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Slowing-In Distribution of Fast Particles Released in Maxwellian Plasma

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SLOWING-IN DISTRIBUTION OF FAST PARTICLES RELEASED IN MAXWELLIAN PLASMA

by

**A. A. Hussein
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ABSTRACT

The time-independent slowing-in distribution is evaluated for a fast beam of test particles released in an infinite Maxwellian plasma background for various situations including quantum-mechanical and classical scattering regimes. The analysis involves considerations of small angle deflection and large angle scattering events. The results are determined for plasma electrons with thermal velocities larger or smaller than the test particle and plasma ions with thermal velocities lower than the test particle. When plasma electrons and ions are slow the slowing-in distribution behaves as $E_i^{1/2}$, where E_i is the test particle energy. When plasma electrons are faster than the test particles, the high-energy tail acquires a $1/E_i$ shape.

The origin of the test particle remains arbitrary and hence the results apply for particles produced by fusion events within the plasma proper and for particles externally injected. The total energy distribution is obtained and normalized, and ranges of validity are determined. The calculations are of the Boltzmann type, which is consistent with binary scattering models.

The slowing-in distributions are also obtained from the Fokker-Planck equation, and as expected, the results agree with those obtained from the Boltzmann equation for large angle scattering only.

I. INTRODUCTION

The slowing-in distribution of fast test particles released in a plasma background is important in thermonuclear applications. Energetic particles are produced in plasma by thermonuclear reactions or are externally injected as in fusion devices using neutral beam heating and in two-component plasma devices.¹⁻³ Evaluation of the heat deposition or thermonuclear burn rates of particles requires characterizing their velocity distribution. Although some effort has been devoted to study the behavior of fast test particles in plasmas, functional deter-

mination of the slowing-in distribution has not yet been examined.

In this report we evaluate the time-independent energy distribution of fast test particles released in an infinite plasma background. Results are valid for plasma confinement times larger than the slowing-down⁴ and the thermalization times⁵ of the test particles in the plasma. The analyses are carried out using Boltzmann-type calculations for the quantum-mechanical and the classical limits in the slowing-down energy range, where the energy of the test particles is larger than the kinetic energy of the plasma particles. In this case the velocity of the test particle

is larger than the thermal velocity of any plasma ion of comparable mass, although the thermal velocity of the plasma electrons can be larger than the velocity of the test particle.

The background plasma components are assumed to have a Maxwellian velocity distribution. This assumption is true unless the plasma particles are being injected at high energies and the particle lifetime is short. Nevertheless, the assumption is still useful if the slowing-in component of the distribution is small compared with the thermal distribution part.

In Sec. II the balance equations are set up in their general form with the definitions of the probability functions involved. The quantum mechanical analysis is then carried out in Sec. III. The energy transfer probabilities are determined using the elastic scattering cross section, which has been evaluated using the first Born approximation. The behavior of the energy transfer probability as function of the after-encounter energy is used to develop a scattering model in which large angle scattering events are excluded. The transport equation is then solved for the slowing-in distribution by using the energy transfer probabilities of test particle scattering caused by plasma particles with thermal velocities smaller than the test particle velocity. The effect of the electrons that are thermally moving faster than the test particles is considered, and the corresponding slowing-in energy distribution is obtained. In Sec. IV the classical results are evaluated using the Rutherford cross section excluding small angle scattering events.

Because the quantum mechanical solutions involve neglecting large angle scattering events, whereas the classical solutions include only these events, we examine in Sec. V the possibility of extending the range of applicability of the results in the classical and quantum mechanical regimes. In Sec. VI the slowing-in distribution is then combined with the thermal distribution of the test particles and the average energy and normalization factors are determined. The main results are summarized in Sec. VII. Using available expressions of the stopping power derived from the Fokker-Planck equations, we compare the slowing-in distribution with that obtained here. Recommendations for further work are given in Sec. VIII.

II. GENERAL FORMULATION

Let us define the encounter density in the slowing-down range $\psi_{j\nu}(E_j, T_\nu)$ as the rate of encounters per

unit volume per unit energy between test particles of species j and plasma particles of type ν . If the cross section for nonelastic scattering between the test particles and the plasma background is negligible compared with the elastic scattering cross section, the total encounter density of the test particles $\Psi_j(E_j)$ in an I -component plasma is

$$\Psi_j(E_j) = \sum_{\nu=1}^I \psi_{j\nu}(E_j, T_\nu) = \sum_{\nu=1}^I \Sigma_s(E_j, T_\nu) n_j(E_j) v_j^1(E_j), \quad (1)$$

where $\Sigma_s(E_j, T_\nu)$ is the macroscopic cross section for elastic scattering between test particles of species j at energy E_j and plasma particles of species ν at kinetic temperature T_ν , $n_j(E_j)$ is the number density of particles of species j per unit energy between energy E_j and $E_j + dE_j$, and $v_j^1(E_j)$ is the velocity of species j in the laboratory system of coordinates. For a multicomponent plasma, the density of encounters per unit energy between the test particles and the plasma particles of species ν can be expressed in terms of the total encounter density using the definition of Eq. (1); that is,

$$\psi_{j\nu}(E_j, T_\nu) = \Sigma_s(E_j, T_\nu) \Psi_j(E_j) / \Sigma_{ST}(E_j), \quad (2)$$

where Σ_{ST} is the total elastic scattering cross section,

$$\Sigma_{ST}(E_j) = \sum_{\nu} \Sigma_s(E_j, T_\nu). \quad (3)$$

Expressions for elastic scattering cross sections have been derived for a wide range of test particle energies and kinetic temperatures of plasma particles.^{5,6}

The space- and time-independent transport equation for test particles in an infinite medium can be written in terms of encounter densities as

$$\Psi_j(E_j) = \sum_{\nu} \int_{E_j}^{E_j/\alpha_{j\nu}} dE_j' \Gamma(E_j' + E_j, T_\nu) \psi_{j\nu}(E_j', T_\nu) + S_j(E_j), \quad (4)$$

where the energy transfer probability $\Gamma(E_j \rightarrow E_j, T_\nu)$ is the probability that the energy of a fast test particle of species j changes from E_j to E_j after an elastic scattering event with a plasma particle of species ν at kinetic temperature T_ν . The lowest energy at which a particle of species j at energy E_j can emerge after an encounter is $\alpha_{j\nu} E_j$. The after-encounter energy can be obtained from the kinematics of collision and is given by

$$E_j' = E_j (m_\nu^2 + m_j^2 + 2m_\nu m_j \cos \theta_{j\nu}) / (m_\nu + m_j)^2, \quad v_j^L > v_\nu^T \quad (5)$$

and

$$E_j' = E_j (1 - \sqrt{\frac{m_e kT_e}{m_j E_j}} \theta_{je}^2), \quad v_j^L < v_e^T < \frac{m_j v_j^L}{m_e} \quad (6)$$

Here m_ν , m_j , and m_e are the masses of a particle of species ν , a particle of species j , and an electron, respectively; $\theta_{j\nu}$ and θ_{je} are the deflection angles from an encounter between a particle of species j with a particle of species ν and with an electron, respectively; T_e is the electron temperature; and v_e^T and v_i^T are the thermal velocities of a particle of species ν and an electron, respectively; that is, $v_i^T = \sqrt{2kT_i/m_i}$ and $v_e^T = \sqrt{2kT_e/m_e}$. Here T_ν is the kinetic temperature of a plasma particle of species ν . Equation (5) is valid for plasmas with relatively cold electrons for which $v_j^L > v_e^T$ and $v_j^L > v_i^T$, where v_i^T is the thermal velocity of a plasma ion of species i . However, if the plasma electrons are of relatively higher but still modest kinetic temperature such that $v_j^L < v_e^T$, Eq. (6) applies to energy change due to encounters with the electrons and Eq. (5) applies for scattering by plasma ions as long as $v_j^L > v_i^T$. Equation (6) is based on the fact that only elastic scattering between fast ions of species j and the faster electrons result in small angles of deflection. The condition $E_j > kT_\nu$ and $E_j > kT_e$ applies for Eqs. (5) and (6). The value of $\alpha_{j\nu}$ is given by the ratio of the energy after a head-on collision and the initial energy of the particle of species j . For encounters with fast electrons the value of α_{je} is given by the ratio between the energy corresponding to $\theta_{je} \approx 1$ and that corresponding to $\theta_{je} = 0$. Thus, from Eqs. (5) and (6)

$$\alpha_{j\nu} = \left(\frac{m_\nu - m_j}{m_\nu + m_j} \right)^2, \quad v_j^L > v_\nu^T, \quad \text{and} \quad (7)$$

$$\alpha_{je} \approx \begin{cases} 1 - \frac{4m_e}{m_j}, & v_j^L > v_e^T \end{cases} \quad (8a)$$

$$\alpha_{je} \approx \begin{cases} 1 - \sqrt{\frac{m_e kT_e}{m_j E_j}}, & v_j^L < v_e^T \end{cases} \quad (8b)$$

The source term $S_j(E_j)$ in Eq. (4) represents the number of test particles of species j that are released per unit time at an energy E_j into a plasma. The functional dependence of the source term on E_j is determined by the extent of the spread in initial energy of the test particles and by their origin, that is, whether they are being produced by fusion reactions in the plasma or being externally injected. In many cases of practical interest the source may be represented by a delta function centered about the initial energy E_{j0} , that is,

$$S_j(E_j) = S_{j0} \delta(E_j - E_{j0}) \quad (9)$$

where S_{j0} , the source strength, is the rate of release of particles of species j in a unit plasma volume per unit energy. Gaussian distributions also are used for source distributions. If the test particles are externally injected the source strength as well as the initial energy E_{j0} is to be specified from the injected beam parameters. In the case of reaction test particles j , that is, test particles produced by fusion reactions within the plasma, E_{j0} is obtained from the reaction kinematics without considering the kinetic energies of the reacting particles. If the reaction test particles are produced by fusion reactions between ions of species μ , the source strength is

$$S_{j0} = \sum_{\mu} n_\mu n_{\mu'} \langle \sigma v \rangle_{\mu\mu'}(j) / (1 + \delta_{\mu\mu'}) \quad (10)$$

where n_μ and $n_{\mu'}$ are the number density of plasma ions μ and μ' , respectively; $\delta_{\mu\mu'}$ is the Kronecker delta; and $\langle \sigma v \rangle_{\mu\mu'(j)}$ is the reactivity of fusion events resulting in the release of particles of species j ; that is, the product of the fusion cross section times the relative velocity averaged over the two distributions of the reacting ion species μ and μ' for reactions leading to the production of particles of species j .

The source strength can also be estimated as that required to balance losses from the distributions according to the continuity equation. Thus, for fissionable ions of species j ,

$$S_{j0} = \sum_{i=1}^N \frac{N_i(N_i - \delta_{ij})}{(1 + \delta_{ij})} \langle \sigma v \rangle_{ij} + \frac{N_j}{\tau_j}, \quad (11)$$

where N_i and N_j are the average number densities of ions of species i and j , respectively; τ_j is the average confinement time of ions of type j ; $\langle \sigma v \rangle_{ij}$ is the reactivity of fusion events involving ions of species i and j ; and the averages are calculated over the velocity distributions including the slowing-in components of the distributions. The term N_j/τ_j is finite for systems where the confinement time τ_j is larger than the slowing down time, and becomes zero for infinite systems because $\tau_j \rightarrow \infty$. If the test particles are electrons or nonfissionable ions, the burnup term is zero and the source strength is then equal to the leakage rate; that is, $S_{j0} = N_j/\tau_j$.

To solve Eq. (4) and obtain the encounter density $\psi_j(E_j)$, the energy transfer probability $\Gamma(E_j' \rightarrow E_j, T_\nu)$ for each of the plasma species must be specified. Thus, let us consider a quasineutral plasma that consists of several ion species. The probability that a test particle of species j at an energy E_j emerges after an encounter with a plasma particle of species ν at a kinetic temperature T_ν with an energy between E_j and $E_j + dE_j'$ is equal to the probability of the test particle scattering into an angle between $\theta_{j\nu}$ and $\theta_{j\nu} + d\theta_{j\nu}$ after an encounter with the same plasma particle, that is,

$$\Gamma(E_j' \rightarrow E_j, T_\nu) dE_j' = \frac{2\pi \Sigma_S(E_j, T_\nu, \theta_{j\nu}) d \cos \theta_{j\nu}}{\Sigma_S(E_j, T_\nu)}, \quad (12)$$

where $\Sigma_S(E_j, T_\nu, \theta_{j\nu})$ is the angular differential elastic scattering cross section. The after-encounter energy is related to the scattering angle by Eq. (5) for $v_j' > v_j^T$, and the elastic scattering cross section can be explicitly written in terms of pre- and after-encounter energies. Hence, the energy-transfer probability can be determined from Eq. (12). In the case of $v_j' < v_j^T$ for which scattering events result only in small deflection angles, Eq. (12) may be approximated to

$$\Gamma(E_j' \rightarrow E_j, T_\nu) dE_j' = \frac{2\pi \Sigma_S(T_\nu, \theta_{j\nu}) \theta_{j\nu} d\theta_{j\nu}}{\Sigma_S(T_\nu)} \quad (13)$$

and Eq. (6) can then be used directly to eliminate the dependence on the angle of deflection from the right-hand side of Eq. (13).

The total probability of changing the energy of the test particle from an energy E_j to any energy E_j' in the range $\alpha_{j\nu} E_j \leq E_j' \leq E_j$ after an encounter with a plasma particle of species ν is unity and is equal to the probability of scattering the test particle in any angle $\theta_{j\nu}$ in the range $0 \leq \theta_{j\nu} \leq \pi$ after an encounter. The same equality follows for an encounter by electrons of $v_e^T > v_j^T$ leading to an angle of deflection in the range $0 \leq \theta_{je} \leq 1$.

To obtain an asymptotic solution for Eq. (4) at energies $E_j < E_{j0}$ we may define the slowing-down density T_j as the number of test particles of species j slowing down past an energy E_j per unit volume per unit time; that is,

$$T_j = \sum_\nu \int_{E_j}^{E_j/\alpha_{j\nu}} dE_j' G_j(E_j', E_j; T_\nu) \psi_{j\nu}(E_j', T_\nu), \quad (14)$$

where $G_j(E_j', E_j; T_\nu)$ is the probability that a test particle at an energy $E_j' > E_j$ emerges after an encounter with a plasma particle of species ν with an energy between $\alpha_{j\nu} E_j$ and E_j ; that is,

$$G_j(E_j', E_j; T_\nu) = \int_{\alpha_{j\nu} E_j}^{E_j} dE_j'' \Gamma(E_j' \rightarrow E_j'', T_\nu). \quad (15)$$

In a nonabsorbing plasma background and in absence of particle leakage;

$$T_j = S_{j0} \quad (16)$$

The relations given above and the related discussion are applicable in both the quantum-mechanical and classical limits. However, the expressions that relate the scattering cross sections to the parameters of the plasma and the test particles differ. The quantum-mechanical cross sections are used when

$$v_j^L > 2.188 \times 10^8 Z_j \text{ cm/s}, \quad v_j^L > v_v^T \quad (17)$$

and

$$v_e^T > 2.188 \times 10^8 Z_j \text{ cm/s}, \quad v_j^L < v_e^T \quad (18)$$

where Z_v and Z_j are the charge numbers for particles of species v and j , respectively. On the other hand, classical cross sections are used when the larger-than signs in Eqs. (17) and (18) are replaced by smaller-than signs. In general, the total cross section is evaluated classically and quantum-mechanically, and the smaller of the two values is used.

III. QUANTUM-MECHANICAL SLOWING-IN DISTRIBUTION

When the velocity of the test particles is large enough to satisfy Eq. (17) and at the same time $v_j > v_v^T$, where v refers to both plasma ions and electrons, the total elastic scattering cross section is given in cgs units by⁴

$$\Gamma_S^Q(E_j, T_v) = \frac{Z_j^2 e^2 m_j k T_v}{2 \hbar^2 E_j} \quad (19)$$

where superscript Q is assigned to quantum-mechanical parameters. The angular differential, elastic scattering cross section is given in the first Born approximation by

$$\Gamma_S^Q(E_j, T_v, \theta_{jv}) = \frac{n_v Z_j^2 Z_v^2 e^4 m_j}{16 \mu^2 E_j^2 \left(\sin^2 \theta_{jv} + \frac{\pi n_v Z_v^2 e^2 \hbar^2}{2 \mu^2 E_j k T_v} \right)^2} \quad (20)$$

$$0 \leq \theta_{jv} \leq \pi$$

where μ is the reduced mass of a pair of particles of species v and j , and n_v is the number density of particles of species v . Actually, the total elastic scattering cross section given by Eq. (19) is obtained by integrating the differential cross section of Eq. (20) over all scattering angles.

Taking the differential of Eq. (5) with respect to the angle of deflection and rearranging the results, we get

$$dE_j' = \frac{2 \mu^2 E_j}{m_v m_j} d \cos \theta_{jv} \quad (21)$$

By substituting Eqs. (19)-(21) into Eq. (12) the energy transfer probability is

$$\Gamma^Q(E_j \rightarrow E_j', T) dE_j' = \begin{cases} \Delta_{jv}^Q / (E_j - E_j' + \Delta_{jv}^Q)^2, \\ \text{for } \alpha_{jv} E_j < E_j' < E_j, \\ 0, \text{ otherwise} \end{cases} \quad (22)$$

where

$$\Delta_{jv}^Q = \frac{\hbar^2}{2 m_v \lambda_{Dv}^2} \quad (23)$$

Here λ_{Dv} is the screening Debye radius defined as

$$\lambda_{Dv}^2 = \frac{k T_v}{4 \pi n_v Z_v^2 e^2} \quad (24)$$

In the case when $v_j < v_j^T < v_e^T$, both the conditions given by Eqs. (17) and (18) must be satisfied to use quantum-mechanical cross sections for scattering events involving the test particles on one hand and plasma ions and electrons on the other. If v_e^T satisfies the condition of Eq. (18) and v_j does not satisfy the condition of Eq. (17), the quantum-mechanical cross sections are used to describe elastic scattering of the test particles by encounters with plasma electrons, and classical cross sections are used to describe elastic scattering of test particles by encounters with plasma ions and vice versa.

For $v_j < v_e^T$, the angular differential cross section is given in the first Born approximation by

$$\Sigma_S^Q(\tau_e, \theta_{je}) \approx \frac{n_e z_j^2 e^4}{(k\tau_e)^2 (\theta_{je}^2 + \theta_{je0}^2)^2}, \quad 0 < \theta_{je} < 1, \quad (25)$$

where θ_{je0} is the often called quantum-mechanical cutoff angle,

$$\theta_{je0}^Q = \frac{\hbar_e}{k\tau_e} \sqrt{\frac{2\pi n_e}{m_e}} \quad (26)$$

where n_e is the electron number density. The total elastic scattering cross section is obtained by integrating the differential cross section of Eq. (25) over all angles of deflection; that is,

$$\Sigma_S^Q(\tau_e) \approx \frac{n_e z_j^2 e^2}{2\hbar^2} \quad (27)$$

Substituting Eqs. (26) and (27) into Eq. (13) and eliminating the dependence on the angle of deflection by means of Eq. (6), the energy transfer probability is obtained as

$$\Gamma^Q(E_j \rightarrow E_j', \tau_e) \approx \begin{cases} \Delta_{je}^Q(E_j) / (E_j - E_j' + \Delta_{je}^Q(E_j))^2, \\ \text{for } \alpha_{je} E_j < E_j' < E_j, \\ 0, \text{ otherwise} \end{cases} \quad (28)$$

where α_{je} is as given by Eq. (8b) and

$$\Delta_{je}^Q \approx \frac{2\pi n_e \hbar^2 e^2}{(k\tau_e)^{3/2}} \sqrt{\frac{E_j}{m_j m_e}} \quad (29)$$

Here the parameter Δ_{je}^Q is a function of E_j , whereas in the case of $v_j > v_e^T$ it is independent of E_j as indicated by Eqs. (23) and (24).

The energy transfer function of Eq. (28) applies only to scattering of test particles with electrons when $v_j < v_j^T < v_e^T$. The energy transfer function for scattering by plasma ions in this case is given by Eq. (22) if the condition of Eq. (17) is satisfied. The results of Eqs. (25)-(29) are applicable only for modest electron kinetic temperatures, that is

$$v_e^T < \frac{m_i}{m_e} v_j^T$$

Considerations of higher plasma kinetic temperatures are not included here because provisions of upward scattering in energy need to be included in that case, and such considerations are outside the slowing-down energy range of interest. Although the results of Eqs. (19)-(24) can be applied to fast test electrons, the approximations involved in the derivation of Eqs. (25)-(29) do not allow for such situations.

A. The Scattering Model

Consider the energy transfer probability function given by Eq. (22). This has a maximum corresponding to no energy transfer, that is, when the test particle emerges from one encounter with the pre-encounter energy E_j . In this case, $\Gamma^Q(E_j \rightarrow E_j, T_\nu) = 1/\Delta Q_\nu$. The probability function then decreases very slowly as the after-encounter energy decreases to an energy $E'_j = E_j - \Delta Q_\nu$, that is, $\Gamma^Q(E_j \rightarrow E_j - \Delta Q_\nu, T_\nu) = 1/4\Delta Q_\nu$. The energy range $(E_j, E_j - \Delta Q_\nu)$ corresponds to distant encounters, which are the most probable events, and the energy transfer probability in this range may be considered as independent of both E_j and E'_j . For after-encounter energies below $E'_j = E_j - \Delta Q_\nu$, the energy transfer probability rapidly drops as $\Delta Q_\nu/(E_j - E'_j)^2$ until it reaches a minimum at an after-encounter energy $E'_j = \alpha_{j\nu} E_j$ corresponding to head-on collision, which is least likely because

$$\Gamma^Q(E_j \rightarrow \alpha_{j\nu} E_j, T_\nu) = \frac{\Delta Q_\nu}{(1 - \alpha_{j\nu})^2 E_j^2}.$$

For $m_j = m_\nu$ it follows that $\alpha_{j\nu} = 0$, and $\Gamma^Q(E_j \rightarrow 0)$ assumes a lower value than that obtained for $m_j \neq m_\nu$, using the same plasma parameters. In the case of scattering by plasma electrons with a velocity $v_e^T < v_j$, Eq. (23) shows that $\Delta Q_e > \Delta Q_i$, where i refers to ions of species i and Eq. (8a) gives

$$(1 - \alpha_{ej}) = \frac{4m_e}{m_j}.$$

Consequently

$$\Gamma^Q(E_j \rightarrow (1 - \frac{4m_e}{m_j})E_j, T_e) = \frac{\Delta Q_e^2}{16m_e^2 E_j^2},$$

which is of higher value compared with scattering by encounters with plasma ions; however, for relatively large values of E_j the slowing-down energy transfer probability function behaves in the same manner as in the case of $m_j = m_\nu$, and hence this function has a relatively lower value in the close-encounter domain, that is, in the range $(E_j - \Delta Q_\nu, \alpha_{j\nu} E_j)$, than its value in

the range $(E_j, E_j - \Delta Q_\nu)$ for scattering by plasma electrons or ions.

If we neglect contributions from close encounters the results of Eq. (22) may be approximated to,

$$\Gamma^Q(E_j \rightarrow E'_j, T_\nu) = \begin{cases} \frac{1}{\Delta Q_\nu}, & E_j - \Delta Q_\nu < E'_j < E_j \\ 0, & \text{otherwise} \end{cases}$$

Although the probability that the test particle acquires an energy $E'_j < E_j - \Delta Q_\nu$ after an encounter is assumed to be zero, the total probability of energy transfer leading to an after-encounter energy between $E_j - \Delta Q_\nu$ and E_j is approximately equal to unity as is the value of the integral over E'_j of Eq. (22) in the range of $\alpha_{j\nu} E_j \leq E'_j \leq E_j$.

By the same argument the energy transfer probability of Eq. (28) for $v_j < v_e^T$ can be approximated to

$$\Gamma^Q(E_j \rightarrow E'_j, T_e) \approx \begin{cases} \Delta E'_j / \Delta Q_e(E_j), & E_j - \Delta Q_e(E_j) < E'_j < E_j \\ 0, & \text{otherwise} \end{cases} \quad (31)$$

This approximation is tantamount to neglecting scattering events leading to large scattering angles, which is expected to be the case for ion-electron encounters in the range

$$v_j^L < v_e^T < \frac{m_j}{m_e} v_j^L.$$

The total probability of energy transfer leading to any after-encounter energy E'_j between $E_j - \Delta Q_e(E_j)$ and E_j after a single encounter is unity, and hence the approximate energy transfer probability of Eq. (31) is normalized as is the probability given by Eq. (28).

The above scattering model is consistent with an approach developed by Lane and Everhart⁷ for calculating the total elastic scattering cross section and was used by Hussein and Sabri⁸ in evaluating the scattering kernel in the thermalization range. In such an approach the Rutherford cross section is used for the scattering angles in the range $\theta_{Q_{j\nu}} \lesssim \theta_{j\nu} \leq \pi$, and the Born approximation formula of Eq.

(20) is used for the range $0 \leq \theta_{j\nu} \lesssim \theta_{j\nu 0}$. Here the angle $\theta_{j\nu 0}$ is defined by Eq. (26) for scattering by electrons of velocity $v_{j\nu}^T > v_{j\nu}^L$ and by

$$\theta_{j\nu 0}^Q = \frac{\hbar}{\mu \lambda_{D\nu}} \sqrt{\frac{m_1}{2E_j}} \quad (32)$$

for scattering by plasma electrons or ions of velocity $v_{j\nu}^T < v_{j\nu}^L$.

The scattering model introduced here is thus equivalent to using the Born approximation formula for small deflection angles and neglecting events leading to scattering angles greater than $\theta_{j\nu 0}$. The parameters $\Delta_{j\nu}^Q$ and $\Delta_{je}^Q(E_j)$ are related to the cutoff angle by the relations

$$\Delta_{j\nu}^Q = \frac{\mu^2 E_j}{m_\nu m_j} (\theta_{j\nu 0}^Q)^2 \quad (33)$$

and

$$\Delta_{je}^Q(E_j) = \sqrt{\frac{m_e E_j}{m_j k T_e}} (\theta_{je 0}^Q)^2 \quad (34)$$

respectively.

B. Slow Plasma Ions and Electrons

For encounters of a test particle of species j and plasma particles of thermal velocities less than $v_{j\nu}^L$, the energy transfer probability is given by Eq. (30) for plasma electrons and ions. At energies E_j below the source energy E_{j0} , Eq. (4) reduces to the form,

$$\psi_j(E_j) \approx \sum_\nu \frac{1}{\Delta_{j\nu}^Q} \int_{E_j}^{E_j + \Delta_{j\nu}^Q} dE_j' \psi_{j\nu}(E_j', T_\nu) \quad (35)$$

which has a solution

$$\psi_j(E_j) = C \quad (36)$$

as can be shown using Eqs. (2) and (3), where C is a constant to be determined from the slowing-down density. From Eq. (15) the probability function $G_j(E_j', E_j, T_\nu)$ is approximately unity for the test particle to have an after-encounter energy between $E_j' \Delta_{j\nu}^Q$ and E_j . Thus, the slowing-down density given by Eq. (14) reduces to

$$T_j \approx \sum_\nu \int_{E_j}^{E_j + \Delta_{j\nu}^Q} dE_j' \psi_{j\nu}(E_j', T_\nu) \quad (37)$$

Substituting Eq. (36) into Eq. (37) and using the definitions given by Eqs. (2) and (3),

$$T_j = C \sum_\nu \frac{Z_S^Q(E_j, T_\nu)}{Z_{ST}^Q(E_j)} \Delta_{j\nu}^Q \quad (38)$$

because the ratio of the cross sections is independent of E_j . The total cross section can be obtained from Eq. (19) using Eq. (3); that is,

$$\Sigma_{ST}^Q(E_j) = \frac{Z_1^2 e^2 n_1}{2n^2 E_j} \sum_\nu k T_\nu \quad (39)$$

From Eqs. (36) and (38),

$$\psi_j(E_j) = \frac{\Sigma_{ST}^Q(E_j) T_j}{\sum_\nu \Sigma_S^Q(E_j, T_\nu) \Delta_{j\nu}^Q} \quad (40)$$

The slowing-in distribution can be explicitly obtained from Eq. (40) using Eqs. (2), (16), (19), (23), (24), and (39): that is,

$$n_j(E_j) dE_j = \frac{s_{j0} E_j^{1/2} dE_j}{\sqrt{2m_j} \pi Z_j^2 e^4 \sum_v \frac{n_v Z_v^2}{m_v}} \quad (41)$$

The sum in the denominator of Eq. (41) is essentially n_e/m_e , and hence the slowing-in distribution is established primarily because of encounters between the test particles and the plasma electrons; thus

$$n_j(E_j) dE_j = \frac{n_e s_{j0} E_j^{1/2} dE_j}{\sqrt{2m_j} \pi Z_j^2 e^4 n_e} \quad (42)$$

Comparing Eq. (41) with relations derived for the slowing-down time⁴ shows that

$$n_j(E_j) \delta E_j \approx s_{j0} / \left(\sum_v \left(\frac{1}{\delta \tau_{sjv}} \right) \right) \quad (43)$$

where $\delta \tau_{sjv}$ is the slowing-down time required to decrease the energy of test particles of species j by an increment δE_j by encounters with plasma particles of species v . However, the mean time $\delta \tau_{sje}$ to transfer an increment of energy δE_j from the test particles to the electrons is smaller than the time to transfer the same energy increment to ions of comparable mass, and hence

$$n_j(E_j) \delta E_j \approx s_{j0} \delta \tau_{sje} \quad (44)$$

which corresponds to the result of Eq. (42).

C. Slow Plasma Ions and Fast Electrons

In case of $v_i < v_j < v_e^T$, the probability of energy transfer to electrons differs from the probability of energy transfer to plasma ions, assuming the inequalities of Eqs. (17) and (18) are satisfied. Because

$$v_e^T < \frac{1}{m_e} v_j^L,$$

upward scattering of the test particle in energy may be neglected, and the slowing-down approximation is valid. Substituting Eq. (31) for the energy transfer probability due to scattering by electrons and Eq. (30) for that due to scattering of the test particle by plasma ions in Eq. (4); and making use of Eq. (29), the encounter density at energies $E_j < E_{j0}$ may be written as

$$\begin{aligned} \psi_j(E_j) \approx & \frac{(kT_e)^{3/2} \sqrt{m_e m_j}}{2\pi m_e^2 h^2} \int_{E_j}^{E_j + \Delta_{je}^Q(E_j)} \frac{\psi_{je}(E'_j, T_e)}{\sqrt{E'_j}} dE'_j \\ & + \sum_i \frac{1}{\Delta_{ji}^Q} \int_{E_j}^{E_j + \Delta_{ji}^Q} dE'_j \psi_{ji}(E'_j, T_i) \end{aligned} \quad (45)$$

Equations (3), (19), and (27) show that the total scattering cross section is approximately given by

$$\Sigma_{ST}^Q(E_j) \approx \frac{Z_j^2 e^2 m_j}{2h^2 E_j} \sum_i kT_i \quad (46)$$

Because the cross section of scattering by electrons is negligible compared with that by plasma ions and because $kT_i (v_e^T)^2 \ll kT_e (v_j^L)^2$, the density of encounters by electrons is approximately given by

$$\psi_{je}(E'_j, T_e) = \frac{\Sigma_S^Q(T_e) E'_j}{\Sigma_{ST}^Q(E_j) E_j} \psi_j(E'_j) \quad (47)$$

and the density of encounters by plasma ions is

$$\psi_{ji}(E'_j, T_i) = \frac{\Sigma_S^Q(E_i, T_i)}{\Sigma_{ST}^Q(E_j)} \psi_j(E'_j) \quad (48)$$

ψ_{ji} does not depend on E_j because the ratio of the cross sections in Eq. (48) is independent of E_j . Substituting Eqs. (47) and (48) into Eq. (45) we get

$$\psi_j(E_j) \approx \frac{\Sigma_S^Q(T_e)}{\Sigma_{ST}^Q(E_j) \Delta_{je}^Q(E_j) E_j^{1/2}} \int_{E_j}^{E_j + \Delta_{je}^Q(E_j)} dE_j' \sqrt{E_j'} \psi_j(E_j') + \sum_1 \frac{\Sigma_S^Q(E_j, T_1)}{\Sigma_{ST}^Q(E_j) \Delta_{j1}^Q(E_j)} \int_{E_j}^{E_j + \Delta_{j1}^Q(E_j)} dE_j' \psi_j(E_j'), \quad (49)$$

which has a solution of the form given by Eq. (36). The probability that an encounter with a plasma ion leads to a change in the energy of the test particle from E_j to an after-encounter energy between $E_j - \Delta_{ji}^Q$ and E_j is $G_j(E_j, E_j, T_1) \approx 1$. Similarly, a particle at energy E_j emerges after an encounter with a plasma electron at an energy between $E_j - \Delta_{je}^Q(E_j)$ and E_j with a probability $G_j(E_j, E_j; T_e) = 1$. Consequently, the slowing-down density is

$$T_j \approx \frac{\Sigma_S^Q(T_e)}{\Sigma_{ST}^Q(E_j) E_j} \int_{E_j}^{E_j + \Delta_{je}^Q(E_j)} dE_j' E_j' \psi_j(E_j') + \sum_1 \frac{\Sigma_S^Q(E_j, T_1)}{\Sigma_{ST}^Q(E_j)} \int_{E_j}^{E_j + \Delta_{j1}^Q(E_j)} dE_j' \psi_j(E_j'). \quad (50)$$

Introducing the solution of Eq. (36) and using Eq. (16),

$$\psi_j(E_j) \approx \frac{\Sigma_{ST}^Q(E_j) S_{j0}}{\Sigma_S^Q(T_e) \Delta_{je}^Q + \sum_1 \Sigma_S^Q(E_j, T_1) \Delta_{j1}^Q}. \quad (51)$$

Equations (19), (23), (24), (27), and (29) can be used to give the slowing-in distribution in the case of $v_j^T < v_j^L < v_e^T < m_j/m_e v_j^L$ as

$$n_j(E_j) dE_j \approx \frac{S_{j0} dE_j}{\sqrt{2} \pi Z_j^2 e^4 \sum_1 \left[\frac{\sqrt{m_e} Z_1 n_1 E_j}{m_j (kT_e)^{3/2}} + \frac{\sqrt{m_1} Z_1^2 n_1}{E_j^{1/2} m_1} \right]} \quad (52)$$

where the quasi-neutrality condition is used to eliminate the electron number density. The first and second terms between brackets in the denominator represent contributions from encounters with the electrons and plasma ions, respectively. Electron encounters are responsible for enhancing the slowing-in tail of the distribution if

$$E_j \gg \frac{m_j}{m_e^{1/3}} \left(\frac{Z_1}{m_1} \right)^{2/3} kT_e \quad (53)$$

or

$$v_j^L > \left(\frac{m_e Z_1}{m_1} \right)^{1/3} v_e^T. \quad (54)$$

In this case,

$$n_j(E_j) dE_j \approx \frac{m_1 (kT_e)^{3/2} S_{j0} dE_j}{\sqrt{2} m_e \pi Z_j^2 n_e e^4 E_j}. \quad (55)$$

On the other hand, if

$$v_j^L < \left(\frac{m_e Z_1}{m_1} \right)^{1/3} v_e^T \quad (56)$$

the ions are dominant in establishing the slowing-in distribution and Eq. (52) reduces to

$$n_j(E_j)dE_j = \frac{s_{j0} E_j^{1/2} dE_j}{\sqrt{2\pi} z_j^2 \sum_1 \frac{z_1^2 n_1}{m_1}}, \quad (57)$$

which is similar to Eq. (41) except the summation in the denominator of Eq. (55) does not include the electrons. The above results show that the rate of energy transfer to the electrons from energetic test particles decreases and the slowing-down time due to encounters with electrons increases sharply with increasing electron temperature.

IV. CLASSICAL SLOWING-IN DISTRIBUTION

The classical cross sections must be used in calculating the encounter density if

$$v_j^L < 2.188 \times 10^8 z_j \text{ cm/s}, \quad v_j^L > v_v^T, \quad (58)$$

where ν refers to plasma electrons and ions; or if

$$v_e^T < 2.188 \times 10^8 z_j \text{ cm/s}, \quad v_j^L < v_e^T. \quad (59)$$

The classical cross section of elastic scattering between a test particle of species j and plasma particles of species ν is

$$\Sigma_S^C(\tau_\nu) = \frac{k\tau_\nu}{z_\nu^2 e^2}, \quad (60)$$

where C is assigned to parameters evaluated classically. The cross section given in Eq. (60) is independent of the test particles parameters, and hence it applies for the case $v_j > v_\nu^T$, as well as for $v_j < v_\nu^T$. This cross section is equivalent to that obtained for elastic scattering between a neutral particle and a hard, uncharged sphere of radius $\lambda_{D\nu}$. Thus, the classical scattering model excludes impact parameters larger than $\lambda_{D\nu}$, and consequently,

scattering events leading to deflection angles smaller than an angle below which diffraction takes place are neglected. This is in contrast to the scattering model developed in Sec. III.A for the quantum-mechanical case wherein only small angle scattering events are included.

A. Slow Plasma Particles

In the classical limit the angular differential elastic scattering cross section is given by the Rutherford cross section; namely

$$\Sigma_S^C(E_j, \theta_{j\nu}) = \frac{z_1^2 z_\nu^2 e^4 m_1^2 n_\nu}{16\pi^2 E_j^2 \sin^4 \frac{\theta_{j\nu}}{2}} \theta_{j\nu} \quad , \quad \theta_{j\nu 0}^C < \theta_{j\nu} < \pi, \quad (61)$$

in the case of $v_j > v_\nu^T$, where a minimum cutoff angle of deflection $\theta_{j\nu 0}^C$ is used to avoid the divergence in calculating the total cross section. Thus, encounters leading to angles of deflection less than $\theta_{j\nu 0}^C$ are neglected. This corresponds to neglecting encounters at impact parameters larger than the screening radius $\lambda_{D\nu}$. The value of $\theta_{j\nu 0}^C$ is given by

$$\theta_{j\nu 0}^C = \frac{z_1 z_\nu e^2 m_1}{\mu E_1 \lambda_{D\nu}}. \quad (62)$$

This cutoff angle corresponds to a maximum after-encounter energy $E_{j,\max}$, which can be obtained from Eq. (5); that is,

$$E_{j,\max} = E_j - \Delta_{j\nu}^C(E_j), \quad \text{where} \quad (63)$$

$$\Delta_{j\nu}^C(E_j) = \frac{z_1^2 z_\nu^2 e^4 m_1}{4\pi E_j \lambda_{D\nu}^2}. \quad (64)$$

Substituting Eqs. (60) and (61) into Eq. (12) and using Eq. (5) to eliminate $\theta_{j\nu}$, we get

$$r^C(E_j + E'_j, T_v) = \begin{cases} \frac{\Delta_{jv}^C(E_j)}{(E_j - E'_j)^2} , \\ \alpha_{jv} E_j \leq E'_j \leq E_j - \Delta_{jv}^C(E_j) \\ 0, \text{ otherwise} \end{cases} \quad (65)$$

The integral of this energy transfer probability over E'_j is approximately unity. Substituting Eq. (65) into Eq. (4), we get for energies $E_j < E_{j0}$

$$\psi_j(E_j) \approx \sum_v \int_{E_j + \Delta_{jv}^C(E_j)}^{E_j/\alpha_{jv}} dE'_j \frac{\Delta_{jv}^C(E'_j) \psi_{jv}(E'_j, T_v)}{(E'_j - E_j)^2} \quad (66)$$

Here, the density of encounters by plasma particles ν is related to the total encounter density by the relation given in Eq. (2), which can be written explicitly by using Eqs. (3) and (60) as

$$\psi_{jv}(E_j, T_v) = \frac{kT_v \psi_j(E_j)}{Z_v^2 e^2 \Sigma_{ST}^C} \quad (67)$$

where

$$\Sigma_{ST}^C = \sum_{\mu} \frac{kT_{\mu}}{Z_{\mu}^2 e^2} \quad (68)$$

By substituting Eqs. (64) and (67) the total encounter density is explicitly given by

$$\psi_j(E_j) = \frac{\pi Z_j^2 e^4 m_i}{\Sigma_{ST}^C} \sum_v \frac{Z_v^2 n_v}{m_v} \int_{E_j + \Delta_{jv}^C(E_j)}^{E_j/\alpha_{jv}} dE'_j \frac{\psi_j(E'_j)}{E'_j (E'_j - E_j)^2} \quad (69)$$

A solution of the form

$$\psi_j(E_j) = CE_j \quad (70)$$

approximately satisfies Eq. (69), where C is a constant to be determined from the slowing-down density as given by Eqs. (14) and (16). The probability that a test particle at an initial energy $E'_j > E_j$ is scattered to an energy between $\alpha_{jv} E'_j$ and E_j in an encounter $G_j(E'_j, E_j, T_v)$ is obtained by substituting Eq. (65) into Eq. (15) and integrating over the energy range of interest. Thus

$$G_j(E'_j, E_j; T_v) = \Delta_{jv}^C(E'_j) \left[\frac{1}{E'_j - E_j} + \frac{1}{(1 - \alpha_{jv}) E'_j} \right] ,$$

$$E_j + \Delta_{jv}^C(E_j) \leq E'_j \leq E_j/\alpha_{jv} \quad (71)$$

and hence the slowing-down density is obtained by substituting Eqs. (70) and (71) into Eq. (14) and adjusting the limits of integration; that is,

$$T_j = \frac{\pi C Z_j^2 e^4 m_i}{\Sigma_{ST}^C} \sum_v \frac{Z_v^2 n_v}{m_v} \int_{E_j + \Delta_{jv}^C(E_j)}^{E_j/\alpha_{jv}} dE'_j \left[\frac{1}{E'_j - E_j} + \frac{1}{(1 - \alpha_{jv}) E'_j} \right] \quad (72)$$

Carrying out the integral, using Eq. (16), and evaluating the value of the constant C , we can obtain the slowing-in distribution from Eq. (70); that is,

$$n_j(E_j) dE_j = \frac{S_{j0} E_j^{1/2} dE_j}{\pi \sqrt{2m_j} Z_j^2 e^4 \sum_v \left[\frac{Z_v^2 n_v}{m_v} \ln \Lambda_{jv}^C(E_j) \right]} \quad (73)$$

where

$$\Lambda_{jv}^C(E_j) = \left(\frac{2}{\phi_{jv0}(E_j)} \right)^2 \quad (74)$$

The distribution of Eq. (73) is insensitive to the dependence of the logarithmic function on E_j . The sum in the denominator of Eq. (73) is proportional to $1/m_e$. The dominant term in this sum is that of the electrons, and hence Eq. (73) becomes

$$n_j(E_j)dE_j = \frac{n_e S_{10} E_j^{1/2} dE_j}{\pi \sqrt{Z_j} Z_j^2 e^4 n_e \ln \Lambda_{je}^C(E_j)}, \quad (75)$$

that is, the slowing-in distribution is primarily established by encounters with plasma electrons.

B. Effects of Fast Plasma Electrons

For plasma electrons with a thermal velocity v_e^T that satisfies the condition given by Eq. (54), classical cross sections must be used in the derivation of the scattering kernel. If $v_j < v_e^T$ the angular differential cross section is obtained from the Rutherford formula by setting the relative velocity $v_r \approx v_e^T$; that is,

$$\sigma_{je}^C(\tau_e, \theta_{jv}) = \frac{Z_j^2 e^4 n_e}{(kT_e)^2 \theta_{jv}^4} \cdot \theta_{jv0}^C \leq \theta_{jv} \leq 1, \quad (76)$$

where scattering of particles of species j by electrons is assumed to result only in small angles of scattering above the cutoff angle. Here, the classical cutoff angle is given by

$$(\theta_{jv0}^C)^2 = \frac{\pi Z_j^2 e^4 n_e}{(kT_e)^3}. \quad (77)$$

The classical elastic scattering cross section is the same for $v_j < v_e^T$ and is given by Eq. (60). Consequently, the energy transfer probability can be obtained by substituting Eqs. (60) and (76) into Eq. (13) and eliminating the angular dependence by using Eq. (6). Thus,

$$\Gamma^C(E_j + E_j', T_e) dE_j' = \begin{cases} \frac{\Delta_{je}^C(E_j) dE_j'}{(E_j - E_j')^2}, & \alpha_{je} E_j < E_j' < E_j - \Delta_{je}^C(E_j) \\ 0, & \text{otherwise} \end{cases} \quad (78)$$

where

$$\Delta_{je}^C = \frac{\pi Z_j^2 e^4 n_e}{(kT_e)^2} \sqrt{\frac{m_e E_j}{m_j kT_e}} \quad (79)$$

and α_{je} is given by Eq. (8b). In deriving Eq. (78) the approximation $\alpha_{je} \approx 1$ is used to simplify the denominator. The use of a cutoff angle below which the scattering cross section vanishes has excluded the possibility of zero angle of deflection, and consequently, the maximum energy with which the test particle emerges after an encounter with an electron is $E_j - \Delta_{je}^C(E_j)$, which corresponds to a deflection by an angle θ_{jv0}^C .

Because in the slowing-down energy range $v_j^L > v_e^T$, when the masses of the test particle and the plasma ion are comparable, the energy transfer probability of Eq. (65) can be used to derive the density of encounters by plasma ions if v_j^L satisfies the condition given by Eq. (58). Substituting Eqs. (58) and (78) into Eq. (4), the total encounter density at energies $E_j < E_{j0}$ becomes

$$\begin{aligned} \nu_j(E_j) = & \frac{\pi Z_j^2 e^4}{\Sigma_{ST}^C} \left\{ \frac{n_e}{(kT_e)^{3/2}} \left(\frac{n_e}{n_j} \right)^{1/2} \int_{E_j + \Delta_{je}^C(E_j)}^{E_j/\alpha_{je}} \frac{dE_j' \nu_1^T(E_j')}{(E_j' - E_j)^2} \right. \\ & \left. + n_j \sum_1 \frac{Z_1^2 n_1}{n_j} \int_{E_j + \Delta_{j1}^C(E_j)}^{E_j/\alpha_{j1}} \frac{dE_j' \nu_1(E_j')}{E_j'(E_j' - E_j)^2} \right\}. \quad (80) \end{aligned}$$

where ΣS_T is given by Eq. (68) and the summation in the second term is only over plasma ions. The solution given by Eq. (70) satisfies Eq. (80) with a constant C that can be determined from the slowing-down density. Evaluation of the probability function $G_j(E_j, E_j; T_e)$ of Eq. (15) for electrons using Eq. (78) gives

$$G_j(E_j', E_j; T_e) = \frac{\Delta_{je}^C(E_j')}{(E_j' - E_j)} \cdot E_j + \Delta_{je}^C(E_j) \leq E_j' \leq \frac{E_j}{\alpha_{je}} \quad (81)$$

The corresponding function for ions of species i is given by Eq. (71). By separating the term representing contributions from electrons in Eq. (14) from the terms for the slowing-down density by the ions and substituting Eqs. (70), (71), and (81) into the result, we get

$$T_j = \frac{\pi Z_j^2 e^4 C}{\Sigma S_T} \left\{ \frac{n_e}{(kT_e)^{3/2}} \left(\frac{n_e}{n_j} \right)^{1/2} \int_{E_j + \Delta_{je}^C(E_j)}^{E_j/\alpha_{je}} \frac{dE_j' \sqrt{E_j' E_j}}{(E_j' - E_j)} \right. \\ \left. + n_j \sum_i \frac{Z_i^2 n_i}{n_1} \int_{E_j + \Delta_{ji}^C(E_j)}^{E_j/\alpha_{ji}} \frac{dE_j'}{(E_j' - E_j)} \right\} \quad (82)$$

By carrying out the integration and evaluating the constant C using Eq. (16), we find that the number density of test particles of type j at energy between E_j and $E_j + dE_j$ is

$$n_j(E_j) dE_j = s_{j0} E_j^{1/2} dE_j \left/ \left[\sqrt{2\pi} Z_j^2 e^4 \left[\frac{n_e \sqrt{n_e}}{(kT_e)^{3/2} n_j} E_j^{3/2} \ln \Delta_{je}^C \right. \right. \right. \right. \\ \left. \left. \left. + \sqrt{n_j} \sum_i \frac{Z_i^2 n_i}{n_1} \ln \Delta_{ji}^C(E_j) \right] \right] \right\} \quad (83)$$

where

$$\Delta_{je}^C = \left(\frac{2}{\theta_{je}^C} \right)^2 \quad \text{and} \quad (84)$$

$$\Delta_{ji}^C = \left(\frac{2}{\theta_{ji}^C(E_j)} \right)^2 \quad (85)$$

with θ_{je}^C as given by Eq. (77) and θ_{ji}^C as given by Eq. (62). The result obtained in Eq. (83) is similar to that of Eq. (52) except for the appearance of the two logarithmic terms in the denominator. The values of the two logarithmic functions are insensitive to moderate changes in E_j or T_e and can be considered to have the same value. Thus, the contribution of the electrons to the slowing-in distribution may be neglected if the condition of Eq. (56) is valid and Eq. (83) reduces to

$$n_j(E_j) dE_j = \frac{s_{j0} E_j^{1/2} dE_j}{\pi \sqrt{2\pi} Z_j^2 e^4 \sum_i \frac{Z_i^2 n_i}{n_1} \ln \Delta_{ji}^C(E_j)} \quad (86)$$

If Eq. (53) or Eq. (54) is satisfied the slowing-in distribution is mainly established by electrons and Eq. (83) reduces to

$$n_j(E_j) dE_j = \frac{n_1 (kT_e)^{3/2} s_{j0} dE_j}{\sqrt{2\pi} \pi Z_j^2 e^4 n_e (\ln \Delta_{je}^C) E_j} \quad (87)$$

which varies as $1/E_j$ as in the quantum-mechanical case.

V. GENERAL FORM OF THE SLOWING-IN DISTRIBUTION

Equations (41) and (73) give the number density of test particles at energies between E_j and $E_j + dE_j$ within the slowing-down range calculated quantum-mechanically and classically, respectively, for $v_j > v_T$. The two equations are the same except for the factor $\ln \Delta_{je}^C(E_j)$, which appears in the

denominator of Eq. (73). Similarly, Eqs. (52) and (83) give the same result except for the logarithmic functions $\ln \Lambda_{je}^C$ and $\ln \Lambda_{ji}^C(E_j)$ for the slowing-in distribution of test particles of species j quantum-mechanically and classically, respectively, for $v_j^T < v_j^L < v_e^T$. The logarithmic functions appear as a result of using the cutoff angles. In addition, the quantum-mechanical results of Eqs. (41) and (52) include only scattering events leading to small angles of deflection below a maximum angle of deflection θ_{jv0}^Q and θ_{je0}^Q as given by Eqs. (32) and (26), respectively. If the Rutherford cross section is used for scattering angles larger than these quantum-mechanical cutoff angles, results similar to those obtained classically in Eqs. (73) and (83) can be obtained with the arguments of the logarithmic terms replaced by the corresponding quantum-mechanical values; that is,

$$\Lambda_{jv0}^C(E_j) + \Lambda_{je0}^Q(E_j) = (2/\theta_{jv0}^Q(E_j))^2, \quad v_j^L > v_e^T, \quad (88)$$

where $\theta_{jv0}^Q(E_j)$ is given by Eq. (32) for ions or electrons.

$$\Lambda_{je}^C + \Lambda_{je}^Q(T_e) = (2/\theta_{je0}^Q)^2, \quad v_j^L < v_e^T, \quad (89)$$

where $\theta_{je0}^Q(E_j)$ is given by Eq. (26) for electrons only. By inspecting Eq. (4) we find that the general solution can be obtained by adding the contributions from scattering through all possible angles; thus adding Eq. (42) to the quantum-mechanical version of Eq. (75) we get

$$n_j(E_j)dE_j = \frac{m_e S_{j0} E_j^{1/2} \left[1 + \ln \Lambda_{je}^Q(E_j) \right]}{\sqrt{2m_j} \pi Z_j^2 e^4 n_e} dE_j, \quad v_j^L > v_e^T, \quad (90)$$

where $\Lambda_{je}^Q(E_j)$ is given by Eq. (88). Contributions from scattering by plasma ions are neglected as indicated in reducing Eq. (41) to Eq. (42). Equation (90) can be obtained directly by using the energy transfer probability derived from the Born formula in the range $0 \leq \theta_{jv} \leq \theta_{jv0}^Q$ and that obtained from the Rutherford formula in the range $\theta_{jv0}^Q < \theta_{jv} \leq \pi$. Scattering probabilities have been derived by Husseiny and Forsen⁴ using this technique.

Similarly, Eq. (52) can be combined with Eq. (83) after making the replacement given by Eqs. (88) and (89); that is,

$$n_j(E_j)dE_j = \frac{S_{j0} dE_j}{\sqrt{2m_j} Z_j^2 e^4} \left\{ \frac{1}{\sum_i \left[\frac{\sqrt{m_e} Z_i n_i E_j}{m_j (kT_e)^{3/2}} + \frac{\sqrt{m_i} Z_i^2 n_i}{E_j^{1/2} m_i} \right]} \right. \\ \left. \frac{1}{\sum_i \left[\frac{\sqrt{m_e} Z_i n_i E_j \ln \Lambda_{je}^Q(T_e)}{m_j (kT_e)^{3/2}} + \frac{n_i Z_i^2 \ln \Lambda_{ji}^Q(E_j)}{E_j^{1/2} m_i} \right]} \right\}, \quad v_j^L < v_j^L < v_e^T \quad (91)$$

where $\Lambda_{je}^Q(T_e)$ is given by Eq. (89) and $\ln \Lambda_{ji}^Q(E_j)$ by Eq. (88).

Examination of Eqs. (42) and (52) shows that they remain unchanged in the limit $\hbar \rightarrow 0$ and hence are valid for classical scattering as well as quantum-mechanical. Thus, equations similar to those given by Eqs. (90) and (91) may be derived, and the results are essentially the sum of Eqs. (42) and (75) for $v_j^L > v_e^T$ and the sum of Eqs. (52) and (83) in the range $v_j^T < v_j^L < v_e^T$. Therefore, Eqs. (90) and (91) can be used for classical cases by choosing the classical arguments of the logarithmic functions. However, because the terms containing the logarithmic functions are small compared with the other terms, they can be neglected. As we have pointed out previously the logarithmic function in the term including contributions from electron scattering in Eq. (91) may have a quantum-mechanical or classical argument independent of the logarithmic function occurring in the terms related to the plasma ions.

VI. ENERGY DISTRIBUTION INCLUDING SLOWING-IN COMPONENT

Charged test ions released in plasmas whether by thermonuclear reaction or by energetic injections, generally appear at energies substantially higher than the Maxwellian energies. Collisions with other plasma particles then slow the test ions until those which survive the slowing-in process become thermalized. In ordinary matter the slowing-in component of the test particle flux is very small, but in

plasmas the slowing-in component can be much larger. The contributions from small angle scattering events to the slowing-in distribution of Eq. (42) can be used for situations in which $v_j > v_v^T$, where v refers to plasma ions and electrons. On the other hand, if $v_i^T < v_j^T < v_v^T$ and if the condition given by Eq. (56) is applicable, the slowing-in component for the test particles is given by Eq. (57). Equations (42) and (57) have the same dependence on E_j and both are valid classically and quantum-mechanically. Thus, the number of test ions of species j with energies in the interval E_j to $E_j + dE_j$ within the slowing-down range is

$$n_j(E_j)dE_j = c_{vj}E_j^{1/2}dE_j \quad (92)$$

where

$$c_{vj} = \begin{cases} \frac{n_e s_{j0}}{\pi^2 s_j^2 a^4 \sqrt{2n_j n_e}} & \cdot v_j^T > v_v^T \quad (93a) \\ \frac{s_{j0}}{\pi^2 s_j^2 a^4 \sqrt{2n_j} \sum_i \frac{z_i^2 n_i}{a_i}} & \cdot v_i^T < v_j^T < \left(\frac{n_e z_j}{n_i}\right)^{1/2} v_e^T \quad (93b) \end{cases}$$

In principle, one can compute the source strength of the test ions from Eq. (11) using data from a previous iterate, then compute a slowing-in distribution for this iterate from Eq. (92), join this slowing-in distribution smoothly to the Maxwellian, and thus obtain an ion distribution of the form

$$n_j(E_j) = N_j \left\{ M_{Tj}(E_j) + f_s(E_j)E_j^{1/2} \right\} \quad (94)$$

where N_j is the number density of the test particles averaged over their total energy distribution, $f_s(E_j)$ is a smoothing function varying over the joining interval, and $M_{Tj}(E_j)$ is the Maxwellian distribution in energy; that is,

$$M_{Tj}(E_j) = \frac{2}{\pi^{1/2}} \frac{E_j^{1/2}}{(kT_j)^{3/2}} e^{-E_j/kT_j} \quad (95)$$

where T_j is the kinetic temperature of the test particles calculated from the contracted second moment of a Gaussian velocity distribution.

For convenience we express the test ion distribution in normalized form; that is,

$$n_j(E_j)dE_j = \begin{cases} (1-a_j)M_{Tj}(E_j)dE_j & \cdot 0 \leq E_j \leq \bar{E}_j \text{ and } \hat{E}_j \leq E_j < \infty \\ N_j \left[(1-a_j)M_{Tj}(E_j) + \frac{3a_j E_j^{1/2}}{2(\bar{E}_j^{3/2} - \hat{E}_j^{3/2})} \right] dE_j & \cdot \bar{E}_j \leq E_j < \hat{E}_j \end{cases} \quad (96)$$

where $M_{Tj}(E_j)$ is a Maxwellian at a shifted kinetic temperature T_j' given by

$$kT_j' = \left[\frac{2}{3} \bar{E}_j - \frac{4c_{vj}}{15N_j} (\bar{E}_j^{5/2} - \hat{E}_j^{5/2}) \right] / (1-a_j) \quad (97)$$

where the constant a_j is defined as

$$a_j = \frac{2}{3} \frac{c_{vj}}{N_j} \quad (98)$$

and $\bar{E}_j$ is the average energy over the total distribution including the slowing-in component. The expression given by Eq. (94) is properly normalized when the slowing-in component is assumed to act only over the interval $(\bar{E}_j, \hat{E}_j)$ as in Eq. (96). A proper choice of the upper limit is $\hat{E}_j \approx E_{j0}$, which is equivalent to neglecting the oscillations near the source energy on the slowing-in distribution. In the case of charged particle scattering, several encounters take place between the test ions and electrons before the energy of the ions changes to values significantly below E_{j0} . Consequently, the oscillations in the ion density are expected to be damped in the vicinity of E_{j0} after the first few encounters, and hence the asymptotic solution of Eq. (92) may be extended to $\hat{E}_j \approx E_{j0}$. The lower limit can be taken as

$$\tilde{E}_j = \frac{m_j}{m_e} kT_e$$

because the scattering probability in the slowing-down range is valid to this limit.⁶

VII. COMPARISON WITH FOKKER-PLANCK METHODS

The use of the RMJ form of the Fokker-Planck equation⁸ corresponds to the classical calculation and to excluding encounter events leading to deflection angles smaller than a certain cutoff angle θ_{jv}^C . The slowing-in distribution may be estimated using the plasma stopping power as obtained from Fokker-Planck type calculations and comparing the results with those obtained above. At test ion energies for which the slowing-down process dominates, the test particles at energy E_j must have been injected at a previous time

$$\int_{E_j}^{E_{j0}} dE_j' / \{v_j(E_j') k_j(E_j')\},$$

where $k_j(E_j)$ represents the total rate of change of ion energy with track length x ,

$$k_j(E_j) = - \sum_v \left(\frac{dE_j}{dx} \right)_v \quad (99)$$

including the effects of ion-electron Coulomb interactions, ion-nucleus Coulomb and strong interactions, elastic interactions, other nuclear interactions, and other (usually negligible) processes. Energy straggling will be ignored. The number of ions of type j with energies in the interval E_j to dE_j is then

$$n_j(E_j) dE_j = \int_{E_j}^{E_{j0}} \frac{dE_j'}{v_j(E_j') k_j(E_j')} \int_{E_j + dE_j}^{E_{j0}} dE_j'' / v_j(E_j'') k_j(E_j'')$$

$$= \frac{S_j(E_{j0})}{v_j(E_j) k_j(E_j)} dE_j \quad (100)$$

For test particle energies for which $E_j > kT_e$, and $v_j > v_{Te}$, $k_j(E_j)$ is given by⁹

$$k_j(E_j) = \sum_v \frac{\pi Z_j^2 e^4}{m_e E_j} \Lambda_{jv}^C(E_j) \frac{m_j n_v}{m_e} \quad (101)$$

where Λ_{jv}^C is related to the classical cutoff angle θ_{jv}^C by Eq. (74). The stopping power of the plasma is dominated by the effect of electrons because the stopping power of the plasma ions is $\sim m_e/m_i$, thus

$$k_j(E_j) = \frac{\pi Z_j^2 e^4}{m_e E_j} \Lambda_{je}^C(E_j) \quad (102)$$

Substituting Eq. (102) into Eq. (100), we get

$$n_j(E_j) dE_j = \frac{n_e S_{j0} E_j^{1/2} dE_j}{\pi Z_j^2 e^4 n_e \sqrt{2m_j} \Lambda_{je}^C(E_j)} \quad (103)$$

which agrees with the classical result given by Eq. (42) for $v_j > v_{Te}$.

The denominator of the right-hand side of Eq. (100) is

$$v_j^1(E_j)k_j(E_j) = -\sum_v \left(\frac{dE_j}{dt} \right)_v,$$

the rate of energy transfer from the test particle of species j to all plasma particles. Using the quantum-mechanical value of this rate of energy transfer results in an expression similar to that of Eq. (103) with $\Lambda_{je}^C(E_j)$ replaced by $\Lambda_{je}^Q(E_j)$, which is given by Eq. (88). This confirms the results of Sec. V.

In the case of $v_j^1 < v_e^T$, the stopping power by plasma electrons is given by⁹

$$k_{je} = \frac{4\sqrt{\pi}Z_j^2e^4n_e}{\sqrt{E_jkT_e}} \ln\Lambda_{je}^C(E_j) \sqrt{\frac{m_e}{m_j}} \left(\frac{E_j}{kT_e} - 1 \right). \quad (104)$$

Substituting Eq. (104) into Eq. (100) and assuming that electrons are dominant in stopping the test particles and that $E_j > kT_e$, the slowing-in component of the distribution is

$$n_j(E_j)dE_j = \frac{n_j(kT_e)^{3/2}S_{j0}dE_j}{3\sqrt{2\pi}m_eZ_j^2e^4\ln\Lambda_{je}^C(E_j)n_eE_j}. \quad (105)$$

This result is

$$\frac{\sqrt{\pi}\ln\Lambda_{je}^C(T_e)}{6\ln\Lambda_{je}^C(E_j)}$$

times the result given by Eq. (87), where $\Lambda_{je}^C(T_e)$ is given by Eq. (85) and $\Lambda_{je}^C(E_j)$ is given by Eq. (74).

Although results based on the Fokker-Planck equation predict the slowing-in distribution in the classical limit when scattering events leading to small deflection angles are neglected, they do not predict the more dominant contributions from deflection angles in the range $0 \leq \theta_{jv} < \theta_{jv0}^C$. The agreement between these results and those obtained from the Boltzmann equation is limited to close-range scattering only. The use of the Boltzmann equation is more straightforward than the use of Fokker-Planck equations. In addition, experience in

using the Boltzmann equations in multigroup neutron calculations may be put to advantage in studying charged particle distributions.

VIII. CONCLUSIONS AND RECOMMENDATIONS

The slowing-in energy distribution of fast test particles released in a Maxwellian plasma was examined. We found that scattering events leading to small deflection angles dominantly contribute to the distribution. Including only the dominant terms in the slowing-in distribution gives Eq. (52) for $v_j^1 \leq v_e^T$ and Eq. (92) for $v_j^1 > v_e^T$. These results are applicable classically and quantum-mechanically. The results estimated from Fokker-Planck calculations underestimate the slowing-in distribution, although they agree with that part of the distribution which is contributed from large angle scattering.

The effect of leakage from the plasma confinement throughout the slowing-down process must be considered especially if the slowing-down time is comparable with the confinement time. Fissionable test particles are likely to become involved in fusion reactions during the slowing-down process, and hence the effect of burn and other types of absorption on the slowing-in distribution need to be examined. The Boltzmann formulation used here makes it possible to use multienergy group calculations to obtain more accurate results for the energy distributions of energetic particles released in multicomponent plasmas. The well-developed neutron distribution codes may be put to advantage in this type of calculation.

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