VIBRATION OF FUEL BUNDLES

by

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NOMENCLATURE

A	Cross-section of rod
a_{jn}	Arbitrary constant to be determined in Eq. (1)
a_{pnqm}	Coefficient given by Eq. (54)
b_{jn}	Arbitrary constant to be determined in Eq. (1)
b_{pnqm}	Coefficient given by Eq. (54)
c_p, c'_p	Torsional spring at the support of cylinder p
C	Viscous damping coefficient
C'	Drag coefficient associated with fluid pressure
$\bar{C}$	Drag coefficient associated with skin friction
c_{mn}	Coefficient given by Eq. (54)
c_{pnqm}	Coefficient given by Eq. (54)
d_{mn}	Coefficient given by Eq. (54)
$\vec{e}_i, \vec{e}_x, \vec{e}_y$	Unit vector
e_{mn}	Coefficient given by Eq. (54)
E	Modulus of elasticity
f	External loading per unit length
f_R	Real part of frequency
f_I	Imaginary part of frequency
F_D	Damping force per unit length given by Eq. (38)
F_L	Axial force per unit length due to fluid drag
F_N	Normal force per unit length due to fluid drag
F_p	Resultant fluid pressure acting on rod surface per unit length
f_{pn}	Generalized force given by Eq. (54)
F_{21}	Hydrodynamic force acting on cylinder 2 due to the motion of cylinder 1
g	Acceleration due to gravity
G, G_x, G_y	Gap

H_1	Fluid dynamic force acting on cylinder 1 in the x direction
h_{pn}	Generalized coordinate in Eq. (51)
I	Moment of inertia
k	Total number of cylinders under consideration
k_p, k'_p	Spring constant at the support of cylinder p
l	Rod length
m	Rod mass per unit length
m_j	Rod mass per unit length of cylinder j
M	Bending moment
$\vec{n}$	Unit vector normal to cylinder surface
n_i	$\vec{e}_1 \cdot \vec{n}$
p	Fluid pressure
p_0	Fluid pressure at upstream
p_1	Pressure gradient
P	Weighted modal matrix
Q	Shear force
r	Radial coordinate
r_j	Radial coordinate associated with cylinder j
R	Radius of a group of cylinders consisting of identical cylinders
R_j	Radius of cylinder j
R_{ij}	Distance between the centers of cylinders i and j
S_1	Surface of cylinder 1
S_0	A surface
t	Time
T	Axial tension
u_1	Displacement in the x direction of cylinder 1
U	Fluid velocity component in radial direction

U_i	Displacement of cylinder i
v_i	Displacement in the y direction of cylinder i
V	Axial flow velocity
V_i	Fluid dynamic force acting on cylinder i in the y direction
x	Cartesian coordinate
x_j	Coordinate in x direction associated with cylinder j
y	Cartesian coordinate
y_j	Coordinate in y direction associated with cylinder j
z	Axial coordinate
α_p	Eigenvector vector
α_{il}	Added mass coefficients
α_{inl}	Constant determined from Eqs. (9)
β_{il}	Added mass coefficients
β_{inl}	Constant determined from Eqs. (9)
γ_{pq}	Added mass of cylinder p due to motion of cylinder q
γ_{inl}	Constant determined from Eqs. (9)
δ_{il}	Kronecker's Delta
δ_{inl}	Constant determined from Eqs. (9)
ζ, ζ_{pn}	Modal damping ratio
$\bar{\theta}$	The angle between the axis of cylinder and flow direction
θ_j	Angular coordinate associated with cylinder j
λ	Eigenvalue of Eqs. (58) and (59)
Λ	Diagonal matrix formed from eigenvalue Ω^2
μ	Internal damping coefficient
μ_p	Principal value of added mass matrix
μ'_p	Principal value of added mass coefficient matrix
ρ	Fluid density
σ_{il}	Added mass coefficients

τ_{12}	Added mass coefficients
ϕ_j	Velocity potential associated with the motion of cylinder j assuming all other cylinders are stationary
ϕ	Total velocity potential
$\psi_m(z)$	Orthonormal function of cylinders in vacuo
ψ_{1j}	Angle between the x-axis and the vector from the center of cylinder 1 to that of cylinder j
ω_n	nth natural frequency of a group of identical cylinders in vacuo
ω_{pn}	nth natural frequency of cylinder p in vacuo
Ω	Vibration frequency ($= \Omega_R + i\Omega_I$)
Ω_p	Natural frequency of the pth mode
Ω_{pn}	Natural frequency of the p mode associated with the axial wave number n

Subscripts

$i = 1, 2, 3, \dots k$
 $j = 1, 2, 3, \dots k$
 $m = 1, 2, 3, \dots \infty$
 $n = 1, 2, 3, \dots \infty$
 $p = 1, 2, 3, \dots 2k$
 $q = 1, 2, 3, \dots 2k$

Superscripts

i = Variables written in terms of coordinates associated with
cylinder i

ABSTRACT

Several mathematical models have been proposed for calculating fuel rod responses in axial flows based on a single rod consideration. The spacing between fuel rods in liquid metal fast breeder reactors is small; hence fuel rods will interact with one another due to fluid coupling. The objective of this paper is to study the coupled vibration of fuel bundles. To account for the fluid coupling, a computer code, AMASS, is developed to calculate added mass coefficients for a group of circular cylinders based on the potential flow theory. The equations of motion for rod bundles are then derived including hydrodynamic forces, drag forces, fluid pressure, gravity effect, axial tension, and damping. Based on the equations, a method of analysis is presented to study the free and forced vibrations of rod bundles. Finally, the method is applied to a typical LMFBR fuel bundle consisting of seven rods.

I. INTRODUCTION

Flow-induced vibrations of fuel rods have been extensively studied both experimentally and analytically; a brief review is given in Reference 1. The objectives of those studies are to understand the dynamic characteristics of fuel rods subjected to axial flows, to predict fuel rod displacements at a given flow velocity, and to minimize the detrimental effects of flow-induced vibrations. Several mathematical models have been proposed for calculating fuel rod responses; those models are based on the vibration of a single rod not coupling with the rods surrounding it.

In a typical design of LMFBR fuel rods, the spacing between fuel rods is small. The vibration of an element will interact with the surrounding ones because of fluid coupling. Therefore, any motion of an element in a fuel bundle will excite all other elements and fuel rods will respond as a group rather than as a single element. To the writer's knowledge, such coupled modes^{*} of fuel bundle vibration have not been analyzed. In order to model flow-induced vibration of fuel bundles, fluid coupling effects must be included. The objectives of this paper are to present a general method of analysis for coupled vibration of a group of parallel circular cylinders subjected to fluid flows and to develop a model for fuel bundle vibration. A detailed study is presented for a typical LMFBR seven-rod bundle.

In addition to nuclear fuel bundles, other structural components consisting of a group of circular cylinders, ranging from heat exchanger tubes, piles, parallel pipelines, and bundled transmission lines, frequently experience vortex-excited oscillations, fluidelastic instability, and other types of flow-induced vibrations. Many investigators have studied the

* A coupled mode refers to a natural mode in which all cylinders in a group vibrate at the same frequency with definite phase relations among the cylinders. On the other hand, an uncoupled mode refers to a natural mode in which only a single cylinder is oscillating, while all others are stationary.

dynamics of various types of structural components consisting of multiple cylinders. Those include two parallel cylinders [2,3,4], two cylinders located concentrically and separated by a fluid [5-8], a row of cylinders [2,9-13], and a group of cylinders [14-20]. Those studies have revealed several complex fluid-structural interaction characteristics. Despite the progress being made on the dynamics of multiple cylinders in a liquid, a general method of analysis is not available. The method of analysis presented in this paper is not only applicable to fuel rod vibrations, but also useful in the study of other structural components.

II. ADDED MASS COEFFICIENTS

A. Formulation and Solution

The added mass coefficients for a row of circular cylinders have been studied previously [13]. The study is based on the potential flow theory. The same method of analysis is applied for rod bundles.

Consider the motion of a group of k circular cylinders vibrating in an ideal incompressible fluid, as shown in Fig. 1. The axes of the cylinders are perpendicular to the x - y plane. Let R_j be the radius of cylinder j and (x_j, y_j) be the local coordinates associated with cylinder j .

The velocity potential associated with the motion of cylinder j , assuming all other cylinders are stationary, can be written

$$\phi_j = \sum_{n=1}^{\infty} \left(\frac{R_j^{n+1}}{r_j^n} \right) (a_{jn} \cos n \theta_j + b_{jn} \sin n \theta_j) , \quad (1)$$

where r_j and θ_j are cylindrical coordinates referred to cylinder j , and a_{jn} and b_{jn} are arbitrary constants to be determined. The total field at a point in the fluid consists of the partial fields generated by all cylinders; i.e.,

$$\phi = \sum_{j=1}^k \phi_j . \quad (2)$$

All ϕ_j can be written in terms of the local coordinates associated with cylinder i using the following relationships [13]:

$$\frac{\cos n \theta_j}{r_j^n} = (-1)^n \sum_{m=0}^{\infty} \frac{(n+m-1)! r_i^m}{m! (n-1)! R_{ij}^{n+m}} \cos[m\theta_i - (m+n)\psi_{ij}] ,$$

and

$$\frac{\sin n \theta_j}{r_j^n} = (-1)^{n+1} \sum_{m=0}^{\infty} \frac{(n+m-1)! r_i^m}{m! (n-1)! R_{ij}^{n+m}} \sin[m\theta_i - (m+n)\psi_{ij}] , \quad (3)$$

where R_{ij} is the distance between the centers of cylinders i and j , and ψ_{ij} is the angle between the x -axis and the vector from the center of cylinder i to that of cylinder j . Let the superscript i denote the

Fig. 1. A group of k circular cylinders vibrating in a liquid

variable written in terms of the local coordinates associated with cylinder

1. Therefore,

$$\phi^1 = \phi_1 + \sum_{j=1}^k \star \phi_j^1, \quad (4)$$

where $\sum^{\star}$ denotes the summation for j from 1 to k except $j = 1$. Using

Eqs. (1) and (3) gives

$$\begin{aligned} \phi_j^1 = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^n (n+m-1)! R_j^{n+1} r_i^m}{m! (n-1)! R_{ij}^{n+m}} \{ a_{jn} \cos[m\theta_i - (m+n)\psi_{ij}] \\ - b_{jn} \sin[m\theta_i - (m+n)\psi_{ij}] \}. \end{aligned} \quad (5)$$

The velocity components of cylinder 1 in the x and y directions are $\frac{\partial u_1}{\partial t}$ and $\frac{\partial v_1}{\partial t}$ respectively. The fluid velocity component in the r direction is U . In terms of the local coordinates of cylinder 1,

$$U^1 = \frac{\partial \phi^1}{\partial r_1}. \quad (6)$$

At the interface of the cylinders and fluid, the following conditions must be satisfied:

$$U^1 \Big|_{r_1=R_1} = \frac{\partial u_1}{\partial t} \cos \theta_1 + \frac{\partial v_1}{\partial t} \sin \theta_1, \quad i = 1, 2, 3, \dots, k. \quad (7)$$

Substituting Eq. (6) into (7) and using (4) and (5), a_{jn} and b_{jn} are determined as follows:

$$a_{in} = \sum_{\ell=1}^k \left(\alpha_{in\ell} \frac{\partial u_{\ell}}{\partial t} + \gamma_{in\ell} \frac{\partial v_{\ell}}{\partial t} \right),$$

and

$$b_{in} = \sum_{\ell=1}^k \left(\delta_{in\ell} \frac{\partial u_{\ell}}{\partial t} + \beta_{in\ell} \frac{\partial v_{\ell}}{\partial t} \right).$$

(8)

$\alpha_{in\ell}$, $\beta_{in\ell}$, $\gamma_{in\ell}$, and $\delta_{in\ell}$ are solutions of the following equations:

$$\begin{aligned}
(-n)\alpha_{inl} + \sum_{j=1}^{k*} \sum_{m=1}^{\infty} \frac{(n+m-1)! R_i^{n-1} R_j^{m+1}}{(n-1)!(m-1)! R_{ij}^{n+m}} (-1)^m \{ \alpha_{jml} \cos[(m+n)\psi_{ij}] \\
+ \delta_{jml} \sin[(m+n)\psi_{ij}] \} = \delta_{nl} \delta_{il} , \\
(-n)\delta_{inl} + \sum_{j=1}^{k*} \sum_{m=1}^{\infty} \frac{(n+m-1)! R_i^{n-1} R_j^{m+1}}{(n-1)!(m-1)! R_{ij}^{n+m}} (-1)^m \{ \alpha_{jml} \sin[(m+n)\psi_{ij}] \\
- \delta_{jml} \cos[(m+n)\psi_{ij}] \} = 0 , \\
(-n)\gamma_{inl} + \sum_{j=1}^{k*} \sum_{m=1}^{\infty} \frac{(n+m-1)! R_i^{n-1} R_j^{m+1}}{(n-1)!(m-1)! R_{ij}^{n+m}} (-1)^m \{ \gamma_{jml} \cos[(m+n)\psi_{ij}] \\
+ \beta_{jml} \sin[(m+n)\psi_{ij}] \} = 0 , \\
(-n)\beta_{jn l} + \sum_{j=1}^{k*} \sum_{m=1}^{\infty} \frac{(n+m-1)! R_i^{n-1} R_j^{m+1}}{(n-1)!(m-1)! R_{ij}^{n+m}} (-1)^m \{ \gamma_{jml} \sin[(m+n)\psi_{ij}] \\
- \beta_{jml} \cos[(m+n)\psi_{ij}] \} = \delta_{nl} \delta_{il} ,
\end{aligned}
\tag{9}$$

$$i, l = 1, 2, 3, \dots k, \quad n = 1, 2, 3, \dots \infty .$$

The fluid forces acting on the cylinders can be calculated from fluid pressure p ;

$$p = -\rho \frac{\partial \phi}{\partial t} , \tag{10}$$

where ρ is fluid density. The two components of fluid force acting on cylinder i in the x and y directions are H_i and V_i respectively;

$$\begin{aligned}
H_i &= - \int_0^{2\pi} p^i \Big|_{r_i=R_i} R_i \cos \theta_i d\theta_i , \\
\text{and} \\
V_i &= - \int_0^{2\pi} p^i \Big|_{r_i=R_i} R_i \sin \theta_i d\theta_i .
\end{aligned}
\tag{11}$$

Using Eqs. (4), (5), (8), (10) and (11) gives

$$H_i = -\rho\pi \sum_{l=1}^k \left(\frac{R_i + R_l}{2} \right)^2 \left(\alpha_{il} \frac{\partial^2 u_l}{\partial t^2} + \sigma_{il} \frac{\partial^2 v_l}{\partial t^2} \right),$$

and

(12)

$$V_i = -\rho\pi \sum_{l=1}^k \left(\frac{R_i + R_l}{2} \right)^2 \left(\tau_{il} \frac{\partial^2 u_l}{\partial t^2} + \beta_{il} \frac{\partial^2 v_l}{\partial t^2} \right),$$

where

$$\begin{aligned} \alpha_{il} &= \frac{4R_i^2}{(R_i + R_l)^2} \left\{ -\alpha_{iil} - \sum_{j=1}^{k*} \sum_{n=1}^{\infty} \left(\frac{R_j}{R_{ij}} \right)^{n+1} (-1)^n \left(\alpha_{jnl} \cos[(n+1)\psi_{ij}] \right. \right. \\ &\quad \left. \left. + \delta_{jnl} \sin[(n+1)\psi_{ij}] \right) \right\}, \\ \beta_{il} &= \frac{4R_i^2}{(R_i + R_l)^2} \left\{ -\beta_{iil} - \sum_{j=1}^{k*} \sum_{n=1}^{\infty} \left(\frac{R_j}{R_{ij}} \right)^{n+1} (-1)^n \left(\gamma_{jnl} \sin[(n+1)\psi_{ij}] \right. \right. \\ &\quad \left. \left. - \beta_{jnl} \cos[(n+1)\psi_{ij}] \right) \right\}, \\ \sigma_{il} &= \frac{4R_i^2}{(R_i + R_l)^2} \left\{ -\gamma_{iil} - \sum_{j=1}^{k*} \sum_{n=1}^{\infty} \left(\frac{R_j}{R_{ij}} \right)^{n+1} (-1)^n \left(\gamma_{jnl} \cos[(n+1)\psi_{ij}] \right. \right. \\ &\quad \left. \left. + \beta_{jnl} \sin[(n+1)\psi_{ij}] \right) \right\}, \\ \tau_{il} &= \frac{4R_i^2}{(R_i + R_l)^2} \left\{ -\delta_{iil} - \sum_{j=1}^{k*} \sum_{n=1}^{\infty} \left(\frac{R_j}{R_{ij}} \right)^{n+1} (-1)^n \left(\alpha_{jnl} \sin[(n+1)\psi_{ij}] \right. \right. \\ &\quad \left. \left. - \delta_{jnl} \cos[(n+1)\psi_{ij}] \right) \right\}. \end{aligned} \quad (13)$$

α_{il} , β_{il} , σ_{il} and τ_{il} are called added mass coefficients. α_{ii} , β_{ii} , σ_{ii} , and τ_{ii} are self-added mass coefficients, which are proportional to the hydrodynamic force acting on cylinder i due to its own acceleration, while the others are mutual-added mass coefficients, which are proportional to the hydrodynamic force acting on a cylinder due to the acceleration of another cylinder.

B. Reciprocal Relations

Consider the case that, in a group of k cylinders, cylinder i is moving with a velocity $\frac{\partial u_i}{\partial t} \vec{e}_i$, where $\vec{e}_i$ is a unit vector, and all other cylinders

are stationary. The mathematical solution for the fluid velocity potential is given by

$$\phi = \frac{\partial U_1}{\partial t} \phi_1, \quad (14)$$

ϕ_1 is the solution of the following problem:

$$\begin{aligned} \nabla^2 \phi_1 &= 0, \\ \nabla \phi_1 \cdot \vec{n} &= \frac{\partial \phi_1}{\partial n} = \vec{e}_1 \cdot \vec{n} = n_1 \quad \text{on } S_1, \\ \nabla \phi_1 \cdot \vec{n} &= 0 \quad \text{on all other } S_j \quad (j \neq 1), \end{aligned} \quad (15)$$

and certain infinity conditions, where $\vec{n}$ is a unit vector normal to the cylinder surface, and S_1 is the surface of cylinder 1. The hydrodynamic force acting on cylinder l in the direction of $\vec{e}_l$ is given by

$$F_{l1} = \left(\iint_{S_l} \rho \phi_1 n_l dS_l \right) \frac{\partial^2 U_1}{\partial t^2}. \quad (16)$$

Similarly, consider the case that all cylinders are stationary except cylinder l moving in the direction of $\vec{e}_l$ with a velocity $\frac{\partial U_l}{\partial t}$. The fluid velocity potential is

$$\phi = \frac{\partial U_l}{\partial t} \phi_l. \quad (17)$$

ϕ_l is the solution of the following problem:

$$\begin{aligned} \nabla^2 \phi_l &= 0, \\ \nabla \phi_l \cdot \vec{n} &= \frac{\partial \phi_l}{\partial n} = \vec{e}_l \cdot \vec{n} = n_l \quad \text{on } S_l, \\ \nabla \phi_l \cdot \vec{n} &= 0 \quad \text{on all other } S_j \quad (j \neq l), \end{aligned} \quad (18)$$

The hydrodynamic force acting on cylinder 1 in the direction of $\vec{e}_1$ is

$$F_{1l} = \left(\iint_{S_1} \rho \phi_l n_1 dS_1 \right) \frac{\partial^2 U_l}{\partial t^2}. \quad (19)$$

Using Eqs. (15), (16), (18) and (19) gives

$$\begin{aligned} F_{1l} &= -\gamma_{1l} \frac{\partial^2 u_l}{\partial t^2} , \\ F_{l1} &= -\gamma_{l1} \frac{\partial^2 u_1}{\partial t^2} , \end{aligned} \quad (20)$$

where

$$\begin{aligned} \gamma_{1l} &= -\rho \iint_{S_1} \phi_l \frac{\partial \phi_1}{\partial n} dS_1 , \\ \gamma_{l1} &= -\rho \iint_{S_l} \phi_1 \frac{\partial \phi_l}{\partial n} dS_l . \end{aligned} \quad (21)$$

Note that ϕ_1 and ϕ_l are two harmonic functions. According to Green's theorem,

$$\iint_{S_0} \phi_1 \frac{\partial \phi_l}{\partial n} dS_0 = \iint_{S_0} \phi_l \frac{\partial \phi_1}{\partial n} dS_0 \quad (22)$$

holds for any surface S_0 enclosing a region in which $\nabla^2 \phi_1$ and $\nabla^2 \phi_l$ are zero. Consider the region between the surface S_r enclosing $\sum_{j=1}^k S_j$ and apply Eq. (22) to the functions ϕ_1 and ϕ_l ;

$$\iint_{S_l} \phi_1 \frac{\partial \phi_l}{\partial n} dS_l - \iint_{S_1} \phi_l \frac{\partial \phi_1}{\partial n} dS_1 = \iint_{S_r} \left(\phi_1 \frac{\partial \phi_l}{\partial n} - \phi_l \frac{\partial \phi_1}{\partial n} \right) dS_r . \quad (23)$$

Let S_r go to infinity, the integral over S_r must vanish. Therefore

$$\iint_{S_l} \phi_1 \frac{\partial \phi_l}{\partial n} dS_l = \iint_{S_1} \phi_l \frac{\partial \phi_1}{\partial n} dS_1 . \quad (24)$$

It follows from Eqs. (21) and (24) that

$$\gamma_{1l} = \gamma_{l1} . \quad (25)$$

Let

$$U_1 \vec{e}_1 = u_1 \vec{e}_x ,$$

and

$$U_l \vec{e}_l = v_l \vec{e}_y ,$$

(26)

then from Eqs. (12) and (20), Eq. (25) is reduced to

$$\sigma_{1l} = \tau_{l1} . \quad (27)$$

Similarly, let $U_1 \vec{e}_1$ and $U_l \vec{e}_l$ be equal to other components of the cylinder displacements, it is shown that

$$\alpha_{1l} = \alpha_{l1} ,$$

and

(28)

$$\beta_{1l} = \beta_{l1} .$$

Equations (27) and (28) are the reciprocal relations. Physically, these mean that the hydrodynamic force acting on cylinder 1 in the $\vec{e}_1$ direction due to a unit acceleration of cylinder l in the $\vec{e}_l$ direction is equal to the hydrodynamic force acting on cylinder l in the $\vec{e}_l$ direction due to a unit acceleration of cylinder 1 in the $\vec{e}_1$ direction.

The added mass coefficients α_{il} , β_{il} , σ_{il} , and τ_{il} can be combined into a single added mass matrix γ_{pq} , where

$$\left[\gamma_{pq} \right]_{2k \times 2k} = \begin{bmatrix} \rho \pi \left(\frac{R_1 + R_l}{2} \right)^2 \alpha_{1l} & \rho \pi \left(\frac{R_1 + R_l}{2} \right)^2 \sigma_{1l} \\ \rho \pi \left(\frac{R_1 + R_l}{2} \right)^2 \tau_{1l} & \rho \pi \left(\frac{R_1 + R_l}{2} \right)^2 \beta_{1l} \end{bmatrix} . \quad (29)$$

Since γ_{pq} is symmetric, for a group of k cylinders, there are $k(2k+1)$ independent added mass components. It is possible to find a group of $2k$ principal axes such that

$$\gamma_{pq} = 0 \quad \text{for } p \neq q . \quad (30)$$

Let the eigenvalues and eigenvectors of $[\gamma_{pq}]$ be μ_p and $\{\alpha_p\}$ ($p = 1, 2, 3 \dots 2k$) respectively; one has the relation

$$[\gamma_{pq}]\{\alpha_q\} = \mu_p \{\alpha_p\} \quad (31)$$

As an example, consider the case of two cylinders with the same radius R . Assume that the cylinders are located on the x -axis. In this case, it is found that [13]

$$\begin{aligned} \sigma_{jl} &= \tau_{jl} = 0, \\ \alpha_{11} &= \alpha_{22} = \beta_{11} = \beta_{22}, \\ \alpha_{12} &= \alpha_{21} = -\beta_{12} = -\beta_{21}; \end{aligned} \quad (32)$$

hence, the added mass matrix can be written

$$[\gamma_{pq}] = \rho\pi R^2 \begin{bmatrix} \alpha_{11} & \alpha_{12} & 0 & 0 \\ \alpha_{12} & \alpha_{11} & 0 & 0 \\ 0 & 0 & \alpha_{11} & -\alpha_{12} \\ 0 & 0 & -\alpha_{12} & \alpha_{11} \end{bmatrix}. \quad (33)$$

It is found that the principal values of the added mass matrix are

$$\begin{aligned} \mu_1 &= \rho\pi R^2(\alpha_{11} - \alpha_{12}) \\ \mu_2 &= \rho\pi R^2(\alpha_{11} + \alpha_{12}) \\ \mu_3 &= \rho\pi R^2(\alpha_{11} + \alpha_{12}) \\ \mu_4 &= \rho\pi R^2(\alpha_{11} - \alpha_{12}) \end{aligned} \quad (34)$$

Those values may be considered as effective added masses for a group of cylinders. The physical meaning is as follows: In a group of cylinders consisting of identical cylinders, the natural frequencies of the group are equal to the frequency of the cylinder in vacuo multiplied by a factor

$\left(\frac{m}{m + \mu_p}\right)^{1/2}$, where m is cylinder mass. This will be shown in IV.B.

When all cylinders are identical, the principal values of added mass matrix are proportional to the principal values of added mass coefficient matrix. In this case, it is only necessary to consider the added mass coefficient matrix. This can be seen from Eqs. (34) that the principal values of the added mass coefficient matrix are $\mu_1' (= \alpha_{11} - \alpha_{12})$, $\mu_2' (= \alpha_{11} - \alpha_{12})$, $\mu_3' (= \alpha_{11} + \alpha_{12})$ and $\mu_4' (= \alpha_{11} + \alpha_{12})$.

C. Numerical Results

Based on the analysis, a computer program, AMASS, is developed for calculating added mass coefficients. This program can be used to calculate all elements of added mass matrix for a group of cylinders, in which the cylinders may have different diameters and may be arranged in arbitrary pattern.

Added mass coefficients are in terms of series solutions. A finite number of undetermined coefficients are determined by inverting the matrix formed by truncating the infinite sets of Eqs. (9). The added mass coefficients for four identical cylinders and a seven-rod bundle are shown in Fig. 2 as functions of the number of terms taken in the calculations. In Fig. 2, the cylinders are closely spaced (the gap-radius ratio is equal to 0.1); the convergent rates are relatively small. As the gap between the cylinders increases, fewer terms are needed. In the following calculations, n is taken to be ten.

Added mass coefficients depend on R_i , R_{ij} , ψ_{ij} and k . For a group of identical cylinders, arranged at equal distance, added mass coefficients depend on the gap only. Figure 3 shows the added mass coefficients as functions of gap-radius ratio for a three-cylinder bundle. For large G/R , α_{11} and β_{11} approach one, and all others approach zero. As G/R decreases, the magnitudes of all added mass coefficients increase.

Fig. 2. Variations of added mass coefficients with the number of terms n used in calculations

Fig. 3. Added mass coefficients of three circular cylinders as functions of gap-radius ratio G/R

Fuel bundles may be arranged in a hexagonal pattern or in an array. The self-added mass coefficient for the central element is of interest. Figures 4 and 5 show the values of the coefficient as functions of gap-radius ratio as well as the experimental data obtained by Moretti and Lowery [21]. In both cases, $\sigma_{11} = \tau_{11} = 0$. However, β_{11} is always equal to α_{11} in a hexagonal bundle and β_{11} is equal to α_{11} only for $G_x = G_y$ in an array. The analytical results and experimental data (the experimental data presented in Fig. 5 are for $G_x = G_y$) agree well over the range of parameters tested except the experimental values are consistently higher.

The principal values of added mass coefficient matrix for a seven-rod bundle and a nine-rod array are presented in Figs. 6 and 7 as functions of gap-radius ratio. Note that the principal values are not distinct; there are five repeated roots in each case. This is due to the symmetry of arrangement. If the rods are arbitrarily arranged and the rods are of different sizes, the eigenvalues will be distinct.

The largest eigenvalues of the added mass coefficient matrix μ'_1 is of particular interest, since it is associated with the lowest natural frequency of coupled modes. The ratios of μ'_1 to the largest value of the self-added mass coefficient σ_{11} are presented in Table 1. The ratio depends on the gap-radius ratio G/R , rod number, and rod arrangement. Moretti and Lowery [21] suggest that $2\sigma_{11} + 1$ can be used as an upper limit for μ'_1 . From Table 1, it is found that $\mu'_1 < (2\sigma_{11} + 1)$ except for the nine-rod array at $G/R = 0.1$.

Fig. 4. Theoretical and experimental values of added mass coefficients α_{11} for a seven-rod bundle

Fig. 5. Theoretical and experimental values of added mass coefficients α_{11} and β_{11} for a nine-rod array

Fig. 6. Eigenvalues of added mass coefficient matrix for a seven-rod bundle

Fig. 7. Eigenvalues of added mass coefficient matrix for a nine-rod array

Table 1. Ratio of the Largest Principal Value of Added Mass Coefficient Matrix to the Largest Self-Added Mass Coefficient for Two Rod Bundles

Rod Bundle	Gap-Radius Ratio G/R	Self-Added Mass Coefficient α_{11}	Principal Value of Added Mass Coefficient Matrix μ_1'	Ratio μ_1'/α_{11}
Seven-Rod Bundle	0.1	3.9355	7.6758	1.9504
	0.2	2.6110	5.0770	1.9445
	0.3	2.0680	3.9751	1.9222
	0.4	1.7704	3.3447	1.8892
	0.5	1.5840	2.9307	1.8502
	0.6	1.4577	2.6358	1.8082
	0.8	1.3006	2.2418	1.7237
	1.0	1.2098	1.9896	1.6446
	1.2	1.1526	1.8145	1.5743
	1.4	1.1145	1.6860	1.5128
	1.6	1.0880	1.5880	1.4596
Nine-Rod Array	0.1	3.1697	7.4923	2.3637
	0.2	2.2330	4.9331	2.2092
	0.3	1.8347	3.8542	2.1007
	0.4	1.6105	3.2403	2.0120
	0.5	1.4672	2.8391	1.9350
	0.6	1.3687	2.5547	1.8665
	0.8	1.2443	2.1768	1.7494
	1.0	1.1714	1.9364	1.6531
	1.2	1.1251	1.7701	1.5732
	1.4	1.0941	1.6484	1.5066
	1.6	1.0724	1.5558	1.4508

III. EQUATIONS OF MOTION OF A GROUP OF CIRCULAR CYLINDERS IN AXIAL FLOW

Consider a rod in a group of k circular cylindrical rods immersed in a fluid flowing at a velocity V parallel to the z -axis (Fig. 8). The rod has linear density (mass per unit length) m , flexural rigidity EI , and total length l . A small element δz of the rod is shown in Fig. 9, where the elastic forces, hydrodynamic forces, damping force and excitation forces associated with the motion in the x - z plane are given.

In Fig. 9, T is axial tension, Q is shear force, M is bending moment, $m \frac{\partial^2 u}{\partial t^2} \delta z$ is the inertia force of the rod (u is rod displacement in the x direction), $f \delta z$ is the external force acting on the rod surface, mg is rod gravity. These forces are quite obvious; however, the others require some explanations.

Fluid Pressure

It is known that the effect of uniform fluid pressure on a cylindrical rod is equivalent to applying an axial tension pA , where p is the fluid pressure and A is the cross-sectional area of the rod. Newland [22] has determined the pressure effect for nonuniform pressure; the force acting on the rod is

$$F_P = \frac{\partial}{\partial z} \left(pA \frac{\partial u}{\partial z} \right) . \quad (35)$$

Hydrodynamic Force

The hydrodynamic forces acting on a cylinder are given in Eqs. (12) for stationary fluid. The resultant force in the x direction is given by

$$H_i = -\rho\pi R^2 \sum_{\ell=1}^k \left(\alpha_{i\ell} \frac{\partial^2 u_{\ell}}{\partial t^2} + \sigma_{i\ell} \frac{\partial^2 v_{\ell}}{\partial t^2} \right) . \quad (36)$$

When the fluid is flowing with a velocity V , the hydrodynamic force can be calculated following Lighthill [23];

$$H_i = -\rho\pi R^2 \sum_{\ell=1}^k \left[\alpha_{i\ell} \left(\frac{\partial}{\partial t} + V \frac{\partial}{\partial z} \right)^2 u_{\ell} + \sigma_{i\ell} \left(\frac{\partial}{\partial t} + V \frac{\partial}{\partial z} \right)^2 v_{\ell} \right] . \quad (37)$$

Fig. 8. Schematic of a group of circular cylindrical rods in axial flow

Fig. 9. An element δz of a cylinder

Viscous Damping Force

The force representing the viscous damping effect can be expressed as

$$F_D = C \frac{\partial u}{\partial t} , \quad (38)$$

where C is an effective viscous damping coefficient. This coefficient can be determined experimentally by exciting the cylinder in a stationary fluid.

Drag Forces

The forces acting on a single rod set obliquely to a stream of fluid were discussed by Taylor [24]. For rough cylinders, he proposed the following expressions for normal and tangential drag forces:

$$F_N = \rho R V^2 (\bar{C} \sin \bar{\theta} + C' \sin^2 \bar{\theta}) , \quad (39)$$

and

$$F_L = \rho R V^2 \bar{C} \cos \bar{\theta} ,$$

where $\bar{\theta}$ is the angle between the axis of cylinder and flow direction, and $\bar{C}$ and C' are drag coefficients associated with skin friction and pressure respectively. Taylor's model is developed for a single cylinder in an infinite fluid. In a group of cylinders, mutual interaction will occur and the drag forces will be different from a single cylinder. To the writer's knowledge, no general method of solution and experimental data are available for drag forces acting on a group of cylinders. Therefore, Taylor's model is used for multiple cylinders. The drag forces in Eqs. (39) can be interpreted as interference drag.

For small oscillations, the angle of incidence can be approximated by

$$\bar{\theta} = \frac{1}{V} \left(\frac{\partial u}{\partial t} + v \frac{\partial u}{\partial z} \right) . \quad (40)$$

Using Eqs. (39) and (40) and neglecting the high order terms yield

$$F_N = \rho R V \bar{C} \left(\frac{\partial u}{\partial t} + v \frac{\partial u}{\partial z} \right) ,$$

and

$$F_L = \rho R V^2 \bar{C} . \quad (41)$$

Having all the force components, we are in a position to derive the equations of motion. The equations for translational equilibrium in the z and x directions, and rotational equilibrium are

$$\frac{\partial T}{\partial z} - mg + F_L = 0 , \quad (42)$$

$$\frac{\partial Q}{\partial z} - F_N + H_1 + F_P + F_L \frac{\partial u}{\partial z} + \frac{\partial}{\partial z} \left(T \frac{\partial u}{\partial z} \right) - F_D - m \frac{\partial^2 u}{\partial t^2} = -f , \quad (43)$$

and

$$Q = - \frac{\partial M}{\partial z} . \quad (44)$$

The rod material is postulated to obey a stress-strain relationship of the Kelvin type. Classical beam theory gives

$$M = EI \frac{\partial^2 u}{\partial z^2} + \mu I \frac{\partial^3 u}{\partial t \partial z^2} , \quad (45)$$

where μ is the internal damping coefficient, and I is the moment of inertia. Using Eqs. (44) and (45) gives

$$Q = -EI \frac{\partial^3 u}{\partial z^3} - \mu I \frac{\partial^4 u}{\partial t \partial z^3} . \quad (46)$$

Integrating Eq. (42) and using Eq. (41) yields

$$T(z) = T(l) + (\rho R V^2 \bar{C} - mg)(l - z) , \quad (47)$$

where $T(l)$ is the axial tension at the downstream end. Substituting Eqs. (46) and (47) into (43) yields

$$\begin{aligned}
& EI \frac{\partial^4 u}{\partial z^4} + \mu I \frac{\partial^5 u}{\partial t \partial z^4} + \rho \pi R^2 \sum_{l=1}^k \left[\alpha_{1l} \left(\frac{\partial}{\partial t} + v \frac{\partial}{\partial z} \right)^2 u_l + \sigma_{1l} \left(\frac{\partial}{\partial t} + v \frac{\partial}{\partial z} \right)^2 v_l \right] \\
& - [T(l) - mg(l-z) + \rho R V^2 \bar{C}(l-z) + pA] \frac{\partial^2 u}{\partial z^2} - \left(mg + \frac{\partial p}{\partial z} A \right) \frac{\partial u}{\partial z} \\
& + \rho R V \bar{C} \left(\frac{\partial u}{\partial t} + v \frac{\partial u}{\partial z} \right) + C \frac{\partial u}{\partial t} + m \frac{\partial^2 u}{\partial t^2} = f \quad . \quad (48)
\end{aligned}$$

Equation (48) is the equation of motion of cylinder 1 in the x-z plane. Similarly the equation of motion in the y-z plane can be derived. For convenience, a subscript p will be used to denote variables associated with cylinder p in the x-z plane, while p+k in the y-z plane. For example, $u_p(z,t)$, $E_p I_p$, C_p , and f_p are the displacement, flexural rigidity, damping coefficient, and excitation in the x direction, and the corresponding quantities in the y direction are u_{k+p} , $E_{k+p} I_{k+p}$, C_{k+p} , and f_{k+p} . Using these notations, we obtain the equations of motion for a group of k cylinders as follows:

$$\begin{aligned}
& E_p I_p \frac{\partial^4 u_p}{\partial z^4} + \mu_p I_p \frac{\partial^5 u_p}{\partial t \partial z^4} + \sum_{q=1}^{2k} \gamma_{pq} \left(\frac{\partial}{\partial t} + v \frac{\partial}{\partial z} \right)^2 u_q \\
& - [T_p(l) - (m_p g - \rho R_p V^2 \bar{C}_p)(l-z) + pA_p] \frac{\partial^2 u_p}{\partial z^2} \\
& - \left(m_p g + \frac{\partial p}{\partial z} A_p \right) \frac{\partial u_p}{\partial z} + \rho R_p V \bar{C}_p \left(\frac{\partial u_p}{\partial t} + v \frac{\partial u_p}{\partial z} \right) \\
& + C_p \frac{\partial u_p}{\partial t} + m_p \frac{\partial^2 u_p}{\partial t^2} = f_p \quad , \quad p, q = 1, 2, 3 \dots 2k \quad . \quad (49)
\end{aligned}$$

The appropriate boundary conditions associated with the equations of motion are:

for $z = 0$,

$$k_p u_p + E_p I_p \frac{\partial^3 u_p}{\partial z^3} = 0 \quad , \quad c_p \frac{\partial u_p}{\partial z} - E_p I_p \frac{\partial^2 u_p}{\partial z^2} = 0 \quad ; \quad (50)$$

and for $z = l$,

$$k_p' u_p - E_p I_p \frac{\partial^3 u_p}{\partial z^3} = 0 \quad , \quad c_p' \frac{\partial u_p}{\partial z} + E_p I_p \frac{\partial^2 u_p}{\partial z^2} = 0 \quad (50)$$

(Contd.)

where c_p and c_p' are torsional spring constants, and k_p and k_p' are displacement spring constants.

Various forms of equations of motion for a single cylinder in axial flow have been derived [25]. Those equations have been successfully employed to study the responses of simple rod to flow excitations. On the other hand, there has been only one attempt by Paidoussis to derive the equations of motion for a group of cylinders [26]. Since the hydrodynamic coupling effect is not included in the equation, it cannot be used to study coupled vibrations of a group of cylinders. Equations (49) include the fluid coupling; hence, it can be used for studies of both stability and response problems of fuel bundles.

IV. ANALYSIS

A. Flowing Fluid

In this analysis, it is assumed that all cylinders are of the same length and have the same type of boundary conditions. Let

$$u_p(z, t) = \sum_{n=1}^{\infty} h_{pn}(t) \psi_n(z) , \quad (51)$$

where $\psi_n(z)$ is the n th orthonormal function of the cylinders in vacuo; i.e.,

$$\frac{1}{l} \int_0^l \psi_m(z) \psi_n(z) dz = \delta_{mn} , \quad (52)$$

where l is the length of the cylinders. Using Eqs. (49), (51) and (52) gives

$$a_{pnqm} \ddot{h}_{qm} + b_{pnqm} \dot{h}_{qm} + c_{pnqm} h_{qm} = f_{pn} , \quad (53)$$

$$p, q = 1, 2, 3 \dots 2k , \quad m, n = 1, 2 \dots \infty ,$$

where

$$\begin{aligned} a_{pnqm} &= (m_p \delta_{pq} + \gamma_{pq}) \delta_{mn} , \\ b_{pnqm} &= \left(\frac{\mu_p}{E_p} m_p \omega_{pn}^2 + \rho R_p V \bar{C}_p + C_p \right) \delta_{pq} \delta_{mn} + 2V \gamma_{pq} c_{mn} , \\ c_{pnqm} &= m_p \omega_{pn}^2 \delta_{pq} \delta_{mn} - \{ [T_p(l) + p_o A_p - (m_p g - \rho R_p V^2 \bar{C}_p) l] d_{mn} \\ &\quad + [m_p g - \rho R_p V^2 \bar{C}_p - p_1 A_p] e_{mn} \} \delta_{pq} \\ &\quad + V^2 \gamma_{pq} d_{mn} + (\rho R_p V^2 \bar{C}_p - m_p g + p_1 A_p) \delta_{pq} c_{mn} , \end{aligned} \quad (54)$$

$$c_{mn} = \frac{1}{l} \int_0^l \frac{\partial \psi_m}{\partial z} \psi_n dz ,$$

$$d_{mn} = \frac{1}{l} \int_0^l \frac{\partial^2 \psi_m}{\partial z^2} \psi_n dz ,$$

$$e_{mn} = \frac{1}{\ell} \int_0^{\ell} \frac{\partial^2 \psi_m}{\partial z^2} \psi_n z dz , \quad (54)$$

(Contd.)

$$f_{pn} = \frac{1}{\ell} \int_0^{\ell} f_p \psi_n dz ,$$

and ω_{pn} is the n th natural frequency of cylinder p in vacuo. The fluid pressure has been assumed to have the form $p = p_0 - p_1 z$, where p_0 corresponds to the fluid pressure at the upstream end and p_1 is the pressure gradient.

Equations (53) consist of an infinite number of differential equations. However, typically, only a finite number of equations are selected from case to case, according to the desired accuracy. It is assumed that the maximum value for m and n in Eqs. (53) is taken to be N . Eq. (53) may be written in matrix form;

$$[B]\{\ddot{S}\} + [D]\{\dot{S}\} + [K]\{S\} = \{L\} , \quad (55)$$

where B , D , and K are square matrices and S and L are column matrices. The dimension is $(2k \times N)$.

When the fluid is flowing, K is not necessarily symmetric and positive definite and D is not necessarily proportional to B or K . Eq. (55) can be written

$$[\bar{\alpha}]\{\dot{\xi}\} + [\bar{\beta}]\{\xi\} = \{\gamma\} , \quad (56)$$

where

$$[\bar{\alpha}] = \begin{bmatrix} -K & 0 \\ 0 & B \end{bmatrix} , \quad [\bar{\beta}] = \begin{bmatrix} 0 & K \\ K & D \end{bmatrix} , \quad (57)$$

$$\{\xi\} = \begin{Bmatrix} S \\ \dot{S} \end{Bmatrix} , \quad \{\gamma\} = \begin{Bmatrix} 0 \\ L \end{Bmatrix} .$$

Then the damped free vibration mode shapes and mode values are obtained by solving the equation

$$[\lambda \bar{\alpha} + \bar{\beta}]\{X\} = 0 . \quad (58)$$

The adjoint eigenvalue system is

$$[\lambda \bar{\alpha}^T + \bar{\beta}^T] \{Y\} = 0 \quad (59)$$

The solutions of Eqs. (58) and (59) can be achieved by standard procedures.

Assuming that the modal matrices obtained from Eqs. (58) and (59) are

$[P]$ and $[\Delta]$ respectively, let

$$\{\xi\} = [P] \{W\} \quad (60)$$

Substituting Eq. (60) into (56) and using the biorthogonality condition, one has

$$[\alpha'] \{\dot{W}\} + [\beta'] \{W\} = [\Delta^T] \{\gamma\} \quad (61)$$

where α' and β' are diagonal and hence Eq. (61) are uncoupled and easily solved.

The eigenvalue obtained from Eq. (58) is equal to $i\Omega$, where Ω is the natural frequency of coupled modes. The dynamic behavior of the system is determined by Ω : (1) When Ω is real, the system performs undamped oscillations; (2) When Ω is complex having a positive imaginary part, the system performs damped oscillations; and (3) When Ω is complex having a negative imaginary part, the system loses stability by buckling or flutter.

If postinstability characteristics are of interest, a nonlinear theory including large deformation has to be developed. However, in the stable range, system responses can be calculated from Eqs. (51), (60) and (61).

B. Stationary Fluid

Considerable simplification can be made when fluid is stationary and internal damping, gravity effect, and fluid pressure are neglected.

In this case, Eqs. (49) become

$$E_p I_p \frac{\partial^4 u_p}{\partial z^4} + C_p \frac{\partial u_p}{\partial t} + m_p \frac{\partial^2 u_p}{\partial t^2} + \sum_{q=1}^{2k} \gamma_{pq} \frac{\partial^2 u_q}{\partial t^2} = f_p \quad (62)$$

$$p, q = 1, 2, 3 \dots 2k$$

Following the procedure as in the case of flowing fluid, using Eqs. (51), (52) and (62) yields

$$m_p \ddot{h}_{pn} + \sum_{q=1}^{2k} \gamma_{pq} \ddot{h}_{qn} + 2m_p \zeta_{pn} \omega_{pn} \dot{h}_{pn} + m_p \omega_{pn}^2 h_{pn} = f_{pn} , \quad (63)$$

$$p, q = 1, 2, 3 \dots 2k , \quad n = 1, 2, 3 \dots \infty ,$$

where ω_{pn} , as defined previously, is the n th frequency of cylinder p in vacuo and

$$\zeta_{pn} = \frac{C_p}{2m_p \omega_{pn}} ,$$

and (64)

$$f_{pn} = \frac{1}{L} \int_0^L f_p \psi_n dz .$$

Note that Eqs. (63) can be applied to all values of n . For each n , there are $2k$ equations which are coupled. However, there is no coupling among the equations for different n . This is true for a group of cylinders with the same type of boundary conditions and of the same length. For a given n , Eqs. (63) may be written

$$[B]\{\ddot{h}\} + [D]\{\dot{h}\} + [K]\{h\} = \{L\} . \quad (65)$$

For free vibration, neglect the damping and forcing terms, and let

$$\{h\} = \{\bar{h}\} \exp(i\Omega t) . \quad (66)$$

Natural frequencies and mode shapes can be calculated from the undamped homogeneous equations

$$[K]\{\bar{h}\} = \Omega^2 [B]\{\bar{h}\} . \quad (67)$$

K is a diagonal matrix and M is symmetric. Let $[P]$ be the weighted modal matrix formed from the columns of eigenvectors. It is easily shown that

$$[P^T B P] = [I] ,$$

and (68)

$$[P^T K P] = [\Lambda] ,$$

where $[I]$ is an identity matrix and $[\Lambda]$ is a diagonal matrix formed from the eigenvalue Ω^2 .

When all cylinders are identical, and have the same properties in the x and y directions,

$$\omega_{np} = \omega_n, \quad m_p = m, \quad (69)$$

$$p = 1, 2, \dots, 2k.$$

In this case, Eqs. (67) may be written

$$[\gamma_{pq}]\{\tilde{h}_q\} = \mu_p\{\tilde{h}_p\}, \quad (70)$$

where

$$\mu_p = \frac{m(\omega_n^2 - \Omega^2)}{\Omega^2}. \quad (71)$$

Equation (70) is identical to Eq. (31); that is, the eigenvectors of the added mass matrix are the same as the mode shapes of the coupled modes and the eigenvalues of the added mass matrix are related to the natural frequencies by Eq. (71). Corresponding to each eigenvalue μ_p , the natural frequency of the coupled mode is given

$$\Omega_p = \left(\frac{m}{m + \mu_p} \right)^{1/2} \omega_n. \quad (72)$$

Equation (72) shows that the natural frequency of the coupled mode is reduced in proportion to $\left(\frac{m}{m + \mu_p} \right)^{1/2}$. This is similar to that of a single structure in a liquid; therefore, μ_p may be interpreted as an effective added mass.

The response to an excitation can be calculated from Eqs. (65) and the associated initial conditions. For practical consideration, the damping matrix $[D]$ may be assumed to be proportional to the stiffness matrix $[K]$. In this case Eqs. (65) can be reduced to a set of $2k$ uncoupled modal equations by letting

$$\{h\} = [P]\{W\} , \quad (73)$$

and premultiplying Eqs. (65) by the transpose $[P^T]$, the result being

$$[P^T B P]\{\ddot{W}\} + [P^T D P]\{\dot{W}\} + [P^T K P]\{W\} = [P^T]\{L\} . \quad (74)$$

In Eqs. (74), the square matrices on the left side are diagonal matrices. Thus, each equation reduces to that of a single oscillator and has the form

$$\ddot{v}_{pn} + 2\eta_{pn}\Omega_{pn}\dot{v}_{pn} + \Omega_{pn}^2 v_{pn} = \sum_{l=1}^{2k} p_{lpn} f_{ln} , \quad (75)$$

where Ω_{pn} is the frequency of coupled mode and η_{pn} is the corresponding damping ratio. Equation (75) is easily solved and the displacement and other quantities of interest can be calculated from Eqs. (51).

If D is not proportional to the stiffness matrix, a damped vibration mode superposition method, as shown in the previous section for flowing fluid, can be used.

C. Numerical Examples

Numerical results presented in this section are based on steel tubes whose outside radius is 1.270 cm (1.5 in.), wall thickness 0.1588 cm (0.0625 in.), and length 1.27 m (50 in.) and are simply supported at both ends. In each group of tubes under consideration, it is assumed that all tubes are identical.

Figures 10 show the normal modes of two groups of cylinders consisting of three and four cylinders for $n = 1$ where $G = G_x = 0.127$ cm (0.05 in.), and $G_y = 0.1905$ cm (0.075 in.). For each n , there are $2k$ normal modes for a group consisting of k cylinders.

Figure 11 shows the frequencies of three tubes as functions of the gap G . As the spacing decreases, the frequencies of lower modes decrease while those of the higher modes increase. When the spacing increases, all frequencies approach that of a single tube in an infinite fluid.

Fig. 10. Normal modes of three and four cylinders vibrating in a liquid

Fig. 11. Natural frequencies of a group of three cylinders as functions of gap distance

Note that there are two repeated frequencies in Fig. 11. Corresponding to the second and third frequencies, there are two modes. The modes presented in the figure are orthogonal to each other. However, those are not the only sets of solution; a linear combination of the two modes of the set also satisfy the conditions of orthogonality.

Figures 12 and 13 present the response of a group of four tubes when tube 1 is subjected to an excitation $g_1 \sin \omega t$ in the x direction, uniformly distributed along the tube. The tubes are of the same size and simply supported at both ends. The magnification factor is defined as the ratio of the displacement at midspan to that of the deflection of tube 1 in the x direction to a static load of the same magnitude. In Fig. 12, $G_x = 0.127$ cm (0.05 in.) and $G_y = 0.1905$ cm (0.075 in.), the tubes are close to one another and the coupling effect is significant. It is seen that although only tube 1 is subjected to external excitation, there is a significant response in the y direction and the responses of tube 4 are comparable with those of tube 1. In Fig. 13, $G_x = 2.54$ cm (1 in.) and $G_y = 3.81$ cm (1.5 in.), the coupling effect is much smaller. For those tubes not directly excited, the responses are small. In this case, the response of tube 1 in the x direction is similar to that of a single tube.

To illustrate the general characteristics of tube bundles to axial flow, Fig. 14 shows the complex frequency ($f_R + if_I$) plotted as an Argand diagram of a two tube system. The frequency is obtained from Eq. (58), in which

$$\lambda = i(2\pi)(f_R + if_I) \quad (76)$$

The parameters used in computation are as follows: gap distance $G = 0.127$ cm (0.05 in.); axial tension $T_p(\xi) = 0.0$; fluid pressure $p_o = 0.0$; pressure gradient $p_1 = 0.0$; drag coefficient $\bar{C}_p = 0.01$; internal damping coefficient $\mu = 0.0$; and external damping coefficient $C_p = 6.8948 \text{ N-s/m}^2$ (0.001 lb-sec/in.²).

Fig. 12. Frequency response of four tubes for $G_x = 0.127$ cm (0.05 in.) and $G_y = 0.1905$ cm (0.075 in.)

Fig. 13. Frequency response of four tubes for $G_x = 2.54$ cm (1 in.) and $G_y = 3.81$ cm (1.5 in.)

Fig. 14. Complex frequencies of two tubes, simply-supported at both ends in axial flow

The first four modes are shown in Fig. 14 where the number in the figure indicates the flow velocity in 30.48 m/s (100 ft/sec). Note that all natural frequencies are located in the upper half of the complex plane when the flow velocity is small, and the system performs damped oscillation in all modes. The effects of the flowing fluid are to reduce the natural frequencies and to contribute to damping; the damping effect is due to drag force. As the flow velocity increases, the real part f_R of the first mode becomes zero, and the imaginary part f_I becomes negative; the system loses stability by buckling. With further increase in the flow velocity, the instability becomes flutter type. The second mode exhibits similar characteristics as the first mode. In the flow velocity range considered, the third and fourth modes are stable. However, as the flow velocity is further increased, they might become unstable.

The effects of various parameters on the dynamic characteristics have also been studied. Internal and external damping forces obviously contribute to damping and axial tension tends to increase natural frequencies of the system. The effects of fluid pressure and drag force are as follows:

(1) Increasing fluid pressure tends to increase natural frequencies; its effect is the same as that applying an axial tension. Therefore, natural frequencies increase with p_0 and decrease with p_1 .

(2) In the practical ranges of flow velocity, drag forces contribute to damping. A system with higher drag coefficient $\bar{C}_p$ will have a larger damping factor.

V. VIBRATION OF A TYPICAL LMFBR FUEL BUNDLE

The fuel in the liquid metal fast breeder reactor is stored in stainless steel tubes with helical wire wraps around the outside surface. The parameters describing a typical design of LMFBR fuel rod are given in Table 2. A seven-rod bundle, as shown in Fig. 15, in sodium is considered in this study; vibrations in stationary sodium and flowing sodium are presented.

In calculations, the stiffnesses of fuel and wire wraps are neglected; however, the masses are included. As an approximation, the rods are assumed to be simply-supported at both ends and the length is equal to the spacer pitch. The system parameters calculated from Table 2 and other parameters used in computation are given in Table 3.

A. Added Mass Coefficients

The added mass coefficients are computed by the program AMASS. The results are presented in Table 4. Several general characteristics are noted:

- (1) The self-added mass coefficient for rod 1 is the largest.
- (2) α_{11} and β_{11} are the same. Calculations also have been made for the self-added mass coefficient of rod 1 in other directions. It is found that the self-added mass coefficient of rod 1 is independent of the direction of motion.
- (3) α_{12} and β_{12} are symmetric, and $\sigma_{12} = \tau_{21}$. This has been proved in Section II.B.
- (4) α_{11} and β_{11} are always positive and larger than one; while σ_{11} and τ_{11} may be negative. For example, $\sigma_{11} = 0$; that is, the motion of rod 1 in the x direction does not induce any force acting on itself in the y direction.

Table 2. Fuel Rod Parameters

Outside Diameter	0.5842 cm (0.23 in.)
Wall Thickness	0.0381 cm (0.015 in.)
Rod-to-Rod Pitch	0.7264 cm (0.286 in.)
Spacer Wire Diameter	0.1422 cm (0.056 in.)
Spacer Pitch	30.48 cm (12 in.)
Fuel Density	9.4708 gm/cm ³ (0.342 lb/in. ³)
Stainless Steel Density	7.9754 gm/cm ³ (0.288 lb/in. ³)
Young's Modulus for Stainless Steel at 1000°F	15.8579 x 10 ¹⁰ Pa (23 x 10 ⁶ psi)

Fig. 15. Schematic of a seven-rod bundle

Table 3. System Parameters Used in Calculations

Rod Moment of Inertia I_p	$2.4474 \times 10^{-3} \text{ cm}^4$ ($5.88 \times 10^{-5} \text{ in.}^4$)
Rod Mass per unit Length m_p	0.2552 Kg/m (0.0143 lb/in.)
Rod Length l	30.48 cm (12 in.)
Flexural Rigidity $E I_p$	9.3245 Pa (1352.4 psi)
Axial Tension $T_p(l)$	0.0
Internal Damping Coefficient μ_p	0.0
External Damping Coefficient C_p	2.0684 N-sec/m^2 ($0.0003 \text{ lb-sec/in.}^2$)
Fluid Pressure p_o	$3.4474 \times 10^5 \text{ Pa}$ (50 psi)
Fluid Pressure Gradient p_1	1.6287 Pa/cm (0.6 lb/in.^3)
Drag Coefficient $\bar{C}_p$	0.01
Sodium Density	0.8205 gm/cm^3 (51.2 lb/ft^3)

Table 4. Added Mass Coefficients for a Seven-Rod Bundle

$[\alpha_{i\ell}] =$	1.6042	0.1405	0.1405	-0.5140	0.1405	0.1405	-0.5140
	0.1405	1.2753	-0.4376	-0.1383	0.0152	0.0843	0.1846
	0.1405	-0.4376	1.2753	0.1846	0.0843	0.0152	-0.1383
	-0.5140	-0.1383	0.1846	1.3709	0.1846	-0.1383	-0.1497
	0.1405	0.0152	0.0843	0.1846	1.2753	-0.4376	-0.1383
	0.1405	0.0843	0.0152	-0.1383	-0.4376	1.2753	0.1846
	-0.5140	0.1846	-0.1383	-0.1497	-0.1383	0.1846	1.3709
$[\sigma_{i\ell}] =$	0.0	-0.3779	0.3779	0.0	-0.3779	0.3779	0.0
	-0.3779	0.0552	0.1282	-0.0904	-0.0952	-0.0382	0.2310
	0.3779	-0.1282	-0.0552	-0.2310	0.0382	0.0952	0.0904
	0.0	-0.1667	-0.4874	0.0	0.4874	0.1667	0.0
	-0.3779	-0.0952	-0.0382	0.2310	0.0552	0.1282	0.0904
	0.3779	-0.0382	0.0952	-0.0904	-0.1282	-0.0552	-0.2310
	0.0	0.4874	0.1667	0.0	-0.1667	-0.4874	0.0
$[\tau_{i\ell}] =$	0.0	-0.3779	0.3779	0.0	-0.3779	0.3779	0.0
	-0.3779	0.0552	-0.1282	-0.1667	-0.0952	0.0382	0.4874
	0.3779	0.1282	-0.0552	-0.4874	-0.0382	0.0952	0.1667
	0.0	-0.0904	-0.2310	0.0	0.2310	0.0904	0.0
	-0.3779	-0.0952	0.0382	0.4874	0.0552	-0.1282	-0.1667
	0.3779	-0.0382	0.0952	0.1667	0.1282	-0.0552	-0.4874
	0.0	0.2310	0.0904	0.0	-0.0904	-0.2310	0.0
$[\beta_{i\ell}] =$	1.6042	-0.2958	-0.2959	0.3587	-0.2958	-0.2959	0.3587
	-0.2958	1.3390	0.3920	0.0101	-0.0947	-0.2125	-0.2302
	-0.2959	0.3920	1.3390	-0.2302	-0.2125	-0.0947	0.0101
	0.3587	0.0101	-0.2302	1.2434	-0.2302	0.0101	0.0701
	-0.2958	-0.0947	-0.2125	-0.2302	1.3390	0.3920	0.0101
	-0.2958	-0.2125	-0.0947	0.0101	0.3920	1.3390	-0.2302
	0.3587	-0.2302	-0.0101	0.0701	0.0101	-0.2302	1.2434

The principal values of the added mass coefficient matrix are given in Table 5. Note that the ratio of the largest principal value μ'_1 and α_{11} is 1.8554.

B. Free Vibration in Stationary Liquid

The effects of fluid pressure, damping, and drag force are temporarily neglected in order to understand more clearly the effect of fluid coupling. The fundamental frequency of the individual rod is 65.949 Hz in vacuo and 63.257 Hz in sodium.

First, consider the case that only a rod in the group vibrates, while all other rods are stationary. This type of motion is known as uncoupled vibration. The frequency of this type of vibration can be calculated using the self-added mass coefficients. For example, the frequency for rod j vibrating in the x direction is given by the following relationship:

$$\begin{aligned} &\text{frequency of uncoupled mode for rod } j \\ &= (\text{frequency in vacuo for rod } j) \cdot \\ &\quad \left(\frac{\text{mass of rod } j}{\text{mass of rod } j + \alpha_{jj} \times \text{displaced mass of fluid by rod } j} \right)^{1/2}. \quad (77) \end{aligned}$$

The frequencies of uncoupled modes for the seven-rod bundle are given in Table 6a.

The frequencies of coupled modes can be calculated from Eq. (67). However, when all rods are identical, as shown in Section IV.B, the following procedure can be used to calculate the frequencies of coupled modes:

$$\begin{aligned} &\text{frequency of coupled mode} \\ &= (\text{frequency in vacuo}) \cdot \left(\frac{\text{mass of rod}}{\text{mass of rod} + \mu'_p \times \text{displaced mass of fluid}} \right)^{1/2}. \quad (78) \end{aligned}$$

The results are given in Table 6b.

Table 5. Principal Values of the Added Mass
Coefficient Matrix for a Seven-Rod
Bundle

$$\mu_1' = 2.9765$$

$$\mu_2' = 2.6800$$

$$\mu_3' = 2.6800$$

$$\mu_4' = 1.7771$$

$$\mu_5' = 1.5803$$

$$\mu_6' = 1.5803$$

$$\mu_7' = 1.0420$$

$$\mu_8' = 1.0420$$

$$\mu_9' = 0.7840$$

$$\mu_{10}' = 0.7840$$

$$\mu_{11}' = 0.6194$$

$$\mu_{12}' = 0.4860$$

$$\mu_{13}' = 0.4860$$

$$\mu_{14}' = 0.3764$$

Table 6. Natural Frequencies of a Seven-Rod Bundle

a. Uncoupled Vibration

Rod Number	Natural Frequencies Assuming all Other Rods are Stationary	
	Vibration in the x Direction	Vibration in the y Direction
1	61.782 Hz	61.782 Hz
2	62.573 Hz	62.417 Hz
3	62.573 Hz	62.417 Hz
4	62.340 Hz	62.651 Hz
5	62.573 Hz	62.417 Hz
6	62.573 Hz	62.417 Hz
7	62.340 Hz	62.651 Hz

b. Coupled Vibration

58.783 Hz	63.151 Hz
59.394 Hz	63.811 Hz
59.394 Hz	63.811 Hz
61.379 Hz	64.242 Hz
61.839 Hz	64.599 Hz
61.839 Hz	64.599 Hz
63.151 Hz	64.896 Hz

The mode shape of an uncoupled mode is the same as that of an isolated rod. However, the mode shape for a coupled mode is much more complicated. The normal modes for the seven-rod bundle are given in Figure 16.

Note that there are 14 natural frequencies which correspond to a single frequency in the case of an isolated rod. Among 14 frequencies, there are five pairs of repeated frequency. These are associated with the symmetry of arrangement. The lowest frequency is associated with the mode in which the central rod is stationary and all peripheral rods vibrate in the direction along the radial line connecting the centers of the rod and the central rod. In this mode, since the fluid has to be displaced considerably, the effective added mass is large. On the other hand, associated with the 14th mode, all peripheral rods vibrate perpendicular to the radial lines and are in phase; the added mass effect is much smaller.

C. The Effects of Sodium Flow

Vibration frequencies for rod bundles subjected to fluid flows can be calculated from Eq. (58). Table 7 gives the vibration frequencies for the seven-rod bundle for several flow velocities. The real and imaginary parts of frequencies, f_R and f_I (see Eq. 76), are given by the upper and lower numbers in each group. f_R and f_I are related to the modal damping ratio ζ , which is given by

$$\zeta = \frac{f_I}{f_R} \left/ \left[1 + \left(\frac{f_I}{f_R} \right)^2 \right]^{1/2} \right. \quad (79)$$

Table 7 shows that f_R decreases with increasing flow velocity, while f_I increases with flow velocity; that is, as the flow velocity increases, natural frequencies of all modes decrease and damping ratios increase. The decrease in natural frequencies is due to fluid centrifugal force, while the increase in damping is attributed to fluid drag force. As it can be

Fig. 16. Normal modes of a seven-rod bundle vibrating in a liquid

Table 7. Natural Frequencies of a Seven-Rod Bundle at Several Flow Velocities

Flow Velocity	0	25 ft/sec	50 ft/sec	75 ft/sec	100 ft/sec
1st Mode	59.361 0.51261	59.094 0.55803	58.287 0.60298	56.920 0.64733	54.957 0.69097
2nd Mode	59.979 0.52333	59.738 0.56974	59.011 0.61574	57.783 0.66125	56.023 0.70615
3rd Mode	59.979 0.52333	59.738 0.56974	59.011 0.61574	57.783 0.66125	56.023 0.70615
4th Mode	61.983 0.55890	61.825 0.60856	61.350 0.65802	60.551 0.70723	59.416 0.75614
5th Mode	62.448 0.56731	62.309 0.61773	61.890 0.66799	61.186 0.71805	60.188 0.76787
6th Mode	62.448 0.56731	62.309 0.61773	61.890 0.66799	61.186 0.71805	60.188 0.76787
7th Mode	63.773 0.59164	63.685 0.64427	63.422 0.69682	62.982 0.74928	62.361 0.80162
8th Mode	63.733 0.59164	63.685 0.64427	63.422 0.69682	62.982 0.74928	62.361 0.80162
9th Mode	64.439 0.60406	64.376 0.65781	64.190 0.71151	63.878 0.76516	63.438 0.81874
10th Mode	64.439 0.60406	64.376 0.65781	64.190 0.71151	63.878 0.76516	63.438 0.81874
11th Mode	64.875 0.61226	64.828 0.66674	64.691 0.72120	64.461 0.77562	64.138 0.83000
12th Mode	65.235 0.61907	65.202 0.67417	65.104 0.72924	64.941 0.78430	64.712 0.83933
13th Mode	65.235 0.61907	65.202 0.67417	65.104 0.72924	64.941 0.78430	64.712 0.83933
14th Mode	65.535 0.62478	65.513 0.68039	65.448 0.73599	65.341 0.79157	65.190 0.84714

seen from Table 7, the effect of flow on natural frequencies is small for practical flow velocity range, say 7.62 m/s (25 ft/sec) to 15.24 m/s (50 ft/sec). If the flow velocity is increased to a certain limit, in this case about 270 ft/sec, the rod bundle becomes unstable by buckling. However, such a high flow velocity will not be encountered in practical situations. Therefore, instabilities associated with parallel flows, in general, are of little concern in the design of practical system components even in the case of rod bundles, except when the unsupported length of the rods is long.

VI. CONCLUSIONS

This paper presents a general method of analysis for coupled vibration of a group of circular cylinders vibrating in liquids; particular emphasis is placed on fuel bundle vibration. This has been the first method ever developed for analyzing rod bundle vibration in liquids, which can be applied to a group of cylinders arranged in any pattern. With this method of analysis, the response of a system consisting of a group of cylinders subjected to fluid flows can readily be analyzed. In particular, the mathematical models for fuel bundles subjected to parallel flows and heat-exchanger tube banks subjected to cross flows can be refined by incorporating the interaction effect, which all existing models fail to take into account. Efforts are being made in developing better mathematical models for both parallel- and cross-flow induced vibrations of tube bundles.

The computer code, AMASS, can be used to calculate the added mass coefficients for a group of cylinders with different diameters. In the code, it is necessary to solve a system of linear equations. The number of equations depends on the number of cylinders and number of terms used in the series solution. The current version of AMASS can be used for many practical system components.

The equations of motion derived for fuel bundles include fluid coupling, damping, drag force, centrifugal force, Coriolis force, fluid pressure, gravity effect, and axial tension. These equations can be used for analyzing the response of fuel bundles to flow noises and establishing stability limits. Several forms of the equations of motion were presented previously; only in one study, the equation was developed for fuel bundles [26]. However, the fluid coupling effect is not included; it can be used for uncoupled vibration only. For coupled vibration, Eq. (49) will give correct results.

Based on the method of analysis for added mass coefficients and the equations of motion for rod bundles, free and forced vibrations of a group of rods are analyzed. Numerical results are presented for rod bundles consisting of two, three, four, and seven rods. It is noted that in all cases, the lowest frequency of coupled modes is always lower than the lowest frequency of uncoupled modes. The implication is that using uncoupled modes for mathematical models of rod bundles is not conservative.

An experiment designed to verify the theory has been planned. A series of tests involving several groups of tubes in stationary fluid and flowing fluid will be performed. One set of tests for two tubes vibrating in a stationary water has been completed. Preliminary results show that the theoretical results and experimental data are in good agreement.

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