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Symmetries Among the Strongly-Interacting Particles

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SYMMETRIES AMONG THE STRONGLY-INTERACTING PARTICLES

Introduction

The spectrum[1-3] of the strongly-interacting particles exhibits many fascinating regularities. (See figures 1 and 2.) These regularities comprise, first, the multiplet structure associated with isotopic spin, and secondly, the apparent clustering together of these multiplets into supermultiplets. This article is concerned with attempts to understand how these regularities reflect the interactions among the particles.

From the time of the neutron's discovery, its mass has been known to be nearly equal to that of the proton. It was natural to suppose that this near identity of mass was associated with a similarity of internal structure. This idea was formalized through the introduction by Heisenberg[4] of the concept of isotopic spin, which was just a mathematical way to state that there was one fundamental structure, called the nucleon, over which two different net amounts of electrical charge could be distributed without greatly distorting it. The name of this concept is a reminder that the mathematical treatment is similar to that of the spin of the electron. But even without going into mathematical details, it can be seen that it is implicit in this idea that the neutron and proton should participate in similar ways in the structure of nuclei and in nuclear reactions, and this has been found to be true, when due account is taken of electromagnetic forces. As the internal structures of the proton and neutron have been examined more closely, the additional particles which have been found have also shown the isotopic spin symmetry, so that now the idea of isotopic spin is usually taken for granted and even used as a tool in the identification of new particles. For a further discussion of isotopic spin, we refer the reader to textbooks or to the survey by Wick[5].

The usefulness of the isotopic spin concept leads naturally to the suggestion that the particles within a supermultiplet might also have similar structures, and that their interactions might be describable by a mathematical formalism similar to that used for isotopic spin. Of course, any such higher symmetry would necessarily be much less accurate than isotopic spin symmetry. However, the alternative to assuming that some symmetry exists is to assume that the way in which the spectrum arises from the dynamical interactions is so complicated that it cannot be comprehended in any simplified manner. It is clear that we would be forced to look for a rough system of classification and a convenient starting point for the description of irregularities, even if we anticipated that the result would be to rule out such symmetry schemes.

In other branches of physics in which phenomenological symmetries arise, it is convenient to distinguish three aspects of the mathematical theory of the regularities. For example, we might think of the regular, ordered structure of a crystal lattice. The first step is to find out what kind of mathematical construct is appropriate to the description of such a regularity. The description of crystal symmetries is provided, as is well known, through a branch of group theory dealing with the "crystallographic groups". The second step is to decide which group pertains to a given substance and to derive the further implications of the symmetry. To explore the influence of the symmetry upon the many physical properties of a crystalline material, one needs to develop certain mathematical tools. The basic mathematical problem is that of working out relations among the representations of a given group. The final and most fundamental aspect of the study of crystalline symmetry is the elucidation of the way the regular structure arises from the internal interactions.

The theory of particle symmetries will naturally also exhibit these three aspects.

Our aim is to describe particle symmetries on an entirely phenomenological level, so it is necessary to keep in mind the nature of the empirical information available about the strongly-interacting particles. This consists of the properties of the isolated particles, their cross sections for production and scattering, and their interactions with the electromagnetic and weak currents. We must likewise keep in mind that the theoretical machinery available for analysis of strong interactions is provided by the dispersion relations among S-matrix elements. These considerations determine the form taken by the mathematical development in the last three sections of this article. Before proceeding with this development, however, we digress in the next section into the problem of trying to understand why a study of the regularities should lead to the consideration of compact Lie groups.

Particle Symmetry Groups

Isotopic spin and the higher symmetry are properties of the strong interactions. In contrast, charge Q and hypercharge Y, as well as parity and charge conjugation, are concepts which apply also to electromagnetism. When we restrict our attention to the strong interactions, the additively conserved quantities Q and Y that distinguish the different members of a supermultiplet are "given concepts" which provide the framework around which the symmetry groups are constructed. For notational succinctness, we shall consider here ℓ independent quantum numbers H_σ, where $\ell = 1$ corresponds to charge-multiplets, $\ell = 2$ to the case of current interest, and $\ell > 2$ to the possibility that in the future other conserved quantities may be found, some of which could arise within the strong interactions and not be conserved as well as Q. Independence of the H_σ means that they commute with each other and are also linearily independent; for example, the component of isotopic spin $I_z = Q - 1/2\, Y$ is not independent of Q or Y, but could replace one of them.

Eigenvalue lattices.- We shall make extensive use of a geometrical picture of a supermultiplet in which each particle is represented by a point on a lattice in ℓ dimensions; the coordinates of the point are given by the eigenvalues of the H_σ[6]. For example, the lattice for the neutron-proton doublet consists of just two points. The lattice for the baryon octuplet is shown in Fig. 3. In technical treatments of Lie groups, these eigenvalue lattices are derived concepts and are usually called "weight diagrams". In the present physical context the lattices are actually more fundamental than the groups.

The particles which are placed on the same lattice have the same spin and parity and nearly the same mass, and are supposed to have a similar internal structure, in a sense that will be defined more explicitly later. There is a standard way to describe the internal symmetries of a quantum mechanical system defined by a set of states, which we follow. We introduce unitary operators U which transform a given state $|a\rangle$ into a "similar" one $U|a\rangle = |a_U\rangle$. Along with any such operators, we include their products and their inverses, so that the entire collection forms a group.

Transformation of the states according to

$$A(\xi) = \exp(i\,\xi^\sigma H_\sigma) \quad \text{(summed with range } \ell\text{)}$$

with arbitrary real ξ^σ is a gauge transformation that merely expresses the fact that the H_σ are additively conserved and independent. By hypothesis, there are other less trivial transformations of the states which describe their supposed similarity. If U is one of these, the product $U^{-1} A(\xi) U = A'(\xi)$ is another operator which transforms states into similar images. Either $A'(\xi)$ is a new continuous transformation, or, if A' has the form (1), (can be written as $A' = A(\xi')$), we find by considering infinitesimal ξ^σ that

$$U^{-1} H_\sigma U = (\partial \xi'^\rho / \partial \xi^\sigma) H_\rho \qquad 2$$

This implies that U can be written as a product U = Ru, where R corresponds to a linear transformation of the lattice coordinates and u is a unitary transformation of the states at each lattice position among themselves. With a suitable choice of coordinates, R can be pictured as a reflection or rotation of the lattice into itself. Operators U which do not have this simple geometrical interpretation necessarily generate additional continuous transformations; the transformations (1) are then embedded in a Lie group with additional parameters.

A simple illustration is given by the neutron and proton. There are only two conceivable symmetry groups: "charge symmetry", which contains a single discrete operation which interchanges the two nucleons, and isotopic spin symmetry (SU_2). The arbitrary linear superpositions of neutron and proton states which are generated by SU_2 have, of course, no physical significance except as they help to describe the way the nucleons resemble each other.

Continuous groups are the primary candidates for particle symmetries. Discrete symmetries involve fewer relations among the structures of the particles, and so are less interesting and useful. Moreover, since they are necessarily related to symmetries of the lattices, the problem of superposing them upon a continuous symmetry can be conveniently postponed.

Properties of Lie groups.- We will now outline a few of the general properties of Lie groups, as they refer to operations on a given lattice. Further information, including the steps omitted from derivations, can be found in the cited literature[7-13]. It will not be necessary to follow all of the details in order to comprehend the later material.

Generators G_a of infinitesimal transformations are defined by

$$U(\epsilon \xi) \approx 1 + i \epsilon \xi^a G_a \quad \text{(summed with range N)} \qquad 3$$

for small ϵ. The mutually commuting operators H_σ (assumed to be complete) form ℓ of these G_a, and it is natural to anticipate that the remaining $(N - \ell)$ G's can be interpreted as displacement operators on the lattice. Since we must be able to express the infinitesimal transformation $U^{-1}(\epsilon \eta)\, U^{-1}(\epsilon \xi)\, U(\epsilon \eta)\, U(\epsilon \xi)$ in the form (3), there must exist a commutator relation:

$$[G_a, G_b] = - F_{ab}{}^c G_c \qquad 4$$

which defines a "Lie algebra". The $F_{ab}{}^c$ so defined are called structure constants. They satisfy the Jacobi identity

$$F_{bc}{}^e F_{ae}{}^d + F_{ca}{}^e F_{be}{}^d + F_{ab}{}^e F_{ce}{}^d = 0 \qquad 5$$

We shall now show how to relate the values of the $F_{ab}{}^c$ to the geometrical structure of the eigenvalue lattices. We construct operators $E_\alpha(\sigma) = K_\alpha{}^a(\sigma)\, G_a$ which satisfy

$$[H_\sigma, E_\alpha(\sigma)] = r_{\alpha,\sigma} E_\alpha(\sigma) \qquad 6$$

The $K_\alpha{}^a(\sigma)$ are eigenvectors of the matrix $F_{\sigma b}{}^c$. Considering two operators, we have

$$[H_\sigma, [H_\rho, E_\alpha(\sigma)]] = [H_\rho, [H_\sigma, E_\alpha(\sigma)]] \qquad 7$$
$$= r_{\alpha,\sigma}[H_\rho, E_\alpha(\sigma)]$$

which implies that $[H_\rho, E_\alpha(\sigma)] \propto E_\alpha(\sigma)$; the E_α are therefore actually independent of σ and

$$[H_\sigma, E_\alpha] = r_{\alpha,\sigma} E_\alpha \qquad 8$$

Since the H_σ have real eigenvalues, $r_{\alpha,\sigma}$ is real and $E_\alpha^+ = E_{-\alpha}$ corresponds to $-r_{\alpha,\sigma}$. The $E_{\pm\alpha}$ are therefore step operators or displacement operators on the lattice, along the directions of the ℓ-dimensional vectors $\pm \underset{\sim}{r}_\alpha$ which are called "roots"; in the state $|a_\alpha\rangle$ defined by $E_\alpha | a \rangle = E_\alpha(a) | a_\alpha \rangle$, the eigenvalues of the H_σ are increased by $r_{\alpha,\sigma}$ from those in $| a \rangle$. The coefficient $E_\alpha(a)$ introduced here is the numerical value of the element of the matrix E_α.

We assume that it is possible to go from each point on the eigenvalue lattice to any other by repeated application of step operators. If this were not true, the structure of the unconnectable states could not be said to be related, and they would have been wrongly included in the same supermultiplet. This requirement implies (in fact, by definition) that the lattice corresponds to an irreducible representation of a semi-simple group.

It can be seen from the geometrical interpretation that if there is a root $\underset{\sim}{r}_{\gamma} = \underset{\sim}{r}_{\alpha} + \underset{\sim}{r}_{\beta}$, then

$$[E_\alpha , E_\beta] = N_{\alpha\beta} E_\gamma \qquad 9$$

(the $N_{\alpha\beta}$ are certain coefficients), and otherwise $[E_\alpha , E_\beta] = 0$; moreover, $[E_\alpha , E_{-\alpha}] = S_\alpha^{\ \sigma} H_\sigma$. It is possible to prove two remarkable additional theorems about the roots $r_{\alpha,\sigma}$. The first is that, except for the H_σ themselves, which all correspond to the root $\underset{\sim}{r} = 0$, there is only one step operator E_α for each distinct root $\underset{\sim}{r}_\alpha$. The second is that it is possible, by judicious choice of the H_σ and of the normalization factors in the E_α, to find a canonical basis in which $S_\alpha^{\ \sigma} = r_{\alpha,\sigma}$. Limitations of space preclude inclusion of proofs here (for further information, see Dynkin's survey[7]), but one of the points in the argument is worthy of special mention.

For any one of the E_α, we can obtain $s_\alpha{}^\sigma = C_\alpha r_{\alpha,\sigma}$ with $C_\alpha = \pm 1$, just by setting up in the lattice a suitable coordinate system. The lattice has a finite number of points, so $H_\alpha = r_{\alpha,\sigma} H_\sigma$ is bounded. Supposing the maxima of $\pm H_\alpha$ to occur for the states $|\pm\rangle$, we have $E_{\pm\alpha}|\pm\rangle = 0$, and hence $C_\alpha \langle \pm | H_\alpha | \pm \rangle = \langle \pm | [E_\alpha, E_{-\alpha}] | \pm \rangle \gtrless 0$ from which $C_\alpha > 0$ follows.

For any α, the three operators $E_{\pm\alpha}$ and H_α satisfy the same commutation rules as do the isotopic spin step operators $I_\pm$ and the operator I_z (apart from an irrelevant positive numerical factor). It is therefore possible to identify among the lattice points SU_2 multiplets extending along any lattice direction $\underset{\sim}{r}_\alpha$. This gives a very severe limitation on the eigenvalue lattices which can possibly exist; in particular, the lattice must have reflection symmetry in each of the $(\ell - 1)$ dimensional planes $H_\alpha = 0$. It is also implicit in the above remarks that the group is compact.

It is helpful to think less abstractly and picture the N operators G_a as a vector. The vector formed by the three isotopic spin operators has a three-dimensional representation in terms of rotations of ordinary spatial vectors. The "vector" representation of the G_a in a general algebra is the representation by the N x N matrices $(G_a)_b{}^c = F_{ab}{}^c$, which satisfies (4) as a consequence of (5). This is called the adjoint or regular representation by different authors. If this representation is irreducible, the group is said to be simple. The eigenvalue lattice for the adjoint representation, which consists of ℓ coincident points at the origin, and one point at the end of each root vector, is called the root diagram. It has the second role of showing graphically the displacements induced by the step operators.

To define a length for the vector G_a we introduce a metric tensor $g_{ab} = F_{ac}{}^d F_{bd}{}^c$ and its inverse g^{ab}. It then follows from (5) that $G^2 = g^{ab} G_a G_b$ (the Casimir operator) is invariant under all transformations of the group. It also follows that $F_{abc} = g_{cd} F_{ab}{}^d$ is totally antisymmetric. In the canonical basis of the Lie algebra we have

$$G^2 = \sum_{\sigma=1}^{\ell} H_\sigma^2 + \sum_{\alpha=1}^{(N-\ell)/2} \left\{E_\alpha , E_{-\alpha}\right\} \qquad 10$$

A total of ℓ invariant operators can be constructed, but G^2 is the only one which is quadratic.

Relations between supermultiplets.- A given set of particles, such as the baryon octuplet, can be associated with many different Lie groups, depending on how many extra conserved quantities and how many step operators are introduced. As long as one looks only at the baryons, these groups are equally valid, and equally useless. The idea of similar structure obtains physical content through the interactions. For example, the eight pseudoscalar mesons occur in the internal structure of baryons, and when the baryons are transformed, these virtual mesons automatically undergo certain transformations, too. We must require that free and virtual mesons transform in the same way, in order to be able to say that the baryons have similar structures. A more precise statement of this requirement is obtained as a mathematical condition upon S-matrix elements.

The simplest transition matrix to consider is the amplitude $\langle bc \mid T \mid a \rangle$ for decay of an unstable particle a into two daughters b and c. If a is stable, this amplitude is the coupling constant referring to the virtual decay. This amplitude is nonvanishing only if

$$H_\sigma \mid a \rangle = H_\sigma \mid bc \rangle = \left\{ H_\sigma(b) + H_\sigma(c) \right\} \mid bc \rangle \quad 11$$

The decay product of a is the state $|\hat{a}\rangle = \sum_{bc} |bc\rangle\langle bc|T|a\rangle$. A transformed state $|a_U\rangle = U|a\rangle$ decays into $|\hat{a}_U\rangle = \sum |bc\rangle\langle bc|T|a_U\rangle$. The separated particles in the state $|bc\rangle$ must transform as $U|bc\rangle = |b_U c_U\rangle$. This is compatible with $|\hat{a}_U\rangle = U|\hat{a}\rangle$ only if

$$\langle b_U c_U | T | a_U \rangle = \langle bc | T | a \rangle \qquad 12$$

If U reflects the lattice, this equation relates directly the amplitudes associated with corresponding lattice points. For infinitesimal transformations, (12) leads to

$$E_\alpha (b_{-\alpha}) \langle b_{-\alpha} c | T | a \rangle + E_\alpha (c_{-\alpha}) \langle bc_{-\alpha} | T | a \rangle$$

$$= E_\alpha (a) \langle bc | T | a_\alpha \rangle \qquad 13$$

The $\langle bc|T|a\rangle$ which are obtained as solutions of (11-13) are proportional (by definition!) to generalized Clebsch-Gordan coefficients for the group.

The degree to which the coupling constants satisfy (12) and (13) is a measure of the degree to which a symmetry may be said to exist.

Similar relations may be obtained for general S-matrix elements. These additional relations are less useful for the approximate higher symmetry than for isotopic spin. The reason is that cross sections tend to be large when peripheral and resonant contributions enhance them, and they are therefore strongly affected by kinematical coincidences. Fortunately, this dominance by individual states also makes it easier to infer the coupling constants from measured cross sections.

Conjectured symmetries.- Many groups whose representations might conceivably accommodate the observed supermultiplets have been proposed as the basis of symmetry schemes. Since information about resonant states is accumulated very slowly, a certain amount of conjecture about unobserved supermultiplet components has been unavoidable. As more data have been obtained, many schemes which once seemed quite promising have had to be discarded. In particular, the Sakata model, global symmetry, and schemes based on C_2 and G_2 are no longer popular. For a discussion of these models and for references to the original literature, the surveys by Behrends, Dreitlein, Fronsdal and Lee[13], Ne'eman[9,17], Tarjanne[15], and D'Espagnat[16] may be consulted.

So far, no serious difficulties, and some striking successes, have arisen in the Gell-Mann-Ne'eman SU_3 model[9,18-20], the "eightfold way", which accommodates naturally the octuplets and the decuplet apparent in the spectrum. The supermultiplets possible in this model are described in the next section.

SU_3 Symmetry

<u>The SU_3 algebra.</u>- Recall that SU_2 is the group which consists of the 2 x 2 unitary matrices which have a unit determinant. Similarly, to get SU_3 we take the 3 x 3 unitary unimodular matrices. The Lie algebra is formed by the eight independent traceless Hermitian matrices. The canonical form of the eight generators is given by the following 3 x 3 matrices:

$$E_1 = \frac{1}{\sqrt{6}}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \frac{1}{\sqrt{6}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad E_3 = \frac{1}{\sqrt{6}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$E_{-\alpha} = E_\alpha^{\dagger} \tag{14}$$

$$H_1 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H_1' = \frac{1}{6}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

We have interchanged the $E_{\pm 2}$ from a more usual notation[14] in order to obtain more symmetrical formulas. The explicit commutation relations obtained from Eq. (14) are

$$[E_\alpha, E_\beta] = -6^{-1/2}\epsilon_{\alpha\beta\gamma}E_{-\gamma}$$

$$[E_\alpha, E_{-\beta}] = 3^{-1/2}H_\alpha\delta_{\alpha\beta} \quad (\text{no summation}) \tag{15}$$

$$[H_\alpha, E_\alpha] = 3^{-1/2}E_\alpha, \quad [H_\alpha', E_\alpha] = 0$$

where

$$H_{2(3)} = -\tfrac{1}{2}H_1 - (+)\sqrt{3}\,H_1'/2 \;, \quad H_{2(3)} = +(-)\sqrt{3}\,H_1/2 - H_1'/2 \,. \qquad (16)$$

The connection with the physical operators I_z , Y, and Q is made as follows:

$$I_z = \sqrt{3}\,H_1 \;, \qquad Y = 2H_1' \,, \; Q = -2H_3 \qquad (17)$$

The eigenvalue lattice for the eight-dimensional adjoint representation (the root diagram) has been shown in Fig. 3. Three equivalent sets of orthogonal coordinates in the lattice are provided by (H_α, H_α'). These coordinate systems are related through reflections in the lines $H_\alpha = 0$ and their products, the 120° rotations[28-32]. The three different ways that SU_2 is contained in SU_3 (which are known colloquially as I-spin, U-spin, and V-spin[33]) are likewise transformed into each other by these Weyl reflections. An explicit element of SU_3 which accomplishes the reflection in $H_1 = 0$, for example, is the rotation through 180° about the y(I)-axis.

The traceless 3x3 matrices λ_a are complete, and therefore satisfy an anticommutation relation

$$\{\lambda_a, \lambda_b\} = g_{ab}/9 + D_{ab}{}^{c}\lambda_c \tag{18}$$

which also serves to define the symmetrical quantity $D_{ab}{}^{c}$. The commutator of λ_d with (18) gives the relation

$$F_{ad}{}^{c}D_{bc}{}^{e} + D_{bd}{}^{c}F_{ca}{}^{e} + F_{ab}{}^{c}D_{cd}{}^{e} = 0 \tag{19}$$

from which it follows that $N_a = D_a{}^{bc}G_bG_c$ behaves as a vector in SU_3-transformations, in the sense that

$$[G_a, N_b] = -F_{ab}{}^{c}N_c \tag{20}$$

These two SU_3-vectors, G_a and N_a, both play important roles in the mathematical structure of the theory.

<u>Irreducible representations of SU_3</u>. Consider a general representation, and imagine each state to be plotted on an eigenvalue lattice. The root diagram (Figure 3) shows that the possible sites lie on a hexagonal lattice on which the values of Y and Q differ by integers. The problem is to determine which sites are occupied, and the multiplicity of occupation. We shall sketch here a geometrical solution to this problem.[15,34,35] The method is a natural extension of that used in the theory of angular momentum.

If the representation contains a finite number of states, there will be a number in which the hypercharge Y takes a maximum value Y_m, and among these one state $|Y_m, Q_m\rangle$ in which Q is largest. We assume (without losing generality, as will become evident) that this state is unique. This state is characterized by the fact that $E_3 |Y_m, Q_m\rangle = E_1 |Y_m, Q_m\rangle = E_{-2} |Y_m, Q_m\rangle = 0$. Other states $|Y_m, Q_m - n\rangle$ may be generated by using the step operator E_{-1} n times. Let us suppose that λ is the greatest number of steps that can be taken along the upper boundary of the lattice; that is, that $E_{-1} |Y_m, Q_m - \lambda\rangle = 0$. These $\lambda + 1$ states constitute an I-spin multiplet with $I = \lambda/2$. In a similar way, a U-spin multiplet consisting of states $|Y_m - k, Q_m\rangle$, $k = 0, 1, \ldots, \mu$ is obtained by stepping in the direction $-\underset{\sim}{r}_3$ with the operator E_{-3}.

Let us suppose that $E_1 |Y_m - k, Q_m\rangle = 0$, which we know to be true for $k = 0$. Since $E_1 E_{-3} |Y_m - k, Q_m\rangle = E_{-3} E_1 |Y_m - k, Q_m\rangle$, we then have $E_1 |Y_m - k - 1, Q_m\rangle = 0$, so the members of this U-spin multiplet are also components of I-spin multiplets with $I_3 = I = (\lambda + k)/2$.

The same argument applies to $E_{-2} \left| Y_m - k, Q_m \right\rangle$, if we use $E_{-2} E_{-3} = E_{-3} E_{-2} + 6^{-1/2} E_1$ and the preceding result. This shows that no lattice points with $Q = Q_m + 1$ can be reached from the states $\left| Y_m - k, Q_m \right\rangle$ (at least, in one step). Another set of states with $I_z = I$ is generated by applying the step operator E_2 to the state $\left| Y_m - \mu, Q_m \right\rangle$.

We now make use of the fact the eigenvalue lattice must be invariant under the three Weyl reflections. It must therefore have the shape of a hexagon whose alternate sides have the lengths λ and μ. Furthermore, the center of the hexagon must be at the point $H_\alpha = H_\alpha' = 0$. A simple geometrical calculation then gives

$$Y_m = \frac{1}{3}\lambda + \frac{2}{3}\mu , \quad Q_m = \frac{2}{3}\lambda + \frac{1}{3}\mu \qquad (21)$$

If either λ or μ is zero, the hexagon is actually an equilateral triangle; if both vanish, it consists of only one point. All sites in the interior of the hexagon are occupied by states, since the boundary states are members of I-spin multiplets.

In general, the interior sites of the hexagon are multiply occupied. To see this, consider the state $E_2 \left| Y_m , Q_m \right\rangle$. This can easily be shown to be linearily independent of the state

$E_{-1} E_{-3} \quad Y_m , Q_m$ provided both and μ are positive. To obtain an orthogonal state we may use, instead of E_2, the operator $\bar{E}_2 = \sqrt{2} \; H_1 E_2 + E_{-1} E_{-3}$[34]; the commutation relations show that $E_1 \bar{E}_2 \; | a \rangle = 0$ if $E_1 \; | a \rangle = 0$. The I-values of $\bar{E}_2 \; | Y_m, Q_m \rangle$ and $E_{-1} E_{-3} \; | Y_m, Q_m \rangle$ therefore are, respectively, $\frac{1}{2}(\lambda - 1)$ and $\frac{1}{2}(\lambda + 1)$. Repeated application of $\bar{E}_2$ to the states $| Y_m - k, Q_m \rangle$ generates new states with diminishing values of I, the process necessarily terminating at a point demanded by the symmetry of the lattice.

To show that all of the states of an irreducible representation can be written in the form $(E_{-1})^n \quad (\bar{E}_2)^{\ell} \quad (E_{-3})^k \; | Y_m , Q_m \rangle$, we first replace these by the equivalent but non-orthogonal set $(E_{-1})^n \quad (E_2)^{\ell} \quad (E_{-3})^k \; | Y_m, Q_m \rangle$. It is then an easy matter to show that the application of any step operator to a member of this set leads to a linear combination of others of the set, by commuting this step operator past the factors in the product. To label the states three quantum numbers are needed; these are naturally taken to be Y, I, and $m = I_z$. The multiplicity of the lattice sites is most easily pictured by Tarjanne's[15] scheme of contour lines connecting the points of equal multiplicity. According to the reflection symmetry, these contours must be symmetrical hexagons nested inside each other, the smallest actually being an equilateral triangle inside of which the multiplicity is constant. This contour diagram is especially convenient for determining the values of I which occur for a given value of Y, and for counting the total number $d(\lambda, \mu)$ of states in a representation (λ, μ). An elementary calculation gives

$$d(\lambda, \mu) = \frac{1}{2}(\lambda + 1)(\mu + 1)(\lambda + \mu + 2) \qquad (22)$$

The Casimir operator can be evaluated by applying the expression given in Eq. (10) to the state $| Y_m, Q_m \rangle$.[3,36,37] The result is

$$9 G^2(\lambda, \mu) = \lambda^2 + \mu^2 + 3\lambda + 3\mu \qquad (23)$$

Another quantity of interest is $\nu = \lambda - \mu$ (Mod 3). According to Eq. (21), the possible values of Y and Q are equal to $\nu/3$ + (an integer). This number ν is evidently conserved (Mod 3) in the decomposition of product representations. It provides a distinction between representations of SU_3 similar to that between the integral- and half-integral-spin representations of SU_2. Only the $\nu = 0$ representations occur in the Gell-Mann—Ne'eman symmetry model.

We have already pictured the adjoint representation (1,1) in Fig. (3). The eigenvalue lattices for (3,0) and (2,2) are shown in Fig. 4. The values of d, G^2 and ν are listed in Table I for some of the simpler representations.

Table I

(λ, μ)	ν	d	G^2
(0 , 0)	0	1	0
(1 , 1)	0	8	1
(3 , 0)	0	10	2
(2 , 2)	0	27	8/3
(4 , 1)	0	35	4
(1 , 0)	1	3	4/9
(0 , 2)	1	6	10/9
(2 , 1)	1	15	16/9

All of the representations of SU_2 are self-conjugate, that is, equivalent to the representation obtained by taking the complex conjugate of the representation matrices. This is not true of SU_3. The conjugate infinitesimal matrices are $U(\epsilon\xi^a)^* = 1 - i\epsilon\xi^a G_a^T$; in particular, a minus sign appears in front of the commuting diagonal operators H_1 and H_1', so the eigenvalue lattice of the conjugate representation is obtained by reflecting the original lattice through the origin. This gives the rule $\overline{(\lambda,\mu)} \cong (\mu, \lambda)$. It is a general principal of quantum mechanics that bras $\langle a|$ and kets $|a\rangle$ transform contragradiently. (We have already used this fact implicitly in deriving Eq. (13).) In order that the transition amplitudes $\langle b_1 \ldots b_n C | T | a_1 \ldots a_m \rangle$ and $\langle b_1 \ldots b_n | T | a_1 \ldots a_m \bar{C} \rangle$ be related in the proper way through crossing symmetry, it is necessary that $\langle C|$ and $|\bar{C}\rangle$ transform in the same way, and therefore if the particles C belong to the representation (λ,μ), their antiparticles $\bar{C}$ must belong to (μ, λ). In particular, bosons and their antiparticles can belong to the same representation only if $\lambda = \mu$.

We should comment at this point that although we have used the electric charge Q in distinguishing among particles and in setting up eigenvalue lattices, we didn't really use any properties of Q except that it is conserved and its values differ by integers. We could add to Q a multiple of any other conserved quantity, such as the baryon number, and use the combination as a lattice coordinate. Now, the coordinate we have called Q so far was chosen to make the eigenvalue lattices look especially symmetrical — at the center of a lattice, Q = 0. It is a remarkable fact that nature actually uses for electromagnetic interactions the Q defined in this symmetrical way. In any continuous symmetry the charge would have to transform as the sum of a scalar and a vector component, but in the Gell-Mann—Ne'eman model only the vector term exists.

Biedenharn[36,37] has shown that a solution of the commutation relations (15) is given, for arbitrary (λ, μ), by the following matrices.

$$\langle I, m+1, Y | E_1 | I\, m\, Y \rangle = [(I-m)(I+m+1)/6]^{1/2}$$

$$\langle I+\tfrac{1}{2}, m+\tfrac{1}{2}, Y+1 | E_{-2} | I\, m\, Y \rangle = \langle I+\tfrac{1}{2}, -m-\tfrac{1}{2}, Y+1 | E_3 | I, -m, Y \rangle$$

$$= 18^{-1} \{ [(I+m+1)/(I+1)(2I+1)] [\lambda - \mu + 3(I + \tfrac{1}{2} Y + 1)]$$
$$\times [\lambda + 2\mu + 3(I + \tfrac{1}{2} Y + 2)][2\lambda + \mu - 3(I + \tfrac{1}{2} Y)] \}^{1/2}$$

$$\langle I-\tfrac{1}{2}, m+\tfrac{1}{2}, Y+1 | E_{-2} | I\, m\, Y \rangle = - \langle I-\tfrac{1}{2}, -m-\tfrac{1}{2}, Y+1 | E_3 | I, -m, Y \rangle$$

$$= 18^{-1} \{ [(I-m)/I(2I+1)] [-\lambda + \mu + 3(I - \tfrac{1}{2} Y)]$$
$$\times [\lambda + 2\mu - 3(I - \tfrac{1}{2} Y - 1)]$$
$$\times [2\lambda + \mu + 3(I - \tfrac{1}{2} Y + 1)] \}^{1/2} \qquad (24)$$

The unwritten matrix elements vanish. When λ and μ are non-negative integers, this is the same finite-dimensional representation we constructed by our heuristic argument, and the rigor missing from our discussion is supplied by examination of this result. The phases in the above formulas are the same ones used in the tables by Tarjanne[38] and by deSwart.[39]

Product Representations. We have already commented on how important product representations are to the physical applications. There are many very elegant ways to determine the irreducible representations contained in a direct product, that is, to catalogue the non-vanishing solutions to Eq. (13).[12,14,39] Instead of describing the general solution to this problem, however, we shall use here the elementary method familiar from the theory of angular momenta. At the same time we can describe the calculation of explicit Clebsch-Gordan coefficients.

The first step is to write down all the product states with the same values of the conserved quantities Y and Q . In the first column of table II we have shown the multiplicities of some of the $[Y, Q]$ lattice sites in the representation $(\lambda, \mu) \otimes (1,1)$, assuming λ and μ are both positive. The state with the largest $[Y, Q]$, which are $[Y_m + 1, Q_m + 1]$, evidently belongs to the representation $(\lambda + 1, \mu + 1)$ (refer to Eq. (21)). If we apply E_{-1} to this state we obtain that linear combination of the two $[Y_m + 1, Q_m]$ states which also belongs to $(\lambda + 1, \mu + 1)$. By continuing with the procedure described in the last section, all components of this representation can be found.

In effect, we solve the Eq. (13) stepwise, using the explicit forms of the representation matrices given by (24). The second column of table II gives the multiplicites of the $[Y, Q]$ sites in $(\lambda + 1, \mu + 1)$.

Table II

Multiplicities in $(\lambda, \mu) \otimes (1, 1)$

site	total	$(\lambda+1, \mu+1)$	$(\lambda-1, \mu+2)$	$(\lambda+2, \mu-1)$	$2.(\lambda, \mu)$
Y_m+1, Q_m+1	1	1	0	0	0
Y_m+1, Q_m	2	1	1	0	0
Y_m, Q_m+1	2	1	0	1	0
Y_m, Q_m	6	2	1	1	2.1
. . .					

Now consider the state at $[Y_m + 1, Q_m]$ which is orthogonal to the component of $(\lambda + 1, \mu + 1)$. Starting from this state, the irreducible representation $(\lambda - 1, \mu + 2)$ is constructed. We next construct $(\lambda + 2, \mu - 1)$, and are then left with two linearily independent combinations of states at $[Y_m, Q_m]$. Since every eigenvalue lattice has a hexagonal shape, and since in all other remaining states either Y or Q is smaller, both of these linear combinations must belong to (λ, μ). The Clebsch-Gordan coefficients for the reduction $(\lambda, \mu) \otimes (1,1) \supset (\lambda, \mu)$ are therefore not unique but contain a free parameter. That is, there are two independent sets of coefficients. The entire reduction can be obtained by continuing this process.

For triangular representations, the reduction is somewhat different: in particular, (λ, μ) is only obtained once, as sketched in Table III.

Table III

Multiplicities in $(\lambda, 0) \otimes (1,1)$

site	total	$(\lambda+1, 1)$	$(\lambda-1, 2)$	$(\lambda, 0)$
Y_m+1, Q_m+1	1	1	0	0
Y_m+1, Q_m	2	1	1	0
Y_m, Q_m+1	1	1	0	0
Y_m, Q_m	4	2	1	1
...				

The last terms in the following special examples can be identified by adding up the dimensions:

$$(1,1) \otimes (1,1) = (2,2) \oplus (3,0) \oplus (0,3) \oplus 2.(1,1) \oplus (0,0)$$

$$(3,0) \otimes (1,1) = (4,1) \oplus (2,2) \oplus (3,0) \oplus (1,1) \qquad (25)$$

They are also required for consistency with the reductions $(0,0) \otimes (1,1) = (1,1)$ and $(1,1) \otimes (1,1) \supset (3,0)$. The general rule is that if $A \otimes B \supset n \cdot C$, then $\bar{A} \otimes C \supset n \cdot B$; this is a consequence of the fact that in each case one deals with the same manifold of solutions to (13). (The practical applications of this rule are complicated by the need to define carefully the phases of the states[38,39] and the desirability of a canonical notation for the independent solutions.[37])

The commutation rules (4) and Eq. (20) have exactly the form that (13) takes in the reduction $(\lambda, \mu) \otimes (1,1) \supset (\lambda, \mu)$. This shows that the two vectors G_a and the component of N_a which is orthogonal to G_a provide the two sets of Clebsch-Gordan coefficients. However, N_a must be proportional to G_a in a triangular representation, since there is then only one independent set of reduction coefficients. In the eightfold representation, where G_a is given by the structure constants $F_{ab}{}^c$, N_a is given by the symmetrical quantities $D_{ab}{}^c$. Thus, in the coupling of pseudoscalar mesons to baryons, there are two independent terms, spoken of as "D" and "F" type[18,19] or as "symmetrical" and "antisymmetrical". These couplings are expressed in Table IV in terms of the components of normalized states. We have used the short hand notation $(N\pi)$ for a normalized $I = \frac{1}{2}$ states of nucleon and pion, etc., and denote by c and s the coefficients of the symmetrical and antisymmetrical couplings ($c = \cos\theta$, $s = \sin\theta$). This table may also be used for other octuplets, just by changing the names of the particles. Note that the Bose statistics require that the coupling of vector mesons to pseudoscalar pairs use only the antisymmetrical coupling ($c = 0$). Similar rules apply to other Boson couplings.

More complete tables of reduction coefficients have been calculated by many authors (References 15, 38-43). We caution the reader that these tables do not all use the same phase conventions. A computer program for the numerical evaluation of these coefficients has been developed by Moshinsky and Brody (private communication).

Table IV

Reduction coefficients for $(1,1) \otimes (1,1) \supset (1,1)$

$$N = c(2\sqrt{5})^{-1}\left[3(N\pi) - (N\eta) - 3(\Sigma K) - (\Lambda K)\right] + s/2\left[(N\pi) + (N\eta) + (\Sigma K) - (\Lambda K)\right]$$

$$\Xi = c(2\sqrt{5})^{-1}\left[+3(\Xi\pi) + (\Xi\eta) - 3(\Sigma\bar{K}) + (\Lambda\bar{K})\right] + s/2\left[-(\Xi\pi) + (\Xi\eta) - (\Sigma\bar{K}) - (\Lambda\bar{K})\right]$$

$$\Sigma = c(10)^{-\frac{1}{2}}\left[\sqrt{3}(N\bar{K}) - \sqrt{2}(\Sigma\eta) - \sqrt{2}(\Lambda\pi) + \sqrt{3}(\Xi K)\right] + 6^{-\frac{1}{2}} s\left[-(N\bar{K}) - 2(\Sigma\pi) + (\Xi K)\right]$$

$$\Lambda = 10^{-\frac{1}{2}} c\left[(N\bar{K}) - \sqrt{6}(\Sigma\pi) - \sqrt{2}(\Lambda\eta) - (\Xi K)\right] + 2^{-\frac{1}{2}} s\left[(N\bar{K}) + (\Xi K)\right]$$

DIRECT EXPERIMENTAL TESTS

The most direct way to confront the Gell-Mann-Ne'eman model with experiment is to see whether the apparent supermultiplets do have the content required by the irreducible representations of SU_3. This seems to be true; octuplets, a decuplet, and occasional singlets appear in the spectrum, with masses that are well separated from those of other particles with the same spin and parity (except in the case of the vector mesons). The spins and parities of all members of some supermultiplets are not yet measured, so SU_3 gives some obvious predictions. We shall comment further upon the assignments when we discuss empirical data.

Electromagnetic interactions. Coleman and Glashow[47] pointed out the importance of electromagnetic phenomena for testing whether the structures of particles in a supermultiplet are actually related as supposed.

A simple way to derive their results is given by a technique used by Wick (private communication) to obtain the Gell-Mann--Okubo mass formula, about which we will say more later. The electrical charge is proportional to $Q = -2H_3$; the form factors, and, in particular, the anomalous magnetic moments induced by SU_3-invariant strong interactions must also be SU_3-vectors pointing in the "Q direction". That is, they must have the form $AQ + BN_Q$, where N_Q is the component of the auxiliary vector $\underset{\sim}{N}$ which commutes with the three U-spin generators. Since N_Q is quadratic in the generators, it must be expressible in the form

$$N_Q = aU(U+1) + bQ^2 + cG^2 \tag{26}$$

In the triangular representations $(0, \lambda)$ and $(\lambda, 0)$, $U = \frac{1}{3}\lambda \pm \frac{1}{2}Q$, and $G^2 = \lambda(\lambda+3)/9$. Since these representations possess only one independent vector, N_Q must reduce to proportionality to Q, which suffices to relate the coefficients in (26):

$$N_Q = a\left[U(U+1) - \frac{1}{4}Q^2 - G^2\right] \tag{27}$$

In the baryon octuplet, p, Σ^+, Σ^-, and Ξ^- have $U = \frac{1}{2}$, and n and Ξ^0 have $U = 1$, but Λ and Σ^0 do not have a definite U-spin. In an octuplet, because it is the adjoint representation, the relation between the I-spin and U-spin eigenstates is exactly the same as the relation between the H_α and $H_\alpha{}^0$ given after Eq. (15):

$$|\Lambda\rangle = \frac{\sqrt{3}}{2}|U=1\rangle - \frac{1}{2}|U=0\rangle, \; |\Sigma^0\rangle = -\frac{1}{2}|U=1\rangle - \frac{\sqrt{3}}{2}|U=0\rangle \tag{28}$$

The relations among the moments of the metastable particles are:

$$\mu(p) = \mu(\Sigma^{+}) \quad , \quad \mu(\Xi^{-}) = \mu(\Sigma^{-})$$

$$\mu(n) = 2\mu(\Lambda) = \mu(\Xi^{0}) = -\left[\mu(\Sigma^{+}) + \mu(\Sigma^{-})\right] \tag{29}$$

Attempts have been made to measure $\mu(\Lambda)$, but the results are not yet conclusive.

Since the electromagnetic contribution to the masses are of second order, they have the tensorial properties of (1,1) $\otimes$ (1,1), but since they are also U-scalars, (3,0) and (0,3) terms don't appear. The most general form in an octuplet therefore has four terms:

$$\Delta M_{EM} = a + bQ + cQ^2 + dU\,(U + 1) \tag{30}$$

which leads to the relation

$$M(\Sigma^-) - M(\Sigma^+) = M(n) - M(p) + M(\Xi^-) - M(\Xi^0) \tag{31}$$

The experimental values,

$$8.0 \pm 0.4 \text{ MeV} \qquad \text{vs.} \qquad 6.9 \pm 1.4 \text{ MeV} \tag{32}$$

are in reasonable agreement. A further discussion of the electromagnetic mass differences is contained in the last section of this article.

Baryon-Meson Coupling Constants. - In the idealized situation in which mass differences within a supermultiplet are ignored, the coupling constants are proportional to Clebsch-Gordan coefficients. But some of the ways in which coupling constants can be defined are more sensitive to the mass differences than others, so judgment must be exercised in comparing predicted couplings with experiment. For example, the pseudoscalar vertex $\bar{u}_2 \gamma_5 u_1 = K$ takes, on the mass shell, the value $K = \sigma_p [\Delta M^2 - \mu^2]^{\frac{1}{2}}$, where ΔM is the difference in the baryon masses, μ is the meson mass, and σ_p is the spin component along the direction of the meson momentum. If the pseudoscalar couplings are assumed to be nearly equal, the residues of pion and kaon poles will differ by an order of magnitude.

When $\Delta M > \mu$, K is approximately equal to the momentum given to the meson when the heavier baryon decays, and in any simple picture of this (P-wave) decay process, the amplitude would be expected to be proportional to the momentum if the particles have a sufficiently small size. We argue that since the difference in the residues reflects an expected kinematical effect of the mass differences, the comparison with SU_3 symmetry should be made in terms of the pseudoscalar couplings. Another argument is that when the particles occur with large virtual momenta, their mass differences should be unimportant. This would occur when they are confined to a small region, as when they are bound together to form a composite. These remarks suggest that vertex functions, far from the mass shell, should be relatively insensitive to mass difference. We adopt this principle throughout this review, but a few comments are necessary about the reconciliation of this principle with polology.

Consider, for example, the pion or kaon exchange terms in baryon-baryon scattering:

$$T(k^2) = (\bar{u}_1 \gamma_5 u_3)(\bar{u}_2 \gamma_5 u_4) F_{13}(k^2) F_{24}(k^2)(k^2-\mu^2)^{-1} \qquad (33)$$

Our assumption is that the form factors $F_{ij}(k^2)$ are insensitive to the mass differences for $k^2 < 0$. Near the meson poles ($k^2 = \mu^2$) the kinematical effects give large departures from SU_3 symmetry, but we assume that these poles and the nearby branch cuts combine in such a way as to reduce the departures for $k^2 < 0$. At the present time, the application of these remarks to the kaon couplings is somewhat academic, since no sufficiently precise data are available for analysis. Kaon photo production data suggest that the kaon-nucleon couplings are too small, but the relation of the cross sections to coupling constants is open to question.

Some evidence about pion-hyperon couplings can be obtained from the study of hypernucleus binding energies. Possible assignments of hypernuclei to SU_3 representations have been considered,[46] but in this case SU_3 is a poor first approximation; as a result of the large sizes of hypernuclei and the universal repulsive cores, binding arises chiefly from pion exchange. Dalitz[47] and DeSwart[48] conclude that the pion couplings (pseudovector form) are: $f_{\Lambda\Sigma\pi} = 0.3 \pm 0.1$ and $f_{\Sigma\Sigma\pi} = 0.1 \pm 0.2$, compared with $f_{NN\pi} = 0.285$. (The errors represent my attempt to estimate the uncertainties produced by neglect of kaons, the vector octuplet, and the baryon resonances.) Supplying the appropriate conventional normalization factors to the entries in Table V, one obtains from SU_3 the relation $2f_{NN\pi} = \sqrt{3}\; f_{\Lambda\Sigma\pi} + f_{\Sigma\Sigma\pi}$. The compatibility of this relation with the hypernucleus information is hardly striking, but if SU_3 is assumed, the numbers may be used to restrict the F-D mixing angle to $(-10^\circ < \theta < 45^\circ)$.

<u>Vector octuplet.</u>— The covariant form of the coupling of vector mesons (V) to pseudoscalar pairs (P) is

$$i f V_\mu [P^+ \partial_\mu P - (\partial_\mu P^+) P] \qquad (34)$$

The decay rate is $\Gamma = \gamma_0 C^2 K$, where C is the Clebsch-Gordan coefficient obtained from Table V, and the kinematical factor is $K = p^3/M_V^2$, p being the barycentric decay momentum. The covariant density of states is $d\rho/dE = p/M_V^2$; the additional factor p^2 again shows the effect of the centrifugal barrier. The data[2] are presented in Table V.

Table V

Comparison of V-decays

Particle	C	Γ	K	γ_0
$\rho \to 2\pi$	2/3	100 MeV ± 10%	75 MeV	2.0
$K^* \to K + \pi$	1/2	50 MeV ± 10%	28.5 MeV	3.5
$\varphi \to K + \bar{K}$	1	3 MeV ± 30%	1.5 MeV	2.0

For a unitary singlet, two-P decay is forbidden. We have assumed, in making the table, that φ is a pure member of the octuplet, so if $\varphi - \omega$ mixing (18, 54-61) is large, $\gamma_0(\varphi)$ is increased, and the discrepancy lies with the ρ rather than the K^*. The data thus seem to indicate the presence of other admixtures as well. However, the widths are least sufficiently alike to show that these states have similar internal structures and should be considered together.

Baryon decuplet. - The $(\frac{3}{2}+)$ resonances are assigned to the representation (3,0). According to this assignment, the N^*(1238 MeV) should be coupled equally to the $(N\pi)$ and (ΣK) channels. However, near the nucleon-exchange pole, the scattering amplitude is purely $(N\pi)$; it cannot be changed much at the position of the resonance, because the Chew-Low plot doesn't show an appreciable curvature. Similarly, the Y_1^*(1385 MeV) should be 25% $(\Lambda\pi)$ and 17% $(\Sigma\pi)$, the remainder consisting of closed channels. Taking account of phase space and the p-wave momentum barriers, the ratio of the kinematical factors is found to be $K(\Sigma\pi)/K(\Lambda\pi) = 0.23$, whence SU_3 predicts $R = \Gamma(\Sigma\pi)/\Gamma(\Lambda\pi) = 0.16$, which is inconsistent with the measured value $R \lesssim 0.06$.

It is clear that SU_3 symmetry works very badly in these states, and that the observations cannot be explained without large admixtures being invoked.

<u>The second baryon octuplet.</u>- We describe under this heading a set of resonances which may actually not form an octuplet: the $(\frac{3}{2}-)$ N' (1512 MeV), the $(\frac{3}{2}-)$ Λ' (1520 MeV), the $(\frac{3}{2}?)$ Σ' (1660 MeV), and an unobserved Ξ'. The parity of the Σ' has been reported as (+)[48], but it is possible that this determination has been affected by interference with a background amplitude. The proposal has been made (Sakurai,[49] Martin,[50] Martin and Wali[51]) that the 1520 MeV Λ' is a unitary singlet, and that the other resonances are either part of an octuplet, or are really just extraneous bumps in the cross section arising from, perhaps, the Peierls[52]-Nauenberg-Pais[53] mechanism.

The kinematical factor governing the decay of a $d_{3/2}$ state is $K = p^{2\ell+1} / [1 + a^2 p^2]^{+n}$ with $\ell = 2$; the extra factor $(1 + a^2 p^2)^{-n}$ has been introduced to allow for a finite radius of interaction.[54,55] The radius is related to a, but only heuristically. Glashow and Rosenfeld[54] used $n = \ell$, $a^{-1} = 350$ MeV. This ad hoc radius correction can be meaningful only if it does not provide a large correction. The only important effect of the parameter a is that it allows the total width of the Λ' to be adjusted to the N' and Σ' widths. If Λ' is a singlet, its width is independent, but in that case the parameter a is not needed for fitting the data.

The factors $C^2(8)$ in Table VI are taken from Table IV. A singlet decays with equal weight into each particle combination, which determines the coefficients $C^2(1)$. We may note that very small admixtures (about 10% in the amplitude) of either the octuplet or singlet amplitude into the other would upset the agreement with the measured widths of the Λ'.

Table VI
(Taken from Glashow and Rosenfeld[54], $\theta' = 35^\circ$)

Particle	p(MeV)	$C^2(8)$	$(C^2(1))$	Γ_{th}(MeV)	Γ_{exp}(MeV)
$N' \to N\pi$	450	.700		67	80
$\Sigma' \to N\bar{K}$	406	.045		3	3
$\Sigma' \to \Lambda\pi$	441	.135		11	11
$\Sigma' \to \Sigma\pi$	386	.220		13	13
$\Lambda' \to N\bar{K}$	244	.445	(2/8)	6 (4)	5
$\Lambda' \to \Sigma\pi$	267	.405	(3/8)	8 (9)	9

The measured widths can be fit satisfactorily using SU_3 coefficients, but this must be considered as a provisional confirmation until the parity of the Σ' and the existence of the Ξ' have been settled, and until the dynamical structures of these states have been clarified.

<u>Cross section relations</u>.- It is difficult to find meaningful cross section relations for two reasons: there are usually many independent amplitudes that contribute to a given type of reaction, and the predictions can be upset by coincidence with resonances. Meshkov, Snow, and Yodh[56] have suggested that the reactions $P + B \rightarrow P + B^*$ may bypass these problems. According to Eq. (25), if $\ell \geq 2$, only the (3,0) and (2,2) representations occur in common, so there must be a relation between any four amplitudes. The relation between the amplitudes for the reactions

$$a = K^+ + p \rightarrow N^{*(++)} + K^0$$

$$b = \pi^+ + p \rightarrow N^{*(++)} + \pi^0$$

$$c = \pi^+ + p \rightarrow N^{*(++)} + \eta$$

$$d = \pi^+ + p \rightarrow Y_1^{*(+)} + K^+$$

is found to be

$$|T_a|^2 + 3|T_d|^2 = |T_b|^2 + 3|T_c|^2 \qquad (35)$$

They compare the cross sections measured at equal energies above the threshold, after dividing out kinematical factors, and find good agreement. It turns out that $|T_a|^2$ and $|T_b|^2$ are nearly equal and large especially about 1/3 BeV above threshold, which implies that the (3,0) and (2,2) amplitudes are nearly equal and not more than 30° different in phase.

For comparison, we may note that the V exchange model gives (using SU_3), $T_c = 0$ and $T_b = 2\ T_a$. However, we see from Table V that the $(\rho\pi\pi)$ coupling constant might really be too small, so it might be possible to interpret the reaction as a peripheral one if the true ρ couplings are used. The difficulty with the peripheral picture is that it is hard to see why reaction d should be so weak.

DYNAMICAL CONSIDERATIONS

The mass formula. - The mass relations introduced by Gell-Mann[18,19] and generalized by Okubo[57] have two important aspects. They are a supplement to the model, in which the regularities apparent among the supermultiplet splittings are correlated. Secondly, the possibility of constructing such a mass formula provides indirect confirmation for the general approximate validity of SU_3.

We describe mass splittings within an isolated super multiplet through their tensorial properties.[58] That is, when the particles a of a supermultiplet (λ, μ) are subjected to an SU_3 transformation, the mass is treated as a matrix $M_a{}^b$ transforming as the direct product $(\mu, \lambda) \otimes (\lambda, \mu)$. With exact SU_3 symmetry, the masses would be exactly degenerate ($M_a{}^b = \bar{M}\, \delta_a{}^b$) and only the one-dimensional component (0,0) of the direct product would be obtained. With broken symmetry, we have a sum of the form

$$M_a{}^b = \bar{M}\, \delta_a{}^b + \sum_{x,r} A(x,r) C_{xa}{}^b(r) \tag{36}$$

where x is a component of the representation $r \subset (\mu, \lambda) \otimes (\lambda, \mu)$, and $C_{xa}{}^b(r)$ is the explicit Clebsch-Gordan coefficient. Some restrictions on the coefficients $A(x,r)$ are given by Hermiticity and charge conjugation invariance. Moreover, since Q and Y are conserved, x must stand for a $Q = Y = 0$ member of the representation, and if we ignore the electromagnetic splittings, it must be the unique component with $I = 0$, which also restricts (r) to be of the form (λ', λ').

Since the S-matrix elements involving the various particles are connected by dispersion relations, a dissymmetry introduced into any amplitude must induce dissymmetries in all the others as well. The splittings of the masses in all the supermultiplets must therefore have the same tensorial properties, provided the splitting is small enough that first order perturbation theory can be used. Similarily, the coupling constant deviations, which can be interpreted as representation admixtures, must be associated with the same dissymmetry representations (r). The representations which can be mixed into a supermultiplet (λ, μ) are those contained in the product $(\lambda, \mu) \otimes (r)$.

The Gell-Mann-Okubo formula rises from the assumption that in Eq. (36) only the (8) dissymmetry term is important. Then we can rewrite (36) as

$$M = \bar{M} + A\,Y + B\,N_Y \qquad (37)$$

The operator N_Y is obtained from N_Q (Eq. (27)) by replacing Q by Y and U by I, so we obtain a formula of the type

$$M(I,Y) = m + a\,Y + b\left[I(I+1) - \frac{1}{4}Y^2 \right] \qquad (38)$$

The baryon ($\frac{1}{2}$ +) octuplet and ($\frac{3}{2}$ +) decuplet satisfy this relation amazingly well. The squared masses of the pseudoscalar mesons also obey the formula. Mixing between the singlet and the I = 0 member of the octuplet (ϕ - ω mixing) could distort the formula for the vector mesons.[59-61]

The mass differences within I-spin multiplets can also be described by Eq. (36) if the $I \neq 0$ tensorial components are retained in the sum. Capps[63] and also Coleman and Glashow[64] have remarked that the (8) tensor components are also the most important terms in these additional splittings, which are supposedly of electromagnetic origin. Coleman and Glashow point out that this characterization is more exact if we first separate out an "external" electromagnetic energy $\mathcal{E}$. This "external" mass may be interpreted as the contribution of the electromagnetic field energy residing outside the particle in question (plus the energy of recoil upon emission of a virtual photon, computed for particles which recoil rigidly, that is, are not virtually excited). This energy, by itself, would make the proton heavier than the neutron, the charged K's heavier than the neutral ones. The "eightfold rule" then applies more accurately to the remaining "internal" part of the electromagnetic mass, as one ought to expect.

Since the $Q = Y = 0$ members of the (8) representation have only the values $I = 0, 1$, the masses in a given multiplet must be displaced in proportion to I_z . Other relations can be obtained from the generalized mass formula, written as follows:

$$M = \bar{M} + AY + BN_Y + A'Q + B'N_Q + \mathcal{E} \tag{39}$$

For the pseudoscalar mesons, we again replace M by M^2. Coleman and Glashow also show that the empirical ratios A'/A and B'/B in the baryon octuplet, as well as B'/B in the meson octuplet, are well approximated by the same universal value $R \gtrsim 1/50$. The mass differences within every supermultiplet are therefore seemingly characterized by a single SU_3 vector pointing in a certain direction.

The fact that the pseudoscalar mesons satisfy a mass-squared relation can be partially explained by the observation that the Bethe-Salpeter equation depends on the squared mass of the bound state.[65] This is an incomplete explanation, because the P mass is also one of the input parameters in the B-S equation; a P can be composed of three P's, and P's can be exchanged among the constituents. In order to not upset the mass-squared relation, we must assume that only P's having very high momenta contribute to the internal structures of themselves and other particles. The reconciliation of a quadratic relation for bosons with a linear relation for baryons is therefore not obvious, but provides some information about the dynamics.

The Gell-Mann-Okubo relation in the decuplet needs further examination. We remember that in the dynamical model of the N^*, continuum states lying relatively low are important, so that higher order perturbations in the masses as well as configuration mixing should be especially strong. In fact, we have already seen that the empirical branching ratios suggest that the decuplet is very impure. We shall return to this point later.

Crossing Symmetry. - Further information about the possible supermultiplets is obtained by making use of the crossing symmetry of scattering amplitudes. In order to do this in a practical way, we have to introduce dynamical models, in which it is assumed that the main properties of the force between two particles can be found by considering the exchange, one at a time, of the lighter bound or resonant states in the crossed channels.

The obvious place to begin such a study is with the ($\frac{3}{2}$+) (PB) resonances, which in the Chew-Low model[66,67] arise from the B exchange force. In this model, the relative strength of the force in different supermultiplets depends only on the mixing angle θ. It turns out that the forces are most attractive in the decuplet (3,0) for $25^\circ \lesssim \theta \lesssim 60^\circ$, while both this decuplet and a singlet of similar energy could occur for $5^\circ \lesssim \theta \lesssim 25^\circ$.[55,68-70] If the Y_0^* should be a ($\frac{3}{2}$+) state, we would conclude that θ lay in the second range; otherwise, the first range is selected. It is noteworthy that resonances in the (2,2) representation could never lie lowest, although there is a weaker attraction in these states.

Another way to view the existence of the decuplet is as follows. From the masses of the four isotopic multiplets, we obtain three equations relating the eleven P-B coupling constant ratios. The non-existence of other ($\frac{3}{2}$+) resonances gives in addition a large number of inequalities among the ratios. These relations are, of course, all consistent with SU_3.

Let us next turn to the vector octuplet, and see how the Chew-Mandelstam[71] bootstrap mechanism for the ρ-mesons generalizes. Here we can use a simple extension of familiar isospin tricks. If a V is exchanged between two P's, the coupling at each vertex is given by the $F_{ab}{}^c$, so to get the relative values of the forces in different representations (r), we have to evaluate the quantity $M(r) = \underset{\sim}{G}(A) \cdot \underset{\sim}{G}(B)$ for the case A and B are both octuplets.[72] Using the fact that $C(r) = G(A) + G(B)$, we have

$$2\,\underset{\sim}{G}(A) \cdot \underset{\sim}{G}(B) = G^2(r) - G^2(A) - G^2(B) \tag{40}$$

Putting in the values of $G^2(\lambda, \mu)$ from Eq. (23) or Table I, we see that M is most negative (the force is most attractive) in the states with small multiplicities. The strongest attraction $(M = -1)$ is in the singlet state, which is symmetrical in the two mesons and can be identified with the f_0 and the Pomeranchuk trajectory. Attraction also occurs for (r) = (8) $(M = -\frac{1}{2})$; the antisymmetric (8) can be identified with the vector mesons themselves (and the symmetric (8) possibly with a scalar octuplet which has not yet been found).

If we include also 2V states, interacting via V exchange and being coupled to 2P states via P exchange, the vertices are still all given by the $F_{ab}{}^c$, so our conclusions about the relative values of the forces in different supermultiplets are unchanged.[72] These graphs are allowed by the Bronzan-Low selection rule,[73] in which P and ω are assigned a quantum number $A = -1$, while V and γ (the photon) are assigned $A = +1$. Similar graphs might play an important role in a bootstrap model of the P's.

The V bootstrap model is actually sufficiently restrictive to determine the coupling constants. If one is given the existence of isobaric octuplets, and neglects other particles, one can obtain a sufficient number of equations to force the coupling constants to have the SU_3 ratios.[74-76]

To account for the ($\frac{1}{2}$+) baryons themselves, we may consider the graphs shown in Fig. 5. We can have two models, according to whether graph A[77] or the graphs B[72] are assumed to be the most important. Undoubtedly, models A and B both contribute, as well as other graphs, but it is not easy to decide on their relative importance, because the vertices must be supplied with form factors which will modify their strength by unknown amounts. The algebra involved in calculating relative values of the forces is more complicated than that which leads to Eq. (40), so we do not go through it here. According to model A which is an extension of Chew's bootstrap model of the nucleons,[77] the potential is most attractive in the ($\frac{1}{2}$+) octuplet, and leads uniquely to $\theta = 33^\circ$.[51,55,78,79]

Model B also leads to an octuplet, and to a slightly different value of Θ which can be calculated easily. The V-B vertex has both F and D terms. The F term alone gives, according to (40), the same potential in both octuplet components, but the D term, being combined with an F at the other end of the line, converts an antisymmetric octuplet to a symmetric one, and vice versa. The self-consistent P-B coupling is then determined by the equation

$$\begin{pmatrix} V(F) & V(D) \\ V(D) & V(F) \end{pmatrix} \begin{pmatrix} \cos\Theta \\ \sin\Theta \end{pmatrix} = V_{eff} \begin{pmatrix} \cos\Theta \\ \sin\Theta \end{pmatrix} \qquad (41)$$

to be $\Theta = 45^{\circ}$. The fact that in both extreme cases one obtains a value of Θ which is compatible with empirical values in comforting, both for SU_3 and for the bootstrap idea.

Different predictions about the ($\frac{3}{2}$ -) states are obtained from models A and B. Martin and Wali[51] and Carruthers[79] show that A plus B1 doesn't lead to a ($\frac{3}{2}$ -) octuplet, but can give a singlet. Model B (being essentially the Ball-Frazer[80]-Cook-Lee[81] model) can give an octuplet[55]; existence of an octuplet resonance could conceivably be "helped along" by the PNP mechanism.[79] [52,53] If we suppose that octuplet ($\frac{3}{2}$ -) resonances do exist, they might decay through the three-step processes drawn in Figure 6. In mechanism A there is a (PB) intermediate state, and in mechanism B a (VB) intermediary, leading to characteristic PB'B mixing angles θ'. The combinations of coupling constants are similar to those occurring in Fig. 5. If we use the fact that when a matrix is multiplied into an approximation to its dominant eigenvector a better approximation results, we obtain from mechanism A along, $|\theta' - 33^\circ| \leq |\theta - 33^\circ|$, and from B alone, $|\theta' - 45^\circ| \leq |\theta - 45^\circ|$. (Reasonable dynamical assumptions have been made. The two mechanisms automatically interfere constructively in order that they may both contribute attractive forces leading to the existence of the resonance.) These estimates of θ' agree reasonably well with the one used in Table VI. We have, therefore, a somewhat paradoxical situation, in which we are not yet sure either theoretically or experimentally whether this octuplet exists, but if it does exist, the decays are reasonably well understood!

The fact that SU_3 symmetry can be incorporated naturally into simple dynamical models provides additional indirect confirmation of its approximate validity, especially as it leads to additional restrictions on the arbitrary parameters in the coupling constants. In contrast, it may be noted that some of the other symmetry schemes proposed in the past are inconsistent with these dynamical models.

<u>Decuplet perturbations.</u>- The $(\frac{3}{2}+)$ resonances are coupled to (BP) continuum states whose thresholds have been so widely separated by the mass differences that these perturbations can be expected to have very large effects. A detailed study of the dependence of resonance pole positions upon the mass differences can be very involved and confusing.[82-87] Fortunately, this question is circumvented if one works directly with the partial wave dispersion relations, which automatically maintain the correct analytic structure of the scattering matrix and also allow a simple treatment of the forces which generate the resonances.[88] The scattering amplitude is then written in the form $F = D^{-1} N$, and for a narrow resonance, the position is given by the energy at which the matrix

$$\mathrm{Re}D\,(E) = 1 - \pi^{-1}\, P \int N'\, \rho' dw' \,(w' - E)^{-1} \qquad (42)$$

has a vanishing eigenvalue, and the reduced widths are given by the corresponding eigenvector ϕ :

$$\phi = \pi^{-1}\, P \int N'\, \rho' dw' \,(w' - E)^{-1}\, \phi \qquad (43)$$

In accordance with the expectation that virtual particles are less influenced by mass differences, it is natural, in preliminary treatments, to assume that N satisfies SU_3 symmetry and to examine the influence of the masses as exerted through ρ.[55,89,90] It has also been usual to take N to be proportional to the one baryon exchange potential. Admixtures of components of the (2,2) representation into the decuplet are then especially large, for two reasons: there is a moderate attraction in these states (about 1/2 of that in the (3,0) states), and the mass differences among the baryons and among the mesons contribute constructively to the matrix elements between these states.

The Y = 0, I = 1 states are

$$\psi(3,0) = 6^{-\frac{1}{2}} \left[(N\bar{K}) + (\Sigma\pi) - (\Xi K) \right] + \frac{1}{2} \left[(\Lambda\pi) - (\Sigma\eta) \right]$$
$$\psi(2,2) = 5^{-\frac{1}{2}} \left[(N\bar{K}) + (\Xi K) \right] + \left(\frac{3}{10}\right)^{1/2} \left[(\Lambda\pi) + (\Sigma\eta) \right] \qquad (44)$$

Note that in a linear combination of the form $\phi = \psi(3,0) + (5/6)^{1/2}\, r\, \psi(2,2)$ the amplitudes of the two most massive components, (ΞK) and $(\Sigma\eta)$, are both reduced when $r > 0$. In fact, for $r = 1$, they are simultaneously eliminated. If the Y_1^* is such a linear combination, the branching ratio is

$$\Gamma(\Sigma\pi)/\Gamma(\Lambda\pi) = 0.16\,(1 + r)^{-2} \qquad (45)$$

which is compatible with the experimental data if $r > 3/5$. For a simple estimate of r, we suggest that $\rho\phi$ might be characterized by $r \approx 1$, and then $N\rho\phi$, or ϕ itself, by $r \approx \frac{1}{2}$. This value is sufficiently close to the tolerable range that it is reasonable to believe that most of the needed reduction in the branching ratio does occur through this simple mechanism. Similar arguments applied to the N^{*} lead to compatibility with the Chew-Low formula. An important problem which remains is that of trying to infer the extent to which the martix N must be shifted from the SU_3-symmetric form. Unfortunately, this would require very refined calculations.

To express Eq. (42) in a form which admits a simple physical interpretation, let us write

$$\pi^{-1} P \int N' dw' \rho' (w' - E)^{-1} = - \bar{V} (\bar{E} - E)^{-1} \qquad (46)$$

where $\bar{V}$ and $\bar{E}$ represent appropriate averages of $(-N)$ and w'. The energy E is then determined as an eigenvalue of the matrix $\bar{E} + \bar{V}$. Reasonable physical assumptions about $\bar{E}$ and $\bar{V}$ then lead to a nearly linear relation between Y and E, even though the admixtures in the eigenvectors are so large that perturbation theory doesn't apply.[90] In fact, Warnock and Wali,[89] by integrating (41) numerically with an SU_3-symmetric N, find a more exact linearity than any approximate discussion would lead one to hope for. In consequence, the agreement between the Ω -mass and that predicted by the mass formula, while hardly a vindication of first order perturbation theory, is nonetheless a considerable triumph for the deeper interpretation of the eightfold way.

Outlook.- Our confidence in the validity of the eightfold way arises primarily from the gross appearence of the mass spectrum, and also from indirect evidence about coupling constants gleaned through dynamical arguments. The direct evidence on coupling constants has shown primarily that the physical states suffer large admixtures. We have mentioned in the Introduction that a powerful motive for investigating symmetry schemes is that our dynamical models are still too crude to enable us to progress without the simplification which a symmetry provides. We see now that the complementary situation also holds: we cannot weigh the evidence for a symmetry scheme without making some use of admittedly crude dynamical calculations.

A more intimate connection between the symmetry groups and dynamical theory occurs in some recent developments. One class of such theories may be described as "elementary particle models" because they have in common the idea that the particles so far observed are all composites of more fundamental objects. A motivation suggested for this idea is the possibility of utilizing all of the representations of SU_3, including the simplest one, which is three-dimensional. Among such theories we have Schwinger's model,[91] the Gürsey-Lee-Nauenberg model,[65] and the models proposed by Gell-Mann,[92] Zweig,[93] and Hara.[94]

The bootstrap concept transcends older approaches to the fusion of ideas about dynamics and symmetry by seeking a physical mechanism for the origin of the symmetry as well as for the particular way the symmetry is broken.[74-76, 90, 95] The so-called "spontaneous breakdown" theories are a variant which share the aim of deriving from dynamical arguments the nature of the departures from symmetry, but start from a postulated underlying exact symmetry.[58, 96, 97]

The elementary-particle and bootstrap approaches can both explain a curious fact about the mass deviations -- that they are characterized by an SU_3-vector which has the same direction in all supermultiplets. In an elementary particle model, one could say that this vector described the mass deviations of an elementary triplet. In a bootstrap model, a characteristic vector emerges which is a property not of any specific supermultiplet, but of the entire complex of particles; that is to say it is a property of the bootstrap mechanism.[64, 98] Both of these approaches also make definite additional predictions. An elementary-particle model of course supposes that the elementary particles will eventually be found. The bootstrap picture does not preclude the existence of new kinds of particles, but it does imply that the ratios of masses can be calculated without additional assumptions or parameters being introduced.

While it is still too early to assert that dynamical calculations have determined the origin of particle symmetries, it is clear that they have provided an essential part of the evidence for the validity of SU_3, and we may safely anticipate that similar dynamical arguments will be equally important in helping to assess super-symmetry schemes which include SU_3 as a subgroup. We close by drawing attention to some of these schemes. In the Bronzan-Low[73] scheme, SU_3 is augmented by a discrete symmetry. Some suggested continuous super-symmetries are the SU_4 models of Tarjanne and Teplitz[99] and of Hara[94], and Schwinger's $SU_3 \otimes SU_3$ model[91].

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CAPTIONS FOR FIGURES

Figure 1. Strongly-interacting boson states. The states which are well established and have a clear interpretation in the eightfold way are denoted by a horizontal line. The spin and parity are also noted. The states which are not so well established, or whose assignment is still in doubt, are denoted by an X. See references 1-3 and 21-27.

Figure 2. Strongly-interacting baryon states. The notation is the same as in Fig. 1.

Figure 3. Left, the eigenvalue lattice of the $j = \frac{1}{2}$ (+) baryon octuplet. Right, the root diagram for SU_3.

Figure 4. Eigenvalue lattices for the representations $(3,0) \equiv (10)$ and $(2,2) \equiv (27)$.

Figure 5. Dynamical models for the $\frac{1}{2}$ (+) baryons. These graphs represent the virtual dissociation of a baryon into its constituents, which subsequently interact by exchanging a particle. Solid lines denote the $\frac{1}{2}$ (+) baryons; the double line, the $\frac{3}{2}$ (+) baryon resonances; the dotted lines, the pseudoscalar mesons; and the wavy lines, vector mesons.

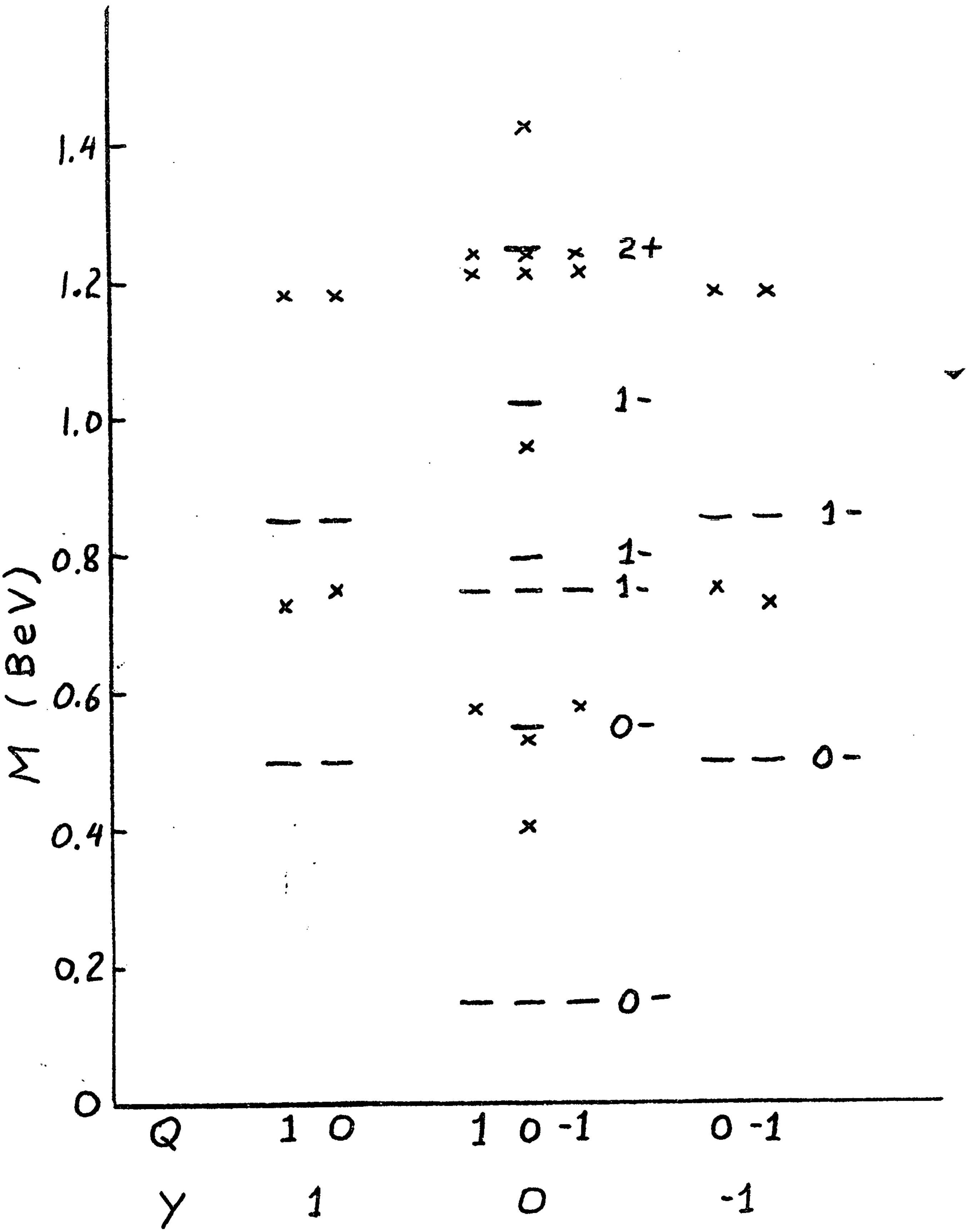

Figure 1

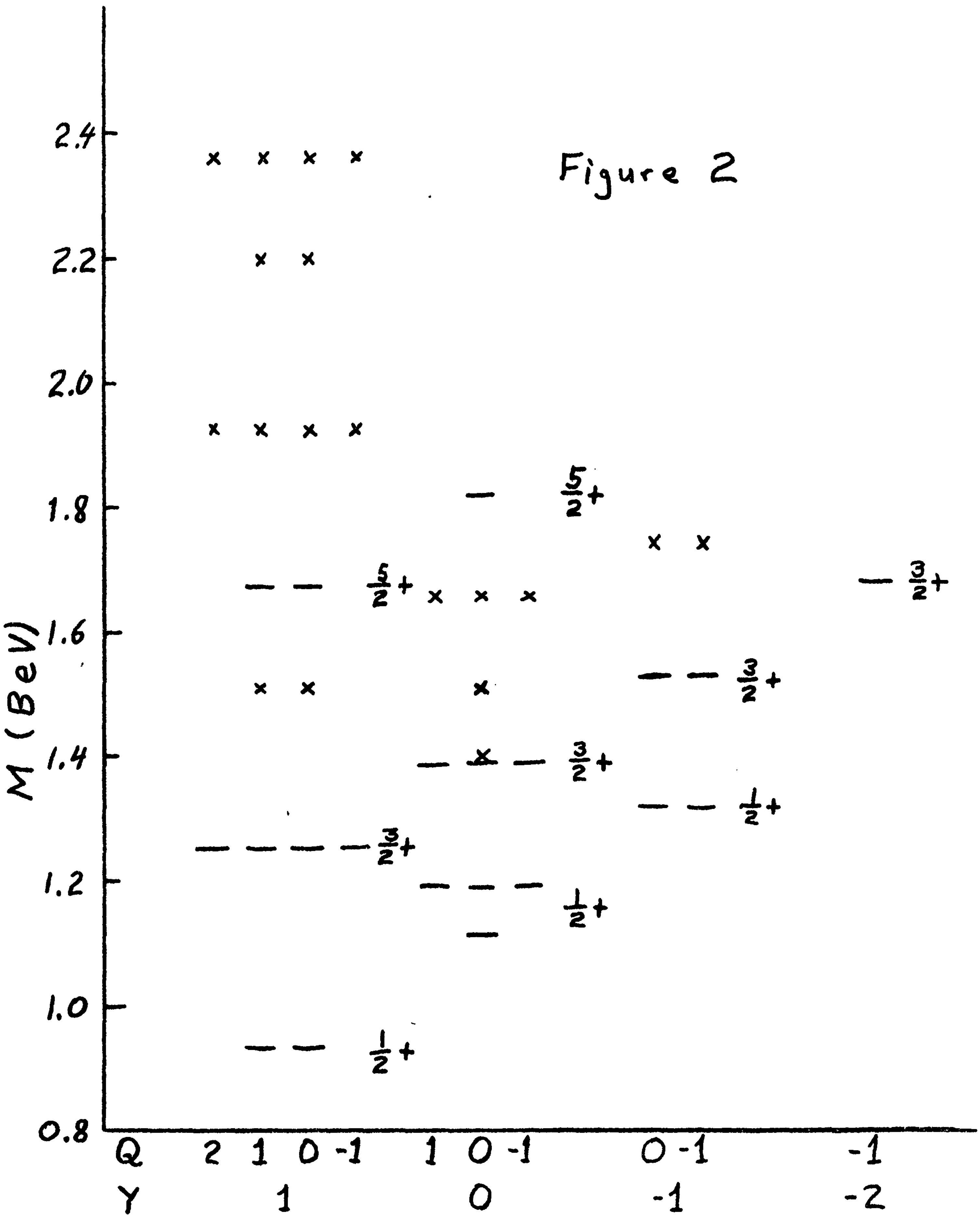
Figure 2
M (BeV)
2.4
2.2
2.0
1.8
1.6
1.4
1.2
1.0
0.8
5/2 +
5/2 +
3/2 +
3/2 +
3/2 +
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1/2 +
1/2 +
1/2 +
Q 2 1 0 -1 1 0 -1 0 -1 -1
Y 1 0 -1 -2

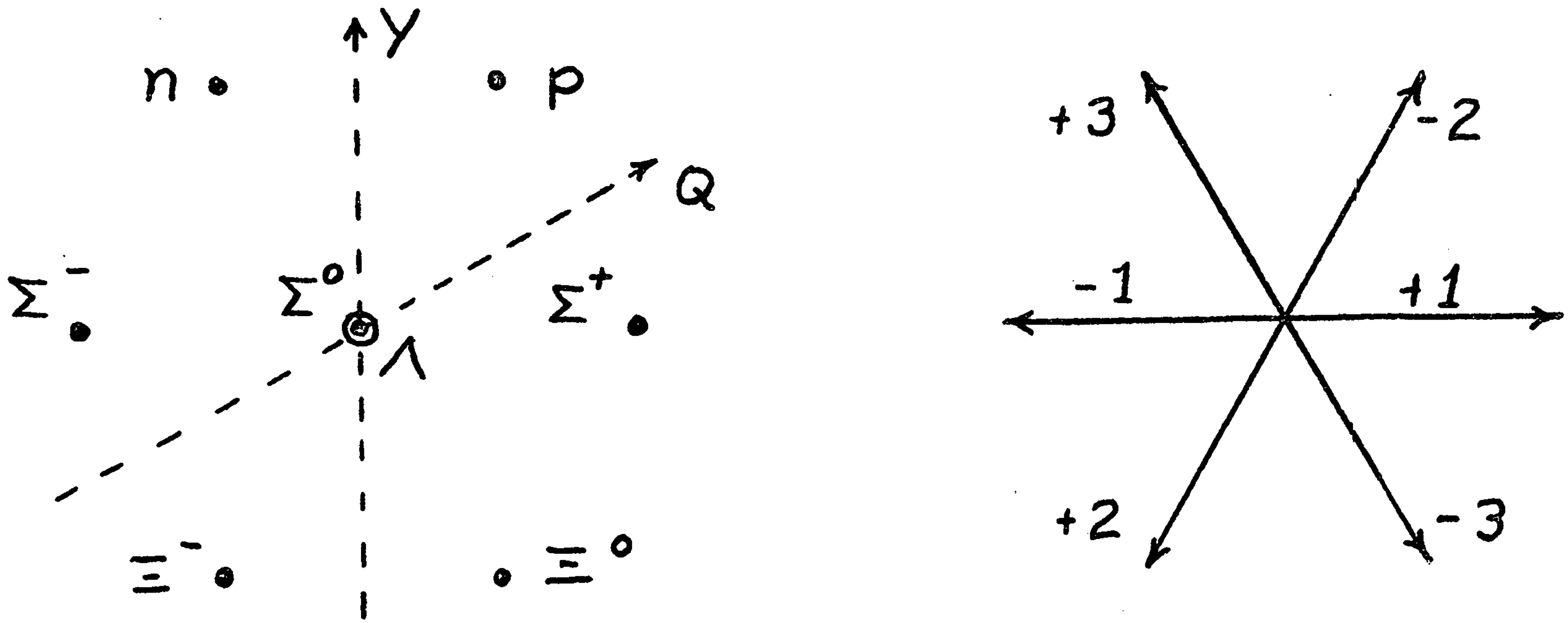

Figure 3

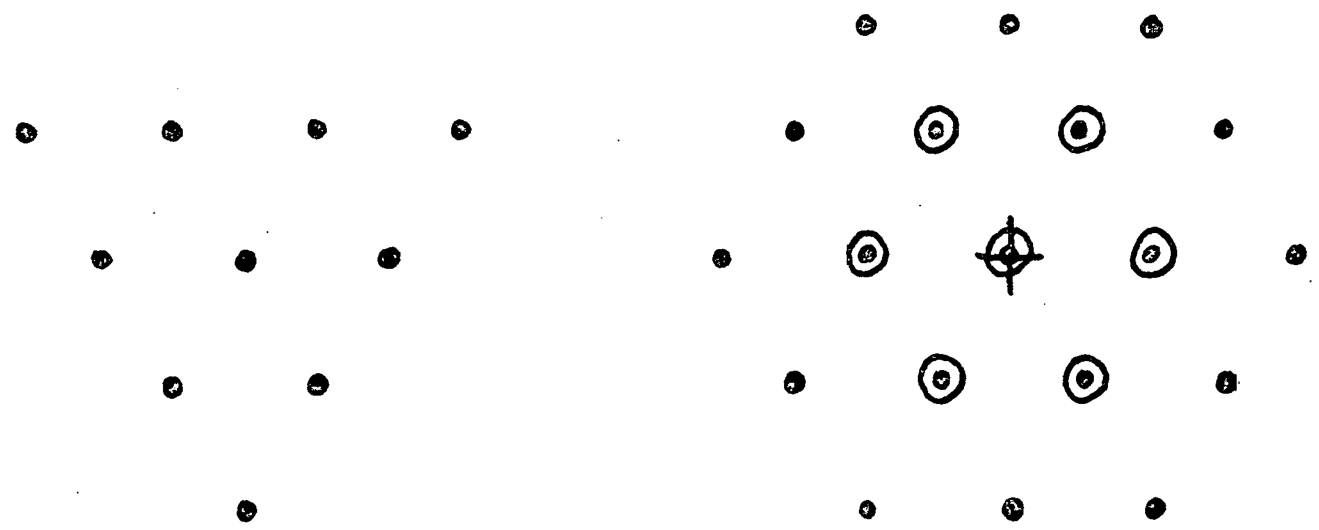

Figure 4

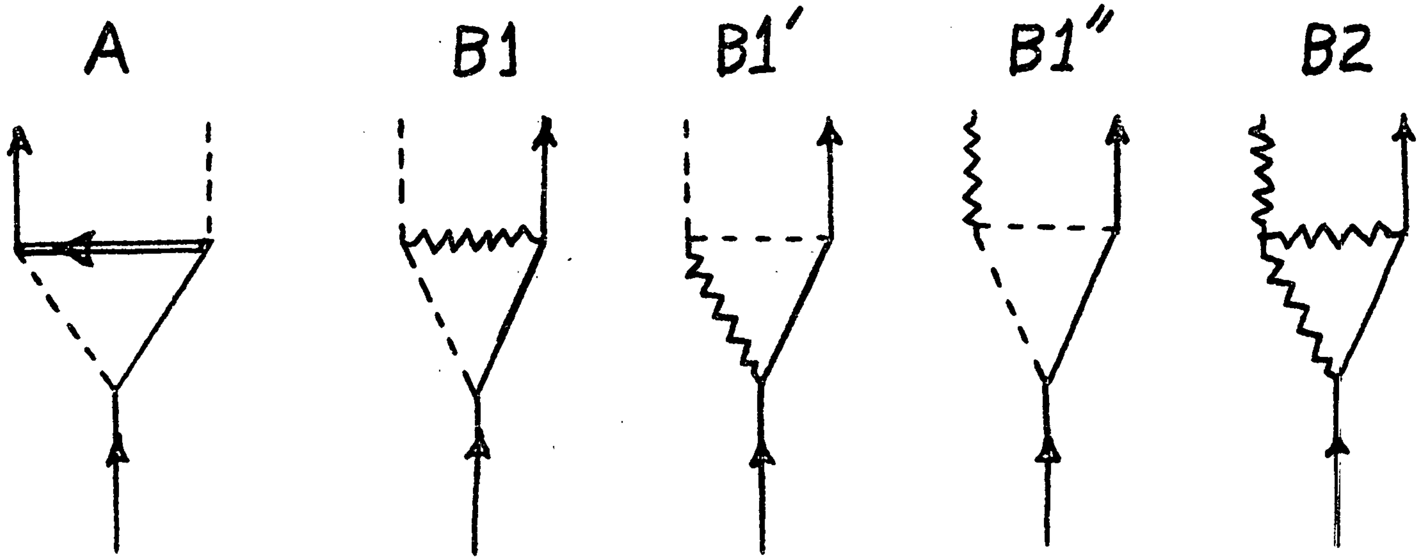

Figure 5