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A COMPARISON OF
NUCLIDIC MASS EQUATIONS
WITH EXPERIMENTAL DATA

by

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A COMPARISON OF NUCLIDIC MASS EQUATIONS WITH EXPERIMENTAL DATA

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ABSTRACT

Existing mathematical expressions for the systematics of the nuclidic masses, binding energies, and nucleon separation energies are summarized. The calculated values of the nuclidic mass equations of Fermi, Román, Bussetti, Cameron, Tsen, Findler, Seeger, Ayres et al., Kümmel et al., Thieberger et al., Zeldes, Green, Levy, Baker, and Wing-Fong (a new mass equation) are compared with the available experimental data. Improvements are suggested for some of the observed deviations in these equations. Some expressions for the isobaric parameters B_A and Z_A (nuclear charge of the most stable isobar), and for the pairing and shell effects are compared with the experimental evidence. An improved empirical formula for Z_A and empirical equations for the deformation parameter β are presented.

CHAPTER I. INTRODUCTION

In the past twenty and some years, more than twenty mathematical expressions have been formulated for the systematics of the relative masses, binding energies, and nucleon separation energies of the atomic nuclides. In general, these mathematical expressions, hereafter collectively called nuclidic mass equations, contain two basic variables, the proton number and the neutron number of the nucleus, and many adjustable parameters or coefficients and constants. In many of the nuclidic mass equations, the numerical values of the parameters and constants have been evaluated. Thus, using an equation of this kind, one can calculate the relative mass, binding energy, and nucleon separation energy of any nuclide within the range of applicability of the equation. In addition, one can predict the energy or threshold of a nuclear reaction involving nuclides of unknown masses. The radioactive decay energy of a nuclide, and thus the approximate lifetime of the nuclide against nuclear disintegration, may also be estimated.

Some of the specific applications of nuclidic mass equations may be mentioned here. The parameters in some of the equations have been used to calculate the nuclear size (58G), surface tension (60D), nuclear deformation (59M, 63J), and pairing energies (62B). Nuclidic mass equations have been used in the calculations of the abundances of nuclides synthesized in stars by neutron capture (59BF, 57BB, 60F), of mass and kinetic energy distributions of fission products (39BW, 56F, 56M, 57HN, 62VT, 58C), and of nuclear reaction cross sections (62BO); and for estimation of the limits of stability of nuclides against nuclear disintegration (47A, 48K, 58WW), and prediction of delayed proton emitters (63V). A complete bibliography on the applications of nuclidic mass equations is beyond the scope of this paper.

In this report, existing nuclidic mass equations in which the values of the parameters and constants have been determined are briefly described in Chapter II. The nuclidic masses, binding energies, and nucleon separation energies calculated with some of these equations are compared with the reported experimental data in Chapter III. A discussion of some of the parameters used in nuclidic mass equations is given in Chapter IV. References are listed at the end of this report. For our convenience, a summary of symbols and definitions to be used in this report is given below:

M	Relative mass of a nuclide.
N	The number of neutrons in a nuclide.
Z	The number of protons in a nuclide.
A	The number of nucleons in a nuclide; mass number $A = N + Z$.
I	Neutron excess $= N - Z = A - 2Z$.
B	Binding energy of a nuclide.
E	Energy.
B_n	Separation energy of the last neutron of a nuclide.
B_p	Separation energy of the last proton of a nuclide.
M_n	The mass of a neutron.
M_p	The mass of a proton.
N_0	The number of neutrons in a closed neutron-shell configuration.
Z_0	The number of protons in a closed proton-shell configuration.
Z_A	The charge (not necessarily an integer) of the most stable isobar for a given A.

P_A	Pairing-effect term.
S	Shell-effect term.
D	Nuclear-deformation term.
δ	Coefficient for the pairing-effect term.
r_0	Coefficient for the nuclear radius.

CHAPTER II. NUCLIDIC MASS EQUATIONS

In this chapter, some existing nuclidic mass equations, both empirical and semiempirical, are briefly described. No detailed derivation of the equations is given, and the theoretical aspects of the nuclear models used in the derivation are mentioned only briefly. The nuclidic mass equations are presented here in expressions for M , $M - A$, or B , in either MeV or mmu (milli mass unit), in the physical scale of $O^{16} = 16,000.000$ mmu. The values of the parameters and coefficients are all taken from the original references.

The nuclidic mass equations which are derived from the general concepts of the same nuclear model are grouped together for convenient description. The nuclear models which have been used in the derivation of these equations include the statistical model, independent-particle (shell) model, and the empirical model. The formulas in each of these three groups are introduced in the order of their appearance in the literature.

It must be emphasized here that the classification of nuclidic mass equations in this way is by no means significant and clear-cut as far as the nuclear models are concerned. In fact, as we shall see later, many of the equations based on different models are equivalent to each other. For example, the Bohr-Wheeler equation and its subsequent modifications are grouped under "Empirical Approach," although they are in essence different versions of the Bethe-Weizsäcker mass formula, which is based on a liquid-drop (statistical) model.

Section A. Statistical-model Approach

In using a statistical-model approach for the derivation of a nuclidic mass equation, one is dealing with the various interactions of the nucleons in the nucleus, such as the potential and kinetic energies of the nucleons and the Coulomb interaction between protons. The liquid-drop model is frequently used. Some of the theoretical expressions are complicated functions.

In this section, the simplified forms of the expressions are presented. Many of the parameters in the mass equations to be described in this section are related to the nuclear properties of the ground state, such as nuclear radius and surface tension. The numerical values of these parameters have been obtained either from empirical fits, or from theoretical estimates or computations, or from independent measurements of the nuclear properties.

1. Bethe-Weizsäcker

Weizsäcker (35W) presented a mathematical description for nuclear energy on the basis of the statistical model of a liquid drop. He divided the total nuclear energy into four components: the volume energy E_V , the surface energy E_S , the Coulomb energy E_C , and the symmetry

energy E_a . For a system containing a large number of rigid spheres held by charge-independent nuclear forces in a spherical, closest-packing arrangement (an incompressible drop), his description stated, in a simplified way, that the volume energy is proportional to A and the surface energy is proportional to $A^{2/3}$. If the protons are uniformly distributed within the nuclear sphere with a radius $r_0 A^{1/3}$, then the Coulomb energy would be proportional to $Z^2/A^{1/3}$.

In a nucleus, there exist strong exchange forces which are attractive for particles in the same quantum state but are repulsive between identical particles in different quantum states, even if these states are degenerate. The most stable nuclear system containing A nucleons would have an apportionment of N and Z nucleons with the greatest number of attractive bonds and the least number of repulsive bonds. Thus, the symmetry energy is proportional to I^2/A .

The total nuclear energy, then, is

$$M(Z, N) = ZM_p + NM_n + E_v + E_a + E_s + E_c$$

$$= ZM_p + NM_n - a_v A + a_a I^2/A + a_s A^{2/3} + a_c Z^2/A^{1/3},$$

where a_v , a_a , a_s , and a_c are parameters. This is the simplified form of Weizsäcker's description presented by Bethe and Bacher (36BB) and is commonly known as the Bethe-Weizsäcker mass formula. The values of a_v , a_a , a_s , and a_c have been evaluated empirically and semiempirically by many people, and are summarized in Table 1. Kumar and Preston (55KP) and Ayres *et al.* (62AH), have shown that these parameters are slowly varying functions of A .

Table 1
VALUES OF THE PARAMETERS IN THE BETHE-WEIZSÄCKER'S MASS FORMULA (in MeV)

Author(s)	Reference	M_n (MeV)	M_p (MeV)	a_v (Volume)	a_a (Symmetry)	a_s (Surface)	a_c (Coulomb)	P (Pairing)
Bethe and Bacher	36BB	7.868	7.514	13.860	19.50	13.20	0.580	
Bohr and Wheeler	39BW				Table	14	0.59	Table
Mattauch and Flüge	46MF	8.329	7.571	14.66	20.54	15.4	0.602	
Fermi	46F, 50F	8.315	7.564	14.00	19.3	13.0	0.583	33.5/A ^{3/4}
Goodman	47G			13.97	19.32	13.0	0.645	
Feenberg ^a	47F	8.306	7.570	14.0	18.10	13.10	0.584	Graph
Albada	47A			14.2	20.75	15.0	0.63	
Fowler ^a	47FH	8.315	7.572	15.30	22.55	16.75	0.690	
Rosenfeld ^b	48R			14.66	20.54	15.4	0.602	
Friedlander and Kennedy ^a	49FK	8.364	7.582	14.1	18.1	13.1	0.585	132/A
Pryce	50P	8.315	7.564	14.05	19.6	14.0	0.61	
Metropolis and Reitwiesner	50MR	8.364	7.582	14.044	19.31	13.0	0.583	33.5/A ^{3/4}
Green and Engler	53GE	8.364	7.582	15.569	23.28	17.226	0.698	10/A ^{1/2}
Green	54G, 55G	8.364	7.582	15.753	23.69	17.804	0.710	12/A ^{1/2}
Bussetti ^a	55B	8.370	7.585	14.838	19.439	12.733	0.627	36/A ^{3/4}
Evans	55E	8.364	7.582	14.1 ± 0.2	19.0 ± 0.9	13 ± 1	0.595 ± 0.02	Graph
Kumar and Preston	55KP			15.73	23.28	18.11	0.70	Table
Tsen ^c	57T	8.366	7.585	14.672	17.177	14.629	0.484	
Wapstra	58W	8.367	7.584	15.835	23.20	18.33	0.714	11.2/A ^{1/2}
Ayres <i>et al.</i>	62AH	8.367	7.585	14.75	15.1-19.3	14.8-16.0	0.63-0.645	Graph
Dutta <i>et al.</i>	62DP			15.568	22.547	17.231	0.699	

^a Coulomb energy proportional to $Z(Z-1)$ instead of to Z^2 .

^b Same as Mattauch and Flüge (46MF).

^c Coulomb energy proportional to $Z^{5/3}$ instead of to $Z^2/A^{1/3}$.

The Bethe-Weizsäcker mass formula has been used, modified, and refined by numerous persons. Many of the subsequent nuclidic mass equations in the literature are modifications of this basic equation.

2. Wigner

Wigner (37W) described the binding energy of a nucleus as the sum of the potential, kinetic, and Coulomb energies. The Coulomb energy, proportional to $Z(Z-1)/A^{1/3}$, was similar to Weizsäcker's, and the kinetic energy of the nucleons was taken as the sum of Weizsäcker's volume and symmetry energies: $c_1A + (c_2I^2/A)$. The potential energy of the nucleus was expressed in terms of two functions, L and L' : Potential energy = $-A(A-1)L'/2 - CL$, where L is the average magnitude of the potential energy from forces which change sign with symmetry, L' is the average magnitude of the potential energy between pairs of nucleons from forces independent of symmetry, and C is the number of symmetric couplings minus the number of antisymmetric couplings, and C is a function of A and I . The binding energy of a nucleus is, in a simplified form:

$$B = c_0Z(Z-1)/A^{1/3} + c_1A + (c_2I^2/A) - A(A-1)L'/2 - CL.$$

This basic formula was further developed by Barkas (39B), Gamow and Critchfield (49GC), Collins et al. (52CN), and Halsted (52H), and the expression becomes (in mmu):

$$B = 0.635 Z(Z-1)/A^{1/3} + 14.64 A + 32.53 \left(I^2/4 - \frac{1}{2} \delta \right) / A \\ - \frac{1}{2} A(A-1)L' - L[2A - (A^2/8) - (I^2/8) - I - (3\delta/4)],$$

where

$$\delta = 0 \text{ for even } Z, \text{ even } N; \quad 1 \text{ for odd } A; \quad 2 \text{ for odd } Z, \text{ odd } N.$$

The values of L and L' have been evaluated empirically by Collins et al. (52CN) and Halsted (52H) at various regions of A , and are listed in Table 2. These workers also indicated the need of shell corrections in this equation.

Table 2

VALUES OF L AND L' IN WIGNER'S FORMULA (in mmu)		
Region	L	L'
$N \leq 20$	40.31/ A	(51.926/ A) + 0.1696
$N \geq 20; Z \leq 20$	55.52/ A	(54.726/ A) + 0.1570
$N \leq 28; Z \geq 20$	0.8989	(46.632/ A) + 0.2814
$N \geq 28; Z \leq 28$	1.2316	(46.212/ A) + 0.3504
$Z \leq 50$	87.42/ A	(68.692/ A) + 0.04129
$Z \geq 50$	0.56992	(50.398/ A) + 0.15760

3. Fermi

Fermi (46F, 50F) derived the following equation (in mmu), which is equivalent to the Bethe-Weizsäcker formula except for the pairing-effect term, $\lambda/A^{3/4}$, and is sometimes called the Fermi-Weizsäcker mass equation:

$$M(A, Z) = 1.01464 A + 0.014 A^{2/3} - 0.041905 Z_A \\ + 0.041905 (Z - Z_A)^2 / Z_A + 0.036 \lambda / A^{3/4},$$

where

$$Z_A = A / (1.98067 + 0.0149624 A^{2/3})$$

and

$$\lambda = +1 \text{ for even } A, \text{ odd } Z; \quad -1 \text{ for even } A, \text{ even } Z; \quad 0 \text{ for odd } A.$$

This equation may be considered as another version of the Bohr-Wheeler mass equation (see Section C of this chapter). Numerical values of $M(A, Z)$ in the ranges from $A = 1$ to 260 and $Z = 1$ to 100 have been calculated by means of this equation and tabulated (50MR, 54M).

Dube and Singh (54DS) have added correction terms to this equation for the calculation of alpha disintegration energies in the rare-earths region.

4. Feenberg

Feenberg (47F) modified the Bethe-Weizsäcker mass formula to include the effects of compressibility of a liquid as well as the odd-even effects (pairing energy) of the nucleus:

$$E = E_V + E_S + E_a + E_C - [(E_S k_S + E_a k_a + E_C)^2 / 2E_V''] + (\delta / A) \\ = -a_V A + a_S A^{2/3} + (a_a I^2 / A) + [a_C Z(Z + 1) / A^{1/3}] \\ - \frac{1}{2E_V''} \{k_S a_S A^{2/3} + (k_a a_a I^2 / A) + [a_C Z(Z + 1) / A^{1/3}]\}^2 + (\delta / A).$$

The term containing the braces arises from the change in volume of the nucleus under the internal pressure produced by the electrostatic force and by the deficiency in binding energy associated with E_a and E_S . The coefficients k_S and k_a are measures of the changes in E_S and E_a , respectively, resulting from these effects. The quantity E_V'' is a compressibility coefficient. A few values of k_S and E_V'' were assumed in the

study, but no unique values of k_s , k_a , and E_V'' were determined. The values of a_V , a_s , a_c , and a_a for several sets of k_s and E_V'' are listed in Table 3. The pairing effects were given in the original reference (47F) in a graphical plot.

Table 3
VALUES (in mmu) OF THE PARAMETERS IN
FEENBERG'S EQUATION

E_V''	k_s	$a_c/4$	a_a	a_s	a_V
50 A	0.0	0.161	19.0	13.9	14.9
50 A	0.5	0.169	19.2	14.5	15.1
30 A	0.5	0.178	19.0	14.8	15.1
∞	$\sim$	0.157	19.4	14.1, 14.8	15.1

Feenberg also correlated some of the parameters in the Bethe-Weizsäcker mass formula with those in the Wigner and Bohr-Wheeler (cf. Chapter II, Section C-1) mass equations.

5. Román

Román (51R) considered the nuclear radius to be

$$r_0 [1 + 0.8 (I^2/A^2)] A^{1/3}$$

instead of $r_0 A^{1/3}$. Starting with a potential function given by $(\Delta E_V - \mu^2 E_V)$, where μ is related to the π -mesonic mass, and with a homogeneous and saturation density distribution, he derived the following expression for the binding energy of a nucleus (in MeV):

$$B = \frac{16}{[1 + 0.8 (I^2/A^2)]^3} \left\{ -A + 1.243 \frac{A^{2/3}}{[1 + 0.8 (I^2/A^2)]} - \frac{0.853}{[1 + 0.8 (I^2/A^2)]^3} \right\} \\ + \frac{0.6084 Z^2}{[1 + 0.8 (I^2/A^2)] A^{1/3}}.$$

The first, second, and last terms are expressions for the volume, surface, and Coulomb energies, respectively. The third term is a small correction term, having no direct physical meaning, the effect of which is to diminish the surface tension by a small amount. No expression was given for the interactions between protons and neutrons, i.e., the symmetry or exchange energy.

6. Flügge and Woeste

Flügge and Woeste (52FW) examined the compressibility of a liquid drop and the mutual displacement of the protons and neutrons in the nucleus. They showed that both the proton density and the neutron density increase toward the nuclear surface with increasing A . As a result, the nuclear radius is not strictly proportional to $A^{1/3}$, as for incompressible matter, but should increase somewhat more rapidly with A , i.e.,

$$r_0 = C_0(1 + C_1 Z A^{-1/3} - C_2 A^{-1/3}).$$

A series of correction terms were derived for the surface energy, Coulomb energy, and symmetry energy in the Bethe-Weizsäcker formula on the basis of this argument. A compression-energy term was introduced. Their expression for binding energy is (in MeV):

$$B = -E_V + 20 [1 - 2(Z/A)]^2 A + 14 A^{2/3} + 0.618 (Z^2/A^{1/3}) \\ + (0.000212 Z^2 + 3.86)A^{1/3} + [(0.017 Z^2 - 0.001872 Z^3)/A^{2/3}],$$

where E_V is the volume energy of an incompressible liquid drop.

7. Cameron

Cameron (57C) modified the volume, surface, and Coulomb energy terms in the Bethe-Weizsäcker formula and introduced a Coulomb exchange energy term, E_{ex} , plus shell-effect and pairing-energy terms. The volume energy was expressed as the sum of the volume saturation energy proportional to A and the volume symmetry energy proportional to I^2/A . The surface energy was modified to include a symmetry energy contribution, as suggested by Feenberg (47F). The Coulomb energy was calculated for a trapezoidal radial model of the nucleus (36BB). The Coulomb exchange energy was introduced on the basis of correlations in the motions of the protons in the nucleus.

The mass-excess equation was written in the following form (in MeV):

$$M - A = 8.367 A - 0.783 Z + E_V + E_S + E_C + E_{ex} + S(Z, N) + P(Z, N),$$

where

$$E_V = -17.0354 [1 - 1.846 I^2/A^2]A,$$

$$E_S = 25.8357 A^{2/3} [1 - 1.712 I^2/A^2] \left[1 - \frac{0.62025}{A^{2/3}} \right]^2,$$

$$E_C = 0.779 \frac{Z(Z-1)}{A^{1/3}} \left[1 - \frac{1.5849}{A^{2/3}} + \frac{1.2273}{A} + \frac{1.5772}{A^{4/3}} \right],$$

and

$$E_{ex} = -0.4323 \frac{Z^{4/3}}{A^{1/3}} \left[1 - \frac{0.57811}{A^{1/3}} - \frac{0.14518}{A^{2/3}} + \frac{0.49597}{A} \right].$$

The values of $(S+P)$, a total number of 310, for different proton and neutron numbers are listed in the original reference and are too numerous to be included here. The calculated values of mass excess, separation energies of proton, neutron, and alpha particle, as well as beta-decay energies for $Z = 11$ to 122 and $A = 22$ to 322, using this equation, are available (57C).

8. Tsen

Tsen (57T) modified the Coulomb energy term in the Bethe-Weizsäcker mass formula to $a_c Z^2/Z^{1/3}$ on the assumption that the charge distribution radius of a nucleus is proportional to $Z^{1/3}$ instead of $A^{1/3}$. His mass equation, without any pairing-effect terms, is (in mmu)

$$M = 8.146 Z + 8.985 N - 15.757 A + 15.711 A^{2/3} + 0.52 Z^{5/3} \\ + 18.447 I^2/A.$$

He claimed that this modification improved the agreement between the calculated binding energies with the experimental data and that this equation predicted very accurately the mass numbers, A^* , of the most stable isotopes. However, the expression given for A^* in the reference (57T) was not for A^* , but for Z_A . The correct expression for A^* should be

$$A^{*2} + 0.8971 A^{*5/3} = 1.580 Z^2.$$

9. Findler

Findler (58F) considered the total nuclear energy as a sum of the kinetic energy E_k , Coulomb energy E_C , Coulomb exchange energy of protons E_r , and nuclear interaction energy E_u , based on a two-body, static, and charge-independent potential:

$$B = E_k + E_C + E_r + E_u.$$

The kinetic energy consists of the Fermi-gas kinetic energy E_f and the Weizsäcker energy E_w of nonhomogeneous density distribution. The

expressions for all these energy terms in the simplified forms, are

$$E_k = [1 - (4/A)] E_f + E_w;$$

$$E_f = C_1 A^{5/3} + C_2 (Z^2/A^{1/3}) + C_2' A^{2/3} Z;$$

$$E_w = C_3 A;$$

$$E_C = C_4 Z^2;$$

$$E_r = C_5 [16 A^{1/3} Z + 8 (Z^2/A^{2/3}) - A^{4/3}];$$

$$\begin{aligned} E_u = & C_6 A^2 + C_7 Z^2 + C_8 A^{7/3} + C_9 A^{8/3} + C_{10} A^3 + C_{11} Z^2 A^{1/3} \\ & + C_{12} Z^2 A^{2/3} + C_{13} Z^2 A + C_{14} (Z^2/A^{1/3}) + C_{15} A^{5/3} + C_{16} A Z \\ & + C_{17} Z A^{4/3} + C_{18} Z A^{5/3} + C_{19} Z A^2 + C_{20} Z A^{2/3}. \end{aligned}$$

The parameter $C_1 \dots C_{20}$ are combined forms of many coefficients given in the original equations. The values of these coefficients were not explicitly stated in the original article.

10. Gunter and Hubbs

Gunter and Hubbs (59GH) revised the Bethe-Weizsäcker mass formula to include the effect of a diffuse nuclear surface. The volume and surface effects were combined by integrating over an assumed trapezoidal density function. A series of terms in the function of the ratio of one-half of the surface depth to $r_0 A^{1/3}$ were added to the volume-surface and the Coulomb energy terms. The Coulomb energy term was also modified by replacing Z^2 by $Z(Z-1)[1 - (0.767/Z^{2/3})]$, as suggested by Cooper and Henley (53CH). Their equation for nuclidic masses in a simplified form is (in MeV)

$$\begin{aligned} M = & ZM_p + NM_n - \frac{a_v A}{1.263} \left[1 - \frac{1.111}{A^{1/3}} + \frac{1.234}{A^{2/3}} - \frac{0.2742}{A} \right] \\ & + \frac{0.7992 Z(Z-1)[1 - (0.767/Z^{2/3})]}{A^{1/3} [1 + (1.234/A^{2/3})]^2} \\ & \left[1 + \frac{1.028}{A^{2/3}} + \frac{0.6855}{A} + \frac{0.254}{A^{4/3}} - \frac{0.04031}{A^{5/3}} \right] + a_a (I^2/A) \pm \delta. \end{aligned}$$

The values of a_v , a_a , and δ were not given.

11. Mozer

Mozer (59M) modified the Coulomb energy term in the Bethe-Weizsäcker formula into the following form (in MeV):

$$E_c = 0.69614 \left(\frac{Z^2}{A^{1/3}} - 0.764 \frac{Z^{4/3}}{A^{1/3}} - 1.707 \frac{Z^2}{A} + \dots \right).$$

He also introduced shell-structure effects and nuclear-deformation energy, and mentioned the effects on the nuclear masses of introducing an isotopic exchange term proportional to I/A , as well as isotopic surface-correction terms proportional to $I/A^{4/3}$ and $I^2/A^{4/3}$. His mass formula is (in MeV)

$$\begin{aligned} M(Z,A) = & ZM_p + NM_n - 15.297 A + 29.188 (I^2/A) + 18.342 A^{2/3} \\ & + \frac{1}{2} \delta + 0.69614 (Z^2/A^{1/3}) - 0.53185 (Z^{4/3}/A^{1/3}) \\ & - 1.1883 (Z^2/A) + S + D, \end{aligned}$$

where

$$\delta = 0 \text{ for odd } A; \quad +1 \text{ for odd } Z, \text{ odd } N; \quad -1 \text{ for even } Z, \text{ even } N.$$

No expression was given for δ .

The expressions for the shell effects and nuclear deformation are

$$S(A) = (V_{\max} - a_v) \left[-\frac{3}{5} A + \sum_A (A_k/A)^{n/3} \right]$$

and

$$D(Z,A) = -C_2 \lambda^2 g + 2C_3 \lambda^3 g,$$

where

$$C_2 = 10 A^{2/3} \left[1 - 0.0155 \frac{Z(Z-1)}{A} \right],$$

$$C_3 = 0.9524 A^{2/3} \left[1 + 0.031 \frac{Z(Z-1)}{A} \right],$$

$$\lambda = \frac{C_2 - (C_2^2 - 3C_1C_3)^{1/2}}{3C_3},$$

and

$$C_1 = n(V_{\max} - a_v) \sum f_m (A_k/A)^{n/3}.$$

The quantity $V_{\max}$ is the interaction potential of the nucleon with the maximum kinetic energy. Here A_k is the mass number of the stable nuclide exactly intermediate between two closed shells with respect to the nuclear type of interest; its values are listed in Table 4. For a harmonic oscillator potential $n = 1$ and $n = 2$ for a deep square-well potential. The values of f_m , a deformation parameter, are not available except that $f_m = 0$ for $N < 82$ and for closed-shell configurations. Typical values of some of the parameters are listed in Table 5.

Table 4

VALUES OF A_k FOR THE m th NUCLEON

m	A_k (Protons)	A_k (Neutrons)
≤ 2	1	1
3 to 8	10	10
9 to 20	29	27
21 to 28	53	44
29 to 50	89	70
51 to 82	161	116
83 to 126	269	176
127 to 184		255

Table 5

VALUES OF THE PARAMETERS IN MOZER'S EQUATION (in MeV)

Well Type	n	a_v	a_a	a_s	a_c	$V_{\max} - a_v$	g
Square well	2	15.282	22.755	15.757	0.69016	8.5183	1.2226
Harmonic oscillator	1	17.980	22.899	15.963	0.69412	17.461	1.2930

12. Maldonado

Maldonado (60M) investigated the effects of introducing a repulsive core to the potential of interaction in a nucleus. In his treatment, the total energy of a nucleus consisted of the kinetic energy, proportional to $A^{5/3}$, and the potential energy, proportional to A^2 . The electrostatic

energy of repulsion of protons has been disregarded. The binding energy of a nucleus is (in MeV)

$$B = 46.549 A^2 \left(\frac{0.87802}{x_0^2 A^{1/3}} + \frac{0.2773}{x_0^2} - \frac{0.9978}{x_0} + 0.5496 - 0.1944 x_0 + 0.0543 x_0^2 + \dots \right),$$

where x_0 is a radius parameter to be adjusted to make B a minimum. For $A = 62, 119$, and 267 , x_0 was found to be $1.1, 1.0$, and 0.9 , respectively. In the above equation, the first term is the kinetic energy, and the rest is the potential energy.

It may be mentioned here that the values of binding energies calculated with this equation are very low compared with the experimental data. For example, for $A = 119$ and $x_0 = 1.0$, the above equation is reduced to

$$B = 40.871 A^{5/3} - 14.477 A^2;$$

then B is $-87,340$ MeV. The experimental binding energies for $A = 119$ are around $1,010$ MeV. These large deviations from the experimental data are not just due to the neglect of the Coulomb energy (proton repulsive force). The Coulomb energy for $A = 119$ and $Z = 50$ is only about 300 MeV.

13. Seeger

Seeger (61S) modified the symmetry term in the Bethe-Weizsäcker formula to include I as well as I^2 , as suggested by Wigner (37W, 52BW), and the parameter a_a to include composition dependence of the surface effects (60D). He also adopted Mozer's modification (59M) of the Coulomb energy term of a trapezoidal charge distribution and exchange effects. His mass formula is (in MeV)

$$\begin{aligned} M - A &= 8.3674 N + 7.5848 Z - 17.06 A \\ &+ [33.61 - (59.54/A^{1/3})] (I^2 + 2I)/A + 25.00 A^{2/3} \\ &+ \frac{0.8076 Z^2}{A^{1/3}} [1 - (0.7636/Z^{2/3}) - (2.29/A^{2/3})] \\ &- S_{jk} (N', Z') + \frac{1}{2} \delta_A, \end{aligned}$$

where

$$\delta_A = + \text{for odd } Z, \text{ odd } N; \quad 0 \text{ for odd } A; \quad - \text{for even } Z, \text{ even } N.$$

However, no mathematical expression was given for δ_A .

The shell-effect terms, which include nuclear-deformation effects, are sine functions of N and Z , with a cross-product term:

$$S_{jk}(N, Z) = a_j \sin N'\pi + a_k \sin Z'\pi + b_j \sin 2N'\pi \\ + b_k \sin 2Z'\pi + (c_j + c_k) (\sin N'\pi) (\sin Z'\pi) + d,$$

where j and k refer to the neutron and proton shells, respectively, and a , b , c , and d are adjustable parameters. Further, N' and Z' are the fractional occupations of the last shells:

$$N' = \frac{N - N_j}{N_{j+1} - N_j}; \quad Z' = \frac{Z - Z_k}{Z_{k+1} - Z_k},$$

where the closed-shell configurations are $N_j, Z_k = 8, 20, 50, 82, 126$ and 184 , and $N_j \leq N \leq N_{j+1}$ and $Z_k \leq Z \leq Z_{k+1}$. For example, if $N_j = 20$, then $N_{j+1} = 50$; etc. The values of the adjustable parameters in the shell-effect terms are listed in Table 6.

Table 6

VALUES OF THE PARAMETERS IN THE SHELL-EFFECT
TERMS OF SEEGER'S MASS FORMULA (in MeV)

$\leq Z \text{ or } N <$	a	b	c	d
8, 20	4.008	-0.428	-1.389	13.51
20, 50	-0.508	2.331	-0.463	13.51
50, 82	-7.636	0.496	1.950	13.51
82, 126	-15.63	-2.284	10.67	13.51
126, 184	-27.59	2.660	26.99	13.51

The values of mass excess, separation energies of neutron, proton, and alpha particle, as well as beta-decay energies for $Z = 11$ to 123 and $A = 22$ to 306 , calculated by means of this equation are available (61S).

14. Ayres, Hornyak, Chan, and Fann

From Mozer's analysis of the Bethe-Weizsäcker mass formula (59M), one notices that the values of the parameters a_v , a_a , a_s , and a_c differ in various mass regions, probably varying with the mass numbers. Ayres et al. (62AH) examined the dependence of these parameters on A . In their expression for the binding energy, the asymmetry term, expressed in other mass equations as $a_a(I^2 - I)/A$, was split into two parts: $a_a(I^2/A)$ and $a_i(I/A)$. A pairing-effect term, a_π/A was also introduced. The A dependence of a_i and a_π was also studied. The Coulomb-energy expression

was taken from the suggestion of Cooper and Henley (53CH). Their expression for the binding energy is (in MeV)

$$B = a_v A - a_s A^{2/3} - a_c [(Z^2 - 0.767 Z^{4/3})/A^{1/3}] - (a_i |I|/A) \\ - a_a (I^2/A) - (a_\pi/A),$$

where a_v was found to be 13.81 MeV and $a_s = 7.115 + (15.864/A^{1/3})$ MeV. The values of a_c , a_i , a_a , and a_π as functions of A are graphically represented in their paper (62AH).

15. Kümmel, Mattauch, Thiele, and Wapstra

Kümmel et al. (63KM) added shell-effect and nuclear-deformation terms to the Bethe-Weizsäcker mass equation. For the shell effects, they used a shell-model approach and considered the mean distance of the single-particle levels, represented by c , the center between two shells, b , and the change of the level distance, d , as a closed-shell configuration was approached. A term describing the mutual influence of the neutron and proton shells was included. Their equation is (in MeV)

$$B(N, Z) = 645.31 - 20.729 A^{2/3} - 18.489 (I^2/A) - 0.57782 [Z(Z-1)/A^{1/3}] \\ - 11.045 (\delta/A^{1/2}) + \sum_i b_i n_i + \sum_j b_j z_j \\ - \frac{1}{2} \sum_{i,j} [(c_i + d_i n_i) n_i \bar{n}_i + (c_j + d_j z_j) z_j \bar{z}_j] \\ + \sum_{i,j} f_{i,j} n_i \bar{n}_i z_j \bar{z}_j + D,$$

where n_i and z_j represent the numbers of neutrons and protons in excess of N_0 and Z_0 of the i th and j th shells, respectively; $\bar{n}_i$ and $\bar{z}_j$ represent the numbers of neutron holes and proton holes (nonoccupied levels) in the i th neutron and j th proton shells, respectively, such that

$$n_i + \bar{n}_i = N_0 (i + 1 \text{ th shell}) - N_0 (i \text{ th shell}),$$

$$z_j + \bar{z}_j = Z_0 (j + 1 \text{ th shell}) - Z_0 (j \text{ th shell}),$$

and

$$\delta = 0 \text{ for even } N \text{ and even } Z; \quad 1 \text{ for odd } A; \quad 2 \text{ for odd } N \text{ and odd } Z.$$

The values of the various parameters and coefficients are listed in Tables 7 and 8.

Table 7

VALUES (in MeV) OF THE PARAMETERS b, c, AND d IN KÜMMEL'S EQUATION

N or Z	21 to 28	29 to 50	51 to 82	83 to 126	127 to 184
i or j	1	2	3	4	5
b	16.021	15.393	15.041	14.607	13.569
c	0.127989	0.043552	0.02134	0.067338	0.027032
d	0.000	-2.25279×10^{-5}	-7.7549×10^{-5}	-2.2779×10^{-5}	7.4172×10^{-5}

Table 8

VALUES (in keV) OF THE PARAMETER $f_{i,j}$ IN KÜMMEL'S EQUATION

i \ j	1	2	3	4
1	4.3715			
2	0.70354	-0.057325		
3		0.11109	0.0000	
4			0.090085	0.026941
5			0.075963	0.075963

The expression for nuclear deformation is a purely phenomenological coupling between protons and neutrons, vanishing at closed-shell configurations:

$$D = \sum_{i,j} \sum_{vuv'u'} g_{ij}^{(vu,v'u')} n_i^v \bar{n}_i^u z_j^{v'} \bar{z}_j^{u'}.$$

The values of g are given in Table 9. All the values in Tables 7, 8, and 9 are preliminary.

Table 9

VALUES (in MeV) OF THE PARAMETER g IN KÜMMEL'S EQUATION

i	j	v	u	v'	u'	g
	2	0	0	2	2	6.5450×10^{-5}
	3	0	0	2	2	-6.1363×10^{-5}
3		1	2	0	0	-4.9242×10^{-4}
3		2	2	0	0	-1.5245×10^{-5}
4	3	1	2	1	2	8.3144×10^{-8}
4		1	3	0	0	-5.8658×10^{-6}
4		2	3	0	0	-1.1699×10^{-6}
4		3	1	0	0	1.3964×10^{-5}
4		3	4	0	0	3.7429×10^{-9}

16. Johansson

Johansson (63J) modified, as Seeger did (61S), the symmetry energy term of the Bethe-Weizsäcker formula to include composition dependence of the surface effects on symmetry (60D) and a linear relationship of I with symmetry (37W, 52BW). A shell-effect term and a nuclear-deformation term, but no pairing effect, were introduced. His mass equation is (in mmu)

$$B = -16.4243 A + 18.6839 A^{2/3} + 0.7050 (Z^2/A^{1/3}) \\ + 145.50[1 - (2.350/A^{1/3})]\left[\frac{I^2 + 4 I}{4 A}\right] + E_{\text{def}} + S.$$

The deformation energy, E_{def} , was taken from Swiatecki's expression for the deformation of a liquid drop (56S):

$$E_{\text{def}} = +C\beta - 7.08 A^{2/3}(1-x)\beta^2 + 0.674 A^{2/3} \beta^3(1+2x),$$

where

$$\beta = \text{nuclear-deformation parameter (60BB)}$$

and

$$x = (Z^2/A)/(Z^2/A)_1; \quad (Z^2/A)_1 = 46.0$$

The shell-effect term includes the influence of deformation:

$$S = -(\beta_0 - \beta)(k\beta - H),$$

that is, nuclear-deformation and shell effects are inversely proportional to each other, and $\beta_0 H$ is a measure of the shell effect of a spherical (nondeformed) nucleus. However, this mass equation is intended to be used for deformed nuclides only. The values of the parameters in this equation are listed in Table 10.

Table 10

PRELIMINARY VALUES OF THE PARAMETERS
IN JOHANSSON'S MASS EQUATION

	C	β_0	k	H
Rare earth region	55	0.32	172	
Actinide region	25	0.26	100	
$A < 160$				55
$A > 160$				54
$A < 240$				45
$A > 240$				10

Section B. Shell-model Approach

In the shell-model approach, one considers the interaction between the core (closed-shell configuration) and the nucleons outside the core, and the interaction between the nucleons themselves outside the core. From the study of systematics of nucleon-separation energies and radioactive-decay energies, one observes approximately linear relationships of these nuclear properties with the neutron or the proton number, within the boundaries of the closed-shell configurations (53WW, 57D, 60D). This leads to the idea that the binding-energy differences and nucleon-separation energies may be expressed by simple mathematical functions of the number of nucleons outside the core. The nuclidic mass equations based on this approach are much simpler than those based on the statistical-model approach.

1. Woeste

Woeste (54W) made use of the Fermi-Weizsäcker mass formula to express the binding energy of a nucleus in terms of a single-particle model. He expressed the kinetic energy E_K in terms of the magic numbers N_0 and Z_0 for neutrons and protons, respectively:

$$E_K = 12.85 A_0 (A_0/A)^{2/3} \left\{ 1 + \frac{5}{9} [(N_0 - Z_0)/A_0]^2 + \frac{5}{243} \left[\frac{(N_0 - Z_0)}{A_0} \right]^4 + \dots \right\},$$

where

$$A_0 = N_0 + Z_0.$$

The potential energy E_p was taken as the difference between the Fermi-Weizsäcker semi-empirical mass equation, with the surface energy neglected, and the above equation for E_K . The potential energy was divided into two parts: one for the neutron well, and one for the proton well:

$$\begin{aligned} E_p &= N_0 V_N + Z_0 V_Z \\ &= N_0 [-26.88 + 12.2(N - Z)/A] + Z_0 [-26.88 - 12.2(I/A) + a_c(Z/A^{1/3})], \end{aligned}$$

where V_N and V_Z are the depths of the neutron and proton wells, respectively. Although Woeste in essence simply used a liquid-drop model expression for binding energy, he had attempted to introduce concepts from the shell model into the binding-energy expression.

2. Thieberger, Talmi, and De-Shalit

The formula of Talmi, Thieberger, and De-Shalit (56TT, 57TD) expresses the binding-energy differences, i.e., the binding energy of a given

isotope (isotone) minus the binding energy of a reference isotope (isotone), by a quadratic equation:

$$\Delta B = nb + \frac{1}{2} n(n-1)a + y(n)D,$$

where n is the number of neutrons (protons) of the isotope (isotone) in excess of the number of the neutrons (protons) of the reference isotope (isotone), b , a , and D are parameters characteristic of Z or N , and

$$y(n) = 0 \text{ for even } n; \quad 1 \text{ for odd } n.$$

The values of b , a , and D for many isotopes and isotones are listed in Table 11. The above equation is essentially a local fitting of the isotopic and isotonic parabolas.

Table 11

VALUES OF THE PARAMETERS IN THIEBERGER'S MASS FORMULA (in MeV)

	b	a	D		b	a	D
30 ≤ N ≤ 40				120 ≤ N ≤ 126			
²⁸ Ni	11.05	-0.42	-1.46	⁸¹ Tl	8.44	-0.42	-0.57
²⁹ Cu	11.40	-0.55	-1.41	⁸² Pb	8.27	-0.28	-0.67
³⁰ Zn	11.41	-0.43	-1.68	N ≥ 126			
³¹ Ga	12.04	-0.43	-1.17	⁸² Pb	4.561 ± 0.008	-0.026 ± 0.004	-0.70 ± 0.02
³² Ge	12.85	-0.43	-1.74	⁸³ Bi	4.955 ± 0.044	-0.092 ± 0.025	-0.368 ± 0.077
40 ≤ N ≤ 50				⁸⁴ Po	5.306 ± 0.018	-0.075 ± 0.008	-0.817 ± 0.045
³² Ge	8.35	-0.15	-1.84	⁸⁵ At	5.528 ± 0.066	-0.049 ± 0.030	-0.562 ± 0.145
³³ As	9.14	-0.25	-1.16	44 ≤ Z ≤ 50			
³⁴ Se	9.56	-0.22	-1.46	N = 60	8.44	-0.61	-1.68
³⁵ Br	10.11	-0.24	-0.98	N = 61	8.84	-0.65	-0.99
³⁶ Kr	10.60	-0.26	-1.67	N = 62	8.83	-0.51	-1.31
³⁷ Rb	10.91	-0.21	-0.90	Z ≥ 50			
³⁸ Sr	10.96	-0.15	-1.57	N = 72	7.94	-0.61	-1.22
N ≥ 50				N = 73	8.10	-0.52	-1.20
³⁸ Sr	7.17	-0.24	-0.78	N = 74	8.25	-0.44	-0.94
³⁹ Y	7.80	-0.36	-0.56	Z ≥ 82			
⁴⁰ Zr	8.06	-0.23	-0.77	N = 130	6.01	-0.45	-1.09
⁴¹ Nb	8.59	-0.28	-0.52	N = 131	6.06	-0.35	-0.78
⁴² Mo	8.89	-0.29	-1.00	N = 132	6.45	-0.37	-1.05
N ≥ 64				N = 133	6.85	-0.40	-0.71
⁴⁸ Cd	7.84	-0.22	-1.34	N = 134	7.12	-0.39	-0.89
⁴⁹ In	8.31	-0.23	-1.34				
⁵⁰ Sn	8.69	-0.18	-1.15				
⁵¹ Sb	9.00	-0.17	-1.21				
⁵² Te	9.11	-0.14	-0.92				
⁵³ I	9.27	-0.13	-1.37				
⁵⁴ Xe	9.72	-0.15	-1.16				

3. Zeldes

Zeldes (58Z, 60D) expressed the nucleon-separation energies for nuclides not containing magic numbers by the following linear functions:

$$B_n(N_0 + n, Z_0 + p) = c + \frac{1}{2} \pi + (n - 1)d + pI^0 - \Delta_n \delta;$$

$$B_p(N_0 + n, Z_0 + p) = C + \frac{1}{2} \Pi + (p - 1)D + nI^0 - \Delta_p \delta,$$

where

$$\Delta_n \delta = \frac{1}{2} (-1)^{n+1} \pi + \frac{1}{2} (-1)^n (1 - (-1)^p) I',$$

$$\Delta_p \delta = \frac{1}{2} (-1)^{p+1} \Pi + \frac{1}{2} (-1)^p (1 - (-1)^n) I',$$

and

$$N_0 + n = N; \quad Z_0 + p = Z.$$

Here N_0 and Z_0 represent the magic numbers, i.e., 20, 28, 40, 50, 64, 82, 126, and 140. The values of the parameters in these equations are listed in Table 12. The values of c and C for the region $Z = 51$ to 64 and $N = 51$ to 82 have been corrected by Zeldes (63Z).

Table 12

VALUES OF THE PARAMETERS (in MeV) IN ZELDES' FORMULA

Z	N	c	C	π	Π	d	D	I^0	I'
21-28	21-28	8.686	3.738	3.269	3.816	-0.331	-0.795	0.732	0.672
21-28	29-40	5.077	9.907	2.844	3.296	-0.514	-0.836	0.624	0.375
29-40	29-40	9.421	1.968	3.564	2.920	-0.445	-0.934	0.642	0.611
29-40	41-50	4.839	8.871	3.344	3.260	-0.240	-0.736	0.511	0.566
29-40	51-82	2.069	17.115	1.514	2.623	-0.225	-1.028	0.427	0.199
41-50	51-82	6.817	5.240	2.452	2.975	-0.207	-0.501	0.351	0.309
51-64	51-82	9.542	-0.026	2.238	2.295	-0.153	-0.521	0.299	-0.011
51-64	83-126	2.918	9.149	2.053	2.441	-0.086	-0.477	0.300	0.163
65-82	83-126	8.160	1.740	1.673	1.832	-0.205	-0.655	0.361	0.307
>82	127-140	3.975	3.781	1.569	1.884	-0.102	-0.433	0.317	0.203
>82	>140	3.409	8.933	1.071	1.438	-0.173	-0.496	0.245	-0.083

Zeldes' formulae may be combined to give an equation for the mass defect:

$$M(N_0 + n, Z_0 + p) - A = M(N_0, Z_0) - N_0 - Z_0 + n \left[(M_n - 1) - c - \frac{1}{2} \pi + \frac{1}{2} d \right] \\ + p \left[(M_p - 1) - C - \frac{1}{2} \Pi + \frac{1}{2} D \right] - \frac{1}{2} n^2 d - \frac{1}{2} p^2 D - npI^0 + \delta.$$

However, for the calculation of mass defects by use of this equation, a knowledge of the values of $M(N_0, Z_0)$ is required. Experimental values of $M(N_0, Z_0)$ for many nuclides with double shell configurations, such as Ni^{56} , Ni^{68} , Ni^{78} , Sn^{100} , Sn^{132} , Gd^{146} , and Pb^{222} , have not been determined.

Section C. Empirical-model Approach

Before we introduce the nuclidic mass equations based on the empirical model, let us review the systematics of the nuclidic masses as shown by the experimental data. A three-dimensional plot of $(M - A)$ vs Z and N , the so-called nuclear mass surface, exhibits a somewhat parabolic shape (53E, 55G). The bottom of the parabola occurs at a mass number around 90. The cross section of the nuclear mass surface for the same $(N+Z)$ values, i.e., isobars, also has a parabolic shape. The beta-stable isotopes lie at and near the bottoms of these isobaric parabolas. Closer examinations of the surface will reveal "dips" at the regions of closed-shell configurations. Nuclides with closed or nearly closed-shell configurations occur at or near the bottoms of these dips. Furthermore, the surface of the odd- Z -odd- N nuclides lies generally slightly above the surface of the odd- A nuclides. The latter in turn lies slightly above the surface of the even- Z -even- N nuclides. The empirical mass equations attempt to describe all these features in the nuclear mass surface.

1. Bohr and Wheeler

In 1939, Bohr and Wheeler (39BW) further simplified the Bethe-Weizsäcker mass formula and, at the same time, introduced a pairing-effect term P_A :

$$M(Z, A) = M(Z_A, A) + B_A(Z - Z_A)^2/2 + \frac{1}{2} P_A.$$

This equation is an exact description of the nuclidic mass surface if the shell and nuclear-deformation effects are implied in all the parameters. The second term on the right-hand side represents the isobaric parabolas, with B_A a measure of the curvature and Z_A of the vertices, i.e., the charges of the most stable (minimum M) isobars. The first term, $M(Z_A, A)$, is the mass for a nuclide with A nucleons and a charge of Z_A , and it describes the general trend of the mass decrements with A . In the last term P_A is the average distance between the nuclidic mass surface of odd- Z odd- N and the mass surface of even- Z even- N . Bohr and Wheeler had estimated the values of B_A , Z_A , and P_A for a few mass numbers (39BW).

Since the Bohr-Wheeler equation is an exact description of the nuclidic mass surface, the parameters B_A and Z_A may be related to the parameters in other nuclidic mass equations. For example, consider the Bethe-Weizsäcker formula; by definition,

$$\left(\frac{\partial M}{\partial Z}\right)_A = 0 \text{ at } Z = Z_A,$$

so that

$$Z_A = \frac{(M_n - M_p + 4 a_a)A}{2(a_c A^{2/3} + 4 a_a)}$$

and

$$B_A = (M_n - M_p + 4 a_a)/Z_A = (8 a_a + 2 a_c A^{2/3})/A.$$

If we consider Zeldes' mass formula (58Z), then

$$B_A = -d - D + 2 I^0;$$

$$Z_A = \left[(M_p - M_n) - (C - c) - \frac{1}{2}(\Pi - \pi) + \frac{1}{2}(D - d) - N_0 d + Z_0 D + (N_0 - Z_0)I^0 + (d - I^0)A \right] / (d + D - 2 I^0).$$

These two so-called "isobaric" parameters have been studied extensively by several authors (39BW, 45J, 47F, 48K, 52CB, 53BR, 53C, 56F, 55KP). Aten (49A) first discussed the shell effects on Z_A .

From a comparison of Fermi's mass equation, which is equivalent to Bohr-Wheeler's, with the experimental data, Fong (56F) obtained empirical corrections of Z_A and M_A to include the shell effects on these two parameters. He also added a shell-effect term, u , for nuclides with the magic numbers only. The modified expression of the Bohr-Wheeler mass formula is

$$M(Z, A) = M_A + \Delta M_A + B_A(Z - Z_A - \Delta Z_A)^2 + \frac{1}{2} P_A + u.$$

The values of ΔM_A and ΔZ_A were given for $A = 70$ to 165 in the graphs (56F), and the values of u are -2.0 , -2.0 , and -1.1 MeV for 50 neutrons, 82 neutrons, and 50 protons, respectively. Fong believed that there is an extra shell stability of 1 to 2 MeV for these nuclides.

2. Green, Engler, and Edwards

Green et al. (53GE, 53GD, 55G) described the nuclear mass surface in a way very similar to that of Bohr and Wheeler. The parabolic shape of the mass decrement as a function of A was expressed as $0.01 (A - 100)^2 - 64$. The isobaric parabola was expressed in terms of the neutron excess as $(I - I_m)^2 J(A)$, where I_m is the value of neutron excess for the vertex and J is the breadth of the parabola. $J(A)$ was found to be $25/A$ and $I_m(A) = 0.4 A^2/(A + 200)$. A pairing-energy term, $\delta H(A)$, was

included, with $H(A) = 12/A^{1/2}$ and $\delta = 1$ for odd-Z, odd-N; $\delta = 0$ for odd-A; and $\delta = -1$ for even-Z, even-N nuclides. Thus, Green's mass formula is (in mmu)

$$M - A = 0.01 (A - 100)^2 - 64 + 12(\delta/A^{1/2}) \\ + \{I - [0.4 A^2/(A + 200)]\}^2 (25/A) + S_{ij}(Z, N),$$

where the shell-effect term is expressed as parabolic functions of Z and N:

$$S_{ij}(Z, N) = -a_i(N - N_i)^2 - a_j(Z - Z_j)^2 + k_{ij},$$

with N_i and Z_j as the vertexes, and a_i and a_j as the breadths of the parabolas. The subscripts i and j refer to the locations of the shells as indicated in Table 13. The values of a_i and a_j and of N_i and Z_j are also listed in Table 13. The estimated values of k_{ij} are listed in Table 14.

Table 13

VALUES (in mmu) OF THE PARAMETERS IN THE
SHELL-EFFECT TERMS IN GREEN'S FORMULA

i or j	N or Z	a_i	N_i	a_j	Z_j
1	0 to 2				
2	2 to 8	0.22	7		
3	8 to 20	0.18	18	0.05	19
4	20 to 28	0.22	25	0.088	21
5	28 to 50	0.023	33	0.023	40
6	50 to 82	0.022	70	0.030	53
7	82 to 126	0.023	104	0.075	88
8	126 to 148	0.048	140		

Table 14

ESTIMATED VALUES OF k_{ij} IN
GREEN'S FORMULA (in mmu)

i	j	k_{ij}	i	j	k_{ij}	i	j	k_{ij}
2	2	-2.5	4	4	3.5	6	6	8.0
3	2	2.5	5	4	2.5	7	6	3.5
3	3	3.0	5	5	5.5	8	7	1.5
4	3	7.0	6	5	6.5			

3. Levy

Levy (57L) described the nuclear mass surface by the following quadratic equation:

$$M - A = a_0 + a_1A + a_2Z + a_3AZ + a_4Z^2 + a_5A^2 + \delta.$$

This equation is equivalent to the Bohr-Wheeler formula, that is,

$$B_A = 2a_4;$$

$$Z_A = -(a_2 + a_3A)/2a_4;$$

$$M(A, Z_A) = a_0 - (a_2^2/4a_4) + [a_1 - (a_2a_3/2a_4)]A + [a_5 - (a_3^2/4a_4)]A^2.$$

Zeldes pointed out that Levy's model is essentially an independent-particle model, with strong pairing between identical nucleons, and orientation-independent residual interaction, resulting in an equation equivalent to his (58Z). In other words,

$$a_0 = \Delta M(N_0Z_0) - N_0 \left[(M_n - 1) - c - \frac{1}{2} \pi + \frac{1}{2} d \right] \\ - Z_0 \left[(M_p - 1) - C - \frac{1}{2} \Pi + \frac{1}{2} D \right] - \frac{1}{2} N_0^2 d - \frac{1}{2} Z_0^2 D - N_0 Z_0 I^0;$$

$$a_1 = (M_n - 1) - c - \frac{1}{2} \pi + \frac{1}{2} d + N_0 d + Z_0 I^0;$$

$$a_2 = (M_p - M_n) - (C - c) - \frac{1}{2} (\Pi - \pi) + \frac{1}{2} (D - d) - N_0 d + Z_0 D + (N_0 - Z_0) I^0;$$

$$a_3 = d - I^0;$$

$$a_4 = -\frac{1}{2} d - \frac{1}{2} D + I^0;$$

$$a_5 = -\frac{1}{2} d.$$

The values of the parameters and the pairing terms in Levy's equation are listed in Tables 15 and 16. The values of mass excess, separation energies of neutron, proton, and alpha particle, as well as beta-decay energies for $Z = 29$ to 110 and $A = 58$ to 290, calculated by means of this equation, are available (56R).

Table 15

VALUES OF THE PARAMETERS IN LEVY'S EQUATION (in mmu)

Z	N	a_0	a_1	a_2	a_3	a_4	a_5
29-40	29-40	-155.91	13.202	-21.956	-0.9707	1.4544	0.11565
29-40	41-50	-150.06	7.359	-10.094	-0.7023	0.9473	0.10340
29-40	51-82	+96.27	3.780	-17.406	-0.5349	0.8150	0.10050
41-50	51-82	-135.41	5.342	-9.712	-0.5570	0.7432	0.09758
51-64	51-82	-133.60	6.399	-13.465	-0.4287	0.6417	0.06583
51-64	83-126	-672.82	13.059	-14.140	-0.4461	0.6492	0.05370
65-82	83-126	-83.72	3.843	-10.680	-0.4644	0.6464	0.08739
>82	127-140	-1746.56	18.067	-10.846	-0.4363	0.6133	0.05171
>82	>140	571.90	-1.407	-12.238	-0.3971	0.5706	0.08613

Table 16

VALUES OF THE PAIRING TERMS IN LEVY'S EQUATION (in mmu)

Z	N	Even Z- even N	Odd Z- odd N	Odd Z- even N	Even Z- odd N
29-40	29-40	0	2.65	1.44	2.20
29-40	41-50	0	3.08	1.84	1.82
29-40	51-82	0	2.02	1.27	0.75
41-50	51-82	0	3.08	1.54	1.44
51-64	51-82	0	2.52	1.12	1.13
51-64	83-126	0	2.09	0.96	0.73
65-82	83-126	0	1.61	0.84	0.76
>82	127-140	0	1.66	1.01	0.88
>82	>140	0	1.33	0.71	0.50

4. Baker

Baker (58B) described the nuclear mass surface by the following equation:

$$B = Ab(I) + a(I) + C(A) + P(A, I) + S(A, I) + R(A, I),$$

where $R(A, I)$ represents a small residual quantity which may be zero. The first and second terms on the right-hand side of this equation describe the isobaric parabolas in terms of I , and the third term describes the gross shape of the mass decrements. In Baker's work, both $R(A, I)$ and $S(A, I)$ were not studied. The other functions are expressed by the following complicated equations (in micro mass units) for $I \geq 0$:

$$b(I) = 5747.5484 + 162.69093I - 3.1653506I^2 - 7.4583345 \times 10^{-2}I^3 + 4.639864 \times 10^{-3}I^4 \\ - 4.776849 \times 10^{-5}I^5 + 125.5480 \sin \left[\frac{1}{10} \pi (I+3) \right] + 3.28630 \left\{ I \sin \left[\frac{1}{6} \pi (I-5) \right] \right\} \theta(33-I),$$

where

$$\begin{aligned}\theta(x) &= 1, & x > 0 \\ &= \frac{1}{2}, & x = 0 \\ &= 0, & x < 0,\end{aligned}$$

and, for $I < 0$,

$$b(-1) = 5830.4731; \quad b(-2) = 5855.2409.$$

The term $a(I)$ is related to $b(I)$ by the following relationship:

$$\begin{aligned}- \frac{a(I+1) - a(I-1)}{b(I+1) - b(I-1)} &= 14.268349 + 10.994719(I)^{1/2} \\ &+ 3.2025866I - 0.061670752I^{3/2} \\ &+ 8.1243904 \left[\sin \frac{1}{6} \pi I \right] \theta(I-6) \theta(34-I).\end{aligned}$$

The pairing correction used is

$$P = h(A, N-Z)[(-1)^N + (-1)^Z],$$

where

$$\begin{aligned}h(A, N-Z) &= 2200, & 6 \leq A \leq 12, & N-Z = 0, \\ &= 1600, & 14 \leq A \leq 28, & N-Z = 0, \\ &= 1200, & 6 \leq A \leq 28, & N-Z \neq 0, \\ &= 800, & 30 \leq A \leq 88, & \\ &= 600, & 90 \leq A \leq 206, & \\ &= 400, & 208 \leq A \leq 256. &\end{aligned}$$

The function $C(A)$ is expressed by various polynomials at different mass regions:

$$A = 6-12,$$

$$C = -201\,850.72 + 1263.8196A + 130.43287A^2,$$

$$A = 13-19,$$

$$C = -216\,819.93 + 5060.1124A - 62.27547A^2,$$

$$A = 20-29,$$

$$C = -223\,411.89 + 4176.6659A + 3.4115430A^2,$$

$$A = 30-89,$$

$$C = -3\,480\,891.6 + 430\,813.54A - 23\,073.453A^2 + 669.39020A^3 \\ - 11.276786A^4 + 0.11053132A^5 - 5.8557791 \times 10^{-4}A^6 \\ + 1.2980668 \times 10^{-6}A^7,$$

$$A = 90-149,$$

$$C = 10\,706\,566 - 465\,910.79A + 8050.6766A^2 - 68.769400A^3 \\ + 0.29153322A^4 - 4.9188510 \times 10^{-4}A^5,$$

$$A = 150-205,$$

$$C = -357\,421.73 + 6407.9783A - 22.297243A^2,$$

$$A = 206-256, \quad X = A-200,$$

$$C = 21\,717.262 + 2848.9818X - 752.86522X^2 + 36.767068X^3 \\ - 0.91722945X^4 + 1.1346801 \times 10^{-2}X^5 - 5.5685564 \times 10^{-5}X^6,$$

where for odd A the above values should be modified as follows:

$$A = 20-80,$$

$$C_{\text{odd}} = C - 464.66,$$

$$A = 99-111,$$

$$C_{\text{odd}} = C - 923.$$

Fortunately, the calculated values of $b(I)$, $a(I)$, and $C(A)$ for $I = -2$ to 58 and $A = 6$ to 256 have been tabulated (58B).

5. Wing and Fong

Wing and Fong (62WF) further developed the Bohr-Wheeler mass formula to include shell-effect terms in a manner suggested by Wapstra (52W), i.e., by Cauchy distribution functions of N and Z :

$$S(N, Z) = \sum_i \frac{a_i b_i^2}{(N - N_{0i})^2 + b_i^2} + \sum_j \frac{c_j d_j^2}{(Z - Z_{0j})^2 + d_j^2},$$

where a_i and c_j are the maximum shell effects of the i th neutron and j th proton shells, respectively, and $2b_i$ and $2d_j$ are the widths at $\frac{1}{2}a_i$ and $\frac{1}{2}c_j$, respectively. The advantage of these shell-effect terms is that the mass equation becomes a continuous function from one shell region to another. Their empirical mass formula is (in mmu)

$$M - A = M_A + B_A(Z - Z_A)^2 + P_A + S,$$

where

$$M_A = 0.009643 A^2 - 1.9067 A + 35.81,$$

$$B_A = 2.51 - (46.35/A^{1/2}) + 300/A,$$

$$Z_A = (A + 0.003 A^2)/(2 + 0.01 A),$$

$$\begin{aligned} S = & -\frac{4 \times 5.76}{(N - 28)^2 + 5.76} - \frac{6.7 \times 12.25}{(N - 50)^2 + 12.25} - \frac{7.6 \times 18.49}{(N - 82)^2 + 18.49} \\ & - \frac{8.2 \times 25}{(N - 126)^2 + 25} - \frac{36 \times 4.5}{(N - 152)^2 + 36} - \frac{4.8 \times 9}{(Z - 28)^2 + 9} \\ & - \frac{5 \times 9}{(Z - 50)^2 + 9} - \frac{5.8 \times 9}{(Z - 82)^2 + 9}, \end{aligned}$$

and

$$P_A = \begin{cases} +12.36/A^{1/2} & \text{for odd } N, \text{ odd } Z \\ 0 & \text{for odd } A \\ -12.36/A^{1/2} & \text{for even } N, \text{ even } Z. \end{cases}$$

Note that the expression for Z_A is equivalent to the function $I_m(A)$ of Green's mass equation (55G).

CHAPTER III. COMPARISON OF NUCLIDIC MASS EQUATIONS WITH EXPERIMENTAL DATA

Section A. General Discussion

The mass defects, binding energies, or nucleon-separation energies calculated by means of the equations of Fermi (50MR, 54M), Román (51R), Bussetti (55B), Cameron (57C), Tsen (57T), Findler (58F), Seeger (61S), Ayres *et al.* (62AH), Kümmel *et al.* (63KM), Thieberger *et al.* (56TT, 57TD), Zeldes (58Z), Green *et al.* (55G), Levy (56R, 57L), Baker (58B), and Wing-Fong (62WF) can be compared with the experimental data. Such comparisons reveal systematic discrepancies with the experimental data in these equations and show the magnitudes of these discrepancies. The experimental data are entirely obtained from the 1961 Nuclidic Mass Table (62KM) in the $0^{16} = 16.000\ 000$ scale.

It may be pointed out here that the values of parameters in many of the earlier nuclidic mass equations were evaluated with the earlier (but fewer and less accurate) mass data before 1961. It seems more appropriate to compare the calculated values of these equations with the earlier sets of experimental data. However, there are two reasons for the use of only the set of experimental data from the 1961 Nuclidic Mass Table. First, it is impractical and sometimes impossible to identify all the experimental data used for the evaluation of the parameters in a given mass equation. Secondly and more importantly, the 1961 Nuclidic Mass Table contains many new data, and a comparison of the calculated values with the new experimental data will reveal the accuracy of prediction by a given nuclidic mass equation.

In the calculation, the corrected values of the parameters in Levy's (56C) and Zeldes' (63Z) equations were used. The evaluations of Zeldes' and Wing-Fong's mass formulae were performed with an IBM-650 computer, and those of Green's, Baker's, and Thieberger's mass equations were performed with a 704 computer. All other calculated values were obtained from the references. Only a few calculated values were listed in the papers of Bussetti, Ayres *et al.*, Tsen, Findler, and Román. Since their mass equations do not have shell-effect terms, no attempt was made to compute more mass or binding-energy values than those appearing in their papers.

The values of the experimental masses minus the calculated masses, in mmu, and the values of the experimental binding (or nucleon-separation) energies minus the calculated binding (or nucleon-separation) energies, in MeV, are plotted against A in Figs. 1 to 16. In Thieberger's equation, the values of the experimental ΔB minus the calculated ΔB were plotted, where ΔB is the binding energy of a given isotope (isotone) minus the binding energy of its reference isotope (isotone). In these figures,

some of the differences between the experimental and the calculated quantities were not plotted because they are too close to the other points in the graphs for the same isobars. The different vertical scales used in these figures should be noted. The masses or binding energies of some nuclides, which are unavailable or cannot be calculated with the nuclidic mass equations, are not compared with the experimental data.

The use of A , the mass number, as the scale for the abscissa in Figs. 1 to 16 is certainly not the best way to reveal systematic deviations in the nuclidic mass equations. A much preferred method would be a three-dimensional plot of $(M_{\text{exp}} - M_{\text{calc}})$ or $(B_{\text{exp}} - B_{\text{calc}})$ vs N and Z . However, the present practical and convenient way of representation still can reveal important systematic deviations of the nuclidic mass equations. For example, in Fig. 1, we can see the failure of the Fermi-Weizsäcker equation to describe the gross parabola of the nuclidic mass surface. In addition, the need for shell-effect terms in this equation is clearly shown in this figure by the "dips" of the plot at and near the closed-shell configurations. Other nuclidic mass equations which do not have shell-effect terms exhibit these "dips." Further, the magnitude and spread of the deviations for a given A may be a measure of the quality of the isobaric description in a nuclidic mass equation.

The pairing effects are generally small compared with the shell effects. It is not easy to observe the systematic discrepancies due to the pairing effects alone in the present way of comparison. Nuclear-deformation effects are also small. We shall discuss all these effects in the next chapter.

Most of the systematic deviations (much larger than the experimental errors) that we see in Figs. 1 to 16 may be attributed to the inadequacy of the nuclidic mass equations to describe the nuclear mass surface. The equations without shell-effect terms generally give poorer agreements with the experimental data than those with shell-effect terms. Relatively larger discrepancies are found for nuclides far away from the beta-stability line than for the beta-stable nuclides. We may also recognize some of the disagreements between the experimental and the calculated masses or binding energies which are due to poor values of the parameters in the equations. Poor values of parameters result from (1) the use of less accurate values and insufficient number of experimental data (very few experimental mass values in the rare earths region were available before 1958); and (2) poor procedures in the evaluation of the parameters. In the region for which $A > 208$, for example, the calculated M of Cameron and Levy and the calculated B of Baker are about one MeV different from the experimental values. This is due to the fact that these authors used the experimental data from Huizenga's mass table (55H) for the evaluation of their parameters for $A > 201$. The binding energies and masses in Huizenga's table were based on the binding energy of Pb^{208} , which is about one MeV higher than the value given in the 1961 Nuclidic Mass Table.

It may be mentioned that although large discrepancies exist between the calculated and the experimental masses and binding energies, much smaller and perhaps fewer discrepancies may be found in the comparison of the nucleon-separation energies. This is because nucleon-separation energies involve the differences between two nuclidic masses, and some systematic errors in the nuclidic mass equations may be cancelled in the differences.

Section B. Discussion of Individual Mass Equations

As we have mentioned earlier, Fermi's formula does not correctly describe the general parabolic shape of the nuclidic mass surface (see Fig. 1). Also, the lack of shell-effect terms results in "dips" at the closed-shell regions. Because these deviations are so large, other possible systematic deviations of smaller magnitude cannot be easily detected. This equation may be improved by a correction for the parabolic deviation approximately in the following form: $(0.003842 N^2 - 0.64 N + 16.615)$ mmu or $(0.01177 Z^2 - 1.341 Z + 27.493)$ mmu, and by shell-effect terms: $(aN^2 + bN + c)$ mmu, where

N	a	b	c
0 to 16	-0.1471	3.373	-22.687
17 to 29	-0.1480	6.823	-79.68
30 to 49	-0.05595	4.883	-104.92
50 to 82	-0.02464	3.179	-103.24
83 to 112	-0.03083	6.122	-301.85
113 to 125	-0.01406	2.525	-106.61
126 to 146	-0.05560	15.722	-1108.94

With these improvements, over 90% of the deviations in this equation will lie within ± 3.0 mmu, which still is a large range. Slight systematic deviations due to proton-shell effects remain after these corrections.

Román's equation gives very poor agreement with the experimental data (see Fig. 2), poorer as A increases. This may be understood by the fact that Román did not evaluate the constants and parameters in his equation by empirical fits with mass data, but obtained their values directly from other independent measurements. Bussetti's formula (55B) also shows larger (but not random) deviations from the experimental data as A increases (see Fig. 3). Probably Bussetti used the large numbers M instead of $(M - A)$ in the evaluation of his constants, so that the accuracy of the constants is greatly reduced, especially in the regions of higher mass.

Tsen has improved the Bethe-Weizsäcker equation greatly, as the deviation from the parabolic shape of the nuclidic mass surface, observed in Fermi's equation, is now absent (see Fig. 4). Tsen's equation is equivalent to Fermi's except for the Coulomb energy, which is a function of $Z^{5/3}$

instead of $Z^2/A^{1/3}$ as in Fermi's equation. The values of the coefficients do not differ greatly in these two equations (see Table 1). It is rather surprising to see the great improvements in Tsen's equation due to the modification of the Coulomb energy term alone.

The original expressions of Findler and of Ayres et al. are for B/A . They evaluated the parameters in their mass formulas through use of the experimental data for B/A . A small error in B/A (of a few %) will bring about a relatively large (several MeV) difference in $(B_{\text{exp}} - B_{\text{calc}})$, especially for large values of A . Part of the random deviations observed in these two equations (see Figs. 5 and 6) may be due to this effect. There is a systematic deviation in Findler's equation in that the discrepancies from the experimental data increase in magnitude with increasing neutron number for the same element.

All the nuclidic mass equations discussed so far in this section contain no shell-effect terms, and, except for Fermi's equation, only a few calculated masses and binding energies are available for comparison. With more calculated values for comparison, perhaps other systematic deviations may be revealed in these equations.

With several modifications to the Bethe-Weizsäcker mass formula and with shell and pairing effects empirically determined, Cameron has obtained a much improved mass equation, although a few notable deviations still exist around the closed-shell regions ($A = 50, 90, \text{ and } 200$) (see Fig. 7). The bell-shaped deviations at $A = 160$ to 195 may be approximately corrected by use of the term

$$\frac{2.0 \times 13^2}{(177 - A)^2 + 13^3} \text{ MeV.}$$

This correction may be considered to represent nuclear deformation in this mass region. With the inclusion of this correction and a one-MeV correction for $A > 210$ into this equation, it is estimated that over 90% of the calculated values will agree within ± 1 MeV of the experimental data.

Seeger's mass equation, with the sine functions and cross-product terms for the shell effects and nuclear deformation, still exhibits deviations around the closed-shell regions (see Fig. 8). Some of the deviations at $A = 50$ may result from Seeger's neglecting 28 as a closed-shell configuration. The relatively large discrepancies around the mass region of 90 may be due to inadequate shell-effect corrections at $N = 50$ and perhaps also due to neglecting 40 as a closed proton shell. The deviations at $A = 150$ to 180 seem to indicate the need of a correction similar to that in Cameron's mass equation:

$$- \frac{2.5 \times 13^2}{(165 - A)^2 + 13^3} + 1.5 \text{ MeV.}$$

In both Seeger's and Cameron's calculated values, many of the least beta-stable isobars have larger deviations than the most stable ones, indicating poor descriptions of the isobaric parabolas of these mass numbers.

Of all the nuclidic mass equations examined in this section, the equation of Kümmel et al., has the least deviations from the experimental data (see Fig. 9). Over 80% of the discrepancies lie within ± 0.80 MeV, and only 5 out of 810 cases are larger than 1 MeV. No obvious systematic deviations are detected, although the authors observed some (63KM).

The local fitting of the isotopic and isotonic parabolas by the simple formulae of Thieberger and De-Shalit (57TD) seems quite good except for a few cases (see Fig. 10). Relatively large discrepancies (greater than 3 MeV) are found for Zn, Ga, Ge, Rb, Sr, I, and Xe isotopes and for isotones of $N = 133$ and 134 . Better agreement with the experimental data could be obtained if the values of the parameters in their equation for these isotopes and isotones are improved. Their equation computes the difference of binding energies between a given isotope (isotone) and a reference isotope (isotone), the latter usually being far away from the beta-stability line. A slight deviation from parabolic shape of the isotopic or isotonic sequence in the nuclear mass surface will give relatively large error in the binding-energy difference. It seems that, for better empirical fitting, one should use the most beta-stable isotope or isotone as the reference nuclide, whose binding energy is more accurately known than the least stable ones.

It may be recorded here that, in order to obtain minimum deviations, the following nuclides were used as the references in the present computations: Ni^{58} , Cu^{59} , Zn^{61} , Ga^{64} , Ge^{65} , Ge^{72} , As^{73} , Se^{74} , Br^{75} , Kr^{77} , Rb^{80} , Sr^{84} , Sr^{88} , Y^{89} , Zr^{90} , Nb^{91} , Mo^{92} , Cd^{112} , In^{113} , Sn^{114} , Sb^{115} , Te^{116} , I^{122} , Xe^{124} , Tl^{200} , Pb^{202} , Pb^{208} , Bi^{209} , Po^{210} , and At^{211} .

Zeldes' simple formulae predict fairly accurately the neutron and proton separation energies (see Figs. 11 and 12). Over 98% of the deviations lie within ± 1 MeV. Zeldes divided the nuclidic mass surface into several regions with the closed-shell configurations as the boundaries. Figure 17 shows the division with the heavy lines as the external boundaries; the solid points indicate the lightest and heaviest known isotopes. The parameters in his equations have not been evaluated for the nuclides outside these external boundaries, that is, nuclides with $Z = 41$ to 44 and $N = 48$ and 49 , and nuclides with $Z = 83$ to 88 and $N = 108$ to 125 . Since Levy used a similar division of the mass surface, his equation has similar limits of applicability. In a few instances, relatively large deviations are found for nuclides far away from the closed-shell configurations. But no large systematic deviations are observed in Zeldes' equations, probably as a result of cancellation of some systematic errors in the calculation

of mass differences. It will be interesting to compare the mass defects calculated with Zeldes' formulae with the experimental data. However, another limit of applicability of these formulae, as noted in Section B of Chapter II, allows only a limited number of calculations.

One disadvantage in Green's mass equation is the abrupt discontinuity of the shell-effect terms. The values of the shell-effect terms differ greatly from one shell region to the next. As a result, relatively large deviations are observed around the closed-shell configurations (see Fig. 13). In addition, there is no specific rule for assigning nuclides with the magic numbers to a particular shell region (i. e., i and j assignments; see Chapter II, Section C-2). For example, Ca^{48} could have the possible i - j assignments of 4-3, 4-4, 5-3 (unlikely), and 5-4. The calculated values of S_{ij} (shell effects) in these i - j assignments for Ca^{48} are (assuming $k_{ij} = 3.0$ for the i - j assignment of 5-3) 4.97, 1.432, 2.375, and 1.837 mmu, respectively. Similarly, Pb^{208} could be placed in the i - j regions of 7-6, 7-7, 8-6 (unlikely), and 8-7. The calculated values of S_{ij} for Pb^{208} in the i - j assignments of 7-6 and 8-7 are -32.862 and -10.644 mmu, respectively. It is not certain whether one should use the average value of S_{ij} of all possible i and j assignments for a given nuclide.

Furthermore, since the values of k_{ij} for i - j regions of 7-7 and 8-8 are not given, the shell effects of the nuclides with $Z \geq 83$ and $N = 82$ to 126 cannot be obtained from Green's mass formula. Similar limitations exist in other mass regions.

In the mass region of $A = 180$ to 208, the magnitude of $(M_{\text{exp}} - M_{\text{calc}})$ is about the same as the negative values of S_{ij} . This indicates that in this mass region the correct values of the shell effects should be close to zero.

Levy's equation has several systematic discrepancies explainable by inadequate consideration of the shell effects (see Fig. 14). Near the closed-shell configurations (boundaries of Levy's division of the mass surface, see Fig. 17), relatively large deviations from the experimental data are seen, i. e., around mass numbers 60 ($Z = 28$), 70 ($N = 40$), 95 ($N = 50$), 115 to 125 ($Z = 50$), 155 ($Z = 64$, $N = 90$), 190 ($N = 116$), and 208 ($N = 126$, $Z = 82$). In the mass regions of 155 and 170 to 200, the large deviations may be due to neglecting the subshell effects, or nuclear deformation, or both. The nuclear deformation is expected to be large around $A = 170$, and the subshell effect of $N = 116$ is expected to be quite small. At any rate, a correction term:

$$\frac{1.5 \times 64}{(190 - A)^2 + 64} \text{ MeV,}$$

may be applied to the deviations here.

As in the case of Cameron's and Seeger's equations, Levy's descriptions for isobaric parabolas are also inadequate. Levy treated the parameter B_A , which is a measure of the shape of the isobaric parabolas, as a constant within a given mass region. As we shall see in the next chapter, the values of this parameter vary considerably from one mass number to another, except in the regions of very large mass numbers and minimum shell effects.

Most of the systematic deviations in Baker's equation (see Fig. 15) may again be ascribed to inadequate description of the shell effects. Large deviations are seen around the same mass regions as those in Levy's equation. Only the magnitudes of the deviations are different in these two equations. This is interesting in view of the fact that Levy and Baker divided the nuclidic mass surface differently. The same arguments for Levy's deviations around the mass regions of 155 and 170 to 200 may be applied to Baker's deviations in the same mass regions.

The very large spread of deviations in Baker's equation indicates poor description of the isobaric parabolas in terms of neutron excess. Since Green *et al.*, also described the isobaric parabolas in terms of neutron excess, the narrower spread in the deviations of Green's formula (see Fig. 13) suggests a superior isobaric description than Baker's.

The observed systematic deviations between regions of closed-shell configurations in the mass equation of Wing and Fong (see Fig. 16) probably result from under- and over-corrections of the shell effects or lack of shell effects on Z_A . Relatively small deviations around mass number of 190 are again observed, as in Levy's and Baker's formulae. The deviations around mass number 95 are probably due to neglecting 40 as a proton subshell. In spite of the inclusion of 152 as a neutron subshell, this equation exhibits large deviations, with wide spread, for heavy nuclides. This wide spread of some 4 MeV may be explained by poor isobaric descriptions for $A > 230$ (B_A values too high).

CHAPTER IV. DISCUSSION OF SOME PARAMETERS IN THE NUCLIDIC MASS EQUATIONS

In this chapter, several expressions for the parameters Z_A , B_A , and P_A , and for the shell effects and nuclear deformation are summarized and discussed. Some of these expressions are compared with the experimental evidence.

Section A. Z_A , B_A , and P_A

The nuclidic mass surface may be represented by the following expression (the Bohr-Wheeler mass formula):

$$M(Z, A) = M_A + \frac{1}{2} B_A (Z - Z_A)^2 + \frac{1}{2} P_A \delta,$$

where M_A represents the gross shape of the surface, B_A is a measure of the shape of the isobaric parabolas, Z_A is the charge of the most stable isobar, and P_A the average distance between the surface of odd- Z , odd- N and even- Z , even- N . The shell effects and nuclear deformation are assumed to be included in all these four parameters. Although there are actually two mass surfaces for the odd- A nuclides, we here use the approximation of one surface for odd A , such that

$$\delta = \begin{cases} +1 & \text{for odd } Z \text{ and odd } N; \\ 0 & \text{for odd } A; \\ -1 & \text{for even } Z \text{ and even } N. \end{cases}$$

It can be shown that, for odd A ,

$$B_A = M(Z, A) - 2M(Z+1, A) + M(Z+2, A);$$

$$Z_A = \frac{1}{2} + Z + [M(Z, A) - M(Z+1, A)]/B_A.$$

For odd- Z and odd- N nuclides:

$$B_A + 2 P_A = M(Z, A) - 2 M(Z+1, A) + M(Z+2, A);$$

$$Z_A = \frac{1}{2} + Z + [M(Z, A) - M(Z+1, A) - P_A]/B_A.$$

For even- Z and even- N nuclides:

$$B_A - 2 P_A = M(Z, A) - 2 M(Z+1, A) + M(Z+2, A);$$

$$Z_A = \frac{1}{2} + Z + [M(Z, A) - M(Z+1, A) + P_A]/B_A.$$

Using these equations, one can calculate the values of B_A , Z_A , and P_A from the experimental masses. To obtain a value of any one of these parameters, at least four consecutive isobaric masses are needed for a given even A , and at least three for a given odd A .

A better way of deriving these three isobaric parameters from the experimental data is by a least-squares analysis of the beta-decay energies. A plot of the total beta-decay energies vs Z results in two parallel and usually straight lines. There is a change of slope in these lines at closed-shell configurations. The average of the intercepts of these lines at zero beta-decay energy gives $(Z_A - \frac{1}{2})$. The slope of the lines is B_A , and the average distance between the two parallel lines gives P_A . Here, P_A is not assumed to be zero for odd- A nuclides.

Dewdney (63D) has computed the values of B_A , Z_A , and P_A from available experimental data by this method. We shall call these values "experimental data." In this method, for a given mass number, at least two beta-decay energies are needed for the computation of a value of B_A , and at least three for Z_A or P_A .

Besides Dewdney, empirical determinations of Z_A have also been made by several workers (39BW, 45J, 47F, 48K, 49S, 53C, 54S).

It should be noted here that the values of B_A , Z_A , and P_A derived by these two methods represent only the average values, with the assumption that all isobaric sequences exhibit a parabolic shape. Since shell effects distort the isobaric parabolic shape and the linear relation of isobaric beta-decay energies with Z , these values of Z_A , B_A , and P_A will become anomalous at and near closed-shell configurations. Bouchez et al., (53BR) suggested that there might be two isobaric parabolas for a given mass number sequence involving a closed-shell configuration. According to this view point, for a given A , with a magic number, there will be two Z_A values separated by an amount related to the deviation from the smooth trend of Z_A . The vertical displacement of these two isobaric parabolas will be a measure of the deviation of P_A from the smooth trend of P_A . The two isobaric parabolas will also give two B_A values.

Several nuclidic mass equations contain expressions for the parameters B_A , Z_A , and P_A . These expressions are summarized in Tables 17, 18, and 19. The expression for Z_A can be derived from the mass equation by differentiating $M(Z, A)$ with respect to Z , keeping A constant, and then equating the resulting expression to zero:

$$\left(\frac{\partial M}{\partial Z} \right)_A = 0 \text{ at } Z = Z_A.$$

The expression for B_A may be obtained by identifying the coefficients in the mass equation with those in the Bohr-Wheeler mass equation without the pairing-effect and the shell-effect terms.

Table 17

EXPRESSIONS FOR P_A (in MeV)

Author(s)	Reference	P_A
<u>Liquid-drop Approach</u>		
Fermi	46F, 50F	$33.5/A^{3/4}$
Friedlander and Kennedy	49 FK	$132/A$
Bussetti	55B	$36/A^{3/4}$
Wapstra	58W	$11.2/A^{1/2}$
Ayres <u>et al.</u>	62AH	$\alpha\pi/2A$
Kümmel <u>et al.</u>	63KM	$11.045/A^{1/2}$
<u>Shell-model Approach</u>		
Thieberger <u>et al.</u>	57TD	-D
Zeldes	58Z	$\frac{1}{4}(\pi + \Pi) - \frac{1}{2} I'$
<u>Empirical Approach</u>		
Green	55G	$11.2/A^{1/2}$
Levy	57L	Table
Baker	58B	2h
Wing-Fong	62WF	$11.51/A^{1/2}$

Table 18

EXPRESSIONS FOR Z_A

Author(s)	Reference	Z_A
<u>Liquid-drop Approach</u>		
Fermi	46F, 50F	$A/(1.98067 + 0.0149624 A^{2/3})$
Ayres <u>et al.</u>	62AH	$(M_n - M_p + 4a_a) A/2(A_c A^{2/3} + 4a_a)$
Johansson	63J	$\frac{\left[\frac{A}{2} + 1 - 1.175 A^{2/3} - \frac{2.35}{A^{1/3}} + (M_n - M_p)A \right]}{\left[0.004845 A^{2/3} + 1 - \frac{2.35}{A^{1/3}} \right]}$
Tsen	57T	$0.86667 Z_A^{2/3} + 147.576 Z_A/A = 74.627$
<u>Shell-model Approach</u>		
Zeldes	58Z	$\frac{1}{(2I^0 - d - D)} \left\{ (M_n - M_p) + (C - c) + \frac{1}{2} (\Pi - \pi) \right. \\ \left. - \frac{1}{2} (D - d) + A(I^0 - d) - (N_0 - Z_0)I^0 \right. \\ \left. + N_0 d - Z_0 D \right\}$
<u>Empirical Approach</u>		
Green	55G	$\frac{1}{2} [A - 0.4 A^2/(A + 200)]$
Levy	57L	$-(A_2 + A_3 A)/2a_4$
Wing-Fong	62WF	$(A + 0.003 A^2)/(2 + 0.01 A)$

Table 19

EXPRESSIONS FOR B_A (in MeV)

Author(s)	Reference	B_A
<u>Liquid-drop Approach</u>		
Fermi	46F, 50F	$78.064 (1.98067 + 0.0149624 A^{2/3})/A$
Ayres <u>et al.</u>	62AH	$2(4a_a + a_c A^{2/3})/A$
Johansson	63J	$271 (0.004845 A^{2/3} + 1 - 235/A^{1/3})/A$
<u>Shell-model Approach</u>		
Zeldes	58Z	$(2I^0 - d - D)$
<u>Empirical Approach</u>		
Levy	57L	$2a_4$
Wing-Fong	62WF	$2.34 - 43.17/A^{1/2} + 279/A$

Note in Table 17 the similarities of the pairing-effect terms among Wapstra, Kümmel, Green, and Wing-Fong. Barrett (62B) found that Thieberger's pairing effects have an average trend of $11.2/A^{1/2}$, in agreement with the above authors. There are a number of sets of values reported for the parameters in the Bethe-Weizsäcker formula (see Table 1). We used Fermi's values here to represent a typical mass equation based on the liquid-drop model with the parameters independent of A . The values of a_a and a_c in Ayres' mass equation are dependent upon A . The expressions for B_A and P_A in Levy's and Zeldes' formulae are discontinuous functions of A . Wing's expression for Z_A is identical with Green's.

One can also obtain values of B_A , Z_A , and P_A by least-squares analysis of the beta-decay energies calculated with the nuclidic mass equations. The values so obtained will contain shell-structure and nuclear-deformation effects, if these are included in the mass equations.

The Z_A values calculated with the expressions of Fermi, Tsen, Levy, Zeldes, and Green are compared with the experimental data in Figs. 18, 19, and 20. Whenever several calculated values are obtained for a given mass number, as in using Levy's and Zeldes' expressions, the one with the least deviation from the experimental value is considered.

One observes in these figures large deviations which may be attributed to shell effects on Z_A . As early as 1949, the shell effect on Z_A was recognized (49A). As Dewdney (63D) recently pointed out, the rate of

increase of Z_A with A is enhanced at neutron-shell configurations and is reduced at proton-shell configurations. The increasing (positive slope) deviations with A in the plots in Figs. 18 and 20 indicate the shell effects of closed neutron configurations (upward arrows), and the decreasing deviations (negative slope) reflect the shell effects of closed proton configurations (downward arrows) in these expressions. Note the change of slope at double closed-shell configurations, i.e., $N = 20$, $Z = 20$ and $N = 126$, $Z = 82$.

Tsen's mass formula differs from Fermi's in that the Coulomb-energy term is proportional to $Z^2/Z^{1/3}$ instead of to $Z^2/A^{1/3}$. In the mass region below $A = 100$, Tsen's predicted Z_A values are very similar to Fermi's. As the mass number increases, however, the differences become larger and larger. No improvement in Z_A is obtained with Tsen's expression.

In spite of the fact that Zeldes' mass formula is based on a shell model, the predicted values of Z_A still show large deviations from the experimental values at closed-shell configurations (see Fig. 19). This is because of the discontinuity of the mass equation at the magic numbers. Levy's mass formula is equivalent to Zeldes', and the Z_A values show deviations similar to those of Zeldes, but with larger magnitudes in most cases, especially at closed-shell configurations.

Green's expression for Z_A may be improved by including a series of correction terms:

$$\sum_i \frac{a_i b_i^2}{(A - A_i^*)^2 + b_i^2},$$

where a and b are constants, and A^* is the mass number at which maximum deviation of Z_A occurs. The values of the constants are listed in Table 20. With these correction terms, the standard deviation of the predicted values of Z_A from the experimental data is reduced to 0.05. It is found that 70% of the calculated values of Z_A agree with the experimental values within ± 0.10 , that 80% are within ± 0.14 , and that 92% are within ± 0.20 unit. The open circles in Fig. 20 are the differences between the experimental and Green's values of Z_A with these correction terms. The experimental errors greater than 0.25 unit are indicated by vertical bars.

Figures 21 and 22 show comparisons of the calculated values of B_A and P_A , obtained from the expressions given in Tables 19 and 17, with the experimental values (solid points), respectively. The experimental values of both B_A and P_A decrease rapidly with increasing A at low-mass region, but vary slightly at high-mass regions. Note the strong shell effects on both B_A and P_A . Because of these strong shell effects, B_A and P_A are not easily representable by simple functions of A , Z , or N . Fermi's expression seems to fit the general trend of B_A quite well. Wing's values of B_A are slightly high at the heavy-mass

region, and Ayres' and Johansson's B_A values are too low at the light-mass region. Zeldes' values of B_A are rather high for $A = 160$ to 200 .

Table 20

VALUES OF THE CONSTANTS IN THE CORRECTION TERMS
FOR GREEN'S EXPRESSION OF Z_A

A^*	a	b	A^*	a	b
40	0.321	1.14	130	-1.559	14.97
48	-0.284	3.06	147	0.965	5.79
57	0.498	3.25	206	-0.607	4.28
83	-1.031	15.74	250	0.610	16.10
105	1.316	24.21			

For the pairing effects, the expressions of Zeldes, Levy, and Wing-Fong (a continuous function of A) are in better agreement with the experimental values than those with other functions of A . However, none of the expressions examined here correctly describes the shell effects of P_A .

It may be mentioned that the pairing effect defined above represents the average interaction between a proton and a neutron. In this way, the effects of the volume and surface energies of the liquid-drop model on P_A are cancelled. However, it is difficult to separate the shell effects on the pairing interactions. Some authors (61YL, 62NA, 63K) prefer the use of the pairing energies of two neutrons and of two protons defined as follows:

$$P_{nn} = B_n(Z, N) - B_n(Z, N-1);$$

$$P_{pp} = B_p(Z, N) - B_p(Z-1, N).$$

In this way, the shell effects may be separated from the pairing effects when one considers, for example, the P_{nn} for isotopes of a closed proton configuration, assuming no interaction between the proton core and addition of neutrons. More recently, the following definitions were suggested (62NA) for the average pairing energies of two neutrons and of two protons:

$$P_{nn} = \frac{1}{2} [2B_n(Z, N) - B_n(Z, N-1) - B_n(Z, N+1)];$$

$$P_{pp} = \frac{1}{2} [2B_p(Z, N) - B_p(Z-1, N) - B_p(Z+1, N)].$$

In this way, the odd-even effects of B_n and B_p are smoothed out.

Section B. Shell Effects and Nuclear Deformation

Nuclides with and near closed-shell configurations have extra stability of up to several MeV. The amount of the shell stability varies from shell to shell. Due to the mutual influence of the proton and neutron shells on each other, the observed shell effects are unsymmetric with respect to N_0 and Z_0 . The so-called magic numbers 8, 20, 28, 50, 82, and 126 are generally considered to indicate the numbers of protons and neutrons in the major shells, and populations of 40, 64, and 152 are sometimes considered to form subshells. There are experimental indications that discontinuities in the general trend of nucleon-separation energies and beta-decay energies around $N = 90$ and $N = 116$ (57JB, 57JN, 60D) suggest 90 and 116 are magic numbers for neutron subshells. The shell effects of the subshells are quite small compared with the major shells.

Nuclides between the closed-shell configurations have extra stability due to nuclear deformation. The deformation energy is generally small compared with the shell-stability energy. In fact, in most of the existing nuclidic mass equations, nuclear deformation is neglected. In general, the deformation energy of a given nuclide is inversely proportional to its shell-stability energy.

The importance of shell effects on the systematics of nuclidic masses was realized in the mid-20th century from the large deviations found between the experimental masses and the calculated values obtained by some of the then-existing mass equations (49A, 50LT, 51W, 51CN, 52H, 52CN, 52W). Several subsequent nuclidic mass equations included shell-effect terms as corrections to the basic mass equations of Bethe-Weizsäcker and of Bohr-Wheeler. The effect of nuclear deformation was later realized and considered (59M, 60GH, 61S, 63KM, 63J).

Several methods have been devised to describe the systematic shell effects. Cameron (57C) empirically determined the combined effects of shell and pairing interactions for each N and Z . A total of 310 values of these effects were determined. Such a large number of constants makes the mass equation inconvenient to use. Levy (57L) and Zeldes (58Z) divided the nuclidic mass surface into sections with the assumed closed-shell configurations as the boundaries (see Fig. 17). The shell effects are inherent in their mass formulae.

Other workers have attempted to describe the shell effects in terms of continuous functions of Z and N . Green (55G) used the following expression for the shell effects between two shells:

$$S_{ij}(Z, N) = -a_i(N - N_i)^2 - a_j(Z - Z_j)^2 + k_{ij},$$

where N_i is a neutron number between two neutron shells, and Z_j is a proton number between two proton shells. This expression represents an inverted parabola for the shell effects between two shells with N_i or Z_j as the vertex of the parabola and a_i or a_j a measure of its width. The quantity k_{ij} indicates the displacement of the shell effects with respect to the nuclidic mass surface. As we have mentioned in Section B of Chapter III, this shell-effect expression has the disadvantage of discontinuities at or around the closed-shell configurations.

Wapstra (58W) suggested a Gaussian distribution for the shape of the shell effects:

$$S(Z) = H_p \exp [-(Z - Z_0)^2/16];$$

$$S(N) = H_n \exp [-(N - N_0)^2/16],$$

where H_p and H_n are the heights of the shell effects at Z_0 and N_0 , respectively. The preliminary estimates for the values of H_p and H_n are:

Z_0 or N_0	H_p (MeV)	H_n (MeV)
28	4	5
50	3	6
82	5	7
126		8

As the author pointed out (58W), the actual shell effects are more pointed in shape than these Gaussian distributions. Furthermore, these expressions are discontinuous functions of Z and N in crossing from one shell to another.

Mozar (59M) derived his shell-effect expression from the consideration of the difference of the nucleon-separation energies computed on a shell model and of those computed on a statistical model:

$$S(Z, A) = c [-3(A/5) + \sum_A (A_k/A)^{n/3}],$$

where c is a constant, A_k is the mass number of the stable nuclide exactly intermediate between two closed shells with respect to the nuclear type of interest (cf Table 4 in Chapter II), and $n = 1$ for the harmonic oscillator potential and $n = 2$ for the square-well potential. This shell-effect expression is an unsymmetric, oscillating function of A , with minima at closed-shell configurations and maxima intermediate between shells. A similar unsymmetric function of A is also included in his expression for nuclear deformation (cf Section A-11 of Chapter II), which is not repeated here.

Seeger (61S) described the shell effects and nuclear deformation by a series of sine functions with cross products:

$$S_{jk}(N, Z) + D = a_j \sin N'\pi + a_k \sin Z'\pi + b_j \sin 2N'\pi \\ + b_k \sin 2Z'\pi + c_{jk}(\sin N'\pi)(\sin Z'\pi) + d,$$

where j and k refer to the neutron and proton shells, respectively, and N' and Z' are the fractional occupancies of the last shells (cf Section A-13 of Chapter II). The cross-product term has the advantage of including the effects of the proton shell on the neutron shell effects, and vice versa, which are usually neglected in other expressions. The last constant, d , is a measure of the maximum shell effect, that is, $(S + D) = d$ when the sine terms are zero. Seeger made the value of this constant the same from one shell configuration to another in order that $(S + D)$ be a continuous function. This, however, does not seem necessary and desirable.

The shell-effect terms in the Wing-Fong mass equation (62WF) consist of a series of Cauchy distribution functions:

$$S(Z, N) = \sum_i \frac{a_i b_i^2}{(N - N_{0i})^2 + b_i^2} + \sum_j \frac{a_j b_j^2}{(Z - Z_{0j})^2 + b_j^2},$$

where a represents the height of the maximum shell effect and $2b$ represents the full width at half-height. This expression is a continuous function of N and Z from shell to shell, and the effect on a given shell becomes small as N or Z varies farther away from N_0 or Z_0 of this shell.

Note the absence of a cross-product term containing N and Z and the symmetric shape of the parabolic, Gaussian, and Cauchy distribution functions in Green's, Wapstra's and Wing-Fong's expressions, respectively.

Kümmel's shell-effect terms (63KM) may be simplified to cubic functions of N and Z , with cross-product terms for the mutual influence of the neutron and proton shells:

$$S(N, Z) = \sum_i (a_i + b_i n_i)^3 + \sum_j (a_j + b_j z_j)^3 + \sum_{ij} c_{ij} (d_i - n_i)(d_j - z_j) n_i z_j,$$

where n_i and z_j are the numbers of neutrons and protons in excess of N_0 and Z_0 in the i th neutron and j th proton shells, respectively. This expression is unsymmetric with respect to N_0 and Z_0 . Their nuclear-deformation term, purely empirical, consists of power functions of N and Z , vanishing at the closed-shell configurations:

$$D(N, Z) = \sum_{ij} \sum_{vuv'u'} g_{ij} n_i^v \bar{n}_i^u z_j^v \bar{z}_j^u,$$

where $v, u, v',$ and u' are integers and $0 \leq v, u, v', u' \leq 4$. Here g_{ij} is a constant, and $\bar{n}_i$ and $\bar{z}_j$ are the neutron holes and proton holes, respectively (cf Chapter II, Section A-15).

Johansson's expression (63J) for nuclear deformation is taken from a liquid-drop model of a deformed nucleus (56S):

$$D = c_1\beta - c_2A^{2/3}(1 - x)\beta^2 + c_3A^{2/3}(1 + 2x)\beta^3,$$

where $x = \text{constant}(Z^2/A)$ and β is the deformation parameter (60BB). Johansson pointed out that, since the deformation energy is so sensitive to the value of β , for high accuracy the experimental values of β should be used. However, experimental values of β are available for only a limited number of nuclides at the present time. The values of β may be estimated by the following empirical formulae:

$$\text{for nuclides with } 82 < N < 96, \quad \beta = \frac{32.3}{(N - 96)^2 + 93.6};$$

$$\text{for nuclides with } 96 < N < 126, \quad \beta = \frac{110}{(N - 96)^2 + 324};$$

$$\text{for nuclides with } N > 126, \quad \beta = \frac{54.0}{(N - 146)^2 + 200}.$$

The predicted values of β obtained from these expressions are accurate to within 10% of the experimental data available at the present, worse for nuclides very close to the closed-shell configurations.

The shell-effect term in Johansson's mass equation also contains the deformation parameter:

$$S = -(\beta_0 - \beta)(k\beta - H),$$

where k is a constant, β_0 is the maximum deformation within a mass region bound by closed-shell configurations, and $\beta_0 H$ is the maximum shell effect. This expression is a parabolic function of β . Since a plot of β values vs N or Z is unsymmetric, this shell-effect term is also unsymmetric with respect to N_0 and Z_0 .

Figure 23 is a comparison of the shell-effect expressions of Cameron, Green, Mozer, Seeger, and Wing-Fong with the "experimental" data. In order to simplify the comparison, only the shell effects for the stable odd- A nuclides are considered, and smooth curves are drawn through the calculated points. The "experimental" shell effects (solid points in Fig. 23) are obtained by subtracting a smooth mass surface (Bohr-Wheeler's expression) from the experimental masses:

$$S = M_{\text{exp}} - M_A - B_A(Z - Z_A)^2,$$

where the expressions for M_A , B_A , and Z_A are taken from Wing-Fong's mass equation (see Chapter II, Section C-5).

In this comparison, the magnitude of the deviations from the "experimental" points is irrelevant, since it is not model-dependent. Only the occurrence of the "dips" (maximum shell effects) with respect to A and the shape of the oscillation of the curves are of interest. Nearly all the expressions predict the correct occurrence of the "dips." Note the discontinuities at closed-shell configurations in Green's expression. The "experimental" dips at $A = 29$ and 103 are not real shell effects, and the expected dips at $A = 40$ and 120 are not large. Only Seeger's curve does not show a dip at $A = 120$.

Cameron's values follow the "experimental" points quite well, as expected, since his values are empirically determined. Although the expression of Wing-Fong follows the general trend of the "experimental" shell effects, systematic deviations are seen at the regions before and after a closed-shell configuration, indicating under- and over-corrections of the shell effects. It is difficult to compare the shape of Seeger's and Mozer's curves at low-mass regions, for which large deviations exist. However, if these two curves are bent upward to match the "experimental" points, Seeger's curve will be in better agreement with the "experimental" points than is Mozer's. Green's expression for shell effects is definitely inferior to any other examined.

Table 21

SUMMARY OF A AND Z FUNCTIONS IN SOME OF
THE NUCLIDIC MASS EQUATIONS

Each of the nuclidic mass equations of Feenberg, Cameron, Findler, and Baker (not presented in Table 21) contains more than 18 functions of A and Z . Possession of so many terms does not necessarily mean superiority of a mass equation. In fact, some of the nuclidic mass equations with fewer and simpler functions, such as Levy's, are just as good as Cameron's insofar as the agreement with the experimental masses is concerned.

Although some of the existing nuclidic mass equations describe the general trends of the nuclidic masses and binding energies fairly well, systematic deviations from the experimental data are found in every mass equation examined. For the prediction of masses or binding energies of nuclides not too far from the beta-stability line, extrapolation or interpolation of the isobaric, isotopic, or isotonic sequence is most convenient. For the nuclides far away from the beta-stability line, one may select a value or an average from the calculations based on different mass equations, as some mass equations have less deviations for a certain mass region than the others. The mass equation of Kümmel *et al.*, seems to have the fewest systematic deviations from the existing experimental data, and the deviations are also small. Table 22 lists the frequency

Table 22

FREQUENCY DISTRIBUTION OF DEVIATIONS

Mass Region	Deviation (MeV)	Cameron	Seeger	Kümmel	Green	Levy	Baker	Wing-Fong
$40 < A \leq 70$	< 0.5	83	44	85	5	16	63	32
	$0.5-1.0$	27	50	18	31	8	37	41
	$1.1-2.0$	7	22	0	39	5	18	38
	> 2.0	3	5	0	44	6	2	10
$70 < A \leq 115$	< 0.5	145	50	177	56	131	115	44
	$0.5-1.0$	45	78	31	54	42	68	90
	$1.1-2.0$	11	51	0	74	14	15	59
	> 2.0	2	24	0	19	7	4	10
$115 < A \leq 162$	< 0.5	96	101	167	0	127	72	63
	$0.5-1.0$	76	70	26	0	55	63	38
	$1.1-2.0$	21	21	0	3	8	45	54
	> 2.0	0	1	0	190	3	13	37
$162 < A \leq 208$	< 0.5	19	65	137	0	40	32	58
	$0.5-1.0$	28	44	12	0	34	33	57
	$1.1-2.0$	41	32	0	0	48	37	31
	> 2.0	57	4	0	132*	11	43	2
$208 < A$	< 0.5	0	106	162	3	1	7	21
	$0.5-1.0$	88	52	7	23	12	104	27
	$1.1-2.0$	81	11	0	34	134	56	44
	> 2.0	0	0	0	88	4	2	75
Number of adjustable parameters		228	24	38	30	81	56	26

*All deviations here are greater than 6 MeV.

distribution of the deviations (in MeV) found in several nuclidic mass equations at different mass regions. The number of adjustable parameters used to obtain the calculated masses is also indicated.

Most of the observed systematic deviations in the mass equations are due to poor descriptions of the shell effects and isobaric sequences. Many of the existing nuclidic mass equations may be improved by better expressions for the nuclear properties of the ground states, including nuclear deformation and shell effects, and by proper evaluation of the parameters with more nuclidic mass data of good quality. In this regard, more experimental mass values for the nuclides in the rare earths region are needed. A study of the properties of the nuclides far away from the beta-stability line is also helpful. Thus, in spite of the existence of so many nuclidic mass equations, the development of a better or improved one may be of great interest to the nuclear scientists and engineers.

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Fig. 1. Deviations of the Calculated Masses of the Fermi-Weizsäcker Equation

Fig. 2. Deviations of the Calculated Binding Energies of Román's Equation

Fig. 3. Deviations of the Calculated Masses of Bussetti's Equation

Fig. 4. Deviations of the Calculated Binding Energies of Tsen's Equation

Fig. 5. Deviations of the Calculated Binding Energies of the Equation of Ayres et al.

Fig. 6. Deviations of the Calculated Binding Energies of Findler's Equation

Fig. 7. Deviations of the Calculated Masses of Cameron's Equation

Fig. 8. Deviations of the Calculated Masses of Seeger's Equation

Fig. 9. Deviations of the Calculated Binding Energies of the Equation of Kümmel et al.

Fig. 10. Deviations of the Calculated Binding Energy Differences of the Equations of Thieberger, Talmi, and De-Shalit

Fig. 11. Deviations of the Calculated Proton Separation Energies of Zeldes' Equation

Fig. 12. Deviations of the Calculated Neutron Separation Energies of Zeldes' Equation

Fig. 13. Deviations of the Calculated Masses of Green's Equation

Fig. 14. Deviations of the Calculated Masses of Levy's Equation

Fig. 15. Deviations of the Calculated Binding Energies of Baker's Equation

Fig. 16. Deviations of the Calculated Masses of the Equation of Wing-Fong

Fig. 17. Division of the Nuclidic Mass Surface by Zeldes and Levy

Fig. 18. Deviations of the Calculated Z_A Values of Fermi's and Tsen's Expressions

Fig. 19. Deviations of the Calculated Z_A Values of Zeldes' and Levy's Expressions

Fig. 20. Deviations of the Calculated Z_A Values of Green's Expression with and without Corrections

Fig. 21. Comparison of the Calculated Values of B_A with the "Experimental" Values

Fig. 22. Comparison of the Calculated Values of P_A with the "Experimental" Values

Fig. 23. Comparison of the Calculated Shell Effects with the "Experimental" Values (Solid Points)