ANL-6254
Physics and Mathematics
(TID-4500, 15th Ed.)
AEC Research and
Development Report

ARGONNE NATIONAL LABORATORY
9700 South Cass Avenue
Argonne, Illinois

A PULSED NEUTRON SOURCE TECHNIQUE

by

R. Siems* and M. Melissaropoulos**

International School of Nuclear Science and Engineering

November 1960

*Now at the Institut für Reaktorwerkstoffe GFKF, Aachen, Germany

**Now at Greek Atomic Energy Center "Demokritos," Athens, Greece

Operated by The University of Chicago
under
Contract W-31-109-eng-38

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency Thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

DISCLAIMER

Portions of this document may be illegible in electronic image products. Images are produced from the best available original document.

TABLE OF CONTENTS

	<u>Page</u>
ABSTRACT	3
INTRODUCTION.	3
I. DETERMINATION OF DIFFUSION CONSTANT D AND DIFFU- SION LENGTH L IN RECTANGULAR GEOMETRY.	5
A. Use of the Lowest Mode λ_{111} of Different Piles.	6
B. Use of the Higher Modes of the Same Pile.	8
II. THEORETICAL DERIVATION OF THE NORMAL MODE AMPLITUDES IN A RECTANGULAR PILE.	14
A. Source of Thermal Neutrons.	14
B. Source of Fast Neutrons	18
1. Narrow Beam of Fast Neutrons	26
2. Parallel Beam of Fast Neutrons, Constant in z-y Plane .	27
3. Point Source of Fast Neutrons at the Surface	28
REFERENCES.	32
ACKNOWLEDGEMENT	32

A PULSED NEUTRON SOURCE TECHNIQUE

by

R. Siems and M. Melissaropoulos

ABSTRACT

For a measurement of the diffusion constant D from the decay of the fundamental mode only, optimal dimensions of the pile are given, leading to a fast decay of the higher modes. Another method is discussed which makes use also of higher modes. Evaluation formulae and optimal pile dimensions for this method are derived. Finally, the amplitudes of the normal modes are calculated for different initial distributions of the neutrons by means of the Fermi age theory. The error introduced by using Fermi age theory with fixed moderator atoms for the slowing down process down to thermal energy is estimated. In one instance, for a simple initial distribution of thermal neutrons, the neutron density was calculated as a function of space and time. The data obtained in this way have been used to check the accuracy of some approximate evaluation formulae derived in this paper.

INTRODUCTION

"Pulsed neutron source technique" is the application of the following experimental procedure: A neutron burst of short duration is injected into the material under study, and then the decay of the neutron population with time is observed, if necessary at different positions, for example, by feeding the counter output into a multichannel analyser. This technique has been used to study the slowing down and diffusion of neutrons. In this paper we are mainly interested in the latter.

The object of pulsed neutron work has been mainly the determination of material properties like the diffusion constant D or the diffusion length L . For this purpose a simple geometry is used so that the geometrical bucklings of the normal modes can be calculated from the dimensions of the pile. There is, however, another application: Once the material constants are known, the buckling of the lowest mode of a complicated geometry, which may be of considerable interest, for example in reactor physics, can be determined experimentally.⁽¹⁾

The diffusion equation for monoenergetic neutrons of velocity v is

$$D\Delta n = \frac{\partial n}{\partial t} - \Sigma_a v n \quad , \quad (1)$$

where n is the neutron density and Σ_a the macroscopic absorption cross section of the material. At the boundary*

$$n = 0 \quad .$$

By factorization of n into one space-independent and one time-independent part, one obtains the eigensolutions or normal modes:

$$n_i = e^{-(\Sigma_a v + D B_i^2)t} R_i(\mathbf{r}) \quad , \quad (2)$$

where R_i satisfies the eigenvalue equation

$$\Delta R_i + B_i^2 R_i = 0 \quad ,$$

and B_i^2 is the geometric buckling of the i^{th} mode. Equation (2) may be written in the form

$$n_i = e^{-\lambda_i t} R_i(\mathbf{r}) \quad ,$$

with

$$\lambda_i = \Sigma_a v + D B_i^2$$

or with

$$L^2 = D/v\Sigma_a \quad ,$$

so that

$$\lambda_i = (D/L^2)(1 + L^2 B_i^2) \quad . \quad (3)$$

Here diffusion cooling⁽¹⁾ is neglected. Equation (3) is the fundamental equation for the pulsed neutron source method. As mentioned above, this equation may be used either to determine D and L^2 from calculated values of B_i^2 (for a simple geometry) and measured values of λ_i ; in this case, at least two pairs of values (λ_i, B_i^2) are necessary to determine the slope D and the intersection D/L^2 of the straight line $\lambda_i(B_i^2)$ of Eq. (3); or it may be used to determine B_i^2 from known material constants D and L^2 and measured decay constants λ_i . In this case one measurement is sufficient.

*Boundary in this paper means the extrapolated boundary of the pile.

The general solution of Eq. (1) is

$$n = \sum_i f_i e^{-\lambda_i t} R_i(\underline{r}) \quad (4)$$

After a sufficiently long time, only the lowest modes contribute appreciably to n .

I. DETERMINATION OF DIFFUSION CONSTANT D AND DIFFUSION LENGTH L IN RECTANGULAR GEOMETRY

For a rectangular pile (Fig. 1), the space part of the diffusion equation separates into three equations for x , y and z , respectively, whose solutions are sines and cosines.

Fig. 1

Rectangular pile. F indicates the incident neutron beam.

Because of boundary conditions and because of symmetry with respect to the x axis, the general solution is

$$n = \sum_{\substack{\ell, m, n \\ m, n \text{ odd}}} f_{\ell mn} e^{-\lambda_{\ell mn} t} \sin \frac{\ell \pi x}{A} \cos \frac{m \pi y}{B} \cos \frac{n \pi z}{C} = \sum f_{\ell mn} e^{-\lambda_{\ell mn} t} R_{\ell mn} \quad (5)$$

with

$$\lambda_{\ell mn} = (D/L^2) (1 + L^2 B_{\ell mn}^2) \quad (6)$$

and

$$B_{\ell mn}^2 = \pi^2 \left(\frac{\ell^2}{A^2} + \frac{m^2}{B^2} + \frac{n^2}{C^2} \right) \quad (7)$$

A. Use of the Lowest Mode λ_{111} of Different Piles

The standard procedure is to consider only the lowest mode (111) of piles of different sizes. This is justified if, after a sufficient time, only the lowest mode contributes appreciably to n , so that finally the measured decay constant is identical with the decay constant of the (111)-mode, i.e., if higher modes decrease to a negligible value while the amplitude of the lowest mode is still large enough to be measurable with the necessary accuracy. We shall determine for the pile in Fig. 1 the optimal ratio of base length B to height A for a fixed volume AB^2 . (In the most favorable case B and C are, of course, equal.)

Let the amplitudes n_1 and n_2 of the first and second mode, respectively, be equal to each other and equal to N at time $t = 0$.[†] The amplitudes at time t are then

$$n_1 = N e^{-\lambda_1 t} \quad ; \quad n_2 = N e^{-\lambda_2 t} \quad ,$$

with the decay constants λ_1 and λ_2 .

We assume that the contribution of the second mode is negligible when n_2 is less than a certain fraction $1/\mu$ of n_1 ($\mu \gg 1$). Let t^* be the time at which $n_2 = n_1/\mu$. This time t^* is the moment at which we can assume that the experimental curve contains only the first mode. It is obvious that we want n_1/N at this time t^* to be as large as possible.

Now at time t^* ,

$$\ln n_2 = \ln n_1 - \ln \mu \quad , \quad \text{i.e. } t^* = \frac{\ln \mu}{\lambda_2 - \lambda_1} \quad ,$$

and

$$\frac{n_1(t^*)}{N} = e^{-\frac{\lambda_1}{\lambda_2 - \lambda_1} \ln \mu} \quad .$$

It follows, as was to be expected, that $(\lambda_2 - \lambda_1)/\lambda_1$ has to be as large as possible.

[†]Actually, the amplitudes of the neutrons just thermalized are not equal (see Fig. 6d, Curve II). We may, however, count the time from an instant before the neutron burst, at which the back-extrapolated amplitudes are of the same order of magnitude. The same remark applies to the discussion of the higher modes method. The extrapolated amplitudes ϕ_{211} and ϕ_{131} of Fig. 6d, Curve II are equal at an instant 2.8 msec before the neutron burst, while the extrapolated amplitudes ϕ_{211} and ϕ_{151} , which are considered in the higher modes method, are equal at an instant 1.95 msec before the burst. These instants are then considered as the time $t = 0$.

By means of the definitions

$$g_{\ell 11} = \frac{\lambda'_{\ell 11} - \lambda_{111}}{\lambda_{111}} \frac{1}{L^2 \pi^2}$$

and

$$g_{1m1} = \frac{\lambda_{1m1} - \lambda_{111}}{\lambda_{111}} \frac{1}{L^2 \pi^2} ,$$

we obtain

$$g_{\ell 11} = \frac{(\ell^2 - 1)/A^2}{1 + L^2 \pi^2 \left(\frac{1}{A^2} + \frac{2A}{V} \right)}$$

and

$$g_{1m1} = \frac{(m^2 - 1)A/V}{1 + L^2 \pi^2 \left(\frac{1}{A^2} + \frac{2A}{V} \right)} .$$

The quantity A is then determined by requiring that for a fixed value of V the smaller of $g_{\ell 11}$ and g_{1m1} shall be as large as possible.

This condition can be satisfied only if $g_{\ell 11} = g_{1m1}$. Proof: $g_{\ell 11}$ is a monotonously decreasing function of A ; on the other hand, g_{1m1} monotonously increases with A . If, now, $g_{\ell 11} \neq g_{1m1}$ and, for example, $g_{\ell 11} < g_{1m1}$, then, by decreasing A , the quantity $g_{\ell 11}$ would increase; i.e., the smaller of the two g 's would increase.

The condition for B is therefore

$$g_{\ell 11} = g_{1m1} ,$$

i.e.,

$$B^2 = \frac{m^2 - 1}{\ell^2 - 1} A^2 \quad \text{with} \quad B^2 = V/A . \quad (8)$$

Now the second lowest x mode is (211) and the second lowest y mode is (131). The condition (8) is, therefore, with $m = 3$ and $\ell = 2$,

$$B^2 = \frac{8}{3} A^2 ,$$

i.e.,

$$B \approx 1.63 A$$

If, however, the detector can be placed in a position where the (211) mode, the (131) mode and the (113) mode are zero (for example, at the position $x = A/2$, $y = z = B/6$), then the lowest modes besides (111) which do not vanish at the detector are (311) and (151). In this case, $m = 5$ and $\ell = 3$, and Eq. (8) reduces to

$$B^2 = 3A^2$$

i.e.,

$$B \approx 1.73 A$$

A pile with this ratio of base length to height is suited best for measuring the (111) mode alone.

B. Use of Higher Modes of the Same Pile

Instead of measuring the decay constant λ_{111} of the lowest mode at different piles, we may measure in addition the decay constants $\lambda_{\ell mn}$ of some of the higher modes at the same pile. It is then necessary to measure the decay curves at different points of the pile to get some information on the spatial distribution.

The most direct approach would be to determine the neutron density $n(r)$ as a function of space at least at two different times t_1 and t_2 , and then to calculate the Fourier coefficients $F_{\ell mn}$ of $n(r)$ for each time. From

$$f_{\ell mn} e^{-\lambda_{\ell mn} t} = F_{\ell mn} \quad , \quad (9)$$

we may then obtain the different values for λ .

In a pile with suitable dimensions ($A > B, C$), after a short time only the first ($\ell 11$) modes remain, since all other modes decrease very rapidly.* In this case we only have to perform the x integrations along a line $z = \text{const}$ and $y = \text{const}$, since the y and z integrations contribute to $F_{\ell 11}$ nothing but a constant factor which has no influence on the

*Let the pile dimensions be, for example, $2m \times 1m \times 1m$. If we place the counter in a position where the modes with $m = 3$ and $n = 3$ are zero, the first m and n modes which we have to worry about are (151) or (115). However, $B_{151}^2 > B_{911}^2$, i.e., the (151) or the (115) modes decrease faster than the first nine ($\ell 11$) modes.

determination of λ according to Eq. (9). We therefore have to measure the neutron population n only along a line parallel to the x axis.

Another method of evaluation is the following: We consider only times which are so large that all but the first four ($\ell 11$) modes have disappeared. In this case we obtain with

$$a_{\ell} = f_{\ell 11} e^{-\lambda_{\ell 11} t} \cos \frac{\pi Y}{B} \cos \frac{\pi Z}{C}, \quad (10)$$

the following equations:

$$n(x) = a_1 \sin \frac{\pi x}{A} + a_2 \sin \frac{2\pi x}{A} + a_3 \sin \frac{3\pi x}{A} + a_4 \sin \frac{4\pi x}{A}$$

$$n'(x) = a_1 \frac{\pi}{A} \cos \frac{\pi x}{A} + a_2 \frac{2\pi}{A} \cos \frac{2\pi x}{A} + a_3 \frac{3\pi}{A} \cos \frac{3\pi x}{A} + a_4 \frac{4\pi}{A} \cos \frac{4\pi x}{A}.$$

By means of the definitions

$$\left. \begin{aligned} n_2 &= n\left(\frac{A}{2}\right) &= a_1 - a_3 \\ n_2' &= \frac{A}{2\pi} n'\left(\frac{A}{2}\right) &= -a_2 + 2a_4 \\ n_+ &= \frac{A}{2\pi} [n'(0) + n'(A)] &= 2a_2 + 4a_4 \\ n_- &= \frac{A}{2\pi} [n'(0) - n'(A)] &= a_1 + 3a_3 \end{aligned} \right\}, \quad (11)$$

we obtain finally

$$\left. \begin{aligned} a_1 &= \frac{1}{4} (3n_2 + n_-') \\ a_2 &= \frac{1}{4} (n_+' - 2n_2') \\ a_3 &= \frac{1}{4} (n_-' - n_2) \\ a_4 &= \frac{1}{8} (2n_2' + n_+') \end{aligned} \right\}. \quad (12)$$

To get the necessary data, we have to measure the neutron density at four positions (see Fig. 2). The quantity $\lambda_{\ell 11}$ is obtained from measurements at different times according to Eq. (10).

Fig. 2

Determination of the amplitudes of the four lowest modes from measurements at 4 different points.

We now proceed to determine what dimensions A, B, C a pile should have to be suited best for this type of measurement. For the beginning, we restrict ourselves to an analysis of the measurement of the diffusion constant D. The aim is, of course, to determine D as accurately as possible.

The diffusion constant D is determined from the decay constants λ_{111} and $\lambda_{\ell 11}$. The first neglected modes are (1m1) and (1lm). If the neutron density is measured along the line $y = B/6$, $z = B/6$, where $a_{\ell 31}$ and $a_{\ell 13}$ are zero, then $m = 5$. Let

$$\lambda_1 \equiv \lambda_{111} = \frac{D}{L^2} + D\pi^2 \left(\frac{1}{A^2} + \frac{2}{B^2} \right) \quad ;$$

$$\lambda_{\ell} \equiv \lambda_{\ell 11} = \frac{D}{L^2} + D\pi^2 \left(\frac{\ell^2}{A^2} + \frac{2}{B^2} \right) \quad ;$$

$$\lambda_m \equiv \lambda_{1m1} = \frac{D}{L^2} + D\pi^2 \left(\frac{1}{A^2} + \frac{m^2 + 1}{B^2} \right) \quad .$$

The neutron distribution is measured at times t_1 and $t_2 > t_1$, where t_2 is chosen in such a way that at this time the error in the amplitude $N_{\ell}(t_2)$ of the ($\ell 11$) mode as introduced by experimental conditions (intensity of the source, background intensity, total time for the experiment) can be kept below a certain value (e.g., 0.001). Let $t_2 = \nu/\lambda_{\ell}$, where ν is determined by the conditions mentioned above. We assume $\nu = 6$ as an example.

On the other hand, t_1 is chosen in such a way that the error introduced by the neglect of higher modes is smaller than the same value (0.001). This error will have the same order of magnitude as the

amplitude of the lowest neglected mode (1m1). It is again assumed that at time $t = 0$ the amplitudes of the modes in question are equal to each other (see footnote on page 6). Then the condition for t_1 is

$$\frac{\Delta a_\ell(t_1)}{a_\ell(t_1)} = e^{-(\lambda_m - \lambda_\ell)t_1} = 10^{-3} \quad ,$$

i.e.,

$$t_1 = \frac{\mu}{\lambda_m - \lambda_\ell}$$

$$\mu = \ln 1000 \approx 7 \quad .$$

At time t_2 the error from neglecting the (1m1) mode and at time t_1 the experimental errors are smaller than $\overset{0.001}{\blacksquare}$ and are neglected. λ_ℓ is then determined from the equation

$$\lambda_\ell = \frac{\ln a_\ell(t_1) - \ln a_\ell(t_2)}{t_2 - t_1} \quad .$$

The error in λ_ℓ is then

$$\Delta \lambda_\ell = \frac{\overset{0.001}{\sqrt{2}(\blacksquare)}}{\frac{\nu}{\lambda_\ell} - \mu \frac{1}{\lambda_m - \lambda_\ell}} \quad ,$$

and the relative error is

$$\frac{\Delta \lambda_\ell}{\lambda_\ell} = \frac{\overset{0.001}{\sqrt{2}(\blacksquare)}}{\nu - \mu \frac{\lambda_\ell}{\lambda_m - \lambda_\ell}} \quad .$$

The diffusion constant D is determined from the equation

$$D = \frac{1}{\pi^2} \frac{A^2}{\ell^2 - 1} (\lambda_\ell - \lambda_1) \quad .$$

The error in D is mainly due to that in λ_ℓ :

$$\Delta D = \frac{1}{\pi^2} \frac{A^2}{\ell^2 - 1} \Delta \lambda_\ell \quad ,$$

and

$$\begin{aligned} \frac{D}{\Delta D} &= \frac{\lambda_\ell}{\Delta \lambda_\ell} - \frac{\lambda_1}{\Delta \lambda_\ell} = \frac{1000}{\sqrt{2}} \left\{ \nu - \mu \frac{\lambda_\ell}{\lambda_m - \lambda_\ell} - \nu \frac{\lambda_1}{\lambda_\ell} + \mu \frac{\lambda_1}{\lambda_m - \lambda_\ell} \right\} \\ &= \frac{1000}{\sqrt{2}} \frac{\ell^2 - 1}{A^2} \pi^2 L^2 \left\{ \frac{\nu}{1 + \pi^2 L^2 \left(\frac{\ell^2}{A^2} + \frac{2}{B^2} \right)} - \frac{\mu}{\pi^2 L^2 \left(\frac{m^2 - 1}{B^2} - \frac{\ell^2 - 1}{A^2} \right)} \right\} \end{aligned}$$

Introducing $A/\pi L = a$ and $AB^2/\pi^3 L^3 = V$, we obtain

$$\frac{D}{\Delta D} = \frac{1000}{\sqrt{2}} (\ell^2 - 1) \left\{ \frac{\nu}{a^2 + \ell^2 + 2a^3/V} - \frac{\mu}{(m^2 - 1) \frac{a^3}{V} - \ell^2 + 1} \right\}$$

The dependence of $\Delta D/D$ on a is shown in Fig. 3 for $\nu = 6$, $\mu = 7$, $\ell = 2$, $m = 5$, when $V = 1$ and $V = 0.1$.

Fig. 3. Relative error in diffusion constant D introduced by neglecting (151) modes. D is determined from (111) and (211) modes.
 O: Dimensions of pile most favorable for obtaining the (111) mode alone. M: Dimensions of pile with minimum B_{111}^2 .
 —: Range of dimensions with $\frac{B_{111}^2 - B_{111}^2 \min}{B_{111}^2 \min} < 10\%$.

For small values of a the error becomes large because the decay constant of the (1ml) mode approaches that of the (ℓ 11) mode. For large values of a the error becomes large because $\lambda_{\ell 11}$ and λ_{111} are almost equal, so that an error in $\lambda_{\ell 11}$ of 1% means a much larger error in D .

So far we have considered, in the discussions of the (111) method as well as of the method of higher modes, only the accuracy of the measurement of D . Now, usually, a determination of the diffusion length L is performed according to Eq. (6). To obtain L^2 as accurately as possible, B_{111}^2 should be small. This favors a shape of the pile not too different from a cube. The relative deviation

$$\rho = \frac{B_{111}^2 - B_{111}^2 \text{ min}}{B_{111}^2 \text{ min}}$$

of B_{111}^2 from its minimum value versus a is given in Fig. 4. In Fig. 3 the range of values of a with $\rho \leq 10\%$ is indicated. Piles for the (111) method should have values of a at the left end of this interval; piles for the higher modes method should have values of a close to the middle.

Fig. 4

Relative deviation of B_{111}^2 from its minimum value.

II. THEORETICAL DERIVATION OF THE NORMAL MODE AMPLITUDES IN A RECTANGULAR PILE

We are going to calculate, on different assumptions about the initial history of the neutrons, the amplitude distribution of the neutron density [see Eq. (5)].

From one of these distributions we shall obtain, by means of an electronic computer, the neutron density as a function of space and time. These calculated data are then used instead of the experimental values to determine amplitudes and decay constants according to Eq. (10) and thus check the accuracy of the method outlined above.

A. Source of Thermal Neutrons

Let a source at the real surface of the pile ($x = d, y = z = 0$) eject a burst of thermal neutrons at time zero. The distribution of these neutrons at time t is given by

$$w(\underline{r}, t) = \frac{1}{(4\pi Dt)^{3/2}} \left[e^{-\frac{(x-d)^2 + y^2 + z^2}{4Dt}} - e^{-\frac{(x+d)^2 + y^2 + z^2}{4Dt}} \right]$$

if t is so small that practically no neutrons have reached the surfaces of $y = \pm B, z = \pm C$. Here d is the extrapolation distance.

The mean square distances in the x, y and z directions from the source at time t are then

$$\overline{y^2}(t) = \overline{z^2}(t) = 2Dt$$

$$\overline{(x-d)^2}(t) = 2Dt \left[1 - \frac{2\Delta^2}{\phi(\Delta)} \left(1 - \phi(\Delta) - \frac{e^{-\Delta^2}}{\Delta\sqrt{\pi}} \right) \right]$$

where $\Delta = d/\sqrt{Dt}$ and ϕ is the error integral.

The total number $n(t)$ of neutrons still present in the pile at time t is $\frac{1}{2}\phi(\Delta)$ if at time zero one neutron leaves the source in arbitrary direction. Now, let the source operate for a time T . The mean square distances from the source of all neutrons still present in the pile at the end of the irradiation period are then given by the equations

$$\overline{y^2}(t) = \overline{z^2}(t) = \frac{\int_0^T \overline{y^2}(t) n(t) dt}{\int_0^T n(t) dt}$$

and

$$\overline{\overline{(x-d)^2(t)}} = \frac{\int_0^T \overline{(x-d)^2(t)} n(t) dt}{\int_0^T n(t) dt}.$$

Using data for graphite, we obtain for $T = 150 \mu\text{sec}$,

$$\overline{Y^2} = \overline{y^2} = \overline{z^2} = 27 \text{ cm}^2$$

and

$$\overline{(X-d)^2} = \overline{\overline{(x-d)^2}} = 39 \text{ cm}^2.$$

Now

$$\overline{X^2} = \overline{(d + X - d)^2} = \overline{(X - d)^2} + d^2 + 2d \overline{(X - d)}.$$

Because

$$\left\{ (X - d) - \overline{(X - d)} \right\}^2 = \overline{(X - d)^2} - \overline{(X - d)}^2 > 0,$$

we obtain

$$0 < \overline{X - d} < \sqrt{\overline{(X - d)^2}}$$

and

$$\overline{(X - d)^2} + d^2 < \overline{X^2} < \left(\sqrt{\overline{(X - d)^2}} + d \right)^2,$$

i.e.,

$$42 \text{ cm}^2 < \overline{X^2} < 65 \text{ cm}^2$$

and

$$96 \text{ cm}^2 < \overline{R^2} = \overline{X^2} + \overline{Y^2} + \overline{Z^2} < 119 \text{ cm}^2.$$

Finally

$$9.8 \text{ cm} < \sqrt{\overline{R^2}} < 10.9 \text{ cm}.$$

In a very rough approximation we assume, instead of the real distribution at the end of the irradiation time, i.e., at time zero, the distribution given by

$$n(\underline{r}, 0) = \begin{cases} c & \text{for } \begin{cases} 0 < x < p \\ -p < y < p \\ -p < z < p \end{cases} \\ 0 & \text{everywhere else.} \end{cases} \quad (13)$$

The mean square distance p^2 from the origin of these neutrons is set equal to the above value of $\sqrt{R^2} \approx 10$ cm. The amplitude of the (lmn) mode at time $t = 0$ is then

$$f_{lmn} = \bar{C} \frac{32}{\ell_{mn}} \frac{1}{\pi^3} \left(1 - \cos \frac{\ell\pi p}{A} \right) \sin \frac{m\pi p}{B} \sin \frac{n\pi p}{C}, \quad (14)$$

where $\bar{C}$ is determined in such a way that the total number of neutrons at the end of the irradiation time,

$$z_0 = \int_0^T \frac{1}{2} \phi(\Delta) \frac{dt}{T} = \frac{1}{2} (0.54) \quad (\text{for } T = 150 \mu\text{sec})$$

is equal to the number $4cp^3$ of neutrons in the distribution (13). In this way we obtain $\bar{C} = 6.75 \times 10^{-5} \text{ cm}^{-3}$.

The main purpose of this simplified treatment was to obtain a simple model with which the approximate formulas of Chapter I could be tested. For this purpose the function $n(\underline{r}, t)$ with $\bar{C} = 1$ has been tabulated for a graphite pile with

$$p = 10 \text{ cm}; \quad A = 152.1 \text{ cm}; \quad B = 89 \text{ cm}; \quad C = 78.6 \text{ cm}.$$

Figure 5 shows some of the results.

We use the data for $t = 3.4$ msec to check the accuracy of the values obtained by means of Eq. (12). With the values

$$n'(0) = 1.153 \times 10^{-5} \text{ cm}^{-1}$$

$$n'(A/2) = -0.470 \times 10^{-5} \text{ cm}^{-1}$$

$$n(A/2) = 0.1195 \times 10^{-3}$$

$$n'(A) = 0.118 \times 10^{-6} \text{ cm}^{-1}$$

we obtain, according to Eq. (12),

$$a_1 = 0.160 \times 10^{-3}$$

$$a_2 = 0.126 \times 10^{-3}$$

$$a_3 = 0.0406 \times 10^{-3}$$

$$a_4 = 0.0061 \times 10^{-3}.$$

Fig. 5. Decay of the neutron population [Eq. (13)] along the line $y = 0, z = 13.8$ cm.

The correct values are

$$a_1 = 0.159 \times 10^{-3}$$

$$a_2 = 0.125 \times 10^{-3}$$

$$a_3 = 0.0405 \times 10^{-3}$$

$$a_4 = 0.00625 \times 10^{-3} \quad .$$

The error in the first three modes is less than 1% though the data were taken at positions where the (113) and the (131) modes do not disappear. The decay constant is obtained from the equation

$$\lambda_{\ell 11} = \frac{1}{t_2 - t_1} \ln \frac{a_{\ell}(t_1)}{a_{\ell}(t_2)} \quad .$$

At the later time t_2 the error due to the neglect of higher modes is practically zero, and the error in λ is

$$\frac{\delta \lambda_{\ell 11}}{\lambda_{\ell 11}} = \frac{\delta a_{\ell}(t_1)}{a_{\ell}(t_1)} \frac{1}{\ln \frac{a_{\ell}(t_1)}{a_{\ell}(t_2)}} \quad .$$

For the pile considered here, $\ln[a_{\ell}(t_1)/a_{\ell}(t_2)]$ can be made larger than 3, i.e., the error in λ would be of the order of a few tenths of a per cent.

For a source of fast neutrons the method is applicable at much lower times because of the fast decay of higher modes during slowing down.

Figure 6a shows in part I the value of $\phi_{\ell mn} = \frac{1}{k} f_{\ell mn}$ [Eq. (14)] for $A = B = C = 129$ cm; $p = 10$ cm; $k = \frac{8}{\pi} \Sigma / BC = 1.96 \times 10^{-5} \text{ cm}^{-3}$; and in part II the values to which the corresponding amplitudes have decreased after 1.51×10^{-3} sec, again with a factor $1/k$ (part II).

B. Source of Fast Neutrons

After this hypothetical case of a slow neutron source, we are now going to discuss some more realistic models with incident fast neutrons. In all cases, we assume isotropic scattering. The points of first collisions of the incident neutrons are taken as sources of fast neutrons for a treatment by the Fermi age theory. These neutrons are assumed to be practically monoenergetic after the first collision.

Fig. 6. $1/k$ · amplitudes of normal modes

- a) For a source of thermal neutrons at the surface; I: at the end of the irradiation period
II: 1.51 msec later.
- b) For an incident narrow beam of fast neutrons.
- c) For an incident parallel beam of fast neutrons,
intensity independent of y and z .
- d) For a point source of fast neutrons at the surface.

I: after first collisions
 II: after slowing down

In Fermi age theory the probability $w(\underline{r}, t) d\underline{r} dt$ that a fast neutron starting at zero time at the origin is slowed down past a certain energy in $d\underline{r} dt$ is $d\underline{r} dt$ times a δ function at t_F in t , times a space function; t_F is independent of $\underline{r}$. For the slowing down range of energies this can be compared with results of a more rigorous treatment by Sykes,⁽²⁾ who applies a polynomial expansion method for the collision density. In Fermi age theory the time $t_F(u_2)$ for slowing down from lethargy u_1 (velocity v_1) to $u_1 + u_2$ ($u_2 \gg 1$) is

$$t_F(u_2) = \frac{2}{\xi} e^{\frac{u_2}{2}} \theta_0 \quad \text{with} \quad \theta_0 = \frac{1}{v_1 \Sigma_2}.$$

(ξ = average logarithmic energy decrement per collision.) In Table I the time t_F is compared with the mean time of arrival $\bar{t}$ according to Sykes for a moderator of atomic weight 11, i.e., $\xi = 0.171$, and with constant mean free path Σ^{-1} . * Table I gives also the square root Δ of the variance of t as calculated by Sykes. Both $\bar{t}$ and Δ were calculated for a plane source of fast neutrons. In the ~~range~~^{range} for which data were given (up to distances of 5 mean free paths from the source), they were practically independent of r .

TABLE I

Times for Slowing Down from Lethargy u_1 to $u_1 + u_2$ in an Element of Atomic Weight 11. Mean Value and Variance

u_2	$t_F(u_2)$	$\bar{t}(u_2)$	$\sqrt{t^2 - \bar{t}^2} = \Delta$	$t_F/\bar{t}$	$\Delta/\bar{t}$
5	$1.30 \cdot 10^2 \theta_0$	$1.11 \cdot 10^2 \theta_0$	$0.55 \cdot 10^2 \theta_0$	1.17	0.5
10	$1.74 \cdot 10^3 \theta_0$	$1.28 \cdot 10^3 \theta_0$	$0.64 \cdot 10^3 \theta_0$	1.36	0.5
15	$2.11 \cdot 10^4 \theta_0$	$1.56 \cdot 10^4 \theta_0$	$0.78 \cdot 10^4 \theta_0$	1.36	0.5

$\theta_0 = 1/v_1 \Sigma = \text{initial lifetime}$

t_F : slowing down time according to Fermi age theory

$\bar{t}$: Mean slowing down time according to Sykes

So the result of Sykes' treatment is: 1) The mean time of arrival $\bar{t}$ at a certain energy is independent of position. The Fermi age theory value t_F is about 30% too high. 2) The arrival times scatter about this $\bar{t}$ with a Δ of about $\frac{1}{2} \bar{t}$. When applying Fermi age theory we therefore have to bear in mind that there will be deviations from t_F up to about 50%.

* This constancy of Σ is observed in carbon over most of the lethargy range.

Usually t_F will be so small that these deviations are negligible for the amplitude calculations. They will become important only for large geometrical bucklings ($DB^2 t_F \gtrsim 1$).*

When the average neutron energy has decreased to a few kT_0 , the thermal motion of the moderator atoms has to be taken into account. This "thermalization" has been treated for a monatomic gas by von Dardel⁽⁵⁾ and for a crystal by Kothari and Singwi.⁽⁷⁾ According to von Dardel, the average neutron temperature T approaches the moderator temperature T_0 as

$$\frac{T - T_0}{T_0} = \frac{2 \left(1 + \frac{m}{M}\right)}{\cosh \left(c + \frac{t}{t_{th}}\right) - 1} \quad (15)$$

for a monatomic gas of moderator atom.** Here m and M are the masses of neutron and moderator atom, respectively, and c is determined by the initial value of T . The time constant for this process is [see von Dardel and Sjöstrand⁽⁶⁾]

$$t_{th} = \frac{3}{16} \sqrt{\pi} \ell_b \left(\frac{2kT_0}{n} \right)^{-\frac{1}{2}} \frac{M}{m} \left(1 + \frac{m}{M} \right)^{\frac{1}{2}}$$

with the mean free path ℓ_b for neutrons in a moderator with rigidly bound atoms. For graphite, $t_{th} = 51 \mu\text{sec}$. In order to estimate the deviations from the fixed nuclei Fermi age theory in the thermalization range, we calculate the decay of the amplitude a of a normal mode first by making use of von Dardel's Eq. (15) and then by the simple theory. In both cases we determine the amplitude at a time t_3 which we choose so large that the neutrons have become practically thermalized according to Eq. (15) (see Fig. 7). We assume that thermalization effects become important only for energies below $E_1 = 9 \cdot \frac{3}{2} kT_0$ and that the transport cross section

* For water Crouch⁽³⁾ finds experimentally a slowing down time to the Cd edge (0.35 ev) of $\bar{t} = 5.2 \mu\text{sec}$ and $\Delta/\bar{t} = 0.54$. A comparison with results of a Monte Carlo calculation by Haynam and Brouch⁽⁴⁾ shows that chemical binding has to be taken into account with a rather high threshold energy of 1.6 ev. This then yields theoretical values of $\bar{t} = 5.3 \text{ sec}$ and $\Delta/\bar{t} = 0.67$.

** In the derivation of Eq. (15), von Dardel assumed a Maxwellian velocity distribution of the neutrons.

is constant in the thermalization range of energies. Choosing as time zero the time t at which the neutrons reach an energy $E_1 = 9 \cdot \frac{3}{2} k T_0$, or a temperature $T_n = 9 T_0$, we obtain for the age from t_1 to t_3 :

$$\tau_{13}' = \frac{1}{3\Sigma_{tr}} \int_0^{t_3} \bar{v} dt$$

Fig. 7. Decay of normal modes

1-4-3: According to Fermi age theory with fixed nuclei

1-2-3': According to thermalization theory

Now in a gas with a Maxwellian energy distribution

$$\bar{v} = \sqrt{\frac{8}{\pi}} \sqrt{\frac{kT}{m}} \quad \text{and, according to Eq. (15), for graphite}$$

$$T = T_0 \left\{ 1 + \frac{13/6}{\cosh\left(c + \frac{t}{t_{th}}\right) - 1} \right\},$$

with $T(t=0) = 9 T_0$ and $c \approx 0.72$. With the substitution $x = c + t/t_{th}$ we obtain

$$\tau_{13'} = R \int_c^{c + \frac{t_3}{t_{th}}} \left(\frac{13/6}{\cosh x - 1} \right)^{\frac{1}{2}} dx$$

with

$$R = \sqrt{\frac{8kT_0}{\pi m}} \frac{t_{th}}{3\Sigma_{tr}} = \frac{\bar{v}t_{th}}{3\Sigma_{tr}} = D t_{th} \quad .$$

For $t_3 = 6 t_{th}$, we obtain

$$\tau_{13'} = 7.2 R = 79 \text{ cm}^2 \quad .$$

The age t_0' from source energies to E_1 is obtained from the experimentally determined indium-resonance age $t_{1.4\text{ev}}$ according to

$$t_0' = t_{1.4\text{ev}} + \frac{1}{\Sigma_3 \xi} \frac{D}{\bar{v}} \ln \frac{T_{1.4\text{ev}}}{T_1} = 311 \text{ cm}^2 + 19 \text{ cm}^2 = 330 \text{ cm}^2 \quad ,$$

where D and $\bar{v}$ are the diffusion constant and average velocity for thermal neutrons.

The age at 3' is therefore

$$\tau_{0123'} = (330 + 79) \text{ cm}^2 = 409 \text{ cm}^2 \quad .$$

In the simplified treatment, on the other hand,

$$\tau_{14} = \frac{D}{\Sigma_4 \xi \bar{v}} \ln \frac{9E_{th}}{E_{th}} = 29.2 \text{ cm}^2$$

and

$$\tau_{014} = 330 \text{ cm}^2 + 29.2 \text{ cm}^2 = 359.2 \text{ cm}^2 \quad .$$

The decay $4 \rightarrow 3$ follows the equations

$$a_3 = a_4 e^{-\frac{D}{L^2} t_{43} - DB^2 t_{43}} \quad (16)$$

and

$$t_{43} = t_{13} - t_{14}$$

where

$$t_{13} = 6 t_{th} = 306 \mu\text{sec}$$

$$t_{14} = \frac{2}{\sum \xi} \left(\frac{1}{\bar{v}_4} - \frac{1}{\bar{v}_1} \right) = \frac{2}{\sum \xi \bar{v}_4} \left(1 - \frac{\bar{v}_4}{\bar{v}_1} \right)$$

$$t_{14} = 83.2 \mu\text{sec}.$$

The first term in the exponent of Eq. (16) can be neglected. With $Dt_{43} = 47.5 \text{ cm}^2$, we then obtain for the decay according to the simplified theory (0143),

$$a_3 = a_0 e^{-406.7 \text{ cm}^2 B^2},$$

and according to thermalization theory (0123')

$$a_{3'} = a_0 e^{-409 \text{ cm}^2 B^2}.$$

The ratio of the two values is $a_3/a_{3'} = e^{2.3 \text{ cm}^2 B^2}$, which is practically 1 for all reasonable values of B^2 , i.e., for the decay of a neutron pulse in a monatomic gas of carbon atoms, the simple slowing down model is quite adequate.

Thermalization in a crystal (Be) has been treated by Kothari and Singwi⁽⁷⁾ using the Debye model for low neutron energies and the Einstein model for high energies. They obtain a generalized age of 101 cm^2 at a time $113 \mu\text{sec}$ after the initial burst. At this time, according to their results, the neutrons have practically obtained the moderator temperature. The initial amplitude a_0 should therefore have decayed to

$$a_{3'} = a_0 e^{(-101 \text{ cm}^2) B^2}.$$

The thermal age for the simplified treatment is given in the literature as 98 cm^2 and the slowing down time [see, for example, Glasstone and Edlund⁽⁸⁾] as $67 \mu\text{sec}$. The diffusion constant D is given by Antonow et al.,⁽⁹⁾ as $1.22 \times 10^5 \text{ cm}^2 \text{sec}^{-1}$. In this case we obtain $113 \mu\text{sec}$ for the amplitude after the burst:

$$a_3 = a_0 e^{-98 \text{ cm}^2 B^2 - 5.6 \text{ cm}^2 B^2} = a_0 e^{-103.6 \text{ cm}^2 B^2},$$

i.e., practically the same decay as in the more exact treatment.

After this estimate of the accuracy of the simple Fermi age treatment, we are now going to calculate the decay of the neutron populations in the 3 experimental arrangements indicated in Fig. 8.

Fig. 8. a) Incident narrow beam of fast neutrons
 b) Incident parallel beam of fast neutrons
 c) Point source of fast neutrons of the surface

e: extrapolated boundary
 r: real boundary
 s: source
 o: origin of coordinate system

1. Narrow Beam of Fast Neutrons

A narrow beam of fast neutrons enters the pile, parallel to the x-axis, at $x = y = z = 0$. The incidence occurs in a momentary burst at time $t = 0$. The probability that $\frac{1}{2}$ of the neutrons are scattered in the volume element $dx dy dz$ is then

$$w(xyz) dx dy dz = \frac{1}{2} \Sigma e^{-\Sigma x} \delta(y) dx dy dz, \quad (17)$$

where Σ is the macroscopic scattering cross section for fast neutrons. Equation (17) gives the distribution, assumed to be isotropic, of neutrons after the first collision. The slowing down of these neutrons follows the age equation for the slowing down density g :

$$\frac{\partial g}{\partial \tau} = \Delta g \quad (18)$$

(absorption during slowing down is neglected).

The general solution of Eq. (18) is

$$g(xyzt) = \sum_{\substack{\ell, m, n \\ m, n \text{ odd}}} F_{\ell mn} e^{-B_{\ell mn}^2 \tau} \sin \frac{\ell \pi x}{A} \cos \frac{m \pi y}{B} \cos \frac{n \pi z}{C} \quad (19)$$

($\tau = 0$ for neutrons after the first collision).

From Eq. (17) we obtain [for $\tau = 0$ in Eq. (19)]:

$$F_{\ell mn} = \frac{1}{2} \frac{8}{ABC} \frac{\Sigma \lambda}{\Sigma^2 + \lambda^2} \left\{ 1 - (-1)^\ell e^{-\Sigma A} \right\}. \quad (20)$$

The number of neutrons arriving at thermal energies in unit volume is given by Eq. (19) with the thermal age value for t . They arrive at thermal energies at time $\tau = \theta$, where θ is the slowing down time. At this time the amplitude of the (ℓmn) mode is

$$F_{\ell mn} e^{-B_{\ell mn}^2 \tau_{th}}.$$

From then on the decay of this neutron population follows the thermal diffusion equation and is given by

$$n = \sum_{\substack{\ell, m, n \\ m, n \text{ odd}}} f_{\ell mn} e^{-\lambda_{\ell mn} t} \sin \frac{\ell \pi x}{A} \cos \frac{m \pi y}{B} \cos \frac{n \pi z}{C},$$

with

$$f_{\ell mn} = \frac{1}{2} \frac{8\Sigma}{ABC} \frac{\lambda}{\Sigma^2 + \lambda^2} \left\{ 1 - (-1)^\ell e^{-\Sigma A} \right\} e^{-B^2_{\ell mn} \tau_{th}} e^{\lambda_{\ell mn} \theta}. \quad (21)$$

For a graphite pile (density of 1.6 g cm^{-3}) and neutrons of 2.45-Mev primary energy, $\Sigma = 0.128 \text{ cm}^{-1}$. For $A > 55 \text{ cm}$ the term $e^{-\Sigma A}$ is smaller than 10^{-3} and can be neglected. With the data for graphite:

$$D = 2.1 \times 10^5 \text{ cm}^2 \text{ sec}^{-1}, \quad L = 41 \text{ cm}, \quad \tau_{th} = 350 \text{ cm}^2,$$

$\theta = 150 \text{ } \mu\text{sec}$, and the pile dimensions $A = B = C = 129 \text{ cm}$, we obtain

$$f_{\ell mn} = \frac{1}{2} k \frac{\ell}{27.5 + \ell^2} e^{-0.19(\ell^2 + m^2 + n^2)} = k \phi_{\ell mn} \quad (22)$$

with

$$k = \frac{8}{\pi} \frac{\Sigma}{BC}. \quad (22a)$$

Figure 6b shows $\phi_{\ell mn} = \frac{1}{k} F_{\ell mn}$ (I) and $\phi_{\ell mn} = \frac{1}{k} f_{\ell mn}$ (II)

2. Parallel Beam of Fast Neutrons, Constant in z-y Plane

The problem is very similar to that in 1). In this case,

$$w(x, y, z) dx dy dz = \frac{1}{2} \frac{1}{BC} \Sigma e^{-\Sigma x} dx dy dz, \quad (23)$$

which may be compared with Eq. (17). Here $\frac{1}{2}$ of the neutrons are incident on the plane $x = 0$. $F_{\ell mn}$ and $f_{\ell mn}$ now contain the additional factor

$$-\frac{4}{\pi^2} \frac{1}{mn} (-1)^{\frac{m+n}{2}}$$

(see Fig. 6c).

3. Point Source of Fast Neutrons at the Surface

A point source at the real boundary of the material (Fig. 8c) emits fast neutrons. The probability for one of the neutrons to suffer its first collision with a moderator atom in $dx dy dz$ is then

$$w(xyz) dx dy dz = \frac{\Sigma e^{-\Sigma \sqrt{(x-d)^2 + y^2 + z^2}}}{4\pi \{(x-d)^2 + y^2 + z^2\}} dx dy dz, \quad (24)$$

where d is the extrapolation distance. Here the normalization condition for w (1 neutron leaving the source) is

$$\int_{\text{infinite medium}} w dV = 1.$$

The Fourier coefficient $F_{\ell mn}$ in an expansion as in Eq. (19) is then

$$F_{\ell mn} = \frac{8}{ABC} \frac{\Sigma}{4\pi} \int_d^A \int_{-C/2}^{C/2} \int_{-B/2}^{B/2} \frac{e^{-\Sigma \sqrt{(x-d)^2 + y^2 + z^2}}}{(x-d)^2 + y^2 + z^2} x \cos \mu y \cos \nu z \sin \lambda x dy dz dx, \quad (25)$$

with $\mu = m\pi/B$, etc.

For a large pile ($A, B/2, C/2 \gg \Sigma^{-1}$) we may extend the limits of the integral to infinity, i.e.,

$$d < x < \infty; \quad -\infty < y < \infty; \quad -\infty < z < \infty.$$

The integrand is an even function in y and z ; with the substitution $G = (1/8 \Sigma) ABC \pi F_{\ell mn}$, we therefore obtain

$$-\frac{dG}{d\Sigma_1} = \int_d^\infty \int_0^\infty \int_0^\infty \frac{e^{-\Sigma \sqrt{y^2 + R^2}}}{\sqrt{y^2 + R^2}} \cos \mu y \cos \nu z \sin \lambda x dy dz dx$$

with

$$R^2 = (x-d)^2 + z^2.$$

According to Bateman, (10)

$$-\frac{dG}{d\Sigma} = \int_d^\infty \int_0^\infty K_0[R \sqrt{\Sigma^2 + \mu^2}] \cos \nu z \sin \lambda x \, dz \, dx \quad (*)$$

$$-\frac{dG}{d\Sigma} = \int_d^\infty \int_0^\infty K_0[\sqrt{(x-d)^2 + z^2} \cdot s] \cos \nu z \sin \lambda x \, dz \, dx$$

with $s^2 = \Sigma^2 + \mu^2$

$$-\frac{dG}{d\Sigma} = \int_d^\infty \frac{\pi}{2} (\nu^2 + s^2)^{-\frac{1}{2}} e^{-(x-d)(\nu^2 + s^2)^{\frac{1}{2}}} \sin \lambda x \, dx$$

Introducing $p^2 = \nu^2 + s^2$ and $\xi = x-d$, we obtain

$$-\frac{dG}{d\Sigma} = \frac{\pi}{2p} \int_0^\infty e^{-\xi p} \left\{ \sin \lambda \xi \cos \lambda d + \sin \lambda d \cos \lambda \xi \right\} d \xi$$

$$-\frac{dG}{d\Sigma} = \frac{\pi}{2p} \left\{ \cos \lambda d \cdot \frac{\lambda}{p^2 + \lambda^2} + \sin \lambda d \frac{p}{p^2 + \lambda^2} \right\}$$

$$-\frac{dG}{d\Sigma} = \frac{\pi/2}{B_{\ell mn}^2 + \Sigma^2} \left\{ \frac{\lambda \cos \lambda d}{\sqrt{\Sigma^2 + \mu^2 + \nu^2}} + \sin \lambda d \right\} \quad (26)$$

with

$$B_{\ell mn} = \lambda^2 + \mu^2 + \nu^2$$

To obtain G we notice that for $\Sigma \rightarrow \infty$

$$D(x, y, z, \Sigma) = \frac{\Sigma e^{-\Sigma \sqrt{(x-d)^2 + y^2 + z^2}}}{4\pi[(x-d)^2 + y^2 + z^2]}$$

has the properties of a δ function:

$$\lim_{\Sigma \rightarrow \infty} D(x, y, z, \Sigma) = \begin{cases} \infty & \text{for } x = d, \quad y = z = 0 \\ 0 & \text{otherwise} \end{cases}$$

(*) K_0 is the modified Bessel function of the third kind of order zero.

and

$$\int_{\text{all space}} D(x, y, z, \Sigma) dV = 1$$

for all values of Σ .

For $\Sigma \rightarrow \infty$, we may therefore evaluate Eq. (25) directly:

$$G_{\Sigma=\infty} = 0.$$

Then

$$\begin{aligned} G &= G_{\Sigma=\infty} + \int_{\infty}^{\Sigma} \frac{dG}{d\Sigma} d\Sigma \\ -G &= \frac{\pi\lambda}{2} \cos \lambda d \int_{\infty}^{\Sigma} \frac{1}{B_{lmn}^2 + \Sigma^2} \cdot \frac{d\Sigma}{\sqrt{\mu^2 + \nu^2 + \Sigma^2}} + \frac{\pi}{2} \sin \lambda d \int_{\infty}^{\Sigma} \frac{d\Sigma}{B_{lmn}^2 + \Sigma^2} \\ -G &= \frac{\pi}{2} \cos \lambda d \cdot \frac{1}{2B_{lmn}} \lg \left[\frac{B_{lmn} \sqrt{\Sigma^2 + \mu^2 + \nu^2} + \Sigma \lambda}{B_{lmn} \sqrt{\Sigma^2 + \mu^2 + \nu^2} - \Sigma \lambda} \frac{B_{lmn}^{-\lambda}}{B_{lmn}^{+\lambda}} \right] \\ &\quad + \frac{\pi}{2} \sin \lambda d \cdot \frac{1}{2B_{lmn}} \left[\text{arctg} \frac{\Sigma}{B_{lmn}} - \frac{\pi}{2} \right]. \end{aligned}$$

With the substitutions $S = \frac{\Sigma}{\sqrt{\Sigma^2 + \mu^2 + \nu^2}}$ and $\gamma = \frac{\lambda}{B_{lmn}}$,

we finally obtain

$$\begin{aligned} F_{lmn} &= \frac{2\Sigma}{ABC B_{lmn}} \cos \lambda d \lg \frac{(1-S\gamma)(1+\gamma)}{(1+S\gamma)(1-\gamma)} \\ &\quad + \frac{4\Sigma}{ABC B_{lmn}} \sin \lambda d \left[\frac{\pi}{2} - \text{arctg} \frac{\Sigma}{B_{lmn}} \right]. \end{aligned}$$

For $A = B = C$ [See Eq. (22a)],

$$\begin{aligned} F_{lmn} &= k \frac{1}{2 \sqrt{\ell^2 + m^2 + n^2}} \left\{ \frac{\cos \lambda d}{2} \lg \frac{(1-S\gamma)(1+\gamma)}{(1+S\gamma)(1-\gamma)} \right. \\ &\quad \left. \sin \lambda d \left[\frac{\pi}{2} - \text{arctg} \frac{\Sigma}{B_{lmn}} \right] \right\}. \end{aligned}$$

Figure 6d shows $\phi_{lmn} = \frac{1}{k} F_{lmn}$ and $\Phi_{lmn} = \phi_{lmn} e^{-B^2_{lmn}(\tau_{th} - D\theta)}$

for a graphite pile with $A = B = C = 129$ cm.

REFERENCES

1. K. H. Beckurts, Nuclear Sci. and Eng. 2, 516-522 (1957).
2. J. B. Sykes, J. Nuclear Energy 2, 31-37 (1955).
3. M. F. Crouch, Nuclear Sci. and Eng. 2, 631-639 (1957).
4. G. E. Haynam and M. F. Crouch, Nuclear Sci. and Eng. 2, 626-630 (1957).
5. G. F. von Dardel, Phys. Rev. 94, 1272-1283 (1954).
6. G. F. von Dardel and N. G. Sjöstrand, Progress in Nuclear Energy, Series I, Vol. 2, 183-221 (1958).
7. L. S. Kothari and K. S. Singwi, J. Nuclear Energy 5, 342-356 (1957).
8. S. Glasstone and M. C. Edlund, The Elements of Nuclear Reactor Theory, D. Van Nostrand Co. (1952), pp. 183-4.
9. A. V. Antonov, A. J. Isakoff, J. D. Murin, B. A. Neupocoyev, J. M. Frank, F. L. Shapiro, I. V. Shtranich, Proceedings of the United Nations International Conference on the Peaceful Uses of Atomic Energy, Geneva (1955), Vol. 5, p. 3.
10. Bateman, Tables of Integral Transforms, 1954.

ACKNOWLEDGEMENT

Our thanks are due to Dr. Leo Seren for his encouragement and for his interest in this work.