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INTERNAL RAREFIED GAS FLOWS WITH BACKSCATTERING
PART 5: FLOW IN CAPILLARIES AND ANNULI

AUTHOR:

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OAK RIDGE GASEOUS DIFFUSION PLANT

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PART 5: FLOW IN CAPILLARIES AND ANNULI

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ABSTRACT

The flow through a long tube of arbitrary but constant cross section is computed using the BGK model equation and a boundary condition which permits a fraction $1 - \alpha$ of the molecules incident on the tube walls to be scattered back along their incident path while a fraction α are diffusely scattered.

For the special cases of circular and annular cross sections, a comparison is made of predicted and observed flows. In the comparison a value of $\alpha = 0.94$, obtained in Part 2 of this sequence of reports, is used. Significantly better agreement between observed and predicted transport results from the inclusion of backscattering.

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INTERNAL RAREFIED GAS FLOWS WITH BACKSCATTERING
PART 5: FLOW IN CAPILLARIES AND ANNULI

INTRODUCTION

In this report we continue our investigation of the effect of backscattering on internal rarefied gas flow. The Bhatnagar-Gross-Krook equation is used to provide an integral equation for the gas velocity at any point in the cross section of a very long tube with arbitrary, but constant, cross section. The integral equation is solved numerically and its solution is used to compute the flow rate through the tube at arbitrary average pressure, for purely diffuse, and for a small amount of backscattering (6%) at the tube walls. These flows are compared with the experimental observations of Knudsen [1] for capillaries, and of Maegley and Berman [2] for annuli.

ACKNOWLEDGMENT

The derivation of the integral equation for the velocity was carried out by Mr. W. Maegley, a student in the Department of Aerospace and Engineering Mechanics at the University of Minnesota, in partial fulfillment of the requirements for the Ph.D. degree.

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The author would also like to acknowledge Mr. Maegley's assistance in the programming and computations required for all five parts of this report and is indebted to the University of Minnesota Computing Center for furnishing, at no cost, a substantial amount of computing time for this work.

FLOW IN INFINITE SYSTEMS WITH
CONSTANT CROSS SECTION AND BACKSCATTERING

Consider the flow in a very long tube of constant, but arbitrary, cross section. Figure 1 indicates some of the variables pertinent to this problem.

The vectors, $\vec{c}_r$, $\vec{r}$, $\vec{r}_0$, and $\hat{n}$ are in the plane of the cross section with $\hat{n}$ the unit normal directed from the surface into the gas. A point P in the tube is specified by $\vec{r}$ and z where the z axis is in the direction of the pressure gradient driving the flow. A molecular velocity at P is specified by $\vec{c} = \vec{c}_r + \vec{c}_z$ where $\vec{c}_z = \hat{k}c_z$ and $\hat{k}$ is a unit vector in the z direction. The gas velocity at P is $\vec{q}$. The flow in the tube is produced by a small pressure gradient, and the temperature of the tube wall, and of the gas outside the tube, is assumed constant and equal to T_0 .

The BGK equation for this flow is

$$\vec{c}_r \cdot \frac{\partial f}{\partial \vec{r}} + c_z \frac{\partial f}{\partial z} + \nu f = \nu f_{eq} \quad (1)$$

where ν is the collision frequency (assumed independent of $\vec{c}$) and

$$f_{eq} = \frac{n}{(2\pi RT)^{3/2}} e^{-(\vec{c} - \vec{q})^2/2RT} \quad (2)$$

in which n , $\vec{q}$, T are the local gas density, velocity, and temperature, respectively, and R is the gas constant per gram.

The boundary condition is, as before,

$$f^+(\vec{c}, \vec{r}_0, z) = \alpha \frac{A}{(2\pi RT_0)^{3/2}} e^{-c^2/2RT_0} + (1 - \alpha) f^-(-\vec{c}, \vec{r}_0, z) \quad (3)$$

For sufficiently small gas velocity, equations (1) and (3) are linearized by a procedure which has been described in previous parts of this report and lead to

Figure 1

GEOMETRY FOR INTERNAL RAREFIED GAS FLOW

$$\vec{\xi}_R \cdot \frac{\partial h}{\partial \vec{R}} + \xi_z \frac{\partial h}{\partial Z} + h = \eta + 2\xi_z w + (\xi^2 - \frac{3}{2})t \quad (4)$$

$$h^+(\vec{\xi}, \vec{R}_0, Z) = \left[\frac{\alpha A(Z)}{n_0} - 1 \right] + (1 - \alpha)[1 + h^-(-\vec{\xi}, \vec{R}_0, Z)] ; \hat{n} \cdot \vec{\xi} > 0 \quad (5)$$

where h is the perturbation to the global equilibrium distribution function

$$f_0 = \frac{n_0}{(2\pi RT)^{3/2}} e^{-\xi^2} \quad (6)$$

so that

$$f = (1 + h)f_0, \quad (7)$$

and where the following dimensionless variables have been introduced.

$$\vec{\xi} = \vec{c}/\sqrt{2RT_0} ; w = q_z/\sqrt{2RT_0} ; \vec{R} = v_0 \vec{r}/\sqrt{2RT_0} ; Z = v_0 z/\sqrt{2RT_0} \quad (8)$$

$$\eta = (n/n_0) - 1 ; t = (T/T_0) - 1. \quad (9)$$

The equations (9) are the dimensionless perturbations in the gas density and temperature produced by the pressure gradient.

If we set the gas pressure $p = p_0(1 + kz) = p_0(1 + KZ)$ where $K = (1/p_0)(dp/dZ)$, and assume K to be small, then as was the case for plane Poiseuille flow (Part 4), a solution to (4) can be found in which $t = 0$ and $w = w(\vec{R})$ by choosing

$$h(\vec{\xi}_R, \vec{R}, Z) = KZ + \xi_z \phi(\vec{\xi}_R, \vec{R}). \quad (10)$$

With (10), the linearized equation (4) becomes

$$\vec{\xi}_R \cdot \frac{\partial \phi}{\partial \vec{R}} + \phi = 2w - K \quad (11)$$

and the boundary condition (5) becomes

$$\phi^+(\vec{\xi}_R, \vec{R}_0) = -(1 - \alpha)\phi^-(-\vec{\xi}_R, \vec{R}_0) ; \hat{n} \cdot \vec{\xi}_R > 0. \quad (12)$$

Here, $A(Z)$ in equation (5) has been taken to be equal to $n_0(1 + KZ)$.

In terms of dimensionless variables, the axial velocity is given by

$$w(\vec{R}) = \pi^{-3/2} \int_{\vec{\xi}} \xi_z^2 \phi(\vec{R}, \vec{\xi}_R) e^{-\xi^2} d\vec{\xi}. \quad (13)$$

Integration of (11) along $\vec{\xi}_R$ yields

$$\phi(\vec{\xi}_R, \vec{R}) = \phi^+(\vec{\xi}_R, \vec{R}_O) e^{S_O} + \int_{S_O}^0 e^S [2w(\vec{R}') - K] dS \quad (14)$$

where $S = -|\vec{R}' - \vec{R}|/|\vec{\xi}_R|$ and $S_O = -|\vec{R}_O - \vec{R}|/|\vec{\xi}_R|$. Physically, S and S_O represent the negative of the time required for a particle moving with speed ξ_R to travel the distance $|\vec{R}' - \vec{R}|$ and $|\vec{R}_O - \vec{R}|$, respectively. Thus, S and S_O depend on the point $\vec{R}$ and on the velocity $\vec{\xi}_R$. The point $\vec{R}_O$ is obtained by extending $\vec{\xi}_R$ at P in the direction of $(-\vec{\xi}_R)$ until it intersects the wall. $\vec{R}'$ is a variable point on the line joining $\vec{R}$ and $\vec{R}_O$.

With equation (14) we obtain, for the fluid velocity w ,

$$w(\vec{R}) = \pi^{-3/2} \int_{\vec{\xi}} \xi_z^2 e^{-\xi^2} d\vec{\xi} \left\{ e^{S_O} \phi^+(\vec{\xi}_R, \vec{R}_O) + \int_{S_O}^0 e^S (2w - K) dS \right\}. \quad (15)$$

The integration over ξ_z can be carried out to obtain

$$w(\vec{R}) = \frac{1}{2\pi} \int_{\vec{\xi}_R} e^{-\xi_R^2} d\vec{\xi}_R \left\{ e^{S_O} \phi^+(\vec{\xi}_R, \vec{R}_O) + \int_{S_O}^0 e^S (2w - K) dS \right\}. \quad (16)$$

In a qualitative way, one can think of the first integral in (16) as the contribution to the perturbation velocity at $\vec{R}$ due to particles which leave the wall at $\vec{R}_O$ and survive the journey to $\vec{R}$ without intermolecular collisions. The second integral gives the contribution to the perturbation velocity w at $\vec{R}$ due to particles which arrive at $\vec{R}$ as a result of intermolecular collisions all along the path between $\vec{R}_O$ and $\vec{R}$. As in the case of plane Poiseuille flow (Part 4), we must eliminate the boundary term $\phi^+(\vec{\xi}_R, \vec{R}_O)$ from equation (16).

First, evaluate (14) at the point $\vec{R}_1$ obtained by extending $\vec{\xi}_R$ at $\vec{R}$ in its own direction until it intersects the boundary (see accompanying figure) to obtain

$$\phi^-(\vec{\xi}_R, \vec{R}_1) = \phi^+(\vec{\xi}_R, \vec{R}_0) e^{S_0(\vec{R}_1)} + \int_{S_0(\vec{R}_1)}^0 e^S [2w(\vec{R}') - K] dS \quad (17)$$

and note that this represents the perturbation to the distribution function for particles striking the wall at $\vec{R}_1$. If the velocity vector $\vec{\xi}_R$ is reversed, then $\vec{R}_0$ and $\vec{R}_1$ interchange their roles and (17) becomes

$$\phi^-(-\vec{\xi}_R, \vec{R}_0) = \phi^+(-\vec{\xi}_R, \vec{R}_1) e^{S_0(\vec{R}_1)} + \int_{S_0(\vec{R}_1)}^0 e^S [2w(\vec{R}') - K] dS. \quad (18)$$

Since $S_0(\vec{R}_1) = -|\vec{R}_1 - \vec{R}_0|/|\vec{\xi}_R|$, it remains unchanged in the above transformation.

If (18) is substituted into the boundary condition (12), the result is

$$\phi^+(\vec{\xi}_R, \vec{R}_0) = -(1 - \alpha)\phi^+(-\vec{\xi}_R, \vec{R}_1) e^{S_0(\vec{R}_1)} - (1 - \alpha) \int_{S_0(\vec{R}_1)}^0 e^S [2w(\vec{R}') - K] dS. \quad (19)$$

However, the boundary condition (12) furnishes

$$\phi^+(-\vec{\xi}_R, \vec{R}_1) = -(1 - \alpha)\phi^-(\vec{\xi}_R, \vec{R}_1) \quad (20)$$

or with (17)

$$\phi^+(-\vec{\xi}_R, \vec{R}_1) = -(1 - \alpha)\phi^+(\vec{\xi}_R, \vec{R}_0) e^{S_0(\vec{R}_1)} - (1 - \alpha) \int_{S_0(\vec{R}_1)}^0 e^S [2w(\vec{R}') - K] dS \quad (21)$$

so that (19) can be written

$$\begin{aligned} \phi^+(\vec{\xi}_R, \vec{R}_0) &= -(1 - \alpha) e^{S_0(\vec{R}_1)} \left[-(1 - \alpha)\phi^+(\vec{\xi}_R, \vec{R}_0) e^{S_0(\vec{R}_1)} \right. \\ &\quad \left. - (1 - \alpha) \int_{S_0(\vec{R}_1)}^0 e^S [2w(\vec{R}') - K] dS \right] - (1 - \alpha) \int_{S_0(\vec{R}_1)}^0 e^S [2w(\vec{R}') - K] dS \end{aligned} \quad (22)$$

or

$$\phi^+(\vec{\xi}_R, \vec{R}_0) \left[1 - (1 - \alpha)^2 e^{2S_0(\vec{R}_1)} \right] = (1 - \alpha)^2 e^{S_0(\vec{R}_1)} \int_{S_0(\vec{R}_1)}^0 e^S \quad (23)$$

$$\cdot [2w(\vec{R}') - K] dS - (1 - \alpha) \int_{S_0(\vec{R}_1)}^0 e^S [2w(\vec{R}') - K] dS.$$

Equation (23) provides the required expression for the boundary term $\phi^+(\vec{\xi}_R, \vec{R}_0)$ in equation (18). However, the integral equation which results for the velocity is too cumbersome so we confine our attention to small amounts of backscattering* and retain only first order terms in the fraction of backscattering $(1 - \alpha)$ in (23). The result is

$$\phi^+(\vec{\xi}_R, \vec{R}_0) = -(1 - \alpha) \int_{S_0(\vec{R}_1)}^0 e^S [2w(\vec{R}') - K] dS. \quad (24)$$

*See the discussion of this approximation in Part 4 of this report.

The integral equation (18) for the velocity becomes

$$\begin{aligned}
 w(\vec{R}) = & - \frac{(1 - \alpha)}{2\pi} \int_{\vec{\xi}_R} d\vec{\xi}_R \int_{S_0(\vec{R}_1)} e^{S_0(\vec{R}) + S - \xi_R^2} [2w(\vec{R}') - K] dS \\
 & + \frac{1}{2\pi} \int_{\vec{\xi}_R} d\vec{\xi}_R \int_{S_0(\vec{R})}^0 e^{S - \xi_R^2} [2w(\vec{R}') - K] dS .
 \end{aligned} \tag{25}$$

It should be noted that in the first integral, the variable of integration S takes on all its values associated with the variable point $\vec{R}'$ ranging from $\vec{R}_1$ to $\vec{R}_0$. In the second integral, S takes on all its values associated with the variable point $\vec{R}'$ ranging from $\vec{R}$ to $\vec{R}_0$. This has been indicated in (25) by the notation $S_0(\vec{R}_1)$ and $S_0(\vec{R})$ in the lower limits of the respective integrals.

We can proceed with the reduction of (25) by introducing the Heaviside functions,

$$H[S - S_0(\vec{R}_1)] = \begin{cases} 0 ; & S < S_0(\vec{R}_1) \\ 1 ; & S \geq S_0(\vec{R}_1) \end{cases}$$

and

$$H[S - S_0(\vec{R})] = \begin{cases} 0 ; & S < S_0(\vec{R}) \\ 1 ; & S \geq S_0(\vec{R}) \end{cases}$$

and extending the range of integration in the integrals from S_0 to $(-\infty)$.

Thus, we can write for (25)

$$\begin{aligned}
 w(\vec{R}) = & \frac{-(1 - \alpha)}{2\pi} \int_{\vec{\xi}_R} d\vec{\xi}_R \int_{-\infty}^0 H[S - S_0(\vec{R}_1)] e^{S_0(\vec{R}) + S - \xi_R^2} [2w(\vec{R}') - K] dS \\
 & + \frac{1}{2\pi} \int_{\vec{\xi}_R} d\vec{\xi}_R \int_{-\infty}^0 H[S - S_0(\vec{R})] e^{S - \xi_R^2} [2w(\vec{R}') - K] dS .
 \end{aligned} \tag{26}$$

We can now interchange the order of integration and write

$$\begin{aligned}
w(\vec{R}) = & - \frac{(1 - \alpha)}{2\pi} \int_{-\infty}^0 \int_{\vec{\xi}_R} H[S - S_0(\vec{R}_1)] e^{S_0(\vec{R}) + S - \xi_R^2} [2w(\vec{R}') - K] d\vec{\xi}_R dS \\
& + \frac{1}{2\pi} \int_{-\infty}^0 \int_{\vec{\xi}_R} H[S - S_0(\vec{R})] e^{S - \xi_R^2} [2w(\vec{R}') - K] d\vec{\xi}_R dS.
\end{aligned} \tag{27}$$

The variable of integration is transformed from velocity space to physical space by using

$$\vec{R}' = \vec{R} + S\vec{\xi}_R$$

$$d\vec{R}' = S^2 d\vec{\xi}_R$$

and replacing $d\vec{\xi}_R$ in (27) by $d\vec{R}'/S^2$. In addition, we note that $H[S - S_0(\vec{R}_1)]$ vanishes whenever $\vec{R}'$ is such that $|\vec{R}' - \vec{R}_1|$ is larger than $|\vec{R}_0 - \vec{R}_1|$, and that $H[S - S_0(\vec{R})]$ vanishes whenever $|\vec{R}' - \vec{R}| > |\vec{R}_0 - \vec{R}|$. This result restricts the range of integration in physical space as will be described later. Equation (27) can now be written

$$\begin{aligned}
w(\vec{R}) = & - \frac{1 - \alpha}{2\pi} \int_{-\infty}^0 \frac{dS}{S^2} \int_{\vec{R}''} [2w(\vec{R}'') - K] \exp \left[S_0(\vec{R}) + S - \frac{|\vec{R}'' - \vec{R}_0|^2}{S^2} \right] d\vec{R}'' \\
& + \frac{1}{2\pi} \int_{-\infty}^0 \frac{dS}{S^2} \int_{\vec{R}'} [2w(\vec{R}') - K] \exp \left[S - \frac{|\vec{R}' - \vec{R}|^2}{S^2} \right] d\vec{R}',
\end{aligned} \tag{28}$$

where we have used the integration variable $\vec{R}''$ in the first integral to emphasize the fact that the ranges of integration are different in the first and second integrals. This difference will be discussed in detail when the final form of the integral equation is obtained.

For the first integral in (28)

$$S_0(\vec{R}) = - |\vec{R}_0 - \vec{R}|/|\xi_R|$$

and

$$S(\vec{R}'') = - |\vec{R}'' - \vec{R}|/|\xi_R|$$

so that we can use the relation

$$S_o(\vec{R}) = \frac{|\vec{R}_o - \vec{R}|}{|\vec{R}'' - \vec{R}|} S. \quad (29)$$

With this, equation (28) becomes

$$\begin{aligned} w(\vec{R}) = & -\frac{1-\alpha}{2\pi} \int_{-\infty}^0 \frac{dS}{S^2} \int_{\vec{R}''} [2w(\vec{R}'') - K] \exp \left[\frac{S}{|\vec{R}'' - \vec{R}_o|} (|\vec{R}_o - \vec{R}|) \right. \\ & \left. + |\vec{R}'' - \vec{R}_o| - \frac{|\vec{R}'' - \vec{R}_o|^2}{S^2} \right] d\vec{R}'' + \frac{1}{2\pi} \int_{-\infty}^0 \frac{dS}{S^2} \int_{\vec{R}'} [2w(\vec{R}') - K] \\ & \cdot \exp \left[S - \frac{|\vec{R}' - \vec{R}|}{S^2} \right] d\vec{R}'. \end{aligned} \quad (30)$$

Now let

$$\begin{aligned} \beta'' &= -\frac{|\vec{R}'' - \vec{R}_o|}{S}; \quad d\beta'' = \frac{|\vec{R}'' - \vec{R}_o|}{S^2} dS \\ \beta' &= -\frac{|\vec{R}' - \vec{R}_o|}{S}; \quad d\beta' = \frac{|\vec{R}' - \vec{R}|}{S^2} dS \end{aligned}$$

so that we can rewrite (30) as

$$\begin{aligned} w(\vec{R}) = & -\frac{1-\alpha}{2\pi} \int_{\vec{R}''} \frac{[2w(\vec{R}'') - K] d\vec{R}''}{|\vec{R}'' - \vec{R}_o|} \int_0^\infty \exp \left[-\beta''^2 \frac{|\vec{R}_o - \vec{R}| + |\vec{R}'' - \vec{R}_o|}{\beta''} \right] d\beta'' \\ & + \frac{1}{2\pi} \int_{\vec{R}'} \frac{[2w(\vec{R}') - K] d\vec{R}'}{|\vec{R}' - \vec{R}|} \int_0^\infty \exp \left[-\beta'^2 - \frac{|\vec{R}' - \vec{R}|}{\beta'} \right] d\beta', \end{aligned} \quad (31)$$

or, in terms of the Abramowitz functions,

$$J_n(x) = \int_0^\infty t^n \exp[-t^2 - (x/t)] dt,$$

we have the integral equation for the velocity

$$\begin{aligned} w(\vec{R}) = & -\frac{1-\alpha}{2\pi} \int_{\vec{R}''} \frac{J_o(|\vec{R}_o - \vec{R}| + |\vec{R}'' - \vec{R}_o|)}{|\vec{R}'' - \vec{R}_o|} [2w(\vec{R}'') - K] d\vec{R}'' \\ & + \frac{1}{2\pi} \int_{\vec{R}'} \frac{J_o(|\vec{R}' - \vec{R}|)}{|\vec{R}' - \vec{R}|} [2w(\vec{R}') - K] d\vec{R}'. \end{aligned} \quad (32)$$

For convenience, one final transformation is made. We define $\psi(\vec{R})$ by the equation

$$w(\vec{R}) = (K/2)[1 - \psi(\vec{R})] \quad (33)$$

and obtain from (32) the integral equation for $\psi(\vec{R})$

$$\begin{aligned} \psi(\vec{R}) = 1 - \frac{1-\alpha}{\pi} \int_{\vec{R}''} \frac{J_0(|\vec{R}_0 - \vec{R}| + |\vec{R}'' - \vec{R}_0|)}{|\vec{R}'' - \vec{R}_0|} \psi(\vec{R}'') d\vec{R}'' \\ + \frac{1}{\pi} \int_{\vec{R}'} \frac{J_0(|\vec{R}' - \vec{R}|)}{|\vec{R}' - \vec{R}|} \psi(\vec{R}') d\vec{R}' . \end{aligned} \quad (34)$$

The ranges for the area integrals in equation (34) can be visualized with the aid of the following figure which might be the cross section of a flow channel.

For both integrals in (34), one chooses a convenient reference direction at $\vec{R}$, the fixed point in the tube cross section. A straight line through $\vec{R}$, making an angle θ with the reference direction, is extended until it intersects the flow cross-sectional boundary. For the second integral [refer to (a) in the figure] the variable point $\vec{R}'$ ranges over those portions of the typical lines inclined at θ_1 and θ_2 which are drawn as solid lines in the figure. In the first integral [refer to (b) in the figure] the variable point $\vec{R}''$ ranges over the solid portion of the typical lines through $\vec{R}$ inclined at angles θ_1 and θ_2 . The point $\vec{R}_0$, which appears in the first integral, is also shown in (b). In both integrals, if θ is the angle in a plane polar coordinate system, then $0 \leq \theta \leq 2\pi$.

The solution $\psi(\vec{R})$ of equation (34) is used with (33) to obtain the variation of gas velocity in the cross section of the flow channel. A suitable quadrature of $\psi(\vec{R})$ gives the gas flow in the tube.

For the tube with circular cross section and for the annulus, the integral equation (32) has been solved numerically using a scheme much like that described by Cercignani [3].* The details of the numerical work are described in the Ph.D. thesis of Mr. W. Maegley [4]. $\psi(\vec{R})$ has also been integrated numerically over the respective cross sections to give the transport in a long capillary and a long annulus.

Figure 2 indicates the effect of backscattering on the velocity profile in an annulus (radius ratio $\sigma = 0.8711$) for various pressure levels, indicated by approximate Knudsen numbers (K_n) defined as the mean free path divided by the spacing. There is little effect on the general shape of the profiles. Backscattering generally decreases the gas velocity at every point in the cross section.

The effect of backscattering on the flow through a capillary and an annulus is discussed in the next section which is devoted to a comparison of the predicted and observed transports in these systems.

*The author wishes to thank Prof. Cercignani for kindly furnishing his Fortran program listing. Our calculations are generally similar to his with somewhat different integration routines and subinterval selections.

Figure 2
TYPICAL VELOCITY PROFILES IN AN ANNULUS

COMPARISON WITH EXPERIMENT

Flow in a Capillary

Knudsen [1] has measured the flow of carbon dioxide through a long capillary (length/radius = 542) with a nominal radius of 0.01084 cm, at a nominal temperature of 25°C. (In his paper this capillary is called tube 1.) As was done in the case of plane Poiseuille flow (Part 4), we assume that Knudsen's measured flows at the four highest average pressures are adequately described by a continuum equation with slip correction. In this way we arrive at a corrected effective tube radius of 0.01097 cm which differs from the nominal value by about one percent. Knudsen's high pressure points represent much lower Knudsen numbers than the high pressure points of Gaede discussed in Part 4 of this report. As a consequence, the slip correction is much smaller and, even though different slip coefficients were used for the cases $\alpha = 1$ and $\alpha = 0.94$, only one effective radius is required.

Figure 3 shows the volume flow as a function of average pressure computed by the method discussed in this report for the cases of diffuse scattering ($\alpha = 1$) and partial backscattering ($\alpha = 0.94$). Also shown in the figure are the results of Knudsen's measurements. As in the case of plane Poiseuille flow, the inclusion of 6% backscattering leads to a marked improvement in the agreement between predicted and observed flows.

Mr. W. Maegley has measured the dependence of the flow of argon and helium through various annuli on average pressure. This work is reported in detail in his thesis which we hope to issue as a K-report shortly.

For a typical comparison with the calculations described in this report, one of his annuli (Annulus Number 2) with a ratio of inner to outer radius of $\sigma = 0.9932$, and a length-to-spacing ratio of 368, has been selected. Comparison will be made with Mr. Maegley's measurements of argon flow through this annulus.

Figure 3
FLOW IN A LONG CAPILLARY

Since data at sufficiently high pressures are available, a nominal spacing correction factor was determined by using the continuum equation for annulus flow. The effective spacing between the inner and outer cylinders was found to be 0.983 times the nominal spacing.

Figure 4 gives the results of the flow computations for pure diffuse scattering ($\alpha = 1$) and partial backscattering ($\alpha = 0.94$) in an infinitely long annulus with radius ratio 0.9932. Also shown in the figure are the results of the argon flow measurements carried out by Maegley in an annulus with radius ratio 0.9932 and a length-to-spacing ratio of 367.7. For this geometry, also, the inclusion of backscattering improves the agreement between predicted and observed flow rates.

Independent calculations have been made by Maegley (to be reported in his thesis) of the expected free molecule flow through an infinite annulus and through the finite annulus used for the experimental work. The results of these calculations are shown in the figure as $G_{od}(\infty)$, $G_{ob}(\infty)$, and $G_{ob}(L)$. These represent, respectively, the expected free molecule flow (1) through an infinite annulus with pure diffuse scattering, (2) through an infinite annulus with 6% backscattering, and (3) through an annulus with length-to-spacing ratio of 367.7 with 6% backscattering.

Since the solution of the BGK equation for a finite annulus is not known, we can only conjecture, from these free molecule calculations, that the inclusion of backscattering improves agreement between observed and predicted flow rates more than might be deduced from the curves in Figure 4.

SUMMARY AND CONCLUSIONS

The internal rarefied gas flow through a long tube with arbitrary but constant cross sections has been studied with the aid of the BGK model equation. For the special cases of a tube of circular cross section, and a tube with an annular cross section, the expected pressure dependence of flow has been calculated for purely diffuse scattering, and for partial backscattering at the tube walls.

Figure 4
FLOW IN AN ANNULUS

Comparison of these expected flows with experimental data indicates that, for both geometries, the inclusion of 6% backscattering (the value obtained in Part 2 of this report) significantly improves the agreement between predicted and observed flows. As a consequence of this, and of the results obtained in previous parts of this report, it seems that the partial backscattering boundary condition offers a reasonable model for improved description of internal rarefied gas flows in systems whose walls have received no special treatment.

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