RECEIVED BY DUE JUL 17 1970

MASTER

SCATTERING PROBABILITY OF FAST TEST PARTICLES IN A PLASMA*

A. A. HUSSEINY AND H. K. FORSEN
Nuclear Engineering Department
The University of Wisconsin
Madison, Wisconsin 53706

An effective total cross section is calculated for elastic scattering of a monoenergetic beam of test particles in a plasma of arbitrary velocity distribution. Closed form solutions are obtained for the special case of Maxwellian field particles taking into account small angle scattering. The results are compared with those obtained using the Debye cut off technique and it is shown that such an arbitrary cut off can lead to erroneous results in the effective cross section and it can lead to different conclusions with regard to the relative importance of plasma ions and electrons in test particle scattering.

PATENT REVIEW APPROVES RELEASE PROCEDURES
GOVERNING PATENT REVIEW AND RELEASE ARE ON
FILE IN RECEIVING SECTION.

*This work is supported in part by the
U.S. Atomic Energy Commission and the Faculty of
Engineering, University of Alexandria, Alexandria,
U.A.R.

LEGAL NOTICE

This report was prepared as an account of work sponsored by the United States Government. Neither the United States nor the United States Atomic Energy Commission, nor any of their employees, nor any of their contractors, subcontractors, or their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness or usefulness of any information, apparatus, product or process disclosed, or represents that its use would not infringe privately owned rights.

DISTRIBUTION OF THIS DOCUMENT IS UNLIMITED

Shid

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency Thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

DISCLAIMER

Portions of this document may be illegible in electronic image products. Images are produced from the best available original document.

1. INTRODUCTION

Recent feasibility studies¹⁻⁴ of possible steady state thermonuclear reactors have shown the need to investigate in detail the behavior of fast charged particles in energetic plasmas. Such studies are important in calculating the thermalization rates of fast test ions^{1,2}, the slowing down time necessary for these ions to transfer a major fraction of their energy to the plasma¹, secondary fusion reactions³ and velocity distributions⁴ which are essential in studying the dynamic behavior of fusion reactors⁵.

Several investigators have used the Rosenbluth, MacDonald and Judd⁶ (RMJ) form of the Fokker-Planck equation in their studies of this problem. For example, Kranzer⁷ has determined numerically the time history of the probability distribution of fast test ions in velocity space for some special cases. More recently Kuo-Petravic, Petravic and Watson have used the RMJ equation to study the energy transfer from reaction produced alpha particles to plasma electrons and ions in mirror reactors⁴. A modified approach had previously been taken by Butler and Buckingham⁸ to calculate the energy loss rate from fast test ions. In all of the above work, the classical Rutherford scattering cross section has been used. To avoid the well known divergence in integrating over the deflection angles, scattering for angles smaller than $\chi_0^{(c)}$ is neglected where

$$\chi_o^{(c)} = \xi / (v_r^2 \lambda_D) \quad (1)$$

and where $\xi = (q_t q_f / 4\pi \epsilon_0 \mu)$. Here q_t and q_f are the charges of the test and field particles respectively, μ is the reduced mass, v_r is the relative velocity, λ_D is the Debye screening length and the rest of the notation is used in the standard manner. The validity criterion for the use of this classical cut off limit rather than the quantum mechanical cut off angle⁹

$$\chi_o^{(q)} = \hbar / (\mu v_r \lambda_D) \quad (2)$$

is discussed by several authors. The use of the Rutherford cross section is limited to situations in which the orbital model for scattering can be used and the validity criterion for the use of this has been discussed by Williams¹⁰; Bohr¹¹; Everhart, Stone and Carbone¹²; and Lane and Everhart¹³. In cases when the particle's wave properties must be considered the Born approximation should be used. The present authors have treated the problem of slowing down of fast test ions in plasmas in the limit of $v_r \approx v_t$ where v_t is the test particle velocity.^{9,14} In this case the Born approximation can be used for deflection angles from 0 to π and no cut off angle need be assigned to get a closed form solution. For the general evaluation of energy degradation of ions

released from thermonuclear reactions in a plasma, both the classical and quantum mechanical formulas for the cross section must be used since they are valid for mutually exclusive velocity ranges. Lane and Everhart¹³ have examined a similar situation in calculating the total cross section for scattering from Coulomb potentials with exponential screening for the interaction of two colliding atoms. Their technique is adopted here to include all angles of deflection in the calculations and to obtain results which are valid for the complete velocity range.

It is the purpose of this paper to calculate the effective total scattering cross section for the interaction between a test particle or a beam of monoenergetic particles in a plasma which has an arbitrary velocity distribution. The thermal motion of the field particles effects the cross section greatly if the test particle moves with a velocity less than or equal to the mean thermal velocity of the field particles. Closed form solutions are obtained for the special case of Maxwellian field particles.

The total cross section obtained can not be measured due to the dominant contribution of immeasurable small angle encounters; however, its use is essential in calculating the steady state distribution of charged fusion reaction products in a thermonuclear plasma from the transport equation¹. In addition, the relative importance of plasma ions and electrons is found for the special case of a Maxwellian plasma.

The results obtained here are compared with the results obtained using different limits for the cut off angle.

The differential cross sections for elastic scattering by an exponentially screened Coulomb potential are taken for the different angular regions as

$$\begin{aligned} \sigma_s^{(c)}(v_r, \chi) &= \left(\frac{e}{2v_r^2} \right)^2 \sin^{-4} \chi/2 & (3) \\ \chi_0^{(q)} &< \chi \leq \pi \end{aligned}$$

and the Born approximate formula:

$$\begin{aligned} \sigma_s^{(q)}(v_r, \chi) &= \left(\frac{e}{2v_r^2} \right)^2 \left(\sin^2 \chi/2 + \frac{\chi_0^2}{4} \right)^{-2} & (4) \\ 0 &\leq \chi < \chi_0^{(q)}. \end{aligned}$$

There is an uncertainty about the transition region in which neither expression is valid. However, extending the range of validity of Eqs. (3) and (4) to $\chi = \chi_0^{(q)}$ leads to an uncertainty in the effective total cross section of factor of 3/2. Since the uncertainty resides within a small numerical factor, the results are qualitatively correct. Quantatively, we have included angular regions which have previously been neglected in the cut off technique and thus the results are expected to be numerically more rigorous than those using the classical scattering cross section and a comparison is made in Section 6.

2. BINARY ENCOUNTERS

Consider the elastic scattering of a charged test particle whose speed lies between v and $v + dv$ by a charged field particle whose speed lies between V and $V + dV$. As viewed in the c.m. frame of reference the test particle will only change its direction after an encounter and therefore we can write its speed in the laboratory system as:

$$v'^2 = v_m^2 + \left(\frac{M}{m+M}\right)^2 v_r^2 + \frac{2M}{m+M} v_r v_m \cos \chi. \quad (5)$$

We use standard notation where m and M are respectively the masses of the test and field particles, v' is the speed of the test particle after an encounter, χ is the deflection angle in the c.m. system and v_m is the velocity of the center of momentum. The interaction between the two charged particles is described by a screened Coulomb potential and the differential cross sections for such an interaction are given by Eqs. (3) and (4).

The probability $g(v, v', V, \cos \theta) dv'$ that a test particle of velocity between v and $v + dv$ will emerge within a velocity interval between v' and $v' + dv'$ after an encounter with a field particle of velocity V whose direction makes an angle θ with the direction of the test particle is given by

$$g(v'; v, V, \cos \theta) dv' = - \frac{\sigma_g(v_r, \chi) d\Omega(\chi)}{\sigma_g(v_r)} \quad (6)$$

where $d\Omega(\chi)$ is the element of solid angle and $\sigma_g(v_r)$ is the total elastic cross section integrated over all deflection angles as given in reference 13. By the use of Eqs. (3) through (5) the probability given in Eq. (6) can be expressed in terms of velocity as:

$$g(v'; v, V, \cos \theta) = \begin{cases} 0 & v' > v_+ \\ \frac{2\pi \int_{v_-}^{v_+} v' (v_+^2 - v_-^2)}{v_r^4 \sigma_g(v_r) [v_+^2 - v'^2 + \frac{\chi_p^2}{4} (v_+^2 - v_-^2)]^2} & v_+ < v' \leq v_+ \\ \frac{2\pi \int_{v_-}^{v_+} v' (v_+^2 - v_-^2)}{v_r^4 \sigma_g(v_r) (v_+^2 - v'^2)^2} & v_- \leq v' < v_+ \\ 0 & v' < v_- \end{cases} \quad (7)$$

where v_+ , v_0 and v_- are the values of v' at $\chi = 0$, $\chi = \chi_0$ and $\chi = 1$, respectively.

3. TOTAL CROSS SECTION FOR ARBITRARY VELOCITY DISTRIBUTION

Consider the interaction between a charged test particle and field particles of number density n with an angular velocity distribution $N(v, \Omega(\theta))$. The encounter rate per unit volume is

$$v_r \sigma_s(v_r) n N(v, \Omega(\theta)) d\Omega(\theta) dV \quad (8)$$

which can also be expressed as

$$v n d\sigma_s(v, V, \Omega(\theta)) \quad (9)$$

Here $d\sigma_s(v, V, \Omega(\theta))$ is the microscopic cross section for a test particle to encounter a field particle whose velocity lies in a direction contained in the solid angle $\Omega(\theta)$ relative to the initial test particle velocity. Equating both results gives an expression for the differential cross section defined by Eq. (9), i.e.,

$$d\sigma_s(v, V, \Omega(\theta)) = \frac{v_r}{v} \sigma_s(v_r) N(v, \Omega(\theta)) d\Omega(\theta) dV. \quad (10)$$

Now the total cross section for elastic scattering between the test particle under question and the field particles is obtained by multiplying Eq. (10) by the probability given in Eq. (7) and integrating over v' , V and $\Omega(\theta)$; that is

$$\frac{\sigma_s(v)}{\sigma_s(v)} = \int_{v'} \int_V \int_{\Omega(\theta)} g(v'; v, V, \cos\theta) d\sigma_s(v, V, \Omega(\theta)) dv'. \quad (11)$$

For a given distribution, Eqs. (7) and (10) can be substituted into Eq. (11) and the solution gives the desired cross section.

4. SCATTERING CROSS SECTION IN A MAXWELLIAN MEDIUM

Consider a single species Maxwellian plasma of energy kT where

Eq. (11) can be written explicitly as:

$$\begin{aligned} \overline{\sigma_s(v)} = & \frac{2\pi q^2}{v} \left(\frac{M}{2\pi kT} \right)^{3/2} \\ & \left\{ \int_{v_-}^{v_0} \int_0^{v_+} \int_{-1}^1 \frac{v' V^2 (v_+^2 - v_-^2)}{v_r^3 (v_+^2 - v'^2)^2} \exp\left(-\frac{MV^2}{2kT}\right) d\cos\theta dv' dV \right. \\ & \left. + \int_{v_0}^{v_+} \int_0^{v_+} \int_{-1}^1 \frac{v' V (v_+^2 - v_-^2) \exp\left(-\frac{MV^2}{2kT}\right) d\cos\theta dv' dV}{v_r^3 \left[v_+^2 - v'^2 + \frac{\hbar^2 (v_+^2 - v_-^2)}{4\mu^2 v_r^2 \lambda_D^2} \right]^2} \right\}. \end{aligned} \quad (12)$$

While the evaluation of this equation is tedious, it is straightforward and gives the result:

$$\overline{\sigma_s(v)} = 6\pi \left(\frac{q_t q_f}{4\pi\epsilon_0} \right)^2 \left(\frac{\lambda_D}{\hbar v} \right)^2 \operatorname{erf}(x) \quad (13)$$

where x is the dimensionless velocity $(Mv^2/2kT)^{1/2}$. In the limit of $x > 1$, the error function asymptotically approaches unity. The expression preceeding the error function is in fact the total cross section for elastic scattering by a fixed scattering center. Since a plasma is composed of two or more species it is more convenient to describe the medium by a total macroscopic cross section $\bar{\Sigma}_s$, that is

$$\bar{\Sigma}_s = 6\pi \left(\frac{Z_e e^2}{4\pi\epsilon_0} \right)^2 \frac{\epsilon_0}{k^2 v^2} \left[kT_e \operatorname{erf}(x_e) + \sum_j kT_j \operatorname{erf}(x_j) \right] \quad (14)$$

where the summation is over all ion species.

5. SCATTERING PROBABILITIES IN A THERMONUCLEAR PLASMA

The probability per unit time that a charged test particle will undergo scattering is related to the total cross section by the relation

$$p_s(v) = v \overline{\Sigma_s(v)} \quad (15)$$

Consider the interaction of an energetic ion produced through a fusion reaction with the background thermonuclear plasma. In this case,

$x_e < 1$ and $x_i > 1$ and the scattering probability is approximately given by:

$$p_s(v) \approx \frac{3 Z_t^2 e^4}{8 \pi \epsilon_0 \hbar^2 v} \left[2 \left(\frac{m_e E k T_e}{\pi m} \right)^{1/2} + \sum_j k T_j \right] \quad (16)$$

because we approximate

$$\text{erf}(x) \approx \begin{cases} \frac{2}{\sqrt{\pi}} x & x < 1 \\ 1 & x > 1. \end{cases}$$

More terms can be included for greater accuracy but here we are only interested in the investigation of the relative probability of scattering by different species. The values of the dimensionless velocity x for

charged fusion reaction products in plasmas of thermonuclear interest are always greater than one for ions and less than one for electrons so long as $T_e > 10$ keV.

For all cases considered here the probability that an ion produced from the fusion reaction will scatter via plasma ions is larger than the probability that it will scatter via electrons. Because the reaction produced particle is degraded in energy due to collisions, Eq. (16) will not hold for $x_i \leq 1$ and $x_e \ll 1$. At these lower test particle speeds the scattering probabilities become:

$$P_i(v) \approx \frac{3Z_t^2 e^4}{4\sqrt{2} \pi^{3/2} \epsilon_0 \hbar^2} \left[\sqrt{m_e k T_e} + \sum_j \sqrt{m_j k T_j} \left(1 - \frac{m_j v^2}{6k T_j} \right) \right]. \quad (17)$$

In practice the plasma gains energy from the hotter charged reaction products and consequently the plasma temperature increases. However, in the steady state, energy losses may balance energy gains and the plasma temperature may be fairly constant. As long as $T_i \sim T_e$ the second term dominates

the first and the probability of scattering via hydrogen ions is 30 to 45 times larger than that via electrons. The same conclusion can be reached by investigating the probability distribution function given by Eq. (7). The probability per unit speed that a test particle emerges after an encounter with a speed v_+ is:

$$g(v_+; v, V, \cos\theta) dv_+ \approx \frac{4\mu m v_r \lambda_D^2 v_+ dv_+}{3 v_m \hbar^2} \quad (18)$$

The probability per unit speed that the velocity of a test particle is reduced to v_- is

$$g(v_-; v, V, \cos\theta) dv_- \approx \frac{m \hbar^2 v_- dv_-}{12 \mu^2 v_r^2 \lambda_D^2 v_m} \quad (19)$$

and this corresponds to a head on collision.

Comparison of Eqs. (18) and (19) reveals that the probability per unit velocity of a small change in the velocity of the test particle is much much larger than the corresponding probability for large changes. As a matter of fact if $v \gg V$ a test particle may lose all its energy in one encounter with an ion of equal mass. However, the probability that a 1 MeV reaction triton loses its energy in a single encounter with 10 keV plasma triton is of the order of 10^{-7} per joule; while the probability that the test triton loses a very small fraction of its energy in a single encounter is about 10^{32} per joule. In addition, Eq. (18) indicates that the velocity of the test particle is more likely to be reduced by an encounter with an ion than with an electron while Eq. (19) shows that electrons are more likely to reduce v to v_- rather than the ions.

Thus we can conclude that the probability of scattering per unit time of test particles by ions exceeds the scattering rate by electrons. In addition the probability that the test particles transfer their energy in small increments to the ions is significantly more probable than transferring energy in large increments to the electrons.

Although Eq. (11) has been applied to a Maxwellian plasma, other velocity distributions can be considered as well. For example, one could consider the loss cone distribution found in mirror machines. The evaluation of the total scattering cross section and the scattering rates using such distributions is straightforward, although it is complicated by the fact that velocity distribution of the plasma in the mirror machine is not isotropic. The degree of anisotropy depends on the mirror ratio and so is the scalar velocity distribution.

It should be emphasized that we have restricted our examples to the case where $T_e = T_i$, however, the result in Eq. (14) is not so restricted.

6. COMPARISON WITH THE DEBYE CUT OFF TECHNIQUE

To calculate the total cross section for elastic scattering between two particles it is rather conventional to integrate the Rutherford differential cross section over all angles above some cut off angle and thereby neglect small angle encounters. The cut off procedure is only meaningful if the Debye sphere contains many plasma particles of the species under consideration¹⁵. This condition is well satisfied in a thermonuclear plasma, however, the choice of the limiting cut off angle is a crucial problem. The rule is to choose the classical cut off angle $\chi_o^{(c)}$ as the lower limit of the integration if $q_e q_f / (4\pi\epsilon_o \hbar v_r) \gg 1$.

If such is not the case the quantum mechanical cut off angle $\chi_o^{(q)}$ must be chosen and both angles are given in Eqs. (1) and (2).

For energetic test particles or interactions with electrons and a low Z plasma, the quantum mechanical angle is the proper angle.

However, in following the degradation of energy of a test particle in a plasma there will be an energy at which $\chi_o^{(q)} \approx \chi_o^{(c)}$ below which the cut off angle has to be $\chi_o^{(c)}$. In studying the total cross section for binary interactions such a choice is not critical but if the field particles constitute a plasma of a specific velocity distribution, the results will be sensitive to such choices.

If it is applicable to use $\lambda_o^{(c)}$, the integrated total cross section for elastic scattering of a test particle by Maxwellian single species field particles is:

$$\overline{\sigma_s^{(c)}(v)} = \sigma_s^{(c)}\left(\frac{1}{\sqrt{\pi}} x^2\right) \left[x \exp(-x^2) + \frac{\sqrt{\pi}}{2} (2x^2 + 1) \operatorname{erf}(x) \right] \quad (20)$$

and $\sigma_s^{(c)}$ is given by the classical scattering cross section in the limit of $v \gg V$, i.e.,

$$\sigma_s^{(c)} = 4\pi \lambda_D^2 \quad (21)$$

Equation (20) is derived from Eq. (11) with the proper choice of $g(v'; v, V, \cos \theta)$.

The scattering probability per unit time for the interaction of a test particle with a plasma composed of electrons of $x_e < 1$ and ions of $x_i \gg 1$ is:

$$P_s(v) = \frac{4\pi\epsilon_o v}{e^2} \left[\frac{1}{\sqrt{\pi}} \sqrt{\frac{m k^3 T_e}{m_e E}} + \frac{k T_i}{Z_i^2} \right] \quad (22)$$

If $T_e = T_i$, scattering by electrons is seen to dominate that by ions. The apparent contradiction between this result and that found in Section 5 is due to the fact that within the velocity range in which Eq. (22) is valid, the cut off angle is less than that determined by the uncertainty principle. Consequently, we have included energy ranges for which classical results are strictly invalid. The use of the proper cross section given in Eq. (4) for angles larger than $\chi_0^{(q)}$ leads to results which agree with those of Eq. (16) within a numerical factor of order unity.

In the limit of $x_e < x_i < 1$

$$p_e(v) \simeq \frac{8\pi}{e^2} \epsilon_0 v^{\frac{3}{2}} \left[\frac{k T_e}{x_e} + \frac{k T_i}{Z_i^2 x_i} + \frac{k T_i x_i}{3 Z_i^2} \right]. \quad (23)$$

From Eq. (23) the scattering probability by electrons is larger than that by ions. The apparent contradiction between the implication of Eqs. (17) and (23) is attributed to the fact that in the derivation of Eq. (23) the classical cut off angle is much larger than $\chi_0^{(q)}$. The choice of the larger cut off angle does not effect the large angle scattering cross section; however, the scattering probability and many other results are sensitive to such a choice. This is because the extremely gentle encounters leading to small angle scattering make up a large part of the total scattering probability. Thus the main difference between Eq. (17) and (23) is that the former takes into account important scattering events which are neglected by the latter. If the

contribution of these events (is added to Eq. (23) we will get scattering probabilities similar to Eq. (17) within a small numerical factor. In addition, the cut off angle $\chi_0^{(c)}$ in the limit of $x \ll 1$ is inversely proportional to the square of the thermal velocity of the field particles under consideration times the reduced mass of the test and field particles. Thus for $T_e \sim T_i$ the cut off angle for electrons is likely to be lower than that for plasma ions and more small angle scattering events will be included in the calculation of the scattering probability via electrons than that via ions.

Generally speaking, extreme caution must be exercised in using the cut off technique. In the study of test particle interactions with thermonuclear plasmas we are dealing with a broad energy spectrum and there is always a danger of cutting off dominant contributions to the results. In some cases the de Broglie wavelength is much less than the impact parameter of closest approach and consequently the scattering distribution is extremely rare for angles less than the limit given by $\chi_0^{(c)}$. Consequently, the error in the use of the cut off technique is not significant. On the other hand if we are dealing with weak interactions, the small angle scattering becomes more frequent than large angle scattering and an improper choice of the low limit may lead to erroneous results. In addition, the particular composition of the plasma affects the cut off angle. Both classical and quantum mechanical cut off angles depend on the reduced mass and the relative velocity. Consequently the limit on the scattering angles for small mass electrons is different from that for ions. Furthermore, the roles of ions and

electrons can not simply be compared to each other since the neglected small angles events may be more significant for one than the other.

Footnotes

1. A. A. Hussein, Ph.D. Thesis (University of Wisconsin, Madison, 1970) and also PLP-323. (Nov. 1969).
2. D. J. Rose, Nucl. Fusion 9, 183 (1969).
3. R. F. Post, International Nuclear Fusion Reactor Conference, Culham, U.K., Sept. (1969) (to be published).
4. L. G. Kuo-Petravic, M. Petravic, C. J. H. Watson, International Nuclear Fusion Reactor Conference, Culham, U.K., Sept. (1969) (to be published).
5. R. G. Mills, Nucl. Fusion 7, 223 (1967).
6. M. N. Rosenbluth, W. M. MacDonald and D. L. Judd, Phys. Rev. 107, 1 (1957).
7. H. C. Kranzer, Phys. Fluids 4, 214 (1961).
8. S. T. Butler and M. J. Buckingham, Phys. Rev. 126, 1 (1962).
9. A. A. Hussein and H. K. Forsen, Phys. Rev., (to be published 1970).
10. E. J. Williams, Revs. Mod. Phys. 17, 217 (1945).
11. N. Bohr, I. Kgl. Danske Videnskab. Selskab, Mat.-Fys. Medd. 18, 8 (1948).
12. E. Everhart, G. Stone and R. Carbone, Phys. Rev. 99, 1287 (1955).
13. G. H. Lane and E. Everhart, Phys. Rev. 117, 920 (1960).
14. A. A. Hussein and H. K. Foreen, Bull. Am. Phys. Soc. 15, 96 (1970).
15. W. P. Allis, "Motions of Ions and Electrons", Handbuch der Physik, Vol. 21, Springer, Berlin (1956).