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Exact Physical Model for Magnets in Storage Rings

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Abstract

This report represents an effort to describe *exactly* the motion of a charged particle in a simplified model of storage ring made of infinite sequence of FODO cells. We have given special attention to the preservation of the kinematic terms, which are usually ignored, and to the correct Maxwellian representation of the magnetic field. Given the complexity of the resulting equations of motion, we have resorted to the model of *thin lenses* as a valid symplectic integration method to be used in numerical tracking to determine the stability of particle motion over long periods of time.

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1 Motivation

It is common practice to perform computer simulation to determine the stability of motion of charged particles circulating in storage rings over long periods of time. For this purpose several computer codes have been written[1]. In the past, great effort was spent to insure that the representation of the motion used during simulation was symplectic[2]; that is, for conservative systems of particles, like protons, antiprotons and heavy ions, the equations of motion not only are to be derived from Hamiltonian, but they also have to satisfy a variety of momentum and energy conservation laws. If these should not become fully satisfied, the results of such computer tracking, especially long ones, could be invalid. Simulated motion could show instability where it should not occur and viceversa.

There are several types of elements that a particle encounters during its motion in a storage ring. Some of them have linear properties, like drifts and pure quadrupoles. In these cases, the approximate status of motion of the particle at the end of the particular element can be easily given as an integrable function of the status of motion at the beginning of that element. Nevertheless, even in these linear or quasi-linear elements the exact equations of motion have kinematic terms which are difficult to integrate.

Other elements which are encountered by the particle during its motion in a storage ring are non-linear, like sextupoles, intentionally used to correct chromatic effects, and the non- linear imperfections of magnets, especially superconducting magnets. The motion in these non-linear elements cannot be generally integrated. Therefore one has to use some numerical technique to perform integration by computer.

One method is to lump each non-linear element in a thin lens of zero length. Motion through such an element is then represented by a kick. The position coordinates of the particle remain unchanged while velocity receives a sudden change which depends on the particle position and on the properties and strength of the non-linear lens. The numerical codes that make use of this method are called *kick codes* or *exact codes.* The motion which they are representing with a kick is truly symplectic so their approximation of reality is physically correct. TEAPOT[3] is one of these codes which also represents integrable elements, like dipoles and quadrupoles, as a series of thin lenses and drifts. Since the motion in a drift can be calculated exactly, it seems that the representation of any accelerator element as a sequence of drifts and thin lenses, linear and non – linear, is indeed very powerful.

It should be understood that in order to predict the stability of the motion over very long periods of time it is mandatory that all kinematic terms be properly included in the model. Neglecting some of them may invalidate the results of very time consuming exercises on the computer.

Adopting the thin lens representation requires a discussion of one more issue. Forces acting on a particle are to be derived exactly and consistently from Maxwell's equations. Generally, ordinary kick codes do not enforce a complete description of the fields and this is to be corrected. Not only magnets exhibit edge effects due to their discontinuity in space, but the thin lens model either of linear or of non-linear elements is a three-dimensional field representation which requires a careful estimation of the field components. Usually only the longitudinal component of the vector potential is retained for the field estimate. Since this has a discontinuous behaviour, transverse components are also to be present, especially at the edges of the elements. These transverse components are usually neglected. For instance, if the transverse components of the field are neglected, the divergence of the vector potential does not vanish and the divergence and the curl of the magnetic field is different from zero, though the corresponding equations of motion remain symplectic. This fact, again, may invalidate results of long computer simulations. Thus, we believe that it is very important that the field representation is exact, complete and that it satisfies Maxwell's equations.

In this report we shall try to make estimates of both kinematic and field effects on the stability of a particle motion, by employing a truly Maxwellian representation of the magnetic field in exact equations of motion. For this purpose we shall adopt a simple FODO cell model, which repeats periodically to infinity. Only quadrupoles and drifts are included in this model, leaving out the bending magnets to avoid the problem of the trajectory curvature. We believe that this model is a physically consistent approximation of a storage ring. We shall derive several models with different levels of approximation and attempt to compare them by evaluating the importance of these effects. The relevance to the long-term stability is in the meantime investigated by comparing the different models with extensive computer simulations. The results will be shown in a subsequent report.

2 The Test Model

To determine the importance and the magnitude of kinematic effects and of the effects of the field components which are required for the complete Maxwellian representation of the field, we shall adopt the following test model. It is made of an infinitely long sequence of FODO cells. All cells are identical as shown in Figure 1. The FODO cell has the following structure:

$$QF/2 \quad L \quad QD/2 \quad QD/2 \quad L \quad QF/2 \tag{1}$$

which begins from the middle of the horizontally focusing quadrupole (QF) and ends in the middle of the following focusing quadrupole. The vertically focusing quadrupole (QD) is located half-a-way. The quadrupoles have the same length l and are separated by a drift of length L. Quadrupole magnets are described by the field gradient. In QF the gradient is $+G$ and in QD the gradient is $-G$. The only other parameter which is required to calculate the motion of a charged particle of charge q and momentum p is the particle magnetic rigidity

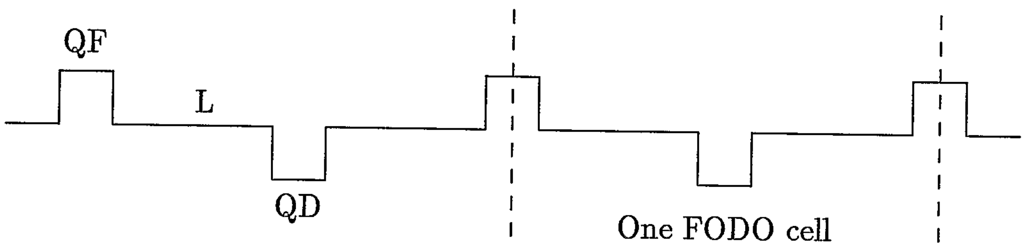

Figure 1: The test model as an infinite sequence of identical FODO cells

$$B\rho = \frac{pc}{q} \tag{2}$$

where c is the speed of light.

Our test model is made only of drifts and quadrupoles. For simplicity we did not include dipole magnets to avoid the issue of the curvature they introduce.

The reference line is the straight line which coincides with the symmetry axis of the quadrupole. We assume that quadrupoles are perfectly aligned so that their axes coincide. The quadrupole field vanish on this axis.

This model is basically an approximation of a storage ring where the reference orbit is a straight line. We shall introduce a longitudinal coordinate s along the reference line and a pair of rectangular transverse coordinates: x - horizontal and y - vertical, which measure the distance of a particle trajectory from the reference line; that is on reference line $x = y = 0$.

3 Equations of Motion

To describe the motion of a particle it is convenient to consider the plane perpendicular to the main axis of motion (the reference orbit) which includes the particle position at the time t. It is on this plane that the distances x and y from the reference axis are evaluated at the time t. To the same instant t one can associate the longitudinal coordinate s, which is given by the interception of the xy plane with the reference orbit. Since there is a unique correspondence between s and t, it is possible and actually more convenient to take the longitudinal coordinate s as the independent variable, instead of the time t. The displacements are then taken as functions of s, that is $x = x(s)$ and $y = y(s)$.

In the following we shall denote with a prime the derivative with respect to s. We shall also denote the components of the magnetic field as B_x, B_y and B_s.

It can be proven[4] that the exact equations of motion, for the FODO test model are

$$x'' = -\frac{q}{cp}\sqrt{1 + x'^2 + y'^2}[B_y(1 + x'^2) - B_x x'y' - B_s y'] \tag{3}$$

$$y'' = \frac{q}{cp}\sqrt{1 + x'^2 + y'^2}[B_x(1 + y'^2) - B_y x'y' - B_s x'] \tag{4}$$

It can also be proven[4] that the following relation holds

$$x'x'' + y'y'' = \frac{q}{p}(1 + x'^2 + y'^2)^{3/2}(y'B_x - x'B_y) \tag{5}$$

The last relation may be found useful in a variety of situations.

4 A Symplectic Integration

The general integration of the system made of Eqs. (3 and 4) is very difficult. It may not be possible to prove their integrability in general. One can see that there are two features that contribute to the difficulty of the problem:

1. There are kinematic terms which are represented by the x' and y' factors which are usually neglected.
2. There are the field contributions, represented by the components B_x, B_y and B_s. Usually the longitudinal component is neglected as well as the edge effects.

We need to find a symplectic integration method which allows us to solve the equations of motion, satisfying all energy and momentum conservation

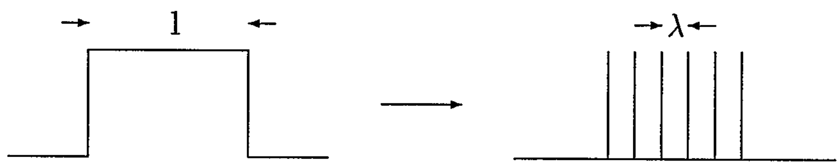

Figure 2: Replacing a full length quadrupole with a sequence of thin lenses and drifts

laws. Conventional methods, like Newton, Lagrange, Runge-Kutta, etc., may not be valid for this purpose. We find the "TEAPOT – like" method of replacing each element with a sequence of thin lenses and drifts, as shown in Figure 2, a valid one.

The length l of the quadrupole magnet is divided in N steps. Each step is made of a drift of length $\lambda = l/N$ with a thin lens quadrupole in the middle. Each of these N thin lenses is a quadrupole magnet with zero length. In the limit of $\lambda \to 0$ or $N \to \infty$, one can recover the original model of full length quadrupole.

In a drift $B_x = B_y = B_s = 0$, so that the equations of motion are simply

$$x'' = y'' = 0 \tag{6}$$

with solution represented by straight line trajectories.

The integration through the thin lens quadrupoles is shown in Figure 3. The trajectory of the particle is made of straight lines before and after the thin lens. The trajectory is deflected by the thin lens leaving the position of the particle unchanged and altering the velocity by an amount which depends on x, y, x' and y' as well as on the field gradient in the magnet.

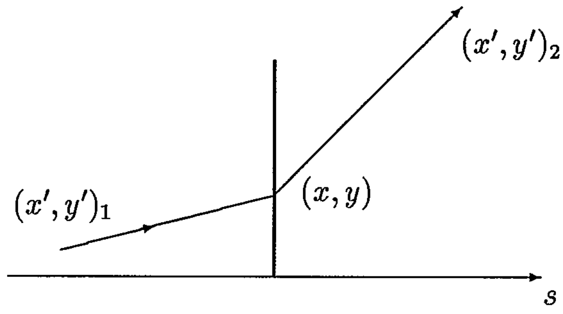

Figure 3: The trajectory of a particle going through a thin lens

5 Exact Field Expression

We wish to calculate the exact expression for the magnetic field distribution of a finite length n-pole magnet, including the distribution at the edges. For this purpose we introduce two functions: the dimensionless function $f(s)$ which defines the longitudinal shape of the magnet and

$$g_n(x, y) = (a_n + ib_n)(x + iy)^n \tag{7}$$

which describes the longitudinal field potential inside the n-pole magnet. It is easily verified that

$$\nabla^2 g_n(x, y) = 0 \tag{8}$$

Let $\mathbf{B}$ and $\mathbf{A}$ be respectively the magnetic field and the magnetic vector potential. They satisfy

$$\nabla \mathbf{A} = 0 \tag{9}$$

$$\nabla^2 \mathbf{A} = 0 \tag{10}$$

and

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{11}$$

Since we are using rectangular coordinate system (x, y, s) equation (10) is equivalent to

$$\nabla^2 A_x = \nabla^2 A_y = \nabla^2 A_s = 0 \tag{12}$$

It can be proven by direct substitution that the following expressions solve Eqs. (9) and (10) for any chosen functions $f(s)$ and $g_n(x, y)$:

$$A_s(x, y, s) = g_n \sum_{m=0}^{\infty} C_m (x^2 + y^2)^m f^{(2m)} \tag{13}$$

$$A_x(x, y, s) = -g_n(x + iy) \sum_{m=0}^{\infty} \frac{C_m}{2(m + n + 1)} (x^2 + y^2)^m f^{(2m+1)} \tag{14}$$

$$A_y(x, y, s) = -g_n(y - ix) \sum_{m=0}^{\infty} \frac{C_m}{2(m + n + 1)} (x^2 + y^2)^m f^{(2m+1)} \tag{15}$$

where the coefficients C_m are related to each other according to

$$C_{m-1} = -4m(m + n)C_m \tag{16}$$

$$C_0 = 1 \tag{17}$$

and

$$f^{(m)} = \frac{d^m f(s)}{ds^m} \tag{18}$$

Inserting Eqs. (13 - 15) into Eq. (11) yields

$$B_x = ig_n \sum_{m=0}^{\infty} C_m[(2m + n)x - iny](x^2 + y^2)^{m-1} f^{(2m)} \tag{19}$$

$$B_y = ig_n \sum_{m=0}^{\infty} C_m[(2m + n)y + inx](x^2 + y^2)^{m-1} f^{(2m)} \tag{20}$$

$$B_s = ig_n \sum_{m=0}^{\infty} C_m (x^2 + y^2)^m f^{(2m+1)} \tag{21}$$

As a special case let us consider a quadrupole magnet in normal orientation, that is $n = 2$, $b_2 = 0$ and $a_2 = -G/2$. After taking the imaginary part to obtain the real field representation, we have

$$B_x = Gyf^{(0)} + G\sum_{m=1}^{\infty} C_m (x^2+y^2)^{m-1}[(2m+1)x^2y + y^3]f^{(2m)} \quad (22)$$

$$B_y = Gxf^{(0)} + G\sum_{m=1}^{\infty} C_m (x^2+y^2)^{m-1}[(2m+1)xy^2 + x^3]f^{(2m)} \quad (23)$$

$$B_s = G\sum_{m=0}^{\infty} C_m xy(x^2+y^2)^{m} f^{(2m+1)} \quad (24)$$

If we retain only the linear terms in x and y, we get

$$B_x = Gyf(s) \quad (25)$$

$$B_y = Gxf(s) \quad (26)$$

$$B_s = 0 \quad (27)$$

6 Linear Model

Let us neglect all the kinematic terms in Eqs. (3 and 4) and retain only the linear terms in the field expansion, that is Eqs. (25-27) where $f(s)$ is a step function equal to 1 in the quadrupole and zero everywhere else.

The resulting equations of motion in a quadrupole are

$$x'' = -Kx \quad (28)$$

$$y'' = Ky \quad (29)$$

where, for convenience

$$K = \frac{q}{cp}G = \frac{G}{B\rho} \quad (30)$$

In the drifts, between quadrupoles, the equations of motion are

$$x' = y' = 0 \quad (31)$$

These equations are linear. They can be easily solved and their solution is usually represented by the matrix notation which involves the lattice functions[5] β_H and β_V as well as the phase advance functions[5] Ψ_H and Ψ_V.

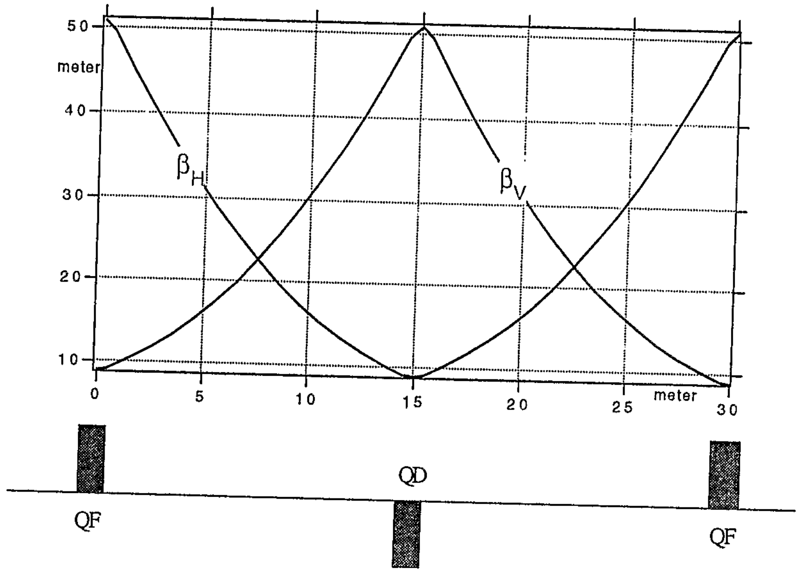

Figure 4: The amplitude lattice functions for the FODO cell

As an example we shall take the following values which approximate regular cells in the arcs of RHIC[6]:

$$B\rho = 840\ Tm \quad (32)$$

$$G = 81\ T/m \quad (33)$$

$$L = 14\ m \quad (34)$$

$$l = 1\ m \quad (35)$$

The lattice functions for this system, obtained with the SYNCH code[7], are plotted in Figure 4. The quadrupole gradient has been chosen to make the phase advances across one FODO cell equal to 90^o in both planes.

7 The TEAPOT Model

Let us now examine the approximation of a quadrupole magnet by a sequence of thin lens magnets and drifts, as shown in Figure 2. We shall still adopt the linear approximation, as it is done in TEAPOT code[3]. The only difference from the equations of motion of the Linear Model is that the function $f(s)$ is a delta function at each thin lens, positioned at $s = s_i$

$$f(s) = \lambda\delta(s - s_i) \quad (36)$$

where $\lambda = l/N$ is the length of a step which is approximated by one thin lens and a drift.

The position coordinates x and y of a particle crossing the thin lens are unchanged, that is

$$\Delta x = \Delta y = 0 \tag{37}$$

At the same time the angles x' and y' receive a change

$$\Delta x' = -K\lambda x \tag{38}$$

$$\Delta y' = K\lambda y \tag{39}$$

This model can be easily integrated and investigated. The results are given for comparison in Table 1. The maximum and minimum values of the amplitude functions are reported as well as the phase advance across the FODO cell, $\Delta\Psi/2\pi$. The results for the Linear Model in Table 1 have been derived with the SYNCH code[7]. The results of the approximation of the whole quadrupole by one thin lens are computed manually, since for this case the following relations hold

$$\Lambda = \frac{L+l}{2} \tag{40}$$

$$\sin(\frac{\Delta\Psi}{2}) = \frac{\Lambda l|K|}{2} \tag{41}$$

$$\beta_{max} = \frac{2}{l|K|}\sqrt{\frac{2+\Lambda l|K|}{2-\Lambda l|K|}} \tag{42}$$

$$\beta_{min} = \frac{2}{l|K|}\sqrt{\frac{2-\Lambda l|K|}{2+\Lambda l|K|}} \tag{43}$$

Remaining entries in the Table 1 are obtained with a TEAPOT type of code, for several different numbers of sequences in which each quadrupole is subdivided. The last three entries correspond to the Simpson-Bode[8] method of integration where magnets are subdivided into thin lenses each carrying a different weighted strength. Several different number of subdividing steps, shown in brackets, are presented.

It can be seen that the single thin lens model provides results which are only few percent from those of the exact Linear Model. Subdividing the

Table 1: Comparison of different Linear Models

MODEL	β_{max}	β_{min}	$\Delta\Psi/2\pi$
Linear Model	50.80254	8.86074	0.24992
Thin Lens	51.75140	8.31240	0.25734
TEAPOT (2)	50.71639	8.77329	0.25176
TEAPOT (4)	50.78029	8.83885	0.25038
TEAPOT (10)	50.79895	8.85724	0.24999
TEAPOT (20)	50.80164	8.85987	0.24994
TEAPOT (100)	50.80231	8.860052	0.24992
TEAPOT (200)	50.80250	8.86071	0.24992
Simpson-Bode (2)	51.15743	8.73205	0.25115
Simpson-Bode (3)	50.78102	8.83882	0.25038
Simpson-Bode (4)	50.75739	8.84859	0.25027

quadrupole in more thin lenses gives a closer approximation. Nevertheless, even if we divide magnet into 200 thin lenses the results are off by an error of 10^{-5} for the amplitude function.

It has been assumed that a more effective way of integrating the motion through a magnet is the Simpson-Bode method. This method is commonly used in tracking particle motion through non-linear elements. Its effectiveness cannot be easily proven except in a linear case where the results can be compared to the exact solution. The results reported in Table 1 are evidence that the Simpson-Bode method has a better convergence property than the method a lá TEAPOT.

Let (x, x') and (y, y') be the particle position and angle in the middle of any of the two quadrupoles, QF and QD. Then the following quantities are invariant[5]

$$\epsilon_H = \frac{x^2}{\beta_H} + x'^2 \beta_H \tag{44}$$

$$\epsilon_V = \frac{y^2}{\beta_V} + y'^2 \beta_V \tag{45}$$

The values of the amplitude functions β_H and β_V vary somewhat from model to model as shown in Table 1. In the following we shall take the Linear Model as the reference one.

8 The Kinematic Model

The kinematic model is defined by the set of exact Eqs. (3 and 4), where kinematic terms are retained to any order but the field expansion is truncated beyond the linear terms, Eqs. (25-27). This model is truly symplectic.

The investigation of this model is relevant to determine the magnitude and the importance of the effects of the kinematic terms. This can be done by comparison with the Linear Model.

We shall continue to approximate a quadrupole with a sequence of thin lenses and drifts. Indeed this is the only method we know for numerical symplectic integration.

The equations of the motion through a thin quadrupole now become

$$x'' = -K\lambda\sqrt{1+x'^2+y'^2}\,(x + xx'^2 - yx'y')\delta(s) \tag{46}$$

$$y'' = K\lambda\sqrt{1+x'^2+y'^2}\,(y + yy'^2 - xx'y')\delta(s) \tag{47}$$

can be integrated exactly yielding kicks

$$\Delta x' = -K\lambda\sqrt{1+x'^2+y'^2}(x + xx'^2 - yx'y') \tag{48}$$

$$\Delta y' = K\lambda\sqrt{1+x'^2+y'^2}(y + yy'^2 - xx'y') \tag{49}$$

where x, x', y and y' at the right hand side correspond to the instant just before entering the lens, as it is proven in the Appendix.

We have not been able to derive invariants similar to those given for the linear model. Nevertheless, it can be noticed that the kinematic terms introduce coupling between horizontal and vertical motion. The coupling disappears only if the motion is purely horizontal ($y = y' = 0$) or purely vertical ($x = x' = 0$). In these cases the equations of the motion reduce to either

$$x'' = -K\lambda x(1+x'^2)^{3/2}\delta(s) \tag{50}$$

$$y'' = 0 \tag{51}$$

or:

$$x'' = 0 \tag{52}$$

$$y'' = K\lambda y(1 + y'^2)^{3/2}\delta(s) \tag{53}$$

Let us take, for instance, the purely horizontal motion. In this case we can decouple our second order differential equation of motion, Eq. (50), into two first order differential equations:

$$x' = \frac{p_x}{\sqrt{1 - p_x^2}} \tag{54}$$

$$p_x' = -K\lambda x\delta(s) \tag{55}$$

which can be derived from the following Hamiltonian:

$$H = 1 - \frac{K\lambda}{2}x^2\delta(s) - \sqrt{1 - p_x^2} \tag{56}$$

9 The Maxwellian Field Model

The Maxwellian Field model is obtained by neglecting all the kinematic terms in Eqs. (3) and (4) but retaining the exact field series representation, Eqs. (22-24). Only B_x and B_y field components enter the equations:

$$x'' = -\frac{q}{cp}B_y \tag{57}$$

$$y'' = \frac{q}{cp}B_x \tag{58}$$

Inserting Eqs. (22-24) we obtain

$$x'' = -K\lambda\{x\delta(s) + \sum_{m=1}^{\infty} C_m(x^2 + y^2)^{m-1}[(2m+1)xy^2 + x^3]\delta^{(2m)}(s)\} \tag{59}$$

$$y'' = K\lambda\{y\delta(s) + \sum_{m=1}^{\infty} C_m(x^2 + y^2)^{m-1}[(2m+1)x^2y + y^3]\delta^{(2m)}(s)\} \tag{60}$$

where again a thin lens representation has been assumed.

The integration of Eqs. (59) and (60) is cumbersome, but it can be exactly derived. If we retain only the linear, $m = 1$ and $m = 2$ terms, we obtain

$$\Delta x' = -K\lambda(x - \frac{1}{2}xy'^2 - yy'x' - \frac{1}{2}xx'^2 + \frac{5}{16}xy'^4 + \frac{5}{4}yy'^3x' + \frac{5}{8}xy'^2x'^2 - \frac{1}{4}yy'x'^3 - \frac{3}{16}xx'^4) \tag{61}$$

$$\Delta y' = K\lambda(y - \frac{1}{2}yx'^2 - xx'y' - \frac{1}{2}yy'^2 + \frac{5}{16}yx'^4 + \frac{5}{4}xx'^3y' + \frac{5}{8}yx'^2y'^2 - \frac{1}{4}xx'y'^3 - \frac{3}{16}yy'^4) \tag{62}$$

The particle coordinates x, x', y and y' appearing at the right hand side have, again, values by which particle was described just before entering the thin lens.

The factors in x' and y' which appear in these equations do not originate from the kinematic terms. They are introduced by the integration across the derivatives of the delta function.

It is important to notice that higher order terms in the field expansion cause higher order terms in the equations of motion. Therefore, raising the truncation in the expansion of the field beyond $m = 2$ will not change the terms in the equations of motion up to the fifth order but will only generate terms higher than fifth order.

We like to point out that this model is also symplectic, whatever is the order of truncation in the expression of the magnetic field.

10 Comparison and Conclusion

Let us introduce the General Model where the equations of motion have the exact representation of the kinematic terms retained and the Maxwellian field series expansion included to a sufficiently high order. In particular we shall retain only the field expansion up to and including the $m = 2$ terms of Eqs. (22-24). Integration through a thin lens then leads to the following changes of the particle velocity

$$\Delta x' = -K\lambda\sqrt{1+x'^2+y'^2}(x + \frac{1}{2}xx'^2 - 3yy'x' - \frac{3}{2}xy'^2 +$$
$$\frac{13}{4}yy'^3x' + \frac{25}{8}xy'^2x'^2 + \frac{3}{4}yy'x'^3 + \frac{13}{16}xy'^4 - \frac{3}{16}xx'^4) \tag{63}$$

$$\Delta y' = K\lambda\sqrt{1+x'^2+y'^2}(y + \frac{1}{2}yy'^2 - 3xx'y' - \frac{3}{2}yx'^2 +$$
$$\frac{13}{4}xx'^3y' + \frac{25}{8}yx'^2y'^2 + \frac{3}{4}xx'y'^3 + \frac{13}{16}yx'^4 - \frac{3}{16}yy'^4) \tag{64}$$

One can see that the square root term is not expanded but retained exactly to preserve the nature of the kinematic terms. This model also preserves the symplectic properties. The terms which appear within the round brackets are proper to the field series representation and truncation.

We have thus developed four models given by the sets of Eqs. (38-39), (48-49), (61-62) and (63-64). Each of these models represent different approximation and can be easily tested against each other to determine the importance of either neglecting or including the kinematic as well as the field expansion terms for the stability of a particle motion over long periods of time.

It is easy to estimate the correction to the usual linear approximation for each of the models. Retaining only the lowest order part without cross terms, we have:

$$\Delta x_1' = -K\lambda x(1 + \frac{3}{2}x'^2) \tag{65}$$
$$\Delta x_2' = -K\lambda x(1 - \frac{1}{2}x'^2) \tag{66}$$
$$\Delta x_3' = -K\lambda x(1 + x'^2) \tag{67}$$

corresponding respectively to the Kinematic, Maxwellian and General model. These equations are not to be used in real simulation but only to estimate the magnitude of the correction introduced, as they are not symplectic.

According to Linear Model

$$\Delta x' = -K\lambda x \tag{68}$$

Using parameters typical of the arc FODO cell in RHIC[6] each model gives a correction of magnitude

$$\frac{\Delta x_i' - \Delta x'}{\Delta x'} \approx 10^{-8} - 10^{-9} \tag{69}$$

These correction factors are not important to determine the motion of only few revolutions and for ring design. They may be very important and cannot be ignored to determine the stability of motion of a particle for a long period of time.

Appendix

We are interested in the integration of the following equation:

$$x'' = F\left[x(s), x'(s)\right]\delta(s) \qquad (A1)$$

where F is an analytical function in a sufficiently large interval around $s = 0$.

Let us replace the delta function with a step function $H(s,\epsilon)$, which is zero everywhere except in the interval from $s = -\epsilon$ to $s = +\epsilon$, where it is equal to one,

$$x'' = F\left(x, x'\right)\frac{H(s,\epsilon)}{2\epsilon} \qquad (A2)$$

The exact solution of the original equation is obtained by taking the limit $\epsilon \to 0$.

In particular

$$\Delta x' = \lim_{\epsilon\to 0}\frac{1}{2\epsilon}\int_{-\epsilon}^{\epsilon} F(s)\, ds \qquad (A3)$$

This integral can be estimated by subdividing the 2ϵ interval into N substeps of length $h = 2\epsilon/N$, so that

$$\Delta x' = \lim_{\epsilon\to 0}\{\frac{1}{2\epsilon}\lim_{N\to\infty}\left[\sum_{n=1}^{N} F(-\epsilon + nh)\right]\} \qquad (A4)$$

Using the analytical characteristics of the $F(s)$ in the interval of integration, we can expand:

$$F(-\epsilon + nh) = F(-\epsilon) + F'(-\epsilon)\, nh + \frac{F''(-\epsilon)}{2}(nh)^2 + ... \qquad (A5)$$

Substituting this expansion into Eq. (A4) yields

$$\Delta x' = \lim_{\epsilon\to 0}\lim_{N\to\infty}\frac{1}{N}\left[NF(-\epsilon) + hF'(-\epsilon)\sum_{n=1}^{N} n + ... + \frac{F^{(m)}}{m!}h^m\sum_{n=1}^{N} n^m + ...\right] \qquad (A6)$$

As $\sum_{n=1}^{N} n^m$ is a polynomial in N of the order $m+1$, in the limit $N \to \infty$ we derive

$$\Delta x' = \lim_{\epsilon \to 0} \left[F(-\epsilon) + 2\epsilon F'(-\epsilon) + ... + \frac{(2\epsilon)^m}{m+1} \frac{F^{(m)}(-\epsilon)}{m!} + ... \right] \qquad (A7)$$

In the limit $\epsilon \to 0$ finally

$$\Delta x' = F(x, x') \qquad (A8)$$

where F is evaluated just at the entrance of the integration interval.

The generalization to a system of two second order differential equations is straight forward.

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