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**APPLICATION OF THE TENSOR CODE
TO THE CALCULATION OF RAYLEIGH WAVES**

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Abstract

Analytical methods are used to obtain two- and three-dimensional solutions for the Rayleigh wave generated by a pressure applied to the surface of an elastic half-space.

The limitations inherent in the TENSOR finite-difference equation are then explored by comparing the TENSOR solutions with those derived by analytic methods.

Introduction

This report describes the results of a comparison of analytic and computer (TENSOR¹) solutions for Rayleigh waves generated by a time-varying pressure applied to the surface of an elastic half-space. Both two- and three-dimensional cases are considered. Certain analytic solutions are convoluted to give solutions for a general distribution of the source in space and time. Some of the limitations on TENSOR calculation of surface waves are discussed.

Analytic Solutions

It is helpful to have the analytic solutions to a few relatively simple cases available when solving a set of complicated partial differential equations by the method of finite differences, since these solutions may be used to check for programming errors and to explore the limitations of the finite-difference method. The problem we have considered is the determination of the response of a uniform elastic half-space to a time-dependent pressure distribution along the surface. Green's functions for this problem have been determined although they do not seem to be noted or referred to as such. The determination is exact in the two dimensional case,² and asymptotic in the three dimensional case³ (Lamb's classic solution).

The two-dimensional Green's function may be used to compute the response to a time-dependent pressure $P(t)$ applied to an infinite strip and given by

$$P(t) = \frac{\tau_0 L}{\pi(\tau_0^2 + t^2)}.$$

This particular time function is the one used by Lamb in his asymptotic solution for the Rayleigh wave. The integral of this function with respect to time gives L , the impulse per unit surface area. The parameter τ_0 has the units of time and specifies the duration of the pressure pulse. Figure 1 shows a section through the half-space. The spatial distribution is specified by

$$P = \begin{cases} P(t) & -a \leq x \leq a \\ 0 & \text{where } a < |x| \end{cases}$$

The half-space is assumed to behave elastically; therefore, the only parameters needed to completely specify its properties are density, bulk modulus, and shear modulus.

The Green's function for the two-dimensional line-source problem is given in terms of the polar coordinates shown in Fig. 1. Let

$$\theta_0 = \begin{cases} \text{arc cosh} \left(\frac{\alpha t}{R} \right) & \alpha t \geq R \\ -1 \text{ arc cos} \left(\frac{\alpha t}{R} \right) & \alpha t < R \end{cases}$$

$$\psi_0 = \begin{cases} \text{arc cosh} \left(\frac{\beta t}{R} \right) & \beta t \geq R \\ -1 \text{ arc cos} \left(\frac{\beta t}{R} \right) & \beta t < R \end{cases}$$

and

$$\Theta = \phi - i\theta_0 \quad \Psi = \phi - i\psi_0 \quad \gamma = \beta/\alpha,$$

where α is the compression wave velocity, β is the shear wave velocity, and ϕ is the polar angle. Then the radial and tangential displacements due to a unit delta-function source are

$$\delta_R = \delta_R^\Delta + \delta_R^W \quad \text{and} \quad \delta_\phi = \delta_\phi^\Delta - \delta_\phi^W,$$

where

$$\delta_R^\Delta = \frac{1}{\pi \rho \beta R} \text{Real} \left\{ \frac{-i\gamma [1 - 2(\gamma \sin \Theta)^2] \cos \Theta \sin \Theta \cot(-i\theta_0)}{[4(\gamma \sin \Theta)^3 [1 - (\gamma \sin \Theta)^2]^{\frac{1}{2}} \cos \Theta + [1 - 2(\gamma \sin \Theta)^2]^2 \sin \Theta]} \right\}$$

$$\delta_\phi^\Delta = \frac{1}{\pi \rho \beta R} \text{Real} \left\{ \frac{-i\gamma [1 - 2(\gamma \sin \Theta)^2] \cos \Theta \sin \Theta}{[4(\gamma \sin \Theta)^3 [1 - (\gamma \sin \Theta)^2]^{\frac{1}{2}} \cos \Theta + [1 - 2(\gamma \sin \Theta)^2]^2 \sin \Theta]} \right\}$$

Fig. 1. Coordinate system and spatial distribution of the pressure pulse.

and

$$\delta_R^W = \frac{1}{\pi \rho \beta R} \operatorname{Real} \left\{ \frac{-i \left[1 - (\gamma^{-1} \sin \psi)^2 \right]^{\frac{1}{2}} \sin \psi \sin 2\psi}{4(\sin \psi)^3 \left[1 - (\gamma^{-1} \sin \psi)^2 \right]^{\frac{1}{2}} \cos \psi + \gamma^{-1} \sin \psi (\cos 2\psi)^2} \right\}$$

$$\delta_\phi^W = \frac{1}{\pi \rho \beta R} \operatorname{Real} \left\{ \frac{i \left[1 - (\gamma^{-1} \sin \psi)^2 \right]^{\frac{1}{2}} \sin \psi \sin 2\psi \cot(-i\psi_0)}{4(\sin \psi)^3 \left[1 - (\gamma^{-1} \sin \psi)^2 \right]^{\frac{1}{2}} \cos \psi + \gamma^{-1} \sin \psi (\cos 2\psi)^2} \right\}.$$

The branch cut for the square root is chosen so that the real part is always positive:

$$\sqrt{\rho e^{i\Phi}} = \sqrt{\rho} e^{i\Phi/2}, \text{ where } -\pi < \Phi \leq \pi.$$

Note that δ_R and δ_ϕ are zero for $at < R$.

To put the Green's function in Cartesian coordinates we have

$$\phi = \arctan(x/z)$$

$$R = x^2 + z^2$$

$$\delta_x = \delta_R \sin \phi + \delta_\phi \cos \phi$$

$$\delta_z = \delta_R \cos \phi - \delta_\phi \sin \phi$$

To obtain the solution for the specified time function and spatial distribution, one must convolute these functions with the Green's function. Thus the displacements are given by

$$\Delta_z(x, z, t) = \frac{1}{2a} \int_{-a}^a d\xi \int_{-\infty}^{\infty} d\tau \frac{\delta_z(x - \xi, z, \tau) \tau_0 L}{\pi [\tau_0^2 + (t - \tau)^2]}$$

and

$$\Delta_x(x, z, t) = \frac{1}{2a} \int_{-a}^a d\xi \int_{-\infty}^{\infty} d\tau \frac{\delta_x(x - \xi, z, \tau) \tau_0 L}{\pi [\tau_0^2 + (t - \tau)^2]}.$$

The Green's function and the double integrals are evaluated numerically by a computer program (COMP) written by L. Barr and the author. On the CDC-7600, approximately 1/10 sec running time is needed to calculate Δ_z and Δ_x at a given point in space and time. Non-dimensional values of δ_R and δ_ϕ obtained from COMP agree with the published results. To check the time-convolution part of the solution, we have computed the displacements at a point very late in time and compared them with Lamb's two-dimensional asymptotic solution for a line source. In this case we have

omitted the spatial convolution. Figure 2 shows the comparison of the vertical displacements at the surface of the half-space. The solid line is the convoluted Green's function, and the small circles are points from Lamb's solution.

Fig. 2. Comparison of the convoluted Green's function solution (solid curve) with Lamb's two-dimensional asymptotic solution (data points). (time = 3.5 msec).

Lamb's asymptotic solution for the surface displacements produced by an infinite line source (the two-dimensional case) is given by

$$\Delta_x(x, t) = \frac{-L H \tau_0}{\mu \left[\tau_0^2 + \left(t - \frac{x}{c} \right)^2 \right]}$$

and

$$\Delta_z(x, t) = \frac{L K \left(t - \frac{x}{c} \right)}{\mu \left[\tau_0^2 + \left(t - \frac{x}{c} \right)^2 \right]},$$

where c is the Rayleigh-wave velocity determined by the equation

$$X^3 - 8X^2 + X \left[24 - \left(\frac{\beta}{\alpha} \right)^2 \right] - 16 \left[1 - \left(\frac{\beta}{\alpha} \right)^2 \right] = 0 \text{ where } c^2 = X\beta^2,$$

and where

$$\begin{aligned} H &= -c^{-1} \left[2c^{-2} - \beta^{-2} - 2(c^{-2} - \alpha^{-2})^{\frac{1}{2}} (c^{-2} - \beta^{-2})^{\frac{1}{2}} \right] / F \\ K &= -\beta^{-2} (c^{-2} - \alpha^{-2})^{\frac{1}{2}} / F \\ F &= 8c^{-1} \left[2c^{-2} - \beta^{-2} - (c^{-2} - \alpha^{-2})^{\frac{1}{2}} (c^{-2} - \beta^{-2})^{\frac{1}{2}} \right] \\ &\quad - 4c^{-3} \left[\left(\frac{c^{-2} - \alpha^{-2}}{c^{-2} - \beta^{-2}} \right)^{\frac{1}{2}} + \left(\frac{c^{-2} - \beta^{-2}}{c^{-2} - \alpha^{-2}} \right)^{\frac{1}{2}} \right]. \end{aligned}$$

Lamb's solution for the surface displacements at large distances from a point source (the three-dimensional case) is given by

$$\Delta_R(R, t) = \frac{-L H \cos^{3/2} \theta \sin \left(\frac{\pi}{4} - \frac{3}{2} \theta \right)}{2 \mu \sqrt{2 R C} \cdot \tau_0^{3/2}}$$

and

$$\Delta_z(R, t) = \frac{L K \cos^{3/2} \theta \cos \left(\frac{\pi}{4} - \frac{3}{2} \theta \right)}{2 \mu \sqrt{2 R C} \tau_0^{3/2}}$$

where $\tan \theta = \tau_0 / (t - \frac{R}{C})$.

Lamb's point-source result can be used for a direct comparison with TENSOR calculations, whereas his line-source solution cannot. This is because the three-dimensional solution approaches the asymptotic form much more rapidly than the two-dimensional solution. Numerical calculation of the two-dimensional case out to the asymptotic limit is not practical.

To make comparisons, we need to extend the Lamb point-solution to a disk of radius a by performing a spatial convolution over the surface of the disk. This can be written as

$$\Delta_R'(\vec{x}, \tau) = \frac{1}{\pi a^2} \iint_{\text{disk}} \Delta_R(|\vec{x} - \vec{x}'|, t) d\sigma'$$

and

$$\Delta_Z'(\vec{x}, t) = \frac{1}{\pi a^2} \iint_{\text{disk}} \Delta_Z(|\vec{x} - \vec{x}'|, t) d\sigma'.$$

For $|\vec{x} - \vec{x}'| \gg a$ these formulas reduce to

$$\Delta_R'(R, t) = \frac{2}{\pi a^2} \int_{-a}^a \Delta_R(R - \xi, t) \sqrt{a^2 - \xi^2} d\xi$$

and

$$\Delta_Z'(R, t) = \frac{2}{\pi a^2} \int_{-a}^a \Delta_Z(R - \xi, t) \sqrt{a^2 - \xi^2} d\xi.$$

Several properties of Lamb's asymptotic solution are interesting. First, we note that the time integral of the source function $P(t)$ is independent of τ_0 , and equal to the constant L . Furthermore, $\lim_{\tau_0 \rightarrow 0} P(t) = \begin{cases} 0 & \text{if } t \neq 0 \\ \infty & \text{if } t = 0. \end{cases}$

Hence $\lim_{\tau_0 \rightarrow 0} \frac{P}{L}(t) = \delta(t)$ is the Dirac delta function, since

$$\lim_{\tau_0 \rightarrow 0} \frac{1}{L} \int_{-\infty}^{\infty} P(t) f(t) dt = \int_{-\infty}^{\infty} \delta(t) f(t) dt = f(0).$$

Therefore, in the limit $\tau_0 \rightarrow 0$ Lamb's point-source solution gives the asymptotic form of the Green's function for the Rayleigh wave in three dimensions. As a result, we can easily find the Rayleigh wave from any spatial distribution of normal surface forces with arbitrary time variation. To get the time variation, we convolve the

given source function with the Green's function. To get the spatial distribution, we perform a double convolution of the Green's function over the surface area of the source. At large distances from the source, this reduces to a single convolution-integral as shown in Fig. 3.

Fig. 3. Reduction of a double spatial convolution to a single integral.

TENSOR Finite-Difference Solutions

There are several things to consider when setting up a finite-difference calculation of an elastic-wave propagation problem. First, there are a number of artificial damping terms added to the difference equations for the purpose of stabilizing them and damping out oscillations. These terms are multiplied by coefficients which, if too large, will cause an excessive and physically unrealistic amount of damping. Second, there is an inherent damping and dispersion due to the truncation error in the difference equations. This effect is proportional to the mesh width and the time step, and because of it one cannot expect to accurately calculate disturbances whose wavelength is less than that of some cutoff value. Both of these factors limit the class of problems one can realistically expect to calculate.

In order to accurately know just what these limitations are, it is wise to begin by calculating a class of problems that have the essential features of the actual physical case one is interested in, but which have known solutions. There are three types of artificial damping (the so-called Q-coefficients) in the TENSOR equations: quadratic scalar, linear tensor, and rotational. Of the three types, the linear Q-coefficient is the one of interest for the small displacement problems we are considering here. The components of the linear Q-coefficient are given by

$$Q_{ij} = C_1 C_s \rho \epsilon_{ij} \Delta x \left(\frac{2\mu}{k} \right)$$

where ϵ_{ij} is the deviatoric strain tensor, C_s is the compression velocity, ρ is the density, Δx is the mean width of the mesh element, μ is the shear modulus, k is the bulk modulus, and C_1 is the adjustable coefficient. We have used the analytic solutions discussed previously to observe the effect of the linear Q-coefficient (C_1) and the mesh spacing upon the numerically calculated surface-wave for the problem shown in Fig. 1. In the two-dimensional case this represents the solution for the pressure given by Eq. (1) applied to an infinite strip, and in the three-dimensional case the

solution is for the pressure applied to a disk. Since the z-axis is a plane or a line of symmetry, one only needs to calculate in the region $x \geq 0$.

The values of the impulse L , time duration τ_0 , and material properties were initially chosen to match the values published in an experimental confirmation of the Green's function for this specific problem.² However, it turned out to be impossible to match the experiment exactly because the pressure measurements were at such a large distance from the source: there was not enough storage space or computer time available to calculate the signal to the same distance. In any case, only the relative magnitudes of the displacements were measured, and in addition the impulse (produced by an explosive cap) could only be estimated. The values of the parameters used were

$$\begin{aligned}\rho &= 2.77 \text{ g/cm}^3 \\ k &= 4.88 \times 10^{11} \text{ dyn/cm}^2 \\ \mu &= 2.74 \times 10^{11} \text{ dyn/cm}^2 \\ P_{\text{max}} &= 1.86 \times 10^8 \text{ dyn/cm}^2\end{aligned}$$

This gives a compressional wave velocity of $\alpha = 5.55 \times 10^5$ cm/sec, a shear wave velocity $\beta = 3.145 \times 10^5$ cm/sec, and a Rayleigh-wave velocity of $C = 2.898 \times 10^5$ cm/sec.

To generate different Rayleigh-wave lengths we used values of the source-time duration given by

$$\tau_0 = 5 \text{ and } 10 \text{ } \mu\text{sec.}$$

The corresponding values of impulse delivered were

$$L = \begin{cases} 1.97 \times 10^4, & 3.94 \times 10^4 \text{ dyn-sec/cm} & \text{(two dimensions)} \\ 1.237 \times 10^5, & 2.475 \times 10^5 \text{ dyn/sec} & \text{(three dimensions).} \end{cases}$$

The spatial half-width of the pulses generated are on the order of 10 cm. We estimate that the size of mesh spacing needed to adequately resolve this is about 1 cm. The maximum number of zones allowed in TENSOR is theoretically unlimited; however, as a practical matter we are limited (on the CDC 7600) to about 40,000. Thus with uniform zoning we are limited to calculating the signal out to a range of ~200 cm and a similar depth. A further complication enters in that there is no really satisfactory non-reflecting boundary condition available; any signal reaching the edge of the mesh will be reflected and will travel back towards the source, thus interfering with the observation of a more slowly-moving outward disturbance. This problem is severe because the compression velocity is almost twice the Rayleigh velocity. The point at which the Rayleigh wave and the reflected compression wave meet is given by

$$X = \frac{2C\tau}{\alpha + C}$$

where l is the horizontal length of the mesh. We economized on zones by increasing the mesh width by 10% per zone when we got past this point. The mesh must also be deep enough so that the returning compression wave reflected from the bottom does not begin to interfere with the Rayleigh wave until it reaches the point X. Roughly, the mesh should be about half as deep as it is wide. It is possible to economize further by increasing the height of the cells by 10% per zone; however, we have not done this. Thus, although some variable zoning has been used, uniform 1-cm zoning is used in the regions where the Rayleigh wave propagates.

A comparison between the exact analytic solution for (1) the two dimensional case $\tau_0 = 5 \mu\text{sec}$, $L = 1.97 \times 10^4 \text{ dyn-sec/cm}$ with (2) the numerical results produced by TENSOR can be seen in Fig. 4. There, the surface vertical displacement is plotted

Fig. 4. Comparison of a TENSOR finite-difference solution with the analytic solution for the Rayleigh wave (two-dimensional case; $C_1 = 0.1$).

against the distance from the source. In this particular case the TENSOR calculation was run with a linear Q-coefficient, $C_1 = 0.1$. A similar set of comparisons for $\tau_0 = 10 \mu\text{sec}$, $L = 3.94 \times 10^4 \text{ dyn-sec/cm}$, $C_1 = 0.05$ is shown in Fig. 5. One can see that the TENSOR and analytic solutions are generally in quite good agreement.

Fig. 5. Comparison of a TENSOR finite-difference solution with the analytic solution for the Rayleigh wave (two-dimensional case; $C_1 = 0.05$).

Comparison of the two pairs of results also shows that doubling the Q -coefficient and reducing the wavelength increases the damping and smoothing of the waves. At $400 \mu\text{sec}$ these problems have just about reached the point where the returning compression wave is about to interfere with the Rayleigh wave. The problems under consideration were quite large in scope, having a total of about 25,000 zones and requiring about 1.5 hours of CDC 7600 time to compute. Although it is a bit difficult to ascribe a definite wavelength to these solutions due to the slowly decaying tail characteristic of two-dimensional spreading waves, one can see that the rapidly oscillating part of the disturbance is confined to a region about 20 cm in length. Comparison of the results in Figs. 4 and 5 show that when the wavelength is of the order of 15 or 20 mesh-widths, numerical dispersion and damping begin to affect the solution.

The effect of varying the linear Q -coefficient can be seen even more clearly in Fig. 6. This figure shows the vertical displacement plotted along the surface for two

Fig. 6. Dependence of the TENSOR finite-difference solution on the choice of linear Q -coefficient (time = 400 μ sec).

calculations which were identical except for a factor of ten difference in the Q -coefficient. From this comparison, it is evident that a constant of 0.1 is too large and causes excessive damping whereas a value of 0.01 is too small to damp out oscillations and overshoot. Reference to Fig. 5 indicates that a value of 0.05 is about right.

The slowly decaying tail-characteristic of the two-dimensional solution is not present in three-dimensional solutions, and a characteristic wavelength is better defined. In this case we have only an asymptotic analytic solution for comparison; however, the wave approaches the asymptotic form much more quickly. In Figs. 7 and 8, TENSOR-computed vertical displacements at the surface are compared with the analytic solution for the cases $\tau_0 = 5$ and 10 μ sec. Both runs have been made with the Q -coefficient equal to 0.05. The analytic solution is for the Rayleigh wave only; however, the TENSOR solution contains both the Rayleigh wave and the

Fig. 7. Comparison of a TENSOR finite-difference solution with Lamb's asymptotic solution for the Rayleigh wave (three-dimensional case; ~25 mesh points per wavelength).

Fig. 8. Comparison of a TENSOR finite-difference solution with Lamb's asymptotic solution for the Rayleigh wave (three-dimensional case; ~18 mesh points per wavelength).

shear wave. These two waves overlap at early times because the difference in velocities is so small. Nevertheless, the comparison between the analytic and TENSOR results is quite good for the longest wavelength case, $\tau_0 = 10 \mu\text{sec}$. Because the pressure is applied to a disk with a 4 cm radius, the wavelength cannot be made smaller by reducing τ_0 without first decreasing the size of the disk. In order to obtain a smaller wavelength for comparison, we ran one more case with the disk radius reduced to 3 cm. This case had the following parameters: $\tau_0 = 5 \mu\text{sec}$; $L = 6.963 \times 10^4 \text{ dyn-sec}$; $C_1 = 0.05$. The results of this calculation are plotted in Fig. 9. A comparison of these three-dimensional cases illustrates very clearly the degeneration of the numerically-calculated signal as the wavelength approaches the mesh size. The artificial damping and dispersion seen in the numerical solution are due to the truncation error in the finite-difference

Fig. 9. Comparison of a TENSOR finite-difference solution with Lamb's asymptotic solution for the Rayleigh wave (three-dimensional case; ~ 12 mesh points per wavelength).

approximation to the partial differential equations. The wavelength in Fig. 9 is about $1/2$ that in Fig. 7; in terms of mesh width, the wavelengths are approximately 12 and 25 zones wide. These plots suggest that if we expect to calculate a wave fairly accurately (about as well as in Fig. 7) we need to allow about 20 mesh points per wavelength.

TENSOR also contains a rotational Q-coefficient to damp out oscillating diagonal disturbances that sometimes develop in the mesh. Calculation of the same case (shown in Fig. 5) with and without the rotational damping produces answers differing by less than 1%, indicating that this Q-coefficient is not necessary for the calculation of surface waves.

Conclusions

1. TENSOR can accurately calculate surface wave phenomena. One should allow approximately 20 mesh points per wavelength, and the linear Q-coefficient should be ~ 0.05 . A rotational Q-coefficient is not necessary.
2. The three-dimensional Lamb-asymptotic Rayleigh-wave solution with appropriate convolution offers a very efficient and convenient means of studying the directional and interference effects produced by a horizontal distribution of sources with arbitrary time dependence.

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