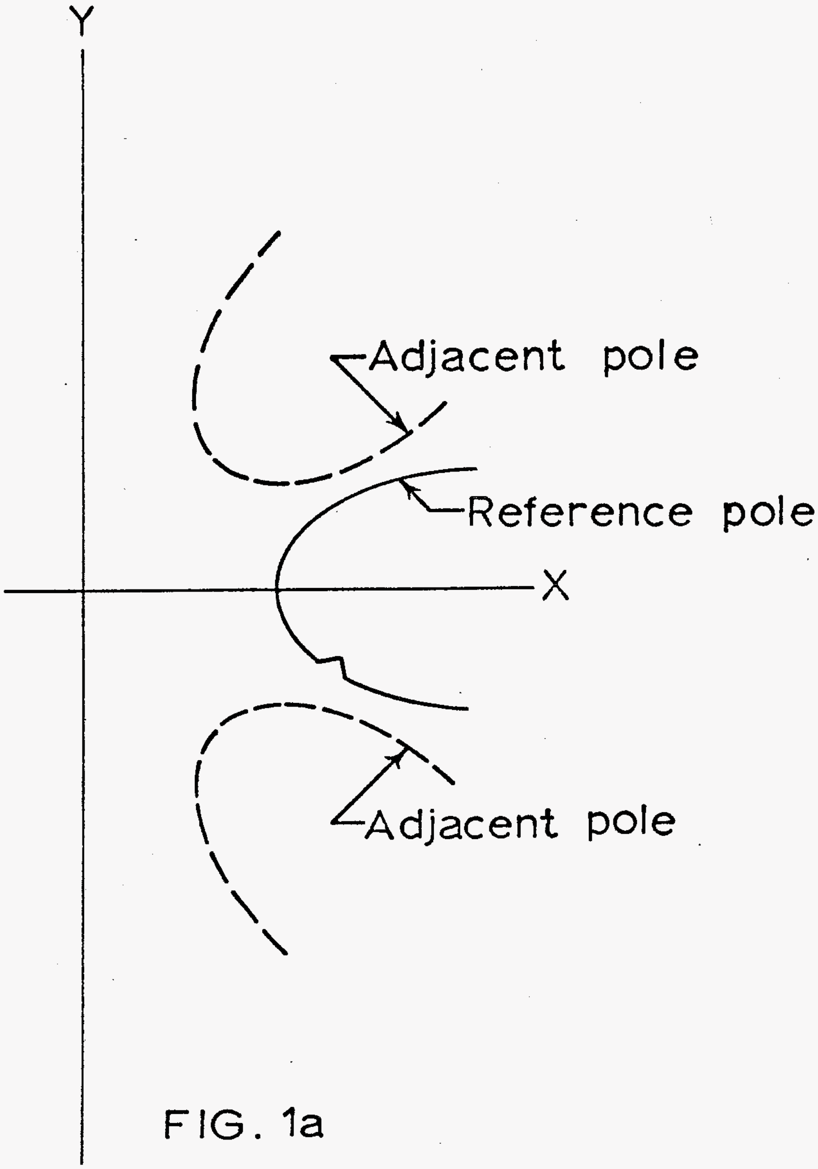

FIG. 1a

XBL 694 4819

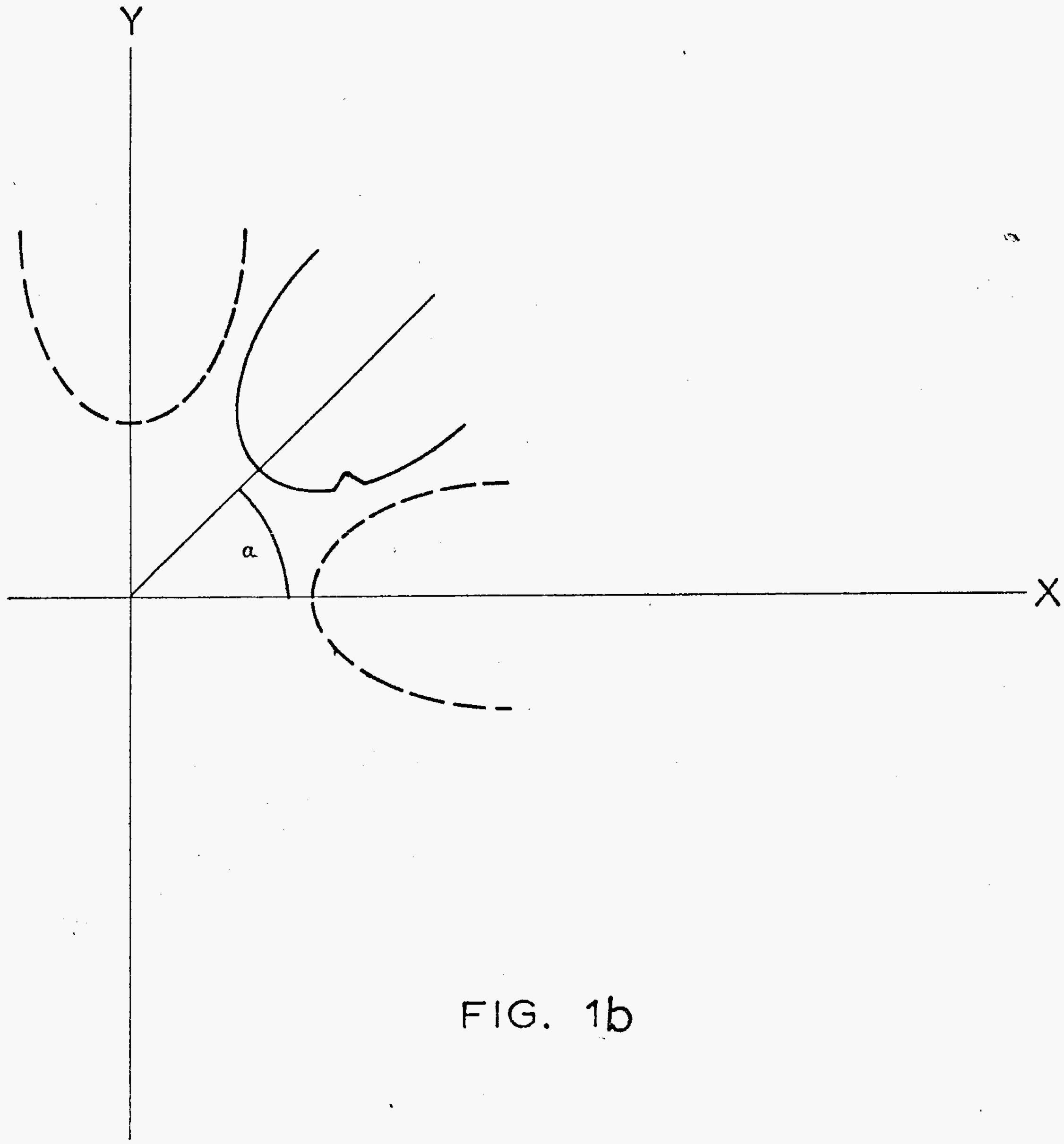

FIG. 1b

XBL 694 4821

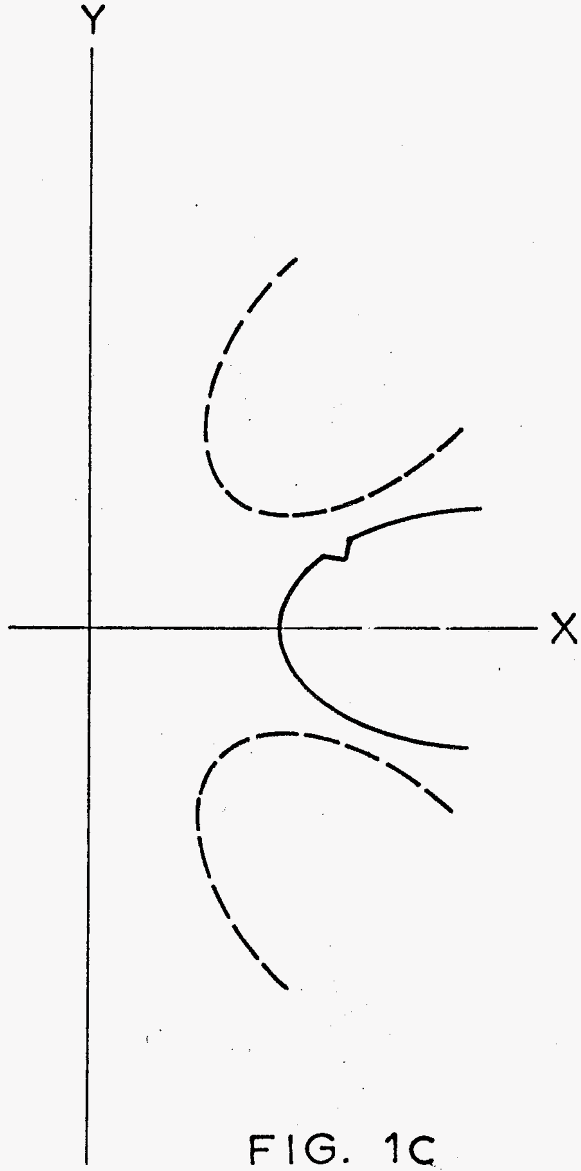

FIG. 1C

XBL 694 4820

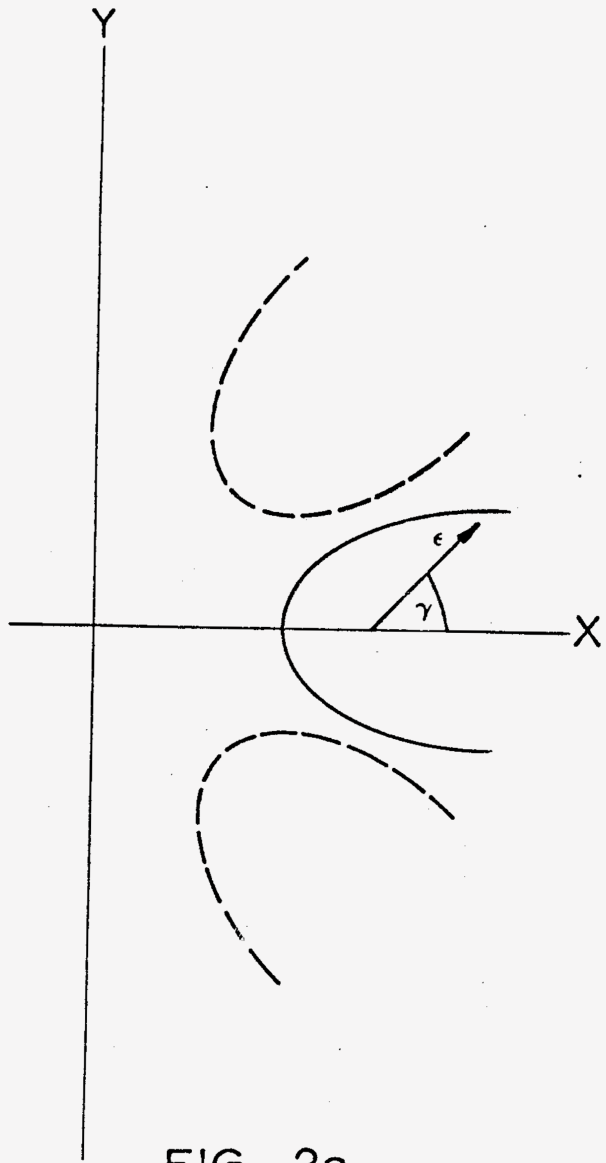

FIG. 2a

XBL 694 4822

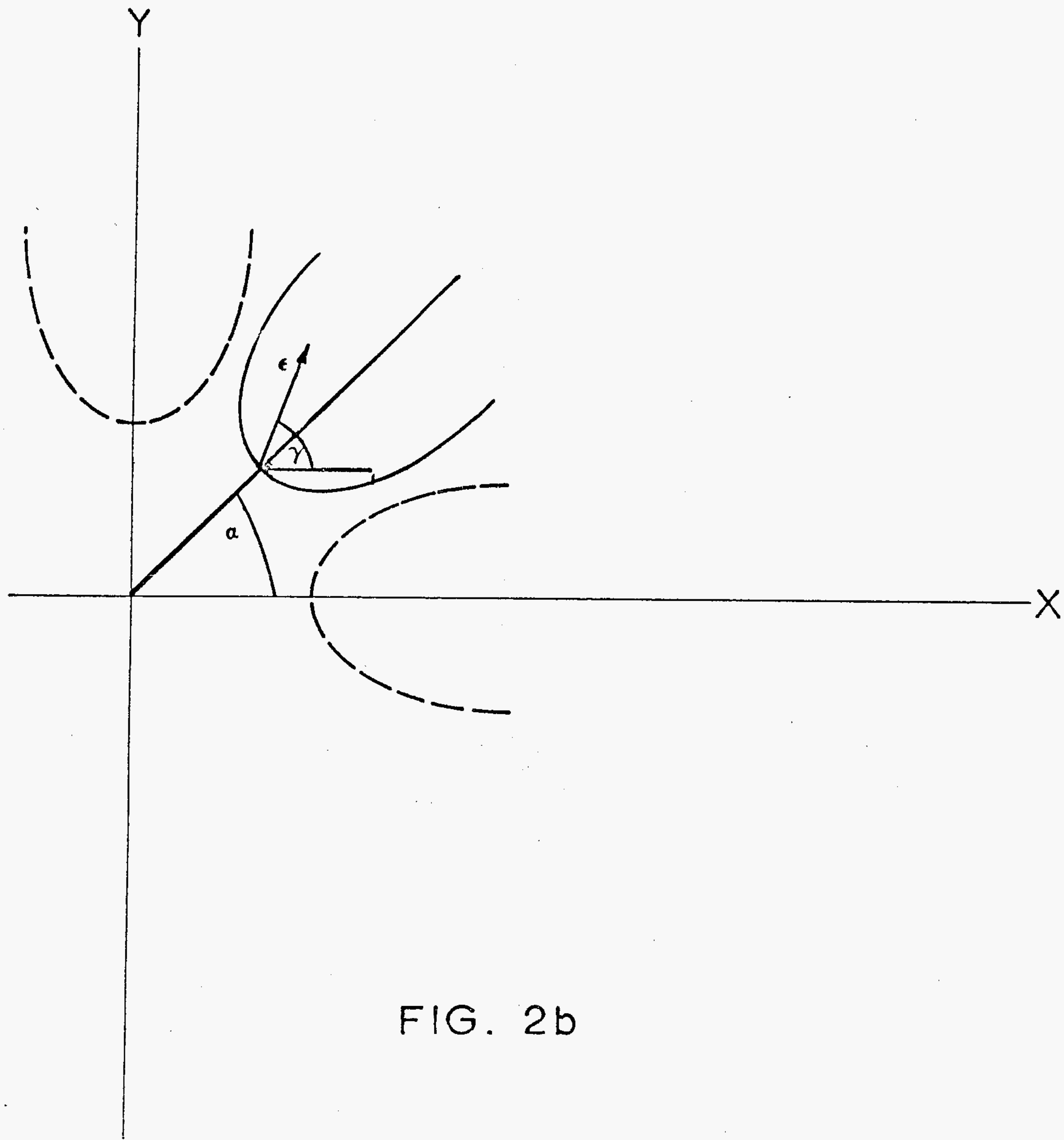

FIG. 2b

XBL 694 4823

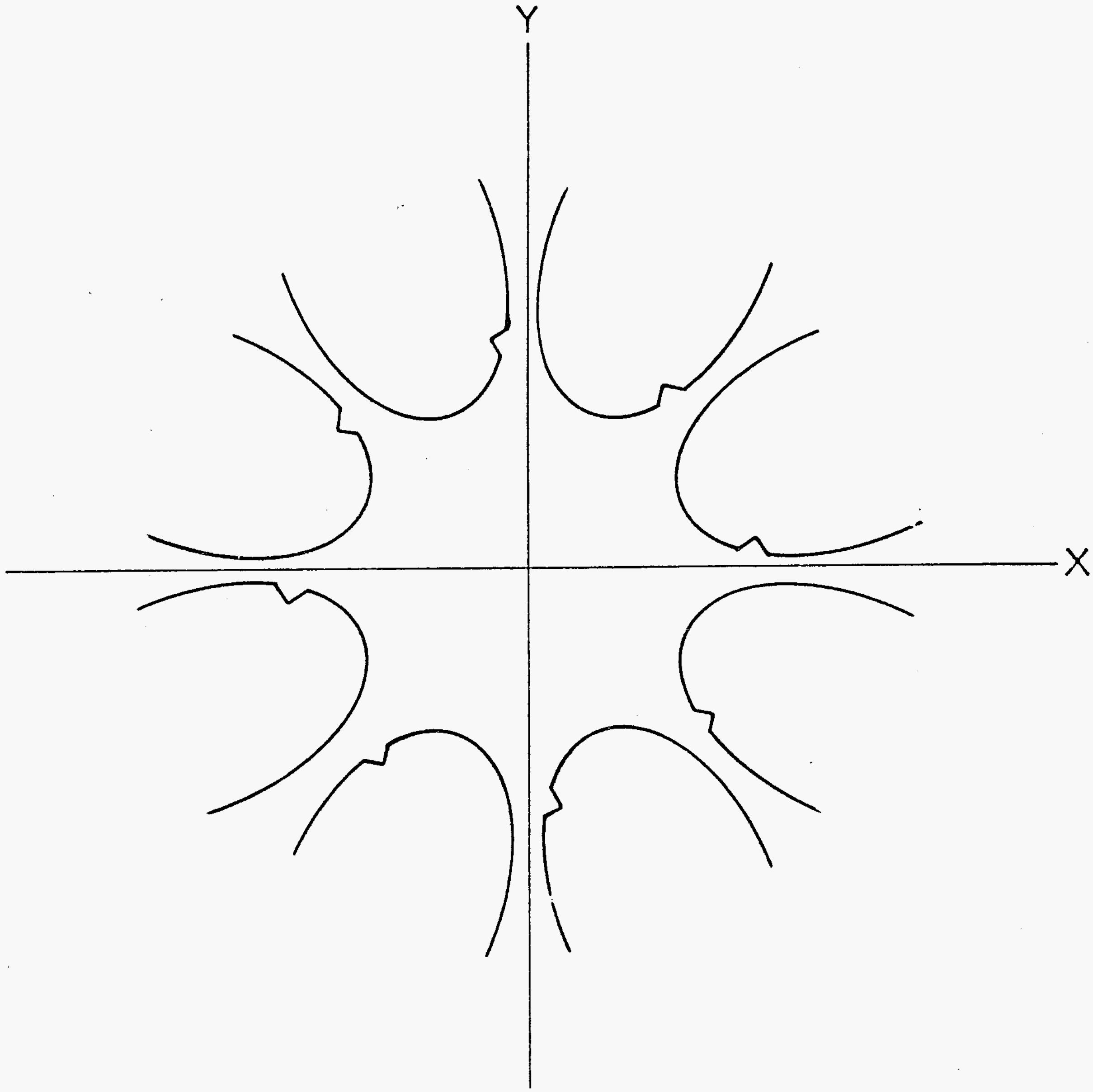

FIG. 3a

XBL 694 4824

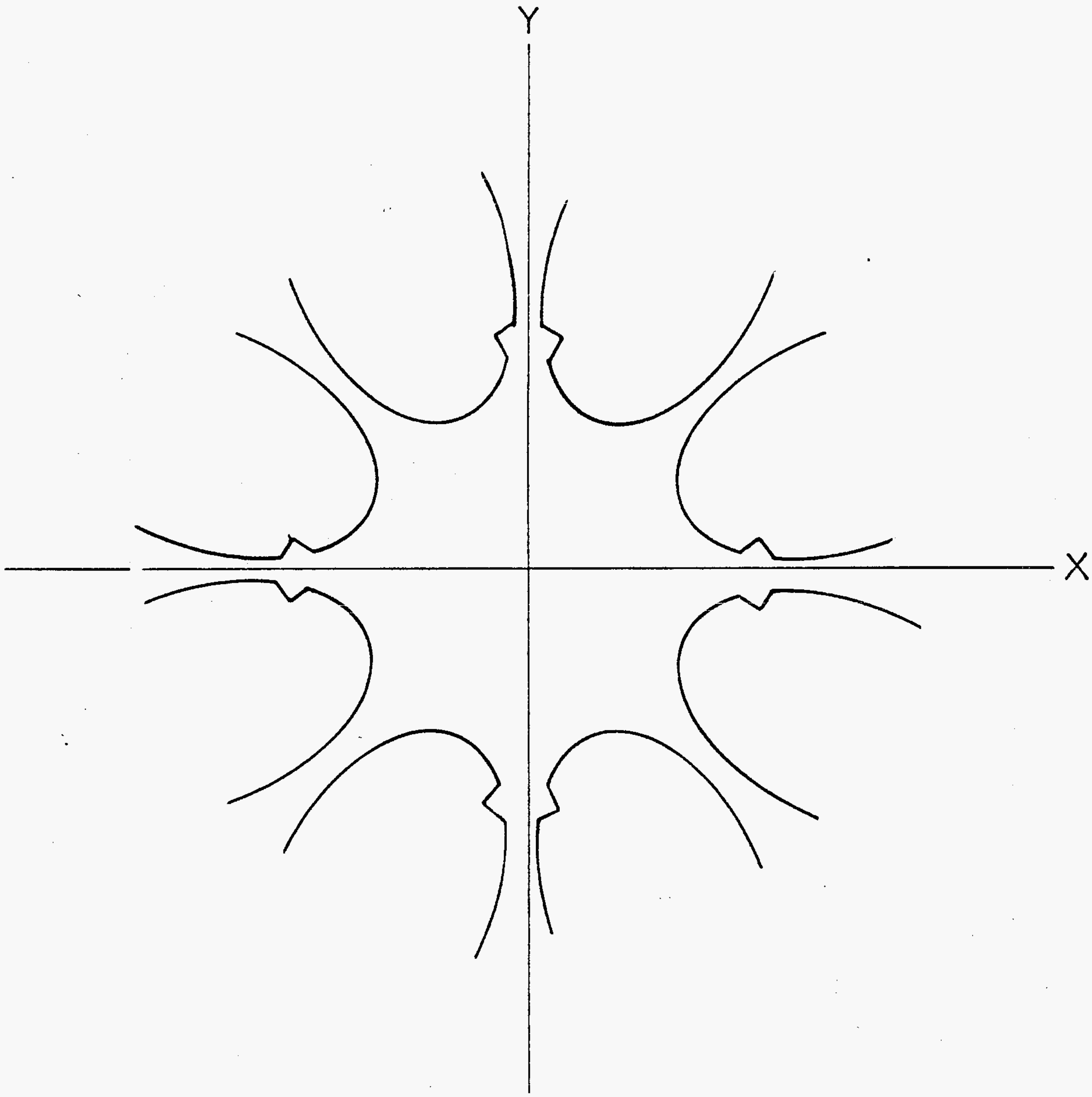

FIG. 3b

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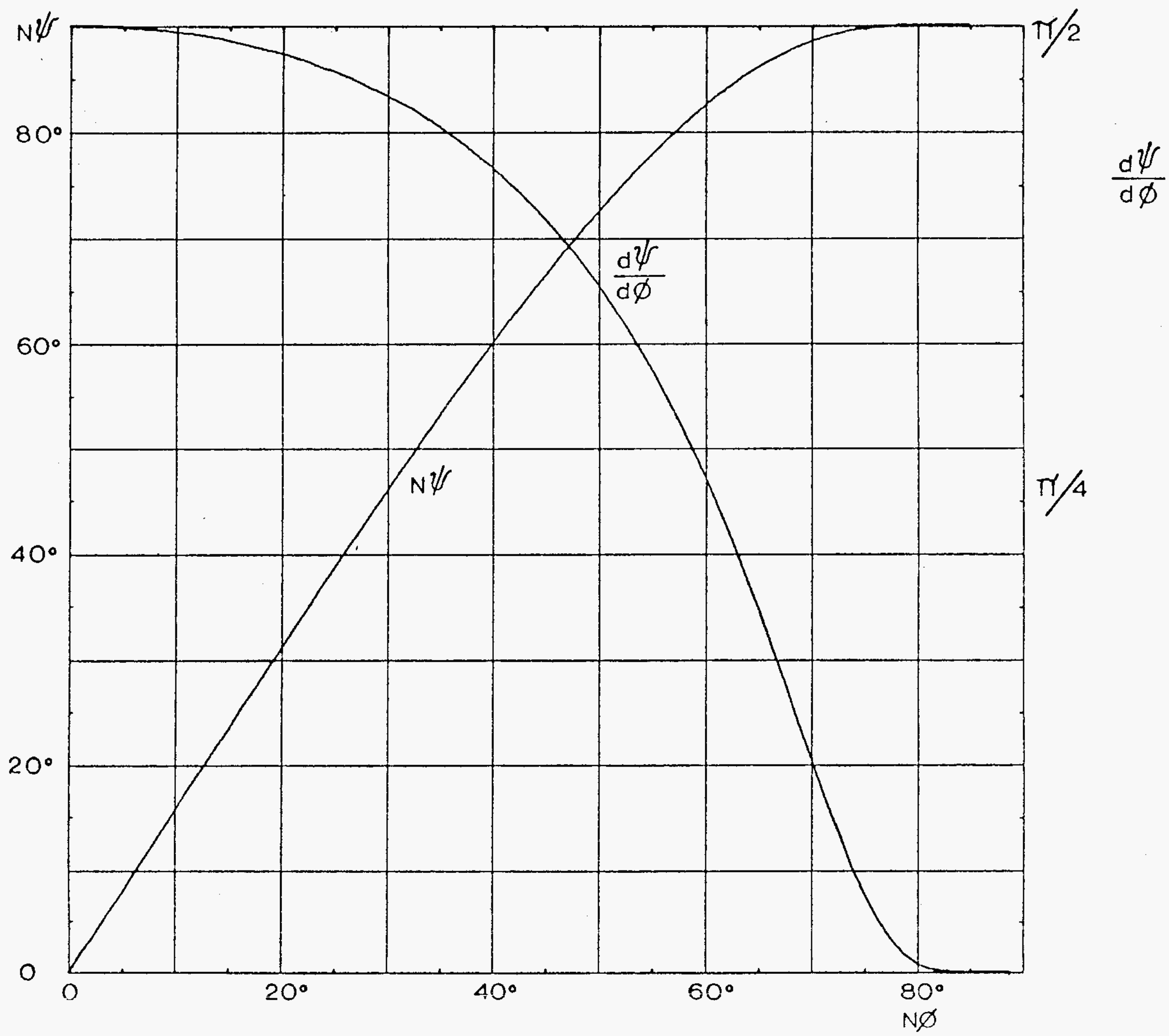

Fig. 4

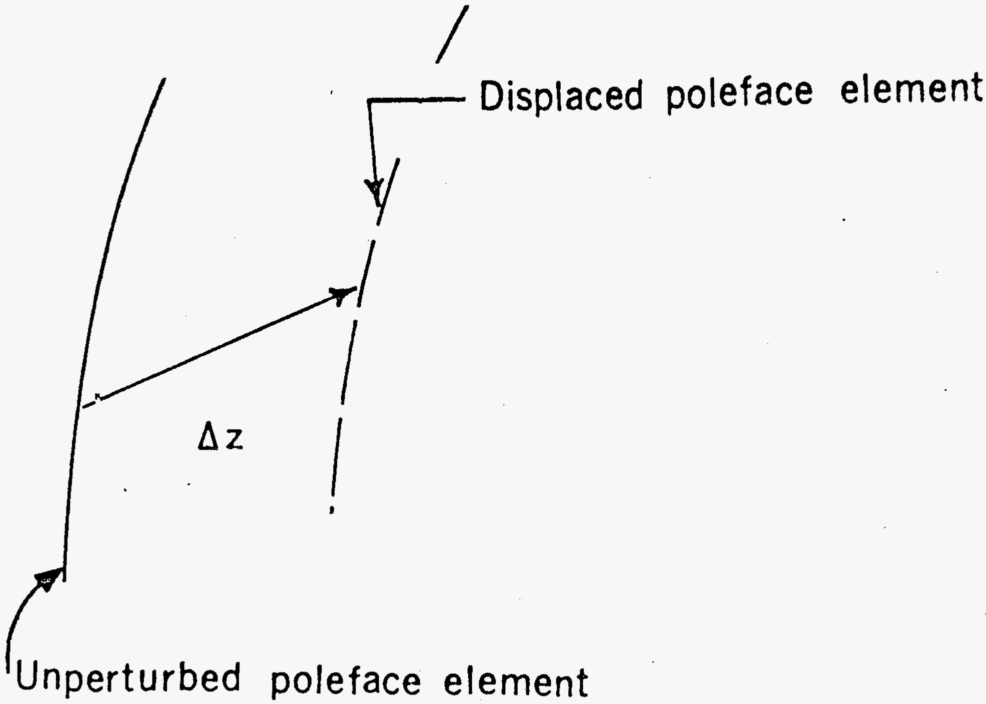

FIG. 5

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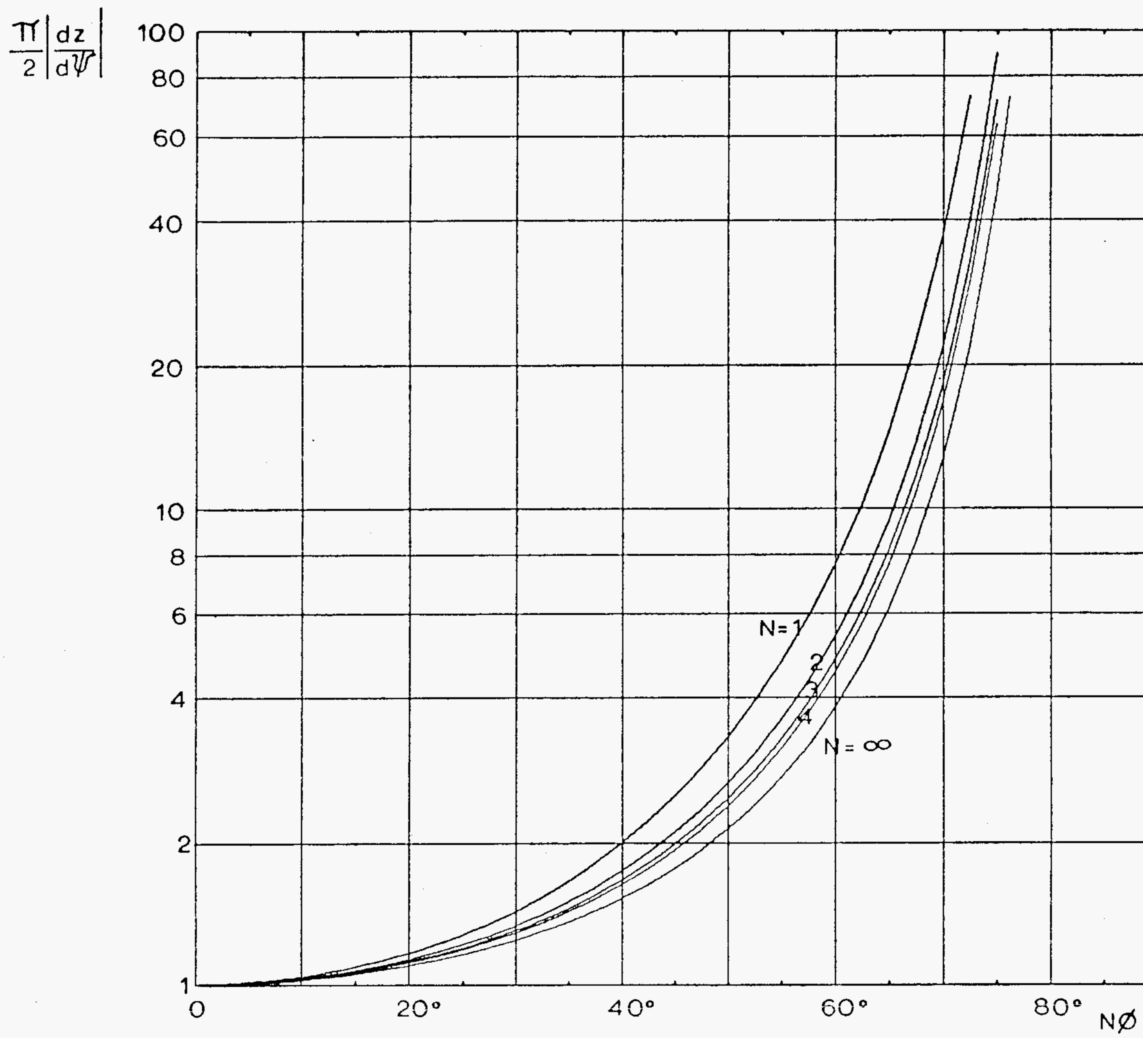

$\frac{\pi}{2}\left|\frac{dz}{d\psi}\right|$
100
80
60
40
20
10
8
6
4
2
1
0
20°
40°
60°
80°
NØ
N=1
2
3
4
N=∞

Fig. 6

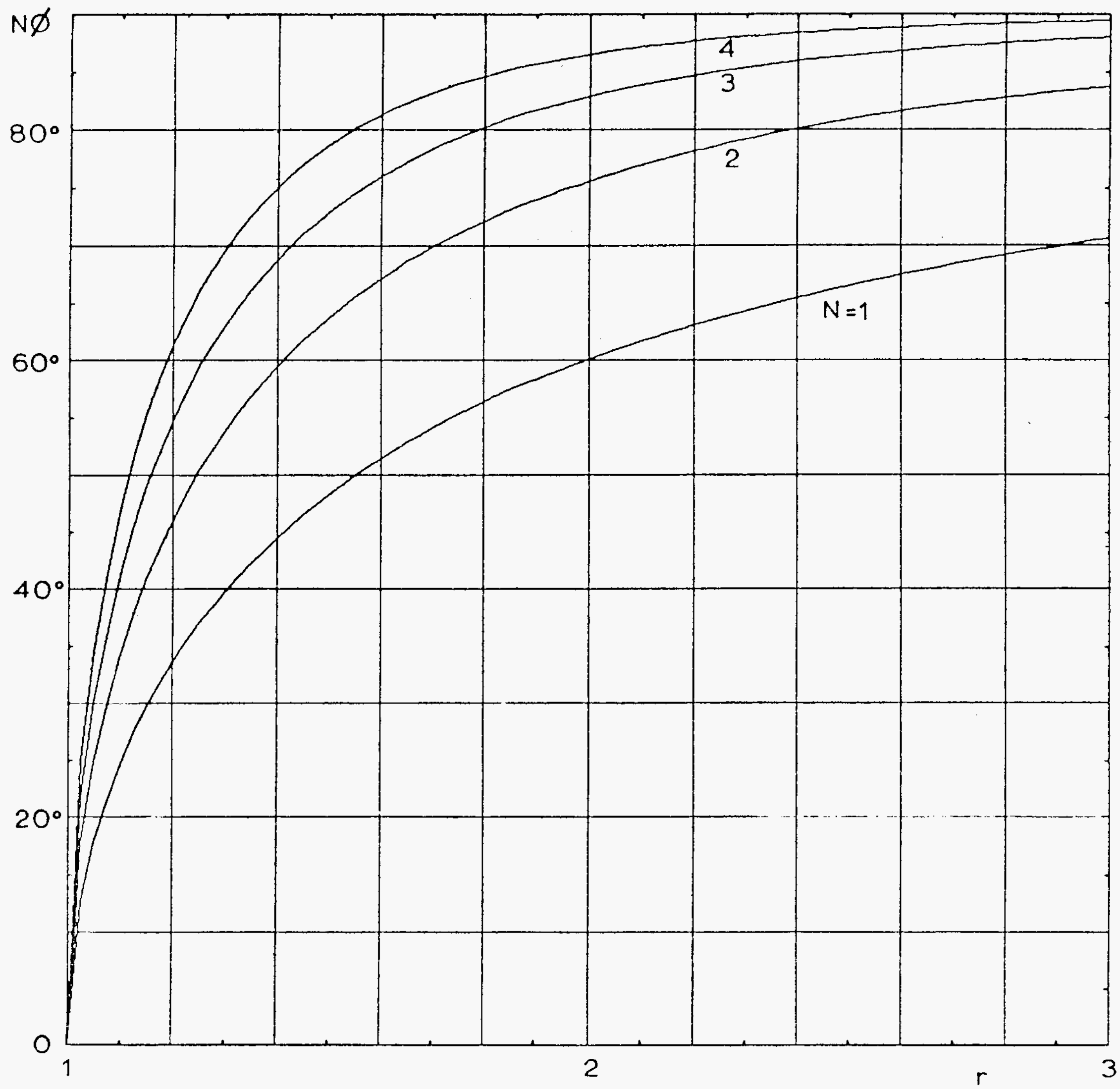

Fig. 7

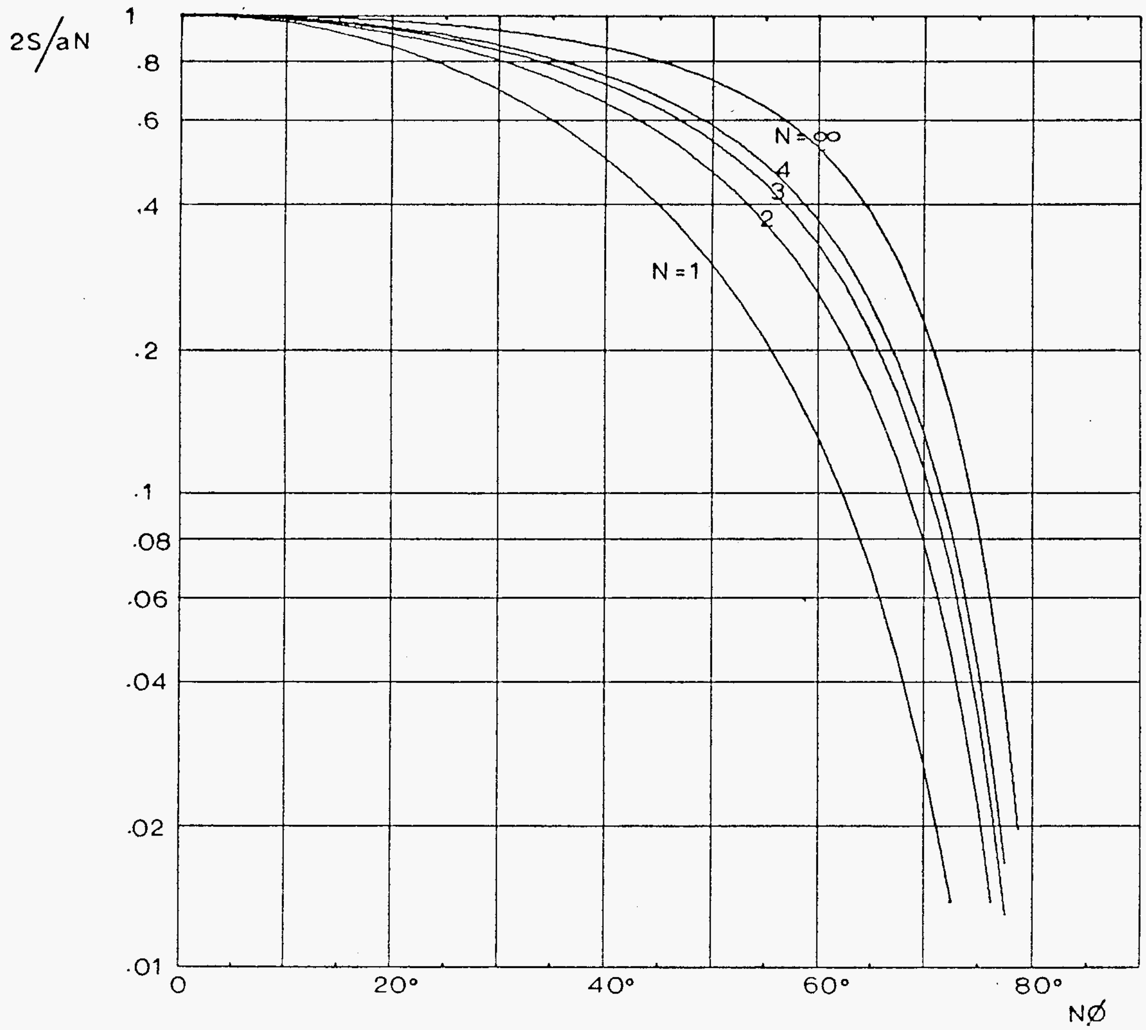

Fig. 8

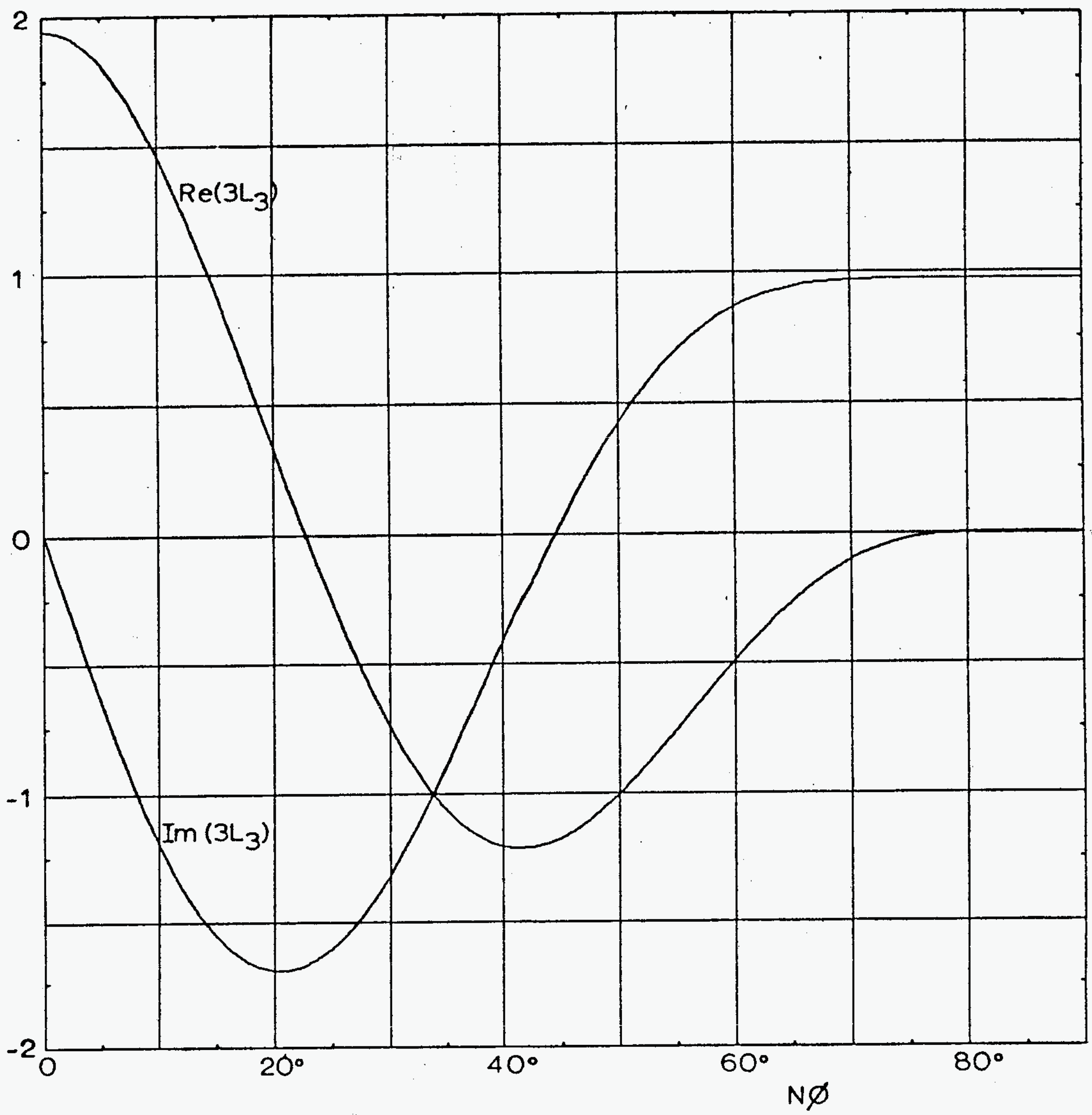

Fig. 9

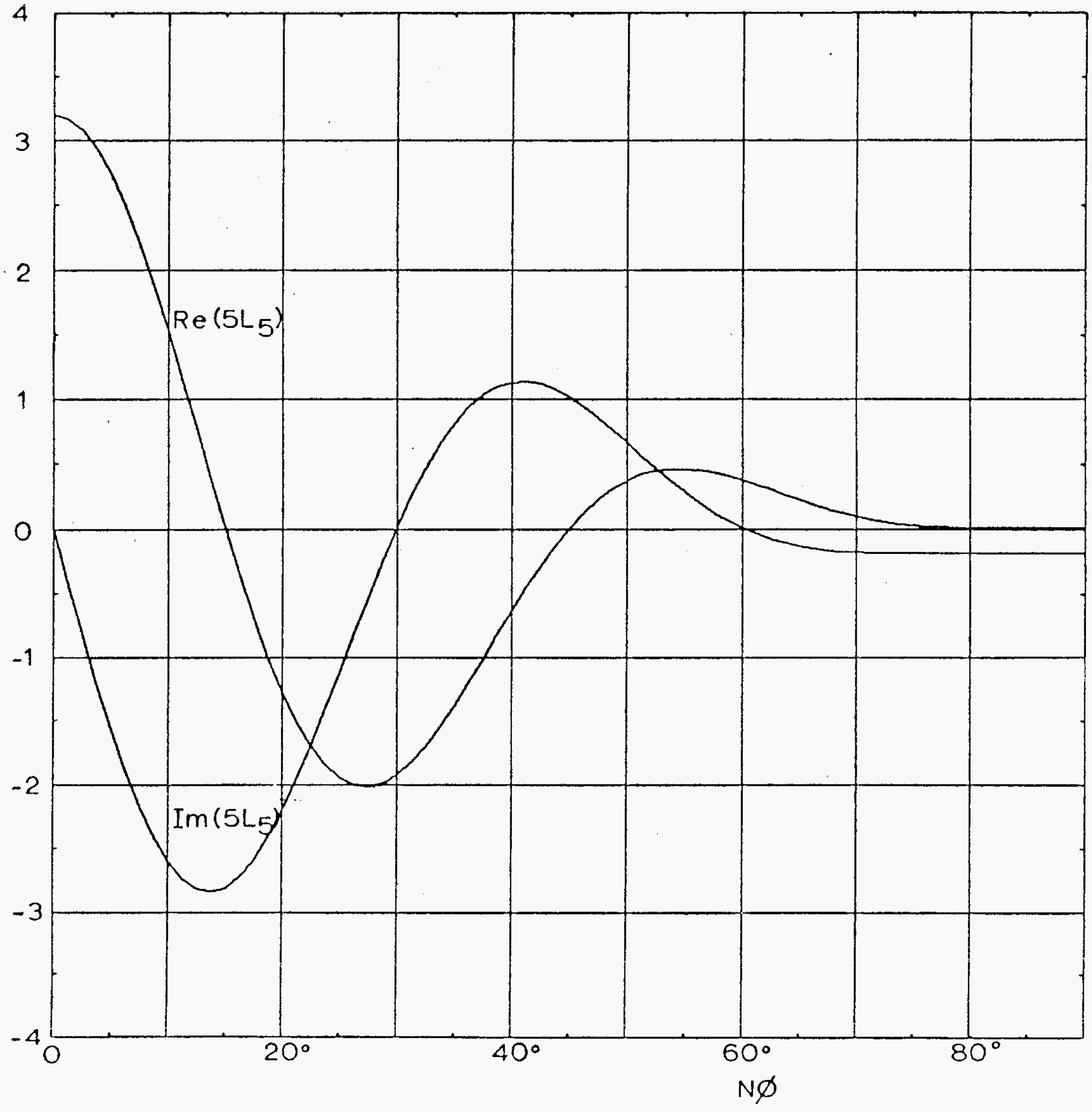

Fig. 10

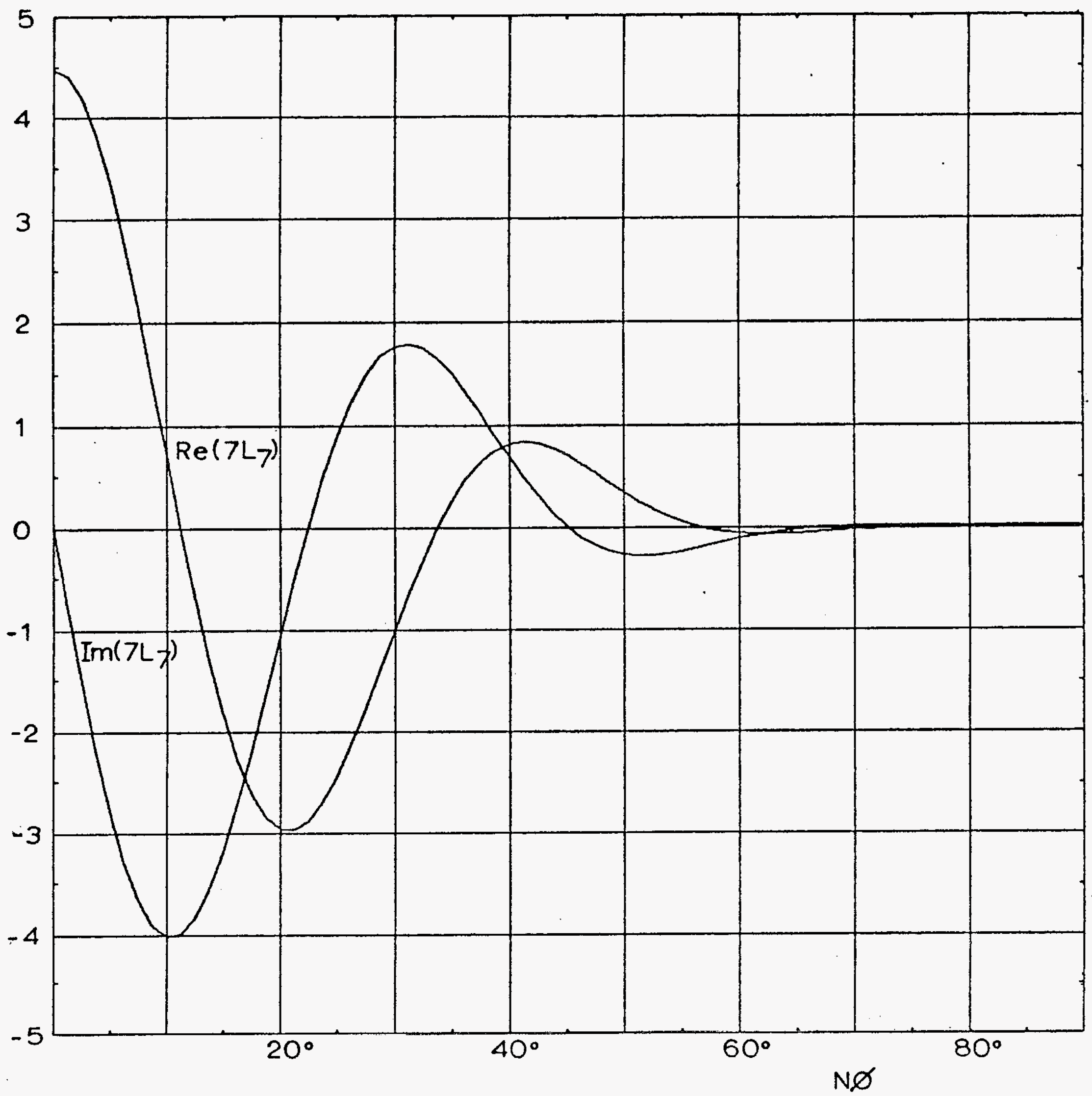

Fig. 11

Table 1

N = 2

n	$\frac{n}{N}\cdot j_n = \frac{n}{N}\cdot\frac{\Delta C_n(x)}{i\epsilon}$	$\frac{n}{N}\cdot b_n = \frac{n}{N}\cdot\frac{\Delta C_n(rd)}{i\epsilon}$	$\frac{n}{N}\cdot a_n = \frac{n}{N}\cdot\frac{\Delta C_n(ad)}{\epsilon}$	$\frac{n}{N}\cdot \rho_n = \frac{n}{N}\cdot\frac{\Delta C_n(r)}{\epsilon}$
1	1.99E-01	-4.25E-01	7.46E-02	1.76E-01
2	2.50E-01	-5.16E-01	2.14E-01	5.00E-01
3	1.57E-01	-2.88E-01	2.88E-01	6.60E-01
4	0.	6.76E-02	2.31E-01	5.00E-01
5	-2.05E-02	1.08E-01	1.08E-01	1.91E-01
6	0.	4.45E-02	2.87E-02	0.
7	1.61E-02	-1.04E-02	1.04E-02	-3.06E-02
8	0.	1.28E-02	1.56E-02	0.
9	-1.90E-03	1.25E-02	1.25E-02	7.53E-03
10	0.	6.37E-03	5.81E-03	0.
11	3.15E-03	-2.44E-03	2.44E-03	-3.62E-03
12	0.	2.66E-03	2.79E-03	0.
13	-2.45E-04	2.27E-03	2.27E-03	9.28E-04
14	0.	1.26E-03	1.23E-03	0.
15	6.69E-04	-5.55E-04	5.55E-04	-6.66E-04
16	0.	5.76E-04	5.82E-04	0.

Table 2

$N = 3$

n	$\frac{n}{N}\cdot j_n = \frac{n}{N}\cdot\frac{\Delta C_n(x)}{i\epsilon}$	$\frac{n}{N}\cdot b_n = \frac{n}{N}\cdot\frac{\Delta C_n(rd)}{i\epsilon}$	$\frac{n}{N}\cdot a_n = \frac{n}{N}\cdot\frac{\Delta C_n(ad)}{\epsilon}$	$\frac{n}{N}\cdot \rho_n = \frac{n}{N}\cdot\frac{\Delta C_n(r)}{\epsilon}$
1	9.79E-02	-3.14E-01	5.09E-02	8.47E-02
2	1.56E-01	-4.95E-01	1.71E-01	2.84E-01
3	1.67E-01	-5.15E-01	3.03E-01	5.00E-01
4	1.33E-01	-3.90E-01	3.90E-01	6.39E-01
5	7.09E-02	-1.73E-01	3.97E-01	6.43E-01
6	0.	6.55E-02	3.18E-01	5.00E-01
7	-1.34E-02	1.08E-01	1.95E-01	2.88E-01
8	-1.07E-02	9.03E-02	9.03E-02	1.08E-01
9	0.	4.16E-02	2.51E-02	0.
10	9.13E-03	-1.90E-03	1.90E-03	-3.38E-02
11	9.72E-03	-1.45E-02	5.49E-03	-2.05E-02
12	0.	1.05E-02	1.31E-02	0.
13	-1.01E-03	1.07E-02	1.36E-02	7.34E-03
14	-1.18E-03	9.85E-03	9.85E-03	5.82E-03
15	0.	5.06E-03	4.56E-03	0.
16	1.63E-03	-1.26E-03	1.26E-03	-3.66E-03
17	2.07E-03	-3.77E-03	1.18E-03	-2.54E-03
18	0.	2.02E-03	2.12E-03	0.
19	-1.12E-04	1.82E-03	2.12E-03	8.18E-04
20	-1.70E-04	1.69E-03	1.69E-03	7.84E-04
21	0.	9.30E-04	9.07E-04	0.
22	3.25E-04	-3.02E-04	3.02E-04	-6.44E-04
23	4.65E-04	-9.34E-04	2.46E-04	-4.79E-04
24	0.	4.13E-04	4.18E-04	0.

Table 3

N = 4

n	$\frac{n}{N}\cdot j_n = \frac{n}{N}\cdot\frac{\Delta C_n(x)}{i\epsilon}$	$\frac{n}{N}\cdot b_n = \frac{n}{N}\cdot\frac{\Delta C_n(rd)}{i\epsilon}$	$\frac{n}{N}\cdot a_n = \frac{n}{N}\cdot\frac{\Delta C_n(ad)}{\epsilon}$	$\frac{n}{N}\cdot \rho_n = \frac{n}{N}\cdot\frac{\Delta C_n(r)}{\epsilon}$
1	5.73E-02	-2.43E-01	3.43E-02	4.93E-02
2	9.97E-02	-4.21E-01	1.23E-01	1.76E-01
3	1.23E-01	-5.12E-01	2.38E-01	3.41E-01
4	1.25E-01	-5.13E-01	3.50E-01	5.00E-01
5	1.09E-01	-4.32E-01	4.32E-01	6.15E-01
6	7.83E-02	-2.91E-01	4.66E-01	6.60E-01
7	3.99E-02	-1.18E-01	4.42E-01	6.20E-01
8	0.	5.76E-02	3.62E-01	5.00E-01
9	-8.84E-03	9.58E-02	2.55E-01	3.40E-01
10	-1.03E-02	9.73E-02	1.53E-01	1.91E-01
11	-6.31E-03	7.25E-02	7.25E-02	7.37E-02
12	0.	3.60E-02	2.08E-02	0.
13	5.59E-03	2.56E-03	-2.56E-03	-3.12E-02
14	8.05E-03	-1.61E-02	-4.69E-03	-3.06E-02
15	6.15E-03	-1.40E-02	3.58E-03	-1.45E-02
16	0.	8.46E-03	1.07E-02	0.
17	-5.91E-04	8.86E-03	1.26E-02	6.37E-03
18	-9.49E-04	9.27E-03	1.13E-02	7.53E-03
19	-7.46E-04	7.74E-03	7.74E-03	4.49E-03
20	0.	4.00E-03	3.58E-03	0.
21	9.48E-04	-6.61E-04	6.61E-04	-3.19E-03
22	1.57E-03	-4.00E-03	-1.41E-04	-3.62E-03
23	1.36E-03	-3.73E-03	7.32E-04	-1.80E-03
24	0.	1.56E-03	1.64E-03	0.

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FIRST ORDER PERTURBATION EFFECTS IN IRON-DOMINATED TWO-DIMENSIONAL SYMMETRICAL MULTIPOLES

K. Halbach

April 1969

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UNIVERSITY of CALIFORNIA BERKELEY