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A GEOMETRY CORRECTION FOR PLANE SOURCE AGE
MEASUREMENTS IN HYDROGENOUS MEDIA

by

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A GEOMETRY CORRECTION FOR PLANE SOURCE AGE MEASUREMENTS IN HYDROGENOUS MEDIA

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ABSTRACT

A method is presented for calculating the age from an indium resonance neutron flux distribution measured along the normal to a large (≥ 10 root ages) plane fission plate immersed in an hydrogenous medium of infinite extent. The calculation is a modification of one suggested by M. E. Rose and A. M. Weinberg for use in age measurements in media which can be characterized by energy independent extrapolation lengths. The modification consists of introducing small correction terms which are integrals over the point displacement kernel at distances greater than the neutron source half-width. The approximate form of the kernel in this region must be inferred from the data. A numerical example of the use of the method is given. The integrals required in the calculation have been evaluated and tabulated for a range of foil exposure geometries.

INTRODUCTION

In the following, age shall be taken as one-sixth of the second moment of the normalized distribution of indium resonance neutrons from a point source immersed in a medium of infinite extent. A variety of theoretical treatments and experimental determinations of the age of fission neutrons in water have been reported.⁽¹⁾ The experimental determinations have not shown the agreement which should be attainable with a proper application of the various corrections to the measured flux distributions, and they are consistently higher than the theoretical determinations. The measurements are subject to significant corrections arising from foil thickness, fission plate thickness, and geometry. Most work has been done using thick, small sources which are intended to approximate a point. Size corrections for these "point" sources diminish as the source and foils become smaller and thinner. Unfortunately, there is an attendant reduction in intensity.

Measurements have recently been reported⁽²⁾ which attempt to avoid the source size-intensity dilemma by using large, thin, plane fission sources and appropriate geometrical corrections. Corrections for the

source and foil thicknesses to the measured distributions were correspondingly reduced. However, the geometrical corrections were still significant - larger than all of the other corrections put together.

In this note we consider the geometrical correction for a foil exposure geometry which appears to be especially useful for age measurements in hydrogenous moderators, i.e., a large, plane source in an infinite medium. The geometry and geometrical corrections are very nearly those suggested by M. E. Rose and A. M. Weinberg⁽³⁾ for age experiments in media which can be characterized by an energy independent extrapolation length. Rose and Weinberg (RW) deduced an expression for the age in the form of a power series in the buckling ($B \equiv \pi/2a$), with coefficients given by the moments of the measured axial distribution. For infinite source size this expression reduces to the well-known result that the age is one-half of the ratio of the second moment of the axial distribution to the zeroth moment.

The boundary conditions assumed by RW cannot be approximated experimentally in hydrogenous moderators. However, it is not difficult to alter the experimental arrangement to get boundary conditions for which the geometrical correction is quite similar in form to that given by RW. An assumption about the behavior of the point displacement kernel at large distances will be necessary, but this behavior may be estimated from the data and the effect of the assumption on the age may be easily tested.

THE PROBLEM

Let $Q(x', y')$ be the strength of an infinitely thin fission neutron source lying in the plane $z = 0$ in a homogeneous moderating material of infinite extent. Let $M(x, y, z)$ be the saturation activation density of an indium resonance detector foil of vanishing area and thickness at the point x, y, z produced by the plane source of strength $Q(x', y')$. Let $P(x, y, z)$ denote the saturation activation density of a similar foil at point x, y, z which would be produced if $Q(x', y')$ were replaced by a unit point source at the origin. Use will be made of negative sources which remove activity from the detector foil. The distribution $M(x, y, z)$ may be expressed in terms of the kernel P and source Q as

$$M(x, y, z) = \iint_{-\infty}^{+\infty} P \left(\sqrt{(x - x')^2 + (y - y')^2 + z^2} \right) Q(x', y') dx' dy' \quad . \quad (1)$$

The moments of the observed axial distribution and of the point displacement kernel are given by the integrals

$$M_{2m} = \int_{-\infty}^{\infty} M(0,0,z) z^{2m} dz, \text{ and } P_{2m} = \int_0^{\infty} P(r) 4\pi r^{2m+2} dr.$$

The age is defined as $\tau = P_2/6P_0$.

The fission neutron source will ordinarily consist of a fission plate placed in the moderator parallel to and a few inches from that surface of the moderator which is exposed to the thermal leakage flux from a source reactor. Foil exposures are made on the side of the fission plate away from the source reactor. The superposition of sources indicated in Eq. (1) is valid only if the fission plate appears thin to neutrons with energies above the cadmium cutoff and only if the moderator is homogeneous and effectively infinite in extent. In the recent measurements⁽²⁾ made with large plane sources in H₂O, it was possible to get adequate intensity at large distances (60 cm) with a thin (0.061 cm) fission plate with thin foils (20 mg/cm²), and with an effectively infinite thickness of moderator (10 cm) between the fission plate and the source reactor.

Effects associated with the detector such as flux depression, self-shielding, and higher resonance activation may affect the shape of the observed axial distribution. If such is the case, the moments of the observed axial distribution must be corrected to the values which would have been obtained with foils of vanishing area, thickness, and higher resonance absorption before these moments can be used in a determination of the age.

Moments M_{2m} of the axial distribution $M(0,0,z)$ can be written in terms of the point kernel and the source as

$$M_{2m} = 2 \int_0^{\infty} z^{2m} dz \int \int_{-\infty}^{+\infty} Q(x,y) P\left(\sqrt{x^2+y^2+z^2}\right) dx dy. \quad (2)$$

A change of variable and order of integration in Eq. (2) yields

$$M_{2m} = 2 \int_0^{\infty} r^{2m} P(r) dr \int \int_{-\infty}^{+\infty} Q(x,y) \left(1 - \frac{R^2}{r^2}\right)^{m-1/2} l(r-R) dx dy, \quad (3)$$

where $r^2 = x^2 + y^2 + z^2$, $R^2 = x^2 + y^2$, and $l(r-R)$ is the unit step function (unity for positive arguments, zero for negative arguments).

It will prove convenient to introduce the function $G_m(r)$, which depends only upon the geometry of the source $Q(x,y)$. Then the moments M_{2m} have the form

$$M_{2m} = \int_0^{\infty} 4\pi r^{2m+2} P(r) G_m(r) dr, \quad (4a)$$

where

$$G_m(r) = \frac{1}{2\pi r^2} \int_0^r \left(1 - \frac{R^2}{r^2}\right)^{m-1/2} R dR \int_0^{2\pi} Q(R, \theta) d\theta, \quad (4b)$$

and $Q(R, \theta)$ is the source distribution expressed in polar coordinates. In Eq. (4a) the moments of the axial flux distribution $M(0, 0, z)$ are given as moments of the product of $P(r)$ and $G_m(r)$. The kernel $P(r)$ is the quantity of interest containing the physics. The geometry factor $G_m(r)$ is peculiar to the particular experiment being analyzed and is not of general interest.

One way of separating the point kernel $P(r)$ and the foil exposure geometry $G_m(r)$ has been suggested by RW. In effect, if an expansion of $G_m(r)$ in powers of r exists, it may be possible to express the moments of the point kernel as convergent sums of moments of the measured axial distribution. If the expansion

$$r^{2m+2} G_m(r) = \sum_{n=0}^{\infty} C_{mn} r^{2n+2} \quad (5)$$

exists for $0 \leq r \leq \infty$, then Eq. (4a) can be written in the form

$$M_{2m} = \sum_{n=0}^{\infty} C_{mn} \int_0^{\infty} 4\pi r^{2n+2} P(r) dr = \sum_{n=0}^{\infty} C_{mn} P_{2n}. \quad (6)$$

Equation (6) gives the moments of $M(0, 0, z)$ as linear combinations of moments of the kernel $P(r)$; the C_{mn} are constants which depend only upon source geometry. To solve for the P_{2m} in terms of the measured M_{2m} , it is possible to cut off the summation in Eq. (6) at some suitable upper limit, say $n = p$, and solve the resulting set of $p + 1$ linear equations. If convergence is sufficiently rapid, the problem may be considered solved; that is,

$$P_{2m} \approx \sum_{n=0}^p C_{mn}^{-1} M_{2n}. \quad (7)$$

According to Eq. (7), moments of the point kernel (and the age) are obtained directly from moments of the experimentally determined axial flux distribution. The coefficients C_{mn}^{-1} depend only upon the foil exposure geometry (the

source distribution) and hold for any kernel of physical interest. No assumptions concerning the kernel are necessary. The existence of an expansion of $G_m(r)$ in the desired form [Eq. (5)] is evidently basic to the scheme.

NONPHYSICAL SOURCES

Consider as an example the uniform plane source of infinite extent: $Q(x, y) = 1$. The geometry factor [see Eq. (4b)] is $G_m(r) = (2m+1)^{-1}$, and it can be expanded in the form given by Eq. (5), in which $C_{mn} = (2n+1)^{-1} \delta_n^m$. The moments of the axial distribution are given by the well known relation

$$M_{2m} = \frac{P_{2m}}{2m+1} \quad (8)$$

It will prove quite useful to consider two nonphysical source distributions which also lead to expansions of $G_m(r)$ in the form given by Eq. (5):

$$Q(x, y) = J_0(BR \sqrt{2}) \quad \text{where } B \equiv j_0/a \sqrt{2} \quad (9a)$$

$$Q(x, y) = \cos Bx \cos By \quad \text{where } B \equiv \pi/2a \quad (9b)$$

The J_0 is a Bessel function and j_0 is its first zero. The cosine product source [Eq. (9b)] can be replaced by an equivalent source which has only radial dependence:

$$Q(R) = \frac{1}{2\pi} \int_0^{2\pi} \cos(BR \cos \theta) \cos(BR \sin \theta) d\theta = J_0(BR \sqrt{2}) \quad (9b')$$

where $b = \pi/2a$. Here the arguments are not restricted, so these sources extend over the whole source plane, oscillating between positive and negative values.

A negative source is interpreted as a source which removes activity from the indium resonance detector foil. Such a source is nonphysical.

The geometry factors for sources [see Eq. (9)] are obtained by introducing Eqs. (9a) and (9b') in Eq. (4b) to get

$$G_m(r) = \frac{1}{r^2} \int_0^r \left(1 - \frac{R^2}{r^2}\right)^{m-1/2} J_0(BR \sqrt{2}) R dR \quad (10)$$

If we make a change of variable such that $R = r \sin \alpha$, we see that Eq. (10) has the same form as Sonine's first finite integral⁽⁴⁾ and we can immediately write

$$G_m(r) = \frac{(2m)! \pi^{1/2} J_{m+1/2}(Br \sqrt{2})}{m! 2^{m+1/2} (Br \sqrt{2})^{m+1/2}} \quad (11)$$

If the Bessel's functions in Eq. (11) are expanded in a power series in r , the geometry factor $G_m(r)$ can be written in the form given in Eq. (5):

$$r^{2m+2} G_m(r) = \sum_{n=m}^{\infty} \left[\frac{(-1)^{n-m} (2m)! n! 2^{n-m} B^{2n-2m}}{m! (n-m)! (2n+1)!} \right] r^{2n+2} \quad (12)$$

where the quantity in brackets is the coefficient C_{mn} .

By substituting the C_{mn} in Eq. (6), we obtain the moments of the axial distribution as expansions in powers of the buckling B with coefficients determined by the moments of the point displacement kernel:

$$M_{2m} = \sum_{n=m}^{\infty} \left[\frac{(-1)^{n-m} (2m)! n! 2^{n-m}}{m! (n-m)! (2n)!} \right] \left[\frac{P_{2n}}{2n+1} \right] B^{2n-2m} \quad (13)$$

where B assumes the appropriate value for a square or circular source symmetry. The quantity $P_{2n}/(2n+1)$ is the $2n$ th moment of the axial distribution which would have been observed if the source were uniform and infinite in extent [cf. Eq. (8)].

The infinite set of linear equations [Eq. (13)] must now be solved to get expansions of the P_{2m} in terms of the M_{2n} in the form given by Eq. (7). A general formula for the coefficient C_{mn}^{-1} can be obtained by observing that the C_{mn} , and consequently the C_{mn}^{-1} , are determined only by the geometry and are independent of the form of the point displacement kernel. Therefore, we can use a Gaussian kernel:

$$\rho(r) = \exp(-r^2/4t) \quad (14)$$

to find an expansion of the ρ_{2m} in terms of the corresponding $\mathcal{M}_{2n}$. The coefficients C_{mn}^{-1} of this expansion are the desired coefficients and will hold for an arbitrary kernel. The use of script ρ and $\mathcal{M}$ implies that in what immediately follows we are confining our attention to a specific kernel (Gaussian).

To find the C_{mn}^{-1} we substitute the kernel [Eq. (14)] and the geometry factor [Eq. (11)] into Eq. (4a) to get an integral which can be recognized as a Hankel transform:

$$\begin{aligned}
\mathcal{M}_{2m} &= \frac{(2m)!}{m!} \frac{4\pi^{3/2}}{(2B\sqrt{2})^{m+1/2}} \int_0^\infty r^{m+1/2} \exp(-r^2/4t) J_{m+1/2}(Br\sqrt{2}) r dr \\
&= 8\pi^{3/2} \left[\frac{(2m)!}{m!} \right] t^{m+3/2} \exp(-2B^2t) \quad .
\end{aligned} \tag{15}$$

We see from Eq. (13) that $\mathcal{M}_{2m}$ approaches $\rho_{2m}/(2m+1)$ as B vanishes. Therefore, for finite B we can rewrite Eq. (15) in the form

$$\rho_{2m} = (2m+1) \mathcal{M}_{2m} \exp(2B^2t) = (2m+1) \mathcal{M}_{2m} \sum_{n=0}^{\infty} \frac{(2B^2t)^n}{n!} \quad . \tag{16}$$

The powers of t can be eliminated by taking ratios of the moments given by Eq. (15):

$$\frac{\mathcal{M}_{2m+2n}}{\mathcal{M}_{2m}} = \frac{m! (2m+2n)! t^n}{(2m)! (m+n)!} \tag{17}$$

and by introducing these ratios into Eq. (16). After making a change of summation index, we can write

$$\rho_{2m} = \sum_{n=0}^{\infty} C_{mn}^{-1} \mathcal{M}_{2n} \quad ; \quad C_{mn}^{-1} = \frac{(2m+1)! n! (2B^2)^{n-m}}{m! (2n)! (n-m)!} \tag{18}$$

Because the inverse coefficients C_{mn}^{-1} are independent of the kernel, we may drop the script notation in Eq. (18) and so obtain an expression in the form of Eq. (7) for an arbitrary kernel. After evaluating some of the inverse coefficients, we can write an expression for the neutron age:

$$\begin{aligned}
2\tau = \frac{P_2}{3P_0} &= \frac{M_2 + \frac{1}{3} B^2 M_4 + \frac{1}{30} B^4 M_6 + \frac{1}{630} B^6 M_8 + \frac{B^8}{22680} M_{10} + \dots}{M_0 + B^2 M_2 + \frac{1}{6} B^4 M_4 + \frac{1}{90} B^6 M_6 + \frac{1}{2520} B^8 M_8 + \dots}
\end{aligned} \tag{19}$$

This is the result given by RW.

One further property of the nonphysical sources, Eqs. (9a) and (9b), worth noting is that they give rise to neutron flux distributions which vanish along planes (or cylinders) normal to the source plane. Equation (1) for $M(x, y, z)$ can be written in the form

$$M(x, y, z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \bar{P}(\sqrt{\eta^2 + \xi^2 + \zeta^2}) \bar{Q}(\eta, \xi) \exp[i(\eta x + \xi y + \zeta z)] d\eta d\xi d\zeta, \quad (20)$$

where $\bar{P}(\sqrt{\eta^2 + \xi^2 + \zeta^2})$ and $\bar{Q}(\eta, \xi)$ are Fourier transforms of kernel and source, respectively. The source transforms are given by

$$\bar{Q}(\eta, \xi) = \bar{Q}(\rho) = \frac{1}{\rho} \delta\left(\rho - \frac{j_0}{a}\right) \quad \text{where } \rho^2 = \eta^2 + \xi^2, \quad (21a)$$

and

$$\bar{Q}(\eta, \xi) = \frac{\pi}{2} \left[\delta\left(\eta - \frac{\pi}{2a}\right) + \delta\left(\eta + \frac{\pi}{2a}\right) \right] \left[\delta\left(\xi - \frac{\pi}{2a}\right) + \delta\left(\xi + \frac{\pi}{2a}\right) \right]. \quad (21b)$$

If these transforms are put in Eq. (20), the resulting flux distributions have the form

$$M(x, y, z) = M(R, z) = \sqrt{2\pi} J_0\left(\frac{j_0 R}{a}\right) \int_{-\infty}^{+\infty} \bar{P}\left(\sqrt{\left(\frac{j_0}{a}\right)^2 + \xi^2}\right) \exp(i\xi z) d\xi, \quad (22a)$$

$$M(x, y, z) = \sqrt{2\pi} \cos \frac{\pi x}{2a} \cos \frac{\pi y}{2a} \int_{-\infty}^{+\infty} \bar{P}\left(\sqrt{2\left(\frac{\pi}{2a}\right)^2 + \xi^2}\right) \exp(i\xi z) d\xi, \quad (22b)$$

and we see that $M(a, z) = 0$ for the circular source and that $M(\pm a, y, z) = M(x, \pm a, z) = 0$ for the square source.

ACTUAL NEUTRON SOURCES

The results of the foregoing discussion are not generally applicable to actual neutron sources because they are finite in extent and non-negative. RW suggest that Eq. (19) may be a good approximation to the age for certain finite source distributions in some moderating media. Suppose that neutron transport in the infinite moderator, infinite source geometry of the above discussion is amenable to a group diffusion type of analysis in which the group extrapolation lengths, d , are energy independent. Then the flux can be made to vanish at $x = \pm a$ and/or $y = \pm a$ (or $R = a$) by replacing the source and moderator in the region $[|x| \text{ and/or } |y|, (a - d)]$ or $[R > (a - d)]$ with a vacuum (air). If the extrapolation length is much smaller than the system size ($d \ll a$), the axial flux $M(0, 0, z)$ will not be very sensitive to the conditions beyond (x and/or $y = \pm a$) or ($R = a$), and the age τ will not be greatly different from that given by Eq. (19). Water and other hydrogenous moderators are not media for which Eq. (19) is

a very good approximation because of the large variation of transport mean free path with energy. The system size would have to be prohibitively large before the ambiguity in extrapolation length became negligible.

This far we have done little other than write down some results obtained by RW. We now extend the discussion to the case of plane sources of finite extent immersed in moderators of infinite extent. The infinite moderator geometry is particularly desirable for the purpose of calculating a geometry correction because difficulties associated with boundaries and energy-dependent transport mean free paths are no longer of concern. In H_2O the source sizes are reasonable ($a \approx 30$ cm or larger). The source distributions to be discussed are given by

$$Q(x,y) = Q(R) = J_0(BR\sqrt{2}) \quad \text{where } B = j_0/a\sqrt{2} \quad \text{for } R \leq a \quad (23a)$$

$$Q(x,y) = Q(R) = 0 \quad \text{for } R > a$$

$$Q(x,y) = \cos Bx \cos By \quad \text{where } B = \pi/2a \quad \text{for } |x| \leq a \text{ and } |y| \leq a$$

$$Q(x,y) = 0 \quad \text{for } |x| > a \text{ and/or } |y| > a \quad (23b)$$

It is practical to set up source distributions such as these by the use of cadmium shims placed between the reactor leakage face and the fission plate. Such shims have a negligible effect upon the epi-cadmium flux distribution. As in the case of source distribution [Eq. (9b)], it will be convenient to replace source [Eq. (23b)] by an equivalent radial source distribution which is an average over the polar angle:

$$Q(R) = J_0(BR\sqrt{2}) \quad \text{where } B = \pi/2a \quad \text{for } R \leq a$$

$$Q(R) = \frac{4}{\pi} \int_{\arccos a/R}^{\pi/4} \cos\left(\frac{\pi R}{2a} \cos\theta\right) \cos\left(\frac{\pi R}{2a} \sin\theta\right) d\theta \quad \text{for } a \leq R \leq a\sqrt{2}$$

$$Q(R) = 0 \quad \text{for } R > a\sqrt{2} \quad (23b')$$

Table I gives the results of a numerical calculation of $Q(R)$ in the interval $a \leq R \leq a\sqrt{2}$.

The geometry factor for the source [Eq. (23a)] is obtained by introducing Eq. (23a) in Eq. (4b) to get

$$G_m(r) \equiv E_m(r) = \frac{1}{r^2} \int_0^r \left(1 - \frac{R^2}{r^2}\right)^{m-1/2} J_0(BR \sqrt{2}) R dR = \frac{(2m)! \sqrt{\pi} J_{m+1/2}(Br \sqrt{2})}{m! 2^{m+1/2} (Br \sqrt{2})^{m+1/2}} \text{ for } r \leq a$$

$$G_m(r) \equiv F_m(r) = \frac{1}{r^2} \int_0^a \left(1 - \frac{R^2}{r^2}\right)^{m-1/2} J_0(BR \sqrt{2}) R dR$$

$$= \sum_{k, \ell=0}^{\infty} \frac{(-1)^{k+\ell} \Gamma\left(m + \frac{1}{2}\right) (aB)^{2\ell} (a/r)^{2k+2}}{\Gamma\left(m + \frac{1}{2} - k\right) k! \ell! \ell! 2^{\nu} (2k + 2\ell + 2)} \text{ for } r > a$$
(24a)

Table I

THE RADIAL SOURCE DISTRIBUTION $Q(R)$
IN THE INTERVAL $a \leq R \leq a \sqrt{2}$

R/a	$Q(R)$	R/a	$Q(R)$
1.00	0.0985	1.25	0.0042
1.05	0.0579	1.30	0.0011
1.10	0.0335	1.35	0.0004
1.15	0.0189	$\sqrt{2}$	0
1.20	0.0095		

Note: $Q(R)$ is obtained by writing the finite square source $\cos(\pi x/2a) \cos(\pi y/2a)$ of width $2a$ in polar co-ordinates and averaging over the angle.

The functions $E_m(r)$ and $F_m(r)$ are defined in the regions $0 \leq r \leq a$ and $a \leq r$, respectively, and they have been introduced in order to minimize confusion in the handling of the functional discontinuity in $G_m(r)$. The integral $E_m(r)$ is the same as that in Eq. (10), and the value of this integral is given by Eq. (11). The integral $F_m(r)$ could not be evaluated in closed form so we expanded the integrand in a double sum and integrated term by term.

In a similar manner we find that the geometry factor for the source [Eq. (23b')] is given by

$$G_m(r) \equiv E_m(r) = \frac{(2m)! \sqrt{\pi} J_{m+1/2}(Br \sqrt{2})}{m! 2^{m+1/2} (Br \sqrt{2})^{m+1/2}} \text{ for } r \leq a$$

$$G_m(r) \equiv F_m(r) = \sum_{k, \ell=0}^{\infty} \frac{(-1)^{k+\ell} \Gamma\left(m + \frac{1}{2}\right) (aB)^{2\ell} (a/r)^{2k+2}}{\Gamma\left(m + \frac{1}{2} - k\right) k! \ell! \ell! (2k + 2\ell + 2) 2^{\ell}} + \frac{1}{r^2} \int_a^r \left(1 - \frac{R^2}{r^2}\right)^{m-1/2} Q(R) R dR \text{ for } r > a$$
(24b)

The only difference between geometry factors (24a) and (24b) is that $F_m(r)$ for the square source has an extra term which is contributed by the "tail" of source distribution (23b') in the interval $a \leq R \leq a\sqrt{2}$.

The nonanalytic nature of sources of finite size leads to geometry factors which cannot be expanded in the form given by Eq. (5) over the interval $0 \leq r < \infty$. Suppose, however, that the point displacement kernel were identically zero for $r > a$. In this case, the form of the geometry factors $F_m(r)$ would be of no consequence, since they would make no contribution to the integral in Eq. (4a). We could arbitrarily let $E_m(r)$ represent $G_m(r)$ over the interval $0 \leq r < \infty$. Then the analysis would proceed exactly as in the case of the nonphysical sources [Eq. (9)] and expansion [Eq. (18)] would be valid for the finite sources [Eq. (23)]. Point displacement kernels of physical interest do not vanish for $r > a$ but they do become small as "r" becomes large. This suggests that the analysis of the finite sources [Eq. (23)] be carried out as a perturbation of the analysis of the nonphysical sources [Eq. (9)].

For geometry factors [Eq. (24)], we can write Eq. (4a) in the form

$$M_{2m} = \int_0^a 4\pi r^{2m+2} P(r) E_m(r) dr + \int_a^\infty 4\pi r^{2m+2} P(r) F_m(r) dr, \quad (25)$$

in which the functional discontinuity in $G_m(r)$ at $r = a$ has been made explicit. Because $E_m(r)$ is a well-behaved function for $r > a$, even though it represents $G_m(r)$ only for $r < a$, we can rewrite Eq. (25) as

$$\begin{aligned} M_{2m} - R_{2m} &= \int_0^\infty 4\pi r^{2m+2} P(r) E_m(r) dr, \\ R_{2m} &\equiv \int_a^\infty 4\pi r^{2m+2} P(r) [F_m(r) - E_m(r)] dr. \end{aligned} \quad (26)$$

The functional form of $E_m(r)$ over the interval $0 \leq r \leq \infty$ is identical with that of the geometry factor $G_m(r)$ given by Eq. (11) for the nonphysical sources [Eq. (9)]. It follows that Eq. (13) and its inverse Eq. (18) are also valid for sources [Eq. (23)] if the M_{2m} are replaced by $M_{2m} - R_{2m}$, that is,

$$P_{2m} = \sum_{n=0}^{\infty} C_{mn}^{-1} (M_{2n} - R_{2n}), \quad C_{mn}^{-1} = \frac{(2m+1)! n! (2B^2)^{n-m}}{m! (2n)! (n-m)!}. \quad (27)$$

Similarly, the neutron age is given by

$$2\tau = \frac{P_2}{3P_0} = \frac{(M_2-R_2) + \frac{B^2(M_4-R_4)}{3} + \frac{B^4(M_6-R_6)}{30} + \frac{B^6(M_8-R_8)}{630} + \frac{B^8(M_{10}-R_{10})}{22680} + \dots}{(M_0-R_0) + B^2(M_2-R_2) + \frac{B^4(M_4-R_4)}{6} + \frac{B^6(M_6-R_6)}{90} + \frac{B^8(M_8-R_8)}{2520} + \dots} \quad (28)$$

Equation (28) is the desired relation between the age and the moments of the measured axial flux distribution $M(0,0,z)$.

The terms R_{2m} correct the moments M_{2m} obtained with sources [Eq. (23)] to the values which would have been obtained with sources [Eq. (9)]. The R_{2m} depend upon the form of the kernel $P(r)$ for $r > a$. If the system size is large enough, the R_{2m} will be small compared to the M_{2m} for values of m which are significant. In this case the R_{2m} can be calculated to sufficient accuracy by using a rough approximation for the kernel $P(r)$ for $r > a$. This approximation of $P(r)$ may be taken directly from age experiments made in the point source geometry or it may be inferred from the large plane source experiment itself. The $P(r)$ may be normalized to the source strength of the plane source distribution by substituting $P(r)$ and the plane source distribution into Eq. (1), calculating $M(0,0,z)$ for some $z > a$ and setting this calculated value equal to the observed value of $M(0,0,z)$. This normalized estimate of $P(r)$ may then be substituted into Eq. (26) for R_{2m} . If the Bessel function in $E_m(r)$ is expanded in a power series, Eq. (26) for R_{2m} can be written in the following form for the purpose of numerical integration:

$$R_{2m} = \sum_{k, \ell=0}^{\infty} \frac{(-1)^{k+\ell} \Gamma\left(m+\frac{1}{2}\right) \pi(aB)^{2\ell} a^{2k+2}}{2^{\ell-2} \Gamma\left(m+\frac{1}{2}-k\right) k! \ell! \ell! (2k+2\ell+2)} \int_a^{\infty} r^{2m-2k} P(r) dr$$

$$- \sum_{k=0}^{\infty} \frac{(-1)^k (2m)! \pi^{3/2} B^{2k}}{k! m! \Gamma\left(m+k+\frac{3}{2}\right) 2^{2m+k-1}} \int_a^{\infty} r^{2m+2k+2} P(r) dr \quad (29a)$$

$$R_{2m} = (\text{same as 29a}) + \int_a^{\infty} 4\pi r^{2m} P(r) dr \int_a^r \left(1 - \frac{R^2}{r^2}\right)^{m-\frac{1}{2}} Q(R) R dR \quad (29b)$$

NUMERICAL EXAMPLE

To illustrate the use of Eqs. (28) and (29) and to give some feeling for the magnitude of the quantities involved, we shall consider the treatment of some representative axial flux distribution data of a preliminary nature. Details of the age measurement other than the geometrical correction factor

will not be considered here. The data were measured⁽²⁾ in H_2O with a U^{235} fission neutron source distribution of the form given in Eq. (23b), where $a = 30.5$ cm. The distance from the edge of the fission plate to the side of the water tank was at least 30.5 cm. This distance is about 5.5 root ages, and so it was assumed that the water moderator was infinite in width. Figure 1 shows the observed axial distribution $M(0,0,z)$ and the function $z^2 M(0,0,z)$. Also shown is $z^2 P(z)$, where $P(z)$ is the neutron flux distribution measured by Hill, Roberts and Fitch⁽⁵⁾ for an approximate point fission neutron source in H_2O . When $z > 30$ cm, the latter data can be fit very well, for the present purposes by $z^{-2} \exp(-z/L)$ with $L = 8.6$ cm. For $z > 30$ cm, the distribution $M(0,0,z)$ has the same form with $L = 8.54$ cm. This similar behavior is to be expected because the finite plane source appears to be a point source at large distances.

Fig. 1. Some indium resonance neutron flux distributions. Here $M(0,0,z)$ is the axial flux distribution obtained with a plane fission neutron source 61 cm wide distributed as $\cos(\pi x/2a) \cos(\pi y/2a)$. The solid line indicates data taken with 12 in. of D_2O and 4 in. of H_2O between source reactor and fission plate; the dashed line indicates data taken with 12 in. of D_2O between reactor and fission plate (see Ref. 2). Further, $P(z)$ is the radial flux distribution observed by Hill, Roberts and Fitch for a point source geometry (see Ref. 4).

We shall assume that for $r > a$ the kernel $P(r)$ is given by

$$P(r) = \frac{P_0}{4\pi a^2} \left(\frac{L}{r}\right)^2 \exp(-r/L) \quad (30)$$

To find the normalization, P_0 , we substitute Eq. (30) into Eq. (1) to get

$$\frac{M(0,0,z)}{P_0} = \frac{1}{4\pi} \left(\frac{L}{a}\right)^2 \iint_{-\infty}^{+\infty} \frac{\exp[-(x^2 + y^2 + z^2)^{1/2} L^{-1}]}{x^2 + y^2 + z^2} Q(x,y) dx dy \quad (31)$$

Equation (32) was integrated numerically for sources (23a) and (23b). Table II gives $M(0,0,z)/P_0$ as a function of the ratio a/L for $z = 1.50 a$ and for $z = 2.0 a$ for the circular source and for the square source.

Table II

NORMALIZATION OF THE APPROXIMATE POINT KERNEL
TO THE OBSERVED AXIAL DISTRIBUTION

a/L	$M(0,0,z)/P_0$ for circular source when		$M(0,0,z)/P_0$ for square source when	
	$z = 3a/2$	$z = 2a$	$z = 3a/2$	$z = 2a$
2.5	1.29×10^{-4}	2.28×10^{-5}	1.51×10^{-4}	2.67×10^{-5}
3.0	4.06×10^{-5}	5.64×10^{-6}	4.75×10^{-5}	6.60×10^{-6}
3.5	1.39×10^{-5}	1.48×10^{-6}	1.63×10^{-5}	1.73×10^{-6}
4.0	4.72×10^{-6}	4.03×10^{-7}	5.52×10^{-6}	4.72×10^{-7}
4.5	1.70×10^{-6}	1.14×10^{-7}	1.99×10^{-6}	1.33×10^{-7}
5.0	6.25×10^{-7}	3.29×10^{-8}	7.31×10^{-7}	3.85×10^{-8}
5.5	2.39×10^{-7}	9.71×10^{-9}	2.75×10^{-7}	1.13×10^{-8}
6.0	9.05×10^{-8}	2.92×10^{-9}	1.07×10^{-7}	3.42×10^{-9}

Note: Normalization of the kernel $P(r) = P_0(4\pi a^2)^{-1} (L/r)^2 \exp(-r/L)$ to the observed axial distribution $M(0,0,z)$ for a circular source of diameter $2a$ distributed as $J_0(j_0 R/a)$ and for a square source of width $2a$ distributed as $\cos(\pi x/2a) \cos(\pi y/2a)$. The normalization is carried out at $z = 3a/2$ and at $z = 2a$.

If the kernel [Eq. (30)] is substituted into Eqs. (29), the quantities $R_{2m}/P_0 a^{2m+1}$ can be expressed as sums of functions of a/L . The coefficients were evaluated and the sums performed with the aid of an IBM-650 computer. Table III gives $R_{2m}/P_0 a^{2m+1}$ as a function of a/L for $m = 0, 1, 2, 3, 4$ and 5 for sources (23a) and (23b).

Table III
THE CORRECTION TERMS R_{2m}

a/L	Values of $10^6 R_{2m}/P_0 a^{2m+1}$ for					
	$m = 0$	$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = 5$
(a) Circular source						
2.8	322	154	264	967	6240	61800
3.0	207	92.3	147	487	2760	24500
3.2	134	56.1	81.6	252	1290	10200
3.4	88.0	34.6	48.0	133	630	4510
3.6	58.3	21.7	28.1	71.9	316	2060
4.0	26.1	8.81	9.79	22.3	83.7	469
4.4	12.1	3.70	3.72	7.28	24.1	117
4.8	5.73	1.63	1.41	2.60	7.4	32
5.2	2.78	0.73	0.57	0.95	2.4	9
5.6	1.35	0.34	0.25	0.35	0.8	0
6.0	0.70	0.16	0.10	0.14	0.3	0
(b) Square source						
2.8	227	122	231	818	5200	50000
3.0	141	77.4	129	448	2530	24000
3.2	88.5	46.3	70.0	230	1220	10000
3.4	55.5	27.6	40.0	117	570	4400
3.6	35.9	17.4	23.4	62.9	283	2000
4.0	14.8	6.82	7.76	18.8	71.5	420
4.4	6.33	2.82	2.87	5.97	20.0	98
4.8	2.69	1.20	1.02	1.60	6.1	27
5.2	1.15	0.51	0.39	0.71	2.2	8
5.6	0.45	0.24	0.17	0.26	0.8	0
6.0	0.19	0.11	0.06	0.10	0.3	0

Note: The correction terms R_{2m} for a circular source of diameter $2a$ distributed as $J_0(j_0 R/a)$ and for a square source of width $2a$ distributed as $\cos(\pi x/2a) \cos(\pi y/2a)$.

For values of a/L less than 2.8, it was not possible to get Eqs. (29) for the R_{2m} to converge with 40 terms. This corresponds to a fission plate radius or half-width of $a = 2.8 \times 8.6 \text{ cm} = 24 \text{ cm}$ in the case of age measurements in H_2O .

The calculation of the age from the axial distribution data shown in Fig. 1 is outlined in Table IV. Let the quantity τ_k be defined as the age given by Eq. (28) when terms of order greater than B^{2k} are set equal to zero. The age τ_k as a function of the index k is shown in Fig. 2. Also shown in Fig. 2 is the age calculated by setting the correction terms R_{2m} equal to zero.

Table IV

SAMPLE CALCULATION OF THE AGE FROM AN
AXIAL FLUX DISTRIBUTION

Given: $M(0,0,z)$ shown in Fig. 2 and the moments M_{2m} listed below, $a = 30.5$ cm, $L = 8.6$ cm, $a/L = 3.55$, $B = 0.0515$, $M(0,0,61) = 4.50 \times 10^{-5}$.				
Calculate:				
(a) $M(0,0,2a)/P_0 = 1.52 \times 10^{-6}$ from Table II, square source; $P_0 = 29.6$.				
(b) From Table III we find the values of $R_{2m}/P_0 a^{2m+1}$ listed below:				
m	M_{2m}	$R_{2m}/P_0 a^{2m+1}$	R_{2m}	$M_{2m} - R_{2m}$
0	3.310×10	3.88×10^{-5}	0.04	3.306×10
1	1.594×10^3	1.97×10^{-5}	1.8×10	1.576×10^3
2	5.15×10^5	2.67×10^{-5}	2.3×10^4	4.92×10^5
3	5.13×10^8	7.40×10^{-5}	6.0×10^7	4.53×10^8
4	9.39×10^{11}	3.30×10^{-4}	2.55×10^{11}	6.84×10^{11}
5	4.74×10^{15}	2.36×10^{-3}	1.74×10^{15}	3.00×10^{15}

To investigate the effect of an error in approximating the behavior of the kernel $P(r)$ for $r > a$, the age τ_4 was determined for a 20% change in the value of L (initially $L = 8.6$ cm). This variation is greater than would be justified by existing point source data in light water (Fig. 1), and so represents a conservative estimate of the sensitivity of the age to the choice of asymptotic kernel. The τ_4 value for $L = 8.6$ cm is 28.32 cm²; that for $L = 6.9$ cm is 28.45 cm², and that for $L = 10.3$ cm is 28.10 cm².

Fig. 2. The age τ_k as a function of the index k . The solid line connects points calculated from Eq. (28) for a source of width $2a$. The dashed line connects points calculated from Eq. (19) for an oscillatory source of infinite extent.

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