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**Particle Motion Inside and Near a
Linear Half-Integer Stopband**

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ABSTRACT

This paper studies the motion of a particle whose tune is inside and near a linear half-integer stopband. Results are found for the tune and beta functions in the stable region close to an edge of the stopband. Results are found for the growth rates and for the exponentially growing particle motion inside the stopband. It is shown that the eigenvalues and the eigenfunctions of the transfer matrix are real inside the stopband. All the results found are also valid for small accelerators where the large accelerator approximation is not used.

1. Introduction

Inside a linear-half integer stopband the particle motion can be unstable and grow exponentially. A result is found for the growth parameter g which is the imaginary part of the tune. It is shown that the real part of the tune inside the stopband is constant at the value $q/2$, where $\nu = q/2$ is the center of the stopband, q being some integer. The proof given does not depend on perturbation theory, and the result follows from the symplectic properties. The eigenvalues and eigenfunctions of the transfer matrix are shown to be real inside the stopband. In the stable region near an edge of the stopband, the tune varies rapidly and the beta function becomes infinite as the unperturbed tune approaches the edge of the stopband. Results are found for the tune and beta function in the stable region

near an edge of the stopband. It is found that the beta function becomes infinite inversely as the square root of the distance of the unperturbed tune from the edge of the stopband. The basic equations used are also valid for small accelerators, where the large accelerator approximations is not used, and all the results found are valid for small accelerators.

2. Results when the tune is not in the stopband

Before treating the interesting case where the tune is inside the stopband, it will be helpful to first treat the case where the tune is not inside the stopband. It will then become clear where the perturbation solution breaks down, when the tune is inside the stopband, and how the perturbation solution can be repaired.

It will be assumed that in the absence of the perturbing fields, the tune of the particle is ν_0 and that the motion is stable when ν_0 is close to $q/2$, where q is an integer.

It is assumed that a perturbing field is present which is given on the median plane by

$$\Delta B_y = -G(s)x \quad (2-1)$$

$G(s)$ is periodic in s and contains the field harmonics that can excite the stopband around $\nu_0 = q/2$.

Introducing η defined by

$$\eta = x/\beta^{1/2}, \quad (2-2)$$

where β is the beta function of the unperturbed field, the equation of motion can be written as

$$\begin{aligned} \frac{d^2\eta}{d\theta^2} + \nu_0^2\eta &= f \\ f &= \nu_0^2\beta^{3/2}\Delta B_y/B\rho \\ f &= -\nu_0^2\beta^2G\eta/B\rho \end{aligned} \quad (2-3)$$

$$B\rho = pc/e, \quad d\theta = ds/\nu_0\beta$$

Eqs. (2-3) are valid for large accelerators, and a small change is required¹ to make them valid for small accelerators (see section 7). However, the final results found below are valid for small accelerators that require the use of the exact linearized equations. This is shown in section 7.

Eq. (2-3) can be written as

$$\begin{aligned}\frac{d^2\eta}{d\theta^2} + \nu_0^2 &= -2\nu_0 b(\theta) \eta \\ b(\theta) &= \frac{1}{2}\nu_0\beta^2 G/B\rho\end{aligned}\quad (2-4)$$

Because $b(\theta)$ is periodic a solution for η will have the form

$$\eta = \exp(i\nu_s\theta) h(\theta) \quad (2-5)$$

where $h(\theta)$ is periodic. It is assumed that the tune ν_0 will change to ν_s because of the perturbing field. Thus η can be assumed to have the form

$$\begin{aligned}\eta &= A_s \exp(i\nu_s\theta) + \sum_{r \neq s} A_r \exp(i\nu_r\theta) \\ \nu_r &= \nu_s + n\end{aligned}\quad (2-6)$$

where n is some integer but $n \neq 0$. For a zero perturbing field the solution for η is $\eta = A \exp(i\nu_0\theta)$. Thus for small perturbing fields it can be assumed that

$$\begin{aligned}\nu_s &\simeq \nu_0 \\ A_r &\ll A_s \text{ for } r \neq s\end{aligned}\quad (2-7)$$

Putting Eq. (2-6) into Eq. (2-4), one obtains a set of equations for the A_r

$$\begin{aligned}(\nu_r^2 - \nu_0^2) A_r &= 2\nu_0 \sum_{\bar{r}} b_{r\bar{r}} A_{\bar{r}} \\ b_{r\bar{r}} &= \frac{1}{2\pi} \int_0^{2\pi} d\theta b(\theta) \exp(-i\nu_r\theta + i\nu_{\bar{r}}\theta) \\ \nu_r &= \nu_s + n\end{aligned}\quad (2-8)$$

Eqs. (2-8) are a set of homogenous equations for the A_r . The condition required for a solution to exist is that the determinant of the coefficients of the A_r should vanish. This condition will determine ν_s , and then Eq. (2-8) can be solved for the A_r in terms of A_s . It is more convenient to solve Eqs. (2-8) using an iterative perturbation procedure. For the initial guess for η in this iterative procedure, one can assume

$$\eta = A_s \exp(i\nu_s\theta) \quad (2-9)$$

One can put this result for η in the right hand side of Eq. (2-8), and solve for A_r which gives

$$\begin{aligned}(\nu_r^2 - \nu_0^2) A_r &= 2\nu_0 b_{rs} A_s \\ \nu_r &= \nu_s + n\end{aligned}\quad (2-10)$$

For $r = s$, Eq. (2-10) gives

$$(\nu_s^2 - \nu_0^2) A_s = 2\nu_0 b_{ss} A_s \quad (2-11)$$

which determines ν_s . One finds

$$\nu_s - \nu_0 = b_{ss} \quad (2-12a)$$

which can be written as

$$\begin{aligned} \nu_s - \nu_0 &= b_0 \\ b_0 &= \frac{1}{2\pi} \int_0^{2\pi} d\theta b(\theta) \\ b_0 &= \frac{1}{4\pi} \int_0^L \beta G / B \rho \end{aligned} \quad (2-12b)$$

This is the well known result^{2,3} for the first order tune shift due to a gradient G . The solution for η to first order can be found using Eq. (2-6) and Eq. (2-8)

$$\begin{aligned} \eta &= A_s \left\{ \exp(i\nu_s \theta) + \sum_{r \neq s} \frac{2\nu_0}{\nu_r^2 - \nu_0^2} b_{rs} \exp(i\nu_r \theta) \right\} \\ \nu_r &= \nu_s + n, \quad n \neq 0 \end{aligned} \quad (2-13a)$$

which can be also written as

$$\begin{aligned} \eta &= A_s \exp(i\nu_s \theta) \left\{ 1 + \sum_{n \neq 0} \frac{2\nu_0 b_n}{n(n + 2\nu_0)} \exp(in\theta) \right\} \\ b_n &= \frac{1}{4\pi} \int_0^L ds \beta G \exp(-in\theta) / B \rho \end{aligned} \quad (2-13b)$$

One can see from Eq. (2-13b) that the above solution will not be valid when ν_0 is close to a half-integer for the denominator $(n + 2\nu_0)$ can become very small.

3. Results when the tune is inside the stopband

It is assumed that ν_0 is near $\nu_0 = q/2$, q being an integer. The stopband is defined as the range of ν_0 for which the tune, ν_s , in the presence of the perturbation given by Eq. (2-2) has a non-zero imaginary part. One can write ν_s as

$$\nu_s = \nu_{sR} - ig \quad (3-1)$$

It will be shown that inside the stopband, where $g \neq 0$, then

$$\nu_{sR} = q/2 \quad (3-2)$$

This may be shown as follows. Let μ be the phase shift for a period, and $\mu = 2\pi\nu_s$ where the period has been assumed to be one turn or 2π in θ . Let T be the transfer matrix for one period. Then one has

$$\cos \mu = \frac{1}{2}(T_{11} + T_{22}), \quad (3-3)$$

and one sees that $\cos \mu$ is real even inside the stopband since the T_{ij} are real. One also has

$$\cos \mu = \cos 2\pi(\nu_{sR} - ig) = \cos(2\pi\nu_{sR}) \cosh(2\pi g) + i \sin(2\pi\nu_{sR}) \sinh 2\pi g \quad (3-4)$$

In order for $\cos \mu$ to be real, one has to have either $g = 0$ or, if $g \neq 0$, $2\pi\nu_{sR} = n\pi$, where n is some integer. Thus, inside the stopband where $g \neq 0$,

$$\nu_{sR} = n/2 \quad (3-5)$$

In order to have continuity when the perturbation goes to zero, one has $n = q$ and

$$\nu_{sR} = q/2 \quad (3-6)$$

For the range of ν_0 which is inside the stopband, ν_{sR} remains constant at $\nu_{sR} = q/2$ while g goes, as will be seen below, from $g = 0$ at the edge of the stopband to a maximum value in the middle of the stopband, where $\nu_0 = q/2$. One may note that this result does not depend on the use of perturbation theory and is valid even for large perturbations. A proof of the result, based on perturbation theory, is given below.

Now let us return to the problem of computing the growth factor, g , using perturbation theory. The iterative perturbation approach used in section 2 appears to breakdown when

ν_0 is close to $q/2$. One sees from Eq. (2-10) that when $\nu_0 \simeq q/2$, then one of the A_r becomes comparable to A_s , and this is the A_r for which $\nu_r = \nu_s - q$, $\nu_r \simeq -q/2$. Thus in the above iterative procedure for finding η , one will assume for the initial guess for η , instead of Eq. (2-9),

$$\begin{aligned}\eta &= A_s \exp(i\nu_s\theta) + A_{\bar{s}} \exp(i\nu_{\bar{s}}\theta) \\ \nu_{\bar{s}} &= \nu_s - q\end{aligned}\tag{3-7}$$

Then Eq. (2-10) is replaced by

$$\begin{aligned}(\nu_r^2 - \nu_0^2) A_r &= 2\nu_0 b_{rs} A_s + 2\nu_0 b_{r\bar{s}} A_{\bar{s}} \\ \nu_r &= \nu_s + n \quad \text{or} \quad \nu_r = \nu_{\bar{s}} + n = \nu_s - q + n\end{aligned}\tag{3-8}$$

For $r = s$ and $r = \bar{s}$ one obtains 2 equations for A_s and $A_{\bar{s}}$

$$\begin{aligned}(\nu_s^2 - \nu_0^2) A_s &= 2\nu_0 b_{s\bar{s}} A_{\bar{s}} \\ (\nu_{\bar{s}}^2 - \nu_0^2) A_{\bar{s}} &= 2\nu_0 b_{\bar{s}s} A_s \\ \nu_{\bar{s}} &= \nu_s - q\end{aligned}\tag{3-9}$$

In Eq. (3-9), it has been assumed, for simplicities sake, that $b_{ss} = 0$. This can be accomplished by redefining ν_0 to be $\nu_0 + b_0$.

In order for Eqs. (3-9) to have a solution, one must have

$$\begin{aligned}(\nu_s^2 - \nu_0^2) (\nu_{\bar{s}}^2 - \nu_0^2) &= 4\nu_0^2 |b_{s\bar{s}}|^2 \\ \nu_{\bar{s}} &= \nu_s - q\end{aligned}\tag{3-10}$$

Eq. (3-10) determines ν_s . If one writes $\nu_s = \nu_{sR} - ig$, one finds

$$\left[(\nu_{sR} - ig)^2 - \nu_0^2 \right] \left[(\nu_{sR} - q - ig)^2 \right] = 4\nu_0^2 |\Delta\nu|^2\tag{3-11a}$$

where $\Delta\nu = b_{s\bar{s}}$

$$\Delta\nu = \frac{1}{4\pi} \int_0^L ds \beta \exp(-iq\theta) G/B\rho\tag{3-11b}$$

$|\Delta\nu|$ will turn out to be the half width of the stopband. Assuming that ν_0 and ν_{sR} are close to $q/2$, then Eq. (3-11a) can be rewritten as

$$(\nu_{sR} - ig - \nu_0)(q - \nu_{sR} + ig - \nu_0) = |\Delta\nu|^2\tag{3-12}$$

The imaginary part of the left hand side gives the equation,

$$g(2\nu_{sR} - q) = 0\tag{3-13}$$

Inside the stopband where $g \neq 0$, one gets

$$\nu_{sR} = q/2 \quad (3-14)$$

Using Eq. (3-14) for ν_{sR} , the real part of Eq. (3-12) gives

$$\begin{aligned} (q/2 - \nu_0)^2 + g^2 &= |\Delta\nu|^2 \\ g &= \pm \left\{ |\Delta\nu|^2 - (q/2 - \nu_0)^2 \right\}^{1/2} \end{aligned} \quad (3-15)$$

Eq. (3-15) shows that the growth parameter g has a maximum at $\nu_0 = q/2$ where $g = |\Delta\nu|$, and decreases to zero at $\nu_0 = q/2 \pm |\Delta\nu|$. The stopband width is then $2|\Delta\nu|$ and extends from $q/2 - |\Delta\nu|$ to $q/2 + |\Delta\nu|$. Thus $|\Delta\nu|$ is the half-width of the stopband.

Now let us find the solutions for η that will give the particle motion inside the stopband. To lowest order, η is given by Eq. (3-7) and inside the stopband

$$\begin{aligned} \eta &= A_s \exp(i\nu_s \theta) + A_{\bar{s}} \exp(i\nu_{\bar{s}} \theta) \\ \nu_s &= q/2 - ig \\ \nu_{\bar{s}} &= -q/2 - ig \end{aligned} \quad (3-16)$$

A_s and $A_{\bar{s}}$ are related by Eqs. (3-9) which gives

$$\begin{aligned} A_{\bar{s}} &= \frac{2\nu_0 b_{\bar{s}s}}{(q/2 + ig)^2 - \nu_0^2} A_s \\ A_{\bar{s}} &= \frac{b_{\bar{s}s}}{(q/2 - \nu_0) + ig} A_s \end{aligned} \quad (3-17)$$

where it has been assumed that ν_0 is close to $q/2$. Eq. (3-17) can be written as

$$\begin{aligned} A_{\bar{s}} &= A_s \exp[-i(\delta_1 + \delta_2)] \\ \delta_1 &= \text{phase } \Delta\nu \\ \delta_2 &= \text{phase } [(q/2 - \nu_0) + ig] \end{aligned} \quad (3-18)$$

where $b_{\bar{s}s} = |\Delta\nu| \exp(-i\delta_1)$, and $|\Delta\nu| = g^2 + (q/2 - \nu_0)^2$ were used.

Putting these results for $A_{\bar{s}}$ into Eq. (3-16) one gets for η and x

$$\begin{aligned} \eta &= A_s \exp(g\theta) \cos(q\theta/2 - (\delta_1 + \delta_2)/2) \\ \delta_1 &= \text{phase } \Delta\nu \\ \delta_2 &= \text{phase } [(q/2 - \nu_0) + ig] \\ x &= \beta^{1/2} \eta \end{aligned} \quad (3-19)$$

where the multiplying constant $\exp[-i(\delta + \delta_2)/2]$ was dropped. Eq. (3-19) give to lowest order the two solutions inside the stopband corresponding to whether g is positive or negative. The first order correction to η could also be found using Eq. (3-8) and the result is

$$\eta = A_s \exp(g\theta) \left[\cos(q\theta/2 - (\delta_1 + \delta_2)/2) + \sum_{n \neq 0, -q} \frac{2\nu_0}{n(n + 2\nu_0)} b_n \cos((q/2 + n)\theta - (\delta_1 + \delta_2)/2) \right] \quad (3-20)$$

$$b_n = \frac{1}{4\pi} \int_0^L ds \beta G \exp(-in\theta) / B\rho$$

The results found above for the stopband width, the growth rate and particle motion in the stopbands will apply equally well to small accelerators where the exact linear equations have to be used, without the approximations used for large accelerators, as shown in section 7.

4. The realness of the eigenvalues and eigenvectors inside the stopband

It will be shown in this section that the eigenvalues and the eigenvectors of the one period transfer matrix are real when ν_0 is inside the stopband. If $\hat{T}(s) = T(s + L, s)$ is the one period transfer matrix, then the eigenvalues and eigenvectors satisfy the equation

$$\hat{T}(s) x(s) = \lambda x(s)$$

$$x = \begin{bmatrix} x \\ p_x \end{bmatrix} \quad (4-1)$$

The symbol x is used to indicate both the column vector x and the first element of this vector. The meaning of x should be clear from the context. The eigenvalue λ is related to ν -value by

$$\lambda = \exp(i\mu)$$

$$\mu = 2\pi\nu \quad (4-2)$$

where, for simplicity, the period is assumed to be one turn or 2π in θ .

In section 3 it was shown that when ν_0 is inside the stopband, ν can be written as $\nu = q/2 - ig$ one finds for x , when ν_0 is inside the stopband,

$$\begin{aligned}\lambda &= \exp(iq\pi + 2\pi g) \\ \lambda &= (-1)^q \exp(2\pi g)\end{aligned}\tag{4-3}$$

and the eigenvalue, λ , is real when ν_0 is inside the stopband.

Now let us consider the eigenvectors of \hat{T} . If λ, x are an eigenvalue and eigenvector of \hat{T} , it is known that $1/\lambda$ is an eigenvalue of \hat{T} , as \hat{T} is symplectic. Since \hat{T} is real, λ^*, x^* are also an eigenvalue and eigenvector respectively of \hat{T} . It appears that one has 3 eigenvalues, $\lambda, 1/\lambda$ and λ^* instead of just two. Two of the 3 apparent eigenvalues must be equal.

When ν_0 is not in the stopband then ν_s is real, $1/\lambda = \exp(-i\mu) = \lambda^*$, and the second solution is λ^*, x^* .

When ν_0 is in the stopband, then ν has an imaginary part, $\nu = q/2 - ig$. Then $1/\lambda \neq \lambda^*$. To avoid having 3 eigenvalues, one needs to have $\lambda^* = \lambda$ and $x^* = x$, or λ and x are real when ν_0 is in the stopband.

One should note that the statement x is real means here that x can be made real by multiplying by a constant. This is because an eigenvector can be multiplied by a constant and still be an eigenvector of \hat{T} . Thus the eigenvector may not appear real because of the presence of a constant complex multiplier, but can be made real by multiplying it with a constant.

In section 3, where η was found by an iterative perturbation procedure for ν_0 in the stopband, the solution found, Eq. (3-17), is real as required by the above theorem, whereas when ν_0 is not inside the stopband, the solution for η found Eq. (2-13) is complex. Since the solution for η was specified to have the form given by Eq. (2-6), then $x = \eta/\beta^{1/2}$ is the first element of the eigenvector, x , of \hat{T} .

5. Tune near the edge of a stopband

In this section, a result will be found for the tune in the stable region outside the stopband but close to one of the edges of the stopband. It will be shown that close to the edge of a stopband,

$$|\nu - q/2| = \{2|\Delta\nu| |\nu_0 - \nu_e|\}^{1/2} \quad (5-1)$$

ν is the tune in the presence of the gradient perturbation, ν_e is the edge of the stopband, $\nu_e = q/2 \pm |\Delta\nu|$. $|\Delta\nu|$ is the half-width of the stopband. Eq. (5-1) shows that when ν_0 is close to an edge of the stopband, ν_e , ν varies rapidly with ν_0 , and the slope of the ν vs. ν_0 curve is vertical at $\nu_0 = \nu_e$.

To find ν in the stable region outside the stopband, where $\nu_0 - q/2 > |\Delta\nu|$, one goes back to the derivation given in section 3 for ν inside the stopband, starting with Eq. (3-12). Eq. (3-15) shows that for $|\nu_0 - q/2| > |\Delta\nu|$ the only acceptable solution is $g = 0$, and Eq. (3-12) can be written as

$$(\nu - \nu_0)(|\nu - q| - \nu_0) = |\Delta\nu|^2 \quad (5-2)$$

where we have put $\nu_s = \nu$.

Assuming that ν_0 is just below the stopband edge $\nu_e = q/2 - |\Delta\nu|$, put $\nu_0 = \nu_e - \epsilon$ and $\nu = q/2 - \delta$ into Eq. (5-2), where ϵ and δ both approach zero as ν_0 approaches the stopband edge. We find

$$\delta = \{\epsilon(\epsilon + 2|\Delta\nu|)\}^{1/2} \quad (5-3)$$

The top edge of the stopband can be treated in the same way and both results can be combined into the one result

$$\begin{aligned} |\nu - q/2| &= \{|\nu_0 - \nu_e|(|\nu_0 - \nu_e| + 2|\Delta\nu|)\}^{1/2} \\ \nu_e &= q/2 \pm |\Delta\nu| \end{aligned} \quad (5-4)$$

Very close to the stopband edge, $|\nu_0 - \nu_e| \ll |\Delta\nu|$, one finds

$$|\nu - q/2| = \{2|\Delta\nu||\nu_0 - \nu_e|\}^{1/2} \quad (5-5)$$

Thus, as ν_0 approaches a stopband edge, ν approaches $q/2$, and $d\nu/d\nu_0$ become infinite like $1/|\nu_0 - \nu_e|^{1/2}$.

6. The beta function near the edge of the stopband

In this section, a result will be found for the beta function in the stable region outside the stopband, but close to one of the edges of the stopband. It will be shown that close to edge of a stopband

$$[(\beta - \beta_0)/\beta_0]_{\max} = [2|\Delta\nu|/|\nu_0 - \nu_e|]^{1/2} \quad (6-1)$$

ν_e is the edge of the stopband, $\nu_e = q/2 \pm |\Delta\nu|$. $|\Delta\nu|$ is the half-width of the stopband. ν_0 , β_0 are the unperturbed tune and beta function. Eq. (6-1) shows that when ν_0 approaches the edge of the stopband, $(\beta - \beta_0)/\beta_0$ becomes infinite like $1/|\nu_0 - \nu_e|^{1/2}$.

The beta function β can be found from the solution for the η function which has the form given by Eq. (2-5). It will be shown below that β is given by

$$\begin{aligned} \frac{\beta}{\beta_0} &= \frac{\nu_0}{\nu} |\eta|^2 \left\langle \frac{1}{|\eta|^2} \right\rangle \\ \left\langle \frac{1}{|\eta|^2} \right\rangle &= \frac{1}{2\pi} \int_0^{2\pi} d\theta \frac{1}{|\eta|^2} \end{aligned} \quad (6-2)$$

ν , β and ν_0 , β_0 are the perturbed and unperturbed values of the tune and the beta function. Thus β can be determined from the η function, if the perturbed tune is known, and in this case the tune was found in section 5.

To derive Eq. (6-2) one notes that

$$x = \beta_0^{1/2} \eta, \quad (6-3)$$

and thus

$$\beta = D \beta_0 |\eta|^2$$

where D is some constant. The constant D can be determined using the relationship

$$\frac{\nu}{\nu_0} = \frac{1}{2\pi} \int_0^{2\pi} d\theta \frac{\beta_0}{\beta} \quad (6-4)$$

which when written in this form, will hold also for small accelerators¹ when the perturbation does not change the closed orbit (see section 7). This gives

$$\begin{aligned} D &= \frac{\nu_0}{\nu} \left\langle 1/|\eta|^2 \right\rangle \\ (\beta/\beta_0) &= (\nu_0/\nu) |\eta|^2 \left\langle 1/|\eta|^2 \right\rangle \end{aligned} \quad (6-5)$$

One now proceeds to find the η function in the stable region near an edge of the stopband. To lowest order, η is given by Eq. (3-7) as

$$\begin{aligned}\eta &= A_s \exp(i\nu_s\theta) + A_{\bar{s}} \exp(i\nu_{\bar{s}}\theta) \\ \nu_{\bar{s}} &= \nu_s - q\end{aligned}\tag{6-6}$$

$A_{\bar{s}}$ is given by Eq. (3-9) which can be written as

$$\begin{aligned}A_{\bar{s}} &= \frac{|\Delta\nu| \exp(-i\delta_1)}{|\nu_s - q| - \nu_0} A_s \\ \delta_1 &= \text{ph}(\Delta\nu)\end{aligned}\tag{6-7}$$

Assuming ν_0 is below the edge of the stopband one puts $\nu_s = q/2 - \delta$ and $\nu_0 = \nu_e - \epsilon$, $\nu_e = q/2 - |\Delta\nu|$. Then

$$A_{\bar{s}} = \frac{|\Delta\nu| \exp(-i\delta_1)}{\delta + \epsilon + |\Delta\nu|} A_s\tag{6-8}$$

The η functions is then given to lowest order by

$$\begin{aligned}\eta &= A_s \exp(i\nu_s\theta) \left[1 + \frac{|\Delta\nu|}{|\Delta\nu| + \epsilon + \delta} \exp[i(-q\theta - \delta_1)] \right] \\ \epsilon &= |\nu_0 - \nu_e|, \quad \nu_e = q/2 \pm |\Delta\nu| \\ \delta &= |\nu - q/2| = \left\{ |\nu_0 - \nu_e| (|\nu_0 - \nu_e| + 2|\Delta\nu|) \right\}^{1/2}\end{aligned}\tag{6-9}$$

Eq. (6-9) has been written so that it holds both above and below the stopband, and Eq. (5-4) was used to replace δ .

One then finds

$$\begin{aligned}|\eta|^2 &= |A_s|^2 (1 + 2C \cos(q\theta + \delta_1) + C^2) \\ C &= \frac{|\Delta\nu|}{|\Delta\nu| + |\nu_0 - \nu_e| + |\nu - q/2|}\end{aligned}\tag{6-10}$$

$$\begin{aligned}\left\langle \frac{1}{|\eta|^2} \right\rangle &= \frac{1}{|A_s|^2} \frac{1}{2\pi} \int_0^{2\pi} d\theta \frac{1}{1 + 2C \cos(q\theta + \delta_1) + C^2} \\ &= \frac{1}{|A_s|^2} \frac{1}{1 - C^2}\end{aligned}\tag{6-11}$$

One can now use Eq. (6-2) to find β

$$\frac{\beta}{\beta_0} = \frac{\nu_0}{\nu} (1 + C^2 + 2C \cos(q\theta + \delta_1)) \frac{1}{1 - C^2}\tag{6-12}$$

One may note that when ν_0 is at the edge of a stopband, $\nu_e - \nu_0 = 0$, $\nu - q/2 = 0$ and $C = 1$. Since one is interested here in the rather large effects due to the $1/(1 - C^2)$ factor

near the edge of the stopband, one can put $\nu_0/\nu = 1$ with only a small error. One finds for β

$$\begin{aligned}(\beta - \beta_0)/\beta_0 &= \frac{2C(C + \cos(q\theta + \delta_1))}{1 - C^2} \\ ((\beta - \beta_0)/\beta_0)_{\max} &= \frac{2C}{1 - C} \\ ((\beta - \beta_0)/\beta_0)_{\max} &= \frac{2|\Delta\nu|}{|\nu_0 - \nu_e| + |\nu - q/2|}\end{aligned}\tag{6-13}$$

using the result $|\nu - q/2| = \{(\nu_0 - \nu_e)(|\nu_0 - \nu_e| + 2|\Delta\nu|)\}^{1/2}$ one gets

$$((\beta - \beta_0)/\beta_0)_{\max} = \frac{2|\Delta\nu|}{|\nu_0 - \nu_e| + \{|\nu_0 - \nu_e|(|\nu_0 - \nu_e| + 2|\Delta\nu|)\}^{1/2}}\tag{6-14}$$

Very close to the stopband, $|\nu_0 - \nu_e| \ll |\Delta\nu|$, one finds

$$((\beta - \beta_0)/\beta_0)_{\max} = \left\{ \frac{2|\Delta\nu|}{|\nu_0 - \nu_e|} \right\}^{1/2}\tag{6-15}$$

Eq. (6-15) shows that as ν_0 approaches the edge of a stopband $((\beta - \beta_0)/\beta_0)_{\max}$ becomes infinite like $\{|\nu_0 - \nu_e|\}^{-1/2}$.

Table 1 below shows how the distortion in the beta function depends on how far ν_0 is from the edge of stopband. The table shows that to keep $((\beta - \beta_0)/\beta_0)_{\max}$ less than 10%, then ν_0 has to be about $10\Delta\nu$ away from the stopband edge.

Table 1: $((\beta - \beta_0)/\beta_0)_{\max}$ versus $|\nu_0 - \nu_e|/|\Delta\nu|$ as computed from Eq. (6-14).

$ \nu_0 - \nu_e / \Delta\nu $.5	1	2	4	6	8	10
$((\beta - \beta_0)/\beta_0)_{\max}$	1.24	.73	.41	.22	.15	.12	.095

7. Comments on the small accelerator results

All the final results found in this paper will also hold for small accelerators where the exact equations of motions have to be used. The exact linear equations have the form¹

$$\begin{aligned}\frac{dx}{ds} &= A_{11}x + A_{12}q_x \\ \frac{dq_x}{ds} &= A_{21}x + A_{22}q_x \\ q_x &= p_x/p\end{aligned}\tag{7-1}$$

In the large accelerator approximation, it is assumed that $A_{11} = A_{22} = 0$, $A_{12} = 1$, and $q_x \simeq dx/ds$. The coefficients A_{ij} are given in reference 1. In particular, A_{12} is given by

$$A_{12} = \frac{1 + x/\rho}{(1 - q_x^2)^{3/2}}.\tag{7-2}$$

where the right hand side is evaluated on the closed orbit.

Although for small accelerators, the differential equations for x and β are different from those usually used for large accelerators, it has been found¹ that the linearized differential equation for $\eta = x/\beta^{1/2}$ is not much different. One finds

$$\begin{aligned}\frac{d^2}{d\theta^2}\eta + \nu_0^2\eta &= f \\ f &= \frac{\nu_0^2\beta_0^2}{A_{12}} \frac{\Delta B_y}{B\rho} \\ B\rho &= pc/e\end{aligned}\tag{7-3}$$

with the difference that θ is now defined by¹

$$d\theta = A_{12} \frac{ds}{\nu_0\beta_0}\tag{7-4}$$

where ν_0, β_0 are the unperturbed tune and beta function. The A_{12} is evaluated on the unperturbed closed orbit.

The relationship between ν and β is somewhat different for small accelerators, and is given by¹

$$\nu = \frac{1}{2\pi} \int_0^C \frac{A_{12}}{\beta} ds\tag{7-5}$$

where C is the circumference of the accelerator. Using Eq. (7-4) this can be written as

$$\frac{\nu}{\nu_0} = \frac{1}{2\pi} \int_0^{2\pi} \frac{A_{12}}{A_{12}} \frac{\beta_0}{\beta} d\theta\tag{7-6}$$

where \bar{A}_{12} is the A_{12} coefficient evaluated on the unperturbed closed orbit. If the perturbation being considered does not change the closed orbit, then $A_{12} = \bar{A}_{12}$ and one has

$$\frac{\nu}{\nu_0} = \frac{1}{2\pi} \int_0^{2\pi} \frac{\beta_0}{\beta} d\theta \quad (7-7)$$

Eq. (7-7) now holds for both large and small accelerators, provided the perturbation does not change the closed orbit.

The results found in this paper are based on Eq. (7-3) for η and Eq. (7-7) that relates ν and β . Although, the perturbation term in Eq. (7-3) now has the extra factor $1/A_{12}$ for small accelerators, this factor of $1/A_{12}$ disappears in the final result when one goes from the variable θ to the variable s according to Eq. (7-4). Thus the result for the stopband width, $\Delta\nu$, as given by Eq. (3-11b) is valid for both large and small accelerators. Keeping the above equations in mind, one can go through the derivations on the previous pages, and show that the final results are valid for both large and small accelerators.

8. Comments on the results

Others have worked on this subject and there is some overlap between the contents of this paper and their work. P.A. Sturrock² obtained results for the stopband width and the growth parameter g at the center of the stopband. E.D. Courant and H. Snyder³ obtained the result for the width of the stopband. H. Bruck⁴ showed that the solutions of the equation of motion at the edge of the stopband are stable. He also states that the real part of ν is constant at $q/2$ across the stopband without giving a proof of this. A.A. Kolomensky and A.N. Lebedev⁵ obtained results for the stopband width and the growth parameter g at the center of the stopband. H. Wiedemann⁶ obtained the result for the width of the stopband using a method similar to that used by Courant and Snyder.

The new results in this paper include the following. The result given for the tune ν near the edge of the stopband, ν_e , $|\nu - q/2| = [2|\Delta\nu||\nu_e - \nu_0|]^{1/2}$. The result given for the beta function near the edge of the stopband, $[(\beta - \beta_0)/\beta_0]_{\max} = [2|\Delta\nu||\nu_e - \nu_0|]^{1/2}$. The result for the growth parameter g across the entire stopband, $g = \{|\Delta\nu|^2 - |\nu_0 - q/2|^2\}^{1/2}$. The proof given showing that the real part of ν is constant over the stopband at $q/2$ does not

depend on perturbation theory, and the result follows from the symplectic properties. The result that all the results found in this paper will also hold for a small accelerator where the large accelerator approximation is not used. The result given for the solutions of the equations of motion when ν_0 is inside the stopband, and the proof that the eigenfunctions and eigenvalues of the transfer matrix are real inside the stopband.

References

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