AD/RHIC-AP-59

Definition of Emittance in Tracking Studies

H. Hahn

February 25, 1988

H. Hahn
?/25/88

DEFINITION OF EMITTANCE in TRACKING STUDIES

1. BEAM PARAMETERS GIVEN BY

   # ions / bunch N_B

   normalized emittance (full coupling)

   $$\epsilon_H = \epsilon_V = \epsilon_N$$

   Actual emittance

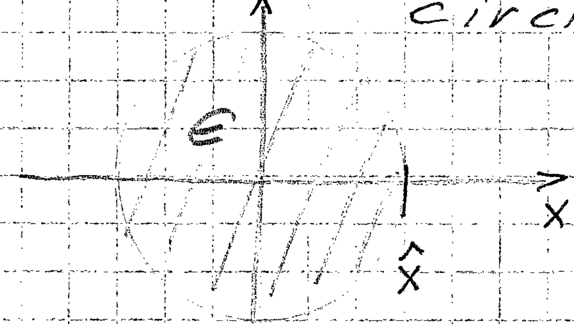
   $$\epsilon = \epsilon_N / (\beta\gamma)$$

2. MOTION of PARTICLES IN PHASE SPACE

   ellipse in (x, x')

   at QF/QD regular ellipse

   circle in $(x, \beta x')$



Emittance defines maximum ellipse

Beam size in (x, y) space:

$$\hat{x} = \sqrt{\epsilon \beta_H}$$

$$\hat{y} = \sqrt{\epsilon \beta_V}$$

3. TRUNCATED GAUSSIAN BEAMS in 2-dimensional phase space

$$N_B = \int_{\epsilon} \frac{N_2}{2\pi\epsilon_{rms}} \exp\left(\frac{-1}{2}\frac{x^2/\beta + \beta x'^2}{\epsilon_{rms}}\right) dx\, dx'$$

N_B = # ions within emittance

N_2 = # ions in infinite phaseplane

Integration over

$$x^2/\beta + \beta x'^2 \leq \epsilon$$

$$\frac{N_B}{N_2} = 1 - \exp\left(-\frac{1}{2}\frac{\epsilon}{\epsilon_{rms}}\right)$$

BNL - CONVENTION

$$\frac{N_B}{N_2} = 95\% \implies \epsilon_{rms} = \frac{\epsilon}{6}$$

at CERN

$$\epsilon = 4\epsilon_{rms} \implies \frac{N_B}{N_{2CERN}} = 86\%$$

<u>HOWEVER</u>: 100% of beam within beam size by definition

4. One-dimensional Gaussian beam

The non-truncated beam has line density

$$\frac{N_1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{1}{2}\frac{x^2}{\sigma^2}\right)$$

with $\sigma = \sqrt{\epsilon_{rms}\,\beta}$

Limiting the beam size to $x \leq \hat{x}$ (scraper) reduces beam to

$$\int_{-\hat{x}}^{\hat{x}} \frac{N_1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{1}{2}\frac{x^2}{\sigma^2}\right) dx$$

$$= 2\,P\left(\frac{\hat{x}}{\sigma}\right) - 1$$

$$= 99\,\% \quad \text{if} \quad \hat{x} = \sqrt{6}\,\sigma$$

$(\neq 95\%)$

<u>Conclusion:</u>

- Line density of truncated Gaussian beam is <u>not</u> Gaussian

- In practice, measurement of linear density <u>distribution</u> yields σ, which is basis for aperture requirement (6σ-rule)

5. 4 Dimensional Gaussian Beams

H/V uncoupled

$$N_B = N_4 \int_{\epsilon_H} \frac{1}{2\pi\,\epsilon_{Hrms}} \exp(\quad)\, dx\, dx'$$

$$\times \int_{\epsilon_V} \frac{1}{2\pi\,\epsilon_{Vrms}} \exp\left(-\frac{1}{2}\,\frac{y^2/\beta_V + \beta_V y'^2}{\epsilon_{Vrms}}\right) dy\, dy'$$

Truncation (i.e. scraping) imposes limits:

$x \leq \hat{x} \rightarrow x^2/\beta_H + \beta_H x'^2 \leq \epsilon_H$

$y \leq \hat{y} \rightarrow y^2/\beta_V + \beta_V y'^2 \leq \epsilon_V$

separate constraints ("intersecting cylinders")

$$N_B = N_4 \left[1 - \exp\left(-\frac{1}{2}\frac{\epsilon_H}{\epsilon_{Hrms}}\right)\right]\left[1 - \exp\left(-\frac{1}{2}\frac{\epsilon_V}{\epsilon_{Vrms}}\right)\right]$$

Conclusion:

2-dimensional treatment is based on truncation in <u>one</u> plane only with <u>uncoupled</u> motion

(then $N_4 = N_2$)

6. Gaussian Beams in 4-dim phase space fully coupled ($\epsilon_H = \epsilon_V = \epsilon$)

$$N_B = \int_{\epsilon_T} \frac{N_4}{(2\pi\,\epsilon_{rms})^2} \exp\left(-\frac{1}{2}\,\frac{x^2/\beta_H + \beta_H x'^2 + y^2/\beta_V + \beta_V y'^2}{\epsilon_{rms}}\right) dx\,dx'\,dy\,dy'$$

Integration over 4 dim ellipsoid defined by beam size:

$$\epsilon_T = x^2/\beta_H + \beta_H x'^2 + y^2/\beta_V + \beta_V y'^2 \leq \hat{\epsilon}_T$$

Determine $\hat{\epsilon}_T$:

$x = \hat{x}$ if $x'^2 = y = y' = 0$

$$\hat{\epsilon}_T = \frac{\hat{x}^2}{\beta_H} \equiv \epsilon$$

$$N_B = N_4\left[1 - \left(1 + \frac{\epsilon}{2\epsilon_{rms}}\right)\exp\left(-\frac{\epsilon}{2\epsilon_{rms}}\right)\right]$$

$= 80\%$ of N_4

Conclusion:

Scraper in one-direction at $x = \hat{x} = \sqrt{6}\,\sigma$ reduces beam by 20% due to coupling!

HOWEVER: 100% of BEAM is within beam size by definition

7. Dynamic APERTURE REQUIREMENTS

- The beamsize grows due to intra beam scattering to

$$\epsilon_N = 30\pi \text{ mm}\cdot\text{mrad} \quad \text{at } \gamma = 30$$

$$\Rightarrow \epsilon = 1\pi \text{ mm}\cdot\text{mrad}$$

- Requirements for dynamic aperture are based on
  "6σ-rule"

- Dynamic stability in tracking studies for initial conditions in 4D-space

$$\epsilon_T = x^2/\beta_H + \beta_H x'^2 + y^2/\beta_V + \beta_V y'^2 \leq 6\epsilon$$

$$\leq 6\pi \text{ mm}\cdot\text{mrad}$$

Usually in tracking studies $x' = y' = 0$

$$\epsilon_T = x^2/\beta_H + y^2/\beta_V \leq 6\epsilon = 6\pi \text{ mm}\cdot\text{mrad}$$

CONCLUSION (coupled beam):

- Intrabeam scattering, space charge determined by maximum density in core of beam
  i.e. $N_4 = N_B/0.8$

- aperture requirement determined by
  $\epsilon_T = 6\epsilon$