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FUNDAMENTAL INVESTIGATION OF LIQUID-METAL LUBRICATED JOURNAL BEARINGS

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Quarterly Technical Report No. 1
Contract No. AT(11-1)-1228
Project Agreement No. 8
Modification No. 7

R. A. Burton
H. J. Carper

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Report No. RS-418

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SOUTHWEST RESEARCH INSTITUTE
8500 Culebra Road, San Antonio, Texas 78206

Department of Aerospace Propulsion Research

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Washington 25, D. C.

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APPROVED:

P. M. Ku, Director
Department of Aerospace
Propulsion Research

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FOREWORD

This report was prepared by Southwest Research Institute under Contract No. AT(11-1)-1228, Project Agreement No. 8, Modification No. 7. The work was administered by the Engineering Development Branch, Division of Reactor Development, U.S. Atomic Energy Commission, Washington 25, D. C. with Mr. Nicholas Grossman as Project Engineer.

This report covers work performed from April 1 through June 30, 1964.

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OBJECTIVE

The objective of this program is to investigate the fundamentals of liquid-metal lubricated hydrodynamic bearings, leading to the development of design criteria and performance estimates. This involves theoretical computations, and appropriate experimentation to provide basic data needed to guide these computations. Primary effort is to be devoted to: (1) turbulent film phenomena, (2) inertia effects, and, insofar as possible, (3) surface tension effects.

INTRODUCTION

The need for this work arises from the fact that liquid-metal lubricated bearings of high reliability are required for advanced nuclear applications. Not only are such bearings expected to operate in regimes of speed and temperature which are unfamiliar, but the working fluids are known to be unforgiving of design errors, when compared with typical oils which have inherent lubricity. For these reasons, it has been judged advisable to undertake fundamental investigations where the goals would be: (1) to make best use of existing theories, (2) to define clearly problem areas where existing theories cannot be extended, (3) to provide new approaches and data as needed, and (4) to develop design procedures and analytical predictions of performance. Because some of the problem areas are known in advance, and because these can be dealt with by relatively straightforward experimentation, the initial effort also involves a program of basic measurements, as well as theoretical analysis.

TECHNICAL DISCUSSION

Background of Problem

(1) Turbulent Film Phenomena. Many liquid-metal bearing applications are expected to call for turbulent film operation. There is considerable divergence as to the effects predicted in the several papers available on this subject. These differences are not due to mathematical errors, but may be attributed to approximations introduced into the equations. Careful critical study of this aspect of the derivations suggests that these approaches may be improved.

(2) Inertial Effects. The same factors which increase the likelihood of turbulence also increase the likelihood of simple inertial or accelerational effects. Three kinds of these are anticipated, one being overall accelerations where the velocity is changing from point to point in the film [i.e., where $(\rho u \partial u / \partial x)$ is significant]. This can directly affect the pressure distribution. The second effect is concerned with changes in velocity profile where boundary configuration changes rapidly. For example, at the entrance to a slider or near a step, this can influence pressure distribution. Thirdly, it is entirely possible that inertial effects can give rise to separation and standing vortices in regions of large negative pressure gradient. Only analysis can show whether conditions would give rise to some likelihood of this effect.

(3) Surface Tension Effects. It has been clearly shown experimentally that changes of surface tension can change bearing friction, and possibly other factors. This can be explained theoretically by noting that surface tension changes the boundary condition for cavity formation under other than "full film" operating conditions. There is no theoretical work of value on this effect, though there appears to be a sufficiently broad understanding of the basic physics to permit a theoretical approach to be made.

(4) Shaft Whip and Whirl. The problem of predicting bearing instability is not restricted to liquid metals. A copious literature exists on this subject, and it is felt that the analytical basis for such analysis has been developed sufficiently for this purpose. The chief task is to exploit existing theory more fully, by providing better estimates of the force-displacement relations in the bearing. This requires a more complete understanding of the factors listed above. Shaft stability will not be studied directly in this program; but, as noted, the work would lead to improved information to deal with this, separately.

In the paragraphs which follow, pertinent aspects of the problems will be discussed. These discussions are intended to show how the areas for continued investigation relate to previous work. They further provide perspective as to the nature of the theoretical work to be undertaken.

Turbulent Film Phenomena

In typical hydrodynamic bearing operation, the fluid movement in the lubricant film is laminar. However, when fluids of low viscosity and high density are used in high speed bearings of relatively large clearance, the fluid movement departs from the simple laminar pattern and may become turbulent. This transition results in changes of pressure distribution, bearing power loss, eccentricity, and other factors. Through these effects

turbulence undoubtedly influences vibratory stability of the shaft-bearing system also.

The initial onset of this transition is not the random, disorderly movement characteristic of so-called "fully-developed" turbulence; it is, rather, an orderly formation of cellular vortices. If bearings are operated at Reynolds numbers considerably above this transition, a second change occurs, and the movement of the fluid becomes analogous to that found in turbulent pipe-flow. This latter type of flow has been dealt with extensively in the literature and can be related to the bearing problem.

In preparation for this work, a paper has been written which summarizes much of the problem of turbulent film behavior. This is attached as Appendix A, since it has not yet appeared in print.

Present work is going into using the methods discussed therein, in the analysis of simple configurations, with and without end leakage. Analyses are also being developed for vortex flows, when a pressure flow acts along or across the primary shear flow. These are to be reported in the second quarterly progress report where they will form the primary subject of discussion

Inertia Effects

It has been shown by several writers^(1, 2, 3) that inertial effects in lubricant films can have significant influence at high Reynolds number. For example, at a Reynolds number roughly double the critical, about 10% increase in load support would be expected. More important, the variation of attitude with load seems to be affected.

Analyses have been based upon averaging inertial terms across the stream, and these appear to be adequate for their purpose as long as inertia effects are not excessive. More exact approaches are available, however, where more effort is justified. Kahlert, for example, has provided an iterative method⁽²⁾ where increased accuracy is provided by improved repetitions of the calculation. Milne has provided what appears to be an improved method⁽³⁾; however, most of the improvements appear to be of academic value only. His method offers to provide no better answers than Kahlert's for the thin film cases dealt with in lubrication studies.

Inertial effects are taken into account in the Reynolds equation of thin films in the following way

$$\rho \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \frac{\partial u}{\partial t} \right) = \frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2}$$

The first two terms on the left side of the equation are of importance even in steadily operating bearings, and it is these which have been accounted for in the above-mentioned derivations. The third term appears in unsteady operation, such as would be encountered in shaft whirl. This effect has not been dealt with to any great degree in the literature, though it is mentioned incidentally in many derivations.

One effect of inertia which has not been studied in detail is its effect on variation of the velocity profile in a lubricant film. Schlichting has shown that velocity profiles with small (or zero) velocity gradient near the walls can more easily develop turbulence than do, say, parabolic profiles⁽⁴⁾. In extreme cases, where there is a reversal of flow near the wall, a state of transitory stall may develop in ducts, and possibly in lubricant films. Such zones would be expected in regions where pressure decreases in the direction of flow, and the resulting stall would be analogous to turbulence. Though such a state can exist in suitable dimensions, it is difficult to visualize its occurring in typical bearing film dimensions. It would be more reasonable to expect the film to become turbulent, according to already known criteria, before inertia is sufficient to lead to separation. Furthermore, even though a tendency toward separation could change the stability of laminar bearing flow when operating below, but near, the usual turbulent transition, this cannot be dealt with theoretically at the present state of the art.

Returning to the analysis of simple inertial effects, the analyses cited have dealt with inertial effects in laminar flows. To proceed with corresponding accuracy in turbulent flows one need not copy these procedures identically. It is entirely possible that only a crude average inertia effect need be dealt with.

One approach may be to simplify the term $\left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right)$ to $\left(\frac{V}{2} \frac{\partial u}{\partial x} \right)$,

where V is a selected velocity typical of the flow. Burton⁽⁵⁾ has shown in laminar boundary layer calculations that this accounts for inertial effects surprisingly well, and even predicts the profile changes leading to separation. This approximation is amenable to use with turbulent flow as well as laminar, though the derivation cited was restricted to the laminar boundary layer.

As to the term $(\partial u / \partial t)$, simplified calculations should show whether or not there are feasible lubrication situations where it may be of importance. If so, analytical methods can undoubtedly be devised in terms of the situation itself.

In summary we note that existing papers on inertia effects in laminar flows appear to deal with these adequately, both as to type and magnitude of

effect. Some of the cruder methods used can be adapted to turbulent flows. It is not felt that much is lost in these "cruder" estimates since they appear to predict both direction and general magnitude of effects in line with more exact predictions. The necessity for refinement here is not so critical as with the turbulence problem where totally incorrect predictions may result from oversimplification. Prediction of inertial effects upon velocity profile in transition films (and other partially turbulent films) does not appear to offer useful information at the present state of understanding of turbulent transition. The inertial term which appears only in unsteady flows has not been dealt with adequately in the literature. Estimations of this effect in cases of shaft whirl are called for.

/ Surface Tension Effects

Surface tension for liquid metals run roughly 10 - 20 times typical values for conventional lubricants. Consequently if surface tension effects are to be expected at all in lubrication, these would be amplified in the case of metals. Campbell⁽⁶⁾ has shown that hydrodynamic bearing friction can be changed by a sizeable amount when a nonwetting monolayer is placed on the journal of a conventionally lubricated bearing. This is not due to slippage of the lubricant relative to the surface. A summary of studies of wall slip is given by Goldstein⁽⁷⁾, and in no case has a significant slippage been detected. Even so, writers continue to hypothesize slip in the case of metals running on nonwetting surfaces⁽⁸⁾.

Since the effect of changing wettability is measureable, and, since evidence is available to indicate that this is not due to slip, another explanation must be sought. This may be found in the end condition of the film, where surface energy must be altered as the film divides, or exposes solid surfaces.

The end condition for films has been dealt with wholly from internal considerations in the literature. Saibel has generalized earlier derivations to a "least energy" condition⁽⁹⁾; however, it would not be correct to attempt to carry over this concept to include surface energy. Saibel's least-energy condition is simply a formalized relation which results from the assumption of concurrence of zero pressure gradient and zero pressure at film ends.

Photographic studies⁽¹⁰⁾ have been made of so-called cavitation conditions in films. "Cavitation" so used should not be confused with the unstable, impulsive cavitation which is associated with surface damage; rather, it is the formation of relatively stable filaments, joining continuously with the end of the unbroken film in many cases. Floberg⁽¹¹⁾ has shown how forces on the filaments can be calculated. For example, he

has shown how shear effects can be accounted for in this region. On the other hand, he has not taken into account surface energies. This factor should not detract from the value of his work, since surface energy would be expected to act mainly at the initial formation of the filaments. Since relative surface areas are not changing rapidly once filaments are formed, surface energy effects would not be expected to be of major importance along the length of the filaments.

In typical sketches⁽¹¹⁾ showing the formation of filaments, the air spaces between are shown to have ogival form with infinite curvature at the tip (initiation of the cavity). Since the pressure beneath a surface is directly determined by the curvature, negatively infinite pressure would be called for at the ogive's tip. This condition is untenable, and must be modified by more detailed consideration of the dynamics in this region. No major obstacle is foreseen in attacking this problem. Apparently the only reason it has been delayed is that its importance has not been realized.

In fact, in many derivations of bearing properties, it is assumed that the film is continuous around the bearing; or else, some arbitrary angle of continuous fluid film is chosen. For example, many derivations treat regions of negative pressure as "regions of air." Others take account of the shear in such regions but do not account for the pressures. Though such assumptions are coarse, they have given, in many cases, sufficiently good estimates of bearing performance for design purposes. This is especially true since bearing design criteria are far from exact, and an uncertainty would be present even if exact film calculations could be made. Furthermore, in a pressurized, lightly loaded, circumferentially fed bearing, breakup of the film need not take place, since low pressures need never exist. Also, in the case of a bearing with known load direction, proper location of oil feed grooves will avoid cavitation.

Summarizing: Irrespective of ways in which various investigators have gotten around the problem of accounting for surface tension effects, Campbell's measurements showed that a 25% change in bearing friction could result, when operating in the hydrodynamic regime and using oils for lubricants. Surface energy (wetting energy) variations with liquid metals could exceed by 20-fold the variations in his experiments. In view of this, surface energy effects could be of significance in liquid-metal lubrication.

Theoretical investigations should be made on surface energy effects influencing the end conditions of lubricant films, using already existing formulations of free-surface curvature and its relation to pressure, drawing upon a steady flow energy equation, and the Navier-Stokes equations. Whatever approximations are required for solution must be made ad hoc, as the derivations develop.

EXPERIMENTAL PROGRAM

Description of Test Apparatus

The completed test apparatus consists of a drum (the "journal") supported on a shaft and rotating inside of a cylinder, (the "bearing") as shown in Figure 1. Measurements of velocities and turbulence quantities will be made using a hot-wire anemometer, with the anemometer probe extending into the journal-bearing clearance as shown in Figure 1. The probe is mounted in a micrometer transversing mechanism which will allow radial as well as axial movement of the probe.

As shown in Figure 1, the bearing is mounted on a sled which can be moved back and forth in the horizontal plane allowing journal eccentricity to be varied since the journal's center of rotation remains stationary. The bearing can also be rotated around the journal on the ball bearing supports; hence the probe can be set to various attitude angles.

A photograph of the journal is shown in Figure 2. The journal is 20 in. wide and is machined to a diameter of 68.250 in., with an out-of-roundness of less than 0.010 in. The drive motor shown in the figure is a 1/2 HP DC motor, with continuously variable speed, and coupled to the journal through a gear reduction unit by a ratchet-type overrunning clutch. Journal speeds of from 0 to over 100 rpm can be obtained with this arrangement. The journal has been equipped with a brake, shown in Figure 3, to facilitate more rapid deceleration of the rather high inertia load.

The bearing (fabricated of steel plate and angle) and the sled mount are shown in Figure 4. The bearing is 20 in. wide and the inside diameter is machined to 69.310 in. with an out-of-roundness of not greater than 0.015 in. The resulting journal-bearing radial clearance is 0.530 in. or a clearance ratio C_r of 0.0155.

The surface finishes of both the journal and the bearing were measured and the centerline average roughnesses are as follows

Journal

Axial traverse	190 μ in
Circumferential traverse	90 μ in.

Bearing

Axial traverse	42 μ in.
Circumferential traverse	20 μ in.

92505 K

FIGURE 1. SCHEMATIC DIAGRAM OF TEST APPARATUS

FIGURE 2. JOURNAL WITH DRIVE MECHANISM

FIGURE 3. JOURNAL WITH BRAKE

FIGURE 4. BEARING WITH SLED MOUNT

In order to compare the roughness height with the thickness of the laminar sublayer, the sublayer thickness was computed using data from Couette⁽¹²⁾ and Emmons⁽¹³⁾ and using the equation

$$u^* = \frac{u_c}{\sqrt{\frac{2}{c_f}}} \quad (1)$$

where

u^* = friction velocity

u_c = velocity at centerline of channel

c_f = friction coefficient

From Couette⁽¹²⁾, for a Reynolds number of 1000, (expected to be the maximum used in the experiments), $c_f = 0.008$. From Emmons⁽¹³⁾, at the boundary of the laminar sublayer $yu^*/\nu = 10$, where y in this case is the thickness of the laminar sublayer. From this relationship and Eq (1)

$$y = \frac{10\nu\sqrt{\frac{2}{c_f}}}{u_c} \quad (2)$$

For a Reynolds of 1000, $u_c \approx 7$ ft/sec. Then substituting the above values into Eq (3)

$$y \approx 0.043 \text{ in.}$$

This shows that the maximum roughness height is less than 0.5% of the minimum thickness of the laminar sublayer and therefore the roughness can be considered to have a negligible effect on the flow.

Velocity and Turbulence Measurements

It is planned to obtain velocity profiles as a function of Reynolds number, axial and angular position, and journal eccentricity in the laminar, vortex, and fully-turbulent regimes for full and sector bearings. According to the data reported by Taylor⁽¹⁴⁾, a Reynolds number range of from 20 to 1000, based on half-film thickness and the mean flow velocity, should include these three flow regimes.

Measurements of turbulence level $\sqrt{u'^2}$ will be made, particularly in the transition regions, and efforts will be made to "trace" specific vortex filaments in vortex flow.

Initial velocity and turbulence measurements will be made with the journal concentric and later with various journal eccentricities with a smooth bearing surface and a radial bearing clearance of 0.530 in. Subsequent velocity and turbulence measurements will be made with altered clearances and artificial roughness introduced on the bearing and/or journal surfaces.

Pressure Measurement

Static Pressure. For an estimate of the minimum static pressure which might be expected in the film the following equation from Fuller⁽¹⁵⁾ for a full journal bearing was used

$$p = \frac{\mu w}{m^2} \times k_p \quad (3)$$

where

μ = absolute viscosity of fluid

w = speed of rotation

$m = \frac{c}{r} = \frac{\text{radial clearance}}{\text{journal radius}}$

$$k_p = 6 \left[\int \frac{k^* d\theta}{(1 - \epsilon \cos \theta)^3} - \int \frac{d\theta}{(1 - \epsilon \cos \theta)^2} \right]$$

where

$k^* = \frac{h^*}{mr} = \frac{h^*}{c}$ (h^* = film thickness at point of maximum pressure)

ϵ = eccentricity ratio

θ = angle from point of minimum film thickness

The following values were used in the calculation:

$\mu = 0.375 \times 10^{-7}$ (absolute viscosity of air at 60°F and 14.696 psia)

$w = 0.1 \frac{\text{rad}}{\text{sec}}$ (speed of rotation corresponding to a Reynolds number of 20 for laminar flow)

$$m = 0.015$$

$$k_p = 0.098 \text{ (minimum value of } k_p \text{ taken from Table 24, page 196, ref. 15, where } \epsilon = 0.4 \text{ and } \theta = 5^\circ \text{)}$$

Substituting these values into Eq (3)

$$p = 1.155 \times 10^{-6} \text{ psig}$$

Thus, the pressure to be measured is exceedingly low.

There are no known instruments available at the present time which will accurately resolve a pressure difference of this order of magnitude. Consequently some effort will be put forth on the development of a method for measuring this pressure using the hot-wire anemometer as the sensing device.

Velocity Pressure. For a Reynolds number of 20, the velocity pressure, $1/2 \rho u^2$ will be of the order of 10^{-7} psi, while for a Reynolds number of 1000, the velocity pressure will be of the order of 10^{-3} psi.

FUTURE PROGRAM

Preliminary velocity profile measurements will be made during the forthcoming period.

Preliminary calculations of turbulent-film, sector, and full bearings will be carried out, taking account of inertia effects. The theoretical bases for computation of combined shear and pressure flows in the vortex-flow regime will be developed. Very preliminary studies of the back-ground theory of surface-tension dynamics will be carried out.

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APPENDIX

TURBULENT FILM BEARINGS UNDER SMALL DISPLACEMENTS*

by

R. A. Burton

ABSTRACT

The experimental and analytical literature is reviewed briefly, and is drawn upon to provide solutions for the interactions of weak pressure flows with a strong Couette flow, as would be experienced in turbulent film bearings. In the pertinent derivations, care has been taken to avoid known pitfalls in the application of mixing-length theory. Insofar as is possible, the results are shown to be in accord with reported experimental measurements. The areas of uncertainty in these analyses are areas where further experimental measurements are called for.

INTRODUCTION

There has been a rapidly growing literature on the measured and analytically predicted behavior of journal bearings operating with superlaminar lubricant films. Due to inherent complexities in the fluid mechanics of such bearings, discrepancies and divergences are apparent among the different investigations. The study reported here has been undertaken as a step toward the assessment of the experimental findings, along with their implications regarding the analytical prediction of turbulent-film bearing behavior. In addition, the principal analytical approaches, themselves, have been subjected to scrutiny. This has led to an evaluation of the consequences of some simplifying assumptions, and has suggested alternative solution techniques which avoid some of the more serious pitfalls. As will be shown, the special case of bearings with very small displacements (very weak pressure flow) lends itself to analysis by existing means with a minimum of restrictive assumptions, and the solution for such a case provides valuable insights into the leakage resistance, load support and friction interrelationships for turbulent film bearings as compared with their laminar equivalents.

*Presented at ASLE Annual Meeting, Chicago, May 27, 1964.

DESCRIPTION OF COMPONENT FLOWS

In its most direct form, the problem dealt with here is that of predicting the interaction between a Couette flow and a pressure flow. By "Couette flow" is meant the simple shear flow such as exists in the annulus between two concentric cylinders, when one is turned and the other is held stationary. By "pressure flow" is meant a pressure-induced fluid motion such as would take place between stationary walls. For the special case where the pressure and Couette flows cross at right angles, this might be thought of as an idealization of the practical problem of end-leakage through a thin annulus around a turning shaft. Other combinations of flow may represent the fluid movement in an actual bearing where the shaft may be displaced from its central position, and pressure may vary both axially and circumferentially.

For laminar motion in bearings, linear superposition is possible with all combinations of such flows. For example, axial leakage is independent of the shaft speed for laminar flow, but it is clearly dependent upon this factor for turbulent flow. This requires that, in turbulent-film bearings, the flows must be dealt with from the outset as combinations; or else account must be taken of some kind of "coupling" effect when superposition is attempted. In some cases the coupling effect would exceed the individual contribution from a flow component.

In any attempt to predict bearing behavior, it is necessary to relate the bearing clearance variation and surface velocities to the pressure variation. This kind of solution has been produced by several investigators, for both slider and journal bearings. Ultimately, the accuracy of such solutions must depend upon how well they account for the stresses on a given point of bearing surface, in terms of the Couette flow and the codirectional and orthogonal pressure flows which exist in the film over the surface point. Once a correct and workable statement of this relationship can be written, an accurate overall solution can follow directly (though not necessarily effortlessly) for simple bearings.

A mathematical idealization of the key problem might be to obtain an accurate solution for a unidirectional pressure flow passing between two infinite and parallel plates, where these plates experience tangential linear sliding in some arbitrary direction relative to the pressure flow. In the real bearing, questions may be asked about: (a) the special effects of accelerations of the fluid and of the plates, (b) the effects of bearing curvature in producing secondary flow, and (c) the effects of departures of surface geometry from the ideal. It is true that all of these must ultimately be taken into account. However, this should not obscure the fact that the idealized problem is, itself, only imperfectly understood at present.

EXPERIMENTAL BACKGROUND

In studies of viscosity of fluids between rotating cylinders, Couette⁽¹⁾ showed the existence of a transition from simple viscous flow in fluid films, and his data have permitted subsequent computations of the resistance of a turbulent fluid to shearing. Taylor⁽²⁾ carried out both theoretical and experimental studies of a similar nature, and showed the influence of the relative film thickness ($2b/r$) on transition and resistance to turning. Though Taylor's film thicknesses were greater than would be expected in bearings, his data nevertheless set the problem into perspective. His measurements show agreement with Wendt⁽³⁾ as well as with Couette's original work (which serves as a thin-film limiting case for Taylor's data). More recently, additional confirmation has been provided by Donnelly⁽⁴⁾, who has assembled a body of data from the literature on transition and shearing resistance, in an effort to validate a theoretical equation derived by Stuart⁽⁵⁾ for resistance to shearing just above the transition. They are also supported by an almost independent wealth of data assembled by Bjorklund and Kays⁽⁶⁾.

Figure 1 summarizes the variation of the transition Reynolds number for the outer cylinder (bearing) turning, and for the inner cylinder (journal) turning. Note that for bearing-turning, stability improves with increased b/r , whereas for journal-turning stability gets worse with increased b/r .

In very thin films ($b/r \ll 1$) the distinction between "outer" and "inner" would disappear. The two variations would converge together at the left of the figure, and the critical Reynolds number should become independent of b/r . Following this line of reasoning and taking note of the fact that the bearing-turning curve has almost leveled out within the range of experimental data available, Taylor suggested that a projection of either "magnitude" or "slope" of his data (broken lines) might correspond to the transition for very thin films. He proposed that the curve for the journal-turning would not continue across the appropriate broken line but would bend and follow it toward $b/r \rightarrow 0$. More recently Reichardt⁽⁷⁾ has measured transition in a parallel flow and found the transition Reynolds number to be 750, somewhat above Taylor's predictions.

As to the physical behavior of a fluid above the observed transitions, when $b/r \ll 1$ the turbulent flow is disordered, much like turbulent flow in straight pipes. However, when the inner cylinder (journal) is turning and b/r is sufficiently large to precipitate early transition, the structure of the flow is initially "cellular" and orderly, and may be termed "vortex flow." If Reynolds number is further increased, the "cellular" structure ultimately gives way to disorder.

FIGURE 1. TRANSITION LOCI FOR COUETTE FLOW BETWEEN CONCENTRIC CYLINDERS

FIGURE 2. UNIVERSAL VELOCITY PROFILE FOR TURBULENT SHEAR-FLOW NEAR WALL, SHOWING PRESSURE FLOW DATA AND POWER-LAW APPROXIMATION

For the case of journal-turning, one would interpret Fig. 1 as calling for an initial transition to fully turbulent flow, only for clearance ratios below, say, 0.001 (i.e., where the "inner cylinder turning" line crosses the appropriate broken line representing "fully turbulent" transition.) Since larger clearances are of interest for high-speed bearings, it is therefore necessary to give consideration both to fully turbulent flow, and to vortex flow.

Fully Turbulent, Plane Couette and Pressure Flows

The limiting case of flow between parallel plates may be termed "plane" flow. When the fluid is turbulent in plane flow, there is a close resemblance between the velocity profiles for pressure flow and Couette flow. The chief difference between these is that the pressure-flow profile is symmetrical about midstream while the Couette-flow profile is reversed on crossing this line. But for either, when $y < b$, it has been shown experimentally that the quantitative relationship between profiles is closer than might be expected⁽⁸⁾.

Of the great many equations for velocity profile and friction coefficient for pressure flow, the simplest are the power laws⁽⁹⁾. To a fair approximation, the "universal velocity profile" may be represented by

$$\frac{u}{u^*} = (8.7) \left(\frac{yu^*}{\nu} \right)^{1/7} \quad (1)$$

where u is the local velocity parallel to a fixed wall, u_c is midstream velocity, y is the coordinate measured normal to the wall, and b is the centerline value of y . The friction velocity $u^* \equiv u_c / \sqrt{2/C_f}$, and the friction coefficient $C_f \equiv 2\tau / \rho u_c^2$, where ρ is fluid density, and τ is the wall shear stress. This may be simplified to give the velocity profile

$$\frac{u}{u_c} = \left(\frac{y}{b} \right)^{1/7} \quad (2)$$

Also for $u = u_c$, Eq (1) reduces to the friction law

$$C_f = 0.0457/R^{1/4} \quad (3)$$

Figure 2 illustrates the agreement between Eq (1) and pressure-flow data. In Fig. 3 the special power-law profile (Eq [2]) is compared with experimental profiles for plane Couette flow as measured independently by Robertson⁽¹⁰⁾ and Reichardt⁽⁷⁾. The experimental profile departs

FIGURE 3. TURBULENT VELOCITY PROFILES FOR PLANE COUETTE FLOW COMPARED WITH POWER-LAW APPROXIMATION

FIGURE 4. PRANDTL-BLASIUS POWER LAW COMPARED WITH EXPERIMENTAL DATA FOR COUETTE FLOW

from the power law near the wall, becoming almost linear. When the slope of the laminar sublayer is computed for Reynolds number of 10^4 (using $M[\partial u/\partial y] = C_f \rho u_c^2/2$) it is found to be such that

$$\frac{b}{u_c} \left(\frac{\partial u}{\partial y} \right) = 23 \quad (4)$$

Thus, it accounts for the flattened region and intersects the power law curve very near to the point of departure of the data (see broken line).

Figure 4 shows the degree of fit of Eq (3) with friction coefficient data given by Robertson and by Couette (as plotted by Taylor and replotted by Robertson). The agreement between Couette- and pressure-flow data and the power-law curves not only illustrates that there is a close relationship between the two types of flow, but also that the power laws can serve as very good "empirical" equations for plane flows. It also illustrates the basic consistency between the universal velocity profile and the available experimental data. That this is not always the case is illustrated in Table 1, which permits comparison of a number of theoretical and empirical profiles. Robertson has also offered other comparisons which further illustrate the range of differences.

TABLE 1. DISPLACEMENT THICKNESS FOR TURBULENT PLANE-COULETTE FLOW

Profile	δ^*/b
Experimental ^(7, 10)	0.124
von Karman ⁽¹⁰⁾	0.128
Power law ⁽⁸⁾	0.125
Reichardt (derived) ⁽⁷⁾	0.121
Constantinescu ⁽²⁸⁾	0.15
Robertson (derived) ⁽¹⁰⁾	0.18
Pai ⁽²¹⁾	0.36
Chou and Saibel ⁽²⁷⁾	0.3
Laminar	0.50

Here δ^* is the displacement thickness, defined as

$$\delta^* = \int_0^b (1 - u/u_c) dy \quad (5)$$

The values shown in the table were planimetered from published profiles as well as was possible, and are representative of $R = 10^4$. Despite inherent errors in this procedure, the values given are believed to reflect the true deviation of each from experiment.

Codirectional, Plane, Pressure and Couette Flows. The fact that a single wall-friction law will encompass both pressure flow and Couette flow acting singly, raises the question as to why it should not also apply to codirectional, combined Couette and pressure flows. Robertson⁽¹⁰⁾ has reported data for such a case, where a pressure flow reduced the midplane velocity u_c from $0.5 V$ (for Couette flow alone) to approximately $0.3 V$, relative to the wall from which the velocity measurements were made. Table 2 illustrates the results of directly comparable measurements of the two types.

TABLE 2. COMPARISON OF COUETTE FLOW WITH COMBINED COUETTE AND PRESSURE FLOW

u_c/V	R	$\left(\frac{b}{u} \frac{\partial u}{\partial y}\right)_c$	C_f	δ^*
0.29	0.675×10^4	0.217	0.0048	0.145
0.279	0.89	0.181	0.0048	0.129
<u>0.290</u>	<u>1.106</u>	<u>0.155</u>	<u>0.0047</u>	<u>0.0987</u>
0.292	0.8903	0.184	0.0048	0.124
0.502	0.82	0.148	0.00485	0.122
0.497	0.998	0.209	0.00467	0.137
<u>0.497</u>	<u>1.18</u>	<u>0.180</u>	<u>0.00450</u>	<u>0.117</u>
0.499	0.999	0.179	0.00467	0.125

The three runs from each class of flow were selected on the basis of having nearly corresponding values of Reynolds number. Comparing the averages for the two types of flow, it is seen that the dimensionless mid-stream slope $(b/u \times \partial u / \partial y)_c$ was increased slightly by the pressure flow. Friction coefficient C_f was also increased slightly (but less than 4 percent), and the displacement thickness δ^* was not changed significantly. These numbers will be referred to further in terms of derived predictions.

Taylor Vortex Flow

The friction variation in the bearing used by Smith and Fuller⁽¹¹⁾ is very close to that for concentric, long cylinders as reported by Taylor and others, for shaft-turning and large clearance ratio. This has been plotted as a broken line in Fig. 4, showing a steeper slope than is characteristic of plane Couette flow, and following the power law:

$$C_f = \frac{0.172}{R^{0.43}} \quad (6)$$

Bjorklund and Kays⁽⁶⁾ have written a more general relationship which follows the data from a number of sources^(2, 12, 13, 14), and takes account of the Reynolds criterion for transition in the case of shaft-turning

$$C_f = 0.4695 \left(\frac{b}{r} \right)^{0.261} / R^{0.478} \quad (7)$$

For the clearance ratio used by Smith and Fuller ($b/r = 0.00146$), this would predict a somewhat lower value than Eq (6) would call for. This difference may be due to the shortness of the bearing used by Smith and Fuller, since a length effect has been indicated by Sneek and Fox⁽¹⁵⁾. It is of interest also that Batchelor⁽¹⁶⁾ has provided a theoretical justification for vortex flow following a friction law such that

$$C_f = \text{Const.} / R^{0.5} \quad (8)$$

The effect of vortex flow on a superposed axial flow is a subject of some controversy. For example, Cole⁽¹⁷⁾ has shown that, for a Reynolds number up to 5-times the transition value, the axial flow behaves as predicted for simple viscous flow. Apparently, the Taylor vortices simply wash out with the axial flow. On the other hand, experiments with bearing configurations suggest an interaction between leakage flow and vortex flow. For example, Wilcock⁽¹⁸⁾ has shown a definite falling off of leakage flow in bearings, following the transition. His clearance ratios (0.00161, 0.0025, 0.00436, 0.00382) overlap the range where vortex transition, rather than full-turbulent transition should dominate. It is possible that the two-dimensional nature of the bearing flow requires the fluid to move circumferentially as well as axially in order to leak out, and an increase in the circumferential resistance would affect axial flow. Another explanation would be that the flow in the bearing had undergone its second transition to full turbulence, whereupon axial resistance would be expected.

The transition to vortex flow is influenced by shaft eccentricity. For a relatively large clearance ratio ($b/r = 0.1$), Cole⁽¹⁷⁾ has demonstrated that the transition speed tends to rise with increased eccentricity, though the increase is only about 10% of that predicted on the basis of minimum clearance. For example, the transition speed was raised from 75 to 90 rpm for an eccentricity change from zero to 50 percent. Taylor's criterion based upon the minimum clearance would have predicted a rise to almost 200 rpm for the same eccentricity change.

Smith and Fuller have given evidence suggesting similar effects in bearings. On the other hand, Schultz-Grunow⁽³⁴⁾ has shown that, for the outer-cylinder-turning, rotating eccentricity could bring about transition at Taylor's predicted value, while improved concentricity could delay the transition significantly. This does not necessarily contradict the above results, since it is implicit that the flow behavior for outer-cylinder-turning is not directly comparable with that for shaft-turning.

DiPrima⁽¹⁹⁾ has carried out stability derivations which support Cole's observations. This work also predicts that an axial flow will substantially increase the Reynolds number of transition to vortex flow.

Second Transition. It has been mentioned that vortex flow can ultimately give way to a second type of flow that is fully turbulent, and is analogous to pipe flow^(2, 19, 20, 21). For the case where the clearance ratio is very small, one would predict the vortex transition to be possible only at very high Reynolds numbers, and the initial transition to be to fully turbulent flow, instead. This would be expected where the line designated "inner cylinder turning" rises above the appropriate broken line in Fig. 1. If the initial transition is of the vortex type, there is no indication that it leads to an early precipitation of full turbulence. Observations indicate that vortex flow can prevail for many multiples of the first-transition Reynolds number⁽²¹⁾.

When friction behavior is considered, the C_f vs R curves for early transition tend to approach the curve for full-turbulent flow at high Reynolds numbers, and they change slope so as to parallel this curve. If the data of Smith and Fuller (see Fig. 4) had been extended further, a similar change of slope might have been observed, as it was in Taylor's work. In the latter work, the only curves which crossed the thin film (full turbulent) curve were for extremely large clearances ($b/r > 0.05$), and this would not be representative of bearing operation. As a rule of thumb, it might be suggested that the transition lies at that point where the extrapolated vortex-flow friction curve intersects the fully-turbulent Couette-flow curve. From the data of Smith and Fuller, this would be predicted to occur at $R \approx 2 \times 10^3$. Undoubtedly, this transition is not instantaneous, since the change of slope (or friction law) is not instantaneous in Taylor's data. Even when it occurs alone the fully turbulent transition itself is seen to cover a ratio of Reynolds numbers of about 3:1 (see Fig. 4).

Shaw⁽²²⁾ has suggested that the second transition should be accompanied by a rise in friction. Taylor's data do not support this, but suggest rather that the second transition consists of a reduction of the slope of the friction curve. It is of interest that even though the slope

ultimately tends to parallel the full-turbulent friction curve the magnitudes tend to remain higher than this curve in many cases. This may be due to experimental error, since there was corresponding scatter for measurements in the laminar regime.

Experiments with Bearing Configurations

Bearings may differ from the simplified experimental apparatus of Taylor, for example, in that the length-diameter ratio is smaller, the shaft is apt to be located eccentrically in the bore, and flow is admitted through slots or holes which may be placed in a variety of arrangements. Bearings may also be in the form of slippers, rather than continuous cylinders. Though bearings have not been investigated in their full variety, there is sufficient reported work to indicate that the expected transitions do appear, essentially as predicted. Reviewing pertinent examples of the reported bearing experiments, we find:

Wilcock⁽¹⁸⁾ has reported measurements of torque and other factors for bearings operating above and below the Taylor transition, for shaft-turning. He indicated a general agreement between his measurements and the predicted transition, with discrepancies which may have been caused by heating of the film. He found transition to influence leakage flow in apparent contradiction to Cole, as discussed. He also found transition to decrease with eccentricity under load. Since the clearances suggest an initial transition to vortex flow, his results indicate that the vortex flow not only increased energy dissipation (i. e., torque), but also served to improve load support.

Abramovitz⁽²³⁾ has measured power dissipation in a tilting-pad thrust bearing, finding a very low transition Reynolds number of about $R = 150$, based upon minimum film thickness. It is likely that this minimum thickness is not the determining factor (see mention of Cole's observation of eccentricity effects), so it cannot be assumed that this observation contradicts the data presented previously. In his discussion Abramovitz notes that roughness may be sufficiently great to play a role in the resistance law.

Smith and Fuller⁽¹¹⁾ have operated a journal bearing well above the transition Reynolds number. The clearance ratio as well as the observed friction law indicate that vortex flow prevailed rather than full turbulence. Pressure distributions were measured as well as the relationship between eccentricity and attitude angle. Evidence was given to suggest that increased eccentricity suppresses the tendency toward transition.

Stahlhuth and Trippett⁽²⁴⁾ have reported the operation of liquid-metal lubricated bearings in the "turbulent" regime. In their difficult measurement situation, they have indicated only fair agreement between the observed transition and that predicted by Taylor. They have also shown a definite relationship between shaft speed and leakage flow, indicating that the transition increases the resistance to leakage. They also have demonstrated that increased load (eccentricity) increases the transition Reynolds number.

ANALYSIS OF BEARING BEHAVIOR

Except for studies of transition, vortex flow has not been dealt with analytically in the computation of bearing forces. In fact, aside from the derivations of Batchelor⁽¹⁶⁾, the techniques available for such computations apply only to fully turbulent flow, and even with this restriction the interaction between pressure flow and Couette flow is not well understood. In the paragraphs which follow, previous work will be reviewed briefly. This will be followed by a detailed examination of the special case where a weak pressure flow acts either across or along a strong Couette flow. As has been stated before, the objective will not be to obtain integrated solutions for bearings, but to clarify the relationships between the component flows and the shearing stress on the boundaries.

Discussion of Related Derivations

To simplify their computations of turbulent bearing performance, Smith and Fuller⁽¹¹⁾ have assumed that the pressure flow in the direction of rotation can be dealt with independently from the Couette flow. This neglects the effects of turning on the pressure-flow resistance law, and it predicts zero "apparent viscosity" for small amounts of pressure flow.

Tao has carried out derivations for turbulent lubrication of an simple slider⁽²⁵⁾ and for a short bearing with end leakage⁽²⁶⁾. He has also assumed independence of the resistance laws of the pressure flow and Couette flow. This accounts for the flattened peak of his pressure distribution function, for example. Furthermore, he has used Pai's function⁽²¹⁾ for the Couette flow profile, this not being one of the more accurate representations of the turbulent-flow profile.

Chou and Saibel have approached the journal bearing problem using mixing length theory⁽²⁷⁾. As applied, this approach accounts for the coupling effects previously neglected, but does not account for end leakage. To provide a continuous variation across the film, they have chosen to let the mixing length vary according to

$$\ell^2 \propto y(2b - y) \quad (9)$$

For this expression $\partial \ell / \partial y \rightarrow \infty$ at the wall and is therefore not in accord with Prandtl's statement that $\partial \ell / \partial y \approx 0.4$ in the turbulent fluid near the wall. Since details near a wall have a predominant effect on the overall solution, the velocity profile must differ from the measured one.

Constantinescu has applied mixing-length theory to the journal bearing problem using a linear variation of mixing length out from the wall⁽²⁸⁾. It is interesting that his results for the variation of C_f (derived from equations for fully developed turbulence) agree closely with experimental results of Smith and Fuller for what appears to be vortex flow.

Constantinescu has extended his scope to deal with three-dimensional flow⁽²⁹⁾, with the tacit assumption that the resistance laws for orthogonal components of a flow are independent. This would correspond in short bearings to the assumption of independence of pressure flow and Couette flow, and would predict that leakage flow is unaffected by the speed of turning, thus, he has omitted account of the important coupling effects in the flow.

Arwas, Sternlicht and Wermick⁽³⁰⁾ have carried out an analysis for two-dimensional full-turbulent flow in bearings and have obtained results which agree closely with the measurements of Smith and Fuller. They have also used the mixing-length approach, assuming that ℓ varies linearly out from the wall. They have derived a power-law expression for the variation of C_f such that $C_f = \text{Const.} / R^{0.4}$ as compared with the empirical laws of Eq (6) and Eq (7).

Mixing Length Approach for Codirectional Plane Flows

In view of the effective application of the mixing-length approach in several of the investigations listed, it is desirable to examine the problems inherent in this technique, in the search for pitfalls which can lead to erroneous results. The background of the mixing-length equation for parallel flows has been discussed in detail⁽³¹⁾, and is not so much subject to question as is the application of the mixing length equations themselves. Dealing with time-mean quantities, the apparent shear stress for

turbulent parallel flows may be expressed in terms of mixing length and other variables as follows, letting x be measured in the direction of flow, and letting y be measured normal to the flow. The time-mean fluid velocity is represented by u , the density by ρ , the viscosity by μ , and apparent shear stress on surfaces parallel to the wall by τ_{xy}^* (this apparent stress may be dealt with as an actual stress so far as the time-mean flow is concerned).

$$\tau_{xy}^* = \rho \ell^2 \left(\frac{\partial u}{\partial y} \right)^2 + \mu \frac{\partial u}{\partial y} \quad (10)$$

Totally aside from theoretical considerations which led to the original writing of this form of equation, it has broad validity as long as no restrictions are placed on the ways in which ℓ can vary. The equation would be of little practical, computational value, however, if ℓ did not behave in some regular manner.

It has been found that the variation of ℓ between two walls is not greatly affected by Reynolds number for pressure flows. In the near vicinity of a wall experimental data indicate that $\ell \approx 0.4y$. Further out (near midstream, for instance) the value of ℓ is dependent upon the details of the flow. However, a second fortuitous circumstance of the solution for the turbulent-flow profile is such that only small differences in the profile itself (and the corresponding wall shear stress) result from relatively large differences in the choice of the midstream value of ℓ -- so long as the variation near the wall is correctly dealt with. Thus Eq (10) would be expected to serve to generate useful "engineering" solutions for combined pressure and Couette flows.

As an aid to solution, Eq (10) may be rewritten in the following form

$$\frac{\partial u}{\partial y} = \frac{-\mu + \sqrt{\mu^2 + 4\tau_{xy}^* \rho \ell^2}}{2\rho \ell^2} \quad (11)$$

This may also be written in terms of dimensionless quantities as follows

$$\frac{\partial (u/u_c)}{\partial (y/b)} = \frac{-1 + \sqrt{1 + 2C_f R^2 (\ell/b)^2 (\tau_{xy}^*/\tau_{xy})}}{2R (\ell/b)^2} \quad (12)$$

For typical Reynolds numbers, the term in Eq (10) involving viscosity is of importance only near the wall; and, for most of the region between walls (surfaces), the equation simplifies to:

$$\frac{\partial(u/u_c)}{\partial(y/b)} = \frac{b}{\ell} \sqrt{\frac{C_f}{2}} \left(\frac{\tau_{xy}^*}{\tau_{xy}} \right) \quad (13)$$

If it is assumed that $\ell = 0.4y$. Eq (13) may be solved as follows for the case of Couette flow, where the stress τ^* has a constant value τ independent of y

$$\frac{u_c - u}{u_c} = - \sqrt{\frac{C_f}{0.32}} \ln \left(\frac{y}{b} \right) \quad (14)$$

When $(u_c - u)/u_c$ is computed for the known friction coefficient and Reynolds number, and is plotted against velocity profile data (Fig. 3), the computed curve is found to pass to the right of the power law curve, and provide a good fit for the data of Reichardt.

A more elaborate approach may be used to allow for the mid-stream departure of the mixing length from the assumed linear variation. Eq (13) may be rewritten to provide a two-term approximation of the mixing-length variation, which permits an even closer fit of the experimental data, except in the region near the wall, where viscous effects become important. It is of interest that the midstream mixing length is related to the dimensionless slope of the velocity profile as follows

$$\ell/b = \sqrt{C_f/2} \left(\frac{b}{u} \frac{\partial u}{\partial y} \right)_c \quad (15)$$

For $\left(\frac{b}{u} \frac{\partial u}{\partial y} \right)_c = 0.179$ (see Table 2), it is found that $\ell = 0.26b$. This falls between the value for the centerline of a pipe $\ell = 0.14b$, and that for a turbulent jet or wake $\ell = 0.414b$.

To provide a complete solution for the velocity profile, extending to the wall, it is possible to solve Eq (12) directly and to integrate the result by machine computation. Such a computation was carried out for $R = 10^4$ and $C_f = 0.00457$, representative of the empirical friction law, and it was assumed that $\ell = 0.4y$. The unexpected result of this computation has been that the dimensionless velocity defect $1 - u/u_c \approx 0.5$ at $y = 0$, rather than going to unity as it should. Improvement of the equation for the mixing-length distribution near midchannel reduces but does not eliminate this large discrepancy.

By way of explanation, Emmons⁽³²⁾ has measured eddy size as a function of distance from the wall in a channel, and has found this to follow a linear variation out from the wall, except very close to the wall

where l varies in proportion to y^3 . Emmons points out that there is a correspondence (though not an identity) between eddy dimension and mixing length; thus, a similar variation would be expected for mixing length itself. Computationally, the effect of this would be that the linear mixing-length distribution would drop to zero for $y < e$, where e is very small distance out from the wall. Emmons' data suggest that $e \approx 0.02$; the thickness of a computed laminar film required to bridge the gap in the above Couette flow calculation is found to be $e \approx 0.01$. This gives support to the applicability of Emmons' observations to such flows. It also suggests that maximum confidence should result from calculations which are independent of the effects of such detailed considerations, near the wall.

Codirectional Weak Pressure Flow and Strong Couette Flow. When an incremental pressure flow is superimposed on a primary Couette flow, account must be taken of the linear shear stress variation of the pressure flow $\left[\text{where } \Delta\tau_{xy}^* = (b - y) \frac{dp}{dx} \right]$ and of the constant shear stress τ_{xy} of the Couette flow. The wall stress contribution due to the pressure flow is $\Delta\tau_{xy}$. If all quantities indicated by the prefix Δ represent the incrementation of the specified quantities as a result of the presence of the pressure flow, Eq (10) may be rewritten

$$\tau \left[1 - \frac{\Delta\tau_{xy}}{\tau_{xy}} \left(1 - \frac{y}{b} \right) \right] = \rho l^2 \left[1 + \frac{\Delta(l^2)}{l^2} \right] \left[\left(\frac{\partial u}{\partial y} \right)^2 + 2 \left(\frac{\partial u}{\partial y} \right)^2 \frac{\partial(\Delta u)/\partial y}{\partial u/\partial y} + \left(\frac{\partial \Delta u}{\partial y} \right)^2 \right] + \mu \frac{\partial(u + \Delta u)}{\partial y} \quad (16)$$

Here τ represents the shear stress due to the Couette flow, and $\Delta\tau$ is the increment of wall stress due to the incremental pressure flow. This allows for the possibility that the mixing-length function l^2 may be incremented by the function $\Delta(l^2)$. This incrementation will disappear only if, for all values of y , $\Delta(l^2)/l^2$ is an order smaller than $\Delta u/u$ when $\Delta u/u \rightarrow 0$. With this latter assumption and the elimination of terms involving $(\Delta u)^2$, Eq (16) simplifies to

$$\frac{\Delta\tau_{xy}}{\tau_{xy}} \left(1 - \frac{y}{b} \right) = \frac{\partial(\Delta u)/\partial y}{\partial u/\partial y} \left[1 + \rho l^2 \frac{(\partial u/\partial y)^2}{\tau_{xy}} \right] \quad (17)$$

If it can be assumed that the term involving l approaches unity except in the sublayer, where it is assumed to be zero, and if the sublayer thickness is 'a', this reduces to the following form upon integration:

$$\frac{\Delta\tau_{xy}}{\tau_{xy}} \left[u - \int_0^y \frac{y}{b} \frac{\partial u}{\partial y} dy \right] = \Delta u + \int_a^y \frac{\partial(\Delta u)}{\partial y} dy \quad (18)$$

To permit evaluation of the integrals remaining, an approximate velocity profile may be substituted. One choice would be the power law profile $u/u_b = (y/b)^{1/7}$ which has been shown to agree closely with experimental data for plane Couette flow. Other profiles might be used to obtain better fits; however, the process of integration tends to eliminate the effects of minor deviations between possible choices. Now let the limit of integration be $y = b$

$$\frac{\Delta\tau_{xy}}{\tau_{xy}} (1 - 1/8) = \frac{\Delta u_c}{u_c} (2 - a/b) \quad (19)$$

then if $a = 0.05b$ as shown in Fig. 2

$$\frac{\Delta\tau_{xy}}{\tau_{xy}} = \frac{8}{7} (2 - 0.05) \frac{\Delta u_c}{u_c} = 2.22 \frac{\Delta u_c}{u_c} \quad (20)$$

The approximation in evaluating the right-hand integral is seen not to be serious, since the entire integral involved is only a small part of the final answer. Only about 2 percent error would result in the answer if this integral was nearly 100 percent in error. To set this into perspective the universal velocity profile, [as approximated by the power law, Eq (13)] may be assumed to apply whether the incrementation of center-line velocity is due to a pressure flow or Couette flow. This has been substantiated experimentally as shown in Table 2 and would lead to the following equation

$$\frac{\Delta\tau_{xy}}{\tau_{xy}} = \frac{17}{8} \frac{\Delta u_c}{u_c} = 2.125 \frac{\Delta u_c}{u_c} \quad (21)$$

It is seen here that there is no large difference between Eq (20) and Eq (21).

From Eq (18) displacement thickness for the incremental pressure flow is found to be

$$\delta^* = 0.059b \quad (22)$$

Whereas, for an incremental Couette flow, it would be

$$\delta^* = 0.125b \quad (23)$$

This would lead to an increase of δ^*/b in a combined flow when the pressure flow acts to decrease the centerline velocity u_c . This is not in accord with Table 2. Thus, it appears that changes in the mixing-length distribution must tend to hold the velocity profile constant. And, it appears that the assumption of the universal velocity profile in the half-channel provides a valid approximation where data are available for checking. Possibly the assumption of "unchanging mixing-length profile" [Eq (20)] and "unchanging velocity profile" [Eq (21)] may be thought of as bracketing cases - and for many purposes the predictions of the two are not really very far apart.

The Coupling Between Orthogonal Couette and Pressure Flows

In the problem of leakage around high-speed shafts and also short journal bearings, a weak pressure flow acts at right angles to a strong Couette flow. For the case where the motion is turbulent, Tao⁽²⁶⁾ has been able to solve the short bearing problem by neglecting any coupling effects between the two component flows. Constantinescu⁽²⁹⁾ has written mixing-length equations for such flows, also assuming the components to be independent. On the other hand, Prandtl has provided a mixing-length equation⁽³¹⁾ which permits accounting for coupling effects. The application of this to bearing problems⁽³³⁾ will be reviewed here and examined as to restrictions involved in its application. The mixing-length equation for three-dimensional stress variation may be written, in Cartesian coordinates, as

$$\tau_{xy}^* = (\rho \ell^2 J + \mu) \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \quad (24)$$

Where J is defined by

$$\begin{aligned} J^2 = 2 & \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 + \left(\frac{\partial w}{\partial z} \right)^2 \right] + \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right)^2 \\ & + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)^2 \end{aligned} \quad (25)$$

Simplifying to apply to parallel flow (which is uniform in the sense that $\partial u / \partial x = \partial w / \partial z = \partial v / \partial y = 0$), this becomes

$$\tau_{xy}^* = \left[\rho \ell^2 \sqrt{\left(\frac{\partial u}{\partial y} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2} + \mu \right] \left[\frac{\partial u}{\partial y} \right] \quad (26)$$

Interchanging coordinates

$$\tau_{zy}^* = \left[\rho \ell^2 \sqrt{\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2} + \mu \right] \left[\frac{\partial w}{\partial y} \right] \quad (27)$$

Here the x and z coordinates are parallel to the boundary surfaces, and the y coordinate is normal to these. The velocity u in the x-direction will represent the Couette flow, and the velocity w in the z-direction will represent the pressure flow.

When τ_{zy} is the wall stress of the pressure flow, it is possible to write, as in the previous case,

$$\begin{aligned} \tau_{zy}(1 - y/b) = \rho[\ell^2 + \Delta(\ell^2)] & \left[\left(\frac{\partial u}{\partial y}\right)^2 \frac{(\partial w/\partial y)}{(\partial u/\partial y)} \right. \\ & \left. + \mu \frac{\partial u}{\partial y} \frac{(\partial w/\partial y)}{(\partial u/\partial y)} \right] \end{aligned} \quad (28)$$

This allows for possible incrementation of the mixing-length function, ℓ^2 . However, if it is assumed that $\Delta(\ell^2)/\ell^2$ is of the same order as w/u , and, if $w/u \ll 1$, this simplifies to

$$\tau_{zy} \frac{\partial u}{\partial y} \left(1 - \frac{y}{b}\right) = \tau_{xy} w \quad (29)$$

where b is half the distance separating the plates. To permit integration, a Couette flow profile must be used. Since it has been shown that the profile $u/u_b = (y/b)^{1/7}$ fits experimental data, and, since small departures are smoothed out by the integration process, Eq (29) may be reduced to

$$\tau_{zy} u \left(1 - \frac{1}{8} \frac{y}{b}\right) = \tau_{xy} w \quad (30)$$

Applying this to a point halfway between plates where $y = b$, this becomes

$$\frac{\tau_{zy}}{\tau_{xy}} = \frac{8}{7} \frac{w_c}{u_c} \quad (31)$$

Quite aside from possible questions concerning the validity of Eq (24) and (25), note that: (a) The solution did not require the viscous effects beneglected. (b) It was not necessary to assume an equation for

mixing-length variation. (c) It was not necessary to assume that the mixing-length function remains constant, so long as the order of $\Delta(l^2)/l^2$ is the same or smaller (i. e., higher order) than w/u , for the case where $w/u \ll 1$. As simplified in Eq (27), it is equivalent to stating that the eddy viscosity of the primary flow also serves as the eddy viscosity of the incremental flow. Furthermore, in Eq (27) it is seen that the eddy viscosity, ϵ , is determined by the vector derivative of the combined velocity with respect to y , hence

$$\epsilon = \rho l^2 \sqrt{\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2} \quad (32)$$

As used, however, it has not been necessary to accept this relationship wholly. One need only accept that in whatever true equation governs, the relative participation of the $\partial w / \partial y$ is such that it tends to vanish as $\partial w / \partial y \rightarrow 0$.

On the basis of the above arguments, one is drawn to conclude that for Eq (31) to be valid, only a very low degree of validity is required of the equations which went into its derivation.

A second question of interest would be how well the universal velocity profile relationship would predict the axial resistance to the flow if it is assumed to apply on the substitution of ($U_c^2 = u_c^2 + w_c^2$) for the centerline velocity term u_c^2 , and if it is assumed that the shearing resistance is colinear with U_c such that $\tau(w_c/U_c) = \tau_{xy}$. Under this circumstance $\tau_{zy}/\tau_{xy} = w_c/u$, which is not far different, numerically, from Eq (31).

Comparisons Between Turbulent and Laminar Films

The chief differences between turbulent- and laminar-film bearing behavior can best be illustrated by comparison of the resistance laws for Couette flow, codirectional pressure flow, and orthogonal pressure flow.

For laminar films (retaining the nomenclature used in the turbulent-film derivations involving weak pressure flows):

$$\text{Resistance to turning,} \quad \tau_{xy} = \mu u_c / b \quad (33)$$

$$\begin{array}{l} \text{Resistance to codirectional} \\ \text{pressure flow} \end{array} \quad \frac{\Delta \tau_{xy}}{\tau_{xy}} = \frac{3 \Delta \bar{u}}{u_c} \quad (34)$$

Resistance to orthogonal pressure flow	$\frac{\Delta \tau_{zy}}{\tau_{xy}} = \frac{3 \Delta \bar{w}_c}{u_c} \quad (35)$
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The turbulent-film equivalents are:

Resistance to turning	$\tau_{xy} = C_f \rho u_c^2 / 2 \quad (3)$
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Resistance to codirectional pressure flow	$\frac{\Delta \tau_{xy}}{\tau_{xy}} = 2.22 \frac{\Delta u_c}{u_c} = 2.59 \frac{\Delta \bar{u}}{u_c} \quad (20b)$
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Resistance to orthogonal pressure flow	$\frac{\Delta \tau_{zy}}{\tau_{zy}} = \frac{8}{7} \frac{w_c}{u_c} = \frac{\bar{w}}{u_c} \quad (31b)$
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Here u_c is the centerline velocity of the Couette flow acting alone and is half the relative sliding velocity of the surfaces, and barred quantities indicate averages across the film.

Of special significance is the fact that, for the turbulent film, resistance to pressure flow in the tangential direction (codirectional with the Couette flow) is essentially double the resistance to axial flow (orthogonal to the Couette flow).

Considering the long bearing, a comparison of Eq (20b) with Eq (34) suggests that ratio of load support to shearing resistance is inferior to that for simple viscous flow, but is essentially of the same order of magnitude.

Regarding end leakage, comparison of Eq (31b) and Eq (35) shows the turbulent film bearing to be inferior. In fact, comparison between Eq (20b) and Eq (31b) would suggest that turbulence causes a bearing to behave as though it were shorter than an equivalent laminar film bearing. At first this seems contradictory since the onset of turbulence reduces leakage in a given clearance. But, on the basis of the larger clearance required to make turning friction equivalent to that of the laminar film, the turbulent-film bearing will show greater leakage than the original laminar-film bearing would.

Though the above observations would suggest that turbulent operation is inferior to laminar operation, it must be remembered that the turbulent film bearing gives greater initial (small displacement) stiffness for a given film thickness, since the stiffness is proportional to the resistance to pressure flow, and since this is, in turn, proportional to the shear-resistance to turning, which in turn may be several times that for laminar flow, depending upon Reynolds number

CONCLUSIONS

In bearing experiments, transitions to vortex flow and to fully turbulent flow occur essentially as predicted from simplified experiments involving long cylinders. There is evidence that eccentricity suppresses transition in both experimental arrangements.

There is contradictory evidence regarding the effects of transition on the resistance to axial flow. In cylinders where the transition was to vortex flow, the transition did not affect the resistance to axial flow. On the other hand, for bearings operating with sufficient clearance to predict transition to vortex flow, there was a definite influence of transition upon resistance to axial flow. Derivations are given to show that such a resistance would be expected in fully turbulent bearing films, and it may be that details of lubricant admission, departures of geometry from the ideal, and the relatively short length/diameter ratio lead directly to the second transition to full turbulence rather than vortex flow. Flow visualization experiments in bearings are called for to clarify this issue.

The resistance law for fully turbulent flow (which would prevail in slider bearings, and journal bearings at high Reynolds number or low film thickness) has been shown to correspond closely to that for a boundary-layer or for pipe flow. The somewhat different resistance law for Couette flow after vortex transition is relatively consistent among numerous investigators, and applies also to at least one example of bearing flow. It is somewhat paradoxical that mixing-length derivations for fully turbulent flow in the corresponding low Reynolds number range have also given a corresponding resistance law. There should be hesitation, however, in assuming that there is a direct analogy between vortex flow and fully turbulent flow.

Pitfalls in the application of mixing length theory have been mentioned, and derivations which sidestep some of these difficulties have been carried out for the special case of a weak pressure flow superposed on a strong Couette flow. On the basis of these, which apply to the fully turbulent bearing under small displacements, comparisons with laminar-flow lubrication have been possible.

One item of major interest is that the use of a single friction law [Eq (3)], applied to midstream velocity, will predict a shear stress (codirectional with the midstream velocity) that is very close to the stresses which have been computed, both for codirectional and crossed flows.

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LIST OF SYMBOLS

b	Half-thickness of film.
C_f	Friction coefficient.
l	Prandtl's mixing length.
R	Reynolds number ub/ν based on centerline velocity.
S	Dimensionless slope of velocity profile $(b/u_c)(\partial u/\partial y)_c$.
u	Velocity parallel to surface in general. For primary Couette flow, velocity in direction of sliding.
u^*	Friction velocity, $\sqrt{\tau/\rho}$.
v	Velocity normal to boundary surface.
V	Relative sliding velocity between bearing surfaces.
w	Velocity parallel to boundary, normal to direction of primary sliding.
x	General coordinate, also coordinate in direction of primary sliding.
y	Coordinate normal to boundary.
z	Coordinate parallel to boundary and normal to direction of primary sliding.
δ^*	Displacement thickness of half-film.
μ	Absolute viscosity.
ν	Kinematic viscosity.
ρ	Fluid density.
τ	Time-mean shear stress on wall.
τ^*	Time-mean local shear stress.
$()_c$	Subscript refers to median surface of film.