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# EXAMPLES OF FOUR-DIMENSIONAL SOLITON SOLUTIONS AND ABNORMAL NUCLEAR STATES

T. D. Lee

Columbia University, New York, N.Y. 10027
and
CERN, Geneva



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## DISCLAIMER

## 1. Introduction

In this talk, I would like to give some examples of four-dimensional quantum soliton solutions. For simplicity, let us first consider only relativistic Boson fields with nonlinear couplings (assuming them to be renormalizable in the usual sense). We define a classical soliton solution to be one that

1. has a finite and nonzero rest mass,

and 2. is confined in a finite region in space at all time.

In quantum mechanics, the corresponding quantum soliton solution then 1) also has a finite and nonzero mass (expressed in terms of the usual renormalized quantities), which reduces asymptotically to the classical expression when the appropriately defined nonlinear coupling constant $g$ is sufficiently small, and 2) has a spatial extension that gives rise to "soft" form factors (i.e., goes to zero at large momentum-transfer). Because of the uncertainty principle, the quantum soliton solution cannot be confined in space at all time. [According to our definition, neither the usual free meson solution nor the hydrogen atom is a soliton, since classically the former is clearly unconfined and the latter, being unstable, emits a radiation field that also is unconfined. ] Our definition <u>differs</u> from that used in some mathematical literature in which the term soliton is restricted only to some extremely specialized nonlinear solitary wave solutions whose shape and velocity remain unchanged even after a head-on collision; such a narrow definition is quite useless in particle physics since even two electrons, after a head-on collision, have to radiate.

In the four-dimensional space, if besides scalar fields there is also the non-Abelian gauge spin-1 field, solitons may be formed by imposing special boundary conditions on the gauge field at infinity, as in the t'Hooft magnetic monopole solution. As we shall see, there is an alternative mechanism to generate soliton solutions without any special boundary condition at infinity. Let us assume that the system has some local current conservation law $(\partial j_\mu^a/\partial x_\mu) = 0$. Consider the sector $Q^a \equiv \int j_0^a d^3r \neq 0$. Since $Q^a$ contains both the field variables and their time derivatives, classically the lowest energy state is, in general, time-dependent. With an appropriate interaction, this lowest energy state may be a soliton, as will be illustrated by the following simple example.

## II. An Example

We assume the system to consist of only spin-0 fields: a complex field $\phi$ and a Hermitian field $\chi$. The interaction Lagrangian density $\mathcal{L}$ is

$$\mathcal{L} = -\frac{\partial\phi^\dagger}{\partial x_\mu}\frac{\partial\phi}{\partial x_\mu} - \frac{1}{2}\left(\frac{\partial\chi}{\partial x_\mu}\right)^2 - (m+g\chi)^2\phi^\dagger\phi - g^{-2}V(g\chi) \qquad (1)$$

where $x_\mu = (\vec{r}, it)$ and $\phi^\dagger$ is the Hermition conjugate of $\phi$. Because of renormalizability, $V$ is assumed to be a 4th-order polynomial in $\chi$. Furthermore, we assume $V$ to have an absolute minimum at $\chi = 0$ and a local minimum at $\chi = -m/g$. Thus, $V$ may be written as

$$V(g\chi) = \frac{1}{2}\left(\frac{\mu}{m}\right)^2\frac{g^2\chi^2}{1+a}\left[(m+g\chi)^2 + am(m+\tfrac{4}{3}g\chi)\right] \qquad (2)$$

where $\mu$ and $a$ are constants. When $\chi \to 0$, $g^{-2}V(g\chi) \to \frac{1}{2}\mu^2\chi^2$; hence, $\mu$ may be regarded as the $\chi$-meson mass. We note that $V'(g\chi) \equiv dV(g\chi)/d(g\chi)$ is zero at $\chi = 0$, $-m/g$ and $-\frac{1}{2}m(1+a)/g$. Thus, $0 \lneqq a \lneqq 1$ in order that $\chi = -\frac{1}{2}m(1+a)/g$ be a local maximum of $V$ and $\chi = 0$ be its absolute minimum. Let $\Delta$ be the magnitude of $V(g\chi)$ at its local minimum, $g\chi = -m$, measured in units of $\mu^2 m^2$: $\Delta \equiv V(-m)/(\mu m)^2 = [6(1+a)]^{-1}a$. Because $a$ is between 0 and 1, one has $0 \leq \Delta \leq \frac{1}{12}$. The current density $j_\mu \equiv -i\left[\phi^\dagger\frac{\partial\phi}{\partial x_\mu} - \frac{\partial\phi^\dagger}{\partial x_\mu}\phi\right]$ satisfies the usual local conservation law. Consequently, the charge $Q \equiv -i\int j_4 d^3r$ is a constant of motion.

For simplicity, we discuss only the radially symmetrical solution. For the classical solution, it is convenient to introduce $\phi(\vec{r}, t) = g^{-1}\rho(r)\, e^{-i\omega t}$ and $\chi(\vec{r}, t) = g^{-1}\theta(r)$ where $\rho$ and $\theta$ are both real functions, depending only on $r = |\vec{r}|$. Therefore, $\rho$ and $\theta$ satisfy

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\rho}{dr}\right) + \omega^2\rho - (m+\theta)^2\rho = 0 \tag{3}$$

and

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\theta}{dr}\right) - 2\rho^2(m+\theta) - V'(\theta) = 0 \tag{4}$$

where

$$V'(\theta) = \left(\frac{\mu}{m}\right)^2 \theta(m+\theta)\left(m + \frac{2\theta}{1+a}\right) . \tag{5}$$

The charge $Q$ is related to $\omega$ and $\rho$ through $Q = 8\pi(\omega/g^2)\int_0^\infty r^2\rho^2\,dr$. The total energy of the system is given by

$$E = (4\pi/g^2)\int_0^\infty r^2\,\mathcal{E}\,dr \tag{6}$$

where

$$\mathcal{E} = \left(\frac{d\rho}{dr}\right)^2 + \tfrac{1}{2}\left(\frac{d\theta}{dr}\right)^2 + \omega^2\rho^2 + (m+\theta)^2\rho^2 + V(\theta) \quad . \tag{7}$$

The radial equations (3) and (4) can be derived by keeping $Q$ fixed and setting the variational derivatives $\delta E/\delta\rho(r) = \delta E/\delta\theta(r) = 0$.

We establish the following two properties for the classical solution:

Theorem 1. The minimum of $E$ when $\chi = 0$ is $E^0_{min} = Qm$; the corresponding $\phi$ for $Q \neq 0$ is not confined within a finite volume. [The superscript 0 in $E^0_{min}$ serves as a reminder of the condition $\chi = 0$.]

Proof. When $\chi = 0$, one has clearly $\theta = 0$, and therefore $\mathcal{E} \geq (m^2+\omega^2)\rho^2$; this leads to $E \geq m^2 I + \frac{1}{4}(Q^2/I)$ where $I = (4\pi/g^2)\int_0^\infty r^2\rho^2\,dr$. By varying with respect to $I$, we derive a lower bound $E^0_{min} \geq Qm$. An upper bound of $E^0_{min}$ can be obtained by assuming a trial function $\rho(r) = \text{constant} \times e^{-r/R}$. We obtain, after minimizing the energy with respect to the multiplicative constant in $\rho(r)$

$$Q m \leqq E^0_{min} \leqq Q(m^2 + R^{-2})^{\frac{1}{2}} \quad . \tag{8}$$

By letting $R \to \infty$, we establish Theorem 1, which is the expression for the usual unconfined solution. [Rigorously speaking, if the entire system is contained within a finite volume with a periodic boundary condition, then the state $E^0_{min} = Q m$ exists. On the other hand, if the volume of the system is infinite, then in order that $Q$ is well defined, we must assume $\rho$ to be square-integrable. Therefore, classically the state $E^0_{min} = Q m$ does not exist. The value $Q m$ is only the limit-point of, but not included in, the set $E(\rho)$, which is evaluated according to (6) over all square-integrable $\rho$, provided $\theta = 0$.]

Theorem 2. When $Q$ is greater than a critical value $Q_c$, there emerges a soliton solution with an energy lower than $Q m$. For small $g$, $Q_c \cong \left(\frac{32\pi^2\mu}{9mg}\right)^2 \Delta$ if $\Delta \neq 0$ and $Q_c \cong \frac{9\pi^3\mu}{2g^2 m}$ if $\Delta = 0$.

Proof. (i) Let us consider first the case $\Delta \neq 0$. We assume a trial function for $\rho(\vec{r})$:

$$\rho(\vec{r}) = \begin{cases} \frac{\rho_0}{r} \sin \omega r & \text{for} \quad r \leqq R \\ 0 & \text{for} \quad r \geqq R \end{cases} \tag{9}$$

where $\omega R = \pi$ and $\rho_0 = (Q g^2/4\pi^2)^{\frac{1}{2}}$. The trial function for $\theta(\vec{r})$ is assumed to be $-m$ for $r \leqq R$, $m(r - R - \lambda)/\lambda$ for $R + \lambda \geqq r \geqq R$ and $0$ for $r \geqq R + \lambda$. An upper bound of the lowest energy value $E_{min}$ can then be derived. By using (6) and (7), we find

$$E_{min} < \frac{\pi Q}{R} + \frac{2\pi m^2}{3g^2} \Big[ 2\mu^2 \Delta (R^3 + \frac{6}{5} R^2 \lambda + \frac{3}{5} R \lambda^2 + \frac{4}{35} \lambda^3) + \frac{1}{10} \mu^2 \lambda (R^2 + R\lambda + \frac{2}{7} \lambda^2) + \lambda^{-1} (3R^2 + 3R\lambda + \lambda^2) \Big] . \tag{10}$$

This inequality holds for arbitrary values of $\lambda$ and $R$. The optimal value of $\lambda$ can be easily seen to be $O(\mu^{-1})$. For $R$ large and $\Delta \neq 0$, the righthand side of (10) becomes

$R^{-1}\pi Q + \frac{1}{3}(4\pi R^3 m^2 \mu^2 \Delta/g^2) + O(R^2 \mu m^2/g^2)$. Taking its minimum, which occurs at $R \cong (2\mu m)^{-\frac{1}{2}}(g^2 Q/\Delta)^{\frac{1}{4}}$, we find

$$E_{min} \leqq \frac{4}{3}\pi\left(\frac{2\mu m}{g}\right)^{\frac{1}{2}} \Delta^{\frac{1}{4}} Q^{\frac{3}{4}} + O(R^2 \mu m^2/g^2) \quad . \tag{11}$$

By comparing this upper bound with $Qm$, one establishes Theorem 2 for $\Delta \neq 0$.

(ii) For $\Delta = 0$ and $R$ large, by applying a similar argument to (10), one derives

$$E_{min} < \left(\frac{6}{5}\right)^{\frac{1}{6}} \frac{3\pi}{2}\left(\frac{4m^2\mu}{3g^2}\right)^{\frac{1}{3}} Q^{\frac{2}{3}} + O(Rm^2/g^2) \quad . \tag{12}$$

Actually, when $\Delta = 0$ and $R$ is large, a better upper bound can be obtained by assuming a different trial function for $\theta(\vec{r})$. We assume $\theta(\vec{r}) = -m\left[1 + e^{\mu(r-R)}\right]^{-1}$, but keeping $\rho(\vec{r})$ the same, still given by (9). The upper bound becomes $E_{min} \leqq R^{-1}\pi Q + \frac{1}{3}(2\pi R^2 m^2 \mu/g^2) + O(Rm^2/g^2)$. By taking its minimum value, which occurs at $R \cong \left(\frac{3g^2 Q}{4m^2\mu}\right)^{\frac{1}{3}}$, we find

$$E_{min} \leqq \frac{3\pi}{2}\left(\frac{4m^2\mu}{3g^2}\right)^{\frac{1}{3}} Q^{\frac{2}{3}} + O(Rm^2/g^2) \quad . \tag{13}$$

This improved upper bound is lower than the previous bound (12) by a factor $\left(\frac{5}{6}\right)^{1/6}$. By directly comparing (13) with $Qm$, we complete the proof of Theorem 2.

It is not difficult to show that when $Q$ is sufficiently large, both the upper bounds (11) and (13) give respectively the correct asymptotic expressions of $E_{min}$ for the cases $\Delta \neq 0$ and $\Delta = 0$. We note that even when $g^2$ is small, $Q_c$ may still be $\sim O(1)$, provided that the ratio $(\mu/m)$ is also small.

The quantization of this soliton solution can be carried out by following the general canonical procedure developed in collaboration with N. Christ. We expand the quantum field operators:

$$\varphi(\vec{r},t) = g^{-1}\rho(\vec{r}-\vec{Z})\,e^{iz} + \sum_n q_n(t)\,\psi_n(\vec{r}-\vec{Z})\,e^{iz}$$

and (14)

$$\chi(\vec{r},t) = g^{-1}\theta(\vec{r}-\vec{Z}) + \sum_n q_n(t)\,\psi'_n(\vec{r}-\vec{Z})$$

where $\rho(\vec{r})$, $\theta(\vec{r})$ are the classical solutions and $\psi_n(\vec{r})$, $\psi'_n(\vec{r})$ are all c.no. functions that satisfy a certain set of orthonormal relations. [The details will be discussed in the talk by N. Christ.] The result is that $\vec{Z}$, $z$ and the $q_n$'s describe a complete set of generalized coordinate variables, all independent. The conjugate momentum of $\vec{Z}$ is the total momentum $\vec{P}$ of the system, and the conjugate momentum of $z$ is the charge operator $Q$. Because the Schroedinger wave function is periodic in $z$ with a period $= 2\pi$, the charge $Q$ is quantized. When $g$ is small, to $O(g^{-2})$ the quantum soliton mass is given by the same classical expression (11) or (13).

## III. Abnormal Nuclear State

The generalization of the above soliton mechanism to include Fermions leads to, among others, the abnormal nuclear state. The details will be omitted in this written report, since it has already been well discussed in a series of papers written in collaboration with G. C. Wick and M. Margulies.

## IV. Remarks

Besides the abnormal nuclear state, solitons involving a few Fermions can also be constructed quite simply, e.g., by first forming a soliton solution out of Bosons, and then assuming an additional attractive interaction which binds the Fermions to the Boson-soliton; the composite is a new soliton solution. When the non-linear coupling $g$ is small, systematic quantum expansions can be carried out for such soliton solutions. [Except for the weak strength of $g$, these examples resemble the bag model considered by Bardeen, Chanowitz, Drell and Yan, and other related models of M. Creutz and of P. Vinciarelli. ]

We note that even when all elementary interactions are characterized by a weak coupling g, since the Boson field strength in the soliton solution is $\sim O(g^{-1})$, the binding force between the additional Fermion and the Boson-soliton remains strong, $\sim O(1)$. For the same reason, the forces between solitons (with or without Fermions) can be very strong, $\sim O(g^{-2})$. Because of the inherently "soft" form factors associated with such strong forces, at small distances only the elementary weak interaction remains. Consequently, there is no ultraviolet divergence connected with such strong forces between solitons. In this sense, these solitons resemble the observed hadrons; their strong forces are "soft", but their weak interactions are "hard" (i.e., point-like).