CHARGED PARTICLE OPTICS OF MAGNETIC SECTOR SPECTROMETERS
WITH \( H = H_0 \propto r^{-1} \)
(z, r, \( \phi \) CYLINDRICAL COORDINATES WITH
THE z-AXIS THE OBJECT-IMAGE LINE)

by

A. H. Jaffey and C. A. Mallmann*
Argonne National Laboratory

J. Suarez-Etchepare
Comision Nacional de la Energia Atomica,
Buenos Aires, Argentina

T. Suter**
Institute of Physics
University of Uppsala, Sweden

*On leave from Comision Nacional de la Energia Atomica, Buenos Aires, Argentina. Part of the work described was performed in Buenos Aires.

**On a fellowship from Concejo Nacional de Investigaciones Cientificas y Tecnicas, on leave from Comision Nacional de la Energia Atomica, Buenos Aires, Argentina.
DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency Thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.
DISCLAIMER

Portions of this document may be illegible in electronic image products. Images are produced from the best available original document.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ABSTRACT</td>
<td>7</td>
</tr>
<tr>
<td>INTRODUCTION</td>
<td>7</td>
</tr>
<tr>
<td>PART A. Analysis Neglecting Fringing-field Effects</td>
<td>11</td>
</tr>
<tr>
<td>#1. Trajectories within a $r^{-1}$ Magnetic Field</td>
<td>13</td>
</tr>
<tr>
<td>Case with $p_\phi = 0$</td>
<td>16</td>
</tr>
<tr>
<td>Case with $p_\phi \neq 0$</td>
<td>20</td>
</tr>
<tr>
<td>#2. Trajectories in the Field when Initial Conditions Are</td>
<td>21</td>
</tr>
<tr>
<td>Set by Point Source on Instrument Axis (in Field-free Region) and by</td>
<td></td>
</tr>
<tr>
<td>Arbitrary Entrance Profile Shape</td>
<td></td>
</tr>
<tr>
<td>#3. Exit Profile Shape Determined by the Imaging Condition</td>
<td>23</td>
</tr>
<tr>
<td>#4. Ghost Peaks</td>
<td>27</td>
</tr>
<tr>
<td>#5. Image of a Finite Source</td>
<td>32</td>
</tr>
<tr>
<td>Rays with $p_\phi = 0$</td>
<td>32</td>
</tr>
<tr>
<td>Rays with $p_\phi \neq 0$</td>
<td>39</td>
</tr>
<tr>
<td>General Case</td>
<td>43</td>
</tr>
<tr>
<td>The $\delta a_0/a$ Approximation</td>
<td>48</td>
</tr>
<tr>
<td>#6. Dispersion and Resolution</td>
<td>50</td>
</tr>
<tr>
<td>Dispersion</td>
<td>52</td>
</tr>
<tr>
<td>Resolution</td>
<td>56</td>
</tr>
<tr>
<td>#7. Transmission of the Spectrometer. Transmission Decrease Due to</td>
<td>66</td>
</tr>
<tr>
<td>Source Size</td>
<td></td>
</tr>
<tr>
<td>Transmission Losses Due to Exit Profile Aperture. $\eta^*_2$</td>
<td>68</td>
</tr>
<tr>
<td>Transmission Losses Including Detector Aperture. $\eta^*_23$</td>
<td>72</td>
</tr>
<tr>
<td>Transmission through Multigap Instruments</td>
<td>74</td>
</tr>
<tr>
<td>Comparison between Iron and Iron-free Toroidal Systems</td>
<td>74</td>
</tr>
</tbody>
</table>
# TABLE OF CONTENTS

PART B. Analysis Considering Fringing-field Effects ........................................... 75

#8. General Characteristics and Effects of the Fringing Fields ........................................... 77

- Structure and Magnitude of Fringe Field ........................................... 77
- Nature of the Trajectory Perturbations ........................................... 81
- \( \psi \)-deflection in the Median Plane ........................................... 82
- Lens Effect of the Fringe Field ........................................... 85

#9. Reduction of Transmission by the Lens Effect ........................................... 87

- Transmission for a Point Source ........................................... 87
- Computation of \( \Delta \beta \)-deflection for Point Source
  - Neglecting Interaction with \( \Delta \psi \)-effect ........................................... 87
- The Lens Property ........................................... 88
- \( \Delta \beta \)-deflection for Point Source and the Equivalent Virtual Line Source ........................................... 92
- Orbit Rotation due to \( \Delta \beta \)-deflection ........................................... 95
- Imaging Property of Point Source with Fringing Field ........................................... 98
- Transmission from Point Source ........................................... 99
- Calculated Transmission Characteristics of Point Sources in Symmetrical Spectrometers ........................................... 102
- Transmission for a Finite Source ........................................... 107
- Transmission Computation from Boundaries of Allowed \( \phi _S \)-regions ........................................... 109
- Computation of \( \eta _2T \) for One Source Point ........................................... 113
- Averaging \( \eta _2T \) over Source ........................................... 118
- Averaging \( \eta _2T \) over \( \psi _S \) ........................................... 124
- Computation of \( \eta _{21T} \) ........................................... 124
- Effect of Fringing Field in Single-sector Instruments ........................................... 130
- Multigap, Tilted-axis, Iron-core Instrument ........................................... 130
- Use of Ring Detector Slit in Multigap Instrument ........................................... 131
- Transmission for \( \psi _S \)-values in the Neighborhood of the Critical \( \psi _S \)-value ........................................... 135

#10. Effect of the \( \Delta \psi \)-deflection on the Lens Property ........................................... 136

- Estimate of \( \theta \) for an \( r^{-1} \)-field ........................................... 138
- Computation of \( \theta \) for Two-dimensional Uniform Fields ........................................... 141
- Example of a Computation of \( \mu _S (eff) \) ........................................... 143
TABLE OF CONTENTS

#11. Effect of the $\Delta \psi$-deflection on the Resolution

- Shift in Entry Angle ($\Delta \mu$) in Uniform Field
- Shift in Entry Point ($\Delta z$) in Uniform Field
- Shift in Entry Point ($\Delta \eta$) in $r^{-1}$-Field
- Shift in Entry Angle ($\Delta \psi$) in $r^{-1}$-Field
- Computation of $\delta \psi_1$
- Computation of $\delta \psi_2$
- Effect of $\Delta \eta$ on the Focusing
- Effect of $\delta \psi_1$, $\delta \psi_2$, and $\Delta \eta$ on the Focusing
- Effect of $\delta \psi_1$, $\delta \psi_2$, and $\Delta \eta$ on the Resolution

#12. Correction of Profiles for the $\Delta \psi$-effect

- Differential Characteristics of the Trajectories
- Angular Magnification
- Circularity Departure of Entrance and Exit Curves
- Magnetic Path Magnification: $ds_f/ds_g$
- Entrance (or Exit) Curve Correction for Axial Source (or Focus) Displacement
- Axial Magnification: $dz_f/dz_g$
- Radial Magnification: $dr_f/dr_1$
- Exit-curve Corrections from Relative Field Measurements
- Entrance-curve Corrections from Relative Field Measurements
- Dispersion: $dz_f/d(\log A_0)$
- Source Dispersion (Effect of Source Movement)
- Correction of a Constructed Instrument

#13. Median-plane Trajectories in the Fringing Field

#14. Graphical and Analytical Calculation of Exit Curves

- Symmetrical Instruments

#15. Calculation of Profiles including Fringing-field Effects
# TABLE OF CONTENTS

| #16. Considerations in the Choice of Design Parameters | 206 |
| Ghost Peaks | 206 |
| Resolution | 206 |
| Luminosity | 207 |
| Detector-size Considerations | 208 |

ACKNOWLEDGEMENTS | 213

REFERENCES | 213

APPENDIX I: Derivation of Bessel Function Expansions | 215

APPENDIX II: Tables of Functions for Idealized Field Spectrometers | 217

Table A-II.1. Trajectory Function $U(K,\psi)$ | 219
Table A-II.2. Trajectory Function $V(K,\psi)$ | 229
Table A-II.3. Magnetic Field Profile | 235
Table A-II.4. Orbit Rotation $F_1(K,\psi_s)$ | 241
Table A-II.5. Dispersion Function $G(K,\psi_s)$ | 245
Table A-II.6. Dispersion Function $\frac{\partial U(K,\psi)}{\partial K}$ | 249

APPENDIX III: Evaluation of $\psi_f^1$ | 257

APPENDIX IV: Table of Functions for Spectrometers with Fringing Fields | 261

Table IV. Table of Functions for Spectrometers with Fringing Fields | 263

APPENDIX V: The Dispersion Formula | 269

APPENDIX VI: Published Data on Constructed Spectrometers | 271
CHARGED PARTICLE OPTICS OF MAGNETIC SECTOR SPECTROMETERS WITH $H = H_\phi \propto r^{-1}$

$(z, r, \phi$ CYLINDRICAL COORDINATES

WITH THE $z$-AXIS THE OBJECT-IMAGE LINE)

by

A. H. Jaffey, C. A. Mallmann, J. Suarez-Etchepare and T. Suter

ABSTRACT

The charged particle optics of symmetric and asymmetric spectrometers is developed, taking into account the effects of fringing fields. Formulae are given for: the image of point and extended sources, and their resolution and transmission. The suppression of ghost peaks arising from multiloop trajectories is studied. Methods and tables for the calculation of instruments are given. A comparison is made between various instruments already constructed, and some general considerations concerning the choice of design parameters are presented.

The correction (for fringing field effects) of sector profiles of instruments already constructed is analyzed, and formulae are given which permit the calculation of modifications of the profiles necessary to insure good focusing with high transmission.

Tables include: general trajectory functions; profile curves; and dispersion and transmission functions for symmetrical instruments.

INTRODUCTION

Since the first proposal by Kofoed-Hansen, Lindhard and Nielsen\(^{(1)}\) of the magnetic sector spectrometer with $H = H_\phi \propto r^{-1}$,\(^*\) there have been proposals for the use of this focusing principle for many different purposes, and about 15 such instruments have been built.

\(*r = \text{distance of field point to the object-image (source-detector) axis, which is labelled the } z\text{-axis.}\)
An important advantage of this kind of instrument is the high transmission achievable, because (i) focusing occurs over a wide range of emission angles, and (ii) many identical instruments can be used in parallel around a toroid. At the same time, the obtainable resolutions are good, because the imaging properties and the dispersion are satisfactory for each one of the instruments. At high transmissions, in fact, the resolution is ultimately limited by the effects of the fringing fields.

The main features of the charged particle optics of symmetric* instruments of this kind have already been discussed in the paper by Kofoed-Hansen, et al. (1) It is the purpose of this report to present the charged particle optics of symmetrical and asymmetrical instruments, including fringing field effects. Part of this work has already been published by L. Lagatta, C. Molina-Vedia, J. J. Peyre and some of us (C.A.M. and J.S.E.). (2-5) In this report, we will not discuss multigap spectrometers with other types of magnetic fields. References to such spectrometers can be found in the review article on beta-ray spectrometers by M. Mladjenovic. (18)

Single-gap spectrometers of this design have been built by Kofoed-Hansen et al. (1) [later used by Huus et al. (6)], by Shea et al. (7) and by Bernstein and Graetzer. (8) A single-gap spectrograph** has been built by J. O'Connell. (9)

Multigap iron spectrometers have been built by Nielsen and Kofoed-Hansen, (10) Nielsen (11) and Mallmann et al. (5, 12) The instrument described in (5) is the first half of a multigap electron-electron coincidence spectrometer (13) The second half will be a multigap tilted axis (see Sec. #9) iron spectrometer

Multigap iron-free spectrometers have been built by Vladimirskii et al., (14) Tretiakov et al., (15) Freedman et al., (16, 17) Burgov, (18) and Bartis. (19) The multigap iron-free spectrometers (14) have been used in the measurement of the electron-neutrino angular correlation in the beta decay of the free neutron. (20) Two such instruments are used in series as the beta spectrometer and the third one as the recoil-proton spectrometer.

Freedman's spectrometer (16, 17) is the first half of an iron-free toroidal electron-electron coincidence spectrometer. It will allow, in addition to the usual work of beta spectrometry, the measurement of $\beta$-e angular

*Reflection symmetry with respect to a plane perpendicular to the z-axis, midway between object and image.

**The profile is a particularly simple one from the construction point of view, since both entrance and exit profiles lie on a single line parallel to the z-axis. Focusing is adequate for a limited range of emission angles. See also Fig. (3.3).
correlation, \( \alpha - e^- \) coincidences, and \( (\beta - e - e), (\beta - e - \gamma), (e - e - \alpha), (\beta - e - X) \) triple coincidences. Both instruments may also be used in series as a single spectrometer in order to reduce the background due to scattered electrons and to improve the resolution.

A summary of the published data on the instruments constructed to date will be found in Appendix VI.

The same focusing principle has been suggested for use in a pair-spectrometer,\(^{(9)}\) but such an instrument has not yet been constructed.

This report has been divided into two parts. Part A, which includes Sections #1-7, gives the charged particle optics neglecting the fringing-field effects. It is assumed that the magnetic field drops abruptly to zero at the boundaries of the polefaces. Part B, which includes Sections #8-15, shows how the results of Part A require modification in order to take into account the fringing-field effects. In Section #16, we provide some general considerations on the choice of optimum design parameters for instruments. Some mathematical formulae and tables are given in the appendices.
PART A

ANALYSIS NEGLECTING FRINGE-FIELD EFFECTS
#1. TRAJECTORIES WITHIN A $r^{-1}$ MAGNETIC FIELD

In this type of charged particle spectrometer, the components of the magnetic field $\mathbf{H}$ referred to a cylindrical coordinate system $(z; r; \phi)$ are

$$H_z = 0; \quad H_r = 0; \quad H_\phi = A_0/r \quad ,$$

(1.1)

where $A_0$ is a constant. The field exists only inside a torus whose symmetry axis is the $z$-axis. Outside of the torus, the field is assumed to be zero. This is a simplified model in which the fringing fields are neglected; these fields will be analyzed in Part B. For the present analysis, therefore, we shall assume that a discontinuity in the magnetic field occurs at the boundaries. Before undertaking to examine the transition at the boundaries, we shall consider the trajectories in a field given by Eq. (1.1), a field in which no boundaries are present.

The magnetic vector potential $\mathbf{A}$ satisfies the equation

$$\nabla \times \mathbf{A} = \mathbf{H} \quad .$$

(1.2)

Expressing this equation in cylindrical coordinates, and making the reasonable assumption that $\mathbf{A}$ is independent* of $z$, we deduce from (1.1) and (1.2) that $A_z = A_z(r); \quad A_r = A_\phi = 0$, and finally that

$$A_z(r) = A_0 \ln (b/r) \quad ,$$

(1.3)

where $b$ is an arbitrary constant.

H. O. W. Richardson(21) studied in detail the equations for the trajectories of charged particles in this magnetic field. In the following we give those of his results which we need in our case.

The relativistic Lagrangian for a particle with rest mass $m_0$ and charge $e$, in electromagnetic units, moving with velocity $\mathbf{v}$ in a magnetic field with magnetic vector potential $\mathbf{A}$, is

$$L = m_0 c^2 \left[ 1 - \sqrt{1 - \frac{v^2}{c^2}} \right] + e (\mathbf{v} \cdot \mathbf{A}) \quad ,$$

(1.4)

where $c$ is the velocity of light. Here $v^2$ is a constant of the motion, where

$$v^2 = r^2 + \dot{z}^2 + r^2 \phi^2 \quad ,$$

(1.5)

*The assumption of independence is already implicitly contained in (1.1), since these conditions assume that no end effects exist, which implies that the field essentially stretches infinitely in the $z$-direction.
because the Lorentz force is perpendicular to the velocity (constant energy). Replacing (1.3) in (1.4) gives

\[ L = m_0 c^2 \left[ 1 - \sqrt{1 - \frac{v^2}{c^2}} \right] - eA_0 \frac{\dot{z}}{b} \ln \frac{r}{a}. \]

(1.6)

From Lagrange's equations, one obtains

\[ \dot{z} = \frac{A_0 e}{m} \ln \frac{r}{a}. \]

(1.7)

and

\[ p_\phi = \text{constant} = mr^2 \phi, \]

(1.8)

with* \( r = a \) when \( \dot{z} = 0 \) and \( m = m_0 \sqrt{1 - (v/c)^2} \).

Fig. 1.1

Putting (1.7) and (1.8) into (1.5) and solving for \( \dot{r} \), we have

\[ \dot{r} = v \sqrt{1 - \left[ \frac{a_0}{r} \right]^2 - \left[ \frac{\ell n (r/a)}{K} \right]^2}, \]

(1.9)

where

\[ K = - \frac{mv}{A_0 e} = - \frac{p}{A_0 e} = \frac{(B \rho C)}{A_0} = \frac{\rho C}{r}, \]

(1.10)

and

\[ a_0 = p_\phi / p, \]

(1.11)

with \( \rho_C \) the radius of curvature of the trajectory.

*In Fig. 1.1, \( r = a \) is the ordinate of the point on the trajectory where the tangent is vertical.
It is desirable at this point to specify the conventions to be used relative to the algebraic signs of $A_0$ and $K$. From the usual right-hand rule, the field $H$ in Eq. (1.1) is taken as positive along the positive $\phi$-direction. Thus, in Fig. (1.1), with the positive $z$-axis going to the right, while $r$ goes up, positive $\phi$-motion comes out of the paper above the $z$-axis. Deflection of particles as shown in Fig. (1.1) will occur for positive charges only if $H$ is along the negative $\phi$-direction, i.e., if $A_0$ is negative; for a negative charge, of course, the same deflection would require positive $A_0$. The quantity $A_0e$ is thus inherently a negative quantity for an instrument designed to give deflections as shown in the figure. Further, $K$ is correspondingly always positive, both for positive and negative charges, the adjustment being made in the sign of $A_0$ if the charge of the particle to be deflected is changed. As to the conventions on $(B\rho C)$, we always take $p$ and $\rho C$ as positive quantities. Since $B$ has the same sign as $A_0$ and these are opposite in sign to that of the charge, $p/e = -(B\rho C)$.

Let $v_{rz}$ be the component of velocity in the $z$-$r$ plane. Thus, from Eq. (1.5),

$$v^2 = v_{rz}^2 + r^2 \phi^2 \quad (1.12a)$$

We are primarily interested in the motion in the $z$-$r$ plane. For this purpose, it is convenient to define the angle $\psi$ by the relation

$$\cos \psi = \frac{z}{v_{rz}} \quad (1.12b)$$

Evidently $\psi$ is the angle formed by the projection of the velocity vector $v$ on the $z$-$r$ plane (i.e., the vector $v_{rz}$) with the positive $z$-axis. We have, then, from (1.9),

$$\dot{r} = v \sin \psi \sqrt{1 - \frac{a_0^2}{r^2}} \quad (1.13a)$$

and

$$\dot{z} = v \cos \psi \sqrt{1 - \frac{a_0^2}{r^2}} \quad (1.13b)$$

From (1.7), it follows that

$$r = K \cos \psi \sqrt{1 - \frac{a_0^2}{r^2}} \quad (1.14)$$

We expand the radical and, assuming that $a_0^2$ is small,* set $r \approx ae^{-K\cos \psi}$ wherever it appears in a term multiplying $a_0^2$. Then, with good approximation,

*Conditions for the validity of this approximation are discussed at the end of Section #5 (p. 48).
From (1.13a), (1.13b), and (1.15a), we have, to the same approximation as in (1.15a),

\[ z = \int \frac{\dot{z}}{t} \, dt + z_m \]

\[ = aK \int \cos^e \, e^{-K \cos^e} \left[ 1 - \frac{1}{2} \left( \frac{a_0}{a} \right)^2 \right] e^{2K \cos^e} \cos^e \, dt + z_m . \]

Similarly, from (1.8), (1.13a) and (1.15a),

\[ \phi = a_0 \eta \int \frac{\, d \tau}{\tau^2} + \phi_m \]

\[ = \frac{a_0 K}{a} \int e^{K \cos^e} \left[ 1 - \frac{3}{2} \left( \frac{a_0}{a} \right)^2 \right] e^{2K \cos^e} \, d\psi + \phi_m . \]

With some modifications, these results correspond to equations found by H. O. W. Richardson (see Ref. 21, page 794). Richardson's results are identical with those presented here for trajectories with \( p_{\phi} = 0 \); however, for cases in which \( p_{\phi} \neq 0 \), evaluation of Richardson's equations requires numerical integration. By using the \( \cos^e \) definition (1.12b), rather than Richardson's \( z/v \), it was possible to develop here approximate analytical expressions which are not only simpler to evaluate, but also make possible an estimate of the error involved in the use of approximations.

**Case with \( p_{\phi} = 0 \).** For particles with \( p_{\phi} = 0 \), from (1.11) we deduce that \( a_0 = 0 \) and then

\[ z = aK \int \cos^e \, e^{-K \cos^e} \, d\psi + z_m ; \]

\[ r = ae^{-K \cos^e} ; \]

and

\[ \phi = \phi_m \text{ (plane trajectory)} . \]

It may be noted, since \( \phi = 0 \), that \( \psi \) is now the angle formed by the velocity vector \( \mathbf{v} \) with the positive \( z \)-axis.

Following Kofoed-Hansen et al.,(1) these equations are transformed into

\[ z = aK U(K, \psi) + z_m \; ; \]

\[ r = ae^{-K \cos^e} \; ; \]

\[ \phi = \phi_m \text{ (plane trajectory)} . \]
and

\[ \phi = \phi_m \quad , \quad (1.17c) \]

upon taking \( z = z_m \) for \( \psi = \pi \) and designating \( U(K, \psi) \) as the following function:

\[ U(K, \psi) = \int_\pi^\psi \cos \psi e^{-K\cos \psi} \, d\psi \quad . \quad (1.18) \]

The \( U \) function may be obtained by a development in a series of Bessel functions:

\[ U(K, \psi) = -iJ_1(iK)(\pi-\psi) + \sum_{\ell=0}^{\infty} \frac{1}{\ell} (i)^{\ell-1} \left[ J_{\ell-1}(iK) - J_{\ell+1}(iK) \right] \sin \ell \psi \quad . \quad (1.19) \]

A table of \( U(K, \psi) \) is given in Table II.1 of Appendix II.

It is evident that the trajectory is periodic in \( r \), but with a net drift of the orbits in the \( z \)-direction [see Fig. (1.1)]. Indeed, we have

\[ z(\psi + 2\pi) - z(\psi) = 2\pi aK J_1(iK) \quad (1.20) \]

as the period, in which \( \cos \psi \)-values (and hence the \( r \)-values) repeat themselves; thus, for points on the \( z \)-axis separated by the period, the \( r \)-values are the same:

\[ r(\psi + 2\pi) - r(\psi) = 0 \quad . \quad (1.21) \]

Because of the periodicity in \( r \), it is possible to restrict detailed considerations to the region \( 0 \leq \psi \leq 2\pi \). The trajectory for other \( \psi \)-values simply repeats the motion within this region; i.e., \( r \)-values repeat while \( z \)-values are displaced by the amount in (1.20) every time \( \psi \) goes through a change of \( 2\pi \). We have, then, as the equations of the trajectories,

\[ z = aK \left[ U(K, \psi) + 2\pi n J_1(iK) \right] + z_m \quad (1.22a) \]

and

\[ r = ae^{-K\cos \psi} \quad , \quad (1.22b) \]

with

\[ 0 \leq \psi \leq 2\pi \quad . \quad (1.23) \]

\[ n = \ldots, -2, -1, 0, 1, 2, \ldots \]

*See Appendix I for proof.
The particular loop of the trajectory is described by the \( n \)-value, \( \psi \) giving the position in the loop.

From the definition of \( U(K, \psi) \) in Eq. (1.18), it is evident that \( U(K, \pi) = 0 \). Thus, in the part of the trajectory for which \( n = 0 \), it is found that \( z = z_{m} \) when \( \psi = \pi \).

It can be shown that the symmetry lines of the trajectories are given by

\[
z = z_{m} \quad \text{at } \psi = \pi \quad (1.24a)
\]

and

\[
z = z_{m} = z_{m} - aK \ J_{1}(iK) \quad \text{at } \psi = 0 \quad (1.24b)
\]

The first lies half-way between two loops; the second bisects a loop (see Figs. 1.1 and 1.2). From Eq. (1.24b), it is evident that Eq. (1.22a) may also be written as

\[
z = aK \ [U(K, \psi) + 2\pi n' \ J_{1}(iK)] + z_{m}' \quad (1.25)
\]

where \( n' = n + \frac{1}{2} \) and has only half-integral values. We may write Eq. (1.22a) and Eq. (1.25) as one equation:

\[
z = aK \ [U(K, \psi) + 2\pi n \ J_{1}(iK)] + Z_{M} \quad , \quad (1.26)
\]

where

\[
Z_{M} = z_{m} \ \text{if } n \text{ is an integer} \\
Z_{M} = z_{m}' \ \text{if } n \text{ is half-integral} .
\]

The characteristic points of these trajectories are given in (1.27) below, and may be observed in Figs. (1.1) and (1.2).

<table>
<thead>
<tr>
<th>( \psi )</th>
<th>( r )</th>
<th>( z )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( r_{\text{min}} = ae^{-K} )</td>
<td>( aK[-\pi iJ_{1}(iK) + 2\pi n iJ_{1}(iK)] + Z_{M} = 2\pi (n - \frac{1}{2}) aK iJ_{1}(iK) + Z_{M} )</td>
</tr>
<tr>
<td>( \frac{\pi}{2} )</td>
<td>( a )</td>
<td>( aK[U(K; \pi/2) + 2\pi n iJ_{1}(iK)] + Z_{M} )</td>
</tr>
<tr>
<td>( \pi )</td>
<td>( r_{\text{max}} = ae^{K} )</td>
<td>( aK[2\pi n iJ_{1}(iK)] + Z_{M} )</td>
</tr>
<tr>
<td>( 3\pi/2 )</td>
<td>( a )</td>
<td>( aK[U(K, 3\pi/2) + 2\pi n iJ_{1}(iK)] + Z_{M} )</td>
</tr>
<tr>
<td>( 2\pi )</td>
<td>( r_{\text{min}} = ae^{-K} )</td>
<td>( aK[\pi iJ_{1}(iK) + 2\pi n iJ_{1}(iK)] + Z_{M} = 2\pi (n + \frac{1}{2}) aK iJ_{1}(iK) + Z_{M} )</td>
</tr>
</tbody>
</table>
From Equations (1.22b) and (1.26), one may deduce:

I. For different a-values, keeping K and Z_M constant, one obtains a family of similar trajectories differing only in the scale factor "a." Thus, (z-Z_M) is proportional to "a," as is r, as well as the distance of any point on the trajectory from (Z_M,0). Hence for any two trajectories in this family, corresponding points (z_1,r_1) and (z_2,r_2) are related by the equation:

\[
\frac{r_1}{r_2} = \frac{z_1 - Z_M}{z_2 - Z_M} = \frac{a_1}{a_2}
\]

(1.28)

We will call this family symmetric because, for a fixed \( \phi \)-value, the loops of the trajectories have \( z = Z_M \) as a symmetry axis.

II. Varying \( Z_M \) by the amount \( \Delta Z_M \) and leaving K and "a" constant, the trajectory is displaced parallel to the z-axis by \( \Delta Z_M \). Thus, the trajectory point \( (z_i,r_i) \) goes into \( (z_i + \Delta Z_M,r_i) \).

III. Varying K and leaving "a" and \( Z_M \) constant, the form of the trajectory varies.

The most general monoparametric family of trajectories of similar form (K = constant) is given by

\[
z = aK \left[U(K,\psi) + 2\pi n iJ_1(iK)\right] + Z_M(a)
\]

(1.29a)

\[
r = ae^{-K\cos \psi}
\]

(1.29b)

where \( Z_M \) is a function of "a." This family we call asymmetric, because it does not have the symmetry properties of the family which has K and \( Z_M \) constant while varying "a."
The families with constant $K$ are the most important ones, because they correspond to the case in which a given magnetic field intensity ($A_0$) acts on particles with equal momenta but with different initial conditions.

Case with $p_\phi \neq 0$. The case of particles with $p_\phi \neq 0$ is studied for small $\phi$-components, so we set $a_0 = \delta a_0$. From Eqs. (1.15a) and (1.15b), it is evident that in the first approximation, i.e., to terms of the order of $\delta a_0$, the $r$ and $z$ equations remain the same as for the case where $p_\phi = 0$ [Eqs. (1.16a) and (1.16b)].* Higher-order approximations appear in Eqs. (1.15a) and (1.15b) and may be evaluated through expansion in Bessel function series by the method of Appendix I, or evaluation may be carried out through numerical integration in an electronic computer. From (1.15c), it may be seen that, to the same approximation,** $\phi$ is

$$\phi = K V(K, \psi) \frac{\delta a_0}{a} + \phi_m,$$

where $^\dagger$

$$V(K, \psi) = \int_\pi^{\psi} e^{K \cos \psi} \ d\psi = J_0(iK)(\psi - \pi) + \sum_{\ell=1}^{\infty} \frac{2}{\ell} (-i)^\ell J_\ell(iK) \sin \ell \psi$$

(1.31)

A table of $V(K, \psi)$ is given in Table II.2. of Appendix II.

From these relations, it may be seen that $\phi$ increases a uniform amount for every period of $\psi$, e.g., with each loop. Thus

$$\phi(\psi + 2\pi) - \phi(\psi) = 2\pi K J_0(iK) \frac{\delta a_0}{a}$$

(1.32)

We shall discuss below (Section #7) the influence of this rotation on the transmission of the instruments.

---

*Conditions for validity of this approximation are discussed at the end of Section #5 (p. 48).

**This is also the second approximation, since there is no term in $(\delta a_0)^2$.

$^\dagger$ For derivation, see Appendix I.
#2. TRAJECTORIES IN THE FIELD WHEN INITIAL CONDITIONS ARE
SET BY POINT SOURCE ON INSTRUMENT AXIS (IN FIELD-FREE
REGION) AND BY ARBITRARY ENTRANCE PROFILE SHAPE

Considering the planes \( \phi = \) constant, the arbitrary sections of the
torus whose segments form the pole pieces of the electromagnet deter­
mine the region in these planes in which
the magnetic field varies as \( A_0/r \) and the
region where the field is zero.

Every particle emitted from the
point source, located at \((z,r) = (z_s,0)\),
describes a straight trajectory until it
enters the magnetic field. Let the curve
limiting the section of the torus (Fig. 2.1)
be given as a function of the parameter
\( \psi_s \) by the equations

\[
\begin{align*}
  r_e &= z_s f_s(\psi_s) \\
  z_e &= z_s [1 + f_s(\psi_s) \cot \psi_s]
\end{align*}
\]  

(2.1a) and

(2.1b)

Fig. 2.1

Then, at the torus boundary, the point of incidence into the magnetic
field is \((z_e,r_e)\). Since \( p_\phi = 0 \) for such particles, the trajectory continues to
lie in the original plane and is given by (1.29), the parameters being deter­
mined by the point and angle at which the trajectory enters the magnetic
field.

Because of the continuity of the velocity across the boundary be­
tween the field-free region and the magnetic field, the straight lines from
\((z_s,0)\) to \((z_e,r_e)\) are tangent to the trajectories within the magnetic field
at the points \((z_e,r_e)\). Thus, the two conditions arising from equating the
point and angle of entry (i.e., from equality of position and derivative) at
the boundary determine the boundary conditions for the trajectory of the
particles inside the field, i.e., they determine the constants of integration,
\( Z_M \) and "a." As a matter of fact, the following relations must hold for
them:

\[
\begin{align*}
  z_s f_s(\psi_s) &= a e^{-K \cos \psi_s} \\
  z_s [1 + f_s(\psi_s) \cot \psi_s] &= a K [U(K,\psi_s) + 2 \pi n_s J_1(iK)] + Z_M
\end{align*}
\]

(2.2a) and

(2.2b)

From (2.2a), it is evident that "a" depends upon the angle of entry, \( \psi_s \); thus
\[ a(\psi_s) = z_s f_s(\psi_s) e^{K \cos \psi_s} \quad (2.3) \]

Similarly, from (2.2b) and (2.3), one deduces that
\[ Z_M(\psi_s) = z_s \left\{ 1 + f_s(\psi_s) \cot \psi_s - f_s(\psi_s) e^{K \cos \psi_s} K \left[ U(K, \psi_s) + 2\pi n_s i J_1(iK) \right] \right\}. \quad (2.4) \]

Inserting (2.3) and (2.4) in (1.29), one obtains the equations of the trajectories:
\[ r = z_s f_s(\psi_s) e^{-K(\cos \psi - \cos \psi_s)} \quad (2.5a) \]
and
\[ z = z_s \left\{ 1 + f_s(\psi_s) \cot \psi_s + K f_s(\psi_s) e^{K \cos \psi_s} \left[ U(K, \psi) - U(K, \psi_s) + 2\pi (n - n_s) i J_1(iK) \right] \right\}. \quad (2.5b) \]

The Eqs. (2.5) give, for constant K and for different values of \( \psi_s \) (different emission angles), a family of trajectories which, in general, is asymmetric, since \( Z_M \) is not constant. The particles will describe these trajectories until they emerge from the magnetic field; after such exit, the trajectories are straight and are tangent at the emergent point to the curves described in Eqs. (2.5).

If all the emergent rays intercept the symmetry axis at the same point \((z_f, 0)\), this point is called the image of \((z_g, 0)\). To achieve this it is necessary that the surface bounding the torus in the exit region obey the conditions given in Section \#3.
#3. EXIT PROFILE SHAPE DETERMINED BY THE IMAGING CONDITION

For focusing to occur, all the emergent rays must pass through the focus \((z_f,0)\); this requirement results in a particular shape for the exit boundary of the magnetic field. We consider the straight lines traced from \((z_f,0)\) to the trajectories of Eqs. (2.5), in such a manner that the straight lines are tangent to the trajectories at the points of intersection. The geometrical condition on the magnetic field exit boundary is that it be the envelope of these points of intersection. As at the entrance curve, the continuity of the velocity (and hence the derivative of the curve) across the boundary insures that the trajectories within the field go into the tangent lines when the field becomes zero. Note that the boundary curve of the magnetic field in the planes \(\phi = \text{constant}\) is not defined solely by the values of \((z_e,r_e)\), but also by the value of \(z_f\), as well as \(K\) and \(z_s\).

The angle between the emergent trajectories and the positive \(z\)-axis is called \(\psi_f\) (see Fig. 3.1), and, as in Eq. (2.1), dimensions are related to \(z_f\) by representing the exit boundary curve as a function of the parameter \(\psi_f\) by means of the relations

\[
\begin{align*}
  r_0 &= z_f f_f(\psi_f) \quad (3.1a) \\
  z_0 &= z_f [1 + f_f(\psi_f) \cot \psi_f] \quad (3.1b)
\end{align*}
\]

Since the trajectory curve and the emergent boundary curve intersect at \((z_0,r_0)\), the two equations (2.5) and (3.1) are equal at this point. Thus,

\[
z_f f_f(\psi_f) = z_s f_s(\psi_s) e^{-K(cos\psi_f-cos\psi_s)}
\]

(3.2a)

and

\[
z_f [1 + f_f(\psi_f) \cot \psi_f] = z_s \left\{ 1 + f_s(\psi_s) \cot \psi_s \right. \\
+ K f_s(\psi_s) e^{K \cos \psi_s \left[ U(K, \psi_f) - U(K, \psi_s) + 2\pi (n_f - n_s) i J_1(iK) \right]}. \quad (3.2b)
\]

From (3.2) one deduces the following implicit relation between \(\psi_s\) and \(\psi_f\):

\[
cot \psi_f e^{-K \cos \psi_f} - K U(K, \psi_f) - K 2\pi n_f i J_1(iK) = \cot \psi_s e^{-K \cos \psi_s} - K U(K, \psi_s) \\
- K 2\pi n_s i J_1(iK) + \frac{z_s - z_f}{z_s} e^{-K \cos \psi_s} f_s(\psi_s). \quad (3.3)
\]
One can give the relation in an explicit form, the details depending upon
the form of the expression for $f_s$. Thus,

$$\psi_f = \psi_f(\psi_s)$$

(3.4)

or we may calculate the relation numerically.

By either method, one uses Eq. (3.2) to obtain $(z_0, r_0)$, the boundary
curve for the field at the emergent side. The remaining boundaries of the
section in which the field is given by Eq. (1.1) are provided by the trajec-
tories for which $\psi_s$ is a maximum and $\psi_s$ is a minimum.

In Fig. (3.2) we give the emergence boundary curves for the case
in which $f_s = \sin\psi_s$; $z_s = -z_f = 1$; and $n_s = n_f = 0$ for $\pi/2 \leq \psi_s < \pi$, and
$K$ takes on the values 0.40; 0.55; 0.60; and 1.00.

We now consider the sym-
metric case in which $Z_M(\psi_s) = b_1 =$ constant. It will be recalled from
Section #1 that for this family of trajectories the maximum $r$-value
in the magnetic field occurs at the
same $z$-value for each member of
the family, namely, at $z = b_1$; hence,
the symmetry. For a given $z_s$, $n_s$, $b_1$ and $K$, the entrance curve is
given (from Eq. 2.2b) by

*As we shall discuss in Section #8, the fringing field enters in two
ways: (i) the magnetic field extends beyond the profile into the nomi-
inally "field-free" region, and (ii) the magnetic field inside the
profile region is decreased below the theoretical value in Eq. (1.1).
The penetration of the fringing effect past the profile into the in-
terior region cannot be avoided, since the sector property of the
spectrometer is essential. However, the penetration past the two
boundaries determined by $\psi_s$(max) and $\psi_s$(min) can be minimized in
effect by placing the real boundaries (of the polefaces) at some dis-
tance beyond the trajectories discussed. At the two bounding tra-
jectories, the fringing effect is then negligible.

**The instrument is a zero-loop one ($n_f - n_s = 0$) in which the entrance
boundary $f_s(\psi_s)$ has been chosen to be a circle with $(z_s, 0)$ as center
and with radius $z_s$. Incident particles enter the magnetic field per-
pendicular to the curve described by $f_s(\psi_s)$, since their paths are
along radii of the circle. It may be noted that the points (A, B, C, D)
and the curve connecting them are not discussed in this section, but
will be discussed in Section #4.
If the image point is placed symmetrically relative to \( z = b_1 \), i.e.,
\[-(z'_f - b_1) = z_s - b_1, \]
and if we consider symmetrical portions of the trajectories \( n_s = -n_f \), then the exit curve is symmetric relative to \( f_s(\psi) \) given in Eq. (3.5). Thus, the exit curve given by \( f_s(\psi) \) is the mirror image of the entrance curve given by Eq. (3.5), the reflection being taken in the plane \( z = b_1 \).

In Fig. (3.3) is illustrated the variation of the entrance (and emergence) curves with the parameter \( K \), for the symmetric case with \( b_1 = 0 \) and \( n_s = 0 \). Because of the symmetry, both entrance and emergence curves have the same form (except for the reflection), so only one is given here. In the figure are shown circles with \( (z_s, 0) \) as center to simplify a comparison of the calculated entrance curves with the corresponding circles. The curves are drawn in terms of \( z_s \) as a unit, the point \( (z_s, 0) \) being shown at 1.00. A table of the coordinates \( (z_e, r_e) \) is given as a function of \( (K, \psi_s) \) in Table II.3 of Appendix II.

It may be noted that the function \( f_s(\psi) \) becomes infinite for a value of \( \psi \) which depends upon \( K \) and \( n_s \). Thus, e.g., consider the case \( n_s = 0 \): Since \( U(K, \psi) > 0 \) for \( 0 < \psi < \pi \), the quantity \( [K e^{K \cos \psi_s} U(K, \psi_s)] \) is positive throughout the \( \psi_s \)-range. Then, since the range of \( \cot \psi_s \) is from \( \infty \) to 0 for \( \psi \) varying from 0 to \( \pi \), there will be a value of \( \psi_s \) for which the denominator is zero. With \( n_s \neq 0 \), the term in \( iJ_1(iK) \) is included. For \( K < 1, iJ_1(iK) > -0.7K \); hence \( [U(K, \psi_s) + 2\pi n_s iJ_1] \) may be negative. In this case, the zero in the denominator occurs in the range \( \frac{1}{2} \pi < \psi_s < \pi \).
For $b_1 = 0$, the coordinates $(z_e, r_e)$ have the value $(0, 0)$ for $\psi_S = \pi$. As $\psi_S$ is decreased from $\pi$ towards zero, the profile starts at the origin and asymptotically becomes tangent to a ray at a critical $\psi_S$-value; at this angle, $(z_e, r_e)$ becomes $(\infty, \infty)$. The equation has no physical solution for values of $\psi_S$ smaller than the critical value, since the $(z_e, r_e)$ coordinates become negative. It is not practical to use the profile in the neighborhood of the critical region, even though the $\psi_S$-values involved exceed the critical value, for reasons discussed in Section #9.* Curves illustrating the asymptotic structure of the profiles are shown in Fig. (3.4), and, as in Fig. (3.3), are given using $z_S$ as the unit of length [see also Table II.3].

*See p. 135.
In the design of an instrument, $z_s$, $z_f$ and the entrance curve $f_s(\psi_s)$ are chosen as geometrical characteristics. Then, choosing the trajectory characteristics $(K, n_s, n_f)$, the exit curve may be determined by the method of Section #3. As $K$ is varied, with a given $f_s$, a family of exit curves are formed, as in Fig. (3.2). A difficulty arises in that with all the design factors fixed, including $K$, it is possible for trajectories characterized by other $(K, n_s, n_f)$ values to pass through $(z_f, 0)$; thus, by virtue of multiloop trajectories, particles of other than the desired momentum may arrive at the detector.

If, with the same values of $(f_s, z_s, z_f)$ and with a particular $K_1$, the exit boundary curve for a different value of $n_s$ and/or $n_f$ is determined, this curve is not a member of the above family. This curve cuts across the previous family, as illustrated [see Fig. (3.2)] by the curve passing through the points (A, B, C, D). For the same $(n_s, n_f)$, another value, $K_2$ yields another such exit curve, displaced relative to the curve ABCD, but also intersecting the original family of curves. A series of $K'$-values yield exit curves which sweep across the family of original emergence curves.

We consider first the family of exit curves (varying parameter $K'$) which intersect the original family of exit curves (varying parameter $K$). The physical significance of the $K'$-family is the following:

An instrument is designed with particular $(z_s, z_f, f_s, n_s, n_f, K)$, so that at a particular $A_0$, it focuses (for every $\psi_s$) all particles of momentum $mv = -KA_0e$. For example, in Fig. (3.2), we choose the exit curve suitable for $K = 0.57$, with $(n_s, n_f) = (0, 0)$; for a given $A_0$, all particles of momentum $p$ pass through $(z_f, 0)$ for any $\psi_s$. In the same magnetic field (i.e., $A_0$), particles of momentum $p_i$ follow different trajectories, which are characterized by different values $(n'_s, n'_f, K'_1)$. To focus these at $(z_f, 0)$ for all $\psi_s$, another exit curve is required than the one shown in Fig. (3.2) for $K = 0.57$; e.g., the required exit curve might be ABCD. Since the exit curve is not ABCD, no focusing occurs for all $\psi_s$. However, for one $\psi_s$-value, the corresponding trajectory passes through the point B, so that the trajectory will pass through $(z_f, 0)$; this is the only trajectory with $(p'_i, n'_s, n'_f)$ to reach the focal point. Thus, for momentum $p$, particles reach $(z_f, 0)$ for all $\psi_s$, but this is true for only one $\psi_s$ for $p_i$. If the source emitted only the two momenta $p$ and $p_i$, then the interfering intensity would be very small indeed.

Using the same $(n'_s, n'_f)$, but another $p_2$ (hence $K_2'$) at the same $A_0$-value, another exit curve $A'B'C'D'$ is required for focusing for all $\psi_s$; the curve $A'B'C'D'$ is similar to, but displaced from, ABCD and intersects the actual exit curve at $B'$. As before, only for one $\psi_s$ does a trajectory
pass through $B'$, and hence through $(z_f, 0)$. Thus, for any particular momentum other than the one focused by the exit curve for $(K, n_g, n_f)$, the intensity reaching the detector is very small.

Each $K'$ corresponds to a different exit curve, and the intersection of this family of curves with the curve for $(K, n_g, n_f) = (0.57, 0, 0)$ provides a class of points, $(B_i)$, along this curve, each of which corresponds to a different $\psi_s$. With a given instrument design (i.e., $K, n_g, n_f$), for the allowed $\psi_s$-range, there is a definite $K'$-range, which depends upon the $(n_g', n_f')$ values considered. [Examples are shown in Figs. (4.3) and (4.4)]. For another choice $(n_g', n_f')$, a similar $K''$-range occurs, and a corresponding contribution to particles of the wrong momenta passing through the focal point.

Thus, if the source emission has a wide range of momenta, then the contribution at $(z_f, 0)$ from each $K'$-value, small though it is, when combined with all the others coming in at a particular $A_0$, gives a considerable background of particles having momenta other than $p$. In fact, if the range of momenta in the source is moderately uniform, and if the range of $K'$ (and hence of $p'$) for a fixed $A_0$ lies within the momentum range of the source, then this background will be comparable in intensity to the desired, focused particles of momentum $p$.

An example is shown in Fig. (4.1). With the chosen entrance curve, the exit curve has been determined for parameter $K$ and $(n_g, n_f) = (-\frac{1}{2}, \frac{1}{2})$, a single-loop trajectory.*

Particles with the appropriate momentum are focused at $(z_f, 0)$. With the same $A_0$, but with a different momentum $p'$ (hence, with a different $K'$), there is one emission angle $\psi_s$ for which a trajectory with $n_g = n_f = 0$

---

*This trajectory is characterized by one loop, and in general the trajectories with half-integral values of $n$ have an odd number of loops $[(n_f - n_g) = \text{odd}].$ in the magnetic field. Integral values of $n$ have an even number of loops; e.g., the trajectory at the bottom of Fig. (4.1) has $n_g = n_f = 0$, and has no loops. A two-loop trajectory has $n_g = -1$, $n_f = 1$ if $Z_M$ is symmetric relative to the orbit loops.
also passes through \((z_f, 0)\). With the same \(A_0\) and the original momentum \(p\) (hence with the original \(K\)), a trajectory in which \(n_s = n_f = 0\) and in which the emission angle is \(\psi_s'\) does not pass through \((z_f, 0)\), but hits the \(z\)-axis at another point, namely, \((z_f', 0)\).

The converse situation also arises. For a mono-momentum \((p)\) source, trajectories with different \((n_s, n_f)\) result in another type of spurious effect. As above, an instrument designed for \((K, n_s, n_f)\) focuses particles of momentum \(p\) at a given \(A_0\). If we vary \(K\) to \(K'\) by changing \(A_0\), we find that another \((n_s, n_f)\) requires another exit curve, which cuts across the \(K\)-parameter family, as in Fig. (3.2). The discussion of the intersection points \((B_i)\) and the relation of each to a single \(\psi_s\)-value is as above, the chief difference being that \(K'\) varies, not by keeping \(A_0\) constant and varying \(p\), but by keeping \(p\) constant and varying \(A_0\). Thus, with a monoenergetic source, not only will particles be counted at a field corresponding to \(K\) (with \(K = p/A_0e\)), but also at \(A_0'\) (with \(K' = -p/A_0e\)).

Although the emitted momentum \(p\) is constant, the fact that particles are detected over a range of values of \(A_0\) leads to an "apparentum momentum distribution" from the source. Thus, a monoenergetic source gives, apart from the main peak, other peaks with smaller peak intensity, called "ghost" peaks. Because a ghost peak corresponds to a range of \(K'\), the necessary variation of \(A_0\) corresponds to an apparent range in \(p\), and the peak is broad, i.e., has poorer resolution than the main peak.

In an instrument designed for \(n_s = n_f = 0\), for the higher values of \(\psi_s\) the trajectories with \(|n_s| > 0\) cannot go through the instrument because they cross the entrance boundary curve for the second time and thus escape from the magnetic field region (e.g., see Fig. 4.2).

As an illustration, consider a symmetric instrument \((n_s = -n_f; b_1 = 0)\) and the cases \(n_s = 0; -1/2; -1; -\frac{3}{2}\), i.e., no loop, one-loop, two-loop and three-loop trajectories, respectively. The function \(f_s(\psi_s)\)

\[
f_s(\psi_s) = \frac{1}{K_0 \cos \psi_s \left[ U(K, \psi_s) + 2\pi n_s i J_1(iK) \right] - \cot \psi_s}.
\]
In the case of this symmetrical instrument, it is convenient to treat the entrance curve as varying with K'. The significance of the variation of K' in terms of bringing particles of momentum other than p to (z_f, 0) or in creating ghost peaks with constant p is the same as described above.

In an instrument designed for (K, n_s), the symmetry condition with b_1 = 0 requires that f_s be given by Eq. (4.1). With this entrance (and exit) curve, particles of momentum p pass through (z_f, 0) for all \( \psi_s \). For another (K', n'_s), the appropriate entrance curve \( f'_s(\psi'_s) \) is calculated by putting \( (K', n'_s) \) into Eq. (4.1). The two entrance curves described by \( f_s(\psi_s) \) and \( f'_s(\psi'_s) \) will, in general, intersect at one point, and at this point (from Eq. 2.1), \( f_s(\psi_s) = f_s(\psi'_s) \). Since the actual instrument has the entrance curve \( f_s(\psi_s) \), it is only for a \( \psi_s \) going through this intersection point that a trajectory involving \( (K', n'_s) \) will pass through \( (z_f, 0) \). We saw that for a symmetric instrument, fixing K' and n'_s determined \( f'_s(\psi'_s) \); conversely, given the value \( f'_s(\psi'_s) = f_s(\psi_s) \) and n'_s, then K' is determined. We have then

\[
\frac{K \cos \psi_s}{K' \cos \psi'_s} = \frac{K e^{-\pi n_s^2 i j_1(iK)}}{K' e^{-\pi n'_s i j_1(iK')}} = \frac{K}{K'},
\]

which allows us to calculate \( K' = K'(K, n_s, n'_s, \psi'_s) \).

Numerical results of such calculations are given in Figs. (4.3) and (4.4); in Fig. (4.3), we have an instrument with K = 0.6 and n_s = 0 (zero-loop trajectory), and in Fig. (4.4), an instrument with K = 0.4 and n_s = -\( \frac{1}{2} \) (one-loop trajectory).

It is inconvenient to show the effect of these trajectories in the case of fixed A_0 and wide range of momenta from the source. However, the effect of a monoenergetic source in forming ghost peaks is readily shown. From K = -p/A_0e, we obtain \( A'_s = A_0(K/K') \) and from this the spectrum of a monoenergetic source [see Figs. (4.5) and (4.6)] is predicted: Consider the instrument
involved in Fig. (4.3), for which $K = 0.6$ and $n_g = -n_f = 0$. The spread in momenta in the satellite peak is determined by the spread in $K'$ over the range of $\varepsilon_S$ considered ($95^\circ$ to $155^\circ$). Thus, for the type of trajectory determined by $(n_g = -\frac{1}{2}, n_f = \frac{1}{2})$, $K'$ varies from 0.44 to 0.32 (see Fig. 4.3), and the resulting apparent momentum is shown in Fig. (4.5) as the first peak beyond the main (narrow) peak.

Since for all of the alternative types of trajectories, $K' > K$, then for a fixed momentum $p$, we have $A'_0 > A_0$, and the ghost peak has an apparently higher momentum than the main peak. Because the whole range of $\varepsilon_S$ is accepted by the instrument and delivered into these single-loop orbits, to the first approximation it is assumed that the intensity in the satellite peak is roughly the same as in the main peak.* The height of the satellite peak has been normalized so that the enclosed area is the same as that of the main peak. Because no details of profile have been evaluated, the peak is shown as of rectangular shape, although this must be incorrect as to detail. The extension along the $mv$-axis is calculated solely by the range of $K'$ evaluated from Fig. (4.3). The remaining satellite peaks are similarly calculated, $n_g = -1$ being the next one. Since the $K'$-values overlap, the apparent $mv$-values also overlap; thus, the satellite peaks are shown as superimposing because of their poor resolution. In practice, these would show as a somewhat smeared-out broad peak.

The instrument involved in Fig. (4.4) yields the apparent momentum distribution of Fig. (4.6). The main peak is that of a single-loop trajectory. For $n_g' = 0$, we have $K' > K$; so $A'_0 < A_0$ and the ghost peak generated by a zero-loop trajectory is of apparently smaller momentum than the main peak.

The elimination of ghost peaks can easily be achieved in instruments with $n_g = n_f = 0$, with a baffle as shown in Fig. (4.2). This is an important reason for choosing an instrument to focus trajectories without loops.

*It follows from the discussion in Section #7 that multiloop trajectories have poorer transmission than zero-loop ones, so this assumption is a gross overestimate of the satellite intensity.
We have seen that the image of a point source \((z_g, 0)\) is the point \((z_f, 0)\). We wish now to determine the nature of the image of a source of finite dimensions. For this, we divide the emitted rays into:

- those with \(p_\phi = 0\) and
- those with \(p_\phi \neq 0\).

Rays with \(p_\phi = 0\). These rays are emitted in a plane \(\phi = \text{constant}\). On this plane we consider as emission points:

a) \((z_g + 6z_g, 0)\)

b) \((z_g, 5r)\).

If we know the image of these points, it is easy to find the image of any point \((z_g + 6z_g, 5r)\), as will be seen below.

a) The particle emitted at \((z_g + 6z_g, 0)\), which enters the magnetic field at \((z_e, r_e)\), forms an angle \(\psi_g + 6\psi_g\) [see Fig. (5.1)] with the positive \(z\)-axis. We relate \(6\psi_g\) to \(6z_g\) as follows: Writing Eq. (2.1b) as

\[
6\psi_g = 6z_g + r_e \cot \psi_g = z_g + r_e \cot \psi_g
\]

and differentiating, keeping \((z_e, r_e)\) constant, from Eq. (3.2a) we find

\[
6\psi_g = \frac{\sin^2 \psi_g}{r_e} \delta z_g = \frac{\sin^2 \psi_g}{r_0} e^{-K(\cos \psi_f - \cos \psi_s)} \delta z_g.
\]  

Fig. 5.1
With this value of the entrance angle, \((\psi + \delta \psi)\), we can return to the equations of motion in the magnetic field, and from Eq. (1.29), we have

\[
z = (a + \delta a) K[U(K, \psi + \delta \psi) + 2\pi n_s j_i(I)] + Z_M(a + \delta a)
\]

\[
r = (a + \delta a) e^{-K \cos(\psi + \delta \psi)}
\]

By means of Eq. (2.3), \((a + \delta a)\) is calculated, and values of \((z, r)\) then give the new trajectory in the field.

We will find it convenient, however, to express the coordinates of the trajectory as "variations" or perturbations from the "unvaried" trajectory of a particle emitted from the central point \((z_s, 0)\).

Before pursuing this line, it is convenient to specify the notation to be used in this variation. As noted before, \(\delta \psi\) represents the variation of entrance angle due to motion of the point source from \((z_s, 0)\) to \((z_s + \delta z_s)\) while keeping the entry point \((z_e, r_e)\) constant. The varied trajectory intersects the exit profile described by \(\ell_f(\psi_f)\) and emerges into the field-free region with exit angle shifted from \(\psi_f\) to \(\psi_f + \delta \psi_f\). The exiting ray intersects the \(z\)-axis at a point displaced from \((z_f, 0)\), i.e., at \((z_f + \delta z_f, 0)\).

The position of \((z_f + \delta z_f, 0)\) is completely specified when both the angle of emergence \((\psi_f + \delta \psi_f)\) and the position of emergence \((z_0 + \delta z_0, r_0 + \delta r_0)\) into the field-free region are known. The position of emergence is given by the intersection of the profile and the varied trajectory, and depends upon the shape of the exit profile in the neighborhood of the unvaried emergence point \((z_0, r_0)\).

For the purposes at hand, it is convenient to describe the shape in the neighborhood of the unperturbed exit point \((z_0, r_0)\) in terms of the differential property \(\frac{d\psi_f}{d\psi_s}\). In this coefficient, \(d\psi_s\) represents a variation of \(\psi_s\) due to variation of \((z_e, r_e)\) while keeping the source point fixed at \((z_s, 0)\): \(d\psi_f\) is the corresponding variation of \(\psi_f\) in which \((z_0, r_0)\) varies but the focal point remains at \((z_f, 0)\). Clearly, \(\delta \psi_s \neq d\psi_s\), and \(\delta \psi_f \neq d\psi_f\). For greater clarity, we shall use the notation: \(d\psi_f = \delta \theta\) [see Fig. (5.1)]. Then, \(\delta \psi_f\) gives the variation in emergence angle, whereas \(\delta \theta\) defines the variation in emergence position.

We consider a particular "varied" trajectory, specified by its parameter \(\psi_s + \delta \psi_s\) and the trajectory coordinates \(\psi + \delta \psi\). The original trajectory in the magnetic field \((z, r)\) is given as a function of \(\psi\) and the entrance angle \(\psi_s\). In the neighborhood of \((\psi, \psi_s)\) the coordinates \((z, r)\) are expanded, and second and higher powers of \((\delta \psi, \delta \psi_s)\) are neglected; the coordinates of the new trajectory \((z + \delta z, r + \delta r)\) are then expressed in terms of \((z, r)\) and \((\delta z, \delta r)\).
The "variations" from the coordinates of the "unvaried" trajectory are calculated as follows: From Eq. (2.1) and (2.5)

\[
r = r_e e^{-K(\cos \psi - \cos \psi_s)}
\]  

(5.2a)

and

\[
z = z_e + r_e e^{K \cos \psi_s} F_0
\]  

(5.2b)

where

\[
F_0 = K \left[ U(K, \psi) - U(K, \psi_s) + 2 \pi (n - n_s) iJ_1(iK) \right].
\]

Then, differentiating and noting [from Eq. (1.18)] that

\[
dU(K, \psi) = \cos \psi e^{-K \cos \psi} d\psi,
\]

we have

\[
\delta r = K r e^{e^{-K(\cos \psi - \cos \psi_s)}} \left[ \sin \psi \delta \psi - \sin \psi_s \delta \psi_s \right]
\]

(5.2c)

and

\[
\delta z = K r e^{e^{-K(\cos \psi - \cos \psi_s)}} \left\{ \cos \psi \delta \psi - e^{K \cos \psi} \left[ F_0 \sin \psi_s + \cos \psi_s e^{-K \cos \psi_s} \right] \delta \psi_s \right\}
\]

(5.2d)

Equations (5.2c) and (5.2d) describe any point on the "varied" trajectory. For a point on this trajectory lying on the exit boundary curve, \( \psi_f \) is substituted for \( \psi \), and (\( \delta z, \delta r \)) for this point are, then, [utilizing Eq. (3.2a) and (3.3)],

\[
\delta r = K r_0 \left[ \sin \psi_f \delta \psi_f - \sin \psi_s \delta \psi_s \right]
\]

(5.3a)

\[
\delta z = K r_0 \left\{ \cos \psi_f \delta \psi_f - e^{K \cos \psi_f} \left[ F \sin \psi_s + \cos \psi_s e^{-K \cos \psi_s} \right] \delta \psi_s \right\}
\]

= \[ Kr_0 \left\{ \cos \psi_f \delta \psi_f - \sin \psi_s \left[ \cot \psi_f - \frac{z_s - z_f}{r_0} \right] \right\} \delta \psi_s \]  

(5.3b)

with
\[ F = K \left[ U(K, \psi_f) - U(K, \psi_g) + 2\pi(n_f - n_g) \right] iJ_1(iK) \]  \hspace{1cm} (5.3c)

Thus, we have a means of calculating any point along the new trajectory, and we wish to find out where this intersects the exit boundary. To do this, we determine the coordinates along \( f_f(\psi_f) \) by expanding \((z,r)\) in terms of \( \delta \theta \), thus giving the new coordinates \((z_0 + \delta z_0, r_0 + \delta r_0)\), with the condition that they lie on the exit boundary. We substitute \( \theta \) for \( \psi_f \) in Eq. (3.1) and differentiate, noting that \( z_f \) is constant; differentiation is with respect to \( \theta \), evaluated at \( \theta = \psi_f \). Thus,

\[ \delta r_0 = z_f f_f'(\psi_f) \quad \delta \theta = r_0 \Lambda_f \delta \theta \]  \hspace{1cm} (5.4a)

\[ \delta z_0 = z_f \left[ f_f'(\psi_f) \cot \psi_f - \frac{f_f'(\psi_f)}{\sin^2 \psi_f} \right] \delta \theta = r_0 \left[ \Lambda_f \cot \psi_f - \frac{1}{\sin^2 \psi_f} \right] \delta \theta \]  \hspace{1cm} (5.4b)

where

\[ f_f'(\psi_f) = \left[ \frac{df_f(\theta)}{d\theta} \right] \theta = \psi_f = \Lambda_f f_f(\psi_f). \]  \hspace{1cm} (5.4c)

The condition that the exit boundary and the new trajectory intersect at \((z_0 + \delta z_0, r_0 + \delta r_0)\) implies that the two equations for \( \delta z \) and those for \( \delta r \) are equal at the intersection. From the equations in \( \delta r \) [Eqs. (5.3a) and (5.4a)], we have

\[ \delta \theta = \frac{K}{\Lambda_f} \left[ \sin \psi_f \delta \psi_f - \sin \psi_g \delta \psi_g \right]. \]  \hspace{1cm} (5.5a)

From the equations in \( \delta z \), Eq. (5.3b) and (5.4b), and from Eq. (3.2a) and (5.5a),

\[ \delta \psi_f = \left[ -\Lambda_f \sin \psi_f \sin \psi_s \frac{z_s - z_f}{r_0} + \frac{\sin \psi_s}{\sin \psi_f} \right] \delta \psi_s. \]  \hspace{1cm} (5.5b)

To introduce \( \delta z_f \), we consider Eq. (3.1b):

\[ z_0 = z_f + r_0 \cot \psi_f. \]  \hspace{1cm} (5.6a)

Then,
\[
\delta z_0 = \delta z_f + \cot \psi_f \delta r_0 - \frac{r_0}{\sin^2 \psi_f} \delta \psi_f . \quad (5.6b)
\]

Introducing (5.4a) and (5.4b), we have
\[
\delta z_f = \frac{r_0}{\sin^2 \psi_f} (\delta \psi_f - \delta \theta) . \quad (5.7a)
\]

From (5.5a) and (5.5b)
\[
\delta \psi_f - \delta \theta = \left[ (\Lambda_f - K \sin \psi_f) \left( \frac{z_s - z_f}{r_0} \sin \psi_s \sin \psi_f + \frac{\sin \psi_s}{\sin \psi_f} \right) \right] \delta \psi_s . \quad (5.7b)
\]

If we differentiate Eq. (3.3) with respect to \( \psi_f \), keeping \( z_s \) and \( z_f \) constant, as noted above, we get
\[
\Lambda_f - K \sin \psi_f = \frac{r_0}{z_s - z_f} \left[ \frac{1}{\sin^2 \psi_f} - \frac{(\psi_s')}{\sin \psi_s} e^{-K(\cos \psi_s - \cos \psi_f)} \right] , \quad (5.7b)
\]

where
\[
(\psi_s') = \frac{d \psi_s}{d \psi_f} = \frac{1}{(\psi_f')} = \frac{1}{\frac{d \psi_f}{d \psi_s}} . \quad (5.8)
\]

Equation (5.8) is the differential coefficient giving the angular variation of the exit profile with \( \psi_s \), while \( \Delta_f \) gives the fractional variation of \( r_0 \) with \( \psi_f \). Then,
\[
\delta \psi_f - \delta \theta = \left[ \frac{\sin \psi_f}{\sin \psi_s}(\psi_s') e^{-K(\cos \psi_s - \cos \psi_f)} \right] \delta \psi_s . \quad (5.9)
\]

Combining this result with Eq. (5.1a) and (5.7a), we have
\[
\delta z_f = \frac{\sin \psi_s}{\sin \psi_f (\psi_f')} \frac{1}{\delta \psi_f} \delta z_s = Q_f \delta z_s . \quad (5.10)
\]

An explicit expression may be given for \( \psi_f \) if \( f_s(\psi_s) \) and \( f_f(\psi_f) \) are known (see Appendix III).
Up to this point we have considered only particles which originate on the z-axis, and hence automatically traverse trajectories lying in planes Φ = constant, i.e., for which p_Φ = 0. We wish to include trajectories arising from an extended source lying in a plane normal to the z-axis at (z_G,0). Since only small sources are treated, if there is also a z-extension of the source, we have only to add the effect of the displacement δz_s as given in Eq. (5.10).

We consider a point A on this source with coordinates (z_s, r_A, Φ_A). The "image" of this point is the intersection of the rays emitted by A with the plane z = z_f; this image need not be a point. A coordinate system closely tied to the direction of an emitted ray simplifies its description, so we define a new pair of coordinates, (r_1, r_2), for each of the rays emitted from A, as follows:

(i). The Φ-plane containing the entrance point (z_e, r_e) of the trajectory is called Φ_s. A given emission point corresponds to many different Φ_s-planes, and, conversely, a given Φ_s-plane (and hence a given set of r_1, r_2 coordinate axes) corresponds to many source points like A.

(ii). r_1 is an axis whose positive direction lies in the plane Φ = Φ_s. The coordinate r_1 is numerically measured in an r-direction, but may have positive or negative values. The coordinate r_1 for any particular ray lies either in the Φ = Φ_s plane or in the plane Φ = Φ_s + π. If the ray emitted from A has a component of motion away from the z-axis (i.e., |Φ_s - Φ_A| < π/2), then r_1 lies in the plane Φ_s and r_1 is positive; if the emitted ray has a component of motion toward the z-axis (i.e., |Φ_s - Φ_A| > π/2), then r_1 lies in the plane Φ_s + π and the coordinate r_1 has a negative value.* In general,

\[ r_1 = r_A \cos (Φ_s - Φ_A). \]  

The magnitude of r_1/r_A thus depends upon the magnitude of |Φ_s - Φ_A|, being largest (and equal to unity) when this is 0 or π.

(iii). r_2 is normal to the r_1- and z-axes. It, too, is measured numerically in an r-direction, but has a sign. Positive r_2 is 90° from the positive r_1-axis (i.e., from the Φ_s-plane), the rotation being taken in the negative Φ-direction.

*If Φ_s lies in the gap of a multigap instrument, then the positive r_1-axis lies in the direction of the gap and the r_1-component either points directly toward or away from this gap. Then r_1 is positive if the point A lies on the side of the z-axis toward the gap [i.e., up in Fig. (1.1)], and negative on the other side.
In general,

\[ r_2 = r_A \sin(\phi_s - \phi_A). \]  

(5.11b)

The magnitude of \( r_2/r_A \) is thus at a maximum for \( |\phi_s - \phi_A| = \frac{1}{2}\pi \).

(iv) If the trajectory is projected on the \( \phi_s \)-plane, then the coordinate \( \psi \) is measured in the usual way on this projection [see Eq. (1.12b)]. Then \( \psi_s \) is the \( \psi \)-value of this projection in the field-free region, i.e., the angle formed by the \( z \)-axis and the line from \((z_s,0)\) to \((z_e,r_e)\).

The rays emitted from the point A have trajectories which can be described in terms of two extreme classifications: those for which \( p_\phi = 0 \) and those for which \( p_\phi \neq 0 \), but \( p_\phi \) has the largest possible value for any ray emitted from A at a given \( \psi_s \) [from Eq. (5.18), this evidently corresponds to the largest possible \( \delta r_2 \)]. The general case, for a given \( \psi_s \), is compounded of the characteristics of both of these classifications. We consider the various rays emitted at a given \( \psi_s \)-value.

Trajectories with \( p_\phi = 0 \). These trajectories occur when the point A lies either on the entry plane or its opposite, i.e., either \( \phi_A = \phi_s \) or \( \phi_A = \phi_s + \pi \). When emission occurs into either plane, there is no \( \phi \)-component of motion. For a given point A, \( r_1 \) may take either of two opposite directions: for A in \( \phi_s \), since positive \( r_1 \) lies in \( \phi_s \), we have \( \delta r_1 = + r_A \); for A in \( \phi_s + \pi \), the \( r_1 \)-coordinate is negative and \( \delta r_1 = -r_A \).

It is evident that the straight line representing the trajectory in the field-free region may be extended and will cut the \( z \)-axis at a point \((z_s + \delta z_s',0)\), and for such a trajectory, the particle will cross the \( z \)-axis on the focus side as though it did indeed originate at \((z_s + \delta z_s',0)\). We have

\[ \delta z_s' = - \cot \psi_s \delta r_1. \]  

(5.12)

It is evident that, when \( \delta r_1 \) is positive, its projection onto the \( z \)-axis is negative when \( 0 \leq \psi_s \leq \frac{\pi}{2} \) and positive when \( \frac{\pi}{2} \leq \psi_s \leq \pi \); the opposite is true when \( \delta r_1 \) is negative. If we again consider the particle to enter the magnetic field at \((z_e,r_e)\), forming an angle \( \psi_s + \delta \psi_s \) with the positive \( z \)-axis, we have [substituting \( \delta z_s' \) for \( \delta z_s \) in Eq. (5.1a)]

\[ \delta \psi_s = - \frac{\sin \psi_s \cos \psi_s}{z_{f'f}(\psi_f)} \left( -K(\cos \psi_f - \cos \psi_s) \right) \delta r_1. \]  

(5.13)

With this expression for \( \delta \psi_s \), and using Eq. (5.7a) and (5.7b), we obtain
\[ \delta z_f = -\frac{\cos \psi_s}{\sin \psi_f} \frac{1}{\psi_f} \delta r_1. \] 

(5.14)

If we take \( \delta r_{1f} \) as the distance of the trajectory from the z-axis at the plane \( z = z_f \) and along the \( r_1 \)-axis, we have, then, as in Eq. (5.12),

\[ \delta z_f = -\cot \psi_f \delta r_{1f}, \] 

(5.15a)

which, when placed into (5.14), yields

\[ \delta r_{1f} = \frac{\cos \psi_s}{\cos \psi_f \psi_f} \delta r_1 = u(\psi_s) \delta r_1, \] 

(5.15b)

with

\[ u(\psi_s) = \frac{\tan \psi_f}{\tan \psi_s}. \] 

(5.15c)

It is now left to determine the image of a point which has extension in both the \( z \) and \( r_1 \) directions, but which emits particles into a plane with \( \phi = \) constant, i.e., a point \( (z_s + \delta z_s, \delta r_1) \). It is evident that for a particle which enters the field at \( (z_e, r_e) \), the effective source point for the particle is obtained by algebraic addition of \( \delta z_s \) and \( \delta z_s' \), the projection of \( \delta r_1 \) on the \( z \)-axis. Hence, the trajectory will intersect the \( z \)-axis on the focus side at \( z_f + \delta z_f \), where \( \delta z_f \) is the algebraic combination of the \( \delta z_f \)-components due to either displacement separately. Thus,

\[ \delta z_f = \frac{1}{\sin \psi_f (\psi_f)} [\sin \psi_s \delta z_s - \cos \psi_s \delta r_1]. \] 

(5.16)

Trajectories with \( p_\phi \neq 0 \). Before taking up the general case of a ray emitted from \( A \) with any value of \( p_\phi \), we first consider a particular case in which the ray has a direction such that the coordinates of \( A \) are \( \delta r_1 = 0 \) and \( \delta r_2 = \pm r_A \), i.e., \( |\phi_s - \phi_A| = \frac{1}{2} \pi \). The general case can then be treated as a superposition of the two special cases: (i) a ray emitted in a direction such that \( A \) has only \( r_1 \)-extension, and (ii) a ray emitted such that \( A \) has only \( r_2 \)-extension.

Let the angle formed by the trajectory with the \( \phi_s \)-plane be \( \delta \zeta_s \) and \( \rho_s \) = the distance from \( (z_s, 0) \) to the profile at \( (z_e, r_e) \). Since \( \sin \psi_s = r_e/\rho_s \), then, to the approximation that \( \tan \delta \zeta_s = \delta \zeta_s \), we have

\[ \delta \zeta_s \approx \frac{\delta r_2}{\rho_s} = \frac{\sin \psi_s}{r_e} \delta r_2 = \frac{\sin \psi_s}{z_s f_s(\psi_s)} \delta r_2. \] 

(5.17)
From Eq. (1.8) and (1.11),

\[ \delta a_0 = a_0 = \frac{p_\phi}{p} = \frac{mr^2\dot{\phi}}{mv} = r \frac{v_\phi}{v}, \]

where \( v_\phi = r \dot{\phi} \) is the \( \phi \)-component of velocity [see Eq. (1.5)]. At the point of entry into the magnetic field, the trajectory makes the angle \( \delta \zeta_S \) with the \( \phi \)-plane; hence,* \( \frac{v_\phi}{v} = \sin \delta \zeta_S \), so from (5.17)

\[ \delta a_0 = r_e \sin \delta \zeta_S = r_e \delta \zeta_S = \sin \psi_S \delta r_2. \tag{5.18} \]

*Note that \( \delta \zeta_S \) is positive when the trajectory is such that \( \phi \) is increasing as the particle enters the magnetic field; also, \( \sin \psi_S \) is positive, since \( \psi_S \) generally lies between 0 and \( \pi \). Then, since \( \delta \zeta_S \) is positive only when \( \delta r_2 \) lies in the direction of decreasing \( \psi \), this is the direction of positive \( r_2 \). In Fig. (5.2), according to the right-handed coordinate system convention of Section #1, it is seen that \( \phi \) increases as it comes out of the paper (above the \( z \)-axis). The trajectory, passing from \((z_S, 0, \delta r_2)\) to \((z_e, r_e)\) in the \( z-r_1 \) plane, has a positive \( \phi \), so \( \delta \zeta_S \) is positive, as is \( \delta r_2 \).
\[ \phi_f - \phi_s = K[V(K,\psi_f) - V(K,\psi_s) + 2\pi(n_f - n_s) J_0(iK)] \frac{\delta a_0}{a}, \]  
(5.19a)

since \( \delta a_0 \) is a constant of the motion [Eqs. (1.5) and (1.8)]. Then, from (2.3) and (5.18),

\[ \Delta \phi = \phi_f - \phi_s = F_1 \frac{\delta r_2}{z_s}, \]  
(5.19b)

where

\[ F_1(K,\psi_f,\psi_s,n_f,n_s,f_s) = K \left[ V(K,\psi_f) - V(K,\psi_s) + 2\pi(n_f - n_s) J_0(iK) \right] \left[ \frac{\sin \psi_s}{f_s(\psi_s)} e^{-K\cos \psi_s} \right]. \]  
(5.19c)

In Fig. (5.3) are plotted curves for \( \frac{z_s}{\delta r_2} \) as a function of \( K \) and \( n_s \) for symmetric spectrometers* for the entry angle \( \psi_s = \pi/2 \).

---

*It may be noted that, since for symmetric spectrometers, \( \psi_f = -\psi_s \) (or \( \psi_f + \psi_s = 2\pi \)), \( V(K,\psi_f) = -V(K,\psi_s) \). Similarly \( n_f = -n_s \), and \( f_s(\psi_s) \) is determined by \( K \). Hence, \( F_1 \) becomes \( F_1(K,\psi_s,n_s) \).
Again, for symmetric spectrometers, in Fig. (5.3a) we show $F_1(\psi_s)$ as a function of $\psi_s$ for various $K$-values with $n_s = 0$.

A table corresponding to this case is given in Table II.4 of Appendix II. It may be noted that there are factors which tend both to increase and decrease $F_1$ as $\psi_s$ is varied; these balance out to a maximum for each $K$-value, as shown in Fig. (5.3a). The various factors are:

(i). $V(K, \psi)$ increases in magnitude as $\psi_s$ decreases to smaller $\psi_s$-values. The influence of this term corresponds to the fact that trajectories entering at small $\psi_s$-values have a much longer path through the magnetic field, and hence tend to cut across more $\phi$-planes.

(ii). The effect of a given displacement $\delta r_2$ is smallest for small and large $\psi_s$, i.e., where $\sin \psi_s$ is small, since under these circumstances the angular momentum is minimal [Eq. (5.18)].
(iii). The factor which varies the least is the term

$$\frac{-K \cos \psi_s}{a} = \frac{e}{z_s f_s};$$

however, the point at which the orbit is vertical has a higher r-value for small \( \psi_s \)-values than for large \( \psi_s \)-values.

Since (ii) tends to depress \( F_1 \) at both ends of the \( \psi_s \)-range, while (i) is largest at the small \( \psi_s \)-end, the maximum is shifted towards smaller angles.

The trajectory emerges from the magnetic field at the point \((z_0, r_0)\); this point and the z-axis define the plane \( \phi = \phi_f \) = constant. As on the source side, the component of the r-extension in this plane is defined by the \( r_{1f} \)-axis, and the corresponding third coordinate, perpendicular to the z- and \( r_{1f} \)-axes, is given by the \( r_{2f} \)-axis. Except for the case \( \Delta \phi = 0 \) (i.e., \( p_\phi = \delta a_0 = 0 \)), the \( r_{1f} \)-axis is not parallel to the \( r_1 \)-axis, nor does \( r_{2f} \) parallel \( r_2 \); both axes are rotated by the angle \( \Delta \phi \). However, the sign of \( \delta r_{2f} \) is also positive in the direction of decreasing \( \phi \), while the \( r_{1f} \)-axis positive direction* lies in the plane \( \phi = \phi_f \). The angle the trajectory makes with the \( \phi_f \)-plane (i.e., \( \delta \zeta_f \)) is related to \( \delta r_{2f} \) in just the same way that \( \delta \zeta_s \) is related to \( \delta r_2 \) [Eq. (5.17)]. Thus, since, as in (5.18), \( \delta a_0 = \sin \psi_f \delta r_{2f} \), we have**

$$\delta r_{2f} = \frac{\sin \psi_s}{\sin \psi_f} \delta r_2 = v(\psi_s) \delta r_2, \quad (5.20)$$

where \( \delta r_{2f} \) is the distance of the trajectory from the z-axis, as measured along the \( r_{2f} \)-axis.

General case. \( p_\phi \neq 0 \) and the ray has any direction; for any particular emission direction, \( A \) has the coordinates \( z = z_s, \delta r_1 \neq 0 \) and \( \delta r_2 \neq 0 \). This case may be considered as a linear combination of the two previous cases. Thus, we get \( \delta r_{1f} \) and \( \delta r_{2f} \) by applying Eqs. (5.14) and (5.20). The coordinate axes \( (r_{1f}, r_{2f}) \) are rotated relative to the source axes \( (r_1, r_2) \) by the angle \( \Delta \phi \) [see Eq. (5.19b)].

* If \( \phi_f \) lies in a gap, then the positive \( r_{1f} \)-axis points into the gap.

** Since \( \psi_f \) generally lies between \( \pi \) and \( 2\pi \), it is seen that \( \sin \psi_f \) is negative where \( \sin \psi_s \) is positive, and \( \delta r_{2f} \) is of opposite sign to \( \delta r_2 \). Thus, in the trajectory whose beginning is shown in Fig. (5.2), it is found that \( \delta r_{2f} \) lies on the opposite side of the z-axis from \( \delta r_2 \) (of course, rotated by \( \Delta \phi \)) and hence has a negative sign.
If the z-coordinate of the source point is \( z_s + \delta z_s \), then the
z-coordinate of the image is similarly displaced to \( z_f + \delta z_f \), as in Eq. (5.10).
This arises because the z-equations of motion, to the first approximation,
are independent of \( \phi \), so \( \delta z_f \) is independent of \( \delta r_1f \) and \( \delta r_2f \). The approxi-
mation that the effect of the \( \delta r_2 \)-coordinate may be treated independently
is appropriate for \( \psi_s \) not close to 0 or \( \pi \).*

For a source at \((z_s,0)\) it has been shown that, for an arbitrary en-
trance curve \( f_s(\psi_s) \), a corresponding exit curve \( f_f(\psi_f) \) can be determined
which will bring all trajectories, independent of their emergent angles \( \psi_s \),
through a preassigned point \((z_f,0)\); i.e., the point source is perfectly imaged
into a single point. With a source of finite dimensions, the question arises
as to whether this arbitrariness in the entrance curve and focal point still
exists. An examination of Eqs. (5.10), (5.15b) and (5.20) indicates that focusing
does not occur, since the position of the trajectory depends upon \( \psi_s \).

This explicit dependence on \( \psi_s \) is removed in a symmetric spectrom-
eter, in which \( \psi_f = -\psi_s \) (or \( \psi_f + \psi_s = 2\pi \)).** Then \( \delta r_1f = -\delta r_1 \) and \( \delta r_2f = -\delta r_2 \),
while \( \delta z_f = \delta z_s \). It is evident that for a source normal to the z-axis, i.e.,
having only \( r_1 \) and \( r_2 \) components, the magnification is \(-1\).

Thus, in a symmetric spectrometer, the ratios \( \delta r_1f / \delta r_1 \) and \( \delta r_2f / \delta r_2 \)
are independent of \( \psi_s \). For a particular ray, the distance \( r_A' \) of the "image"
point \( A' \) (intersection of ray with the plane \( z = z_f \)) from the point \((z_f,0)\)
is equal to \( r_A \), since \( r_A^2 = (\delta r_1)^2 + (\delta r_2)^2 = (\delta r_1f)^2 + (\delta r_2f)^2 = r_A'^2 \).
Although the magnitude of this distance is preserved in the symmetric spectrometer, the
\( \phi \)-coordinate of \( A' \) varies with the \( \phi_s \) of the emitted ray, thus preventing
perfect focusing.

This variation arises from the orbit rotation \( \Delta \phi \) of Eq. (5.19b).
Suppose we consider all rays emitted at a constant \( \psi_s \), but variable \( \phi_s \).
The variation in \( \phi_s \) results in a variation in the magnitude of \( \delta r_2 \), and (since
\( F_1 \) is constant) \( \Delta \phi \) is proportional to \( \delta r_2 \). When \( A \) lies on \( \phi_s \) or \( \phi_s + \pi \), so

\[ \frac{\delta r_1 \delta \xi_s}{\sin \psi_s} \]

* The source point is projected on the z-\( r_2 \) plane along the line of flight
and the intersection with this plane is the virtual source. To the first
approximation, the \( r_2 \)-extension of the projection is the same as the \( \delta r_2 \-
coordinate of the point, but only if \( \psi \) is not too close to 0 or \( \pi \), since

** It is to be noted that Eq. (5.16) does not become independent of \( \psi_s \).
The projection of \( \delta r_{1f} \) onto the z-axis does remain dependent upon
\( \psi_s \), being larger for entry angles closer to 0 or \( \pi \). Although this has
no significance relative to the imaging property, the effect is of
importance for resolution considerations (see Section #6).
does \( \delta r_1 \); then \( \delta r_2 = \delta r_{1f} = \Delta \phi = 0 \), and \( \delta r_{1f} \) lies on \( \phi_s + \pi \) or \( \phi_s \), respectively. When \( A \) is not on \( \phi_s \) or \( \phi_s + \pi \), then \( \delta r_2 \neq 0 \), but the nature of division of the vector radius \( r_A \) into the vector components \( \delta r_1 \) and \( \delta r_2 \) depends upon the angular separation \( (\phi_A - \phi_s) \), as in Eq. (5.11). In a symmetric spectrometer, when \( \delta r_2 \neq 0 \), the radius vector \( r_A \) is rotated from the position taken when \( \delta r_2 = 0 \) by the angle \( \Delta \phi \), i.e., the "image" radius vector is derived by rotating* the source radius vector through the angle \( \pi + \Delta \phi \).

Suppose, on the other hand, we consider only rays emitted into the same \( \phi_s \) from \( A \), but at any \( \psi_s \). All such rays will have the same \( \delta r_2 \), but, since \( F_1 \) is a function of \( \psi_s \) [see Fig. (5.3a)], \( \Delta \phi \) varies with \( \psi_s \) even when \( \delta r_2 \) is constant. The rotation \( \Delta \phi \) thus varies both with \( \psi_s \) and with \( (\phi_s - \phi_A) \).

It is evident, therefore, that the "image" points of \( A \) will lie on a circle of radius \( r_A \), and will cover a segment of the circle whose magnitude is determined by the range of \( \Delta \phi \)-values. For a given \( \psi_s \), the largest \( \Delta \phi \)-value is determined by \( r_A \), since \( \Delta \phi = \delta r_2 \), and the largest \( |\delta r_2| \)-value is \( |\delta r_2| = r_A \). For the entire \( \psi_s \)-range, the largest \( \Delta \phi \)-value occurs for the highest \( F_1 \)-value, as shown in Fig. (5.3a).

This discussion is illustrated in the figures, which show these effects in a symmetric spectrometer. The \( z \)-axis is taken as perpendicular to the paper at \( O \), the projection is taken looking from \( z = z_g \) toward \( z = z_f \), and both the image and source space are shown in the same projection. Figure (5.4a) shows an example of the case in which \( A \) lies on the plane \( \phi_s \). The \( r_2 \)-axis is shown at 90° clockwise rotation from the \( r_1 \)-axis, although there is no \( \delta r_2 \)-component. Since \( \delta r_2 \) is positive, so is \( \Delta \phi \), for the case in which \( \phi_A \) is perpendicular to \( \phi_s \), so that the \( r_1 \)-component is zero. Because \( \delta r_{1f} \) is positive, so is \( \Delta \phi \),

![Fig. 5.4a](image-url)

For the case in which \( A \) is in \( \phi_s \), the trajectory is equivalent to one which originates on the \( z \)-axis at \((\delta z_g, \delta z_g, 0)\) with positive \( \delta z_g \) (when \( \psi_s > 90° \)). The trajectory within the field is tilted relative to the trajectory originating at \((\delta z_g)\) so that the ray crosses the \( z \)-axis before hitting the plane \( z = z_i \) at the image point \( A' \). Hence, the ray, which has been in the plane \( \phi_s \) up to the \( z \)-axis, crosses over into \( z = z_i + \pi \). Thus, \( \phi_A \) lies 180° away from \( \phi_s \). For \( \psi_A = \phi_s + \pi \), the trajectory is equivalent to one from the \( z \)-axis with \( \delta z_g \) negative (when \( \psi_s < 90° \)), as in Fig. (5.1). The 180° transition is made in going from \( \phi_A \) to \( \phi_s \), the trajectory then remaining in \( \phi_s \) until it hits \( z = z_f \) at \( A' \). The sign of \( \delta z_g \) is inverted for \( \psi_s < 90° \), but the analysis is the same. For the case in which \( \phi_A \) is normal to \( \phi_s \), the transition from \( \phi_s \) to \( \phi_A \) changes the \( \gamma \)-angle by 90°, the angle of the trajectory with the \( \phi_s \)-plane being \( \delta z_g \). Within the field \( \gamma \) rotates through \( \Delta \gamma \). At the exit, the trajectory makes the angle \( \delta z_{1f} = \delta z_{2f} \) with the \( \psi_f \)-plane, since the spectrometer is symmetric. The sign convention for \( \delta z_{1f} \) (related to the fact that positive \( r_{1f} \) and \( r_{2f} \) are in the same general directions) is such that this represents a continuation of the ray in the same direction. Hence, because of the symmetry of the instrument, the ray emerges from \( \psi_f \) and hits \( A' \) on \( z = z_e \) with \( \phi_{1f} = \phi_{2f} = 90° \) and the total rotation is \( \Delta \phi + \pi \). In the general case, the argument for the \( r_1 \)-component is the same as for the first case, except that the rotation of \( \phi_f \) through \( \Delta \phi \) results in a rotation of the \( r_1 \)-component by \( + \Delta \phi \). The argument for the \( r_2 \)-component is identical to that for the later case. Since the two components rotate through \( \pi + \Delta \phi \), the point \( A \) does also in going over to the image point \( A' \).
and in this projection this corresponds to a counterclockwise rotation of $\phi$. Thus, the image point is located by reflecting $A$ in the origin to $A'$ [from Fig. (5.4a)] and rotating through the positive angle $\Delta \phi$ to $A''$. If we consider the arc length $A'A''$, we have, from Eq. (5.19b), since $\delta r_2 = r_A$,

$$A'A'' = r_A \Delta \phi = r_A^2 \left( \frac{F_1}{z_s} \right).$$ (5.21a)

If $\phi_S$ is a plane 180° away from the one shown in Fig. (5.4b), then $\delta r_2' = -r_A$, and $\Delta \phi$ would also be negative, so that

$$A'A'' = r_A \Delta \phi = -r_A^2 \left( \frac{F_1}{z_s} \right).$$ (5.21b)

Figure (5.4c) shows an example of the general case. In the example, $\delta r_1''$ is negative, while $\delta r_2'' > 0$; since $\Delta \phi = \delta r_2$, then $\Delta \phi$ is also positive, and hence counterclockwise. The vectors $(r_A, \delta r_1'', \delta r_2'')$ form the triangle OBA. The position of the final vectors $(r_A'', \delta r_1'', \delta r_2'')$ may be determined by reflecting OBA in the origin and then rotating it through $\Delta \phi$ to OB''A''.

Since $A'A'' = r_A \Delta \phi$, we have

$$A'A'' = r_A^2 \sin(\phi_S - \phi_A) \frac{F_1}{z_s}.$$ (5.21c)

Thus, for a given magnitude of $\delta r_2$, the two corresponding image points are separated by the arc distance

$$2 r_A^2 \left| \sin(\phi_S - \phi_A) \right| \left( \frac{F_1}{z_s} \right).$$ (5.22a)

The maximum value occurs for $\phi_S$ normal to $\phi_A$ ($\delta r_1 = 0$), and the arc distance between the two extreme points [see Eqs. (5.21a) and (5.21b)] is

$$2 r_A^2 \left( \frac{F_1}{z_s} \right).$$ (5.22b)

The two rays corresponding to $\phi_S = \phi_A$ and $\phi_S = \phi_A + \pi$, at a given $\psi_S$, have coincident image points, the separation in Eq. (5.22a) equalling zero. The entire arc in between the extremes is filled with image points corresponding to various $\phi_S$-planes. This is true if it is assumed that the entire torus is available to provide $\phi_S$-planes, neglecting the fact that in a real spectrometer some of the $\phi_S$-planes will be occupied by poles or conductors. The latter effect simply introduces gaps in the arc corresponding to the $\phi_S$-planes occupied by matter, as well as those $\phi_S$-planes for which the rotation $\Delta \phi$ is large enough as to cause the trajectories to end up in a poleface.
For \( r_A \ll r_e \) (the entrance coordinate), all \( \phi_s \)-values are equally likely for the emission from \( A \) at a given \( \psi_s \). Eq. (5.22a) then implies that the density of image points is greatest at the extremes of the arc, i.e., nearest the points described in Eqs. (5.21a), (5.21b), and (5.22b).

Since \( F_1 = F_1(\psi_s) \), it is evident that the length of this arc varies with \( \psi_s \) according to Eq. (5.22b). We compute the magnitude of the arc for an extreme case. From Fig. (5.3a), it is evident that for the most useful \( K \) and \( \psi_s \) ranges \( F_1 \) lies between \( \frac{1}{2} \) and 2. If \( r_A/z_s = 0.02 \)* then for \( F_1 = 2 \), we have \( 0.08 r_A \) as the distance between the two extreme image points.

It is evident that a source consisting of a circle of points \( A \) at constant radius \( r_A \) will image into a circle, also of radius \( r_A \), for any emission angle. A circular source area will then image into a circular region of the same area. A line source, on the other hand, when emitting into the full 360° of \( \phi_s \), but at a fixed \( \psi_s \), will yield an area as shown in Fig. (5.4d). Taking the chord length as approximately equal to the arc length for small \( A'A''/r_A \), the outer boundaries are parabolas [Eqs. (5.21a) and (5.21b)] whose curvature increases with \( F_1/z_s \).

The actual image consists of a superposition of similar areas, one for each \( \psi_s \). The outer boundary is then due to the \( \psi_s \) with the largest \( F_1 \)-value. Because of this nesting of areas, the inner regions of small arc length tend to increase in density of points, at a given \( r_A \), thus counteracting the effect of Eq. (5.22a). If the effects of the polefaces are included, the outer parabolas tend to be cut off as \( r_A \) increases, because \( \Delta \phi \) becomes large enough to cause collision of rays with polefaces (see Section #7).

Thus far, we have discussed a source emitting into a spectrometer with available \( \phi_s \)-values covering the entire \( 2\pi \)-range, i.e., a multigap sliced-orange type of instrument. We now examine the effect of rotation on the imaging property of a single sector instrument.

In Fig. (5.5), an unusually wide gap is shown for illustrative purposes, with gap boundaries \((\phi_0, \phi_0)\) and gap angle \( \varphi = \phi_0 - \phi_0 \). Here \( AA_1 \) is a line source normal to the midplane \( \phi_M \), the half-length being such that,

\*For \( z_s = 30 \) cm, the point \( A \) corresponds to an outer point of a 12-mm diameter source.
at the $\psi_s$-angle considered, a particle leaving A and entering the profile at $\phi_M$ is rotated through $\frac{1}{2} \psi$ and emerges from the profile at $\phi_0'$ (i.e., $\Delta \phi = \frac{1}{2} \psi$). On OA, it is seen that $\delta r_2 < 0$; the resulting negative $\Delta \phi$ corresponds to clockwise rotation of the image points. The maximum $\Delta \phi$ for a given distance from O occurs when $\delta r_1 = 0$, i.e., for $\phi_s = \phi_M$; thus, the lower boundary of the image (OC'B'A') corresponds to rays entering $\phi_M$. Correspondingly, the minimum $\Delta \phi$, and hence the upper boundary (OB''A''), arise from points entering $\phi_0$. The image points between A' and A'' arise from $\phi_s$-planes between $\phi_0$ and $\phi_M$; no transmission occurs for $\phi_M > \phi_s > \phi_0$, because the rays are rotated into the poleface. As the source point moves in from A toward O, the critical limiting $\phi_s$-plane, $\phi_C$ (i.e., no transmission for $\phi_C > \phi_s > \phi_0'$), moves toward $\phi_0$, because $\Delta \phi$ decreases in magnitude. If the source line were extended to Q, a ray from Q to $\phi_M$ would not be transmitted, so the lower boundary would not lie on A'Q', although the upper boundary would lie on A''Q'', since a ray from Q to $\phi_0$ would be transmitted.

If the entire $\psi_s$-range used is included, the resulting parabolic strip is increased in width, the lower boundary being due to the $\phi_M$-rays for the $\psi_s$ with the largest $F_1$-value, and the upper boundary due to the $\phi_0$-rays for the $\psi_s$ with the smallest $F_1$-value.

Thus, the imaging property in the $z_f$ plane of a finite source is not linear, nor even point to point. However, for small $\delta r_2$, it is almost point to point, even though nonlinear. It may be noted that the rotation $\Delta \phi$ of the image does not depend upon the gap angle $\psi$ but only upon $F_1$ and $\delta r_2/z_s$. However, $\psi$ determines the maximum sample dimension for which a trajectory can pass through without colliding with a poleface. For such a $\delta r_2$-value, the rotation from an extreme sample point is $\Delta \phi = \psi$; then $\delta r_2/z_s = \psi/F_1$. If, e.g., $\psi = 3^\circ = 0.052$ radians, then for $K = 0.65$ and $\psi_s = \frac{1}{2} \pi$, it is found that $\delta r_2/z_s = 0.052/1.6 = 0.033$; then, if $z_s = 50$ cm, when rays are emitted into a single gap with $\psi_s = \frac{1}{2} \pi$ from portions of the sample for which $\delta r_2 > 1.6$ cm, none of these rays are transmitted.

The $\delta a_0/a$ approximation. It will be recalled that in Section #1, we postponed until this section the question of the validity of approximations in assuming $a_0 = \delta a_0$ small. We are now in a position to consider the meaning of this assumption in terms of sample size. We neglected all powers of $\delta a_0/a$ greater than the first. We have, from Eqs. (2.3) and (5.18), that
\[
\frac{\delta a_0}{a} = \frac{\sin \psi_s}{r_e} e^{-K \cos \psi_s} \delta r_2 = e^{-K \cos \psi_s} \frac{\delta r_2}{\rho_s}.
\] (5.23)

Since \( K \) is generally smaller than unity, for \( \psi_s \)-values generally used, the exponential term has a maximum value between 1 and 2. Since \( \rho_s \) is generally approximately equal to \( z_s \), we find that the approximation is justified when \( \alpha \left( \frac{\delta r_2}{z_s} \right)^2 < 1 \), where \( \alpha \) is a small number determined by the value of the exponential above or the coefficients in Eqs. (1.15a), (1.15b) and (1.15c).
Up to this point, we have considered only monoenergetic charged particles of momentum \(mv\). We turn now to a consideration of the image of particles emitted from the same source, i.e., at \((z_s, 0)\), whose momentum differs from the original momentum by a small amount and has the value \(mv + \delta(mv)\).

For a particular instrument, and hence a given source point \((z_s, 0)\) and image point \((z_f, 0)\), the constant \(K\) [see Eq. (1.10)] is a characteristic property of the instrument. Thus, an increase in the momentum of the electrons to be focused at \((z_f, 0)\) involves a proportionate increase in \(A_0\) [see Eq. (1.1)] to the value \(A'_0\). We have then

\[
K = \frac{mv}{A_0} = \frac{mv + \delta(mv)}{A'_0} \quad (6.1)
\]

We may, however, consider a related situation. Suppose we keep the field at the value given by \(A_0\), and hence focus electrons of momentum \(mv\) at the point \((z_f, 0)\). Electrons of momentum \(mv + \delta(mv)\) will approximately focus and form an "image" at another point \((z_f + \delta z_f, 0)\). For these electrons, if we consider the source and image points to be \((z_s, 0)\) and \((z_f + \delta z_f, 0)\), respectively, the instrument "constant" will in effect have been changed to \(K + \delta K\). Thus, with the magnetic field which focuses electrons of momentum \(mv\) at \((z_f, 0)\), we have

\[
K + \delta K = \frac{mv + \delta(mv)}{A_0} \quad (6.2)
\]

The trajectory of one of these particles will differ from the previous ones only after entrance into the magnetic field. It will intercept the exit boundary curve \(f_s(\psi_f)\) at the point \((z_0 + \delta z_0, r_0 + \delta r_0)\), where \((z_0, r_0)\) is the intersection point for an electron of momentum \(mv\) and the same initial emission angle \(\psi_s\). At \((z_0 + \delta z_0, r_0 + \delta r_0)\), the trajectory will make the angle \(\psi_f + \delta \psi_f\) with the positive z-axis [see Fig. (6.1)].

The calculation is made by a variation procedure similar to that used in Section #5. The trajectory of a particle with momentum \(mv + \delta(mv)\) is determined as a variation of the trajectory of a particle of momentum \(mv\). In Section #5, a similar variation was performed, in which the coordinates of the varied trajectory were determined with \(\delta \psi\) and \(\delta z_s\) (or the equivalent \(\delta \psi_s\)) as the varied quantities, while \(r_e\) was kept...
constant. In the present case, \( z_s, \psi_s, \) and \( r_e \) are kept constant, while the variations considered are \( \delta K \) and \( \delta \psi \). In Eq. (2.5), if we substitute \( K+\delta K \) for \( K \), we have the new trajectory in Fig. (6.1). As before, we consider the variation around a point \((z,r)\) on the old trajectory characterized by \( K \); this point has a particular \( \psi \)-value. Any point \((z+\delta z, r+\delta r)\) in the neighborhood of \((z,r)\) can be reached by first varying \( K \) to \( K+\delta K \), and then varying \( \psi \) to \( \psi +\delta \psi \). By expanding \( z+\delta z \) and \( r+\delta r \) around \( K \) and \( \psi \), then \( \delta z \) and \( \delta r \) are calculated in terms of \( \delta K \) and \( \delta \psi \) [These relations are equivalent to those in Eq. (5.3)].

As before, \( \delta z \) and \( \delta r \) are evaluated at the point that the varied trajectory hits the exit boundary, so that \( \psi \) becomes \( \psi_f \). The variation of the exit point \((z_0,r_0)\) is performed exactly as in Section #5, so that \( \delta z_0 \) and \( \delta r_0 \) are given as functions of \( \delta \theta \) by Eq. (5.4). Equating \( \delta r = \delta r_0 \) and \( \delta z = \delta z_0 \), we get the relations

\[
\delta \theta = \frac{1}{A_f} \left[ (\cos \psi_s - \cos \psi_f) \delta K + K \sin \psi_f \delta \psi_f \right] \tag{6.3a}
\]

and

\[
\delta \psi_f = \left[ \left( A_f \cos \psi_f - \frac{1}{\sin \psi_f} \right) (\cos \psi_s - \cos \psi_f) - F_2 A_f \sin \psi_f e^{K \cos \psi_f} \right] \frac{\delta K}{K}, \tag{6.3b}
\]

where \( A_f \) is defined in Eq. (5.4c),

\[
F_2 = F \cos \psi_s + \frac{\partial F}{\partial K}, \tag{6.4a}
\]

and \( F \) is defined in Eq. (5.3c) as

\[
F = K \left[ U(K, \psi_f) - U(K, \psi_s) + 2\pi (n_f-n_s) \text{i}J_1(iK) \right]. \tag{6.4b}
\]

Since the cause of the variation of the trajectory is immaterial after the orbit reaches the exit boundary, Eq. (5.7a) relates \( \delta z_f \) and the quantity \( (\delta \psi_f - \delta \theta) \). We have, then, from Eqs. (6.3a), (6.3b) and (5.7b),

\[
\delta z_f = \frac{r_0}{\sin^2 \psi_f} \left\{ \frac{r_0}{z_s-z_f} \left[ \frac{1}{\sin^2 \psi_f} - \frac{1}{(\psi_f) \sin^2 \psi_s} \right] e^{-K(\cos \psi_s - \cos \psi_f)} \right\}
\]

\[
x \left[ \cos \psi_f (\cos \psi_s - \cos \psi_f) - F_2 \sin \psi_f e^{K \cos \psi_f} \right]
\]

\[
- \left[ \frac{\cos \psi_s - \cos \psi_f}{\sin \psi_f} \right] \frac{\delta K}{K}, \tag{6.5}
\]

with \( \psi_f \) defined in Eq. (5.8).
Using Eq. (3.3), this may be written as

\[ \delta z_f = G(z_f - z_s) \frac{\delta K}{K}, \]  

where

\[ G = G(\psi_s, \psi_f, n_s, n_f, K, \psi_f') = \frac{\delta z_f}{z_s - z_f} = \frac{\delta z_f}{\delta p} \]

\[ = \frac{e^{-K \cos \psi_f}}{F_3 \sin^2 \psi_f} \left[ \frac{e^{-K \cos \psi_s}}{F_3 \sin^2 \psi_s} - \frac{e^{-K \cos \psi_f'}}{F_3 \sin^2 \psi_f'} \right] \]

\[ \times \left[ \cos \psi_f (\cos \psi_s - \cos \psi_f) - F_2 \sin \psi_f e^{K \cos \psi_f} \right] - \left[ \frac{\cos \psi_s - \cos \psi_f}{\sin \psi_f} \right] \]

with

\[ F_3 = \frac{(z_s - z_f)}{r_0} e^{-K \cos \psi_f} = \cot \psi_s - \frac{e^{-K \cos \psi_s}}{\cot \psi_f - e^{-K \cos \psi_f}} + F \]

From Eq. (6.4b) we have

\[ \frac{\partial F}{\partial K} = K \left[ \frac{\partial U(K, \psi_f)}{\partial K} - \frac{\partial U(K, \psi_s)}{\partial K} + 2\pi(n_f - n_s) i \frac{\partial J_1(iK)}{\partial K} \right] + \frac{F}{K} \]

and, by the methods of Appendix I,

\[ \frac{\partial U(K, \psi)}{\partial K} = - \int_\pi^\psi \cos^2 \psi e^{-K \cos \psi} d\psi = \frac{1}{2} [J_0(iK) - J_2(iK)][\pi - \psi] \]

\[ - \frac{1}{2} \sum_{\ell=1}^\infty \frac{\ell - 2}{\ell} \left[ J_\ell - 2J_\ell(iK) + J_{\ell+2}(iK) \right] \sin \ell \psi \]

and

\[ \frac{\partial J_1(iK)}{\partial K} = \frac{1}{2} i [J_0(iK) - J_2(iK)] \]

Dispersion. For a particular instrument, it is usually more convenient to use the dispersion rather than the G-function, where the dispersion is defined as
so that

\[ \sigma_f = (z_s - z_f) \frac{G}{G} \]  \hspace{1cm} (6.10b)

It is often convenient to consider the percentage rather than the fractional change in momentum. We then use another dispersion measure:

\[ \sigma_\% = \frac{\sigma_f}{100} \]  \hspace{1cm} (6.10c)

In the symmetrical case, since \( \psi_f + \psi_s = 2\pi \), it is found that \( G \) is given by the equation

\[ G = -2 \frac{e^{-K \cos \psi_s}}{F_3 \sin^3 \psi_s} F_2 \]  \hspace{1cm} (6.11)

with

\[ F_3 = 2 \cot \psi_s e^{-K \cos \psi_s} + F = -\left(2z_s/r_e\right)e^{-K \cos \psi_s} \]  \hspace{1cm} (6.12a)

and*

\[ F = K \left[ U(K, -\psi_s) - U(K, \psi_s) - 4\pi n_s J_1(iK) \right] \]
\[ = -K \left[ 2U(K, \psi_s) + 4\pi n_s i J_1(iK) \right] \]  \hspace{1cm} (6.12b)

and

\[ \frac{\partial F}{\partial K} = -2K \left\{ \frac{\partial U(K, \psi_s)}{\partial K} + \frac{F}{K} \right\} \]
\[ \frac{\partial F}{\partial K} = K \left\{ 2 \int_0^{\psi_s} \cos^2 \psi e^{-K \cos \psi} d\psi + 2\pi n_s \left[J_0(iK) - J_2(iK)\right] \right\} + \frac{F}{K} \]
\[ = K \left\{ -[J_0(iK) - J_2(iK)] \left[\pi - \psi_s\right] \right. \]
\[ + \sum_{\ell=1}^{\infty} \frac{i^{\ell-2}}{\ell} \left[ J_{\ell-2}(iK) - 2J_\ell(iK) + J_{\ell+2}(iK) \right] \sin \ell \psi \]
\[ + 2\pi n_s \left[J_0(iK) - J_2(iK)\right] \right\} + \left(\frac{F}{K}\right) \]  \hspace{1cm} (6.12c)

*Since, from Eqs. (1.18) or (1.19), \( U(K, \psi_f) = U(K, -\psi_s) = U(K, 2\pi - \psi_s) = -U(K, \psi_s) \).
Finally,

\[ \sigma_f = 2 \frac{z_s G}{z} \]  \hspace{1cm} (6.13)

It is evident that \( G \) is a more universal function than \( \sigma_f \), involving as it does only the shape of the instrument rather than including the size.

In Fig. (6.2) are given values of \( 2G = \frac{\sigma_f}{z_s} \) as a function of \((K,n_s)\) for symmetric spectrometers, but only for \( \psi_s = \pi/2 \). In Fig. (6.3) is shown the variation of \( \sigma_f/z_s \) with \( \psi_s \) for two special symmetrical cases. Figures (6.4) and (6.4a) show the variation of \( G \) with \( \psi_s \) for various \( K \)-values in symmetrical spectrometers. A table of the function is given in Table II.5 of Appendix II. For nonsymmetrical spectrometers, \( \partial U/\partial K \) is the only function involved which is not already tabulated, and for this Table II.6 is given in Appendix II.

Dispersion considerations alone would dictate that the most favorable instrument would have large \( n_s \). However, several considerations lead us to choose instruments with \( n_s = 0 \) instead of large \( n_s \); these are (i) loss of transmission from a finite source due to rotation of the particles into the pole faces, and (ii) only with \( n_s = 0 \) can one suppress the satellite peaks by using baffles.
Resolution. Using the dispersion computed as above, it is in principle possible to evaluate the resolving power, which is defined as the relative width of the line profile at half the height of the line. A computation of this type is quite involved for this kind of instrument. The shape of the profile for a given \( \psi_0 \) must first be determined. Then, since the dispersion varies with \( \psi_0 \), the superposition of the profiles over \( \psi_0 \) adds further complexity. In an actual instrument, the fringing field (see Sections #8 and #9) further complicates the problem because of a number of effects, the most important of which involves changes in the detector dimensions in order to conserve transmission. In the following, however, the fringing field effect is not considered.

For purposes of computation, the following simplifying assumptions are made to start with:

(i) no fringing-field effect;

(ii) the source is uniformly distributed;

(iii) the source is extended axially with dimensions \( dz_0 \). The basic treatment is made on this basis because of simplicity. As was made evident in Section #5 [e.g., see Eq. (5.12)], an \( r_1 \)-extension may be
transformed into an equivalent axial extension by projection. Sources with r-extension will be examined following the simpler treatment involving only axial source extension;

(iv) the detector slit opening is axial. As in (iii), r-extension of the detector slit will be introduced following the preliminary simpler treatment;

(v) solid angle dω with an emission interval dψS in the neighborhood of the fixed value ψS. Following a computation at fixed ψS, the effect of integrating over ψS is determined; and

(vi) detector slit at z_f with infinitely narrow axial extension. The effect of finite detector extent is introduced later.

The resolution is evaluated starting with these six simplifying assumptions, and then re-evaluated relaxing several of the assumptions.

From Eq. (5.10), the axial dimensions of the image is given by

\[ \delta z_f = Q_f \delta z_S ; \quad Q_f = \frac{dz_f}{dz_s} \]  \hspace{1cm} (6.14)

Because the detector is infinitely narrow, only particles arriving at (z_f,0) are detected. At a given ψS, only one point on the source emits rays which are focused and detected at (z_f,0) and variation of the field intensity is used to sweep over all of the source points. If δA_0 is the range of specific field intensity required to sweep over the entire source length δz_S, then the resolution is the relative A_0-range over which particles are detected at a particular ψS. Thus,

\[ \mathcal{R}_0' = \left| \frac{\delta A_0}{A_0} \right| = \left| \frac{-\delta K/K}{\delta z_f} \right| \delta z_f = \left| \frac{\delta z_f}{\sigma_f} \right| = \left| Q_f \frac{\delta z_S}{\sigma_f} \right| . \]  \hspace{1cm} (6.15)

The line profile is rectangular, with relative base width \( \mathcal{R}_0' \).

The first simplifying assumption to be dropped is (vi), with D_z = axial dimension of the detector slit. It is evident from Fig. (6.5) that the relative base width of the line is

\[ \mathcal{R}_b' = \left| \frac{\delta z_f}{\sigma_f} + D_z \right| . \]  \hspace{1cm} (6.16a)

For convenience of notation, we shall write \( \delta z_f \) for \( |\delta z_f| \), so that Eq. (6.16a) becomes
\[ R' = \frac{\delta z_f + D_z}{\sigma_f} \]  \hspace{1cm} (6.16b)

In the following, absolute values are taken for all lengths.

The line profile is a trapezoid, the region of constant transmission being the range in which \( \delta z_f \) lies entirely within the slit (if \( \delta z_f < D_z \)) or the slit lies entirely within the image (if \( \delta z_f > D_z \)).

For \( \delta z_f > D_z \), the relative length of the region of constant transmission is \( (\delta z_f - D_z)/\sigma_f \), so that the relative width at half-height is

\[ R' = \frac{1}{2} \left[ \frac{\delta z_f - D_z}{\sigma_f} + \frac{\delta z_f + D_z}{\sigma_f} \right] = \frac{\delta z_f}{\sigma_f} = R'_0 \]  \hspace{1cm} (6.17)

i.e., the same as for \( D_z = 0 \).

The height of the peak is proportional to \( d\omega \) and to the fraction of the source overlapping the slit, i.e., the transmission is

\[ t' = d\omega \frac{D_z}{\delta z_f} \]  \hspace{1cm} (6.18)

The area \( S' \) under the profile is a useful measure of relative transmission, and this is the product of the half-width and height \( t' \), so

\[ S' = d\omega \left( \frac{D_z}{\sigma_f} \right) \]  \hspace{1cm} (6.19)

For \( \delta z_f < D_z \), the line profile is also a trapezoid, in which the relative length of the region of constant transmission is \( (D_z - \delta z_f)/\sigma_f \), so that the resolution is

\[ R = D_z/\sigma_f \]  \hspace{1cm} (6.20)

At the peak, the entire source is detected, so that the peak height and profile area are

\[ t' = d\omega = t'_M \]  \hspace{1cm} (6.21a)

and

\[ S' = d\omega \left( D_z/\sigma_f \right) \]  \hspace{1cm} (6.21b)

where \( t'_M \) is the maximum peak height. Since \( S' \) is the same for both \( \delta z_f < D_z \) and \( z_f > D_z \), it is a more usually used measure of relative intensity.
In the intermediate limiting case, δzf = Dz and the profile is triangular. In this case,

\[ t' = \omega = t'_M \quad (6.22a) \]

and

\[ S' = \frac{1}{2} \int d\omega \left[ \frac{Dz + \delta zf}{\sigma f} \right] = \omega \frac{Dz}{\sigma f} \quad (6.22b) \]

These quantities are shown as a function of the detector width Dz in Fig. (6.6). The optimum value of Dz is at Dz = δzf; at this value, the best (minimum) resolution, \( R'_M \), is achieved at the same time as the maximum peak height, \( t'_M = \omega \).
The next simplifying assumption to be dropped is (v). When all the \( \psi_s \)-values in the acceptance range are included, the line profile is made up of the partial peaks obtained for the \( d\psi_s \)-intervals at each of the various \( \psi_s \)-values. An exact calculation involves a summation of the partial peaks, for which we give an approximate solution.

From Eqs. (5.10), (6.7) and (6.10), \( \delta z_f \) and \( \sigma_f \) are, in general, both functions of \( \psi_s \). Over the entire range of \( \psi_s \), we shall have a maximum and minimum value of \( \delta z_f \), namely \( \delta z_{fM} \) and \( \delta z_{fm} \), respectively. Similarly, there are a maximum and minimum value of \( \sigma_f \), namely, \( \sigma_{fM} \) and \( \sigma_{fm} \), respectively. These occur at four different \( \psi_s \)-values.

We first consider the case in which \( D_z \leq \delta z_{fm} \). Then each partial peak has its minimum resolution:

\[
\mathcal{R}_m'(\psi_s) = \mathcal{R}_0' = \frac{\delta z_f}{\sigma_f} \quad (6.23)
\]

It is readily shown that, to a good approximation, the minimum resolution of the entire peak is also independent of \( D_z \), for \( D_z \leq \delta z_{fm} \). It is known for many types of line shapes that it is a fairly good approximation to take the area as the product of peak height and half-width.* Thus, we define

\[
\mathcal{R}_S = S/t \quad , \quad (6.24a)
\]

where \( S \) is the area of and \( t \) the height of the entire peak. If \( \mathcal{R} \) is the actual resolution of the entire peak, then, approximately,

\[
\mathcal{R} \approx \mathcal{R}_S \quad . \quad (6.24b)
\]

Since the area represents all the particles detected over all \( \psi_s \), from Eq. (6.19) we find

\[
S = \sum \psi_s' (\psi_s) = D_z \int \frac{d\omega}{\sigma_f} = D_z \mathcal{R} \int \frac{\sin \psi_s d\psi_s}{\sigma_f (\psi_s)} \quad . \quad (6.25)
\]

* The relation is exactly true for rectangular, triangular or trapezoidal profiles. Even for a Gaussian \( \mathcal{R}'(t') \), area = 0.939 if the area is integrated to \( \infty \). Since, however, the base width is limited in a spectrometer, it is more appropriate to consider a truncated Gaussian. If the truncation is performed by extrapolating the slope at the steepest part of the sides to the base-line, the intersection is at \( 2\delta_z' \), and the ratio

\[
\mathcal{R}'(t') \text{, truncated area} = 0.984 \quad .
\]

The same type of truncation (extrapolation of steepest slope) for a Lorentz shape, intersecting the base at \( \frac{2}{3} \mathcal{R}'(t') \), gives the ratio

\[
\mathcal{R}'(t') \text{, truncated area} = 0.955 \quad .
\]

Thus, the relation is exactly true for each \( \delta \psi_s \)-region, and is probably true to a good approximation for the entire \( \gamma_s \)-region.
If the spectrometer has perfect, or almost perfect, $\psi$-focusing, then the partial peaks have their centers coincident or very closely overlapping. Since the partial peak heights represent transmission, the total transmission over all $\psi_s$ is, from Eq. (6.18),

$$t = \sum t'(\psi_s) = D_z \int \frac{d\omega}{\delta z_f(\psi_s)} = D_z \varphi \int \frac{\sin \psi_s d\psi_s}{\delta z_f(\psi_s)} \quad (6.26)$$

Thus, as was the case for each $d\psi_s$-region [see Eqs. (6.18), (6.19) and Fig. (6.6)], both $S$ and $t$ are proportional to $D_z$ for $D_z \leq \delta z_{fm}$. Hence, $\varrho_s$ is independent of $D_z$ and takes on the minimum value (relative to $D_z$ variation) which we may call $\varrho_m$. With the approximation given by Eq. (6.24b), then

$$\varrho \approx \varrho_m \text{ for } D_z \leq \delta z_{fm} \quad (6.27)$$

To evaluate $\varrho_m$, it is necessary to compute the integrals of Eqs. (6.25) and (6.26), or we may write these as

$$\varrho \approx \varrho_m = \frac{\int \frac{\sin \psi_s}{\sigma_f(\psi_s)} d\psi_s}{\int \left| \frac{\sin \psi_s}{Q_f(\psi_s)} \right| d\psi_s} \quad (6.28)$$

from Eq. (5.10). We note that outside limits may be set in the form of the inequalities

$$\frac{\delta z_{fm}}{\sigma_f M} \leq \varrho_m \leq \frac{\delta z_{fM}}{\sigma_f M} \quad (6.29)$$

It is evident that the maximum transmission occurs when the $\psi$-focusing is perfect, if the largest image lies within the slit, i.e., $D_z \geq \delta z_{fM}$. Eq. (6.21) applies for $t'$ and $S'$, so that the resolution $\varrho t_M$ corresponding to maximum transmission is

$$\varrho \approx \varrho t_M = \frac{S}{t} = \frac{\int \frac{\sin \psi_s}{\sigma_f} d\psi_s}{\int \sin \psi_s d\psi_s} D_z \text{ for } D_z \geq \delta z_{fM} \quad (6.30)$$

Thus, for $D_z \geq \delta z_{fM}$, $t_M$ is proportional to $D_z$. It is evident, then, that the value of $D_z$ which optimizes resolution and transmission lies between limits indicated by

$$\delta z_{fm} \leq D_z \leq \delta z_{fM} \quad (6.31)$$

These points are illustrated in Fig. (6.7).
The symmetric instruments are much simpler to consider. Then \( Q_f = 1 \) and \( \delta z_f = \delta z_s = \text{constant} \). When \( D_z = \delta z_s \), then both resolution and transmission are optimized, since \( R_m = R_{tM} \approx R \), from Eqs. (6.28) and (6.30). A figure illustrating this result would be similar to Fig. (6.6) if \( \psi \)-focusing were perfect.

If, in addition, the \( \psi_s \)-range is chosen so that \( \sigma_f \) is a monotonic function of \( \psi_s \) [see Fig. (6.4)], the half-width is readily computed.

For \( D_z \leq \delta z_s \), we find \( R_m = \delta z_s / \sigma_f(\psi_s) \), with a trapezoidal profile, for a region \( d\psi_s \) around \( \psi_s \). Then \( t' = C'd\psi \), where \( C' = D_z / \delta z_s \) and is unity when \( D_z = \delta z_s \). The trapezoids have monotonically increasing (or decreasing) half-widths as \( \psi_s \) increases. The line profile may be obtained [see Fig. (6.8)] by placing one trapezoid above the previous one in the order given by the corresponding \( \psi_s \)-values, beginning with the one of maximum half-width. The height of the peak is given by

\[
t = C'\int d\psi = C'\varphi \int_{\psi_{SM}}^{\psi_s} \sin \psi_s \ d\psi_s = C'\varphi \left[ \cos \psi_{SM} - \cos \psi_s \right]_M
\]

(6.32)

The half-height of the profile obtained in this way corresponds to an angle \( \psi_{sh} \) given by the relation

\[
\frac{1}{2} \int_{\psi_{SM}}^{\psi_{sh}} \sin \psi_s \ d\psi_s = \int_{\psi_{SM}}^{\psi_{sh}} \sin \psi_s \ d\psi_s
\]

(6.33a)

or

\[
\cos \psi_{sh} = \frac{1}{2} (\cos \psi_{SM} + \cos \psi_{SM'})
\]

(6.33b)

Because the trapezoids are only infinitesimally high, the half-width of the
trapezoid corresponding to $\phi_s$ at $\psi_{sh}$ gives the half-width of the entire line. Then

$$\tau = \delta z_s / \sigma_f(\psi_{sh})$$  \hspace{1cm} (6.34)

A similar argument may be given if $\sigma_f$ is not monotonic over the $\psi_s$-range used but has only one minimum (or maximum). The results are somewhat more complicated and are not given here.

We now modify assumptions (iii) and (iv), taking a linear source and detector normal to the $z$-axis. The source has extension $\delta r_1$ in the $r_1$-direction, while the detector has extension $D_{r_1}$ in the $r_1$-direction. (We neglect the variation in the $r_1$-direction over a gap.) Considering $dz$ as the projection of $\delta r_1$ onto the $z$-axis, taken along $\psi_s$ or $\psi_f$, we have, from Eqs. (5.12) and (5.15a),

$$\delta r_1 = - \delta z_s \tan \psi_s \text{ and } \delta r_{1f} = - \delta z_f \tan \psi_f$$  \hspace{1cm} (6.35)

In analogy to Eq. (6.10a), we introduce the radial dispersion:

$$\sigma_f^r = \frac{\delta r_{1f}}{\delta A_0/A_0} = - \sigma_f \tan \psi_f$$  \hspace{1cm} (6.36)

From Eqs. (5.15b) and (5.15c),

$$u(\psi_s) = \frac{\delta r_{1f}}{\delta r_1} = Q_f \frac{\tan \psi_f}{\tan \psi_s}$$  \hspace{1cm} (6.37)

The previous results with an axial source may be extended to the radial source by considering the projection of $\delta r_1$ onto the axis and substituting this projection into the corresponding equation. Thus, from Eq. (6.15), with $D_{r_1} = 0,*$

$$\tau' = \left| \frac{- \delta r_{1f}}{\sigma_f \tan \psi_f} \right| = \left| \frac{- \delta r_1}{\sigma_f \tan \psi_s} \cdot Q_f \right| = \left| \frac{\delta r_{1f}}{\sigma_f^r} \right| = \left| u(\psi_s) \frac{\delta r_1}{\sigma_f} \right|$$  \hspace{1cm} (6.38)

From Eq. (6.16), with $D_{r_1} \neq 0$, the relative base width is

$$\tau'_b = \frac{(\delta r_{1f} + D_{r_1})}{- \sigma_f \tan \psi_f} = \frac{\delta r_{1f} + D_{r_1}}{\sigma_f^r}$$  \hspace{1cm} (6.39)

*As above, only absolute values of lengths, coefficients and projection functions ($\tan \psi_s$ and $\tan \psi_f$) are taken.
For $\delta r_1 \geq D_{r_1}$,

- Half-width $R' = \delta r_1 / \sigma_{r_1} = R_0'$
- Peak height $t' = d\omega (D_{r_1} / \delta r_1)$
- Profile area $S' = d\omega (D_{r_1} / \sigma_{r_1}^2)$

For $\delta r_1 \leq D_{r_1}$,

- Half-width $R' = D_{r_1} / \sigma_{r_1}^2$
- Peak height $t' = d\omega = t_M$
- Profile area $S' = d\omega (D_{r_1} / \sigma_{r_1}^2)$

Similarly, for the complete $\psi_s$-range, the analogs of Eqs. (6.23) to (6.34) are made by substituting the radial quantities for the axial ones. It should be pointed out in using the method involved in Eqs. (6.32) to (6.34), that $\sigma_f$ is not necessarily monotonic over the $\psi_s$-range used, even when $\sigma_f$ is, because of the $\tan^2 \theta$-factor. From Eq. (6.38),

$$R_0' = \left| Q_f \frac{\delta r_1}{\sigma_f \tan \psi_s} \right|$$

For the range over which $|\tan \psi_s| > 1$, i.e., for $45^\circ < \psi_s < 135^\circ$, it is evident from Eq. (6.15) that the resolution of a radial source is better than that of an axial source of the same dimensions.

The study of sources and detectors which have both $z$- and $r_1$-extension may be done by considering the equivalent $z$- (or $r$-) projections. For example, the equivalent axial image of such a source is*

$$\delta z_f = (\delta z_s - \cot \psi_s \delta r_1) Q_f$$

For the $\psi_s$-range $45^\circ < \psi_s < 135^\circ$, a radial source is preferable because it gives a smaller image on the $z$-axis than an axial source of the same extension.

For a symmetric instrument, it is desirable that both the source and detector have the same extension (either in the $z$-extension or $r_1$-extension). For a symmetric instrument, the image size is constant, independent of $\psi_s$, and when the detector has the same size, maximum transmission is attained with minimum resolution.

*In this relation, the signs of the various terms are included.
Strong fringing fields may affect the transmission conditions to a great extent, as will be discussed in detail in Section #9. They influence only the image size and distribution of rays in the $r_2$-direction, if we disregard the slight deviation of point-to-point focusing in the $r_1$-direction because of trajectory rotation (Section #5). However, the resulting shift in the required $r_2$-dimension of the detector (for maximum transmission) affects the resolution because the $r_2$-direction for one gap is the $r_1$-direction for a gap $90^\circ$ around the torus. The required shift in the detector dimension thus leads to a worsening of the resolution. In this case, it is not feasible to maximize transmission without worsening resolution simultaneously. However, once having chosen $D_{r_1}$, the calculation proceeds as before, provided $D_{r_1}$ is constant for all $r_{2f}$-values. When this is not true, a detailed integration (as in Section #9) is required to consider the entire line profile. Of course, instead of using $d\omega = \theta \sin \psi \, d\psi$ as a weighting factor, it is necessary to use the transmission factor appropriate for fringing fields, e.g., $\eta_d d\omega$. Methods for improving transmission in the presence of fringing fields without worsening resolution are discussed in Section #9.
The transmission \( T \) of the instrument may be defined as the ratio between the number of particles detected per unit time and the number of particles emitted during that time. Such a general definition, however, does not allow computation of numerical values from the structure of the instrument, allowing for such factors as geometrical efficiency, detector efficiency, source-size effect, or fringing-field effect. It is clear that the maximum transmission is equal to the geometrical efficiency, other effects serving only to decrease the transmission. We shall find it convenient to express the transmission as a product of separate terms, the dominating term being the geometric efficiency, while the others are independent factors which express the decrease in transmission due to other effects. Thus,

\[
T = \frac{\Omega_0}{4\pi} \eta_1 \eta_{23}, \tag{7.1}
\]

where \( \Omega_0 \) is the theoretical solid angle subtended by the instrument, \( \eta_1 \) is the mean detector efficiency, and \( \eta_{23} \) is an average transmission effect determined by properties of the instrument which will be discussed shortly. In general, \( \eta_1 \) is energy-dependent, at least to some extent, but is usually \( \phi \)-and \( \psi_s \)-independent. We shall treat it as independent of the spectrometer properties. If other \( \eta_1 \) terms are required, they may be included in Eq. (7.1).

If calculated for a centered point source, \( \Omega_0 \) is given by the equation

\[
\Omega_0 = 4\phi \int_{\psi_{s_{\text{min}}}}^{\psi_{s_{\text{max}}}} \phi \sin \psi_s \, d\psi_s = j\phi \left[ \cos \psi_{s_{\text{min}}} - \cos \psi_{s_{\text{max}}} \right], \tag{7.2}
\]

where

- \( j \) = number of gaps
- \( \phi \) = angular aperture of the gaps
- \( \psi_{s_{\text{min}}} \) = the smallest emission angle considered
- \( \psi_{s_{\text{max}}} \) = the largest emission angle considered.

In computing \( \Omega_0 \), the entrance aperture was defined by the boundaries of the entrance profiles. Some of the particles passing through this aperture may, however, be stopped by two apertures further along the trajectory: (i) the exit aperture, defined by the boundaries of the exit profiles, and
(ii) the detector aperture. The numerical measure of these two effects is given by \( \eta_2 \) and \( \eta_3 \), which are defined for rays emitted from a single point on the source, at a given \( \psi_s \). Here \( \eta_2 \) is the fraction of those particles which have passed through the entrance aperture and which then also pass through the exit aperture of the same gap; the fraction of those transmitted through the exit aperture and which also enter the detector is \( \eta_3 \). Then

\[
\eta_{23} = \eta_2 \eta_3
\]

(7.3)

where \( \eta_{23} \) is also defined at a given \( \psi_s \)-value and emitting point. The quantity \( \eta_{23} \) is determined by integrating \( \eta_{23} \) over the entire sample and \( \psi_s \)-range.

Up to this point, trajectories have been computed neglecting fringing-field effects. Under these conditions, for a point source \( \eta_2 = \eta_3 = \eta_{23} = 1 \), i.e., only the entrance aperture acts as a limiting baffle, since all trajectories remain on \( \phi \)-planes (\( p_\phi = 0 \)) and end at \( (z_f, 0, 0) \). However, where a trajectory has a \( p_\phi \)-component, the resulting orbit rotation \( \Delta \phi \) may cause \( |\phi_f - \phi_M| \) to exceed \( \frac{1}{2} \phi \), implying that the trajectory does not penetrate the exit aperture, e.g., colliding with the poleface. In this case, \( \eta_2 < 1 \).

Such a \( p_\phi \)-component may be introduced either by source extension away from \( (z_s, 0) \), as discussed in Section #5, or by fringing-field deflection (Sections #8 and #9). It is shown in Section #9, that the fringing-field effect may indeed be treated as a virtual source extension. The effect of source extension in causing orbit rotation was developed in Section #5, and it is this theory which is used to evaluate the transmission losses.

When the source is large, the effect of its extension may be considerably larger than the fringing-field effect of Section #9. On the other hand, for a small source, the fringing-field effect may dominate. We shall, therefore, develop in this section the theory of transmission losses due to source extension alone, with no fringing-field effect; the transmission factors are labelled \( \eta_2^*, \eta_3^* \) and \( \eta_{23}^* \). We shall then carry out the computation of fringing-field transmission losses for a point source in the first part of Section #9, labelling the transmission factors as \( \eta_2, \eta_3 \) and \( \eta_{23} \). For many purposes, the results of these relatively simple computations may suffice, as, for example, when only approximate values are required to facilitate choice of instrument design or of sample size. The complete result, including both source extension and fringing-field effects, is covered in the second part of Section #9. The factors in this case are \( \eta_{2T}, \eta_{3T} \) and \( \eta_{23T} \). This theory, of course, results in considerably more complicated computations, but is required where more accurate results are required in cases where fringing-field and source-extension effects are comparable.
Transmission Losses Due to Exit Profile Aperture. $\eta_i^*$. We shall consider a single gap of the spectrometer and take the source to have uniform density of emission. If this is not true, but the variation of density over the source is known, then the results may be modified in an obvious manner; the integrations cannot then be performed explicitly, but must be done numerically. We take the source to lie on a plane perpendicular to the z-axis at $(z_s,0)$, with a shape symmetrical around $\phi_M$ and around the line perpendicular to $\phi_M$, as in Fig. (7.1). The coordinates of a point on the source are given by the $(r_1,r_2)$ components as in Section #5.

To simplify computation, an approximation will be made by taking $r_1$ as always lying in $\phi_M$, rather than in $\phi_s$; then $r_2$ is always perpendicular to $\phi_M$. This approximation leads to the neglect of the following effects:

(i) The magnitudes of $r_1$ and $r_2$ are in error, but only insofar as $\cos(\phi_p - \phi_M)$ differs from $\cos(\phi_p - \phi_s)$ and $\sin(\phi_p - \phi_M)$ from $\sin(\phi_p - \phi_s)$, where $\phi_M + \frac{1}{2}\varphi \geq \phi_s \geq \phi_M - \frac{1}{2}\varphi$, and $P$ is the source point [see Fig. (7.2)]. For small gap angle $\varphi$, this difference is small.*

(ii) For $\phi_p$ within the gap $[|\phi_p - \phi_M| < \frac{1}{2}\varphi]$, the sign of $r_2$ may be changed, depending upon the $\phi_s$-value. Thus, if $\phi_s > \phi_p > \phi_M$, then $r_2$ should be positive, but, with the approximation made, it is taken as negative. This has the effect of changing the direction of orbit rotation, i.e., the sign of $(\phi_k - \phi_s)$. The effect is not important, both because of the limited fraction of the

*For the largest $(\phi_s - \phi_M)$ difference, $\frac{1}{2}\varphi$, 

\[
\cos(\phi_p - \phi_M) - \cos(\phi_p - \phi_s) \simeq \frac{\varphi^2}{8} \cos(\phi_p - \phi_M) + \frac{1}{2}\varphi \sin(\phi_p - \phi_M); \\
\sin(\phi_p - \phi_M) - \sin(\phi_p - \phi_s) \simeq \frac{\varphi^2}{8} \sin(\phi_p - \phi_M) - \frac{1}{2}\varphi \cos(\phi_p - \phi_M). 
\]

At maximum, the difference is $\frac{1}{2}\varphi$, which for $\varphi = 3^\circ$ is 0.026, and for $\varphi = 10^\circ$ is 0.087. For most source points and $\phi_s$-values, the errors are quite a bit smaller.
source involved (for small $\varphi$), and because only small orbit rotations are involved, since $|r_2|$ is small (hence, the likelihood of loss of the ray through rotation is small).

Because the $r_2$-component alone determines the orbit rotation which leads to transmission losses, these losses are the same for each quadrant of a source which has the symmetry of Fig. (7.1). We choose to consider only the upper left-hand quadrant of the source.

If $S$ is the specific activity (number of particles isotropically emitted per unit time) and $d_0$ the surface element on the source, then the collecting power or geometry subtended by the entrance profiles of a single aperture at the source is given by

$$
C_S = \frac{\omega S}{\frac{4\pi}{d_0}} = \frac{1}{4\pi} \int \int S \, d\sigma \, d\Omega \frac{1}{D} \int \int S \, d\sigma \, \phi \sin \psi \, d\psi \, d\sigma
$$

where $d\Omega$ is an element of solid angle and $D$ the disintegration rate of the entire source. To a good approximation $\psi$ may be taken as determined only by the point of entry into the field and independent of the source point. The transmission losses due to orbit rotation may be described as an effective decrease in gap aperture, those $\phi_\text{g}$-values for which trajectories do not pass through the apertures being excluded. The effective geometry may then be computed by substituting $(\phi - \Delta \phi)$ for $\phi$ in Eq. (7.4). Since these losses are a function of $\psi_\text{g}$ (through $F_1$) and of the emitting point $P$, then $\Delta \phi = f(\psi_\text{g}, r_1, r_2)$. The effective geometry is then

$$
C_S \, \overline{\eta_{23}} = \frac{1}{4\pi D} \int \int S \, d\psi_1 \, d\psi_2 \sin \psi_\text{g} \, d\psi_\text{g}
$$

where $\overline{\eta_{23}}$ is the average over the entire sample and $\psi_\text{g}$-range. Since

$$
\eta_{23}^* \, \phi = \phi - \Delta \phi
$$

$\eta_{23}^*$ is a function of $(\psi_\text{g}, r_1, r_2)$.

The use of $\overline{\eta_{23}}$ in Eq. (7.5) implies that losses due to both the exit and detector apertures are included. If it is desired to evaluate the fraction of particles emitted which emerge from the exit aperture, without regard to the detector aperture, then $\eta_{23}^*$ is substituted in Eq. (7.5), $\eta_{23}^*$ in Eq. (7.4), and, in general, $\Delta \phi$ is a different function of $(\psi_\text{g}, r_1, r_2)$ than when $\eta_{23}^*$ and $\eta_{23}^*$ are considered.
It is convenient to take $\phi_M = 0$, so that the gap boundaries are $\phi = \frac{1}{2} \varphi$ on the left [see Fig. (7.2)] and $\phi = -\frac{1}{2} \varphi$ on the right.

We consider first the evaluation of $\eta_2^*$ for a single point $P$ in the source quadrant. Since $r_2$ is the only factor which determines, at a particular $\psi_s$, the $\phi_s$-planes which allow transmission through the exit aperture, $\eta_2^*$ is not a function of the $r_1$-coordinate of $P$. From Eq. (5.19b), the rotation is

$$\Delta \phi = \phi_f - \phi_s = F_1 \left( \frac{r_2}{z_s} \right) \quad ,$$

(7.7)

$r_2$ being negative in the left quadrant, $\Delta \phi < 0$; i.e., $\phi_f < \phi_s$, or the rotation is to the right. At a given $r_2$-value, $\Delta \phi$ is constant for a particular $\psi_s$. There is, then, a critical $\phi_s$-value, $\phi_C$, such that for $\phi_s$ to the right of $\phi_C$ in Fig. (7.2) ($\phi_s < \phi_C$), then $\phi_f < -\frac{1}{2} \varphi$ in Eq. (7.7). Trajectories entering on these $\phi_s$-planes will then not be transmitted through the exit aperture.

From Eq. (7.7),

$$\phi_C = - \left[ \frac{1}{2} \varphi + F_1 \left( \frac{r_2}{z_s} \right) \right] \quad (7.8a)$$

and

$$\eta_2^* = \frac{\frac{1}{2} \varphi - \phi_C}{\varphi} = 1 + \frac{F_1 r_2}{\varphi z_s} \quad . \quad (7.8b)$$

When $\phi_C = \frac{1}{2} \varphi$, then the corresponding $r_2 = r_0^{*+}$ is a boundary value,* such that no transmission occurs at any $r_2$ for which $|r_2| > |r_0^{*+}|$. From Eq. (7.8b)

$$r_0^{*+} = - \frac{\varphi z_s}{F_1} \quad . \quad (7.8c)$$

It may be noted that $\eta_2^*$ is always less than units for a source point away from $(z_s, 0)$. This is due to the fact that a nonzero $r_2$ leads to a definite orbit rotation, which, for some $\phi_s$, gives rise to collision with the poleface (i.e., results in $\phi_f < -\frac{1}{2} \varphi$).

Since $\eta_2^*$ is not a function of $r_1$, this transmission loss is constant over the strip of the sample of constant $r_2$. If the boundary of the sample is described by the relation $r_1 = R_1(r_2)$, and if the specific activity $S$ is constant over the sample, then the transmission through the exit aperture at constant $\psi_s$, averaged over the sample, is

$$\eta_2^*(\psi_s) = \frac{1}{4} \frac{A_s}{S} \int_{r_2m}^0 \left[ R_1 \left[ 1 + \frac{F_1 r_2}{\varphi z_s} \right] \right] dr_2 \quad , \quad (7.9)$$

*The reasons for this notation will be evident in Section #9 [see Table (9.2) and Fig. (9.15)].
where $a_s$ is the area of entire sample. If $R_2^*$ is the magnitude of the farthest $r_2$-extension of the source, then $(-r_{zm})$ is the smaller of $(-r_{0m}^*)$ and $R_2^*$.

If the source is circular, of uniform density, then,

$$|R_1| = \sqrt{R^2 - r_2^2}$$

where $R = \text{source radius} = R_2^*$. If $r_{zm} = -R$ [i.e., $r_{0m}^*$ is outside the source], then,

$$\eta_2^*(\psi_s) = 1 - \frac{4}{3\pi} \frac{F \rho}{\varphi \rho_a} = 1 - \frac{4}{3\pi} \frac{1}{\alpha} \left[ 1 - \left(1 - \alpha^2\right)^{3/2} \right] \tag{7.10a}$$

where

$$\frac{1}{\alpha} = \frac{R}{|r_0^* - |} \frac{R F_1}{\varphi \rho_a} \tag{7.10b}$$

and

$$\alpha > 1.$$

If $r_{zm} = r_{0m}^*$, then

$$\eta_2^*(\psi_s) = \frac{2}{\pi} \left[ \alpha \sqrt{1 - \alpha^2} + \arcsin \alpha \right] - \frac{4}{3\pi} \frac{1}{\alpha} \left[ 1 - \left(1 - \alpha^2\right)^{3/2} \right] \tag{7.10c}$$

where $a$ is defined in Eq. (7.10b), but $\alpha < 1$.

For a square source, whose half-side is $R$, with $\alpha > 1$, for $r_{zm} = -R$,

$$\eta_2^*(\psi_s) = 1 - \left(1/2\alpha\right) \tag{7.11a}$$

For $r_{zm} = r_{0m}^*$, then, with $\alpha < 1$,

$$\eta_2^*(\psi_s) = \frac{1}{2} \alpha \tag{7.11b}$$

After $\eta_2^*(\psi_s)$ is computed, as a function of $\psi_s$, numerical integration may be used to determine

$$\bar{\eta}_2^* = \frac{C_s \eta_2^*}{C_s} = \frac{1}{4} \int \int \int S \varphi \sin \psi_s \sin \psi_s \, d\psi_1 \, d\psi_2 \, d\psi_3$$

$$= \frac{1}{4} \int \int \int S \varphi \sin \psi_s \, d\psi_1 \, d\psi_2 \, d\psi_3$$

$$= \frac{1}{4} a_s \int \sin \psi_s \, d\psi_s$$

$$\int \sin \psi_s \, d\psi_s$$

since $\psi_s$ is not a function of the source point.

The averaging may also be performed over $\psi_s$ prior to the integration over the sample. Thus,
\[
\eta_2^* = 1 + \frac{\iiint (F_1 r_2 / r z_s) \sin \psi_s \, d\psi_s \, dr_1 \, dr_2}{\iiint \sin \psi_s \, d\psi_s \, dr_1 \, dr_2}
\]

\[
= 1 + \frac{1}{4 \phi_s z_s} F_1(\psi_s) \int r_2 \, dr_1 \, dr_2 ,
\]

(7.13a)

the second term being negative. Here \( F_1(\psi_s) \) is defined as

\[
F_1(\psi_s) = \frac{\int F_1(\psi_s) \sin \psi_s \, d\psi_s}{\int \sin \psi_s \, d\psi_s} .
\]

(7.13b)

Some results of such averaging are shown in Figs. (7.3) and (7.4), in the first of which \( \phi = 10^\circ \) and \( F_1(\psi_s) \) varies, and in the second of which \( F_1(\psi_s) = 1.09 \) and \( \phi \) is variable. The following table gives values of \( F_1(\psi_s) \) as a function of \( n_s \) and \( K \), calculated for \( \psi_s \) in the range from 95° to 145°.

<table>
<thead>
<tr>
<th>( n_s )</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
<th>0.8</th>
<th>1.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.24</td>
<td>0.59</td>
<td>1.09</td>
<td>1.73</td>
<td>2.54</td>
</tr>
<tr>
<td>(-1/2)</td>
<td>1.20</td>
<td>3.89</td>
<td>9.37</td>
<td>19.2</td>
<td>36.6</td>
</tr>
</tbody>
</table>

For a symmetrical spectrometer, \( u = v = -1 \), so that the image is identical* to the source, except for being reflected in the origin. For an unsymmetrical spectrometer, \( u \neq v \), but, since \( \psi_s \) is independent of the source

*We neglect here the lack of perfect \( \phi \)-focusing, discussed in Section 5. The smearing of the image of a point source into a small arc is a very small effect when the gap angle \( \phi \) and \( r_2 \) are small, as has been assumed here.
point, the magnification of \( r_1 \) is uniform over the entire source as is that of \( r_2 \), although the magnifications are unequal. A circle then goes into an ellipse, although the size and shape of the ellipse vary with \( \psi_s \).

Because of the essential point-to-point imaging, at constant \( \psi_s \), all of the rays from a single source point which emerge from the exit aperture are counted if the image point is within the detector aperture, or none of them are counted if the image point is outside, i.e., \( \eta_3^* = 1 \) or \( \eta_3^* = 0 \). If the image is entirely within the detector aperture, then \( \eta_3^* = 1 \) for all image points, and

\[
\eta_{23}^*(\psi_s) = \eta_2^*(\psi_s) \quad (7.15)
\]

\( \eta_2^*(\psi_s) \) being given by Eqs. (7.9), (7.10), and (7.11).

Where the image does not lie within the detector aperture, \( \eta_{23}^*(\psi_s) \) is computed by means of the integral in Eq. (7.9), but redefining \( R_1 \) and \( r_{2m} \). If the detector aperture is described by the curve \( r_{1f} = \Gamma_1(r_{2f}) \), then \( R_{10} \) is the smaller of \( |R_1| \) and \( |\Gamma_1/u(\psi_s)| \), at a particular value of \( r_2 \) (and hence \( r_{2f} \)). If \( D_{r_2} \) is the magnitude of the farthest \( r_{2f} \)-extension of the detector aperture, then \( (-r_{2m}) \) is the smallest of \( |r_0^*| \), \( R_2^* \), and \( D_{r_2}/|\psi(\psi_s)| \).

We have, then,

\[
\eta_{23}^*(\psi_s) = \frac{1}{4} \frac{1}{d_s} \int_{r_{2m}}^0 R_{10} \left[ 1 + \frac{F_1 r_2}{\psi z_s} \right] dr_2 \quad (7.16)
\]

In general, \( R_{10} \) may be defined by \( R_1 \) for some \( r_2 \)-values, and by \( \Gamma_1 \) for others. The symmetrical case, however, may be expressed in some greater detail, because the imaging is, to the approximation considered, perfect, and the magnification is the same for both \( r_1 \) and \( r_2 \). We consider the case of a circular source of radius \( R = R^*_2 \) and a circular detector aperture of radius \( R_d = D_{r_2} \). If \( R_d > R \), then Eq. (7.15) holds. If \( R_d < R \), then \( R_{10} = \sqrt{R_d^2 - r_{2f}^2} = \sqrt{R_2^2 - r_2^2} \), and \( |r_{2m}| \) is the smaller of \( |r_0^*| \) and \( R_d \). If \( r_{2m} = -R_d \), then

\[
\eta_{23}^*(\psi_s) = \left[ 1 - \frac{4}{3\pi} \frac{F_1 R_d}{\psi z_s} \right] \frac{R_d^2}{R^2} = \left[ 1 - \frac{4}{3\pi} \frac{1}{\alpha_d} \right] \frac{R_d^2}{R^2} \quad (7.17)
\]

with

\[
\alpha_d = \psi z_s/R_d F_1 = \alpha(R/R_d) \quad (7.18)
\]

If \( r_{2m} = r_0^* \), then the result is the same as in Eq. (7.10c), except that the result is multiplied by \( (R_d/R)^2 \) and \( \alpha_d \) replaces \( \alpha \).

Averaging over \( \psi_s \) is performed as in Eq. (7.12) or Eq. (7.13).
Transmission through Multigap Instruments. The above discussion gave the transmission losses for a single gap. For a multiple-gap instrument, the total transmission is the summation of the transmissions of the individual gaps. For sources and detectors having circular symmetry, the summation is simply a multiple of the single-gap transmission.

Comparison between Iron and Iron-free Toroidal System. In a spectrometer consisting of a series of gaps in an iron magnetic circuit, with coils appropriately placed for creating magnetomotive force,\(^{(1,10)}\) when a trajectory is forced into an equipotential surface defined by a poleface, it is lost to the imaging process. It is either absorbed or scattered by the iron poleface; the latter process may be with or without energy loss. In either kind of scattering, the coherent relation to the original source point is lost, and no imaging can occur. Obviously, baffles and slits must be arranged to avoid interference of the scattered particles with the true image. In an iron-free instrument,\(^{(14-18)}\) the gaps are simply spaces between the wires comprising the torus, and the equipotential surfaces constituting the extension of the gaps are not blocked by matter. Hence, the scattering problem is less troublesome, and particles deflected into an equipotential surface defining a gap may pass through and emerge through the exit side of an adjacent gap.

The discussion of \(\eta_2\) and \(\eta_3\) for a point source in a fringing-field will be given in the first part of Section #9, following a description of the fringing-field and its deflecting properties in Section #8.
PART B

ANALYSIS CONSIDERING FRINGING-FIELD EFFECTS
Thus far, we have assumed that the magnetic field within the spectrometer started abruptly at the profile, following the relation Eq. (1.1), and was zero outside of the profile. As is known, this is only a rough approximation, since all physical gaps have fringing fields which: (i) extend outside the profile and attenuate to essentially zero only at some distance from the profile, and (ii) decrease the magnitude and distort the direction of the field for a short distance within the profile. Because the fringing field does not follow Eq. (1.1), the trajectories of particles passing through this field are not evaluated correctly with the theory developed thus far. It is necessary, therefore, to compute the effect of the fringing field on the trajectories and to correct the previous results for this effect. Under suitable restrictions, the fringing field can be made small enough so that the effect can be treated as a perturbation problem. The main features of the trajectory are then computed as before, with the fringing field effect added as a correction. It will be found that the wide flexibility previously allowed in the choice of K-values, profile curves, and \((n_s, n_f)\)-values is considerably constrained by the necessity for minimizing the fringe-field effect.

Structure and Magnitude of Fringe Field. In principle, if the magnitude and direction of the magnetic field are specified at all points in the region of interest, the trajectories may be computed, even if only numerically. It is not feasible for most magnetic structures to compute the fringe field from the configuration of iron and/or current distribution. It is generally necessary to construct the magnet and then to measure its fringe field by any of a number of field-probing techniques. With such a detailed measurement, the numerical integration of trajectories may be performed. Little information of this type is available for the sliced-orange type of system, so we consider some general features of other magnetic structures in order to develop some qualitative considerations.

The simplest type of structure is that of a pair of parallel polefaces, with a straight-line profile. The lines of force are emitted in a plane perpendicular to the profile. The intensity of the field at a given distance from the profile scales with the gap width (provided the coils are simultaneously changed appropriately). The perturbation of the interior field is relatively small within one-half gap width inside the profile, and is essentially zero at one gap width inside. The external fringing field drops steeply at first, the drop (on the median plane) to about 10% of the interior field value being roughly approximated by a straight line, whose intercept is approximately 2 gap widths out. The field then has a slowly decreasing component which goes out a number of gap widths, but whose rate of fall-off is very sensitive to the structure of the poles, placement of coils, distance to other magnetic materials, etc.
If we apply these general considerations to the $r^{-1}$-field case, we note that

(i) the fringe lines emerge perpendicular to a poleface (if iron) and tend to be perpendicular to the profile as in the above case;

(ii) in the uniform field case, the density of lines of force is uniform as one moves along the profile at a particular distance from profile and median plane. In the $r^{-1}$-field magnet, however, the density of lines of force decreases with $r$, so that the "mutual repulsion" of lines tends to make the lines of force bend up toward increasing $r$ as they leave the poleface. For small gap angles this effect is small and will not be considered at this point; and

(iii) the merging of the fringe field with the interior field should be relatively insensitive to the exterior structure and should be much the same as described for the case of a uniform field.

Some measurements and computations have been made for the sliced-orange type of field, but only on the rate of fall-off on the median plane. Where information is needed concerning the field and deflections in the off-median region, this has been estimated by an indirect method (Sections #10 and 11). Certain aspects of the fringing-field effects are not sensitive to the detailed structure of the field, and these results are computed with the greatest reliability. Some machine computations have been made of the fringing field on the median plane of a 100-gap iron-free spectrometer, with $\phi = 3^\circ$, and symmetrical profiles designed for $K = 0.59$ (Section #3), (16,17) $\psi_s = 110^\circ$ to $150^\circ$. Measurements were made on the median plane of an 8-gap iron-core unit, (5,12) with $\phi = 10^\circ$, $K = 0.60$, and $\psi_s = 95^\circ$ to $145^\circ$. The entrance curve was a circle with radius $\rho$ of 27.0 cm, whereas the appropriate exit curve for focusing was computed via Eq. (3.2a) and (3.3). Some of the results will be quoted after some definitions have been made.

(i) The fringing field $\mathbf{H}$ at the point $(z',r', \phi')$ is decomposed into two orthogonal components:

$$\mathbf{H} = H_\phi + H_{rz}$$

where $H_\phi$ is the component normal to the plane $\phi = \phi'$ at the point, with positive direction corresponding to increasing $\phi$, and $H_{rz}$ is the component lying in the plane $\phi = \phi'$. If the fringe line lay in a plane perpendicular to the profile, then $H_{rz}$ would be parallel to the normal to the profile. Because of the bowing of the lines of force toward larger $r$ as they leave the poleface, this is not quite true, but for our purposes it may be taken as approximately correct. On the median plane $\phi_M$, from symmetry considerations $\mathbf{H} = H_\phi$. Although the direction is the same as that given in Eq. (1.1), the magnitude is not. Inside the profile, the $H_{rz}$ component for any $\phi'$ rapidly decreases to zero as a trajectory passes toward the interior, and $\mathbf{H} \rightarrow H_\phi$ and comes closer to the value given by Eq. (1.1).
(ii) If we consider only trajectories starting from \((z_s,0)\), then, at a given \(\psi_s\), the trajectory lies in the same plane as \(H_{rz}\) as it approaches the profile,\(^*\) so \(H_{rz}\) may be resolved as

\[ H_{rz} = H_t + H_{\perp} \]

(8.2)

where \(H_t\) lies along the trajectory and \(H_{\perp}\) is normal to it.

These components are shown in Figs. (8.1) and (8.2). Figure (8.1) is a perspective view relative to the plane containing the trajectory and the normal to \(\phi_M\); Fig. (8.2) is a view on the \(\phi'\)-plane.

(iii) Let \(\mu\) be the smaller angle \((\mu \leq 90^\circ)\) between the normal to the profile and the trajectory. If \(\nu_s\) is the angle between the normal on the entrance side and the \(z\)-axis, measured in the same manner as the \(\psi\)-angle [see Fig. (8.3)], then at the entrance profile

\[ \mu_s = \psi_s - \nu_s \]

(8.3a)

On the exit side [see Fig. (8.3)],

\[ \mu_f = \nu_f - \psi_f \]

(8.3b)

On the entrance side, \(\mu_s\) is positive if the rotation from the normal to the trajectory is counterclockwise; at the exit, positive rotation is clockwise.\(^**\) In Fig. (8.3), both exit and entrance \(\mu\)-angles are negative. In general, for a symmetrical spectrometer, the respective \(\mu\)-angles are equal in sign as well as in magnitude. This choice is arbitrary in that Eq. (8.3a) might be used for both entrance and exit, but it would lead to opposite signs of \(\mu\) on both sides of a symmetrical system. This point will be discussed further below.

At this point, we introduce the information known concerning the fringing field of the two spectrometers mentioned previously. The fringing

\(^*\)When an extended source is involved, the trajectory does not necessarily lie in the same plane as \(H_{rz}\), even if the fringe lines lie in a plane perpendicular to the profile, because \(p_{\phi} \neq 0\). However, the deviation is small, and we shall neglect it later when the extended source is treated.

\(^**\)See also Table 12.1.
fields were determined only on the median plane $\Phi_M$. As in the cases of uniform fields, the bulk of the dropoff of field was found to be linear, and as this part of the fringing field is the chief source of the deflection, we shall take the entire dropoff as linear, with a corresponding intercept. The fringing field on $\Phi_M$ was found to be approximately like that shown in Fig. (8.4), which represents the field along a trajectory (for given $\psi_0$) vs $s$, the distance along the trajectory. In the absence of fringing fields, $H$ varies according to the curve $EFBPCGT$; along this curve, $H = A_0/r$, as in Eq. (1.1). The lines $EF$ and $TG$ represent the field at the profile, and are greater than the field on the remainder of the trajectory, because the trajectory $r$-values within the profile are greater than the corresponding profile $r$-values when $n_s = n_f = 0$.

When fringing fields are present, the field is nonzero at some distance from the profile and does not follow Eq. (1.1) until the trajectory has gone beyond the profile (at $B$ and $C$). The actual field approximately follows $ABPCD$. The linear variation of the magnetic field in the transition regions ($AB$ and $CD$) is only an approximation, the transitions at $A, B, C,$ and $D$ actually being gradual rather than abrupt. However, the approximation is quite close to the measured values.

As shown in Fig. (8.4), we define the quantity $d_0$ as the total distance along the part of the trajectory for which the fringing field is significant; this distance is reckoned using the straight-line approximation for the fringing field. The other distances shown in Fig. (8.4) are discussed in Section #13.

It is to be noted that $(z_0, r_0)$ is taken as the point at which the fringing field merges with the $r^{-1}$-field. Strictly speaking, this is a poorly defined point, since the fringing field merges only gradually with the interior field. It is made definite here by taking it as the intersection of the extrapolated
straight-line portion of the fringing field with the field curve for the idealized system \((H = 0 \text{ outside profile}, H = A_0/r \text{ inside})\). The notation is consistent with the previous nomenclature, since the \(r^{-1}\)-field starts at the profile in the idealized case. The value of the field at \((z_e, r_e)\) is \(H_e\); evidently, \(H_e = A_0/r_e\), from Eq. (1.1).

As mentioned above, for uniform field magnets with straight profiles, the fringe field tends to scale with the gap width. If this were true for the \(r^{-1}\)-field magnets, then it would be found that \(d_e \propto g_e = r_e \phi\), where \(\phi\) is the gap angle, \(r_e\) the \(r\)-value at \(B\) [see Fig. (8.4)], and \(g_e\) the gap width. In the actual magnet, however, \(g_e/d_e\) varied with \(\psi_s\). In the 8-gap iron-core instrument, \(g_e/d_e\) varied between 0.85 and 0.60 for \(\psi_s\) between 95° and 145°, respectively, the fringe field extent \(d_e\) relative to the gap width being smaller for the larger gap widths. The air-core spectrometer had a similar variation of \(g_e/d_e\). In this case, because the conductors defining the torus had a depth comparable to the gap width, the position of the profile was taken as the center of the conductor (i.e., part way into the actual gap). The ratio \(g_e/d_e\) was found to vary from 0.64 to 0.44 over the respective range of \(\psi_s\) from 110° to 150°.

Nature of the Trajectory Perturbations. The fringing field causes perturbations to the trajectories so that they do not follow the equations described in Sections #1 through 3. It is convenient to analyze these effects into components. Because the effects are small, it is possible, to the first order, to consider the entire perturbation as a linear superposition of these effects. They are:

(a) The \(\psi\)-deflection due to the \(H_\phi\)-component of the fringe field. The \(H_\phi\)-component is in the same direction as in the interior field, and hence tends to cause a change in the \(\psi\)-coordinate in the same direction as does the focusing field. [The effect of the other fringe-field components may be neglected, since, to the approximation that \(B_\phi\) remains small, the \((z, r)\) motion is dependent only on \(H_\phi\).] The added \(\psi\)-deflection may be considered as equivalent to having a larger field region (of the \(r^{-1}\) type) than that defined by the profiles, so the effective boundaries are outside the real boundaries. Hence, with a given profile, the value of \(A_0\) [see Eq. (1.1)] required to focus a particle with given \(P\) and \(\psi_s\) differs from that given by the theory previously discussed. In addition, since the fringing field has different effects for different \(\psi_s\)-values, the profiles computed from the theory of Section #3 will not have the theoretical focusing power (i.e., for all \(\psi_s\)) for any \(A_0\)-value.* This effect may be

*In the 8-gap iron-core instrument, it was found that when the profiles were constructed according to the theory of Section #3, the specific field intensity \(A_0\) required to focus trajectories at \(\psi_s = 95°\) was 11% greater than that required for \(\psi_s = 145°\). This difference follows from the fact that the ratio of the deflecting power of the fringing field relative to that of the interior field [see Eq. (1.1)] is larger for larger \(\psi_s\). Hence, a smaller interior value of \(A_0\) brings the 145° trajectories into focus.
corrected by changing the profile shape and reducing the region enclosed by the profile. The calculations will be described later (Sections #12 and 15).

(b) Variation of the ψ-deflection with φ. In the plane φ = φM, we have H = Hφ. On each of the nonmedian planes, the value of the Hφ-component is different. Hence, trajectories passing through the median plane have different ψ-deflections in the fringing field than do nonmedian trajectories, as well as suffering different displacements of the entry points (ze, re) into the interior field [see Eq. (1.1)]. Although the profiles may be corrected for the change in ψ-deflection on the median plane, as in (a), this correction does not remove the variation of ψ-deflection with φ. An estimate of the magnitude of the effect is given in Section #11.

(c) Lateral deflections due to H⊥. This component of the fringing field acts as a cylindrical lens and transforms a point source to a line image. We shall find that the deflecting power of this lens increases as |μ| increases.

We shall first discuss general features of the deflection by the Hφ and the H⊥ components. In Section #9, the detailed analysis of the lens effect will be presented.

ψ-deflection in the Median Plane. At a given point in the fringing field, the deflecting force equals the centrifugal force around the instantaneous center of rotation. Since the velocity v is normal* to H = Hφ, the magnitude of the deflecting force is:

\[ |e_v \times H| = |e_v H_\phi| = \left| \frac{mv^2}{\rho_{\text{inst}}} \right| = \left| mv \omega_\psi \right| = \left| p \omega_\psi \right|, \]  

\( (8.4a) \)

where \( \omega_\psi = \frac{d\psi}{dt} \) is the instantaneous angular deflection velocity whose axis is colinear with Hφ. The algebraic sign is taken from the considerations following Eq. (1.10), in which it was pointed out that eH is taken as negative when the instrument focuses, i.e., when \( \frac{d\psi}{dt} > 0 \). Since \( p > 0 \),

\[ -evH_\phi = p \frac{d\psi}{dt}. \]  

\( (8.4b) \)

Taking ds as positive along the trajectory, v = ds/dt, and

\[ d\psi = -(e/p)H_\phi \, ds \]  

\( (8.4c) \)

*For the general case in which \( v \) is not normal to Hφ (i.e., does not lie in a φ-plane), see Ref. 22 (also see Section #11). The results are essentially the same as Eqs. (8.4c) and (8.5) as long as pφ remains small.
The quantity $\Delta \psi$ may be computed if $H_\phi$ is known along the trajectory. Taking the linear approximation of Fig. (8.4) as valid,

$$H_\phi = H_e \left(\frac{d}{\varphi e}\right),$$  \hspace{1cm} (8.6)

where $d$ is the distance along the trajectory measured from $A$. Then, since $ds = d(d)$,

$$\psi_e - \psi_s = \Delta \psi_e = -\frac{e}{p} \int_0^d H_e \frac{d}{\varphi e} d(d) = -\frac{1}{2} \frac{e}{p} H_e \varphi e$$

from Eq. (1.1) and (1.10). This may also be expressed as

$$\Delta \psi_e = \frac{\varphi}{2K} \frac{d_e}{\varphi e} = \frac{\varphi}{2K} \frac{d_e}{\varphi e}$$  \hspace{1cm} (8.7a)

Since $d_e/\varphi e$ is roughly constant with $r$, then $\Delta \psi_e$ is roughly proportional to $\varphi$. For the 8-gap iron-core instrument, since $d_e/\varphi e$ actually decreased somewhat as $\varphi e$ increased, $\Delta \psi_e$ was smaller for the largest $\varphi e$-values (smallest values of $\varphi_s$).

We return now to the difference in the interior $A_\phi$-value required to focus as compared to the value computed from the theory of Section #3. From Eq. (8.5), it is evident that the areas under the curves in Fig. (8.4) represent the $\Delta \psi$-values for the field configurations involved. If $H_i$ and $\Delta \psi_i$ are the field and overall $\Delta \psi = \psi_f - \psi_s$ for the idealized (no fringe field) case, while $H_a$ and $\Delta \psi_a$ are the corresponding actual values, including the fringe-field effect, then let

$$I = \frac{\Delta \psi_a}{\Delta \psi_i} = \frac{\int H_a \, ds}{\int H_i \, ds} = \frac{ABPCD}{EFPGT}$$

$$= 1 + \frac{(EAU - BFU) + (VDT - GCV)}{EFPGT},$$  \hspace{1cm} (8.8)

the last terms indicating the areas in Fig. (8.4).
In the 8-gap iron-core instrument, the profiles had been designed so that with \( H_1 \) alone, all \( \psi_s \)-values between \( 95^\circ \) and \( 145^\circ \) would focus at \((z_f,0)\). Experimentally it was found that

(i) For all angles \( I > 1 \) when \((H_a,H_i)\) are fields corresponding to the same internal \( A_0 \)-value [see Eq. (1.1)]; thus, the deflecting power of the instrument was increased by the fringe field. Using the same profiles, to bring the actual ray into focus, i.e., to make \( I = 1 \), the \( A_0 \)-value corresponding to \( H_a \) must be made smaller than that for \( H_i \).

(ii) For the same \( A_0 \)-value \( I(145^\circ) > I(95^\circ) \). With an \( A_0 \)-value which focused \( 145^\circ \) rays, the \( 95^\circ \) rays fell beyond \((z_f,0)\), and the \( A_0 \)-value required an 11% increase to cause focusing.

(iii) A graphical evaluation of the areas in Eq. (8.8) for the various \( \psi_s \)-angles showed that \( I(145^\circ) \) exceeded \( I(95^\circ) \) by 11%, for the same \( A_0 \)-value. A decrease of \( A_0 \) by 11% decreased \( H_a(145^\circ) \) by this amount, and hence equalized \( I(145^\circ) \) and \( I(95^\circ) \), where \( I(95^\circ) \) corresponded to the original \( A_0 \)-value.

(iv) At each \( \psi_s \), focusing is achieved through a decrease in the \( A_0 \)-value such that \( I \) is the same for all \( \psi_s \)-values. The fact that different \( A_0 \)-values are necessary for the various \( \psi_s \)-angles indicates that a change is necessary in the profiles; this change is discussed below.

This difference in the focusing \( A_0 \)-value is to be expected, because \( \Delta \psi_i(95^\circ) > \Delta \psi_i(145^\circ) \). Since the fringing field penetrates the idealized field only for a short distance, we may take the deflection \( \Delta \psi_i \) as remaining very approximately the same when fringing field is included, so the major effect is to add a fringe-field deflection; thus, very approximately, \( \Delta \psi_a = \Delta \psi_i + \Delta \psi_e + \Delta \psi_0 \), the last term arising from the exit fringe field. Since \( \Delta \psi_e + \Delta \psi_0 \) is approximately independent of \( \psi_s \) [see Eq. (8.7b)], it is a larger fraction of \( \Delta \psi_a \) at \( 145^\circ \).

This relation is only very roughly correct. For the detailed correction of the profile, it is necessary to take account of the fact that the interior field [see Eq. (1.1)] is decreased by the fringing effect in the region where it is largest, i.e., near the profiles. As shown in Eq. (8.8), it is the difference \( \text{EUA} - 	ext{BFU} \) that gives the actual increase in \( \Delta \psi \) over the idealized situation, rather than the area \( \text{EUA} = \Delta \psi_e \).

Further details of the fringing-field \( \psi \)-deflection effect are discussed in Section #13.

*E.g., in a symmetrical instrument, \( \Delta \psi_i(95^\circ) = 170^\circ \), but only \( 70^\circ \) for \( \Delta \psi_i(145^\circ) \).
Lens Effect of the Fringe Field. Since this effect is due to $H_rz$, it is zero at $\phi_M$, and increases with $|\phi - \phi_M|$. The bending effect of the fringe field is resolvable into that due to the $H_\phi$-component and that due to the $H_rz$-component. The result for the $H_\phi$-component, as pointed out above is the same [see Eq. (8.5)] provided $p_\phi$ remains small. We define $\beta$ as the angle the trajectory makes with the $\phi$-plane, i.e.,

$$|\sin \beta| = \left| \frac{p_\phi}{p} \right|,$$  \hspace{1cm} (8.9)

the sign of $\beta$ being defined below. The analysis is similar to that of the $\Delta \psi$-case; as above, the magnitudes will be treated first, and the signs determined after suitable conventions are made. Then

$$|ev \times H_rz| = |ev \times H_\perp| = |evH_\perp| = |evH_{tan}\mu| = |mv\omega_\beta| = |p\omega_\beta|,$$  \hspace{1cm} (8.10)

where $\omega_\beta = d\beta/dt$ is the instantaneous angular deflection velocity with axis colinear with $H_\perp$.

Two alternatives are available for sign conventions; we shall choose the following:*  

(i) At the entrance, $\beta$ is positive when $p_\phi < 0$; further, $d\beta_e > 0$ when $p_\phi$ is decreased. At the exit, $\beta$ is positive when $p_\phi > 0$; further, $d\beta_e > 0$ when $p_\phi$ is increased.

(ii) Let $H_t$ be positive when it is in the same direction as $\nu$; then $(H_t \cdot \nu) = H_t \frac{ds}{dt} > 0$. Then

$$d\beta = (e/p) \tan \mu H_t \frac{ds}{dt} \hspace{1cm} (8.11)$$

*The sign of $H_t$ is determined by (a) the convention in Section #1 that $eH_\phi$ is negative for focusing, and (b) the condition that the line integral $\int H \cdot ds$ be zero in Eq. (9.2a) of the next section. The reversal of the convention for $\beta$ at the entrance and exit can be avoided if either one of Eq. (8.3a) or (8.3b) are used at both entrance and exit. In this case however, $\mu$ is of opposite sign at entrance and exit in a symmetrical spectrometer, which proves to be a greater inconvenience than the change of sign convention for $\beta$. In addition, the use of the above convention allows the use of the close analogy to optical lenses.
where $ds$ is taken along the direction of $\mathbf{v}$. Integrating from the external region to the interior field ($H_t$ vanishes at both endpoints),\(^*\)

\[
\Delta \beta = \frac{e}{p} \int \tan \mu \ H_t \ ds
\]

(8.12)

Evaluation of this integral requires information as to the variation of $\mu$ and $H_t$ along the trajectory, the calculation is left to the next section.

\(^*\)Equation (8.10), and hence (8.11) and (8.12) stemming from it, contains the assumption that $\mathbf{v}$ lies in a $\phi$-plane. This is no longer true once the lateral deflection has begun, since $\beta \neq 0$. However, $\beta$ is always a very small angle in all practical situations, so that Eq. (8.12) is closely held. See also Ref. 22 and Section #11.
#9. REDUCTION OF TRANSMISSION BY THE LENS EFFECT

Transmission for a Point Source. Because of the lateral deflections \( \Delta \beta \) of Section #8, trajectories which start out on a \( \phi \)-plane are given a \( p \phi \)-component which may lead to collision of the trajectory with the poleface or may lead to the trajectory missing the detector slit. The \( \Delta \beta \)-deflection leads to a transmission loss from a point source in the same way that transmission is lost from a spread source in a field free of fringing, namely, through the orbit rotation resulting from the \( p \phi \)-component. This identity will be used in evaluating the loss. Before pursuing this aspect, we carry on the evaluation of \( \Delta \beta \) started in Section #8.

Computation of \( \Delta \beta \)-deflection for Point Source, Neglecting Interaction with \( \Delta \psi \)-effect. Because of the variation of \( \psi \) through the fringing field, \( \mu [\text{see Eq. (8.3)}] \) and \( \beta \) vary simultaneously. We shall make the approximation that the initial value of \( \mu \) remains constant during the entire \( \Delta \beta \)-deflection, and shall compute the transmission loss on this basis. It will be shown (see Section #10) that the effect of variation of \( \psi \) through the fringing field may be accounted for through substitution of \( \mu \text{effective} \) for \( \mu \text{initial} \) in all the formulae, \( \mu \text{effective} \) being computed from \( \mu \text{initial} \) and \( \Delta \psi \). With this approximation,

\[
\Delta \beta = \frac{e}{p} \tan \mu \int H_t \, ds ,
\]

(9.1)

with \( \mu = \mu \text{initial} \).

This integral may be evaluated without making any detailed assumptions as to the structure of the fringing field. To do this, we utilize the fact that for a closed path which does not link any current, we have

\[
\oint H \cdot ds = 0 .
\]

(9.2a)

The integration loop is taken as shown in Fig. (9.1):

(a) along the trajectory from \((z_e, 0)\), on a single* \( \phi \)-plane, up to \((z_e, r_e, \phi_R)\), at which point Eq. (1.1) is obeyed and \( H_t = 0 \); (b) from \( \phi_R \) to \( \phi_M \) along the arc \( r = r_e \); and (c) from the point \((z_e, r_e, \phi_M)\) to \((z_e, 0)\) on the median plane \( \phi_M \). Along the part (c), since \( H = H_\phi \), we see that \( \int H \cdot ds = 0 \). The loop is taken in such a sense that ds is along \( v \) on the trajectory itself [e.g., clockwise in Fig. (9.1)]. Hence, the loop is reversed on the exit side, being taken from \((z_f, 0)\) to \((z_0, r_0, \phi_M)\) on the median plane.

*The effect of the cumulative \( \beta \)-deflection is neglected. Because of the gradual shift in \( \beta \), there is a change in \( \phi \) by the time \((z_e, r_e)\) is reached. However, since in all practical situations \( \beta \) is quite small, the change of \( \phi \) is small, and we may treat it as though all of the deflection \( \Delta \beta \) were inserted at one point.
plane, from \( \phi_M \) to \( \phi_f \) at constant \( r = r_0 \) to the point \((z_0, r_0, \phi_f)\), and from this point to \((z_f, 0)\) along the trajectory. At the entrance, then,

\[
\int H_t \, ds + H_e r_e (\phi_M - \phi_s) = 0 \tag{9.2b}
\]

or

\[
\int H_t \, ds = H_e r_e (\phi_s - \phi_M) = H_e r_e \Delta \phi_e \tag{9.3a}
\]

At the exit,

\[
\int H_t \, ds = H_0 r_0 (\phi_M - \phi_f) = H_0 r_0 \Delta \phi_0 \tag{9.3b}
\]

If we define

\[
\Delta x_e = r_e (\phi_s - \phi_M) = r_e \Delta \phi_e
\tag{9.4a}
\]

at the entrance, and

\[
\Delta x_0 = r_0 (\phi_M - \phi_f) = r_0 \Delta \phi_0
\tag{9.4b}
\]

at the exit, then

\[
\Delta \beta = \left( \frac{eH}{p} \right) \tan \mu \Delta x \tag{9.5a}
\]

when \( H \) is the field intensity at \( r = r_e \) (or \( r = r_0 \)) and is given by Eq. (1.1). Since \( eH/p = -1/Kr = -1/\rho_C \), where \( \rho_C \) is the radius of curvature of the trajectory [see Eq. (1.1) and (1.10)],

\[
\Delta \beta = -\frac{\tan \mu}{Kr} \Delta x = -\frac{\tan \mu}{\rho_C} \Delta x \tag{9.5b}
\]

or

\[
\Delta \beta = -\frac{\tan \mu}{K} \Delta \phi \tag{9.5c}
\]

The proportionality of the deflection to the angular separation \( \Delta \phi \) of \( \phi_s \) (or \( \phi_f \)) from the median plane is to be expected, since the curvature of the field lines leads to a larger \( H_{rz} \) component with increasing \( \Delta \phi \). Correspondingly, the proportionality to \( \tan \mu \) follows from the fact that a larger \( \mu \) leads to a larger fraction of \( H_{rz} \) appearing as \( H_z \).

The Lens Property. Because of the insight provided, it is convenient to consider the action of the fringing field as that of a lens, even though the specific lens property is not used in computing the transmission. In a simple optical lens system, the angular deflection of a light ray from a single point is proportional to the distance of the contact point from the optic axis, where the optic axis is the direction along which a ray suffers no
angular deflection. Since $\Delta \beta \ll \Delta x$, it is evident that the $\Delta \beta$-deflection may be treated as due to a lens in which the optic axis is the path taken by a trajectory, at a given $\psi_s$, travelling through $\phi_M$.

In light optics, we have the relationship $P = 1/f = \text{ratio of angular deflection to distance of contact point from optic axis}$, with $f$ the focal length of the lens and $P$ the power of the lens. For a diverging lens, $f < 0$ and $P < 0$; for a converging lens, $(f, P) > 0$. Then,

$$P = \frac{1}{f} = -\frac{\tan \mu}{Kr} = -\frac{\tan \mu}{\rho_C}.$$  \quad (9.6)

With the sign conventions as chosen above, when $\Delta \beta$ represents a deflection toward the median plane, $P > 0$; similarly, for deflection away from $\phi_M$, it is found that $P < 0$. This checks with the usual notions of convergent and divergent lenses, respectively; this check is a useful byproduct of the particular convention chosen.

As an example of these considerations, we may examine the cases shown in Figs. (8.1) and (8.2) and (9.1). Here $90^\circ < \psi_s < 180^\circ$, so that, on looking from the source into the gap in Fig. (9.1), $H = H_\phi$ is negative, i.e., arranged for focusing positive particles. The direction of $H$ in the interior determines the fringe-field direction, as in Fig. (8.1). In Fig. (8.2), $\nu_s > \psi_s$ (rotation of normal to trajectory is clockwise), so $\mu < 0$. Using the force equation $ev \times H$ when the trajectory enters the left side of the gap, as in the figures, we note that the deflection is toward $\phi_M$ (i.e., in the negative $\phi$-direction); hence $\Delta \beta > 0$. Since the line integral [see Eq. (9.2a)] is taken clockwise in Fig. (9.1), $\Delta x_e = r_e(\phi_s - \phi_M) > 0$ (positive $\phi$-direction to left). Then $P$ is positive, implying a convergent lens, which checks with the observation that trajectories are bent toward the median plane.

Now $P$ is independent of the sign of the particle being focused, i.e., a converging lens converges for either positive or negative particles. This is evident from Eq. (9.5a), since the sign of $eH$ is always the same; positive $H$ is required to focus for negative $e$, and negative $H$ for positive $e$.

From Eq. (9.5b), it is evident that negative $\mu$ corresponds to a converging lens. In a symmetrical spectrometer, a trajectory passes through lenses of the same $P$-sign and magnitude on entering and leaving the gap.

It is sometimes convenient to take the rotation direction of the exit $\psi$-angle such that it represents a mirror rotation relative to that of the entrance angle. To distinguish between this angle and that defined in Eq. (1.12b), we use $\Psi$ rather than $\psi$. Then,

$$\Psi_s = \pi - \psi_s,$$  \quad (9.7a)
\( \psi_f = \psi_s - \pi \quad , \) \hspace{1cm} (9.7b)

and

\[
d\psi_s = -d\psi_s \quad ; \quad \frac{d\psi_f}{d\psi_s} = d\psi_f \quad . \hspace{1cm} (9.7c)
\]

Positive rotation of \( \psi_s \) corresponds to rotation from the negative z-direction, while that of \( \psi_f \) corresponds to rotation from the positive z-direction. For both \( \psi_s \) and \( \psi_f \), we have \( 0 < \psi < \pi \). For a symmetrical spectrometer, \( \psi_f = \psi_s \) and \( \psi_f = d\psi_f/d\psi_s = 1 \).

From Fig. (9.2), it is evident that \( \tan \mu \approx \frac{\Delta \rho_s}{\rho \Delta \psi_s} \), or \( \tan \mu = \frac{1}{\rho_s} \frac{d\rho_s}{d\psi_s} \), and similarly on the exit side. As to algebraic sign, when \( \Delta \psi_s \) is positive and \( \Delta \psi_f \) negative, \( \Delta \rho \) has the same sign as \( \tan \mu \). Then,

\[
\text{Entrance: } \tan \mu_s = \frac{1}{\rho_s} \frac{d\rho_s}{d\psi_s} = -\frac{1}{\rho_s} \frac{d\rho_s}{d\psi_s} \\
\text{Exit: } \tan \mu_f = -\frac{1}{\rho_f} \frac{d\rho_f}{d\psi_f} = -\frac{1}{\rho_f} \frac{d\rho_f}{d\psi_f} \quad , \hspace{1cm} (9.8a)
\]

where \( \mu_s \) and \( \mu_f \) are defined in Eq. (8.3), \( \rho_s \) is the distance from \((z_s, 0)\) to the trajectory crossing of the entrance profile and \( \rho_f \) is the distance from \((z_f, 0)\) to the trajectory crossing of the exit profile. However, in this section we make the approximation that no \( \psi \)-deflection occurs outside the profile, so that \( \rho_s \) and \( \rho_f \) are the straight-line distances from \((z_s, 0)\) and \((z_f, 0)\), respectively, to the profiles along the \( \psi_s \)- and \( \psi_f \)-directions. Taking \( \rho_e \) as the distance from \((z_s, 0)\) to the interior point \((z_e, r_e)\) and \( \rho_0 \) as the corresponding distance from \((z_f, 0)\) to the interior point \((z_0, r_0)\), then

\[
r_e = \rho_e \sin \psi_s = \rho_e \sin \psi_s
\]

and

\[
r_0 = -\rho_0 \sin \psi_f = \rho_0 \sin \psi_f \quad ,
\]

\[
P = -\frac{1}{\rho_s} \frac{d\rho_s}{d\psi_s} \frac{1}{K \rho_e \sin \psi_s} = \frac{1}{\rho_s} \frac{d\rho_s}{d\psi_s} \frac{1}{K \rho_e \sin \psi_s} \\
\]

\[
P = -\frac{1}{\rho_f} \frac{d\rho_f}{d\psi_f} \frac{1}{K \rho_0 \sin \psi_f} = \frac{1}{\rho_f} \frac{d\rho_f}{d\psi_f} \frac{1}{K \rho_0 \sin \psi_f} \quad , \hspace{1cm} (9.8b)
\]

where \( \frac{1}{\rho_s} \frac{d\rho_s}{d\psi_s} \) and \( \frac{1}{\rho_f} \frac{d\rho_f}{d\psi_f} \) are measured at the profiles and \( \rho_e \sin \psi_s \) and \( \rho_0 \sin \psi_f \) at the interior points \((z_e, r_e)\) and \((z_0, r_0)\), respectively. Thus, another
criterion for a convergent lens is that the source-profile or profile-image distance decreases as $\psi_s$ increases or as $\psi_f$ decreases, respectively, i.e.,

$$\frac{d\rho_s}{d\psi_s} < 0 \quad \text{or} \quad \frac{d\rho_f}{d\psi_f} > 0 \quad \text{and} \quad \frac{d\rho_s}{d\psi_s} > 0 \quad \text{or} \quad \frac{d\rho_f}{d\psi_f} > 0 .$$

In Table (9.1) are shown the various conditions which indicate the convergence or divergence of the magnetic lens.

<table>
<thead>
<tr>
<th>Table 9.1</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Converging Lens</strong></td>
</tr>
<tr>
<td>$\tan\mu &lt; 0$</td>
</tr>
<tr>
<td>$\mu_s$: rotation from n to t* clockwise</td>
</tr>
<tr>
<td>$\mu_f$: rotation counterclockwise</td>
</tr>
<tr>
<td>$P = \frac{\Delta \beta}{\Delta x} = \frac{\Delta \beta}{r \Delta \phi} &gt; 0$</td>
</tr>
<tr>
<td>Entrance: $\frac{d\rho_s}{d\psi_s} &lt; 0$, $\frac{d\rho_s}{d\psi_s} &gt; 0$</td>
</tr>
<tr>
<td>Exit: $\frac{d\rho_f}{d\psi_f} &gt; 0$, $\frac{d\rho_f}{d\psi_f} &gt; 0$</td>
</tr>
</tbody>
</table>

*n = normal to the profile; t = trajectory

Curves giving values of $\frac{1}{z_s} \frac{d\rho_s}{d\psi_s}$ are shown in Fig. (9.3) for the symmetrical magnetic profiles of Fig. (3.3) and Table II.3 (Appendix II), for various K-values; $\psi_s$ is in radians. The corresponding numerical values of $\frac{1}{z_s} \frac{d\rho_s}{d\psi_s}$ are given in the first column of Table IV.1, and the equation describing it in Eq. (A-III.12). Corresponding numerical values of $\frac{1}{\rho_s} \frac{d\rho_s}{d\psi_s}$, computed from Eq. (A-III.11), are given in Column 2 of the same table.

The fringing-field lens is a cylindrical lens, since the deflecting property ($\Delta \beta$) is directed only in the $\phi$-direction. The long axis of the cylinder lies on the median plane and follows the profile. An important property of a cylindrical lens is to image a point into a line, and the
fringing field also has this effect. An important difference between an ordinary optical cylindrical lens and this magnetic lens is the fact that the focal length of the latter may vary with \( \psi_s \) through variation of \( \mu \). In fact, with an appropriate profile shape, the magnetic lens may have \( P > 0 \) for some \( \psi_s \)-values and \( P < 0 \) for other \( \psi_s \)-values.

\[ \Delta \beta \text{-deflection for Point Source and the Equivalent Virtual Line Source.} \]

The computation of transmission loss is based upon the fact that from the deflection \( \Delta \beta \) there arises a \( p_{\phi} \)-component which was not introduced at the source; the trajectory may thus be considered as analogous to one arising from a source point with a \( \delta r_2 \)-component in a system with no fringing-field effect. We may treat the fringing-field effect by calculating the equivalent virtual source position and then applying the results of Section #5.

Both entrance and exit lenses may have important effects on the transmission. Thus, if the entrance lens is divergent, the incident beam will be spread out, and some of the particles will be lost because of collision with the walls of the gap, thus giving rise to the factor \( \eta_2 \). Apart from this effect, both lenses will produce a lateral broadening of the image (in the \( \delta r_2 \)-direction), and it will then not enter completely into the detector (whose size is chosen for reasons of resolution, see Section #6), producing another loss in transmission (\( \eta_3 \)).
Before calculating these quantities, we examine some illustrative figures. Figure (9.2) shows a trajectory projection viewed in the direction of the field. It can be one passing through $\phi_M$ or through a nonmedian trajectory, since to the approximation that the virtual $\delta r_2$ is small, all trajectories starting from $(z_s, 0)$ pass (see Section #5) through the line perpendicular to $\phi_M$ at $(z_f, 0)$.

Other views of the trajectories are given in Figs. (9.4) and (9.5), in which a family of orbits of constant $\psi_S$ is transformed so that the median orbit goes into a straight line.

The transformation of the constant $\psi_S$ cone to the plane surface is nonlinear, so the trajectories in this projection have a distortion which increases with $|\phi - \phi_M|$. In Fig. (9.4), $OO'$ is the median trajectory, the distance $OP$ of any point $P$ on the trajectory being $s$. The trajectory starts at $O$, increases the $r$-coordinate linearly with $s$ from $O$ to $AA'$ (the gap boundary), at which point, $L$, we have $s = \rho_S$, and (neglecting the penetration into the gap) $r = r_e$. From $L$ to $K$, at the boundary $BB'$, $r$ varies nonlinearly with $s$, returning to linearity from $K$ to $O'$. For the nonmedian orbits, the vertical coordinate is $r(\phi - \phi_M)$, the distortion of $s$ increasing with $r\Delta\phi$. The most highly deformed orbits in the figure are those which represent the extreme trajectories in the absence of fringing fields, i.e., which almost graze the polefaces ($OABO'$ and $OA'B'O'$). Actually, the total path lengths along the orbits $OO'$, $OABO'$ and $OA'B'O'$ are all the same.

The symmetrical system of Fig. (9.4) has converging lenses at both entrance and exit. A trajectory, which would be an extreme one ($OABO'$) with no fringe field present, is bent toward $\phi_M$ through the angle $\Delta\beta$ at the
lens AA', and follows OAG. Projecting the direction (within the field) of the initial trajectory element $\delta s$ backward toward the source, we get a virtual source point at C, with $OC = \text{virtual } \delta r_2$ and $OAC = |\Delta \beta_e|$. Using this value of $\delta r_2$, from Eq. (5.19b) we can calculate the rotation of the orbit from the plane $\phi = \phi_{\text{extreme}}$ toward a $\phi$-plane closer to $\phi_M$. The exit point is at G rather than at B, which would be the exit point in the absence of fringing field. At G, the ray is subject to another deflection $\Delta \beta_0$, which is also toward $\phi_M$. Since $GK = |\Delta x'|$ is smaller than $AL = |\Delta x|$, then $|\Delta \beta_0| < |\Delta \beta_e|$. The trajectory passes through the plane $z = z_f$ at D, and $\delta r_{2f} = O'D$. Similarly, the symmetrical ray OA' acts as though it came from a virtual source (with no fringing field) at G', with $\delta r_2 = OC'$, arrives at G', and then to D', here $\delta r_{2f} = O'D'$. A ray arriving at the profile at a $\phi$-plane such that $|\phi_s - \phi_M| < |\phi_{\text{extreme}} - \phi_M|$ will have a smaller $|\Delta \beta_e|$, a smaller $OC'' = |\delta r_2|$, a smaller $|\Delta \beta_0|$ and a smaller $|\delta r_{2f}| = OD''$.

All rays starting at O cross at X; hence this is the image point for the particular value of $\psi_s$ involved. Thus, the image of the source lies on the central ray, but displaced from $(z_f, 0, 0)$ towards the profile. Since the central ray (for this $\psi_s$-value) intersects the $z$-axis only at $(z_f, 0, 0)$, this means that the image does not occur on the $z$-axis, but lies above the $z$-axis by the amount determined by $\psi_f$ and the distance $O'X$.

To the approximation considered here, all the rays intersect the line DOD', which is perpendicular to $\phi_M$. This is, however, only an approximation, suitable for small $\phi$. The constant $\psi_s$ cone of rays, on emerging from the field, also forms a conical surface with apex (image point) above the $z$-axis and between the profile and $(z_f, 0, 0)$. After passing through the image point, the rays fan out again into the conical surface, and strike the $z$-$r_{2f}$-plane in the arc of a curve rather than in a straight line. These arcs vary with $\psi_s$: the greatest curvature (but in opposite directions) arising from $\psi_s$-values close to 0° or 180°; rays for which $\psi_s = 90°$, on the other hand, actually pass through the $r_{2f}$-axis perpendicular to the $\phi_M$-plane.

The position of the image point X varies with $\psi_s$, even if the lens power $P$ were constant for the entire $\psi_s$-range; this series of image points constitutes a curve. The imaging of the point source into a curve provides another feature of the cylindrical nature of the magnetic lenses, since an optical cylindrical lens images a point source into a line.

Thus, the double-focusing ($\psi$ and $\phi$) property of the hypothetical instrument without fringing field is lost when the fringing field is included. The $\psi$-focusing remains, since the rays still pass through $(z_f, 0)$ on any $\phi$-plane; this, however, is not true for the $\phi$-focusing.

Figure (9.5) illustrates an instrument with divergent entrance and exit lenses. Several differences from the convergent lens case may be noted:
(i) Whereas all rays entering the system in Fig. (9.4) emerge from the magnetic field, this is not true for a divergent input lens. Thus, a ray arriving at \( \phi_s \), lying on AH or A'H' is rotated away from \( \phi_M \) into the pole-faces and is lost. At H (and similarly at the symmetrical H'), we have \( \phi_s = \phi_{\text{critical}} = \phi_C \); a ray arriving at \( \phi_C \) just barely escapes collision with the pole face and emerges at B.

(ii) For a given value of |\( P \)|, it is found that |\( \Delta \beta_e |_{\text{max}} \) is smaller for the divergent lens, since |\( \Delta x_e \)| at the extreme trajectory (i.e., at \( \phi_C \)) is smaller than the |\( \Delta x_e \)| for the convergent lens at the extreme trajectory. Correspondingly, \( |\Delta \beta_0|_{\text{max}} \) at the exit is larger for the divergent lens. Since \( |\Delta \beta_e|_{\text{max}} \) is smaller, the maximum value of the virtual |\( \delta r_2 \)| is also smaller for the divergent lens.

(iii) While the convergent lens forms an image between the profile and \((z_f, 0, 0)\), so that the distance from image to profile is less than \( \rho_f \), the divergent lens forms an image beyond \((z_f, 0, 0)\) with an image-profile distance greater than \( \rho_f \). The convergent lens image is "above" the z-axis; the divergent lens image is "below" the z-axis, when the orientation of Fig. (9.2) is used.

(iv) For the convergent lens, the virtual source point is on the same side of \( \phi_M \) as is \( \phi_s \), while the point on the \( r_2f \)-axis is on the opposite side of \( \phi_M \). For the divergent lens, the virtual source point is on the opposite side of \( \phi_M \) from \( \phi_s \), but \( \delta r_2 \) is on the same side as \( \phi_s \). Thus, as in Section #5, the quantity \( \delta r_2f \) is always on the opposite side from \( \delta r_2 \).

**Orbit Rotation Due to \( \Delta \beta \)-deflection.** In calculating the effect of the fringing field, it is convenient to take \( \phi_M = 0 \). In most cases of interest, then, \( \phi_f \) and \( \phi_s \) have the same sign. This is obvious in the case of the divergent entrance lens, since the ray is always driven away from \( \phi_M \), so that |\( \phi - \phi_M \)| can only increase within the field. With a very strong convergent lens, however, the rays may form a real image point within the magnetic field. After reaching the image point on \( \phi_M \), the rays cross over and diverge, and may strike the polefaces if the lens is strong enough. Even if the lens is not this strong, \( \phi_f \) is of opposite sign to \( \phi_s \). However, spectrometers are usually designed so as to minimize |\( P \)|, so we shall not, for the moment, treat convergent lenses of such strength.

We have, from Eq. (5.19b),

\[
\phi_f = \phi_s + F_1(\psi_s) \frac{\delta r_2}{z_s} \quad .
\]

(9.9)

At the entrance, \( \Delta \beta_e > 0 \) for deflection toward negative \( \phi \); from Section #5, \( \delta r_2 > 0 \) in the negative \( \phi \)-direction. Figures (9.4) and (9.5) show \( \Delta \beta_e \) and \( \delta r_2 \) to be of opposite sign. Since \( \Delta \beta_e \) is equivalent to \( \delta \xi_s \) [see Eq. (5.17)] in magnitude, we define \( \delta r_2 \) as

...
\[ \delta x_2 = -\rho_s \Delta \beta_e \quad , \]  

with \( \rho_s \) taken as the source-profile distance. It is to be noted that, as an approximation, we consider the entire \( \Delta \beta \)-deflection to be inserted at the profile.

From Eqs. (9.4a), (9.5c) and (9.7) with \( \phi_M \) set equal to zero,

\[ \delta r_2 = N_s(\psi_s)z_s \phi_s \quad , \]  

with

\[ N_s(\psi_s) = \frac{\rho_s}{z_s} \frac{1}{K} \tan \mu_s = \frac{f_s}{K \sin \psi_s} \tan \mu_s \]

\[ = \frac{1}{K z_s} \frac{d\rho_s}{d\psi_s} = \frac{1}{K z_s} \frac{d\rho_s}{d\psi_s} \quad , \]  

since

\[ \rho_s = \frac{z_s f_s(\psi_s)}{\sin \psi_s} \quad . \]

Then,

\[ \phi_f = W_s \phi_s \quad . \]  

where

\[ W_s(\psi_s) = 1 + F_1 N_s \quad . \]

Since \( F_1 > 0 \), then \( W_s > 1 \) for a divergent lens, \( \phi_f \) and \( \phi_s \) are of the same sign, and \( |\phi_f| > |\phi_s| \). For a convergent lens, if \( |F_1 N_s| \) is not too large, \( 0 < W_s < 1 \), and \( (\phi_f, \phi_s) \) are of the same sign, but \( |\phi_f| < |\phi_s| \). If, however, the lens is so strongly convergent that \( 2 > |F_1 N_s| > 1 \), then \( -1 < W_s < 0 \), and \( \phi_f \) is of opposite sign to \( \phi_s \), but \( |\phi_f| > |\phi_s| \). In this case, imaging occurs within the field and crossover occurs, but all rays emerge from the magnetic field. If \( |F_1 N_s| > 2 \), then \( W_s < -1 \) and \( |\phi_f| > |\phi_s| \); some rays, then, collide with the polefaces and do not emerge.

Values of \( W_s \) have been computed for the symmetrical magnetic profiles of Fig. (3.3) and Table A-II.3, and are given in the third column of Table A-IV; the corresponding curves are shown in Fig. (9.6) for \( -2 < W_s < 2 \).
Imaging Property of Point Source with Fringing Field. To the approximation we have been considering, namely, that the \( \psi - \) and \( \beta - \) deflections in the fringing field are separable, the \( \psi - \) focusing remains perfect. Thus, all rays from \((z_s, 0)\) pass through \((z = z_f, r_1 = 0)\). The \( \phi - \) focusing, however, is perturbed, so that \( r_2 \neq 0 \) at \( z = z_f \). We shall evaluate the effect of the rotation on the imaging property in the \( r_2 \)-direction, using a converging entrance lens for convenience in presentation, although the same equations apply for a diverging lens. The convergent lens is only of moderate power, i.e., there is no crossover within the field \((W_s > 0)\).

Following the notation of Fig. (9.4), the exit side is shown in Fig. (9.7). If the exit lens had \( P = 0 \), the ray would hit the \( z_f \)-plane at a distance from the axis equal to \( \delta r_{zf} \), computed as in Section #5. In Fig. (9.7), \( \delta r_{zf} = O'H \) and \( GH \) is the ray in the absence of an exit fringing field effect. From Eqs. (5.20) and (9.11a),

\[
\sin \psi_s \frac{\delta r_{zf}}{\sin \psi_f} = \frac{\sin \phi_s}{s} = \nu N_s z_s \phi_s .
\]  

(9.13)

However, since the exit lens has deflecting power, it bends the ray through the angle \( \Delta \beta_0 \), so that the path is actually \( GD \) [see Fig. (9.7)]. The change in sign convention for both the \( r_{zf} \)-component and \( \Delta \beta_0 \) on the exit side gives, from Eq. (9.10),

\[
\delta r_{zf} = -\rho_f \Delta \beta_0 ,
\]  

(9.14)

where \( \delta r_{zf} = HD \) in Fig. (9.7). From Eq. (9.4b), (9.5c) and (9.7),

\[
\delta r_{zf} = N_f z_f \phi_s = N_f W_s z_f \phi_s
\]  

(9.15)

with

\[
N_f(\psi_f) = -\frac{\rho_f}{K z_f} \tan \mu_f = \frac{f_f}{K \sin \psi_f} \tan \mu_f
\]  

(9.16)

\[
= \frac{1}{K} \frac{d \rho_f}{d \psi_f} = \frac{1}{K} \frac{d \rho_f}{d \psi_f} ,
\]

since

\[
\rho_f = \frac{z_f}{\sin \psi_f} .
\]

The total distance of the "image" point from the \( z \)-axis, \( O'D \), is given by the algebraic sum of \( \delta r_{zf} \) and \( \delta r_{zf} \), the summation holding whether the exit lens is diverging or converging. Thus,
\[ R_{zf} = \hat{r}_{zf} + \varphi_{zf} = [v N_s + \frac{z_f}{z_s} N_f W_s] z_s \phi_s \]  

(9.17)

With this relation, the effectiveness of the detector in catching the rays emitted at various \( \phi_s \)-values may be computed.

**Transmission from Point Source.** As in Section #7, we shall first treat the effect of the exit aperture in reducing transmission, i.e., we shall evaluate \( \eta_2 \), following which the effect of the detector aperture is treated through the evaluation of \( \eta_3 \).

For a convergent entrance lens, trajectories are bent toward \( \phi_M \), so that for those particles which would pass through in the absence of fringing field, there is no collision with the polefaces, and \( \eta_2 = 1 \). This is true even for fairly strongly convergent lenses, provided \( W_s > -1 \). Thus, for \( W_s > 0 \), since \( W_s < 1 \) for a convergent lens, \( \phi_f \) approaches \( \frac{1}{2} \varphi \) more slowly than \( \phi_s \) approaches \( \frac{1}{2} \varphi \), so that no ray hits the poleface, and \( \eta_2 = 1 \). Similarly, when \( 0 > W_s > -1 \), crossing of the median plane occurs within the field, but \( \phi_f \) approaches \( -\frac{1}{2} \varphi \) more slowly than \( \phi_s \rightarrow \frac{1}{2} \varphi \) so again \( \eta_2 = 1 \). The case for \( W_s < -1 \) will be considered after examination of the diverging lens.

With a diverging entrance lens, \( \eta_2 < 1 \), because not all the rays passing through the entrance aperture can avoid hitting the poleface, i.e., the exit aperture causes a reduction in transmission. When \( \phi_f = \frac{1}{2} \varphi \), the largest possible \( \phi \)-value, then \( \phi_s = \phi_C = \phi_{\text{critical}} \), such that for \( |\phi_s| > |\phi_C| \), the rays collide with the polefaces. Now \( \phi_C \) is calculated from Eq. (9.12):

\[ \phi_C = \pm \frac{1}{2} \varphi W_s \]  

(9.18)

Because of the uniformity of ray density with the \( \phi \)-angle of emission,

\[ \eta_2(\psi_s) = \frac{|\phi_C|}{\frac{1}{2} \varphi} = \frac{1}{W_s} \]  

(9.19)

with \( W_s > 1 \).

For a very strongly convergent lens, with \( W_s < -1 \), the results are similar, with only sign changes added. Thus,

\[ \phi_C = \pm \frac{1}{2} \varphi W_s \quad ; \quad \eta_2 = |\phi_C|/\frac{1}{2} \varphi = -1/W_s \]  

(9.20)

This result has been included for completeness, but optimum spectrometer designs will not include such values of \( W_s \).

It is evident that, at a given \( \psi_s \), \( \phi \)-focusing occurs at a particular point [X in Figs. (9.4) and (9.5)]. Because the \((z,r)\) coordinates for the image points vary considerably with \( \psi_s \), the detector cannot be placed in
any one position for point-to-point focusing. Some compromise position
must be chosen, which is suitable for all $\psi_s$ in the acceptance range. The
detector is most conveniently placed in the neighborhood of $(z_f, 0, 0)$; this
is particularly true for a multigap instrument, since the image points
from other gaps then occur on the other side of the z-axis. Considerations
of resolution (see Section #6) also dictate placement at $(z_f, 0, 0)$. Suppose
the detector is a circular one with radius $R_d$. Since $r_{1f} = 0$, the only rays
accepted are those for which

$$|\delta R_{2f}| \leq R_d$$

Since the density of rays, for a given $\psi_s$, is uniform with $\phi_s$, and since
$\delta R_{2f}$ is proportional to $\phi_s$, we find that the fraction of rays accepted by the
detector is

$$\eta_3(\psi_s) = \frac{R_d}{|\delta R_{2f}(\text{max})|}$$

for

$$R_d \leq |\delta R_{2f}(\text{max})|,$$

since $\eta_3 \leq 1$. If $R_d > |\delta R_{2f}(\text{max})|$, then $\eta_3 = 1$, i.e., the detector accepts all
the rays passed by the exit aperture.

Now, $\delta R_{2f}(\text{max})$ is the $\delta R_{2f}$-value corresponding to $\phi_s = \phi C$, which
is $\frac{1}{2}$ for a converging entrance lens, and $\frac{1}{2} / |W_s|$ for a diverging entrance
lens or a converging lens with $W_s > -1$. Hence, from Eqs. (9.17) and (9.21),

$$\eta_3(\psi_s) = \frac{2R_d}{\nu N_s + \frac{z_f}{z_s} N_f W_s}$$

for a convergent entrance lens (9.22a)

$$\eta_3(\psi_s) = \frac{2R_d |W_s|}{\nu N_s + \frac{z_f}{z_s} N_f W_s}$$

for a divergent entrance lens. (9.22b)

Equation (9.22b) also applies to the case in which $W_s < -1$. Equations (9.22a)
and (9.22b) are valid only for $\eta_3 \leq 1$. If either ratio is greater than 1, then
$\eta_3 = 1$.

The combined transmission through the exit and detector aperatures
is

$$\eta_{23}(\psi_s) = \eta_2(\psi_s) \eta_3(\psi_s) = \frac{2R_d}{\nu N_s + \frac{z_f}{z_s} N_f W_s}$$

(9.23)
This result is valid only for \( \eta_{23} \leq \eta_2 \); if the expression in Eq. (9.23) exceeds \( \eta_2 \), then \( \eta_{23} = \eta_2 \).

Equation (9.23) applies to cases in which either converging or diverging entrance lenses are present. Other things being equal, however, the numerator is larger for the divergent case. This is evident in an examination of the symmetrical case, which we now consider.

For a symmetrical instrument, all the results are simplified, since
\[
\psi_f = -\psi_s, \quad z_s f_s = z_f f_f, \quad \rho_s = \rho_f, \quad \mu_s = \mu_f, \quad \frac{d\rho_s}{d\psi_s} = -\frac{d\rho_f}{d\psi_f}, \quad z_s = -z_f.
\]

Then,
\[
N_f(\psi_f) = N_s(\psi_s) \quad (9.24a)
\]
\[
\delta r_{2f} = -\delta r_2 = -N_s z_s \phi_s \quad (9.24b)
\]
\[
\delta r_{2f} = -N_s W_s z_s \phi_s \quad (9.24c)
\]
\[
\delta R_{2f} = -(1 + W_s) N_s z_s \phi_s \quad (9.24d)
\]

and
\[
\eta_3(\psi_s) = \frac{2R d}{|1 + W_s| |N_s| z_s \phi} \quad \text{for } \eta_3 \leq 1 \quad (9.25a)
\]
\[
\eta_3(\psi_s) = \frac{2R d |W_s|}{|1 + W_s| |N_s| z_s \phi} \quad \text{for } |W_s| < 1 \quad (9.25b)
\]
\[
\eta_{23}(\psi_s) = \frac{2R d}{|1 + W_s| |N_s| z_s \phi} \quad \text{for } \eta_3 \leq \eta_2 \quad (9.26)
\]

If the right side of Eqs. (9.25a) or (9.25b) is greater than 1, then \( \eta_3 = 1 \). If the right side of Eq. (9.26) exceeds \( \eta_2 \), then \( \eta_{23} = \eta_2 \).

We note the following:

(i) For the divergent entrance lens, \( \eta_2 \) depends upon the shape of the boundary curves and trajectories (through \( F_1 \) and \( \mu \), and hence \( P \)), but not on the size of the instrument (\( z_s \)) nor on \( \phi \). This follows from the result of Section #5 that the rotation depends upon the sample size (\( \delta r_2 \)) rather than on the gap size.

(ii) On the other hand, \( \eta_3 \) depends upon \( \phi \) and \( R_d / z_s \), as well as on \( \mu \) and \( F_1 \). Thus, if the detector slit remains constant, \( \eta_3 \) depends upon the size of the instrument. It is seen that \( \eta_3 \) is inversely proportional to
\(\varphi\) (for \(\eta_3 < 1\)), so that as the gap is opened up with a detector of fixed size, the fraction of the gap which gives useful transmission decreases. In fact, since \(\eta_3 \varphi\) is independent of \(\varphi\) as long as \(\eta_3 \lesssim 1\) [i.e., as long as \(|\delta R_2f(\text{max})| > R_d\)], it is of no value to increase \(\varphi\) beyond the value for which \(|\delta R_2f(\text{max})| = R_d\).

Calculated Transmission Characteristics of Point Sources in Symmetrical Spectrometers. A semiquantitative illustration of the effect of \(\eta_2\) and \(\eta_3\) on the transmission is shown in Figs. (9.8), (9.9) and (9.10), in which \(R_d/z_s = 0.01\). The full width of each figure represents the relative transmission as a function of \(\psi_s\), in the absence of a fringing field, and, from Eq. (7.2), varies as \(\sin^{-2}\psi_s\). The shaded area represents the actual transmission when the ideal transmission is multiplied by \((\eta_2 \eta_3)\).

For \(K = 0.5\), it may be seen from Fig. (9.3) that, for a symmetrical spectrometer, \(d\rho_s/d\psi_s\) is negative and is fairly large over most of the \(\psi_s\)-region considered, so that \(P > 0\) and the lens effect is strong. Because the lenses are convergent, \(\eta_2 = 1\), but because they are strong, the image points are distant from \((z_2, 0, 0)\) and the spreading out at \(r_2f\) is great, leading to small \(\eta_3\). Close to 140°, \(d\rho_s/d\psi_s\) becomes small; the resulting small value for \(P\) leads to \(\eta_3\) being almost unity.

For \(K = 0.6\), Fig. (9.3) shows that \(d\rho_s/d\psi_s\) is negative for the smaller \(\psi_s\)-values, comes close to zero at 120°, and then becomes positive as \(\psi_s\) increases. However, it never becomes large.* Hence, for \(\psi_s\) smaller than approximately 120°, we have \(\eta_2 = 1\) and \(\eta_3\) approaches unity as \(\psi_s\) approaches 120° and \(P\) approaches zero. As \(\psi_s\) increases beyond 130°, the value of \(d\rho_s/d\psi_s\) increases, \(|P|\) increases, so that both \(\eta_2\) and \(\eta_3\) decrease in value. For \(K = 0.8\), we have \(d\rho_s/d\psi_s\) positive and large, so \(\eta_2 < 1\) and \(\eta_3 < 1\) and are relatively small over the entire \(\psi_s\)-range.

*For values of \(\tan \mu_s\), see Fig. (10.1)
Other forms of presentation of $\eta_2$ and $\eta_{123}$ are shown in Fig (9.11), (9.12) and (9.13) for the case of the symmetrical magnetic profiles of Fig. (3.3) and Table II.3 of Appendix II.
Figure (9.11) shows $\eta_2$ as a function of $\psi_s$ for various $K$-values. The corresponding numerical values are given in Table IV.1, Column 4. As pointed out above, the lenses converge when $\frac{d\rho_s}{d\psi_s} < 0$. It is evident from Fig. (9.3), as well as from Fig. (3.3) and Table IV.1, that the lenses for symmetrical systems are convergent for all $K$-values less than $K_0$, where $K_0$ lies between 0.50 and 0.52. For this reason, all of these $K$-values correspond to $\eta_2 = 1$, as shown in Fig. (9.11) for $K = 0.20$ and $K = 0.50$. The lower limit on the $\psi_s$-values covered corresponds to the practical values for the critical angles discussed in Section #3. For $K > 0.52$ we have $\frac{d\rho_s}{d\psi_s} > 0$ close to $\psi_s = 180^\circ$, becoming negative as $\psi_s$ decreases, with the consequence that $\eta_2 < 1$ close to $180^\circ$ and $\eta_2 = 1$ over the remainder of the $\psi_s$-range.
As $\psi_S$ decreases, the divergent lens first becomes stronger, than less divergent and finally convergent. Then $\eta_2$ remains equal to unity until the small $\psi_S$-values are reached, at which point the lens becomes so convergent that $\eta_2$ begins to decrease again due to the crossover of the converging rays within the interior field. The region in which $\eta_2 = 1$ is indicated by the brackets at the side of the figure.

Figure (9.12) illustrates the quantity

$$Y = \frac{1}{\frac{1}{\eta_2} \frac{R_d}{z_s}} \eta_{23}, \quad (9.27)$$

which is a universal quantity for this type of spectrometer and is converted to $\eta_{23}$ through multiplication by specific values of $\frac{R_d}{z_s}$. Numerical values of $Y$ are given in Table IV.1, Column 5.

Small values of $Y$ indicate that the rays make a very broad focal area in the region of $(z_f,0,0)$, and hence lead to small $\eta_{23}$-values unless $R_d$ is very large. Large values of $Y$, on the other hand, indicate that the rays form a very small focal area in the neighborhood of $(z_f,0,0)$, and hence even detectors with very small diameters will accept all the rays. It is evident from the form of $Y$:

$$Y = \frac{1}{|1 + W_S||N_s|}, \quad (9.28a)$$

with

$$\frac{1}{|N_s|} = \frac{K_s}{\rho_s |\tan \mu_s|} = \frac{K_s}{dp_s |d^2 s|}. \quad (9.28b)$$

that there are two cases in which $Y$ becomes large:

(i) In this case, $\left| \frac{dp_s}{d\psi_s} \right|$ is small. Hence, $|\tan \mu_s|$ and $|P|$ are small. Because of the resulting small lateral defocusing, the point source tends to go into a point image, and even a detector with small $R_d$ gathers most of the rays.

(ii) In this case, $|1 + W_S|$ is small. This means that the convergent lens is quite powerful, so that $W_S$ is negative and close to one in magnitude. The rays of the trajectories thus cross over within the magnetic field and
emerge from the field tending to diverge. The convergent lens at the exit, however, bends the rays towards \((z_f,0,0)\). Thus, a small \(R_d\) will still allow the detector to accept all the rays. For this to occur, however, \(|W_s|\) cannot be too far from unity. If \(|W_s|\) is too small, then the exit lens doesn't converge the rays enough to bring them close to \((z_f,0,0)\); on the other hand, if \(|W_s|\) is too large, the rays are bent excessively and pass at some distance from \((z_f,0,0)\).

Figure (9.13) illustrates \(\gamma_{23}\)-values for a particular case in which \(R_d/z_s = 0.01\) and \(\phi = 10^\circ = 0.175\) radians; then \(\gamma_{23} = 0.115\). Since \(\gamma_{23} \leq \gamma_2 \leq 1\), the value 0.115 holds only for \(0.115 < \gamma_2\). When \(0.115 > \gamma_2\), \(\gamma_{23}\) is set equal to \(\gamma_2\). 0.115 \(\gamma_2\) implies only that \(R_d\) is larger than is necessary to catch all of the rays emerging from the exit aperture.

Since the distance to the profile \(P_s\) [or the equivalent \(f_s(\psi_s)\)], the rotation term \(F_1(\psi_s)\) and the strength of the fringing field lens (measured by \(\tan\psi_s\)) are all functions of \(\psi_s\), the over-all value for \(\gamma_1\) requires averaging over \(\psi_s\). We have, then, for either converging or diverging systems, for symmetrical or asymmetrical systems:

\[
\overline{\gamma_2} = \frac{\int \gamma_2(\psi_s) \sin\psi_s d\psi_s}{\int \sin\psi_s d\psi_s} \quad (9.29)
\]

\[
\overline{\gamma_3} = \frac{\int \gamma_3(\psi_s) \sin\psi_s d\psi_s}{\int \sin\psi_s d\psi_s} \quad (9.30)
\]

\[
\overline{\gamma_{23}} = \overline{\gamma_2} \overline{\gamma_3} = \frac{\int \gamma_2(\psi_s)\gamma_3(\psi_s) \sin\psi_s d\psi_s}{\int \sin\psi_s d\psi_s} \quad (9.31)
\]

where the integration is taken from \(\psi_s(\text{min})\) to \(\psi_s(\text{max})\). In general, it is necessary to use Eq. (9.31), since:

\[
\overline{\gamma_2} \overline{\gamma_3} \neq \overline{\gamma_2 \gamma_3} \quad (9.32)
\]

unless \(\gamma_2\) or \(\gamma_3\) are essentially constant over the entire \(\psi_s\)-range.

Because the losses in transmission may be considerable, it is important to choose the K-value (and hence the profile design) in such a
way as to maximize $\eta_2\eta_3$ and thus minimize the losses. It is evident from the figures that $P$ should be kept small, and this is most readily accomplished if part of the lens is convergent and part divergent, as in Fig. (9.9).

Of course, $P = 0$ if $\frac{\partial \rho_s}{\partial \psi_s} = \mu_s = 0$ over the $\psi_s$-range of interest, i.e., the profile is perpendicular to all rays, and hence is a circular arc with $(z_s, 0, 0)$ as center. This point is discussed further in Section #16.

Transmission for a Finite Source. When the transmission losses $\eta_2^s$ or $\eta_2^s$ computed for source extension alone and the losses $\eta_23^s$ or $\eta_23^s$ computed for fringing field and point source are comparable, then neither result is valid, nor can the correct result be derived from a combination of the two.

For the evaluation of the approximate transmission characteristics of a particular instrument design or of a specified sample size, computation of $\eta_23$ or $\eta_2^s$ may be adequate. However, where both effects are comparable in magnitude, and where accurate results are required, both effects must be combined before integration is carried out over the sample. The transmission coefficients for this case are designated $\eta_2T$, $\eta_3T$, and $\eta_23T$.

As in Section #7, the transmission is computed only for the upper left quadrant of the source, and the approximation is again made that the $r_1$-axis is kept fixed in $\Phi_M$. Although only a uniformly distributed source is considered, this is not an essential restriction. If the source intensity is nonuniform, but known, indicated integrations may be carried out numerically, returning to the basic integral, Eq. (7.5) or (7.12). The bending effect of the fringing field is taken to be the same for an off-center ray as for one from $(z_s, 0, 0)$; thus, $\Delta \epsilon_e$ depends only upon $\Phi_s$ and not upon the source point.

As in Section #7, $\eta_2T$ for a source point is a function of the $r_2$-coordinate alone. If a cut is taken through the sample at constant $r_2$, all of the rays from the points involved have the same $\eta_2$-losses, and, to the first order, they may be treated as though they emerged from $(z_s, 0, r_2)$. Then, as in Section #7, the integration over the source is only over $r_2$, using as a weighting factor the $r_1$-boundary of the source for a given $r_2$-value.

Because of the astigmatic lens effect of the fringing field, the imaging at constant $\psi_s$ is point to line rather than point to point, as in Section #7. The $r_{1f}$-coordinate has the same value as in Section #5 and #7, but the $r_{2f}$-coordinate is smeared out by the fringing field, being distributed uniformly in the line $r_{1f}$ = constant. The distribution is symmetrical around the image point position of Section #7 (i.e., the median ray, or focus point with no lens effect) only when $\eta_2T = 1$; for $\eta_2T < 1$, the line distribution is truncated on one or both sides.
The coefficient \( \eta_3 T \) can take on a continuum of values rather than only \((0,1)\), as in Section #7, since it depends on the dimensions of the truncated line relative to the detector aperture boundaries. When \( \eta_3 T < 1 \) for a single source point, its magnitude is determined by the \( r_2 \)-coordinates of the detector aperture boundary at \( r_{1f} \). In fact, because the density of rays is uniform over the truncated line, \( \eta_3 T \) is the fraction of the truncated line lying within the aperture boundaries. Although the length of this line is a function only of the \( r_2 \)-coordinate of the source point, the aperture boundaries vary with the \( r_{1f} \)-coordinate of the image. Hence \( \eta_3 T \) is a function of \( r_1 \) as well as of \( r_2 \).

To determine the combined effect of source spread and fringing field, we use the theory of Section #5 for the \( r_2 \)-effect (as in Section #7) and the results of the first part of this section for the fringing field. Since both effects result in small \( P_\phi \)-values, the two image shifts are small and are added linearly. Both convergent and divergent lenses are treated by the same equations.

As before, the fringing field effect is computed by considering the incidence of a ray at \( \phi_s \) to be equivalent to a virtual source point displaced from the actual point source by a distance \( \delta r_2 \). A single point on the source is then equivalent to a line source spread out symmetrically and uniformly on both sides of \((z_s, r_1, r_2)\). For a given incidence plane \( \phi_s \), we have, as in Eq. (9.9)

\[
\phi_f - \phi_s = F_1(\psi_s) \left( \frac{\delta r_2 + r_2}{z_s} \right), \tag{9.33}
\]

where the algebraic sum of \( \delta r_2 \) and \( r_2 \) is taken. Here \( \delta r_2 \) is the virtual source displacement due to the fringing field, whereas \( r_2 \) is the actual source point displacement (in the \( r_2 \)-direction) from the center \((z_s,0,0)\), both being measured as positive in the negative \( \phi \)-direction. Thus, in the quadrant chosen in Section #7, every point has a negative \( r_2 \) [see Fig. (7.2)]. Using \( \delta r_2 \) as given by Eq. (9.11), we have [see Eq. (9.12)]

\[
\phi_f - \phi_s = F_1 N_s \phi_s + F_1 \left( \frac{r_2}{z_s} \right) \tag{9.34}
\]

or

\[
\phi_f = \phi_s W_s + F_1 \left( \frac{r_2}{z_s} \right). \tag{9.35}
\]

For a source point at \((z_s, r_1, r_2)\), the image formation at the plane \( z = z_f \) is computed in the same manner as for the source point at \((z_s,0,0)\):
\[ \delta r_{zf} = \frac{\sin \psi_s}{\sin \psi_f} (r_2 + \delta r_2) = \frac{\sin \psi_s}{\sin \psi_f} (r_2 + N_s z_s \phi_s) \quad (9.36a) \]

and

\[ \delta R_{zf} = \delta r_{zf} + \delta r_{zf}^s \]

\[ = \left[ v(\psi_s) + \frac{N_f z_f F_1}{z_s} \right] r_2 + \left[ v N_s + \frac{z_f}{z_s} N_f W_s \right] z_s \phi_s \]

\[ = g(\psi_s) r_2 + h(\psi_s) z_s \phi_s. \quad (9.37) \]

As in Section #7 [Eq. (7.14)], the \( r_{1f} \)-coordinate of the image line is given by Eq. (5.15b):

\[ r_{1f} = \frac{1}{\psi} \frac{\cos \psi_s}{\cos \psi_f} r_1 = u(\psi_s) r_1. \quad (9.38) \]

With \( r_2 = 0 \), Eq. (9.37) reduces to Eq. (9.17). For \( r_2 = 0 \), the values of \( \delta R_{zf} \) were spread out uniformly and symmetrically on both sides of \( (z_f,0,0) \). For \( r_2 \neq 0 \), the spread for a given \( r_2 \) is still uniform on the line \( r_{1f} = \) constant, but is no longer symmetric around the \( z \)-axis, nor indeed around the image point \( (z_f,r_{2f},r_{1f}) \) in the absence of fringing field, where \( r_{2f} \) is given by Eq. (5.20). The lack of symmetry arises both from the \( r_{2f} \)-term and from the fact that the range of \( \phi_s \)-values transmitted is not symmetric around \( \phi_M \).

Transmission Computation from Boundaries of Allowed \( \phi_s \)-regions. Equation (9.37) describes the \( r_{2f} \)-component of the image (for a given \( r_2 \) and \( \psi_s \)) as a function of \( \phi_s \). Whether a particular \( \delta R_{zf} \)-value is allowed (i.e., transmitted) depends upon whether the ray incident at \( \phi_s \) traverses the three apertures: entrance, exit, and detector. Because the incidence of rays, at a given \( \psi_s \) and \( r_2 \), has a uniform density with \( \phi_s \), the transmission problem is essentially solved when the boundary \( \phi_s \)-values are correctly chosen.

The condition that a trajectory does not pass the plane of the profile (e.g., collide with the poleface in an iron-core instrument) may be stated as

\[ |\phi - \phi_M| \leq \frac{1}{2} \varphi \quad (9.39) \]

within the gap.
Because of the conservation relation Eq. (1.8), \((\phi - \phi_M)\) changes only monotonically, so that it is sufficient to express the condition in Eq. (9.39) only at the boundaries:

\[
\left| \phi_s - \phi_M \right| = \left| \phi_s \right| \leq \frac{1}{2} \varphi
\]  
(9.40a)

\[
\left| \phi_f - \phi_M \right| = \left| \phi_f \right| \leq \frac{1}{2} \varphi .
\]  
(9.40b)

The condition in Eq. (9.40b), when placed into Eq. (9.35), determines the critical \(\phi_s\)-values beyond which no rays pass through the exit aperture. Thus, setting \(\phi_f = \pm \frac{1}{2} \varphi\), transmission through \(\phi_s\) cannot occur unless the following is true (since we consider only cases in which \(W_s > 0\)):

\[
\frac{1}{W_s} \left[ \frac{1}{2} \varphi + F_1 \frac{r_2}{z_s} \right] \leq \phi_s \leq \frac{1}{W_s} \left[ \frac{1}{2} \varphi - F_1 \frac{r_2}{z_s} \right] .
\]  
(9.41)

The boundary of the detector aperture is described by the curve \(r_1f = \Gamma_1(r_2f)\), or by the inverse \(r_2f = \Gamma_2(r_1f)\). We consider only detector apertures which have reflection symmetry in \(\phi_M\) as well as in the plane perpendicular to \(\phi_M\); in addition, the aperture is entirely open within its boundary. Then, at the image line \((r_1f = \text{constant})\), the boundary points are \(r_2f = \pm R_2f\), where \(R_2f = |\Gamma_2|\). We shall describe the farthest \(r_2f\)-extension of the aperture boundary, i.e., the maximum of \(R_2f\), as \(D_{r_2}f\). Similarly, the farthest \(r_1f\)-extension of the boundary, or the maximum of \(|\Gamma_1|\), is \(D_{r_1}\).

From Eq. (9.37), the requirement that \(|\Delta R_2f| \leq R_2f\) leads to

\[
\frac{1}{z_s h(\psi_s)} [R_2f + g(\psi_s) r_2] \leq \phi_s \leq \frac{1}{z_s h(\psi_s)} [R_2f - g(\psi_s) r_2]
\]  
(9.42)

as an inequality which must be obeyed for transmission through \(\phi_s\).

\(R_2f\) is a function of \(r_1f\), and hence of \(r_1\). For \(h < 0\), the inequality is reversed.

It is convenient to express these boundaries as fractions of the gap width \(\varphi\). Thus, for \(h(\psi_s) > 0\),

\[
E^-_e = - \frac{1}{2} \leq \frac{\phi_s}{\varphi} \leq \frac{1}{2} = E^+_e
\]  
(9.43a)

*It is to be noted that we are examining only the transmission of a single gap. The argument is complete for an iron-core instrument, in which particles entering a gap can only emerge from the same gap if at all. For an air-core instrument, transmission may be increased through particles entering one gap and emerging from another gap, but we do not examine this question quantitatively.*
\[
E_0^- = -\frac{1}{W_s} \left[ \frac{1}{2} + \frac{F_1}{\varphi z_s} r_2^2 \right] \leq \frac{\varphi S}{\varphi} \leq \frac{1}{W_s} \left[ \frac{1}{2} - \frac{F_1}{\varphi z_s} r_2^2 \right] = E_0^+ \quad (9.43b)
\]
\[
E_d^- = -\frac{1}{h(\psi_s)} \left[ \frac{R z}{\varphi z_s} + \frac{g(\psi_s)}{\varphi z_s} r_2 \right] \leq \frac{\varphi S}{\varphi} \leq \frac{1}{h(\psi_s)} \left[ \frac{R z}{\varphi z_s} - \frac{g(\psi_s)}{\varphi z_s} r_2 \right] = E_d^+ \quad (9.44c)
\]

Now,
\[
h(\psi_s) = \frac{\nu(\psi_s)}{K} \frac{\rho_s}{z_s} \tan \mu_s - \frac{W_s}{K} \frac{\rho_f}{z_f} \tan \mu_f, \quad (9.44a)
\]
which may be positive or negative. In fact, for a symmetrical spectrometer,
\[
h(\psi_s) = -\frac{1}{K} \frac{\rho_s}{z_s} \tan \mu_s (1 + W_s) = -N_s(1 + W_s) \quad (9.44b)
\]
has (for \(W_s > -1\)) the sign of \((-\mu)\). Hence, for symmetrical systems, Eq. (9.43c) applies only for a convergent lens. For a divergent lens, the boundaries must be inverted. For an unsymmetrical system, the relation between the sign of \(h(\psi_s)\) and the nature of the entrance and exit lenses cannot be simply assigned. However, in general, when \(h(\psi_s) < 0\),
\[
E_d^- = \frac{1}{h(\psi_s)} \left[ \frac{R z}{\varphi z_s} - \frac{g(\psi_s)}{\varphi z_s} r_2 \right] \leq \frac{\varphi S}{\varphi} \leq \frac{1}{h(\psi_s)} \left[ \frac{R z}{\varphi z_s} + \frac{g(\psi_s)}{\varphi z_s} r_2 \right] = E_d^+. \quad (9.45)
\]

For a very strong convergent lens, for which \(W_s < 0\), the \(\leq\) signs in Eq. (9.43b) should also be changed to \(\geq\), thus reversing the roles of \(E_0^-\) and \(E_0^+\) as upper and lower limits. Because this is not a usual condition, we shall neglect this possibility in further discussion, although the appropriate treatment can always be carried through by the above reversal of terms.

It is evident that any of the three apertures may cut off the \(\varphi_s\)-range, at either end. We take
\[
E_M(r_2, r_1) = \text{smallest of the three upper limits}
\]
\[
E_m(r_2, r_1) = \text{largest of the three lower limits},
\]
the comparison being taken algebraically, rather than with absolute values. The range of transmitted \(\varphi_s\) then lies in the interval determined by \((E_m, E_M)\). If we are interested only in \(\gamma_2 T\), then we take
\( E'_M = \text{smallest of the upper limits in Eq. (9.43a) and (9.43b)} \)

\( E'_m = \text{largest of the lower limits.} \)

Because of the uniform density over \( \psi_s \), the transmission at a given \( \psi_s \), for a given \( (r_1, r_2) \), is the ratio of the acceptance range of \( \psi_s \) to the entire \( \psi_s \)-range (\( \varphi \)). Thus

\[
\gamma_{2T} (r_2, \psi_s) = \frac{E'_M - E'_m}{E'_m - E'_m} \quad (9.46)
\]

\[
\gamma_{23T} (r_1, r_2, \psi_s) = E'_M - E'_m \quad (9.47)
\]

The particular aperture that is effective in limiting the \( \psi_s \)-range varies with \( r_2 \), and may differ at the upper and lower limits. The possible combinations are best illustrated with a graph showing the transmission properties as a function of \( r_2 \), at a fixed \( r_1 \) (and hence fixed \( r_1f \) and \( R_2f \)).

The boundaries \( E_0 \) and \( E_d \) have the form

\[
E = \pm k - q r_2. \quad (9.48)
\]

The boundaries are then parallel lines, of slope \( q \), and separated by the distance \( 2k \).

For \( E_0 \), the slope is

\[
q = -\frac{1}{W_s \varphi z_s} F_1, \quad (9.49)
\]

which is always negative, for lenses which are not strongly convergent (i.e., when \( W_s > 0 \)). The vertical separation is

\[
2k = \frac{1}{W_s} . \quad (9.50)
\]

For \( E_d \), the slope is

\[
q = -\frac{1}{\varphi z_s h(\psi_s)} = \frac{1}{h(\psi_s)} \left[ -v(\psi_s) - \frac{z_f}{z_s} N_f F_1 \right] = \frac{1}{\varphi z_s} . \quad (9.51a)
\]

For a symmetrical spectrometer,

\[
g(\psi_s) = -1 - N_s F_1 = -W_s . \quad (9.51b)
\]

In this case,
\[ q = \frac{1}{h(\psi_s)} \frac{W_s}{\phi z_s} = -\frac{1}{N_s z_s} \frac{W_s}{1 + W_s}. \]  

(9.51c)

Except for spectrometers in which the deviation from symmetry is very large, \(-v = \left( \frac{\sin \psi_s}{\sin \psi_f} \right) > 0\), and of the order of magnitude of unity. From Eq. (9.15b), it is evident that \(-\frac{z_f}{z_s} N_f F_1 = \left( \frac{\rho_f}{h} \frac{F_1}{z_s} \right) \tan \mu_f\) has the sign of \(\mu_f\) and the order of magnitude of \(|\tan \mu_f|\). Hence, if \(|\tan \mu_f|\) is small,* \(q\) has the sign of \(h(\psi_s)\). (This relation may be inverted in unusual cases, as when \(-v < 0\) or when \(|\tan \mu_f|\) is large and \(\mu_f < 0\).) The vertical separation is

\[ |2k| = \left| \frac{2}{\phi z_s} \frac{1}{h(\psi_s)} R_{2f} \right|. \]  

(9.52)

In Fig. (9.14), the boundary curves are shown for a case in which \(h(\psi_s) > 0\), which, for a symmetrical system with \(|\tan \mu_s|\) not large (\(W_s > 0\)), corresponds to a convergent lens. It is evident that for the symmetric detector considered, the transmission is the same for positive and negative \(r_2\), at any particular \(|r_2|\)-value. As previously, only the left-hand upper quadrant is considered. The graphs are drawn for a particular \(r_1\)-value, since the curves \((E_d, E_d^-)\) are a function of the detector dimension \(R_{2f}\), which varies with \(r_{1f}\). The maximum value of \(|r_2|\) is OD, which represents the extent of the source.

The region ABCJFH represents the E-boundaries which enter into the computation of \(\eta_{2T}\). The shaded region ABCJGH represents the boundaries entering into \(\eta_{3T}\).

Computation of \(\eta_{2T}\) for One Source Point. The properties of these curves can be considered in greater detail in Fig. (9.15), which shows only the \(E_6\) and \(E_0\) curves; hence, only the factors governing computation of \(\eta_{2T}\) are covered. The various possible entrance lenses are considered:

*For symmetrical systems \(q\) has the sign of \(h(\psi_s)\) for \(W_s > 0\), from Eq. (9.51b).
\( P > 0 \) (convergent), \( P = 0 \) (no lens, the case of Section \#7), and \( P < 0 \) (divergent). The \( r_2 \)-values at which the various \( E_0 \)-curves intersect the \( E_\infty \)-curves are of some importance. The complex nomenclature involved is given in Table 9.2 and partially illustrated in Fig. (9.15).

It is worth mentioning the following features of these curves:

(i) The intersection points [at \( r_2 = (r_{0+}, r_{0-}) \)] of \( E_0 \) and \( E_\infty \) with the line \( E = 0 \) are independent of the type or strength of the lens. This is evident from Eq. (9.43b), which indicates that the roots of the equations \( E_0^+ = 0 \) and \( E_\infty^- = 0 \) are independent of \( W_\infty \). Thus,

\[
\begin{align*}
\eta_{2T} &= E_{0+}^I - E_{0-}^I = \frac{1}{2} - 0 = \frac{1}{2}. \\
\end{align*}
\]

(ii) From Eq. (9.49), the slopes of the \( E_0 \)-curves are all negative, and are largest in magnitude for a convergent lens \( (P > 0) \), for which \( W_\infty < 1 \); next in steepness is the \( P = 0 \) case, with \( W_\infty = 1 \), and least steep is the divergent lens case \( (P < 0) \), since \( W_\infty > 1 \). The areas shown as convergent only or divergent only (or both) apply only to the cases illustrated. The areas included for the general case depend upon the slope of the \( E_0^- \) and \( E_\infty^+ \) lines. For \( P < 0 \), the slope may vary from zero (horizontal lines), when \( W = \infty \) up to the \( P = 0 \) line; for \( P > 0 \), the slope may vary from infinity (vertical line) when \( W = 0^* \) down to the line \( P = 0 \).

\* It will be recalled that the case of \( W < 0 \) was excluded as an unrealistic value for well-designed spectrometers.
Table 9.2.

*r*-VALUES* AT INTERSECTION OF E₀-CURVES WITH Eₑ-CURVES AND WITH E = 0

<table>
<thead>
<tr>
<th>Intersection of:</th>
<th>with</th>
<th>r₂-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>E₀⁺ (P &gt; 0)</td>
<td>Eₑ⁺</td>
<td>r₀⁺C⁺</td>
</tr>
<tr>
<td>E₀⁺ (P = 0)</td>
<td>Eₑ⁺</td>
<td>r₀⁺*⁺</td>
</tr>
<tr>
<td>E₀⁺ (P &lt; 0)</td>
<td>Eₑ⁺</td>
<td>r₀⁺D⁺</td>
</tr>
<tr>
<td>E₀⁻ (P &gt; 0)</td>
<td>Eₑ⁻</td>
<td>r₀⁻C⁻</td>
</tr>
<tr>
<td>E₀⁻ (P = 0)</td>
<td>Eₑ⁻</td>
<td>r₀⁻*⁻</td>
</tr>
<tr>
<td>E₀⁻ (P &lt; 0)</td>
<td>Eₑ⁻</td>
<td>r₀⁻D⁻</td>
</tr>
<tr>
<td>E₀⁺ (P &gt; 0)</td>
<td>Eₑ⁻</td>
<td>r₀⁻C⁻</td>
</tr>
<tr>
<td>E₀⁺ (P = 0)</td>
<td>Eₑ⁻</td>
<td>r₀⁻*⁻</td>
</tr>
<tr>
<td>E₀⁺ (P &lt; 0)</td>
<td>Eₑ⁻</td>
<td>r₀⁻D⁻</td>
</tr>
<tr>
<td>E₀⁺ (all P)</td>
<td>E = 0</td>
<td>r₀⁺</td>
</tr>
<tr>
<td>E₀⁻ (all P)</td>
<td>E = 0</td>
<td>r₀⁻</td>
</tr>
</tbody>
</table>

*The notation rᵢⱼ *= refers to:

i: Aperture intersection. In this case, it is always 0, because only the exit aperture is considered.

j: refers to sign of exit aperture curve involved, i.e., E₀⁻ or E₀⁺.

k: nature of entrance lens. C = convergent (P > 0), D = divergent (P < 0), * = no lens (P = 0), as in Section #7. Omitted for intersection with E = 0.

m: refers to sign (±) of entrance aperture curve involved [i.e., Eₑ⁺ = ±½ or Eₑ⁻ = ±½], or to 0 for E = 0.
(iii) For the source quadrant considered, \( E^'_M = \frac{1}{2} \) when \( P = 0 \) (no lens effect), but \( E^'_m = -\frac{1}{2} \) only when \( r_2 = 0 \), i.e., \( E^'_m > -\frac{1}{2} \) for \( |r_2| > 0 \). Then \( \eta^*_2 T = \eta^*_2 < 1 \) whenever \( r_2 \neq 0 \), as mentioned in Section #7. This arises because the orbit rotation \( \Delta \phi = (\phi^f - \phi^s) \) results in \( \phi^f < -\frac{1}{2} \phi^s \) for \( \phi^s \)-values in the range \( [-\frac{1}{2} \phi^s + |\Delta \phi|] \), where \( \Delta \phi < 0 \), and, of course, increases in magnitude with \( |r_2| \).

(iv) When \( P > 0 \), the convergent lens effect opposes the \( r_2 \)-caused rotation in (iii), so that for \( |r_2| < |r^C^-| \) all rays pass through the exit aperture and \( \eta^*_2 T = E^'_M - E^'_m = 1 \). As \( |r_2| \) increases beyond this point, the converging lens can compensate only partially for the \( r_2 \)-rotation, and the most negative \( \phi^s \)-rays are blocked by the exit aperture. (\( E^'_m \) increases towards \( E = \frac{1}{2} \)). At the positive \( \phi^s \)-end, there is no cutoff, except by the entrance aperture, since all rotations are towards negative \( \phi \) (\( r_2 \) and \( \delta r_2 \) both negative). Evidently, the steeper is \( E^C_0 \), the greater is the range of \( r_2 \) for which \( \eta^*_2 T = 1 \). From this point of view, the strongest possible convergent lens is desirable for maximum transmission with large \( |r_2| \), provided \( |r_2| < |r^0_0^-| \), i.e., the critical cutoff value of \( \phi^s \) is negative. For \( |r_2| > |r^0_0^-| \), however, the critical value of \( \phi^s \) is positive, and the transmission increases with \( W_0 \), so that a lensless system is better than a convergent lens, and a divergent lens best of all. This is true only for points with large \( r_2 \)-values and is not true when the entire source area is considered. In addition, as pointed out below, strong lenses are inconsistent with high transmission through the detector aperture.

(v) The boundary-value \( r^C^+ \) is computed by setting \( E^0_0 = -\frac{1}{2} \) [see Eq. (9.43b)].

Thus, \( \eta^*_2 T = 1 \) for a convergent lens when

\[
|r_2| < \left| \frac{1}{2} \varphi z_s N_s \right| = |r^C^-| \quad (9.55)
\]

For \( |r_2| > \left| \frac{1}{2} \varphi z_s N_s \right| \), the transmission for the convergent lens is

\[
\eta^*_2 T = E^'_M - E^'_m = \frac{1}{2} + \frac{1}{W_0} \left[ \frac{1}{2} + \frac{F_1}{\varphi z_s} r_2 \right] \quad (9.56)
\]

(vi) For any value of \( P \), complete cutoff by the exit aperture can occur only for very large \( |r_2| \)-values, such that \( |r_2| > |r^0_0^-| \), where \( r^0_0^- = (r^C^+, r^{0+} \text{ or } r^{D+}) \). If the source contains \( |r_2| \)-values larger than these limiting values, no transmission at all can occur for such points. It is evident from Fig. (9.15) that this maximum \( |r_2| \)-value is least for the convergent lens and largest for the divergent lens. The reason is that in
the region of positive \( \phi_s \)-values, the \( \phi_{r_2} \)-value for a divergent lens is positive, thus opposing the rotation effect of the negative \( r_2 \). The boundary value \( r_{0+}^+ \) is computed by setting \( E_0^+ = \frac{1}{2} \). Thus

\[
|r_{0+}^+| = \left| \frac{\varphi_s}{F_1} (1 + W_s) \right| = \left| \frac{\varphi}{F_1} + \frac{1}{2} \varphi N_s \right| z_s, \tag{9.57}
\]

and \( \eta_2 T = 0 \) for \( |r_2| > |r_{0+}^+| \). It is evident that \( |r_{0+}^+| \) increases monotonically with \( W_s \).

(vii) For the divergent case \( P < 0, W_s > 1 \), \( E_0^- > -\frac{1}{2} \varphi \) in the left quadrant, so that \( E_{M} = E_0^- \) for all \( r_2 \). The intersection of \( E_{0+}^+ \) with \( E_{e}^+ = \frac{1}{2} \) is given by

\[
|r_{0+}^D|^+ = \frac{1}{2} \varphi z N_s, \tag{9.58}
\]

which is the same as the magnitude of the limit \( |r_{C}^C| \) for a convergent lens of the same power. For \( |r_2| < |r_{0+}^D|^+ \), we have \( E_{0}^+ < \frac{1}{2} \), so that \( E_{M} = E_{0}^+ \)

for this range. Hence, \( \eta_2 T \) is the vertical separation \( 2k \) between \( E_{0}^+ \) and \( E_{0}^- \). From Eq. (9.50), then,

\[
\eta_2 T = \frac{1}{W_s}, \tag{9.59}
\]

i.e., independent of \( r_2 \) for small \( |r_2| \). This is identical to Eq. (9.21), as might be expected, since \( r_2 = 0 \) in that case. This independence is related to the fact that the critical \( \phi_{s'} \)-planes \( (\phi_s = \pm \frac{1}{2} \varphi/W_s) \) are not at the exit aperture \( (\phi_s = \pm \frac{1}{2} \varphi) \) when \( r_2 = 0 \). Since the rotation is linear in \( (r_2 + \delta r_2) \), an increase in \( (-r_2) \) serves to increase the rotation for \( \phi_s < 0 \) (since \( \delta r_2 < 0 \)) and to decrease it for \( \phi_s > 0 \) by just the same amount. Thus, the increased number of \( \phi_{s'} \)-planes excluded for \( \phi_s < \phi_M \) is compensated by the decreased number excluded for \( \phi_s > \phi_M \). The compensation operates as \( |r_2| \) increases until the positive \( \phi_{s'} \)-boundary moves into \( \phi_s = \frac{1}{2} \varphi \), i.e., until the positive \( \Delta \phi \)-rotation of the lens at \( \phi_s = \frac{1}{2} \varphi \) is just compensated by the negative \( \Delta \phi \)-rotation due to \( r_2 = r_{0+}^D \).

(viii) For \( |r_2| > |r_{0+}^D| \), we have \( E_{M} = E_{e}^+ = \frac{1}{2} \). Then

\[
\eta_2 T = \frac{1}{2} + \frac{1}{W_s} \left[ \frac{1}{2} + \frac{F_1}{\varphi z} r_2 \right], \tag{9.60}
\]

which is the same as Eq. (9.56)
(ix) Using Eq. (7.8b) for the transmission $\eta_2^*$ with no fringing field,

$$\frac{\eta_2^* - \frac{1}{2}}{\eta_2 T - \frac{1}{2}} = W_s$$  \hspace{1cm} (9.61)

for the range of $r_2$ in which Eqs. (9.56) and (9.60) hold for both converging and diverging lenses, i.e., for $|r_2| > \frac{1}{2} \theta z_s N_s$, but $|r_2| < |r_0^-|$. For $|r_2| < |r_0^-|$, both $\eta_2^*$ and $\eta_2 T$ exceed $\frac{1}{2}$; hence, $\eta_2^* > \eta_2 T$ for a divergent lens but is less for a convergent lens. Since $\eta_2^*$ and $\eta_2 T$ are less than $\frac{1}{2}$ for $|r_2| > |r_0^-|$, for a divergent lens $\eta_2^* < \eta_2 T$, but exceeds it for a convergent lens. Hence, for these $r_2$-values, $\eta_2 T$ is greater for a divergent lens than for a convergent lens.

**Averaging $\eta_2 T$ over Source.** The above equations for $\eta_2 T$ apply only for a single point on the source. A complete computation of the transmission through the exit aperture involves averaging over the sample quadrant and then over $\psi_s$. With the assumptions made, all points of the source with the same $r_2$-value have the same $\eta_2 T$-value. As in Section #7, the sample boundary is described by $r_1 = R_1(r_2)$. Hence, integration over the sample requires only integration over $r_2$, weighting each $r_2$-value with the sample height $|R_1|$, yielding the average $\eta_2 T(\psi_s)$, at a fixed $\psi_s$:

$$\eta_2 T(\psi_s) = \frac{1}{4 d_s} \int_{-R_2^*}^{0} \eta_2 T(r_2, \psi_s) |R_1(r_2)| \, dr_2,$$  \hspace{1cm} (9.62)

where, for a converging entrance lens,

(i) $\eta_2 T(r_2, \psi_s) = 1$ for $|r_2| < \frac{1}{2} \theta z_s N_s = |r_0^-|$ \hspace{1cm} [Eq. (9.55)]

(ii) $\eta_2 T(r_2, \psi_s)$ given by Eq. (9.56) for $|r_0^-| < |r_2| < |r_0^+|$, the latter limit being given in Eq. (9.57).

(iii) $\eta_2 T(r_2, \psi_s) = 0$ for $|r_2| > |r_0^+|$.  

For a diverging entrance lens,

(i) $\eta_2 T(r_2, \psi_s) = 1/W_s$ for $|r_2| < \frac{1}{2} \theta z_s N_s = |r_0^+|$ \hspace{1cm} [Eq. (9.58)]

(ii) $\eta_2 T(r_2, \psi_s)$ given by Eq. (9.60) for $|r_0^+| < |r_2| < |r_0^-|$, the latter limit being given in Eq. (9.57).

(iii) $\eta_2 T(r_2, \psi_s) = 0$ for $|r_2| > |r_0^-|$.  

Here $Q_s$ is the area of source, and $R^*_2$ is the magnitude of the farthest $r_2$-extension of the source. This relation involves the assumption (as in Section #7) that the source-activity distribution is uniform.

The simplest case is that of a rectangular source of dimensions $2R^*_2$ by $2R_1$. This is not a useful shape for a multigap instrument in which $0 \leq \phi < 2\pi$, although it is an important case for single-sector instruments or in those using a few sectors over a restricted $\phi$-range. However, even for the general multigap case, the relative simplicity of the integrals allows for a ready examination of qualitative features not as easily evident in an integration for a circular source.

Then, from Eq. (9.62),

\[ \eta_2 T(\psi_s) = \frac{1}{R^*_2 R_1} \int_{-R^*_2}^{0} \eta_2 T(r_2, \psi_s) R_1 \, dr_2 = \frac{1}{R^*_2} \int_{-R^*_2}^{0} (E_M - E_m) \, dr_2, \quad (9.63) \]

which is determined directly from the areas in Fig. (9.15) enclosed by $E_M$ and $E_m$, and by $r_2 = -R^*_2$ and $r_2 = 0$.

For a converging lens, for $R^*_2 > |r_{0^+}|$,

\[ \eta_2 T(\psi_s) = \frac{1}{R^*_2} \left\{ \int_{-C^-}^{0} \left[ \frac{1}{2} + \frac{1}{W_s} \left( \frac{1}{2} + \frac{F_1}{\varphi z_s r_2} \right) \right] \, dr_2 + \int_{C^-}^{0} \, dr_2 \right\} \]

\[ = \frac{1}{R^*_2} \left\{ \frac{1}{2} \left( 1 + \frac{1}{W_s} \right) (r_{0^+} - r_{2m}) + \frac{1}{W_s} \frac{F_1}{\varphi z_s} \left[ (r_{0^+})^2 - (r_{2m})^2 \right] - r_{0^+} \right\} \]

\[ = \frac{1}{2R^*_2} \left\{ \frac{1}{W_s} \left[ \frac{1}{2} \varphi z_s N_s - r_{2m} \right] \left[ 1 + \frac{F_1}{\varphi z_s} \left( \frac{1}{2} \varphi z_s N_s + r_{2m} \right) \right] \right\}, \quad (9.64) \]

where $(-r_{2m})$ is the smaller of $R^*_2$ and $r_{0^+}$, and $r_{0^+} = \frac{1}{2} \varphi z_s N_s$.

If $R^*_2 > |r_{0^-}|$, the part of the source beyond $r_2 = r_{0^-}^C$ does not transmit any rays through the exit aperture at the $\psi_s$-value involved,
Then, with
\[
\rho_m = \rho_0^C = -\left[ \frac{\varphi z_s}{F_1} + \frac{1}{2} \varphi z_s N_s \right],
\]
we find
\[
\eta_2 T(\psi_s) = \frac{1}{2 R_2^*} \frac{\varphi z_s}{F_1} = \frac{1}{R_2^*} (-r_0^0 - \frac{1}{2} \alpha), \tag{9.65}
\]
with
\[
\alpha = \frac{\varphi z_s}{F_1 R_2^*}. \tag{9.66}
\]
For \( R_2^* < |r_0^C| \), then \( \eta_2 T = 1 \), so that
\[
\eta_2 T(\psi_s) = 1. \tag{9.67}
\]
From Fig. (9.15), for a diverging lens, when
\[
R_2^* < \frac{1}{2} \varphi z_s N_s = |r_D^+|
\]
we have
\[
\eta_2 T(\psi_s) = \frac{1}{R_2^*} \int_{-R_2^*}^{0} \frac{1}{W_s} \, \mathrm{d}r_2 = \frac{1}{W_s}, \tag{9.68}
\]
which is the same as Eq. (9.21) for a point source. For \( R_2^* > |r_D^+| \), on the other hand,
\[
\eta_2 T(\psi_s) = \frac{1}{R_2^*} \left\{ \int_{r_2m}^{r_D^+} \left[ \frac{1}{2} + \frac{1}{W_s} \left( \frac{1}{2} + \frac{F_1}{\varphi z_s} \right) r_2 \right] \, \mathrm{d}r_2 + \int_{r_2m}^{r_D^+} \frac{1}{W_s} \, \mathrm{d}r_2 \right\}
\]
\[
= \frac{1}{2 R_2^*} \left\{ \left[ r_D^+ - r_2m \right] \left[ 1 + \frac{1}{W_s} \frac{F_1}{\varphi z_s} (r_D^+ + r_2m) \right] \right. \right.
\]
\[
- \frac{1}{W_s} \left[ r_D^+ + r_2m \right] \right\}, \tag{9.69}
\]
\[\dagger\] In this case, the integral in (9.64) is evidently the area [see Fig. (9.15)] bounded by \( E_0^C \) and \( r_2 = 0 \), which area is \( |r_0^C| \).
where \((-r_{2m})\) is the smaller of \(R_2^*\) and \(|r_0^{D+}|\), and \(r_0^{D+} = -\frac{1}{2} \varphi z_s N_s\). If \(R_2^* > |r_0^{D+}|\), then

\[
r_{2m} = -\left(\frac{\varphi z_s}{F_1} + \frac{1}{2} \varphi N_s z_s\right),
\]

and

\[
\eta_2 T(\psi_s) = \frac{1}{2R_2^*} \frac{\varphi z_s}{F_1} = \frac{1}{2} \alpha.
\] (9.70)

When no lens is present (i.e., \(P = 0, W_g = 1\)),

\[
\eta_2 T(\psi_s) = \frac{1}{R_2^*} \int_{r_{2m}}^0 \left(1 + \frac{F_1}{\varphi z_s} r_2\right) dr_2 = -\frac{1}{R_2^*} \left(r_{2m} + \frac{1}{2} F_1 r_{2m}^2\right),
\] (9.71)

where \((-r_{2m})\) is the smaller of \(R_2^*\) and \(|r_0^{*-}|\). For \(R_2^* > |r_0^{*-}|\), we have \(r_{2m} = -R_2^*\) and

\[
\eta_2 T(\psi_s) = 1 - \frac{1}{2} \frac{F_1 R_2^*}{\varphi z_s} = 1 - \frac{1}{2} \alpha,
\] (9.72)

which is the same as Eq. (7.11a). For \(r_{2m} = r_0^{*-} = -\frac{\varphi z_s}{F_1}\), we have

\[
\eta_2 T(\psi_s) = \frac{1}{2} \alpha,
\] (9.73)

which is the same as Eq. (7.11b).

The magnitude of \(\eta_2 T(\psi_s)\) depends both upon \(R_2^*\) and the nature and magnitude of the lens power. Using the notation \(C, D,\) and \(*\) as superscripts for convergent, divergent, and zero power lenses, respectively, the following relations hold [as is evident from an examination of the areas in Fig. (9.15) or from the analytical expressions above]:

\[
\eta_2^C(\psi_s) > \eta_2^*(\psi_s) > \eta_2^D(\psi_s) \quad \text{for } 0 \leq R_2^* < |r_0^{*-}|
\] (9.74a)

\[
\eta_2^C(\psi_s) = \eta_2^*(\psi_s) = \frac{1}{2} \quad \text{for } R_2^* = |r_0^{*-}|
\] (9.74b)
It is evident from Eq. (9.74d) that, for very large sources, the transmission is independent of the power of the lens. For smaller sources, the convergent lens gives a higher transmission, and since (for other reasons) only small sources are of practical interest, the convergent lens is preferable when the goal is a maximizing of $\eta_{2T}(\psi_S)$.

It may be noted that for the convergent case, the part of the source with $r_2$-values such that $|r_2| > |r_0^+|$ provides a contribution to the transmission which is less than one-fourth the total.† For this reason, it may be a useful rule of thumb to use, as a practical limit for the source size,

$$\max R_2^* \leq \frac{1}{2} |r_0^-| = \frac{1}{2} \varphi z_g/F_1. \quad (9.75)$$

An example may be useful: for a symmetric instrument with a typical K-value ($K = 0.6$), with $n_S = n_f = 0$, we have $F_1(\psi_S = 150^\circ) = 0.61$ and $F_1(100^\circ) = 1.26$. Taking an average $F_1$ as about one, then, in units of $z_g$,

$$2R_2^* = \text{Maximum source width} \approx \varphi.$$

Commonly used values for some constructed instruments have been $\varphi = 0.17 \ (10^\circ)$ and $\varphi = 0.05 \ (3^\circ)$. Other considerations, such as resolution, sets a value $2R_2^*/z_g < 0.05$. Thus, the $\eta_2$-factor is not important for instruments with gaps in the $10^\circ$-range, but may be significant for instruments with gaps as small as $3^\circ$.

It is evident from Fig. (9.14) and Eq. (9.52) that, if the detector opening $2R_2f$ is very large, then $E_d^+$ and $E_d^-$ are not limiting and $\eta_{2T} = \eta_{23T}$. In this case, it is convenient to have a convergent lens, the optimum value of the lens power occurring when $R_2 = |r_0^+| = \frac{1}{2} \varphi z_g N_S^*$, i.e.,

$$N_S = \frac{2R_2}{\varphi z_g} = \frac{1}{2} \frac{R_2}{\varphi z_g}. \quad (9.76)$$

† This is evident from Fig. (9.15) if we consider $P = 0$. The maximum contribution for the part beyond $r_0^-$ is less than one-third the contribution of the part with $|r_2| < |r_0^-|$. The difference is even greater for $P > 0$. 

$$\eta_{2T}(\psi_S) = \eta_{2T}(\psi_S) = \eta_{2T}(\psi_S) \quad \text{for} \ R_2^* \leq \frac{1}{2} |r_0^-|. \quad (9.74d)$$
When the source is circular, as is usual for a multigap instrument, the transmission is not given simply by the areas in Fig. (9.15). Suppose the source radius to be \( R \). In Eq. (9.62), \(|R_1(r_2)|\) is then equal to \( \sqrt{R^2 - r_2^2} \). Since more weight is thus given to the smaller \( r_2 \)-values, the convergent lens will always give better transmission, independent of the source size. Then,

\[
\eta_2 T(\psi_S) = \frac{1}{\frac{1}{4} \pi R^2} \int_{-R}^{0} \sqrt{R^2 - r_2^2} \eta_2 T(r_2, \psi_S) \, dr_2
\]

\[
= \frac{4}{\pi R^2} \left[ \int_{r_2}^{0} \sqrt{R^2 - r_2^2} \left[ \frac{1}{W_\Phi} \left( \frac{1}{2} + \frac{F_1}{\Phi Z_S} r_2 \right) \right] \, dr_2 \right] + \frac{1}{W'} \int_{r_2}^{0} \sqrt{R^2 - r_2^2} \, dr_2
\]

\[ (9.77a) \]

\[
\eta_2 T(\psi_S) = \frac{4}{\pi R^2} \left\{ \frac{1}{4} \left[ 1 + \frac{1}{W_\Phi} \right] \left[ \frac{r_2''}{2} \sqrt{R^2 - r_2''^2} - r_2 r_2'' \sqrt{R^2 - r_2^2} \right] \right. \\
- \frac{1}{3} \frac{1}{W_\Phi} \frac{F_1}{\Phi Z_S} \left[ \left( R^2 - r_2''^2 \right)^{3/2} - (R^2 - r_2^2)^{3/2} \right] \\
- \frac{1}{2} \frac{1}{W'} \left[ r_2'' \sqrt{R^2 - r_2''^2} + R^2 \arcsin \frac{r_2''}{R} \right] \right\},
\]

\[ (9.77b) \]

where, for a convergent lens, \((-r_2m)\) is the smaller of \( \left| r_0^{C+} \right| \) and \( R \), \( r_2'' = r_0^{C-} = \frac{1}{2} \Phi Z_S N_S \), and \( W' = 1 \); for a divergent lens, \((-r_2m)\) is the smaller of \( \left| r_0^{D-} \right| \) and \( R \), \( r_2'' = r_0^{D+} = -\frac{1}{2} \Phi Z_S N_S \), and \( W' = W_\Phi \). For \( R < \left| r_0^{C+} \right| \) (or \( R < \left| r_0^{D-} \right| \)), Eq. (9.77b) is simplified by the fact that \( R^2 - r_2^2 = 0 \) and \( -\arcsin \left( \frac{r_2}{R} \right) = \frac{1}{2} \pi \).

For \( P = 0 \), we have \( W_\Phi = 1 \) and \( r_2' = 0 \); then Eq. (9.77b) is the same as Eq. (7.10a) for \( r_2m = -R \) and the same as Eq. (7.10c) for \( r_2m = r_0^{*+} \).
Averaging \( \eta_2 T \) over \( \psi_s \). Using the usual solid-angle weighting factor, we have, as in Eq. (9.29),

\[
\overline{\eta_2 T} = \frac{\int \eta_2 T(\psi_s) \sin \psi_s \, d\psi_s}{\int \sin \psi_s \, d\psi_s}
\]

(9.78)

Numerical integration is required, even for a uniform source.

Computation of \( \eta_{23} T \). When the detector width \( 2R_{2f} \) is not very large compared to the image, then for some \( \phi_s \)- and \( r_2 \)-values, the detector boundary will cut off the rays. In such a case, \( E_M \) and \( E_m \) are the appropriate boundaries to use rather than \( E_M' \) and \( E_m' \), which were used in the computation of \( \eta_2 T(\psi_s) \) in Eqs. (9.62) and (9.63).

We now consider the case in which \( R_{2f} \) is comparable to the \( r_2 \)-extension of the image. The value of \( R_{2f} \) which may be used depends upon the shape of the detector to be used. If a circular detector baffle is used, which is common to all the gaps, then the maximum of \( R_{2f} \), i.e., \( D_{r2} \), is the same as \( D_{r1} \) (see Section #7). Since the resolution is a function of \( D_{r1} \), as discussed in Section #6, the detector diameter is limited by resolution considerations. In Section #6, it was found that the optimum value of \( D_{r1} \) was about equal to the radius \( R \) of the source. If \( D_{r2} \) is to be made larger than \( R \), the alternatives are to lose in resolution or to look for solutions in which \( D_{r1} \) is not closely coupled to \( D_{r2} \).

Since \( R_{2f} = R_{2f}(r_1) \), then, from Eq. (9.47), the value of \( \eta_{23} T \), averaged over the source, is

\[
\eta_{23} T(\psi_s) = \frac{1}{4} \frac{1}{A_s} \int_{-R_2^*}^{0} \int_{0}^{R_1(r_2)} \eta_{23} T(r_1, r_2, \psi_s) \, dr_1 \, dr_2
\]

\[
= \frac{1}{4} \frac{1}{A_s} \int_{-R_2^*}^{0} \int_{0}^{R_1(r_2)} \left\{ E_M(r_1, r_2, \psi_s) - E_m(r_1, r_2, \psi_s) \right\} \, dr_1 \, dr_2,
\]

(9.79)

where the source boundary is \( r_1 = R_1(r_2) \). An explicit evaluation of the integral is not feasible, because of the multiplicity of possible combinations of \( E_0^\perp \) and \( E_d^\perp \) entering into \( E_M \) and \( E_m \). This is true even when \( R_{2f} \) is

\[\dagger\] Two such solutions are discussed at the end of this section.
constant, as it would be in a rectangular detector baffle. Any particular case, however, specifies the lens power at entrance and exit apertures as a function of $\psi_s$; under these circumstances, the integral is readily evaluated.

It may be noted at this point that the dependence of $E_M$ and $E_m$ on $\psi_s$ enters not only through $W_g$ and $F_1$ in $E_0$ or through $g$ and $h$ in $E_d$, but also through $R_{2f}$ in $E_d$. Because of the finite thickness required in any aperture baffle used for energetic electrons, the effective shape of the baffle varies with $\psi_s$ and with energy.

It is useful, however, to illustrate some of the features affecting $\eta_3T$, and, for this purpose, we shall consider only the $E_d$ and the $E_0$ lines. Neglect of the $E_0$ lines is equivalent to reducing the available $\phi_s$-range, as, for example, by using a baffle at the entrance. The reduction is great enough so that no ray passing this baffle will hit the polefaces. Under these circumstances, the $E_0$-lines are never limiting. This is an unrealistic case, but useful for the purpose at hand. The transmission is then called $\eta_0^T$.

In computing this quantity, it is desirable to consider the various possibilities in the $(E_d^+, E_d^-)$, analogous to the considerations developed in Fig. (9.15) and Table (9.2). The intersection points of the $(E_d^+, E_d^-)$ lines with $(E_e^+ = \frac{1}{2}, E_e^- = -\frac{1}{2}, E = 0)$ are computed from Eqs. (9.43c) and (9.45) for both $h > 0$ and $h < 0$; the results are given in Table (9.3) and illustrated in Fig. (9.16). Neither Eqs. (9.43c) nor (9.45) can be used when $h = 0$. Going back to Eq. (9.37), we have

$$-R_{2f} \leq g \leq R_{2f}. \quad (9.80)$$

Because of our assumption that the exit aperture is not limiting, the $\phi_s$-range is not cut off; the $r_2$ cutoff is then independent of $E$. The slope of $(E_d^+, E_d^-)$ lines is thus infinite [as is evident from Eq. (9.51a)], and the choice of which line to call $E_d^+$ is arbitrary. We choose to label the positive $r_2$-limit as $E_d^+$; this corresponds to the $(-R_{2f})$ limit in Eq. (9.80), since $g < 0$ for the exit lens not too strong, as has been our assumption. It may be noted that the coincidence of the intersection points of $E_d^+(h > 0)$, $E_d^+(h = 0)$, $E_d^-(h > 0)$ and $E = 0$ is true only when the value of $g$ is chosen to be the same for the three cases. In general, when the $g$'s are not so chosen, there are twelve intersection points rather than the two illustrated.

Figure (9.16) illustrates* the results for $h > 0$, for $h = 0$, and for $h < 0$ under conditions in which $\eta_3T = 1$, at least for part of the $r_2$-range.

*The areas shown as belonging to $h > 0$ or $h < 0$ (or both) apply only to the cases illustrated. The areas included for the general case depend upon the slope of the $E_d^-$ and $E_d^+$ lines which can vary considerably on both sides of the vertical at $h = 0$. 
This is true when \( 2R_{2f} > \varphi z_s |h| \), i.e., when neither exit nor entrance lens is very strong. Also shown in Fig. (9.16) is the case when \( h \gg 0 \), i.e., when both entrance and exit lenses are fairly strongly convergent (but not enough to invert the sign). In this case, there is no \( r_2 \)-region for which \( \eta_{3T}^0 = 1 \). The dividing line is readily determined from Eq. (9.52); we have

\[
\eta_{3T}^0 = 1 \text{ only for the value } r_2 = 0 \text{ when the vertical separation is unity, i.e., } |h| = 2R_{2f}/\varphi z_s.
\]  

(9.81a)

When the vertical separation is less than unity, then \( \eta_{3T}^0 < 1 \) for all \( r_2 \). Then

\[
|h| > 2R_{2f}/\varphi z_s.
\]  

(9.81b)

When the vertical separation is greater than unity, \( \eta_{3T}^0 = 1 \) for some \( r_2 \); then

\[
|h| < 2R_{2f}/\varphi z_s.
\]  

(9.81c)

An examination of Fig. (9.16) and Table (9.3) shows that this corresponds to

\[
r_{d-}^+ > 0 \text{ or } r_{d-}^- < 0.
\]  

(9.82)

From Fig. (9.16), it is evident that, averaged over the sample,

\[
\eta_{3T}^0(\psi_g) = 1 \text{ if } R_s^2 \leq |r_{d+}^+| \text{ or } R_s^2 \leq |r_{d-}^-|,
\]  

(9.83a)

\( \dagger \) The lines, as drawn, are for the same \( g \)-value used for the other types of \( E_d \)-lines.
### Table 9.3

**r²-Values at Intersection of E⁻-Curves with E⁺-Curves and with E = 0**

<table>
<thead>
<tr>
<th>Intersection of E⁺ with E⁻</th>
<th>r²-value</th>
<th>r²-magnitude</th>
<th>r²-sign</th>
</tr>
</thead>
<tbody>
<tr>
<td>E⁺(h &gt; 0)</td>
<td>r²⁺⁺</td>
<td>1/2g(−ψz₂h + 2Rzf)</td>
<td>B⁺</td>
</tr>
<tr>
<td>E⁺(h = 0)</td>
<td>r²⁺⁻</td>
<td>−Rzf/g</td>
<td>+</td>
</tr>
<tr>
<td>E⁺(h &lt; 0)</td>
<td>r²⁻⁻</td>
<td>1/2g(−ψz₂h + 2Rzf)</td>
<td>B⁺</td>
</tr>
<tr>
<td>E⁺(h &gt; 0)</td>
<td>E⁻ = 1</td>
<td>r²⁺⁺</td>
<td>B⁺</td>
</tr>
<tr>
<td>E⁺(h = 0)</td>
<td>E⁻</td>
<td>r²⁻⁻</td>
<td>B⁺</td>
</tr>
<tr>
<td>E⁺(h &lt; 0)</td>
<td>E⁻</td>
<td>r²⁻⁻</td>
<td>B⁺</td>
</tr>
<tr>
<td>E⁺(h &gt; 0)</td>
<td>E⁻ = 0</td>
<td>r²⁺⁺</td>
<td>B⁺</td>
</tr>
<tr>
<td>E⁺(h = 0)</td>
<td>E⁻</td>
<td>r²⁻⁻</td>
<td>B⁺</td>
</tr>
<tr>
<td>E⁺(h &lt; 0)</td>
<td>E⁻</td>
<td>r²⁻⁻</td>
<td>B⁺</td>
</tr>
</tbody>
</table>

† The notation r²⁺⁻ refers to 1 aperture intersection. In this case, it is always d, because only the detector aperture is considered. j refers to sign of detector aperture involved, i.e., E⁺ or E⁻ k sign or value of h, i.e., (−, 0, +). m refers to sign of entrance curve involved [i.e., E⁺ or E⁻] or to 0 for E = 0.

‡ A⁺ = 1/2|z²| (2Rzf + ψz₂|h|), B⁺ = 1/2|z²| (2Rzf − ψz₂|h|) The magnitudes of all the intersection points (other than those having Rzf/g) are either A⁺ or B⁺, provided the g- and h-magnitudes are chosen to be the same. The magnitude resulting depends upon whether the 2Rzf-term has the same or opposite sign as the ψz₂h-term.

_only the case is considered in which 2Rzf > |h| |ψz₂|, i.e., neither entrance nor exit lenses are very strong. Since g is negative for the usual spectrometer considered, i.e., where |ψ(ψ₂)| is not too different from unity and the exit lens is not strongly convergent, the sign of an A⁺- or B⁺-term is opposite to that of the 2Rzf-term in the fourth column._
i.e.,

\[ R_2^* \leq \frac{1}{2|g|} \left[ -\varphi z_s |h| + 2R_{2f} \right] \]  \hspace{1cm} (9.83b)

or

\[ R_{2f} - R_2^* \geq \frac{1}{2}\varphi z_s |h| + R_2^* \left[ |g| - 1 \right]. \]  \hspace{1cm} (9.83c)

For a symmetrical spectrometer, this corresponds to

\[ R_2^* \leq \frac{1}{2 |W_s|} \left[ -\varphi z_s |N_s| (1 + W_s) + 2R_{2f} \right] \]  \hspace{1cm} (9.84a)

or

\[ R_{2f} - R_2^* \geq \frac{1}{2}\varphi z_s |N_s| (1 + W_s) + R_2^* N_s F_1 \]  \hspace{1cm} (9.84b)

which may also be written as

\[ R_{2f} - R_2^* \geq \frac{1}{2}\varphi z_s |N_s| \left[ (1 + W_s) \pm \frac{R_2^*}{2\varphi z_s} \right] \{ + \text{for } P < 0 \} \{ - \text{for } P > 0 \}. \]  \hspace{1cm} (9.84c)

If Eq. (9.83) is not satisfied, then \( \gamma_{3T}(\psi_s) < 1 \). Also \( \gamma_{3T} \) decreases rapidly with \( |h| \), and for large values of \( |h| \), i.e., when Eq. (9.81b) holds,

\[ \gamma_{3T}(\psi_s) = E_d^+ - E_d^- = |2k| = 2R_{2f}/\varphi z_s |h|, \]  \hspace{1cm} (9.85)

from Eq. (9.52).

It is evident that a decrease in transmission due to the detector aperture can be prevented by:

(i) use of large \( R_{2f} \)-values. In Fig. (9.16), this corresponds to increasing the vertical separation \( |2k| \) of the lines \( E_d^+ \) and \( E_d^- \), as in Eq. (9.52), without changing their slope. As mentioned above, considerations of resolution argue against having \( R_{2f} \) much greater than \( R_2^* \), so that it is desirable to keep \( (R_{2f} - R_2^*) \) small,

(ii) small \( \varphi \)-values. Limiting the \( \varphi \)-range decreases the spread of the image caused by the fringing field, but, of course, cuts the \( \gamma_{3T} \)-transmission.
(iii) small $|h|$-values. The quantity $h$ affects only the slope of the $E_d$-lines, but leaves the intersection $\pm R_{2f}/g$ with $E = 0$ unchanged (provided $g$ remains fixed). Since the influence of $h$ is quite large, we try, in general, to have boundary values which give the smallest values of $|h|$;

(iv) for a symmetric spectrometer (or one which is fairly close to symmetry), Eq. (9.84b) shows that a convergent lens is preferable, for systems of equal lens power (i.e., $|N_g|$), since in this case $(1 + W_g)$ is smaller and the second term is subtracted from the first, thus leading to a smaller value of $R_{zf} - R_2^*$. It was noted earlier that convergent lenses also gave better results insofar as the $\eta_2$-transmission was concerned;

(v) from the condition in Eq. (9.75) derived from $\eta_2$-considerations, we have the result that usually

\[
\frac{R_2^*}{\frac{1}{2} \Phi z_g} \ll 1,
\]

so that, as a rough approximation, this term may be neglected in Eq. (9.84c), and we may write as an approximate condition for $\eta_{2T}^0 = 1$ that

\[
R_{2f} - R_2^* > \frac{1}{2} \Phi z_g |h|;
\]

and

(vi) joining this condition with the $\eta_{2T}$-condition, we have

\[
\text{Maximum of } \frac{R_2^* F_1}{z_s} < \frac{1}{2} \Phi < \frac{R_{2f} - R_2^*}{z_s |h|}.
\]

This relationship should hold over the main part of the $\psi_s$-range accepted, if the transmission is to be reasonably high.

The choice between symmetric and asymmetric instruments on transmission considerations alone is not readily determined from the above equations, since $N_g$ and $N_f$ are not simply related. Only a detailed computation of $\eta_{2T}(\psi_s)$ can give a definite answer, and this requires examination of particular cases.

Averaging $\eta_{2T}(\psi_s)$ over all $\psi_s$ is carried out as in Eq. (9.78).
Thus,

$$\bar{T}_{23} = \frac{\int \eta_{23} T(\psi_s) \sin \psi_s \, d\psi_s}{\int \sin \psi_s \, d\psi_s},$$

with $\eta_{23} T(\psi_s)$ computed from Eq. (9.79).

**Effect of Fringing Field in Single-sector Instrument.** Some of the detector problems arising in the multigap instrument do not occur when a single sector is used. For the multigap instrument, the detector slit discussed above is placed at $(z_f, 0, 0)$ and is made circular, in order to receive rays from all the gaps with equal efficiency. Since the slit has extension in both the $r_2f$ and $r_1f$ directions for each gap, the diameter of the opening is limited because of resolution requirements (see Section #6), and if the lens effect of the fringing field is large, the losses due to spreading of the beam may be large, i.e., $\eta_{23}$ may be considerably smaller than $\eta_2$.

For the single-sector instrument, the detector slit need not be circular. It need have very little $r_1f$-extension and can have its primary extension in the $r_2f$-direction. Its optimum shape would be approximately that of a thin rectangular slot placed at $(z_f, 0, 0)$ with the long dimension running in the $r_2f$-direction,* and with the short dimension lying along the $z$-axis. Since this slit could be made as long as desired, it is not necessary that $\eta_{23}$ ever be smaller than $\eta_2$. Thus, the only losses from the fringing field would be due to the effect of divergent input lenses, which would drive particles into the polefaces.

An examination of the distribution of rays coming into the $r_2f$-direction indicates that the optimum slit would have a shape slightly different from a rectangular slot. The conical shape of the family of rays of constants $\psi_s$ was described above in connection with the discussion of Fig. (9.4), where it was pointed out that the family strikes the $z$-$r_2f$ plane in an arc rather than in a line. Thus, an optimum slit would be narrowest at the $z$-axis and would fan out at the ends to an extent determined by the $\psi_s$-range covered. It is to be noted, of course, that this curvature is marked only when the gap angle is large, since the extent of the curvature is determined by the curvature of the cone, which in turn is controlled by the magnitude of the gap angle.

**Multigap, Tilted-axis, Iron-core Instrument.** A modification of the multigap instrument makes possible the utilization of the virtue of minimum $r_1f$-extension in the detector of the single-gap instrument. In the multigap instrument considered thus far, the $z$-axes of the various gaps are coincident, so that a detector placed at the point $(z_f, 0, 0)$ for one gap must serve

---

*This would be modified somewhat if the source were large; see Section #5.*
in the same way for all the other gaps. If, however, the z-axes of the various gaps are tilted, so that they lie on the surface of a cone with vertex at \((z_g,0,0)\), then the points \((z_f,0,0)\) for the various gaps are not coincident. This makes it possible to use individual detector slits for each gap, each slit having little or no \(r_1\)-extension, and as large an \(r_2\)-extension as is allowed by the degree of tilting used. The various independent slits may open into the same, or different, detectors, depending upon the space made available by the tilting of the axes, thus offering another degree of freedom. Since it is still necessary to use the toroidal structure, such tilting is possible only with an iron-core instrument. Iron-free toroidal instruments must be built with coincident z-axes.

**Use of Ring Detector Slit in Multigap Instrument.** In our discussion of resolution and transmission in Section 6 and in the earlier part of this section, we have assumed that the detector slit is an open circle, with center on the z-axis at \((z_f,0,0)\). We now consider the possible virtues of displacing the detector slit for each gap to a position off the z-axis.

If it were possible to form a \(\psi\)-focus for finite sources at \((z_f,r_f)\) instead of \((z_f,0)\), then it would be possible to use a detector slit for each gap which has its major extension in the \(r_2\)-direction and minimum extension in the \(r_1\)-direction. Each gap might have its own slit instead of all having one common slit, so that the high-transmission requirements of one gap for large \(r_2\)-extension do not automatically involve a large \(r_1\)-extension for a gap 90° around the toroid. The large extension of the slit in the \(r_2\)-direction would result in \(\eta_{23}\) closely approaching \(\eta_2\); a corresponding increase in transmission results without a worsening of the resolution, which varies primarily with the \(r_1\)-extension of the slit.

The most convenient form of extended slit for the ensemble of gaps is made using a ring baffle, as in Fig. (9.17). We have here a circular ring of mean radius \(r_f\), with the detector located immediately behind it. If the focusing is as shown in Fig. (9.18), then the detector would have dimensions smaller than the inner circle of the ring.

The slits for the various gaps are not necessarily completely independent, unless \(r_f\) is fairly large. Thus, with an iron-free instrument, the images from the various gaps would overlap, because the gaps are close together. In the case of an iron-core instrument, in which the polepieces occupy an appreciable angle, the situation may be more favorable, because each gap may have its own slit shaped to the image-shape, and extending to the plane \(\phi_P\) passing through the middle of the iron section, as in Fig. (9.19).
The usefulness of this approach depends upon the possibility of forming a $\psi$-focus off the z-axis for finite sources. One possibility is that profiles designed for focusing on the z-axis may be utilized for focusing off the z-axis by a change in the specific field intensity $A_0$ [see Eq. (1.1)] used for any particular momentum $p$. A general analysis of this possibility has not been carried out, but a numerical investigation was made for a symmetrical system with $K = 0.59$ [see Fig. (3.3) and Table II.3 of Section #3]. This particular profile design is close to an optimum value for certain kinds of symmetrical instruments, and has been constructed in the form of an air-core instrument.\(^{16,17}\)

In Fig. (9.20) are shown the profile and the rays emerging at the two extreme angles of the $\psi_g$-range considered, as well as the median angle ($\psi_g = 110^\circ, 130^\circ, 150^\circ$). Both the $z$- and $r$-coordinates are given in units of $z_g$. Trajectories were computed for these entrance angles at various values of $A_0$, denoted by the corresponding values of $K$. [From Eq. (1.10), variation of $K$ at fixed $p$ corresponds to a variation of $A_0$.] The variations of $K$ considered around $K = 0.59$ are $\Delta K = \pm 0.01, \pm 0.02, \pm 0.03$.

Figure (9.21) shows the rays as they come together near the z-axis. It is evident that there is no true focal point off the z-axis, and that the width of the "image" for a point source, at its narrowest region, increases as $|\Delta K|$ increases. The $\Delta z$- and $\Delta r$-range of the "image" in this narrowest region is plotted against $K$ in Fig. (9.22), and over this $K$-range the relation is close to linear.*

*The sign of $\Delta z$ and $\Delta r$ is arbitrarily taken by subtracting the $z$-value (or $r$-value) for the $\psi_g = 130^\circ$ ray from the corresponding $(z,r)$ of the $\psi_g = 150^\circ$ ray. This convention gives opposite signs for $\Delta z$ and $\Delta r$ corresponding to the $K$-values on both sides of $K = 0.59$. 
At the top of Fig (9.22) is plotted the mean z-value within this narrowest region against K; another curve shows the mean r-value. Figure (9.23) shows the mean r-value plotted against the mean z-value. For a multigap toroidal system, this curve, rotated around the z-axis, constitutes the focusing surface for a multi-energy source. For use of the instrument as a spectrograph, the optimum focusing would occur if the detector were distributed over this "cone." It is evident that the spectrographic property with this particular profile design is not a good one, due to the rapid deterioration of the focusing property as the momentum deviates from that which focuses at \((z_f,0)\). A more useful design for spectrographic use
would probably involve profiles designed for higher K-values [see Fig. (3.3)]; with these, ψs-angles close to 90° are useable, and for these angles the spread in the image region would be smaller. Of course, with the higher K-values, one must accept the reduced transmission due to the larger fringing-field effects.

Because the use of a ring slit has the advantage of making η23 almost equal to η2, a better focusing arrangement off the z-axis is desirable. It should be possible to use the methods of Section #3 to compute an exit profile (for a given entrance profile) which would give ψ-focusing for an arbitrary focal point (z_f, r_f). If (z_f, r_f) is not too far from (-z_s, 0), then the imaging property of symmetrical systems may not be perturbed too much (see Section #5), and it may be possible to get suitable approximate ψ-focusing for spread sources. The φ-focusing, even for a point source and in the absence of fringing fields, is also perturbed by moving the image point off the z-axis, but this spread is small, since both r_f and φ are small. No general theory for this mode of focusing has been developed.

It should be pointed out that, with an iron-core instrument, the advantage of increasing η23 through extending the slit (for each gap) in the r_2-direction can more readily be attained by tilting the z-axis as discussed in the previous portion of this section. This approach has the advantage of preserving unperturbed φ-focusing. A ring focus, however, if achieved, would be useful for air-core instruments, since it would make it possible to achieve a considerably improved transmission (through increasing η23) at high resolutions.

Transmission for ψ_s-values in the Neighborhood of the Critical ψ_s-value. In Section #3 it was pointed out that the profile for a symmetric instrument becomes infinite for a critical ψ_s-value. Values of ψ_s in the neighborhood of and larger than the critical value cannot be used because \[ \left| \frac{d \phi_s}{d \psi_s} \right| \] for the profile is very large, leading to large values of \[ |\tan \mu_s| \] and \[ |N_s| \]. Imaging within the gap occurs and hence low values of η23 arise [see also Figs. (3.4) and (9.3)].
We have computed the \( \Delta \beta \)-deflection of the trajectory in Eq. (8.12):

\[
\Delta \beta = \frac{e}{p} \int H_t \tan \mu \, ds,
\]

the integral being taken along the trajectory from the field-free region outside the gap to a point \((z_e, r_e)\) on the trajectory where the magnetic field is described by Eq. (1.1) as \( H_e = A_0/r_e \). From Eq. (1.10) and (9.4),

\[
\frac{e}{p} = -\frac{1}{KA_0} = -\frac{1}{K H_e r_e} = -\frac{1}{K H_e} \left( \Delta x / \Delta \phi \right)
\]

Then

\[
\Delta \beta = -\frac{\int H_t \tan \mu \, ds}{K H_e (\Delta x / \Delta \phi)}
\]

From Eq. (9.2) and (9.3), at the entrance,

\[
\int H_t \, ds = H_e \Delta x = H_e r_e \Delta \phi = H_e r_e (\phi_s - \phi_M),
\]

where \( \phi_s \) is the \( \phi \)-plane containing \((z_e, r_e)\) and \( \phi_M \) is the median plane of the gap. Then

\[
\Delta \beta = -\frac{\Delta \phi}{K} \frac{\int H_t \tan \mu \, ds}{\int H_t \, ds}
\]

If \( \mu \) is constant through the fringing field, and equal to the initial value \( \mu_s \), then

\[
\Delta \beta = -\frac{\Delta \phi}{K} \tan \mu_s = -\frac{\Delta x}{Kr_e} \tan \mu_s,
\]

which is the result in Eq. (9.5).

This equation was used in Section 9 for computing transmission, despite the fact that it was known that \( \mu \) was not constant through the fringing field, because of the simultaneous \( \psi \)-deflection. We now propose to compute the necessary correction. If \( \Delta \psi_e \) is the change in \( \psi \) through the fringing field and \( \mu_e \) is the value of \( \mu \) at \((z_e, r_e)\), then

\[
\mu_e = \mu_s + \Delta \psi_e,
\]

in which \( \Delta \psi_e \) is always positive.
We define

\[ \Delta \beta_1 = \frac{e}{p} \tan \mu_s \int H_t \, ds = -\frac{\Delta \phi}{K} \tan \mu_s \]  

(10.8a)

and

\[ \Delta \beta_2 = \frac{e}{p} \tan \mu_e \int H_t \, ds = -\frac{\Delta \phi}{K} \tan \mu_e \]  

(10.8b)

At any point on the trajectory at which the \( \mu \)-angle has the value \( \mu \),

\[ \mu_s < \mu < \mu_e; \quad \tan \mu_s < \tan \mu < \tan \mu_e \]  

(10.9)

Hence, the true \( \Delta \beta \) lies between \( \Delta \beta_1 \) and \( \Delta \beta_2 \), and may be written as

\[ \Delta \beta = \frac{e}{p} \tan [\mu_s(\text{eff})] \int H_t \, ds = -\frac{\Delta \phi}{K} \tan [\mu_s(\text{eff})], \]  

(10.10)

where

\[ \mu_s(\text{eff}) = \mu_s + \theta \Delta \psi_e \text{ and } 0 < \theta < 1 \]  

(10.11)

and

\[ \tan [\mu_s(\text{eff})] = \frac{\int H_t \tan \mu \, ds}{\int H_t \, ds} \]  

(10.12)

For small angles and deflections, the relation can be simplified. Let \( d \) be the distance along the trajectory from the beginning of the integration. If \( \psi_d \) is the \( \psi \)-value and \( \mu_d \) the \( \mu \)-value when the particle is at \( d \), then

\[ \Delta \psi_d = \psi_d - \psi_s = \mu_d - \mu_s \]  

(10.13)

and

\[ \mu_d = \mu_s + \Delta \psi_d \]  

(10.14)

Taking \( \tan \mu_s \) as being of moderate size and \( \tan \Delta \psi_d \) as small, we have

\[ \tan \mu = \tan \mu_d = \tan (\mu_s + \Delta \psi_d) \approx \frac{\tan \mu_s + \tan \Delta \psi_d}{1 - \tan \mu_s \tan \Delta \psi_d} \]  

\[ \approx \tan \mu_s + \Delta \psi_d \]  

(10.15)

Then, from Eq. (10.11) and (10.12),

\[ \tan [\mu_s(\text{eff})] = \tan (\mu_s + \theta \Delta \psi_e) \approx \tan \mu_s + \theta \Delta \psi_e \]  

\[ \approx \tan \mu_s + \frac{\int \Delta \psi_d \, H_t \, ds}{\int H_t \, ds}, \]  

(10.16)
so that
\[ \theta \approx -\frac{\int \Delta \psi_d}{\Delta \psi_e} \frac{H_t}{H} ds \quad \text{and } 0 < \theta < 1. \]  

(10.17)

Since $\Delta \psi_e$ is always positive and $\tan \mu$ is negative for a convergent lens, this effect weakens a convergent lens and strengthens a divergent lens.

It is not feasible to compute $\theta$ accurately. Its determination involves a knowledge of $H_\phi$ and $H_t$ along the trajectory through the fringing field. Since these depend in detail upon the shape of the polepieces, location of coils, etc., it is not feasible to provide a general equation from which $\theta$ may be readily computed. However, it is feasible to compute an approximate* value for $\theta$, because the result is only moderately dependent upon the detailed structure of the fringing field.

**Estimates of $\theta$ for an $r^{-1}$-field.** We shall use two approaches for making approximate estimates of $\theta$. One involves the linear approximation to the field, discussed in Section #8, while the other involves numerical integration of trajectories, using a calculable magnet model. In the first approach, added assumptions are necessary. We make the simplest possible ones, some of which are demonstrably inaccurate, but which serve as crude approximations. The modest agreement between this approach and the second one indicate that the result is only moderately sensitive to these approximations.

The assumptions are:

(i) The magnitude of the fringing field $H$ [see Fig. (8.1)] at any $\phi$-plane varies in the same manner as the value on the median plane, i.e., as described in Eq. (8.6). The linear variation has been shown to be quite a good approximation on the median plane, but has not been so shown for other $\phi$-planes. Even if the linear variation were a good approximation, the value of $d_e$ varies with the $\phi$-plane, being somewhat smaller for $\phi$-planes close to the polefaces. However, we shall take $d_e$ as independent of $\phi$.

(ii) For points well within the polefaces, $H = H_\phi$ and $H_r = H_t = 0$. As $d$ decreases from $d = d_e$, the increase in the curvature of the fringing field leads to an increase in the ratio $H_t/H$. We take this increase to be

*Even though the field values ascribed to particular points are inaccurate, the integration tends to average out the deviations. This is analogous to the situation relative to the $\Delta \psi$-deflection. In Section #11, it is shown that the $\Delta \psi$-deflections on various $\phi$-planes are almost identical, despite the rather considerable differences in the $H_\phi$-values seen by the various trajectories.*
approximately linear and proportional to \((d_e - d)\), and normalized as
\((d_e - d)/d_e\). On a particular \(\phi\)-plane, then,

\[
H_t = H \left[ \frac{H_t}{H} \right] \phi = \text{constant}
\]

\[(10.18)\]

In this relation, \(H_t = 0\) at \(d = 0\), rising to a maximum at \(d = \frac{1}{2}d_e\) and falling to zero at \(d = d_e\).

(iii) At a given \((z,r)\), but variable \(\phi\), the field \(H\) increases toward the poleface, because of the crowding together of the lines of force. In addition, the change in inclination of \(H_t\) toward the poleface also tends to increase \(H_t\). We take the variation to be approximately linear in \(\Delta x\). This is not only a mathematical convenience, but also gives the appropriate limiting form if \(\tan \mu\) is taken as constant.

(iv) We assume that \(\Delta x\) remains constant as the trajectory passes through the fringing field. This is not actually true, since even when the trajectory lies in a \(\phi\)-plane, i.e., \((\phi - \phi_M)\) remains constant, the \(r\)-value of the trajectory changes through the fringing field. In addition, since \(\Delta \beta\) involves a continuous bending of the trajectory in the \(\phi\)-direction, then \((\phi - \phi_M)\) does in fact change through the fringing field. This effect is less important than the other approximations.

Thus \(H_t\) is the product of three factors, the effects being taken as linear and separable:

\[
H_t = D_0 H_e \frac{d_e - d}{d_e} \frac{d}{d_e} \Delta x,
\]

\[(10.19)\]

where \(D_0\) is a constant to be evaluated.

Inserting Eq. \((10.19)\) into \((10.3)\), and taking \(ds = d(d)\),

\[
\Delta \beta = -\frac{D_0}{d_e^2} \frac{\Delta \phi}{K} \int_0^{d_e} \left( d_e - d \right) d \tan \mu \frac{d(d)}{d_e}.
\]

\[(10.20)\]

As a limiting case, we consider the computation as in Section #9, i.e., with \(\mu = \text{constant}\) through the fringing field; then

\[
\Delta \beta = -D_0 \left[ \frac{\Delta \phi}{K} \tan \mu_s \right] \frac{d_e}{d_e}.
\]

\[(10.21)\]

Comparing this result with Eq. \((9.5c)\), we have

\[
D_0 = 6/d_e.
\]

\[(10.22)\]
Evaluating $\Delta \psi_d$ as in Eq. (8.7a), we have

$$\Delta \psi_d = \frac{1}{K_{re}} d_e \int_0^d d d(d) = \frac{d^2}{2 K_{re} d_e} .$$

Substituting (10.14), (10.15), (10.22), (10.23) and (8.7a) into (10.20), we have

$$\Delta \beta = -\frac{\Delta \phi}{K} [\tan \mu_s + 0.3 \Delta \psi_e] \approx -\frac{\Delta \phi}{K} \tan [\mu_s + 0.3 \Delta \psi_e] . \quad (10.24)$$

This is evidently the same as Eq. (9.5c), with the substitution of $\tan [\mu_s (\text{eff})]$ and with $\theta = 0.3$.

As an example of the effect of a variation in the assumptions, if we take the field increase in (ii) above to be zero [i.e., substitute a constant for $(d_e - d)/d_e$ in Eq. (10.18)], then $\theta = 0.5$. Although this assumption is evidently qualitatively deviant from the true situation, since it neglects the increasing curvature of lines of force with distance from the magnet, its use tends to set a limit on the value $\theta$. It seems likely, therefore, that a suitable value of $\theta$ is probably around 0.3 and is certainly $< 0.5$.

A limit of approximately 0.5 can be evaluated from qualitative considerations, without an evaluation of the details of the fringing field. Except for the region close to the poleface edge, $H_0$ is not much smaller than the magnitude $H$; thus, $H_\phi$ varies monotonically with $H$. If $H_{rz}$ varied in the same way, then $H_\phi ds \approx H_{rz} ds$, approximately. Since $d \psi = d\psi H_\phi ds$ and $d\beta \approx \tan \mu H_t ds = \sin \mu H_{rz} ds$, then, with $\mu \approx \sin \mu \approx \tan \mu$, we find $d \beta \approx \mu d \mu$.

Hence,

$$\Delta \beta \approx \frac{1}{2}(\mu_e^2 - \mu_s^2) = \frac{1}{2}(\mu_e + \mu_s) \Delta \mu \approx (\mu_s + \frac{1}{2} \Delta \psi_e) \int H_\phi ds$$

$$\approx (\mu_s + \frac{1}{2} \Delta \psi_e) \int H_t ds \approx (\mu_s + \frac{1}{2} \Delta \psi_e) H_e \Delta x \quad , (10.25)$$

since $\mu_e - \mu_s = \Delta \mu = \Delta \psi_e$. Thus, $\theta = 0.5$.

However, $H_{rz}$ does not, in fact, increase with $H$ along the trajectory as $H_\phi$ does, so that this result tends to overemphasize the effective change in $\mu$, insofar as the $\Delta \beta$-effect is concerned. This arises from the fact that the value of $\mu$ is important only where $H_{rz}$ is appreciable. Since $H_\phi$ increases steadily along the trajectory (except near the poleface, and even here the perturbation is moderate), $d \mu$ increases in the same manner into the interior field. However, $H_{rz}$ has a maximum outside of the profile, and rapidly decreases to zero inside the profile. Thus, instead of having $d \beta$ growing with $d \mu$, as was assumed in the calculation, $d \beta$ is significant over only part of the integration range for $\Delta \mu$, so that the effective value of $\mu$ in $\Delta \beta$ is smaller than the mean, $\frac{1}{2}(\mu_e + \mu_s)$, and $\theta < 0.5$. 
Computation of $\theta$ for Two-dimensional Uniform Fields. We now come to the alternative method for evaluation of $\theta$. If the field values were known in the fringing region, then numerical computation of the trajectories would provide $\Delta \phi$ and $\Delta \psi_e$ values, from which $\theta$ could be evaluated through Eq. (10.10) and (10.11). Such detailed information of the fringing field is not known for any of the toroidal ("orange") type $r^{-1}$ spectrometers, nor even for a single sector of this design. However, we may choose a magnetic model which can be computed analytically, and which may not differ too much from the spectrometer of interest. If the fringing field of the model is similar in structure, although differs in magnitude from the spectrometer, we can expect that the effect of changes in the magnitude of $\Delta \phi$ and $\Delta \psi_e$ will tend to cancel each other in the computation of $\theta$. To the extent that the cancellation is incomplete, the estimate of $\theta$ will be in error.

The model chosen is one that we shall use again in Section #11, with a similar approximation in mind. The model is taken as one with a uniform interior field, and with two-dimensional structure, i.e., the extent of the poleface along the gap is very large in comparison to the essential extent of the fringing field away from the poleface. It is well known that the fringing field of this type of structure may be computed through the use of the conformal transformation technique. Numerical values of the fringing field have been determined through the use of a 704 electronic computer, and applied to the computation of trajectories.

Two magnetic models are considered: Case I. Illustrated in Fig. (11.1), the poles extend indefinitely into and out of the paper, as well as vertically and to the left; Case II. The poles extend indefinitely into and out of the paper and to the left, but consist of infinitely thin sheets with no vertical extension. For trajectories which start out on planes parallel to the median plane, Eq. (10.1) applies with good approximation. Such planes are the analogs of the constant $\phi$-planes in the $r^{-1}$ field. The quantity $\Delta \psi$ was computed numerically through an equation analogous to Eq. (10.1) and $\Delta \psi_e$ from one analogous to Eq. (8.5).

Both $\Delta \phi$ and $\Delta \mu = \Delta \psi_e$ were computed by integrating from a starting point 7.5 gap widths out from the magnetic profile. For a real magnet, this is far enough out so that the magnetic field at this point may be considered negligible. The two-dimensional structure of the model, however, makes its fringing field die off more slowly than that of a real magnet, and this requires some consideration in using the line integral analogous to Eq. (9.2a).

As in Eq. (10.10), we may write

$$\Delta \phi = \frac{e}{p} \tan \left[ \mu_s \left( \text{eff} \right) \right] \oint H_t \, ds$$

(10.26)

*Rigorously, the integrand should include the factor $\cos \beta$, where $\beta$ is the angle of the trajectory to the initial plane. Numerical computations show that $\beta$ is always very small, so the $\cos \beta \approx 1$. 
In evaluating \( \int H_t \, ds \) through the line integral, we must allow for the field intensity at the beginning of the integration. Thus,

\[
\int_{y_i}^{y_m} H_y \, dy + \int_{y_m}^{y_f} H_t \, ds + \int_{y_f}^{y_i} H_y \, dy = 0 ,
\]  

(10.27)

where \( y_m \) denotes the \( y \)-coordinate of the median plane, \( y_i \) the \( y \)-coordinate of the initial plane of the trajectory, and \( y_f \) the \( y \)-value of the trajectory at the end point of the \( \Delta \beta \)-integration, taken to be a suitable distance into the gap. The first integral is evaluated at a constant distance, 7.5 gap widths, out from the magnet face. The third integral is evaluated at a constant distance inside the gap; the choice of the distance is discussed later. We write

\[
\int_{y_m}^{y_i} H_y \, dy = -\gamma_i \, H_e \, \Delta y_i ; \quad \Delta y_i = |y_m - y_i| ,
\]

(10.28a)

\[
\int_{y_f}^{y_m} H_y \, dy = \gamma_f \, H_e \, \Delta y_f ; \quad \Delta y_f = |y_m - y_f| ,
\]

(10.28b)

where \( H_e \) is the magnitude of field far into the interior. In effect, Eq. (10.28) defines \( \gamma_i \) and \( \gamma_f \). Here \( \Delta y_i \) and \( \Delta y_f \) are the distances of the trajectories from the median plane at the beginning and end, respectively, of the trajectory integration. In practice, \( \Delta y_i \neq \Delta y_f \). Then

\[
\int H_t \, ds = -H_e \left[ \gamma_f \, \Delta y_f - \gamma_i \, \Delta y_i \right]
\]

(10.29)

and

\[
\Delta \beta = -\frac{e \, H_e}{p} \tan [\mu_s (\text{eff})] \left[ \gamma_f \, \Delta y_f - \gamma_i \, \Delta y_i \right]
\]

(10.30)

\[
= -\frac{1}{\rho_C} \tan [\mu_s (\text{eff})] \left[ \gamma_f \, \Delta y_f - \gamma_i \, \Delta y_i \right]
\]

Using various values of \( \rho_C \), \( \mu_s \), and \( \Delta y_i \), computation of trajectories through the calculated magnetic field yielded values of \( \Delta \beta \), \( \Delta y_f \), and \( \Delta \mu \), from which are evaluated

\[
\tan [\mu_s (\text{eff})] = -\Delta \beta \, \frac{\rho_C}{\gamma_f \Delta y_f - \gamma_i \Delta y_i}
\]

(10.31a)

and

\[
\theta = \frac{\mu_s (\text{eff}) - \mu_s}{\Delta \mu}
\]

(10.31b)

Both \( \gamma_f \) and \( \gamma_i \) were computed by a numerical averaging of \( H_y \) as in Eq. (10.28).
The $\theta$-value calculated depends upon the endpoint chosen in the integration. We have chosen to compute $\theta$-values for two endpoints. The first is one gap width inside the profile, at which point the $H_y$-component (the analog of $H_0$) differs from the interior field by $< 0.05\%$. For this case, $\gamma_f = 1$. The other endpoint is at 0.3 times the gap width. This is the approximate value at which an extrapolation of the straight-line portion of the fringing field on the median plane intersects the interior field curve, as in Fig. (8.4). The value derived from similar extrapolations for several experimental measurements on electromagnet and permanent magnet structures is also approximately 0.3 times the gap width. For this case, $\gamma_f < 1$, and is a function of $\Delta y_f$.

Also, $\rho_C$ was varied from 4 to 50 gap widths and $\mu_s$ from $-20^\circ$ to $+9^\circ$.

The results of these computations were as follows:

(i) $\tan [\mu_s (\text{eff})]$ is fairly constant for various $\Delta y$-values, at given $\mu_s$ and $\rho_C$. It is more closely constant for large values of $\rho_C$, which corresponds to weak fringing fields. This may be expected from the optical analogy discussed in Section #9.

(ii) When the integration was carried into the gap a distance of one gap width ($\gamma_f = 1$), Case I yielded $\theta$-values close to 0.35 and approximately constant with $\Delta y_f$. Case II yielded $\theta$-values close to 0.30, with similar constancy over $\Delta y_f$.

(iii) When the integration was carried into the gap a distance of 0.3 gap width, Case I yielded $\theta$-values close to 0.54, while Case II yielded $\theta$-values close to 0.50, both having approximate constancy with $\Delta y_f$.

The change in $\theta$ for the two integration endpoints is to be expected, since $\Delta \beta$ increases very slightly between 0.3 and 1.0 gap width inside, while $\Delta \mu$ increases considerably.

The applicability of the calculated model to actual cases is uncertain, since the fringing field differs from that of real magnets, particularly in its rate of fall-off outside of the gap. It seems likely, however, that $\theta$ may be taken as lying between 0.3 and 0.5, even when the straight-line approximation to the magnetic field is used.

Example of a Computation of $\mu_s (\text{eff})$. To illustrate the magnitude of the effect, we have computed $\mu_s (\text{eff})$ for the case of a symmetrical spectrometer with $K = 0.6$ and $\phi = 10^\circ$, as shown in Fig. (10.1). The initial values of $\tan \mu_s$ were computed from Eq. (5.34) and are tabulated in Appendix IV, column 2.
These are shown as the solid curve in the figure. In computing the angular deflections, the straight-line approximation was taken for the magnetic field. The values of $d_e(\psi_s)$ were computed using the measurements described in Section #8: $\varphi r_e/d_e = 0.85$ at $\psi_s = 95^\circ$ and $\varphi r_e/d_e = 0.6$ at $145^\circ$; an approximate linear interpolation was used for $r_e$-values in between. Computing $\Delta \psi_e$ from Eq. (8.7b) and using $\varphi = 0.3$, values of $\mu_s(\text{eff})$ were determined and are shown as the dashed curve in the figure. It is evident that the change from $\mu_s$ to $\mu_s(\text{eff})$ lies between $3^\circ$ and $4^\circ$ from one end of the $\psi_s$-range to the other. Since $0.3 \Delta \psi_e$ is small and does not vary much over the $\psi_s$-range, it is apparent that the curve of $\mu_s(\text{eff})$ is almost a translated version of the $\mu_s$-curve. As pointed out above, the effect is unidirectional, since $\Delta \psi_e > 0$: where the lens is converging, it is made less convergent, a slightly convergent lens is made slightly divergent, and a divergent lens is made more divergent.
#11. EFFECT OF THE $\Delta\psi$-DEFLECTION ON THE RESOLUTION

The increased $\psi$-deflection due to the fringing field ($\Delta\psi$) has been discussed in Section #8. The variation of $\Delta\psi$ with $\psi_g$ degrades the focusing quality, if the profiles are constructed as described in Section #3 for an idealized field with no fringing effects. As discussed in Sections #13 and #15, the profile shape may be changed such that the increased $\Delta\psi$ is compensated, at each value of $\psi_g$, by a decrease in the net bending power of the interior idealized field. In an iron-core instrument, for example, iron may be removed at the profile, adjusting the amount removed at each $\psi_g$ to the requirement that all rays leaving $(z_g,0)$ arrive at $(z_f,0)$.

At the same $\psi_g$-value, however, a ray passing near a poleface experiences a different fringing field than does a ray passing through the median plane. It may be expected, in general, that a different value of $\Delta\psi$ will result, as well as a different point of entry into the idealized (interior) field. The adjustment of the profile can compensate only for the fringing-field $\Delta\psi$-deflection of rays lying in the median plane; insofar as the effect varies with $\phi_g$, the profile adjustment is less effective for $\phi_g \neq \phi_M$. The resulting perturbation of the trajectories disturbs the focusing property, with a corresponding loss in resolution. We propose to examine the difference in the focal position of the two types of trajectories in order to estimate the approximate effect on the resolution.

To pursue this question, it is necessary to have a detailed knowledge of the direction and magnitude of the entire fringing field. This information is not available for most magnets, and certainly not for any of the type with a $r^{-1}$-field. It was proposed, therefore, to consider, first, a simpler type of magnet, which could be treated more easily.

Since fringing fields differ very markedly for magnets of different designs, general computations on the absolute values of $\Delta\psi$ or of the point of entry into the interior field are not feasible. Indeed, accurate adjustment of the profile requires construction of the magnet, followed by an empirical correction after the $\Delta\psi$-effect has been experimentally evaluated. Although the absolute values are so sensitive to the particular form of the fringing fields, it is much less likely that the differences between the median and off-median trajectories are sensitive in this manner, particularly since the difference between the $\Delta\psi$-effects of the two types of trajectories is much smaller than the absolute values. It is for this reason that we consider it justifiable to evaluate the differences using a model which does not actually represent the $r^{-1}$-field spectrometer.

The magnet model is the one discussed in Section #10. As mentioned there, the primary features of the model are: first, the field is taken to be uniform in the interior; and second, the fringing field is taken to be two-dimensional, i.e., the extent of the front of the poleface is very
large in comparison to the essential extent of the fringing field away from the poleface. With these two assumptions, and with the condition that magnetic leakage is negligible (i.e., practically all of the lines of force originating on one pole end up on the other pole), it can be shown that $\Delta \psi$ is essentially independent of the distance of the trajectory from the poleface. (In the $r^{-1}$-spectrometer, this represents independence of the $\phi$-plane.) No detailed information of the structure of the fringing field is necessary for this computation.

Since the interior trajectory is determined not only by the angle of entry into the interior field, but also by the coordinates of the entry point, it is necessary to evaluate both the difference in $\Delta \psi$ and the difference between the displacements of the entry points. The latter requires a numerical computation of the trajectory through the fringing field into the interior field, and such computation involves a detailed knowledge of the value and direction of the fringing magnetic field at all points along the trajectory. Thus, for computing the displacement, specific assumptions must be made concerning the structure of the poles and of their fringing fields.

We consider, first, the general features of the $\Delta \psi$-computation, which is described in detail in Ref. 22. The coordinate $(x,y,z)$ system for the model is shown in Fig. (11.1). The entrance profile lies on the $z$-axis, positive $z$ going into the paper, and extends infinitely in the $\pm z$-direction. The two polefaces are on the planes $y = 0$ and $y = -h$; the median plane is $y = -\frac{h}{2}$. The planes $y = \text{constant}$ with $0 > y > -h$, are analogs of the planes $\phi = \text{constant}$ within one gap of the $r^{-1}$-spectrometer. Trajectories lying on constant-$\phi$ planes in the spectrometer correspond to trajectories lying on constant-$y$ planes in the model magnetic field. The model considered is Case I of Section #10.

Two trajectories are considered: (I) entering the magnetic field on the median plane, and (II) approaching at the same angle relative to the normal to the profile, but lying on an off-median plane. Through the fringing field, both trajectories are bent away from the original direction, but the median trajectory (I) is bent a greater distance. At a distance of the order of one gap width inside the profile (MNP), the magnetic field is essentially uniform. Both trajectories would have arrived at $M$ in the absence of a fringing field; they are, in fact, displaced the distance MN from this arrival point, the entry points differing by the slight distance NP, with $NP \ll MN$. 
Shift in Entry Angle (Δμ) in Uniform Field. Practically all of the trajectory bending by the fringing field occurs in the x-z plane, and it is this bending which affects the resolution. Figure (11.2) represents the projection of the trajectory onto the x-z plane; the magnitude of the bending is measured (as in Section #8) by the angle μ between the tangent to the projected trajectory and the perpendicular to the profile (i.e., the x-axis).* As shown, μ is taken as positive when directed from the x-axis to the tangent in a counterclockwise direction. To be evaluated is the change Δμ suffered by the trajectory in passing through the fringing field, Δμ corresponding to the quantity Δψ_e of the r^-1-field spectrometer.

It is shown that along the trajectory (see Ref. 22)

\[ dμ = - \left( \frac{e}{p} \right) \left[ \cosμ \tanβ H_ξ ds + H_τ ds \right] \]  (11.1a)

and

\[ Δμ = μ - μ₀ = - \frac{e}{p} \int H_τ ds - \frac{e}{p} \int \cosμ \tanβ H_ξ ds \]  \quad (11.1b)

where μ₀ is the original angle of orientation, ds is an element of the trajectory, and β is the angle made by the trajectory relative to the x-z plane, i.e., \( \sin β = - (dy/ds) \). In fact, β is the analogue of the angle β in Section #8, and, as in that section, the initial value of β is taken to be zero. We have

\[ dβ = \left( \frac{e}{p} \right) H_ξ \sinμ ds \]  \quad (11.2a)

and

\[ Δβ = β - β₀ = \frac{e}{p} \int H_ξ \sinμ ds \]  \quad (11.2b)

The connection between the values of Δμ for various trajectories with the same μ₀ is made by considering the number of lines of force passing through the region neighboring the trajectory. The number of such lines is defined as

\[ dN = H \cosθ \; dS \]  \quad (11.3)

where dN is the number of lines passing through the area dS, H is the field magnitude, and θ is the angle between H and the normal to dS. We consider

---

*This angle corresponds to that shown in Fig. (8.2).
an area adjacent to a trajectory and extending one unit in the z-direction; for trajectories which are parallel to the x-axis (i.e., at constant z and y, and $\beta = 0$) over the entire path,

$$N = \int H \cos \theta \, dx = \int H_y \, ds = \left| \frac{P}{e} \Delta \mu \right|.$$  \hspace{1cm} (11.4)

Two such trajectories at different y-values, but at the same z, pass through the same lines of force, and if the integration is taken into the uniform interior field, the integration ends on the same line of force. Hence, $N$ is independent of y, the equality of $N$ for the various trajectories involved implying that $\Delta \mu$ is the same for all of them, median and off-median trajectories alike.

These considerations assume that $\mu = 0$ throughout the trajectory, whereas $\Delta \mu$ is actually accumulated along the entire path through the fringing field. We consider, then, a trajectory at constant y over a curve $z = z(x)$. Now dS is no longer equal to ds; instead, $dS = dx = (dx/ds) ds = \cos \mu \, ds$. Then

$$dN = H \cos \theta \, dx = H_y \cos \mu \, ds = \left| \frac{P}{e} \cos \mu \, d\mu \right|.$$  \hspace{1cm} (11.5a)

We compare two such trajectories, $z_1(x)$ and $z_2(x)$, which lie on different y-planes ($y_1$ and $y_2$). If $z_1$ and $z_2$ were the same curves, then it would be clear that if all lines of force starting on one pole ended on the other, then the same lines of force would thread both trajectories. It is evident that, since the extension of the surface elements in the z-direction is unity for both curves, and since the field intensity is not a function of z, the displacement of $z_1(x)$ from $z_2(x)$ does not change the equality of lines of force. We integrate $dN$ over each trajectory from the same $x_{initial}$ to the same $x_{final}$, where $x_{final}$ is well within the uniform field, so that the integration paths terminate on corresponding lines of force (i.e., lines which originate at the pole at the same x-value). For each trajectory,

$$N_i = \left| \frac{P}{e} (\sin \mu_0 - \sin \mu_i) \right|.$$  \hspace{1cm} (11.5b)

Since $N_1 = N_2$ and the initial $\mu_0$'s are the same, then $\mu_1 = \mu_2$ and $\Delta \mu$ is independent of y.

One feature of this analysis requires further examination. From Eq. (11.2b), it is evident that, when $\mu \neq 0$, then $\beta \neq 0$, provided that $H_x \neq 0$. Thus, the relation $H \cos \theta = H_y$, utilized in Eq. (11.5a), does not apply except on the median plane, where $\beta = 0$ because $H_x = 0$. Analysis\(^{(22)}\) shows that, when $\beta \neq 0$,

$$dN = (p/e) \cos \beta \cos \mu \, d\mu - \sin^2 \mu \sin \beta \, H_x \, ds = (p/e) \cos \beta \cos \mu \, d\mu - (p/e) \sin \mu \sin \beta \, d\beta.$$  \hspace{1cm} (11.6)
and when $\beta$ is small, $\cos \beta \approx 1 - \frac{1}{2} \beta^2$ and $\sin \beta \approx \beta$. Accordingly,

$$dN = dN' - d\epsilon,$$  \hspace{1cm} (11.7a)

where

$$dN' = |(p/e) \cos \mu \, d\mu|$$  \hspace{1cm} (11.7b)

and

$$d\epsilon = \frac{1}{2} (p/e) \beta^2 \cos \mu \, d\mu + \beta \sin^2 \mu \, H_X \, ds$$

$$= (p/e) \left( \frac{1}{2} \beta^2 \cos \mu \, d\mu + \beta \sin \mu \, d\beta \right).$$  \hspace{1cm} (11.7c)

When $\mu$ is not large, $d\mu$ is considerably greater than $d\beta$, because of the $\sin \mu$ factor, as well as the fact that, for $-\frac{1}{2} h < y < 0$, we have $|H_y| > |H_X|$. Thus, $|d\epsilon| < < dN'$ and may be neglected.* With this approximation, Eq. (11.7b) is the same as Eq. (11.5a), and the remainder of the analysis follows. In actual computations of trajectories, it was found that the values of $\Delta \mu$ were quite accurately the same even for $\mu_0$ as large as $\pm 20^\circ$ (Ref. 22).

The proof essentially depends upon: (i) the fact that $\beta$ remains small, so that the motion is primarily in the $(x,z)$ directions, and (ii) upon the fact that the number of lines of force threading the regions neighboring each trajectory does not depend upon the displacement of the trajectory in the $z$-direction. Point (ii) depends partly upon the fact that the magnetic field is two-dimensional, and not a function of $z$.

The application of this result to the r⁻¹-field spectrometer will be developed later.

Shift in Entry Point ($\Delta z$) in Uniform Field. The separation of the two trajectories along the $z$-direction is now considered, i.e., we evaluate the magnitude of $NF$ in Fig. (11.2). A method for making a general computation of the type described for $\Delta \mu$ was not evident, so an actual numerical calculation of trajectories was carried out using an IBM 704 computer. To get the necessary values of $(H_y,H_X)$ as a function of $(y,x)$, a particular pole shape was assumed. If the polefaces are taken as extending infinitely in the $\pm y$-directions (Case I), then the field may be evaluated(23) by means of the technique of conformal transformation. Trajectories were computed for a number of $(y,\mu_0,\rho_C)$-values, where $\rho_C$ is the radius of curvature

---

*As an example, in the numerical computation of trajectories of Ref. (22), for the case in which the radius of curvature $(\rho)$ in the interior field is 10 gap widths [$\rho = 10h$, $\mu_0 = -9^\circ$, and $y_0 = -0.05h$ (close to poleface), it was found that $\Delta \beta/\Delta \mu = 0.016$. Using this as the approximate value of $d\beta/d\mu$, the ratio $d\epsilon/dN' = 0.000015$.  

---
within the uniform field.* It was found that when the orbits reached one
gap width inside the profile, then the difference in the z-coordinates of the
trajectories could be empirically fitted by the expression

\[ |\Delta z| = 0.46 \left( h' \right)^2 \rho_C (1 + 2 \mu^2) \quad , \]

(11.8)

where \( h' \) is the distance of the y plane containing the trajectory from the
median plane (i.e., \( h' = y + \frac{1}{2} h \)), and \( \mu \), the approximate mean value of
\( \mu \) (radians) along the trajectory, is \( \frac{1}{4} (\mu_0 + \mu_{\text{final}}) \). The direction of \( \Delta z \) is
given by the fact that the displacement from the orbit that would have
occurred in the absence of the fringing field is greater for the median
trajectory than for the non-median trajectory.

Shift in Entry Point (\( \Delta \eta \)) in \( r^{-1} \)-Field. Equation (11.8) is used in
considering the effect of the displacement on the resolution of the \( r^{-1} \)-field
spectrometer. In the homogeneous magnet, \( \Delta z \) was a displacement in a
direction parallel to the magnet profile. In the \( r^{-1} \) spectrometer, a dis­
placement along the profile is called \( \Delta \eta \), and on the en­
trance side it is
taken as positive in the direction of decreasing \( \Psi_S \) (increasing \( \psi_S \)). The
median trajectory then enters the interior field at an \( \eta \)-value correspond­
ing to a smaller \( \Psi_S \)-value than that involved for the off-median trajectory,
i.e., at a greater \( \eta \)-value. On the exit side, positive \( \Delta \eta \) is in the direction
of decreasing \( \Psi_f \), and the median trajectory tends to leave at a greater
\( \eta \)-value.

Shift in Entry Angle (\( \Delta \psi \)) in \( r^{-1} \)-Field. The above result on the
equality of \( \Delta \mu \) for trajectories on median and off-median planes is only
approximately true for the \( r^{-1} \)-field. The result was found in the case of
a two-dimensional field, and depended upon the fact that the fringing field
did not vary with \( z \). Due to the \( r^{-1} \)-variation in the spectrometer, however,
the intensity of the fringing field does vary with \( \eta \), so that \( \Delta \mu \) is not the
same for all values of \( \phi_S \).

The difference in \( \Delta \mu \) (i.e., \( \Delta \psi \)) is small, so that the deviation
from equality can be treated, approximately, as a perturbation. Since \( \Delta \psi \)
[see Eq. (11.1a)] is proportional to \( H_y \) (for small \( \beta \)), the relative variation
in \( \Delta \mu \) may be taken as equal to the relative variation in the intensities of
the fringing magnetic field occurring at the median trajectory and the off-
median trajectory.

---

*Variation of \( \rho_C \) essentially varies the relative importance of the
interior field and fringing field deflection. Large \( \rho_C \) corresponds to
weak interior field and hence weak fringing field. An alternative
description is that large \( \rho_C \) corresponds to a long interior trajectory,
so that the fraction of the path occupied by fringing field is small.
The first case considered is an idealized situation analogous to that discussed in Eq. (11.4). Thus, a median ray and an off-median ray of the same \( \psi_s \) approach the magnetic profile at perpendicular incidence (\( \mu_0 = 0 \); for simplicity, the range is \( 90^\circ < \psi_s < 180^\circ \), or, using the notation of Eq. (9.7), \( 0 < \psi_s < 90^\circ \). As before, at first the change in the angle of incidence due to the accumulated \( \Delta \psi \)-deflection is neglected. It is assumed at this point that the fringe lines are not perpendicular to the profile, but follow the cone of constant \( \psi_s \), as in Fig. (11.3). With these two assumptions, both the median and off-median rays cut the same lines of force at the same angle to the profile, and hence, as above, the \( \Delta \psi \)-values are the same.

Figure (11.3) shows a bundle of rays of constant \( \psi_s \) as they approach the magnetic profile, only a short segment of which is shown. A single line of force, MQN, of the fringing field is presented. As illustrated, this is considered to be an ordinary line emerging perpendicularly to the poleface, but bent upwards (towards increasing \( r \)) as it swings around, so that it lies on the constant-\( \psi_s \) cone. Using the coordinate system of Section #1 (i.e., \( z,r,\phi \)), it is evident that these lines of force merge naturally with the interior lines which follow circles of constant \( r \), at any particular \( z \)-value. As the field emerges from the interior, the lines must change to curves which leave the poleface perpendicularly.

The argument may also be carried through when the rays are not incident perpendicularly to the profile, i.e., when \( \mu_0 \neq 0 \). In this case, the lines of force lie on cones perpendicular to the profile, as in Fig. (11.3), but these cones are not constant-\( \psi_s \) surfaces. The trajectories cut through these cones, but, if the projection of these orbits onto constant-\( \phi \) planes involve the same curve \( z = z(r) \) for median and off-median trajectories, then both rays pass through the corresponding lines of force. Hence, \( \Delta \mu \) is again the same for both types of trajectories.

This argument, however, is only approximately correct, since there are two perturbation effects:

(1) Lines of force tend to be perpendicular to the profile rather than following the cones described above. Hence, the median trajectory fringing field intensity becomes smaller than that of an off-median trajectory. The resulting difference in the entry angle is called \( \delta_1 \).
If \( \frac{dr}{d\eta} \) along the profile is negative, as it is for large \( \psi_s \) (small \( \Psi_g \)), then the \( \Delta \eta \)-effect drives the median trajectory towards smaller \( r \)-values than occur for the non-median trajectory, and hence causes a relative increase in its fringing-field intensity. The effect is reversed when \( \frac{dr}{d\eta} \) is positive, as it is for small \( \psi_s \). The resulting difference in the entry angle is called \( \delta \psi_2 \).

Computation of \( \delta \psi_1 \). We consider Case (I) first. The lines of force tend to lie in a plane perpendicular to the profile, with some modification due to inhomogeneity of the field within the gap.* This modification, however, was not evaluated quantitatively and is neglected in the estimate below.

A crude estimate as to the magnitude of the change in \( \Delta \psi \) between median and off-median rays is based on the following assumptions:

(i) The fringe lines are considered to lie on a plane perpendicular to the profile, bowing due to the unsymmetrical repulsion effect being neglected. This will result in an overestimate of \( (\Delta \psi_{OM} - \Delta \psi_M) \), because the intensity of the fringing field through which the median ray passes is underestimated.

(ii) The incident trajectories are perpendicular to the profile. This simplifies the calculation, but, as indicated below, is not essential.

(iii) The intensity of the fringing magnetic field at a particular point is determined by the intensity of the field in the part of the gap from which the corresponding fringe lines start. Thus, Fig. (11.4a) shows the cone of rays of constant \( \Psi_g \) as emitted from a point source.** The projection of this system onto the farthest off-median plane is shown in Fig. (11.4b), whereas the projection onto the median plane is shown in Fig. (11.4c). The line ST is a short segment of the profile approximately perpendicular to the bundle of rays. The farthest off-median ray passes through S, while the median ray passes through T'.

* The modification arises because, within the gap, the field intensity, and hence the density of lines of force, decreases as \( r \) increases; the same decrease in density operates for the fringing field as well. The fringe lines are prevented from lying in a plane perpendicular to the profile by the "repulsion" of lines of force. In a symmetrical system, such as the idealized uniform field magnet discussed above, the intensity of the fringing field is constant for a given \( (x,y) \), so that the repulsion is balanced on both sides of a line of force, and, hence, fringe lines lie in a plane perpendicular to the profile. In the \( r^{-1} \)-field, on the other hand, the gradient of the magnetic field leads to unequal repulsions from the two sides, causing a bowing of fringe lines towards increasing \( r \). Thus, this bowing effect tends to make the fringe lines follow the cones to some extent.

** The planes TT'P' and SS'P are perpendicular to the z-axis (MPP'), and are separated by the distance PP'. The line TS is a short segment of the profile; the line TSM is the extension to the z-axis, but does not represent the profile, which actually curves away from this line. Note that \( \angle OSM = \frac{1}{2} \pi \) and \( \angle SMP = \left( \frac{1}{2} \pi - \Psi_g \right) \).
The fringe lines affecting the farthest off-median ray lie in the plane perpendicular to the profile at S, i.e., the plane SOS'. The fringing lines affecting the median ray lie in the plane approximately parallel to SOS' and passing through TT'. The fringing magnetic fields affecting the two rays then correspond in relative intensities to the interior fields appropriate for the points S and T, respectively.

(iv) The structure of the fringing field at neighboring r-values is considered to be the same, differing only in the intensity resulting from the effect assumed in (iii). Hence, trajectories passing through two sets of fringe lines originating at different r-values will have \( \Delta \psi \)-values differing only insofar as the in-gap field differs at the two r-values.

Since the interior field varies with r, the difference in the r-values for S and T, i.e., \( \Delta r = TQ \) [see Fig. (11.4b)], is calculated from the geometric properties of the system, such as the angular opening of the gap (\( \varphi \)), the angle of emission of the rays (\( \psi_s \)), the angle between the profile and the incident rays (\( \varphi \), in this example).

Thus, \( \angle US'T' = \angle STQ \equiv \psi_s \), and from Figs. (11.4b) and (11.4c), \( \Delta r = TQ = TS \cos \psi_s \) and \( T'S' = US' \cos \psi_s \). From Fig. (11.4e), we have \( T'S' = TS \cos \frac{1}{2} \varphi' \), and from Fig. (11.4f)

\[
\cos \frac{1}{2} \varphi' = \frac{\cos \frac{1}{2} \varphi}{\sqrt{1 - \sin^2 \psi_s \sin^2 \frac{1}{2} \varphi}} \tag{11.9}
\]

From Fig. (11.4d), \( US' = r (1 - \cos \frac{1}{2} \varphi) \), where \( r = UP = SP \).

Then

\[
\frac{\Delta r}{r} = \frac{(1 - \cos \frac{1}{2} \varphi) \cos^2 \psi_s \sqrt{1 - \sin^2 \psi_s \sin^2 \frac{1}{2} \varphi}}{\cos \frac{1}{2} \varphi} = \frac{\varphi^2}{8} \cos^2 \psi_s \left\{ 1 + \frac{\varphi^2}{8} \left[ \frac{5}{6} - \sin^2 \psi_s \right] \right\} \approx \frac{\varphi^2}{8} \cos^2 \psi_s . \tag{11.10a}
\]
Since
\[
\left( \frac{dH}{dr} \right)_{r=r_0} = -\frac{H(r_0)}{r_0}
\]
we have
\[
H(T) \cong H(S) + \frac{dH}{dr} \Delta r = H(S) \left[ 1 - \frac{\Delta r}{r} \right]
\]
(11.10b)
or
\[
H(T) \cong H(S) \left[ 1 - \frac{\varphi^2}{8} \cos^2 \psi_S \right]
\]
(11.10c)

From assumption (iv) above,
\[
\frac{\Delta \gamma_{OM} - \Delta \gamma_M}{\Delta \gamma_{OM}} \cong \frac{H(S) - H(T)}{H(S)} = \frac{\varphi^2}{8} \cos^2 \psi_S
\]
(11.11a)

For \( \varphi = 10^\circ \) and \( \psi_S = 135^\circ \), this ratio is approximately 0.001; for \( \psi_S = 100^\circ \), the ratio is 0.00006.
Then, if \( \delta \psi_1 \) is the difference in the angle \( \psi \) on arrival of the two trajectories into the interior field, we have

\[
\delta \psi_1 = \Delta \psi_{OM} - \Delta \psi_M = \frac{\phi^2}{8} \cos^2 \psi_s \Delta \psi \quad .
\]

(11.11b)

The same result applies when the off-median trajectory involved is not the farthest off-median one. In this case, the quantity \( \phi_{OM} - \phi_M \) is substituted for \( \frac{1}{2} \phi \). Thus, taking \( \phi_M = 0 \), as in Section #9,

\[
\phi_{OM} - \phi_M = \phi_s \quad \text{or} \quad \phi_{OM} - \phi_M = \phi_f \quad .
\]

(11.11c)

Then

\[
\delta \psi_1 = \Delta \psi_{OM} - \Delta \psi_M = \frac{1}{2} \phi_s^2 \cos^2 \psi_s \Delta \psi
\]

(11.11d)

at the entrance, while \( \phi_f \) is substituted at the exit. It should be noted that the \( \Delta \psi \)-value referred to is the \( \Delta \psi_e \) of Eq. (8.7).

Under these assumptions, the off-median ray traverses a more intense fringing field, and thus suffers a slightly larger \( \Delta \psi \)-value. The greatest effect occurs for the largest values of \( \psi_s \).

When the incident rays are not perpendicular to the profile, the magnitude of \( \Delta \psi \) is greater, but if \( \tan \mu \) is small, the relative difference in Eq. (11.12) should remain roughly the same. Although the source points (in the gap) for the fringing field lines change as the trajectories pass through the fringing field, the relative intensities of the fields at the source points remain roughly the same for the median and off-median rays.

The computation for this case is similar: \( \Delta r = TS \cos(\psi_s - \mu) \) and \( TS' = US' (\cos \psi_s / \cos \mu) \). Then

\[
\frac{\Delta r}{r} = \frac{(1 - \cos \frac{1}{2} \phi) \cos(\psi_s - \mu) \cos \psi_s \sqrt{1 - \sin^2 \psi_s \sin^2 \frac{1}{2} \phi}}{\cos \frac{1}{2} \phi \cos \mu}
\]

(11.12a)

\[
\delta \psi_1 = \frac{\phi^2}{8} (\cos^2 \psi_s + \cos \psi_s \sin \psi_s \tan \mu) \quad \Delta \psi \quad ,
\]

(11.12b)

and for any off-median trajectory

\[
\delta \psi_1 = \frac{1}{2} \phi_s^2 (\cos^2 \psi_s + \cos \psi_s \sin \psi_s \tan \mu) \Delta \psi \quad .
\]

(11.12c)
As mentioned above, $\delta \psi_1$ is a rough approximation, in the nature of an upper limit; the formula is an overestimate, since the relative fringing-field intensity for the median ray was underestimated when the bowing of the fringe field was neglected.

**Computation of $\delta \psi_2$.** We now consider the effect described in Case (II). The median trajectory comes into the interior field at a lower $r$-value than the off-median trajectory when $dr/d\eta < 0$, but a higher $r$-value when $dr/d\eta > 0$. A difference in the corresponding fringing field intensities arises from the difference in $r$-value:

$$\Delta r = r_M - r_{OM} \quad (11.13)$$

which is computed from the corresponding value of $\Delta \eta$. This is taken to have the same form as that of $|\Delta z|$, computed for the homogeneous field [see Eq. (11.8)]. Thus from Eq. (11.8) and Eq. (1.10),

$$\eta_M \cdot \eta_{OM} = \Delta \eta = \frac{1}{8} \frac{(r \phi)^2}{p \lambda} = \frac{1}{8} \frac{r^2 \phi^2}{r K} = \frac{1}{8} \frac{r \phi^2}{K} \quad (11.14a)$$

since $|h^*| = r(\phi^2)$ for the farthest off-median plane. Taking $\mu$ to be fairly small the variation with $\mu$ is neglected. For any plane other than the farthest off-median one, we have

$$\Delta \eta = (1/2K) r \phi_s^2 \quad (11.14b)$$

where $\phi_s$ is defined in Eq. (11.11c) and $\phi_f$ is substituted at the exit. Since the median ray displacement is greater, $\Delta \eta$, as defined in Eq. (11.14a), is positive.

The difference corresponding to $\Delta \eta$ is zero before the median and off-median trajectories enter the fringing field, but increases as the trajectories pass through the fringe field, and finally takes the value in Eq. (11.14). Since the effective difference in the $r$-values over the entire trajectory changes from zero to that computed from $\Delta \eta$ within the gap, an average value of $\Delta r$ is taken over the entire path, so that roughly

$$\Delta r_{\text{effective}} = \frac{1}{2} \Delta r \quad (11.15)$$

We see from Fig. (11.5) (for sign convention of $\mu$, see Section #8),

$$\frac{\Delta r}{\Delta \eta} = - \cos(\psi_s + \mu) = \cos(\psi_s - \mu) \quad (11.16)$$
Then
\[ \frac{\Delta r_{\text{eff}}}{r} = \frac{1}{4K} \phi_s^2 \cos(\psi_s - \mu) \quad . \quad (11.17) \]

The sign of \( \Delta r_{\text{eff}} \) depends on whether \( (\psi_s - \mu) < \frac{1}{2} \pi \) (then \( \Delta r_{\text{eff}} \) is positive) or \( (\psi_s - \mu) > \frac{1}{2} \pi \). [In Fig. (11.5), the angle is greater than \( \frac{1}{2} \pi \), so \( \Delta r_{\text{eff}} \) is less than zero.] Then, as in Eq. (11.11a),

\[ H(M) = H(OM) + \frac{dH}{dr} \Delta r_{\text{eff}} = H(OM) \left[ 1 - \frac{\Delta r_{\text{eff}}}{r} \right] \]
\[ = H(OM) \left[ 1 - \frac{1}{4K} \phi_s^2 \cos(\psi_s - \mu) \right] \quad , \quad (11.18) \]

and, as in Eq. (11.12),

\[ \delta \psi_2 = \Delta \psi_{\text{OM}} - \Delta \psi_M = \frac{1}{4K} \phi_s^2 \cos(\psi_s - \mu) \Delta \psi \quad . \quad (11.19) \]

For the commonest type orbits, i.e., with small \( \mu \) and \( \psi_s > \frac{1}{2} \pi \), it is found that \( \delta \psi_2 \) is negative, whereas \( \delta \psi_1 \) [see Eq. (11.12b)] is always positive. As before, the \( \Delta \psi \)-value referred to is \( \Delta \psi_0 \).

**Effect of \( \Delta \eta \) on the Focusing.** Given the differences \( \delta \psi_1, \delta \psi_2 \) and \( \Delta \eta \) at a given \( \psi_s \), it is necessary to evaluate their effect on the focusing property. We start by considering only the effect of \( \Delta \eta \), assuming for the moment that \( \delta \psi = 0 \).

To determine the magnitude of the effect, only an approximate calculation will be made. We consider a spectrometer in which the profile has been adjusted so that a median ray from \((z_s,0)\) focuses at \((z_f,0)\). Thus, in Fig. (11.6a), the median ray is bent by the fringing field and arrives one gap width inside the profile at the point \( P \), at which point the magnetic field follows the form in Eq. (1.1). The off-median ray, after passing through the fringing field, arrives one gap width inside the profile at \( N \), displaced (in a direction parallel to the profile) from the median ray by the distance \( NP = \Delta \eta \). Because \( \delta \psi = 0 \), the two rays are parallel at \( N \) and \( P \), i.e., their \( \psi \)-values are equal.
To simplify the calculation, we consider it equivalent to a situation in which no fringing field exists, but in which two parallel rays arrive at the profile, separated by the distance $N_0 P_0 = \Delta \eta$ along the profile, as in Fig. (11.6b).* In effect, a source point has been axially displaced by the distance $\delta z_g$; this type of perturbation has been treated in Section #5.

From Fig. (11.6c) it is evident that

$$\delta z_g \sin \Psi_g = \delta z_g \sin \Psi_g = \Delta \eta \cos \mu \quad .$$

Then

$$\delta z_g = \Delta \eta \frac{\cos \mu}{\sin \Psi_g} \quad .$$

(11.20)

![Fig. 11.6b](image1)

![Fig. 11.6c](image2)

If we consider only a symmetrical spectrometer, then, from Eq. (5.10), since $\Psi_f = -\Psi_g$ and $\Psi_f = -1$, if there is no fringing field at the exit,

$$\delta z_f = \delta z_g \quad ,$$

(11.21)

both being positive.

Since, however, an exit fringing field does exist, its effect must be included; to do so, it is necessary to evaluate the position and angle of the off-median trajectory just before it hits the exit fringing field, by the method of Section #5.

*Although $\Psi_g > \frac{1}{2} \pi$ in the figure, $\delta z_g > 0$ for $\Psi_g < \frac{1}{2} \pi$ as well, because $\Delta \eta$ is always positive, the median ray having a greater displacement than the off-median one.
In Fig. (11.7a), we consider three rays:

(I) Median ray, in the approximation discussed above, emitted from \((z_s,0)\).

(II) Off-median ray, in this approximation emitted from \((z_s+\delta z_s,0)\).

(III) A pseudo-ray, "emitted" from \((z_s,0)\) and "entering" the profile at the same "point" as (II). It does not exist in the actual spectrometer, and is considered in the approximate model being used, because the theory of Section #5 can be applied only to a pair like (II) and (III). The relation to (I) is easily carried through, because (I) and (III) are both subject to the focusing and symmetry conditions in a symmetric spectrometer (see Section #3).

Rays (I) and (II) are emitted at the angle \(\psi_s\) while (III) is "emitted" at the angle \(\psi_p\), where \(\psi_s = \psi_p + \delta \psi_p\), or \(\psi_p = \psi_s - \delta \psi_s\), with \(\delta \psi_s = -5\psi_s/\psi_p\). Rays (I) and (III), having been "emitted" from \((z_s,0)\) are focused into \((z_f,0)\), and since the profiles are symmetrical, we have equal distances along the profiles: \(N_0P_0 = N_1P_1 = \Delta \gamma\). Correspondingly, the exit angles are symmetrical, so \(\psi_f = 2\pi - \psi_s = -\psi_s\) and \(-\psi_{pf} = \psi_p = \pi - \psi_{pf}\) (using the notation of Eq. (9.7)), with \(\psi_f = \psi_{pf} - \delta \psi_p\). The goal is to evaluate \(N_1N_2\) and \(\delta \psi_{pf}\). For the moment, the exit fringing field effect is neglected, and the rays and angles on the exit side of Fig. (11.7b) are treated as though they were completely determined by the intersection of the trajectories with the exit profile, as in Section #5.

Using the symmetry condition, we have from Eqs. (3.1a) and (5.7b):

\[
\Lambda = \frac{r_0 - Kz_s \sin^3 \psi_p}{z_s \sin^2 \psi_p},
\]
Then, from Eq. (5.5b),

$$
\delta \psi _{pf} = \left[ 1 - \frac{2Kz_s}{r_0} \sin^3 \psi _{p} \right] \delta \psi _{p} = B_1 \delta \psi _{p} \quad .
$$

(11.22a)

From Eq. (5.9),

$$
\delta \psi _{pf} - \delta \theta = \delta \psi _{p}
$$

and

$$
\delta \theta = - \frac{2Kz_s \sin^3 \psi _{p}}{r_0} \delta \psi _{p} = C_1 \delta \psi _{p} \quad ,
$$

(11.22b)

where the notation from Section #5 has been changed: $\psi _f$ to $\psi _{pf}$ and $\psi _s$ to $\psi _p$.

Since $\delta z_s > 0$, then $\delta \psi _p > 0$; hence, $\delta \theta < 0$ (since $C_1 < 0$), and $N_2$ always lies to the right of $N_1$. From Fig. (11.7b), since $\delta \theta < 0$ and $N_1N_2 > 0$,

$$
N_1N_2 \cos \mu \neq -\rho \delta \theta
$$

and

$$
r_0 = - \rho \sin \psi _{pf} = \rho \sin \psi _{pf} = \rho \sin \psi _p \quad .
$$

Then,

$$
N_1N_2 = \frac{-r_0}{\sin \psi _{pf} \cos \mu} \delta \theta \quad .
$$

(11.23)

Similarly, from Fig. (11.7c),

$$
\delta \psi _p = \frac{\Delta \eta \cos \mu \sin \psi _p}{r_0} \quad (11.24)
$$

Then, from Eqs. (11.22), (11.23) and (11.24), and taking $r_e \equiv r_0$,

$$
N_1N_2 = -C_1 \Delta \eta = \frac{2Kz_s \sin^3 \psi _p}{r_0} \Delta \eta \quad .
$$

(11.25)
Since [see Fig. 11.7a)] \(N_1P_1 = \Delta \eta\) (from symmetry), then the distance (along the profile) between the off-median ray (II) and the median ray (I) is 
\[
N_1P_1 - N_1N_2 = N_2P_1 = \Delta \eta + C_1 \Delta \eta = B_1 \Delta \eta = \left[1 - \frac{2Kz_s \sin^3 \psi_p}{r_0}\right] \Delta \eta \quad (11.26)
\]
Here \(N_2P_1\) is positive if the median ray (I) leaves the profile to the right of the off-median ray (II), whereas \(N_2P_1\) is negative when \(P_1\) is to the left of \(N_2\).

Superimposed upon these consequences of the entrance fringing-field separation of median and off-median rays is the separation caused by the exit fringing field. The latter results in a movement of the median ray \((P_1)\) further to the right of the off-median ray \((N_2)\) by the amount \(\Delta \eta'\). The net separation along the profile is then 
\[
\Delta \tau = N_2P_1 + \Delta \eta' \quad (11.27)
\]
If \(\phi_f\) for the off-median ray is the same as its \(\phi_s\)-value, then \(\Delta \eta' = \Delta \eta\). However, since there is either a converging or diverging lens at the entrance, the value of \(\phi_f^2\) is respectively smaller or larger than \(\phi_s^2\). From Eq. (11.14b),
\[
\Delta \eta' = \left(\frac{\phi_f^2}{\phi_s^2}\right) \Delta \eta = C_2 \Delta \eta \quad , \quad (11.28)
\]
where \(C_2 < 1\) for a moderately converging input lens and \(C_2 > 1\) for a diverging input lens.

Then,
\[
\Delta \tau = (B_1+C_2) \Delta \eta = C_3 \Delta \eta \quad , \quad (11.29)
\]
Except for large values of \(K\) with \(\psi_s\) close to 90°, in general, \(C_3\) is positive, i.e., including the exit fringing-field effect, we find that the median ray usually lies to the right of the off-median ray along the profile.

Now, we evaluate the angle of emergence of the off-median ray. The relation between \(\delta \psi_p\) and \(\delta \psi_p\) is given in Eq. (11.22a), in which \(B_1\) may be positive or negative. Since \(C_2 > 0\), then \(B_1 < 0\) when \(C_3 < 0\); however, positive \(C_3\) may correspond to positive or negative \(B_1\). Since we are treating the case where \((\Delta \psi_M - \Delta \psi_{OM}) = 0\), the exit fringing field does not change the relative orientation of the median and off-median rays. Thus, the relations between the angles are as shown in Figs. (11.7a) and (11.8a). Figure (11.8a) shows the various rays arriving at (or near) the z-axis if no exit fringing field were present. If the median ray \((P_1O_2)\) has the \(\psi\)-value \(\psi_M\) and the off-median ray \((N_2O_3)\) has the value \(\psi_{OM}\), then, since 
\[
\psi_{OM} = \psi_{pf} + \delta \psi_{pf} = \psi_{pf} + B_1 \delta \psi_p
\]
and
\[ \psi_M = \psi_f = \psi_{pf} - \delta \psi_p, \]
it follows that
\[ \psi_{OM} = \psi_M + (1 + B_1) \delta \psi_p = \psi_M + \delta \psi_0, \tag{11.30a} \]
where
\[ \delta \psi_0 = (1 + B_1) \delta \psi_p. \tag{11.30b} \]

Since the angular relations do not change after the rays pass through the exit fringing field, Eq. (11.30) is still applicable. Further, \((1 + B_1)\) and \(C_3\) may be separately positive or negative. Figure (11.8b) represents the case in which \(C_3 > 0\) and \((1 + B_1) > 0\) (i.e., \(\Delta \tau\) and \(\delta \psi_0\) are positive); in Fig. (11.8c), however, \(C_3 < 0\) and \((1 + B_1) > 0\) (i.e., \(\Delta \tau\) negative and \(\delta \psi_0\) positive).

In Fig. (11.8b), since \(\delta z_f < 0\),
\[ \Delta \tau \cos \mu \geq O_2 A \sin \psi_f \]
\[ \rho \delta \psi_0 \leq O_3 A \sin \psi_f \]
\[ |\delta z_f| = O_2 A - O_3 A = \frac{\Delta \tau \cos \mu - \rho \delta \psi_0}{\sin \psi_f}. \]
Then, from Eqs. (11.14), (11.31) and (11.32),
\[ \delta z_f = \frac{\rho \delta \psi_0 - \Delta \tau \cos \mu}{\sin \psi_s} \quad (11.31) \]
the relation applying for all values of $(1+B_1)$ and $C_3$.

From Eqs. (11.24), (11.29) and (11.30b) and the relations
\[ \sin \psi_p \simeq \sin \psi_s \quad \text{and} \quad \rho \simeq \rho \sin \psi_s, \]
\[ \frac{\rho \delta \psi_0}{\sin \psi_s} \simeq (1+B_1) \frac{\rho}{r_e} \cos \mu \Delta \eta \quad ; \]
\[ \frac{\Delta \tau \cos \mu}{\sin \psi_s} \simeq (B_1+C_2) \frac{\rho}{r_e} \cos \mu \Delta \eta. \quad (11.32) \]

Fig. 11.8c

Then, from Eqs. (11.14), (11.31) and (11.32),
\[ \delta z_f = \frac{\rho}{2K} (1-C_2) \rho \cos \mu \phi_s^2 \quad (11.33) \]

It may be noted that:

(i) Although $\psi_s$ does not appear explicitly in Eq. (11.33), $\delta z_f$ is
implicitly a function of $\psi_s$, moderately through the variation of $(\rho, \mu)$ with
$\psi_s$, but more markedly through the variation of $(1-C_2)$.

(ii) If the entrance lens is neither divergent nor convergent, then
$C_2 = 1$, and $\delta z_f = 0$. For only slight divergence or convergence of the input
lens, $C_2$ differs only moderately from unity, so $\delta z_f$ is relatively small.
For $C_2 = 1$, $O_2$ and $O_3$ in Fig. (11.8b) coincide, the difference in exit
points $(N_3, P_2)$ being compensated by the difference $\delta \psi_0$ in exit angle.
Figure (11.8c) is not possible when $C_2 = 1$, since $(B_1+C_2) = (1+B_1)$ and $\Delta \tau$
and $\delta \psi_0$ must have the same sign. Numerical values for typical situations
will be given below.

Effect of $\delta \psi_1$, $\delta \psi_2$ and $\Delta \eta$ on the Focusing. We now consider the
more realistic situation in which $\delta \psi_1$ and $\delta \psi_2$ [see Eq. (11.12b) and
(11.19)] are included. The analysis is similar to that following Fig. (11.7a)
except that ray (II) is not parallel to (I). The off-median ray (II) is
"emitted" at the angle
\[ \psi_{s1} = \psi_p + \delta \psi_{p1} \quad \text{with} \quad \delta \psi_{p1} = \delta \psi_p + \delta \psi_1 + \delta \psi_2 \quad ; \quad (11.34a) \]
where $\delta \psi_p$ is shown in Fig. (11.7a), and

$$\delta \psi_1 = \Delta \psi_{OM} - \Delta \psi_M = \frac{1}{2} \phi_s^2 (\cos^2 \psi_s \cos \psi_s \sin \psi_s \tan \mu) \Delta \psi \quad (11.34b)$$

$$\delta \psi_2 = \Delta \psi_{OM} - \Delta \psi_M = \frac{1}{4K} \phi_s^2 \cos (\psi_s - \mu) \Delta \psi \quad . \quad (11.34c)$$

Equation (11.22) becomes

$$\delta \psi_{pf} = B_1 \delta \psi_{p1}; \quad \delta \theta = C_1 \delta \psi_{p1} \quad , \quad (11.35)$$

whereas Eq. (11.23), not involving $\delta \psi_p$, remains unchanged. Equation (11.24) still describes $\delta \psi_p$, but this angle is now only part of the variation [see Eq. (11.34a)]. Substituting Eqs. (11.24), (11.34a) and (11.35) into (11.23),

$$N_1N_2 = - C_1 \Delta \eta - \Gamma (\delta \psi_1 + \delta \psi_2) \quad (11.36a)$$

and

$$N_{2P_1} = B_1 \Delta \eta + \Gamma (\delta \psi_1 + \delta \psi_2) \quad , \quad (11.36b)$$

with

$$\Gamma = \frac{r_0 C_1}{\sin \psi_p \cos \mu} - \frac{r_0 C_1}{\sin \psi_s \cos \mu} = \frac{\rho C_1}{\cos \mu} \quad , \quad (11.36c)$$

while

$$\Delta \tau = (C_2 + B_1) \Delta \eta + \Gamma (\delta \psi_1 + \delta \psi_2) = C_3 \Delta \eta + \Gamma (\delta \psi_1 + \delta \psi_2) \quad . \quad (11.36d)$$

Using Fig. (11.8a) as before, the emergence angles are, omitting the effect of the exit fringing field,

$$\psi_{OM} = \psi_{pf} + \delta \psi_{pf} = \psi_{pf} + B_1 \delta \psi_{p1} = \psi_{pf} + B_1 \delta \psi_p + B_1 (\delta \psi_1 + \delta \psi_2)$$

and

$$\psi_M = \psi_{f} = \psi_{pf} - \delta \psi_p \quad .$$

Then,

$$\psi_{OM} = \psi_M + \delta \psi_{o1} \quad , \quad (11.37a)$$

with

$$\delta \psi_{o1} = \delta \psi_0 + B_1 (\delta \psi_1 + \delta \psi_2) = (B_1 + 1) \delta \psi_p + B_1 (\delta \psi_1 + \delta \psi_2) \quad . \quad (11.37b)$$
Table 11.1

FRINGING-FIELD EFFECTS ON RESOLUTION IN SYMMETRICAL SPECTROMETER WITH $K = 0.6$

<table>
<thead>
<tr>
<th>$\psi_S$</th>
<th>$\varphi$</th>
<th>$\alpha_1$</th>
<th>$\alpha_2$</th>
<th>$\alpha_3$</th>
<th>$\frac{(\delta z_f)_1}{z_s}$</th>
<th>$\frac{(\delta z_f)_2}{z_s}$</th>
<th>$\frac{(\delta z_f)_3}{z_s}$</th>
<th>$G$</th>
<th>$\frac{\delta p_1}{p}$</th>
<th>$\frac{\delta p_2}{p}$</th>
<th>$\frac{\delta p_3}{p}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>110°</td>
<td>3°</td>
<td>0.0606</td>
<td>0.0268</td>
<td>-0.104</td>
<td>0.00017</td>
<td>0.0000038</td>
<td>-0.000015</td>
<td>0.1572</td>
<td>-0.00053</td>
<td>-0.000012</td>
<td>0.000047</td>
</tr>
<tr>
<td></td>
<td>10°</td>
<td></td>
<td></td>
<td></td>
<td>0.0018</td>
<td>0.00014</td>
<td>-0.00055</td>
<td></td>
<td>-0.0059</td>
<td>-0.00045</td>
<td>0.0018</td>
</tr>
<tr>
<td>130°</td>
<td>3°</td>
<td>-0.0322</td>
<td>0.1748</td>
<td>-0.197</td>
<td>-0.000088</td>
<td>0.000025</td>
<td>-0.000028</td>
<td>0.1211</td>
<td>0.00036</td>
<td>-0.00010</td>
<td>0.00012</td>
</tr>
<tr>
<td></td>
<td>10°</td>
<td></td>
<td></td>
<td></td>
<td>-0.00098</td>
<td>0.00093</td>
<td>-0.00105</td>
<td></td>
<td>0.0041</td>
<td>-0.0038</td>
<td>0.0043</td>
</tr>
<tr>
<td>150°</td>
<td>3°</td>
<td>-0.0293</td>
<td>0.493</td>
<td>-0.432</td>
<td>-0.000080</td>
<td>0.000071</td>
<td>-0.000062</td>
<td>0.1062</td>
<td>0.00038</td>
<td>-0.00033</td>
<td>0.00029</td>
</tr>
<tr>
<td></td>
<td>10°</td>
<td></td>
<td></td>
<td></td>
<td>-0.00089</td>
<td>0.0026</td>
<td>-0.0023</td>
<td></td>
<td>0.0042</td>
<td>-0.012</td>
<td>0.011</td>
</tr>
</tbody>
</table>

$\varphi = \text{gap angle.}$

$\frac{(\delta z_f)_i}{z_s} = \text{relative shift of focal point between median and farthest off-median rays, due to one of the effects discussed.}$

$\frac{\delta p_i}{p} = \text{corresponding shift in apparent momentum.}$

$G = (\text{dispersion}) \times \frac{1}{2z_s}$
With this difference must be included the effects of the exit fringing field. The median and off-median rays are further separated angularly by the effects discussed in Eqs. (11.12c) and (11.19) [or the equivalent Eqs. (11.34b) and (11.34c)]. The sign of the separation is the same as at the input side, but the magnitude differs because the lens effect changes the separation of the $\phi$-planes. If $\delta \psi_1$ and $\delta \psi_2$ are the differences at the entrance, then, at the exit the corresponding differences are $C_2 \delta \psi_1$ and $C_2 \delta \psi_2$, where $C_2$ is defined in Eq. (11.28). Then,

$$\psi_{OM} = \psi_M + \delta \psi_{02} \quad (11.38a)$$

$$\delta \psi_{02} = \delta \psi_{01} + C_2 (\delta \psi_1 + \delta \psi_2) = (B_1+1) \delta \psi_p + C_3 (\delta \psi_1 + \delta \psi_2) \quad (11.38b)$$

Equation (11.31) is still valid, with $\delta \psi_{02}$ substituted for $\delta \psi_0$ and with $\Delta \tau$ given by Eq. (11.36d).

Thus, from Eqs. (11.24) and (11.38b) and from (11.36c) and (11.36d), since $r = \rho \sin \psi_s$,

$$\frac{\rho \, \delta \psi_{02}}{\sin \psi_s} = (B_1+1) \frac{\rho}{r} \cos \mu \Delta \eta + \frac{C_3 \rho^2}{r} (\delta \psi_1 + \delta \psi_2)$$

and

$$\frac{\Delta \tau \cos \mu}{\sin \psi_s} = C_3 \frac{\rho}{r} \cos \mu \Delta \eta + \Gamma \frac{\rho}{r} \cos \mu (\delta \psi_1 + \delta \psi_2)$$

$$= C_3 \frac{\rho}{r} \cos \mu \Delta \eta + C_1 \frac{\rho^2}{r} (\delta \psi_1 + \delta \psi_2) \quad .$$

Then,

$$\delta z_f = (1-C_2) \frac{\rho}{r} \cos \mu \Delta \eta + (1+C_2) (\delta \psi_1 + \delta \psi_2) \frac{\rho^2}{r} \quad ,$$

or

$$\delta z_f = \frac{1}{K} \rho \phi_s^2 \left\{ (1-C_2) \cos \mu + \frac{(1+C_2)}{\sin \psi_s} \left[ K \, T(\psi_s, \mu) + \frac{1}{2} \cos \psi_s \sin \psi_s \tan \mu \right] \Delta \psi \right\} \quad ,$$

(11.39)

with

$$T(\psi_s, \mu) = \cos^2 \psi_s + \cos \psi_s \sin \psi_s \tan \mu \quad .$$

It may be restated that this computation is a rough one, whose aim is simply to evaluate the magnitude of the perturbation of the resolution.
In addition to the previously mentioned approximations, such effects as
the shift in the r-value of the trajectory between the input and exit points* have been completely neglected, as well as the difference between $\Delta \eta$
one gap width within the profile and the corresponding projection onto theprofile. However, because the effect computed is small, these approxima­tions lead only to second-order corrections which are negligible.

**Effect of $\delta \psi_1$, $\delta \psi_2$ and $\Delta \eta$ on the Resolution.** To translate this re­sult into a resolution effect, we use the dispersion. Thus, from Eq. (6.6), we see that for a symmetric spectrometer

$$\frac{\delta p}{p} = -\frac{1}{G} \frac{\delta z_f}{2z_s}, \quad (11.40)$$

where the form of $G(K, \psi_s)$ is given in Eq. (6.7), and numerical values inFig. (6.4) and (6.4a) and in Table II.5 (Appendix II).

It is evident that the magnitude of the effect is quite dependent upon the spectrometer design, in particular, upon the values of tan $\mu$, which affect the maximum $\phi_s$ or $\phi_f$-values and the value of $C_2$. Some numerical computations were made for $K = 0.6$, a $K$-value close to the optimum for certain applications (see Section #16), and specifically for a symmetrical spectrometer, for which tabulated values are available in the appendix for $(r_e, W_s, \tan \mu)$.

The assumptions made concerning the fringing-field structure were reasonable ones, but there is no information as to the magnitude of error introduced by these assumptions. Because of this uncertainty, the above computations must be considered as rough ones, almost order-of-magnitude estimates. Summation of the three effects, with possible can­cellation of those of opposite sign, is then not justified. For this reason, they have been computed separately, and are presented in Table (11.1).

We have, then,

(A) From the $\Delta \eta$-effect alone:

$$\frac{\delta z_f}{z_s} = \frac{1}{2K} \frac{\rho}{z_s} \phi_s^2 (1-C_2) \cos \mu = a_1 \phi^2.$$

*For example, it has been assumed that $r_e = r_0$, that $\rho_e = \rho_0$, and that $\Delta \eta$ is symmetric at entrance and exit, so that only the modifying $C_2$ is involved.
(B) From the perpendicularity of lines of force, leading to a lower fringe field for the median rays:

\[
\frac{(\delta z_f)_2}{z_S} = \frac{1}{2} \frac{\rho}{z_S} \, \Phi_S^2 \left( 1 + C_2 \right) \left[ \frac{\cos^2 \psi_S}{\sin \psi_S} - \cos \psi_S \tan \mu \right] \Delta \psi = a_2 \phi^3.
\]

(C) From the \( \Delta \eta \)-effect driving the median plane trajectory to relatively larger \( \eta \)-values, and hence to smaller or greater fringing field, depending upon the sign of \( \frac{d \psi}{d \eta} \):

\[
\frac{(\delta z_f)_3}{z_S} = \frac{1}{4K} \frac{\rho}{z_S} \, \Phi_S^2 \left( 1 + C_2 \right) \frac{\cos (\psi_S - \mu)}{\sin \psi_S} \Delta \psi = a_3 \phi^3,
\]

where [from Eq. (9.12a)]

\[
C_2 = \frac{\phi_f^2}{\phi_S^2} = 1/W_S^2.
\]

The computation is made for the farthest off-median ray, i.e., for the largest possible value of \( \Phi_S \). This is \( \frac{1}{2} \phi \) for a convergent lens and \( \frac{1}{2} \phi/W_S \) for a divergent lens. Taking \( g_\psi/d_\psi \) empirically as approximately 0.6 in Eq. (8.7b),

\[
\Delta \psi = \phi/1.2 \, K,
\]

hence the \( \phi^3 \)-dependence in (B) and (C).

The apparent relative change in momentum is computed from Eq. (11.40), and is shown in the last three columns of the table.

The three \( \psi_S \)-values computed span most of the useful range of a symmetrical spectrometer using this \( K \)-value. Two conclusions may evidently be made from the tabulated results:

(i) For highest resolution, the effects discussed in this section are negligible when \( \phi = 3^\circ \), but must be taken into consideration when \( \phi \) is as large as \( 10^\circ \).

(ii) Even when \( \phi = 10^\circ \), the effect is more serious at larger \( \psi_S \)-values, indicating that improvement may be made in resolution by excluding this part of the \( \psi_S \)-range. This is also the \( \psi_S \)-range in which the \( r_1 \)-projection of the source leads to the greatest loss of resolution.
#12. CORRECTION OF PROFILES FOR THE \( \Delta \psi \)-EFFECT.

As pointed out in Section #8, profiles constructed according to the theory of Section #3 will not, in fact, result in focusing at \((z_f, 0)\), because of increased bending due to the fringing field.

The perturbation of focusing may, indeed, be quite severe unless the gap angle \( \beta \) is small. Lack of detailed information of the fringing-field structure makes it difficult to compute accurately the \( \Delta \psi \)-effect prior to the construction of an instrument. A suitable approach is to design the profiles with the best available estimate of the fringing-field effect (as in Section #15), to construct the instrument, and then to modify the profiles with data based on detailed measurement of the focusing properties. Since the corrections are generally small, they are most easily computed through the use of differential coefficients which give the variation of the trajectories with variation of any of the important parameters, such as \( z_e, r_e, \) and \( K \). Before the profile corrections are considered, therefore, a method is developed for evaluating some of the important differential relations. Some of these have been evaluated previously by a somewhat different technique.

Differential Characteristics of the Trajectories. Suppose all of the parameters of a trajectory are kept fixed, with the exception of two of them (e.g., \( z_f \) and \( K \)). Their differential variation may then be determined with all other parameters kept constant; thus, \( \frac{\partial z_f}{\partial K} \). It is convenient to use the notation of Eq. (9.7); using \( \Psi_s \) and \( \Psi_f \), it is found that the same equations often apply to the entrance and exit points, so that the sub-indices \((s,f)\) may, in these cases, be dropped if desired. The relations of the angles are shown, for \( \Psi_s < 90^\circ \), in Fig. (12.1), in which \( b_s b_s \) is the entrance profile and \( b_f b_f \) the exit profile. Similar angles may be drawn for \( \Psi_s > 90^\circ \).
Variations are considered in the neighborhood of \((z_e, r_e)\). Since only first-order differential variations are involved, the profile and trajectory curves may be replaced by their tangents, and rays which differ by a differential angle, by parallel lines.* The \(\psi\)-focusing effect of the fringing field is neglected, and use is made of the optics and notation of Section #1, 2, and 3.

Consider the trajectory with \(\psi_g = \psi_g(z_e, r_e)\) [i.e., \(\psi_g = \psi_g(z_e, r_e)\)]. Any trajectory with \(\psi_g < \psi_g(z_e, r_e)\) must travel through the magnetic field for some distance, increasing its \(\psi\)-value, before it reaches a point \((z'_e, r'_e)\) at which \(\psi = \psi_g(z_e, r_e)\). The envelope of the points \((z'_e, r'_e)\) is the curve \(\Gamma_g\) [see Fig. (12.2)], which lies within the gap for \(\psi_g > \psi_g(z_e, r_e)\). A similar curve exists on the exit side: for \(\psi_f > \psi_f(z_0, r_0)\), \(\Gamma_f\) is the envelope of all points \((z'_0, r'_0)\) at which the \(\psi\)-value on the trajectory equals \(\psi_f(z_0, r_0)\); \(\Gamma_f\) lies within the gap for \(\psi_f > \psi_f(z_0, r_0)\). Consideration of the \(\Gamma\)-curves is basic to the method to be used here for computing the differential coefficients.

![Fig. 12.2](image)

The various angles involved in the analysis are shown in Fig. (12.2) and defined in Table (12.1).

To determine the position of \(\Gamma_g\), it is necessary to calculate the distance \(ds\) from profile to \(\Gamma_g\), which is the distance along the trajectory in which a ray with initial angle \(\psi_s + d\psi_s\) is bent through the angle \(d\psi = -d\psi_s\), so that its \(\psi\)-angle equals \(\psi_g = \pi - \psi_s\). Now, as a particle travels along its trajectory, \(d\psi > 0\) and \(ds > 0\). If \(\rho_C\) is the instantaneous radius of curvature, then from Eq. (8.4c) and (1.10):

\[
ds = \rho_C \frac{d\psi}{\sin \psi_s} = K r_e d\psi = K \rho_s \sin \psi_s \frac{d\psi}{\psi_s}.
\]  

(12.1a)

*Thus, \(\theta + d\theta\) is replaceable by \(\theta\) in any expression in which differences [e.g., \(\sin(\theta + d\theta) - \sin\theta\)] are not involved.
where $d\Psi_s$ is the increase in the $\Psi_s$-entrance angle and $d\psi$ the increase in $\psi$ in going from $\psi_s + d\psi_s [\angle \psi_s(z_e, r_e)]$ to $\psi_s(z_e, r_e)$; hence, $d\psi = d\Psi_s$. Note that $ds > 0$ going from the profile to $\Gamma_s$. On the exit side, $ds$ is the distance along the trajectory from $\Gamma_f$ to the profile, in which a ray with angle $\psi = \psi_f$ is bent through $d\psi = d\Psi_f$, so that the emergent angle is $\Psi_f + d\Psi_f$. As before,

$$ds = \rho C \, d\psi = K \, r_0 \, d\psi = K \, \rho_f \, \sin \Psi_f \, d\Psi_f \quad . \quad (12.1b)$$

Here, $ds$ is measured from $\Gamma_f$ to the profile.

From Fig. (12.2)* and Table (12.1), for both exit and entrance,

$$\gamma = \Psi - \lambda \quad . \quad (12.2)$$

From Fig. (12.2),

$$\tan \lambda + \tan \mu = \frac{BM + MN}{P_s N} = \frac{ds}{\rho d\Psi} = K \sin \Psi$$

or

$$\tan \lambda = K \sin \Psi - \tan \mu \quad . \quad (12.3)$$

From Eq. (12.2),

$$\tan \gamma = \frac{\tan \Psi - \tan \lambda}{1 + \tan \Psi \tan \lambda} = \frac{\tan^2 \Psi + 1 - 1 - \tan \lambda \tan \Psi}{\tan \Psi (1 + \tan \Psi \tan \lambda)}$$

$$= \frac{\sec^2 \Psi}{\tan \Psi (1 + \tan \Psi \tan \lambda)} - \frac{1}{\tan \Psi}$$

$$= \frac{1}{\sin^2 \Psi [\cot \Psi + K \sin \Psi - \tan \mu]} - \cot \Psi \quad . \quad (12.4)$$

Also,

$$\tan \mu = K \sin \Psi - \tan(\Psi - \gamma) \quad . \quad (12.5)$$

Since $\mu$ is generally made small, and $K$ is often in the neighborhood of 0.6, from Eq. (12.3), $\lambda \approx K \sin \Psi - \mu$, so that, to an accuracy of the order of 10%, the relation

$$\gamma \approx \Psi - K \sin \Psi + \mu \quad . \quad (12.6)$$

may be used for rough computations.

---

*The geometric arguments following are illustrated in Fig. (12.2) and subsequent figures for particular orientations of profiles, $\Gamma$-curves, and $\Psi$-angles. The same results are true for other possible orientations.
The expression \((\cot \psi + K \sin \psi - \tan \mu)\) occurs frequently in the following discussion. It is convenient to define, therefore,

\[
A_s = \cot \psi_s + K \sin \psi_s - \tan \mu_s
\]  

\[
A_f = \cot \psi_f + K \sin \psi_f - \tan \mu_f
\]  

Then,

\[
\tan \gamma_s = \frac{1}{A_s \sin^2 \psi_s} - \cot \psi_s
\]  

\[
\tan \gamma_f = \frac{1}{A_f \sin^2 \psi_f} - \cot \psi_f
\]  

Here \(A_f\) is a quantity which has arisen in another form in the variation approach of Section #5 and #6. Thus, from Eq. (A-III.3a), (A-III.6) and (9.8a), it is evident that \((A_f - K \sin \psi_f)\), a term which first arose in Eq. (5.7b), obeys the relation

\[
A_f - K \sin \psi_f = -K \sin \psi_f + \cot \psi_f - \tan \mu_f
\]  

\[
= K \sin \psi_f + \cot \psi_f - \tan \mu_f = A_f
\]  

Similarly,

\[
A_s - K \sin \psi_s = A_s
\]  

and, from Eq. (5.7b),

\[
A_f = \frac{1}{F_3} \left[ e^{-K \cos \psi_f} - \frac{e^{-K \cos \psi_f}}{\psi_f \sin^2 \psi_f} \right]
\]  

where, from Eq. (6.7b) and (3.2a),

\[
F_3 = -\frac{(z_s - z_f)}{r_0} e^{-K \cos \psi_f} = -\frac{(z_s - z_f)}{r_e} e^{-K \cos \psi_s}
\]  

For a symmetrical spectrometer,

\[
A_s = A_f = \frac{r_e}{z_s} \frac{1}{\sin^2 \psi_s}
\]  

Equations (12.1) to (12.5) relate the various parameters either on the entrance side or on the exit side. It is necessary, however, to relate the quantities between both sides, and for this, reference is made to the equations of motion: Eq. (2.5). The entrance angle, \(\psi_s\), is varied.
changing the coordinates \((z_e^i, r_e^i)\) along \(\Gamma_s\), but \(\psi = \psi_s(z_e, r_e)\) is kept constant on \(\Gamma_s\). The trajectory that intersects the \(\Gamma_s\)-curve at \((z_e^i + dz_e^i, r_e^i + dr_e^i)\) crosses the \(\Gamma_f\)-curve at \((z_0 + dz_0, r_0 + dr_0)\). Using the notation [in Eq. (2.5)] \(\psi = \psi_s(z_e, r_e) = \psi_s = \) the \(\psi\)-value of the trajectory on the \(\Gamma_s\) curve, or \(\psi = \psi_f(z_0, r_0) = \psi_f = \) the \(\psi\)-value of the trajectory on the \(\Gamma_f\) curve, while the entrance angle \(\psi_s\) is set equal to \(\Theta\) (since it is considered variable), then, from Eq. (2.5b),

\[
z_e^i - z_e^j = F(z_e) \frac{K \cos \Theta}{e^{K \cos \Theta}},
\]

with \(F\) defined in Eq. (5.3c):

\[
\log (z_e^i - z_e^j) = [\log(F(z_e))] + \log f_s(\Theta) + K \cos \Theta
\]  

(12.11a)

From Eq. (2.5a),

\[
r_e^i = z_s f_s(\Theta) e^{-K \cos \psi_s} e^{K \cos \Theta}; r_0 = z_s f_s(\Theta) e^{-K \cos \psi_f} e^{K \cos \Theta}
\]

\[
\log r_e^i = [\log z_s - K \cos \psi_s] + \log f_s(\Theta) + K \cos \Theta
\]

(12.11b)

\[
\log r_0 = [\log z_s - K \cos \psi_f] + \log f_s(\Theta) + K \cos \Theta
\]

(12.11c)

Since the relations in Eq. (12.11) differ only by constant quantities, the logarithmic derivatives are equal:

\ [
\frac{d(z_e^i - z_e^j)}{z_e^i - z_e^j} = \frac{dr_e^i}{r_e^i} = \frac{dr_0}{r_0}
\]  

(12.12)

From Table (12.1), the horizontal displacement along the \(i^i\)-curves is, evidently [see also Fig. (12.2)],

\[
d(z_e^i - z_0^i) = dz_e^i - dz_0^i = dr_e^i \tan \gamma_s + dr_0^i \tan \gamma_f
\]

\[
= \left[\frac{r_e^i}{r_0^i} \tan \gamma_s + \tan \gamma_f\right] dr_0^i = \frac{z_e^i - z_0^i}{r_0^i} dr_0^i
\]

(12.13)

Then:

\[
z_e^i - z_0^i = r_e^i \tan \gamma_s + r_0^i \tan \gamma_f
\]

(12.14)

Given \(\psi_s\) and \(\mu_s\), the quantity \(\gamma_s\) may be calculated through Eq. (12.4) or (12.5); from Eq. (12.14), \(\gamma_f\) is computed and thus \(\mu_f\); for an approximate value, Eq. (12.6) may be employed. Geometrical methods may also be used to determine \(\mu\) from \(\gamma\) approximately, as well as \(\mu_f\) from \(\mu_s\), by constructing the profiles and measuring the angles, as in Fig. (12.2).
It may be noted that in a symmetrical instrument

\[ r_e' \tan \gamma_s = r_0' \tan \gamma_f = \frac{1}{2} (z_e' - z_0') \quad (12.15) \]

Having fixed the \( \Gamma \)-lines through evaluation of \((\gamma_s, \gamma_f)\), the differential characteristics can be readily determined.

**Angular Magnification.** \( \Psi' = \frac{d \Psi_f}{d \Psi_s} \).

d \( \Psi_s \) is a variation in the emission angle and d \( \Psi_f \) the corresponding variation in the exit angle. From Fig. (12.3a),

\[ AB = AO_s + O_s B = \Delta r_e' [\cot \Psi_s + \tan \gamma_s] = P_s F = \frac{r_e d \Psi_s}{\sin^2 \Psi_s} \]

and

\[ d \Psi_s = \frac{\Delta r_e'}{r_e} (\cot \Psi_s + \tan \gamma_s) \sin^2 \Psi_s \quad (12.16) \]

The quantity*

\[ r_e \frac{d \Psi_s}{dr_e'} = (\cot \Psi_s + \tan \gamma_s) \sin^2 \Psi_s \quad (12.17a) \]

may be calculated directly by substituting for \( \tan \gamma_s \) from Eq. (12.4), or it may be deduced from Fig. (12.3a). Using Eq. (12.3),

\[ BF = BM + MF = P_s M (\tan \lambda_s + \cot \Psi_s) \]

\[ = \frac{r_e d \Psi_s}{\sin \Psi_s} (\cot \Psi_s + K \sin \Psi_s - \tan \mu_s) = \frac{\Delta r_e'}{\sin \Psi_s} = \frac{r_e d \Psi_s}{\sin \Psi_s} A_s \quad (12.17b) \]

Then,

\[ r_e \frac{d \Psi_s}{dr_e'} = \frac{1}{A_s} \quad (12.17b) \]

*Note that \( dr_e'/d \Psi_s \) is not directly related to the derivative in Eq. (A-III.5), \([dr_e/d \Psi_s]\), since \( dr_e \) is measured along the profile, while \( dr_e' \) is measured along the \( \Gamma_s \)-curve.
Similarly, from Fig. (12.3b),
\[ CA' = Q_f A' + C Q_f = \Delta r_0 \left( \cot \Psi_f + \tan \gamma_f \right) = P_f F' = \frac{r_0 d\Psi_f}{\sin \Psi_f} \]

\[ d\Psi_f = \frac{\Delta r_0}{r_0} \left( \cot \Psi_f + \tan \gamma_f \right) \sin^2 \Psi_f \]

\[ r_0 \frac{d\Psi_f}{dr_0} = \left( \cot \Psi_f + \tan \gamma_f \right) \sin^2 \Psi_f = \frac{1}{A_f} \quad (12.18) \]

Hence,*
\[ \Psi' = \frac{d\Psi_f}{d\Psi_s} = \frac{\left( \cot \Psi_f + \tan \gamma_f \right) \sin^2 \Psi_f}{\left( \cot \Psi_s + \tan \gamma_s \right) \sin^2 \Psi_s} = \frac{A_s}{A_f} \quad (12.19) \]

If the \( \Gamma \)-curves have been calculated, the geometrical determination of \( \mu \) and \( \gamma \) from the construction of Fig. (12.2) allows an immediate approximate determination.

It is evident that, for a symmetrical instrument, \( \Psi' = 1 \).

*It may be noted that Eq. (12.19) is identical to Eq. (A-III.6). This follows directly from Eq. (A-III.3a) and (12.9).
Circularity Departure of Entrance and Exit Curves. In Section #9, it was pointed out that transmission losses increase with \(|\frac{d\rho}{d\psi}| = |\rho \tan \mu|\), and hence it is desirable to keep this quantity small over the entire profile. Since \(d\rho/d\psi = 0\) for a circle, it would be the optimum profile, except for the fact that a circle at input and exit will not provide \(\psi\)-focusing. Nevertheless, under some circumstances, the transmission effects of the fringing field may be minimized by modifying the profile appropriately. However, any modification of the entrance profile requires, through the \(\psi\)-focusing requirement, a corresponding change in the exit profile.

Suppose a portion of the entrance curve is changed by rotating a small region around the entrance point \((z_e, r_e)\) through the angle \(d\mu_s\); in the resulting rotation of the exit curve around \((z_0, r_0)\) the angle \(\mu_f\) will vary by \(d\mu_f\). The two are related by differentiating Eq. (12.14) and the combination (12.17) and (12.18), evaluating at \((z_e, r_e) = (z_0, r_0)\), these quantities, as well as \((\psi_s, \psi_f)\), being constant through the rotations.* Then:

\[
\begin{align*}
\rho_e \, d(\tan \gamma_s) + r_0 \, d(\tan \gamma_f) &= 0 \quad (12.20a) \\
\sin^2 \psi_s \, d(\tan \gamma_s) &= d(\tan \mu_s/A_s^2) \quad (12.20b)
\end{align*}
\]

Calculating the \(f\)-equation corresponding to Eq. (12.20b),

\[
\frac{d(\tan \mu_f)}{d(\tan \mu_s)} = \frac{A_s^2 \sin^2 \psi_f \, r_e}{A_s^2 \sin^2 \psi_s \, r_0} = \frac{1}{(\psi_f^2)^2} \frac{\sin^2 \psi_f}{\sin \psi_s} \frac{r_e}{r_0} \quad (12.21)
\]

Now, \(\rho_s\) and \(\rho_f\) are constant, since they are not affected by the rotation, so that

\[
\frac{d(\rho_f \tan \mu_f)}{d(\rho_s \tan \mu_s)} = -\frac{1}{(\psi_f^2)^2} \frac{\sin \psi_f}{\sin \psi_s} \quad (12.22a)
\]

*For the trajectory through \((z_e, r_e)\) with input angle \(\psi_s\), the rotation causes no shift in trajectory; hence, \((z_0, r_0)\) and \(\psi_f\) remain the same.
where, from Eq. (9.8a),

\[
\frac{d(\rho_f \tan \mu_f)}{d(\rho_s \tan \mu_s)} = \frac{\frac{d(\rho_f)}{\frac{d\Psi_f}{d\rho_f}}}{\frac{d(\rho_s)}{\frac{d\Psi_s}{d\rho_s}}} \quad (12.22b)
\]

It may be noted that these derivatives are always negative.* For the symmetrical spectrometer, the derivatives are equal to -1. It should be noted that rotation of the profile around \((z_s, r_s)\) leads to a simple differential rotation only near this point.

**Magnetic Path Magnification.** \(\frac{ds_f}{ds_s} \). The entrance curve \(b_s b_s\) is translated parallel to itself and along the direction of \(\Psi_s\), so that \(P_s\) goes into \(P'_s\) [see Fig. (12.4)], and \(b_s b_s\) goes into the parallel line \(b'_s b'_s\). Also, \(\Gamma_s\) goes into \(\Gamma'_s\), which is parallel** to \(\Gamma_s\).

![Fig. 12.4](image)

---

*This implies that when the entrance profile is rotated toward the source (positive \(d\mu_s\)), the exit profile is rotated away from the image (negative \(d\mu_f\)).

**Because the field values involved for the trajectories with the new profile \(b'_s b'_s\) are smaller by an amount of the order of \(dr/r_s^2\) than for the corresponding points on trajectories with the old profile, \(\Gamma'_s\) is not quite parallel to \(\Gamma_s\), its inclination differing by a first-order term from that of \(\Gamma_s\). As pointed out above, however, unless differences are involved, the two lines may be considered as parallel.
Two trajectories are considered: (i) "a" starts at $\Psi = \Psi_s$, with the new profile, arriving at $P_s'$; (ii) "b" starts at $\Psi_s + d\Psi_s$, with the old profile, and arrives at $B$. At $P_s'$ and $B$, respectively, the two trajectories have the same $r$-values and $\Psi$-values; since both are within the field, the trajectories from these two points onward are identical, except for the translation $\Delta z' = P_s'B$. Further, "b", on reaching the exit side, hits $\Gamma_f$ at $C$, at which point, of course, $\Psi = \Psi_f$; it emerges from the profile $b_f b_f$ with the exit angle $\Psi_f + d\Psi_f$. Because both "b" and "a" are similar within the field, with only a $\Delta z$-displacement differing between them, "a" arrives at $r = r_0 + \Delta r_0'$ (the point $G$) having $\Psi = \Psi_f$, i.e., the same that "b" had at $C$. Since this is the exit point for "a", the new exit profile $b_f b_f$ passes through $G$, but is not shown in Fig. (12.4). Also, $H$ is the intersection of "a" with $\Gamma_f$.

In Fig. (12.5a) and (12.5b),

\[
\frac{HG}{P_f D} = \frac{GC}{CD};
\]

\[
\frac{r_0 d\Psi_f}{r_0 d\Psi_s} = \frac{\sin^2 \Psi_f}{\sin^2 \Psi_s} = \frac{\Psi_f}{\Psi_s} \frac{r_0}{r_0} \frac{\sin^2 \Psi_s}{\sin^2 \Psi_f};
\]

\[
P_f D = \frac{P_f Q_f}{\sin \Psi_f} = \frac{\Delta r_0'}{\sin \Psi_f};
\]

\[
GC = P_s' B = \Delta z';
\]

and

\[
\frac{ds_f}{ds_s} = \frac{HG}{P_s P_s'} = \frac{P_f D}{CD} = \frac{\Delta r_0'}{r_0} \frac{1}{\Psi'} \frac{\sin \Psi_f}{\sin \Psi_s};
\]

From Eq. (12.8)*

\[
\frac{ds_f}{ds_s} = \frac{1}{\Psi'} \frac{\sin \Psi_f}{\sin \Psi_s};
\]

(12.23)

*The argument based upon Fig. (12.5) develops the case in which $ds_s > 0$. It is evident that for $ds_s < 0$ a reverse displacement from $b_s b_s$ to $b_s b_s$ may be made, and the relationship (12.23) must still hold.
It is evident that we have defined $\mathrm{ds}_s$ as the distance along the trajectory by which $\Gamma_s$ is displaced to $\Gamma'_s$, and that $\mathrm{ds}_f$ is the distance along the trajectory by which $\Gamma_f$ is displaced to $\Gamma'_f$.

For a symmetrical spectrometer, Eq. (12.23) indicates that $\mathrm{ds}_f/\mathrm{ds}_s = 1$. This is also evident from Fig. (12.5), since (from symmetry) $r_e = r_0$; further, $\mathrm{CD} = P_S B = \mathrm{GC}$ and $\mathrm{HG} = P_T D = P_S P'_S$.

**Entrance (or Exit) Curve Correction for Axial Source (or Focus) Displacement.** If the source point is displaced from $(z_s, 0)$ by an amount $\delta z_s$, then the entrance angle of the ray entering at $(z_s, r_e)$ will be varied by the amount $\delta \psi_s/\partial z_s \delta z_s = \delta \psi_s$. From Fig. (12.2), since a positive $z_s$-displacement corresponds to $\delta \psi_s < 0$,

$$\rho_s \delta \psi_s = - \sin \psi_s \delta z_s . \quad (12.24a)$$

Hence,

$$\delta z_s = - \frac{\rho_s}{\sin \psi_s} \delta \psi_s . \quad (12.24b)$$

If the trajectories within the field are to be kept the same, then the $\psi$-value at $(z_e, r_e)$ must be restored to its pre-displacement value by adding (or subtracting)* a region containing magnetic field, so as to produce a deflection $-\delta \psi_s$ between the displaced profile and $(z_e, r_e)$. Since $\mathrm{ds}$ is positive along the trajectory, when part of the magnetic field region is removed, the displacement of the profile $\mathrm{ds}_s$ is positive. But removal of

---

*Add region of magnetic field if $\delta \psi_s$ is positive (i.e., $\delta \psi_s$ and $\delta z_s$ negative); subtract field region for $\delta \psi_s < 0$. Within the field, the angular shift $\delta \psi$ is always negative.
magnetic field region is associated with negative $\delta \Psi_s$. Hence, from Eq. (1.10),

$$ds_s = -\delta \Psi_s \rho_C = -\delta \Psi_s Kr_e .$$

where $\rho_C$ is the instantaneous radius of curvature of trajectory near $(z_e, r_e)$. Then

$$\frac{ds_s}{dz_s} = K \sin^2 \Psi_s . \quad (12.25)$$

At the exit side, a positive $\delta z_f$ corresponds to $\delta \Psi_f > 0$, so that

$$\delta z_f = \frac{\rho_f}{\sin \Psi_f} \delta \Psi_f . \quad (12.26a)$$

Removal of field region corresponds to negative $ds_f$, negative $\delta \Psi_f$, and hence negative $\delta \Psi_f$. Then,

$$ds_f = \delta \Psi_f \rho_C = \delta \Psi_f Kr_0 . \quad (12.26b)$$

Hence,

$$\frac{ds_f}{dz_f} = K \sin^2 \Psi_f . \quad (12.27)$$

It may be noted that a displacement $\delta z_s$ (or $\delta z_f$) also results in a change of $\rho_s$ (or $\rho_f$) due both to the shift in the profile and to the change in the center from which $\rho$ is measured. Now,

$$\frac{d\rho_s}{dz_s} = \frac{\partial \rho_s}{\partial z_s} \frac{dr_e}{dz_s} + \frac{\partial \rho_s}{\partial \Psi_s} \frac{d\Psi_s}{dz_s} .$$

Then, from Eq. (A-III.4) and (12.24b),

$$\frac{d\rho_s}{dz_s} = \frac{1}{\sin \Psi_s} \frac{ds_s \sin \Psi_s}{dz_s} - \frac{r_e \cos \Psi_s}{\sin^2 \Psi_s} \frac{-\sin \Psi_s}{\rho_s}$$

and

$$d \rho_s = ds_s + \cos \Psi_s dz_s . \quad (12.28a)$$
In effect, $\rho_s$ is stretched (and/or compressed) at each end, due to the shift of the profile and the origin of the vector. At the exit side,

$$\frac{d\rho_f}{dz_f} = \frac{1}{\sin\Psi_f} \frac{-ds_f \sin\Psi_f}{dz_f} - \frac{r_0 \cos\Psi_f}{\sin^2\Psi_f} \frac{\sin\Psi_f}{\rho_f},$$

since $d\rho_0/ds_f$ is negative ($ds_f > 0$ in the direction of the trajectory) and $ds_f/dz_f > 0$ from Eq. (12.27). Then,

$$d\rho_f = -ds_f - \cos\Psi_f \frac{dz_f}{dz_f}.$$

Equations (12.25) and (12.27) may be used to adjust the profiles of an instrument which has already been constructed. Suppose it is found experimentally that for different entrance angles: $\Psi_s1, \Psi_s2, \Psi_s3, \ldots$ the trajectories cross the $z$-axis at $z_{f1}, z_{f2}, z_{f3}, \ldots$, which differ in the quantity

$$\Delta z_i = z_{fi} - z_f$$

from the desired $\Psi_s$-independent focus point $(z_f, 0)$. If the quantities are small, the exit curve may be corrected by the amount $\Delta s_c$ at each measured angle:*

$$(\Delta s_c)_i = -\frac{ds_f}{dz_f} \Delta z_i = -K \sin^2\Psi_f \Delta z_i.$$

**Axial Magnification:** $dz_f/dz_s$. This quantity may be determined by first considering profile displacements necessary to keep a focus at $(z_f, 0)$ when the source is displaced by $dz_s$, and then determining the image displacement required to cancel the profile displacements.

Consider profiles which, at fixed $p$ and value of $K$, focus from a source at $(z_s, 0)$ to an image at $(z_f, 0)$. Displacements of the profiles from the initial positions are described by $s_s$ and $s_f$. Movements of either or both source and image points result in changes of $s_s$ and/or $s_f$. The focusing condition requires that there be a relation

$$M(s_s, s_f, z_s, z_f) = 0.$$

Using the notation

$$M_{z_s} = \left(\frac{\partial M}{\partial z_s}\right)_{s_s, s_f, z_f},$$

*It may be noted that Eq. (12.29a) is the negative of the usual variation, since $\Delta z_i = (\text{initial z-value}) - (\text{final z-value})$, and hence the negative sign in Eq. (12.29b).
external subscript quantities being held constant,

\[
A = \left( \frac{\partial s_f}{\partial z_f} \right)_{z_s, s_s} = -\frac{M_{z_f}}{M_{s_f}}; \quad B = \left( \frac{\partial s_f}{\partial s_s} \right)_{z_s, z_f} = -\frac{M_{z_s}}{M_{s_s}};
\]

and \( C = \left( \frac{\partial s_s}{\partial z_f} \right)_{z_f, s_f} = -\frac{M_{z_s}}{M_{s_s}} \). \hfill (12.31b)

Then

\[
\frac{BC}{A} = -\frac{M_{z_s}}{M_{z_f}} = \left( \frac{\partial z_s}{\partial z_f} \right)_{z_s, s_f} \hfill (12.31c)
\]

It may be noted that the differential coefficients in Eqs. (12.23), (12.25) and (12.27) are the partial derivatives of Eq. (12.31b). Hence, the differential coefficient desired is given by Eq. (12.31c), and

\[
\frac{dz_f}{dz_s} = \frac{1}{\psi_f} \frac{\sin \psi_s}{\sin \psi_f} = Q_f , \hfill (12.32)
\]

which is the same as Eq. (5.10).

In the symmetrical instrument, the coefficient is 1.

Radial Magnification: \( dr_{1f}/dr_1 \). This quantity is readily determined from the axial magnification. The \( r_1 \)-extensions \( dr_{1f} \) and \( dr_1 \) are related to equivalent "apparent" \( dz_f \)- and \( dz_s \)-extensions by projection:

\[
dr_{1f} = -dz_f \tan \psi_f; \quad dr_1 = dz_s \tan \psi_s . \hfill (12.33)
\]

Then, from Eq. (12.32),

\[
\frac{dr_{1f}}{dr_1} = -\frac{1}{\psi_f} \frac{\cos \psi_s}{\cos \psi_f} = u(\psi_s) , \hfill (12.34)
\]

which is the same as Eq. (5.15b).

Exit-curve Corrections from Relative Field Measurements. In place of measurements of the quantities \( z_i \) for use in computing the profile corrections through Eq. (12.29b), it is possible to use measurements of the fields required to focus particles into \((z_f, 0)\). Thus, the field intensity factor \((A_0)_i\)
is measured for each angle $\Psi_1$ in the uncorrected instrument. In general, the interesting quantities are only the relative deviations

$$\frac{\Delta A_0}{A_0} = \frac{A_{01} - A_0}{A_0}$$

(12.35)

from a chosen standard value $A_0$. Hence, any relative measure of the field intensity can be used, and this is actually what the usual field-measuring system of a spectrometer provides.

The exit curve correction will be*

$$\left(\Delta s_f\right)_c = - \frac{ds_f}{dA_0} \frac{\Delta A_0}{A_0}$$

(12.36)

where $ds_f$ is the distance the exit profile must be moved (along the trajectory) to continue focusing after a variation $dA_0$ in the specific field intensity. Equation (12.34) may also be used to compute the precision with which it is necessary to calculate and construct the profile curves. From Eq. (1.10),

$$d(\log K) = \frac{dK}{K} = - \frac{dA_0}{A_0} = - d(\log A_0)$$

(12.37a)

Then,

$$\frac{ds_f}{d(\log A_0)} = - \frac{ds_f}{d(\log K)}$$

(12.37b)

Consider the trajectory which passes through $(z_e, r_e)$ and $(z_0, r_0)$ in Fig. (12.6).

---

*The negative sign arises for the same reason as in Eq. (12.29b) i.e., due to the order of terms in Eq. (12.35).
After a change in magnetic field intensity (measured by $dK$), the point on the trajectory at which $\psi = \psi_f$ [i.e., the original $\psi(z_0, r_0)$] is displaced from $(z_0, r_0)$ by $(dz_fK, dr_fK)$ to the point $S$. If this point lay on the original $\Gamma_f$-curve, the new trajectory would still focus at $(z_f, 0)$. However, this is not true in general; the point is displaced** from $\Gamma_f$ by an amount $ds_f$ measured along the trajectory. Hence, it is necessary to add (or subtract) this distance to the path within the magnetic field, in order to get focusing at $(z_f, 0)$. From Fig. (12.6),

$$ds_f = ST = TR/\sin \Psi_f$$

Now, $d\xi = SQ = SR + RQ = TR(\cot \Psi_f + \tan \gamma_f)$. Then, from Eq. (12.18),

$$ds_f = \frac{d\xi}{(\cot \Psi_f + \tan \gamma_f) \sin \Psi_f} = (d\xi \sin \Psi_f)A_f$$

(12.38)

and

$$d\xi = SN - QN = - dz_rK - dr_rK \tan \gamma_f$$

(12.39)

Rewriting the equations of motion in Eq. (12.11), and setting $\theta$ back to $\psi_s$,

$$\log (z'_0 - z'_c) = \log F + \log [z_s f_s(\psi_s)] + K \cos \psi_s$$

(12.40a)

and

$$\log r'_0 = \log [z_s f_s(\psi_s)] + K(\cos \psi_s - \cos \psi_f)$$

(12.40b)

Differentiating, keeping $\psi_s$ and $\psi_f$ constant, and evaluating at $(z_0, r_0)$,

$$\frac{dr'_0}{r_0} = (\cos \psi_s - \cos \psi_f) dK$$

(12.41a)

and

$$\frac{d(z'_0 - z'_c)}{z_0 - z_c} = \left(\cos \psi_s + \frac{F'}{F}\right) dK$$

(12.41b)

where $F$ is defined in Eq. (5.3c) and $F' = \partial F/\partial K$ is given in Eq. (6.8).

*Since the differential shift in field, operating over a differential distance, would lead only to a second-order shift in focus position.

**The point $S$ may or may not lie within the field defined by the original profile. If it does not, then the path through the magnetic field must be increased in length, since the new $\Gamma_f$-curve must lie within the profile boundary.
It may be noted that \( dz_0 K \) and \( d\phi K \) are taken with \( \psi_f \) constant, as well as the initial angle \( \psi_s \); since this also applies to the increments in Eq. (12.41), the two variations are respectively equal. Since the entry point is not varied (\( \psi_s \) constant), \( d(z_0 - z_0') = dz_0 \). Hence from Eq. (12.39) and (12.41),

\[
 d\Sigma = \left\{ \cos\psi_s \left[ (z_e - z_0) - r_0\tan\psi_f \right] + r_0\cos\psi_f\tan\psi_f + \left( F'/F \right)(z_e - z_0) \right\} dK.
\]

Applying Eq. (12.14) and transforming from \( \psi \) to \( \Psi \):

\[
 d\Sigma = \left\{ (z_e - z_0)(F'/F) - \left[ r_e\tan\gamma_s\cos\Psi_s + r_0\tan\gamma_f\cos\Psi_f \right] \right\} dK. \tag{12.42}
\]

Introducing Eq. (12.42) into (12.39),

\[
 \frac{ds_f}{d(\log K)} = K\sin\Psi_f \left[ (z_e - z_0) \frac{F'}{F} - r_e\tan\gamma_s\cos\Psi_s + r_0\tan\gamma_f\cos\Psi_f \right] A_f
\]

\[
 = A_f K\sin\Psi_f \frac{d\Sigma}{dK}. \tag{12.43}
\]

For a symmetric instrument, it is evident from Eq. (12.14), that

\[
 \frac{d\Sigma}{dK} = (z_e - z_0) \left[ \frac{F'}{F} - \cos\Psi_s \right]. \tag{12.44}
\]

**Entrance-curve Corrections from Relative Field Measurements.**

This correction is derivable from the result for the exit profile. If \( z_s \) and \( z_f \) are held constant, then there must be the relation

\[
 Q(K, s_s, s_f) = 0. \tag{12.45}
\]

As above, \( \left( \frac{\partial s_f}{\partial K} \right)_{s_s} = -\frac{Q_f}{Q_s} \) ; \( \left( \frac{\partial s_s}{\partial K} \right)_{s_f} = -\frac{Q_f}{Q_s} \).

Then,

\[
 \left( \frac{\partial s_f}{\partial s_s} \right)_K = -\frac{Q_s}{Q_{sf}} = -\left( \frac{\partial s_s}{\partial K} \right)_{s_f} = -\left( \frac{\partial s_f}{\partial (\log K)} \right)_{s_s}. \tag{12.46}
\]

*See also Eq. (A-V.8) in Appendix V.*
Now,

\[
(\Delta s_s)_c = \left( \frac{\partial s_s}{\partial (\log K)} \right)_{s_f} \frac{\Delta K}{K}; \quad (\Delta s_f)_c = \left( \frac{\partial s_f}{\partial (\log K)} \right)_{s_s} \frac{\Delta K}{K}.
\]  

(12.47)

Hence, from Eq. (12.46),

\[
\frac{(\Delta s_s)_c}{(\Delta s_f)_c} = -\left( \frac{\partial s_s}{\partial s_f} \right)_K
\]

or

\[
(\Delta s_s)_c = -\left( \frac{\partial s_s}{\partial s_f} \right)_K (\Delta s_f)_c.
\]  

(12.48)

Also, from Eq. (12.37a) and (12.47),

\[
(\Delta s_s)_c = - \left( \frac{\partial s_s}{\partial (\log K)} \right) \frac{dA_0}{A_0}.
\]  

(12.49)

This differential coefficient may be computed from Eqs. (12.17), (12.19), (12.23) and (12.46). Thus,

\[
\frac{d s_s}{d (\log K)} = - \frac{K}{\sin \Psi_s} \frac{1}{(\cot \Psi_s + \tan \gamma_s)} \frac{d \xi}{\Delta K}
\]

\[
= - A_s K \sin \Psi_s \frac{d \xi}{\Delta K}.
\]  

(12.50)

**Dispersion**: \( dz_f/d(\log A_0) \). If \( z_s \) and \( s_s \) are held constant, then between the quantities \( K, s_f, \) and \( z_f \) there must be a relation

\[
R(K, s_f, z_f) = 0.
\]  

(12.51a)

Hence,

\[
\left( \frac{\partial z_f}{\partial K} \right)_{s_f} = - \frac{R_K}{R_{z_f}} = - \frac{R_K}{R_{z_f}} \left( \frac{\partial s_f}{\partial K} \right)_{z_f} \left( \frac{\partial z_f}{\partial s_f} \right)_K.
\]  

(12.51b)
Then,
\[ dz_f = \left( \frac{\partial z_f}{\partial (\log K)} \right) s_f \frac{dK}{K} = - \left\{ \frac{ds_f}{d(\log K)} \frac{dz_f}{ds_f} \right\} \frac{dK}{K}, \quad (12.51c) \]

where the differential coefficients are given in Eqs. (12.27) and (12.38) or (12.43). As in Eq. (6.10a), the dispersion is given by

\[ \sigma_f = \frac{dz_f}{d(\log A_0)} = - \frac{dz_f}{d(\log K)} = \frac{1}{K \sin^2 \Psi_f} \frac{ds_f}{d(\log K)} \]

\[ = \frac{A_f}{\sin \Psi_f} \frac{d\Sigma}{dK} = \frac{1}{\sin^3 \Psi_f} \frac{1}{(\cot \Psi_f + \tan \gamma_f)} \frac{d\Sigma}{dK}, \quad (12.52) \]

In Appendix V it is shown that Eq. (12.52) is equivalent to Eq. (6.6). The difference in the form of the two expressions for \( \sigma_f \) arises from the difference in the nature of the variations considered. In Section #6, variations were taken only along the exit profile. In this section, on the other hand, variation of the profile position with \( K \) was considered, as well as the variation of the image position necessary to cancel the profile movement. The same difference in the variations taken was involved in the identical pair of equations Eqs. (A-III.6) and (12.19), as well as in the pair Eqs. (5.10) and (12.32), and the pair (5.15b) and (12.34).

**Source Dispersion (Effect of Source Movement).** If the image is to be kept at \((z_f, 0)\) during a variation of \( K \), then the source is moved a distance

\[ dz_s = - \sigma_s \frac{dK}{K} = \sigma_s \frac{dA_0}{A_0}. \]

As in Eq. (12.47b),

\[ - \left( \frac{\partial z_s}{\partial (\log K)} \right)_{s_s} = \left( \frac{\partial s}{\partial (\log K)} \right)_{z_s} \left( \frac{\partial z_s}{\partial s} \right)_{K}. \quad (12.53) \]

*Equation (12.54) may also be computed using the similar relation

\[ - \left( \frac{\partial z_s}{\partial (\log K)} \right)_{z_f} = \left( \frac{\partial z_s}{\partial (\log K)} \right)_{K} \left( \frac{\partial z_f}{\partial (\log K)} \right)_{z_s} \]

so that, from Eqs. (12.19), (12.32) and (12.50),

\[ \sigma_s = \left( \Psi_f \sin \Psi_f \sin \Psi_s \right) \left( - \frac{A_f}{\sin \Psi_f} \right) \frac{d\Sigma}{\Delta K} = - \frac{A_s}{\sin \Psi_s} \frac{d\Sigma}{\Delta K} \]
Then, from Eqs. (12.21) and (12.46),

$$
\sigma_s = - \frac{dz_s}{d(\log K)} = - \frac{A_s}{\sin \Psi_s} \frac{d\delta}{\Delta K} .
$$

(12.54)

**Correction of a Constructed Instrument.** It is convenient to express the corrections necessary to apply to an instrument with defocusing $dA_0/A_0$ in terms of the dispersion coefficients. Thus, from Eqs. (12.51b), (12.52) and (12.27),

$$
\left( \frac{\partial (\log A_0)}{\partial \delta f} \right)_{s_f} = \left( \frac{\partial (\log K)}{\partial \delta f} \right)_{s_f} = - \left( \frac{\partial z_f}{\partial \delta f} \right)_{s_f} K_{\sin^2 \Psi_f} .
$$

(12.55)

Similarly, from Eqs. (12.53), (12.54) and (12.25),

$$
\left( \frac{\partial (\log A_0)}{\partial \delta s} \right)_{s_s} = \left( \frac{1}{\sigma_s K_{\sin^2 \Psi_s}} \right) .
$$

(12.56)

Also,

$$
\left( \frac{\partial (\log A_0)}{\partial \delta z_f} \right)_{s_f} = \frac{1}{\sigma_f} \quad \text{and} \quad \left( \frac{\partial (\log A_0)}{\partial \delta z_s} \right)_{s_s} = \frac{1}{\sigma_s} .
$$

(12.57)

Corrections may be made for the defocusing by displacing the exit curve by the amount $(\Delta s_f)_c$ at each $\Psi_s$. Thus,

$$
\frac{dA_0}{A_0} = - \frac{(\Delta s_f)_c}{\sigma_f K_{\sin^2 \Psi_f}} .
$$

(12.58)

It is evident, however, from Eqs. (12.56) and (12.57), that other changes may be made which are equivalent to $(\Delta s_f)_c$. Thus, the relative defocusing $dA_0/A_0$ can be corrected by combining a correction at the exit profile, a correction at the entrance profile, a movement of the source point or the image point, and a change in the operating $A_0$-value of the instrument.
Then,

$$\frac{dA_0}{A_0} = -\left[ \frac{\Delta s_f(\Psi_f)}{\sigma_f K \sin^2\Psi_f} + \frac{\Delta s_s(\Psi_s)}{\sigma_s K \sin^2\Psi_s} + \frac{\Delta z_f}{\sigma_f} + \frac{\Delta z_s}{\sigma_s} \right] = C_0 .$$

(12.59a)

With the corrections involved in $C_0$, it is possible to change all of the trajectories so that they focus at the image point, using the operating point $A_0$. In addition to these corrections, it is possible to shift the operating point from $A_0$ by the amount $\Delta A_0$. Then

$$\frac{dA_0}{A_0} = \frac{\Delta A_0}{A_0} + C_0 .$$

(12.59b)

It is possible to choose the most convenient set of values ($\Delta s_f$, $\Delta s_s$, $\Delta z_f$, $\Delta z_s$, and $\Delta A_0$) which fit Eq. (12.59). Only one value can be given to each of $\Delta z_f$, $\Delta z_s$, and $\Delta A_0$, and these will be chosen in such a manner as to minimize the values of $\Delta s_f$ and $\Delta s_s$, which will, of course, vary with $\Psi_s$. 
### Table (12.1)

<table>
<thead>
<tr>
<th>Angle</th>
<th>Description</th>
<th>Angle is positive when:</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mu_s )</td>
<td>IA</td>
<td>Between line PM normal to input ray and profile (in same ( \phi )-plane)</td>
</tr>
<tr>
<td></td>
<td>IB</td>
<td>Between line normal to (and away from) profile and the input ray (in same ( \phi )-plane)</td>
</tr>
<tr>
<td>( \gamma_s )</td>
<td>IIA</td>
<td>Between positive ( r )-axis (in same ( \phi )-plane) and ( \Gamma_s )</td>
</tr>
<tr>
<td></td>
<td>IIB</td>
<td>( \gamma_s = \frac{1}{2} \pi - \alpha_s ) ( (\alpha_s = \text{inclination angle of } \Gamma_s) )</td>
</tr>
<tr>
<td></td>
<td>IIC</td>
<td>( \tan \gamma_s = \frac{dz}{dr} )</td>
</tr>
<tr>
<td>( \lambda_s )</td>
<td>IIIA</td>
<td>Between ( \Gamma_s ) and normal to input ray</td>
</tr>
<tr>
<td></td>
<td>IIIIB</td>
<td>( \lambda_s = \gamma_s - \mu_s )</td>
</tr>
<tr>
<td>( \mu_f )</td>
<td>IA'</td>
<td>Between line normal to output ray and profile</td>
</tr>
<tr>
<td></td>
<td>IB'</td>
<td>Between line normal to profile and output ray</td>
</tr>
<tr>
<td>( \gamma_f )</td>
<td>IIA'</td>
<td>Between positive ( r )-axis and ( \Gamma_f )</td>
</tr>
<tr>
<td></td>
<td>IIB'</td>
<td>( \gamma_f = \alpha_f - \frac{1}{2} \pi )</td>
</tr>
<tr>
<td></td>
<td>IIC'</td>
<td>( \tan \gamma_f = \frac{dz}{dr} )</td>
</tr>
<tr>
<td>( \lambda_f )</td>
<td>IIIA'</td>
<td>Between ( \Gamma_f ) and normal to output ray</td>
</tr>
<tr>
<td></td>
<td>IIIIB'</td>
<td>( \gamma_f = \varphi_f - \lambda_f )</td>
</tr>
</tbody>
</table>

* The direction of \( \Gamma_s \) is from \((\psi_0, \phi_0)\) to larger \( \varphi_s \)-values than \( \varphi_s(\psi_0, \phi_0) \). For \( \varphi_s > 90^\circ \), \( \Gamma_s \) may be directed down and to the right [using the same orientations as Fig. (12.2)], in which case \( \alpha_s < 0 \).
Table 12.2
SUMMARY OF DIFFERENTIAL COEFFICIENTS

| Angular magnification | Equation 12.19 | \[
\frac{dV_f}{dV_s} = \frac{A_g}{A_f}
\]
| Rotation | 12.22a | \[
\frac{d(\rho_f \tan \mu_f)}{d(\rho_s \tan \mu_s)} = -\frac{1}{(\psi_f')^2} \sin \psi_f \sin \psi_s
\]
| Magnetic path magnification | 12.23 | \[
\frac{ds_f}{ds_s} = \frac{1}{\psi_f'} \frac{\sin \psi_f}{\sin \psi_s}
\]
| Profile movement with shift of source or focus | 12.25 | \[
\frac{ds_f}{dz_s} = K \sin^2 \psi_s
\]
| | 12.27 | \[
\frac{ds_f}{dz_f} = K \sin^2 \psi_f
\]
| Axial magnification | 12.32 | \[
\frac{dz_f}{dz_s} = \frac{1}{\psi_f'} \frac{\sin \psi_s}{\sin \psi_f'}
\]
| Radial magnification | 12.34 | \[
\frac{dr_f}{d\rho_1} = \frac{1}{\psi_f'} \cos \psi_s \cos \psi_f'
\]
| Profile movement with shift of \(A_0\) | 12.43 | \[
\frac{ds_f}{d(\log K)} = A_f K \sin \psi_f \frac{dC}{dK}
\]
| | 12.50 | \[
\frac{ds_s}{d(\log K)} = -A_s K \sin \psi_s \frac{dC}{dK}
\]
| Dispersion | 12.52 | \[
\sigma_f = -\frac{dz_f}{d(\log K)} = \frac{A_f}{K \sin \psi_f} \frac{dC}{dK}
\]
| | 12.54 | \[
\sigma_s = -\frac{dz_s}{d(\log K)} = \frac{-A_s}{K \sin \psi_s} \frac{dC}{dK}
\]
| Correction of constructed instrument | 12.55 | \[
\frac{d(\log A_0)}{ds_f} = -\frac{1}{\sigma_f} K \sin^2 \psi_f
\]
| | 12.56 | \[
\frac{d(\log A_0)}{ds_s} = -\frac{1}{\sigma_s} K \sin^2 \psi_s
\]
| | 12.57 | \[
\frac{d(\log A_0)}{dz_f} = -\frac{1}{\sigma_f}
\]
| | 12.57 | \[
\frac{d(\log A_0)}{dz_s} = -\frac{1}{\sigma_s}
\]
#13. MEDIAN-PLANE TRAJECTORIES IN THE FRINGING FIELD

Some of the considerations concerning the median-plane trajectories were introduced in Section #8. The basic deflection Eq. (8.5) was calculated and evaluated on the median plane, using an approximate form for the fringing field on this plane [see Eq. (8.7)]. In this section the trajectories in the fringing field are examined in greater detail. Because of the finite extent of the fringing fields, it is necessary to consider the relationship of the trajectory to the various portions of this field - its external boundary and its internal boundary - as well as to the profile itself. Following is a list of the definitions of various points and distances relative to a trajectory which starts from \((z_s,0)\) at the angle \(\psi_s\) [see Figs. (8.4) and (13.1)]:

The points:

\((z_{1e},r_{1e})\) the point on the trajectory where the fringing field (in the linear approximation) becomes zero, i.e., where the particle passes from field-free space into the fringing field [A in Fig. (13.1)].

\((z_{pe},r_{pe})\) the point on the trajectory at the crossing of the profile computed as in Section #3 [F in Fig. (13.1)].

\((z_e,r_e)\) the point on the trajectory where the fringing field (in the linear approximation) merges with the interior or idealized form of Eq. (1.1) [B in Fig. (13.1)].

\((z_{2e},r_{2e})\) the point of normal projection of \((z_e,r_e)\) onto the initial tangent of the trajectory (i.e., the extension of the original ray with inclination \(\psi_s\)).

The distances:

Up to this point, it has not been important to distinguish between distances to profiles and to the interior field point. At this point:

\(\rho_s\) from \((z_s,0)\) to \((z_{pe},r_{pe})\). [OF in Fig. (13.1)].

\(\rho_e\) from \((z_s,0)\) to \((z_e,r_e)\). [OB in Fig. (13.1)].

\(\rho_{1e}\) from \((z_s,0)\) to profile along the initial tangent. [OE in Fig. (13.1)].

\(d_{1e}\) from \((z_{1e},r_{1e})\) to \((z_{pe},r_{pe})\) measured along the trajectory [AF in Fig. (13.1); AE in Fig. (8.4)].

\(d_{2e}\) from \((z_{pe},r_{pe})\) to \((z_e,r_e)\) measured along the trajectory [FB in Fig. (13.1); EM in Fig. (8.4)].

\(d_e = d_{1e} + d_{2e}\) Same as quantity defined in Section #8. i.e., measured along trajectory [AFB in Fig. (13.1); AM in Fig. (8.4)].
from \( z_{1e}, r_{1e} \) to profile, along initial tangent to trajectory [AE in Fig. (13.1)].

from \( z_{2e}, r_{2e} \) to profile, along tangent [CE in Fig. (13.1)].

t_\text{e} = t_{1e} + t_{2e} = \text{distance from } \( z_{1e}, r_{1e} \) to \( z_{2e}, r_{2e} \) [AC in Fig. (13.1)].

from \( z_{2e}, r_{2e} \) to \( z_{e}, r_{e} \) [CB in Fig. (13.1)].

All distances are taken as magnitudes; hence, always positive.

\( \psi_e \)  
\( \Delta \psi_e = \psi_e - \psi_s \)

On the exit side, the corresponding quantities are designated by changing the index "e" to "o", or "s" to "f"; the exception to this rule is:

\( \Delta \psi_0 = \psi_f - \psi_0 \).

Pursuing now the evaluation of the trajectory perturbation, it is evident that it is necessary to know the coordinate displacement measured by \( \Sigma_e \) [see Fig. (13.1)], as well as the angular displacement \( \Delta \psi_e \) already computed in Section #8. To the approximation that \( \Sigma_e \) is small, at each point along the trajectory in the fringing field,

\[
d\Sigma_e = (d_e - d) \, d\psi
\]  \hspace{1cm} (13.1a)

and, in the linear approximation to the fringing field, from Eq. (8.7),
Then

\[ \Xi_e = \frac{1}{K r_e d e} \int_0^{d e} [(d e - d) d] d(d) = \frac{d_e^2}{6 Kr_e} = \frac{1}{3} \Delta \psi_e \quad d_e. \quad (13.2) \]

Rather than making a direct measurement of \(d_e\) along the trajectory, it is simpler to make the corresponding measurement along the initial tangent. For \(d_e\), then, \(t_e\) is substituted, which is a fair approximation as long as \(\Xi_e\) is small. From Eqs. (8.7) and (13.2),

\[ \Delta \psi_e = \frac{1}{2} \frac{t_e}{K r_e} = \frac{1}{2} \frac{\varphi}{K} \frac{t_e}{s_e} \quad (13.3a) \]

\[ \Xi_e = \frac{1}{6} \frac{t_e^2}{K r_e} = \frac{1}{3} \Delta \psi_e \quad t_e. \quad (13.3b) \]

It may be noted that \(\Delta \psi_e\) is first order in \(t_e\) (or \(d_e\)), whereas \(\Xi_e\) is second order. The position of the point \((z_e, r_e)\) can be readily determined, and from it can be calculated the coordinates \((z_e, r_e)\). Thus, from Fig. (13.1),

\[ r_e = r_{2e} - \Xi_e \cos \psi_s = r_{2e} + \Xi_e \cos \psi_s \quad (13.4a) \]

\[ r_{2e} = (\rho_{1e} + t_{2e}) \sin \psi_s. \quad (13.4b) \]

Now \(r_e\) is evaluated by successive approximations, utilizing the fact that \(\Xi_e\) is small. Thus, \(r_{2e}\) is substituted for \(r_e\) in Eq. (13.3a) and \(\Xi_e\) then calculated from Eq. (13.3b). The value of \(r_e\) from Eq. (13.4a) is then put back into (13.3a), etc. For the \(z\)-coordinate, from Fig. (13.1),

\[ z_e = z_{2e} - \Xi_e \sin \psi_s = z_s + [\rho_{1e} + t_2 \cos \psi_s - \Xi_e \sin \psi_s]. \quad (13.5) \]

From Eqs. (13.3a), (13.4) and (13.5) and the relation

\[ \psi_e = \psi_s + \Delta \psi_e, \]

all the necessary information is available for computing the trajectories. Substituting into Eq. (1.27), the constants \(a\) and \(Z_M\) may then be calculated as a function of \(\psi_s\) and \(K\).
A general procedure for computing a focusing exit curve for any given entrance curve was given in Section #3 (neglecting fringing-field effects). In this section is presented a method for making a fairly rapid determination, using a procedure of interpolation from a master curve. This approach is particularly valuable when a number of alternative profiles are to be examined, without great demands on precision, particularly in the case of asymmetric instruments.

The calculation will be carried out for the case in which \( n_s = n_f = 0 \), since this is the case of greatest usefulness as well as being the simplest; however, the method can be readily generalized.

The basis of the method is the determination of the symmetry point \((Z_M, r_M)\) for each trajectory, i.e., the point at which \( \psi = \pi \). Because fringing fields are neglected, \((z_e, r_e)\), the entrance point, is on the profile, and the distance from \((z_0, 0)\) to \((z_e, r_e)\) is \( \rho_s \). It is convenient to take \( z_0 = 0 \). From Eqs. (1.26) and (2.3),

\[
\begin{align*}
  z_e &= r_e e^{K \cos \psi_s} K U(K, \psi_s) + Z_M = \rho_s \cos \psi_s & \text{(14.1)} \\
  r_e &= \rho_s \sin \psi_s. & \text{(14.2)}
\end{align*}
\]

Then,

\[
Z_M = \rho_s \left[ \cos \psi_s - \sin \psi_s e^{K \cos \psi_s} K U(K, \psi_s) \right] & \text{(14.3)}
\]

and

\[
r_M = a e^K \rho_s \sin \psi_s e^{K(1 + \cos \psi_s)} & \text{(14.4)}
\]

Consider, first, a circular entrance curve of radius \( \rho_s = 1 \). It will be convenient to compute the properties of this case and then to use simple transformations to get the general case. For this case, set \((Z_M, r_M)\) equal to \((Z_0, r_0)\). In Table (14.1) are listed \((Z_0, r_0)\) for several \((K, \psi_s)\) values. The results are shown in Fig. (14.1) in the form of a family of curves. The solid curves represent what may be called \( M_0 \)-curves, joining points with constant \( K \) and variable \( \psi_s \). The dashed curves join points of constant \( \psi_s \) and variable \( K \). By making a fine enough mesh, it is feasible to use the set of curves for interpolation.
It is evident that, because

\[ Z_{\text{Ms}} = \rho_s(\psi_s) Z_{\text{Mo}}(\psi_s) \quad \text{and} \quad r_{\text{Ms}} = \rho_s(\psi_s) r_{\text{Mo}}, \]

(14.5)

it is possible to use these curves to construct the \( M_s \)-curve for any entrance profile defined by \( \rho_s(\psi_s) \). Some examples are evident in Fig. (14.2),
in which are shown the \( M_1 \)-curve for a circular entrance curve with \( \rho_s = \rho_1 \) and the \( M_g \)-curve for an arbitrary noncircular entrance curve \( b_g \). Clearly \( M_1 \) is formed by increasing the radius vector\(^*\) from \([z_g,0]=(0,0)\) to a point on \( M_0 \) by a constant factor, as in going from \( A \) to \( B \), i.e., \( OB/OA = \rho_1 \) for any \( B \) on \( M_1 \). In the same way, \( M_g \) is formed, as in going from \( A_1 \) to \( B_1 \); the factor, however, varies with \( \psi_g \), so that \( OB_1/OA_1 = \rho_s(\psi_g) \).

![Diagram](image)

At the exit, the \( M \)-curves are constructed in the same manner. Using \( z_g \) as the origin, as before, from Eqs. (1.29a), (2.3), and (3.2a),

\[
z_0 = r_0 e^{K \cos \psi_f} K U(K, \psi_f) + Z_{Mf} = z_f - \rho_f \cos \psi_f
\]  

(14.6)

\(^*\)It may be noted that the radius vector corresponding to \( \psi_g \) does not have \( \psi_g \) as an inclination, as is evident from the example in Fig. (14.2).
Then,
\[ r_0 = - \rho_f \sin \psi_f. \quad (14.7) \]

Then,
\[ Z_{Mf} - z_f = -\rho_f \{ \cos \psi_f - \sin \psi_f \mathbf{K} \cos \psi_f \mathbf{K} U(K, \psi_f) \} \quad (14.8) \]

\[ r_{Mf} = a \mathbf{e} K = -\rho_f \sin \psi_f \mathbf{e} K (1 + \cos \psi_f) \quad . \quad (14.9) \]

Again taking \( \rho_f = 1 \), a set of \( M_f \)-curves are derived, each curve passing through the points \((Z_{M_0}^{1}r_{M_0}^{1})\) for the various \( \psi_f \)-values at constant \( K \). These are almost identical to the \( M_0 \)-curves, differing only in being reflections in a plane \( z = \text{constant} \). The \( M_0 \)-curves are not identical to the \( M_f \)-curves (i.e., not superimposable), which simply means, as was pointed out in Section #3, that a symmetric instrument does not have circles for profiles.

From an arbitrary exit curve \( b_f b_f \), the \( M_f \)-curve may be constructed giving \((Z_{Mf}^{1}r_{Mf}^{1})\) values for the same \( K \)-value as was used for the entrance curve \( b_s b_s \). As before, the radius vector to the appropriate point (for a given \( \psi_f \)) on \( M_f \) is multiplied by \( \rho_f \) to give the corresponding point on \( M_f \). In general, for an arbitrary entrance curve \( b_s b_s \) and an arbitrary exit curve \( b_f b_f \), the curve \( M_f \) is not superimposable on \( M_s \). If the curves were superimposable, then the exit curve would satisfy the focusing condition; this follows, because for each preassigned \( \psi_s \) and \( \rho \)-value (hence \( r_e \) or \( \rho_s \)), there would result a unique \( Z_M \) and \( \psi_f \), such that the ray would arrive at \((z_f,0)\).

Inversely, consider the \( M_s \)-curve drawn for the entrance profile \( b_s b_s \) as the \( M_f \)-curve for the to-be-determined exit profile \( b_f b_f \). The radius vector is drawn from \((z_f,0)\) through the point on \( M_s \) corresponding to a particular \( \psi_f \), and the vector continued until it intersects \( M_s \), as, e.g., the extension of \( \mathbf{O} \mathbf{A}' \) to \( \mathbf{O} \mathbf{B}' \) in Fig. (14.2); then the appropriate \( \rho_f \) for this \( \psi_f \) is \( \mathbf{O} \mathbf{B}'/\mathbf{O} \mathbf{A}' \). The \( \psi_s \) corresponding to this \( \psi_f \) is determined from the intersection point \( \mathbf{B}' \) on \( M_0 \). In general, \( \psi_s \neq 2\pi - \psi_f \); for the case in Fig. (14.2), \( \psi_s \approx 95^\circ \) for \( 2\pi - \psi_f = 120^\circ \).

This graphical evaluation of the intersection point \( \mathbf{B}' \) (and the resulting value of \( \rho_f \)) is a fairly rapid method for constructing the exit curve. Alternatively, by the use of approximating formulae for \( M_s \) and \( M_0 \), the intersection points may be determined analytically. This is a procedure which is alternative to the method described in Section #3, and is essentially equivalent in computational difficulty.

Symmetrical instruments. The characteristic feature of a symmetrical instrument is that the \( M_s \)-curve is the vertical line \( Z_M = \text{constant} \), situated midway between \( z_f \) and \( z_s \). With the "inverse" method,
working from this $M_g$-curve and the $M_0$-curve appropriate for the $K$-value considered, $\rho_g$ may be determined as a function of $\psi_g$. The exit curve is, of course, determined by a mirror reflection in $z = Z_M$. Since the explicit form of the curves is available in Eq. (3.5), this method is no faster than the analytical approach unless the family of $M_0$-curves is already at hand.
201

**#15. CALCULATION OF PROFILES INCLUDING FRINGING-FIELD EFFECTS**

It is presumed that the fringing field is known, at least on the median plane, so that \( \Delta \psi_e \) and \( E_e \) may be computed as in Sections #8 and #13, and hence the coordinates \( (z_e, r_e, \psi_e) \) as in Eqs. (13.4), (13.5) and (13.6). With this information, it is possible to use the approach of Section #14 to compute an "equivalent" entrance curve. Thus, through Eq. (1.29), \( (Z_M, r_M) \) may be determined for a number of values of \( z_e, r_e, \) and \( \psi_e \), giving the \( M_s \)-curve for the system, including the fringing-field effect. Working backwards, it is then possible to determine an equivalent entrance profile \( (z_{ec}, r_{ee}) \); this profile is one which would, in the absence of a fringing field [magnetic field described entirely by Eq. (1.1)], give the same set of trajectories, i.e., the same \( M_s \)-curve. Using the equivalent entrance profile, the equivalent exit profile may be computed by the method of Section #3 or of Section #14, and from the latter, the real exit profile may be computed by a reverse computation of \( \Delta \psi_0 \) and \( E_0 \).

It is possible, however, to find the equivalent entrance curve directly from the real entrance curve by an approximate method. The approximation is an adequate one when one considers the fact that \( \psi_e \) and \( E_e \) are generally not well known (being derived from an estimation of the fringing-field magnitude), and in addition are rather small quantities.

The quantities to be considered are illustrated in Fig. (15.1), which is similar to Fig. (13.1). At A, we have \( \psi = \psi_s \), and the fringing field starts to bend the trajectory, and continues up to \( (z_e, r_e) \), i.e., to B, lying within the actual profile \( b_gb_g \), with \( \psi = \psi_e \). If no fringing field existed and the interior field followed Eq. (1.1), then a ray starting at G on a profile \( b_gb_g \) with \( \psi = \psi_s \) will bend through the same change in \( \psi \), arriving at \( (z_e, r_e) \) with \( \psi = \psi_e \). From this point to the \( M_s \)-curve, the trajectories are identical in the idealized and real cases. Because the field intensity within the profile is greater in the idealized case, the rate of bending is greater; hence, the trajectory GB is always at a larger \( \eta \) (see Section #11 for definition) than the real trajectory AFB. From Eq. (1.29b), since \( \log(r/a) = -K \cos \psi \) within the idealized field, then

\[
ds = \frac{dr}{\sin \psi} = Kr \ d\psi,
\]

(15.1)
where \( ds \) is positive along the trajectory. To the first approximation, \( r \) may be taken as constant along the trajectory GB and equal to \( r_e \). Then, from Fig. (15.1),

\[
\Delta s_1' = s(B) - s(G) = Kr_e (\psi_e - \psi_s) = Kr_e \Delta \psi_e .
\]

(15.2)

The necessary displacement of the profile \( b_g' b_s' \) from \( b_g b_s \) along the trajectory is the negative of \( \Delta s_1' \), so that

\[
\Delta s_1' = -Kr_e \Delta \psi_e .
\]

(15.3)

In the second approximation, allowance is made for the variation of \( r \), using Eq. (2.5a):

\[
r = (r_e e^{K\cos \psi_e}) e^{-K\cos \psi}
\]

(15.4)

\[
\Delta s_1 = s(G) - s(B) = \left( \frac{ds}{d\psi} \right)_{\varphi = r_e} \Delta \psi + \frac{1}{2} \left( \frac{d^2s}{d\psi^2} \right)_{\varphi = r_e} (\Delta \psi)^2,
\]

(15.5a)

where

\[
\Delta \psi = \psi_s - \psi_e = -\Delta \psi_e , \quad \text{and} \quad \frac{ds}{d\psi} = Kr
\]

(15.5b)

from Eq. (15.1). Since

\[
\frac{d^2s}{d\psi^2} = (K\sin \psi) (Kr_e) e^{K(\cos \psi_e - \cos \psi)} ,
\]

we have

\[
\Delta s_1 = -Kr_e [1 - \frac{1}{2} (K\sin \psi) \Delta \psi_e] \Delta \psi_e .
\]

(15.6)

As in Eq. (13.1a), to the approximation that \( \Xi_1 \) is small, at each point along the trajectory portion GB,

\[
d\Xi_1 = (\Delta s_1 - s_G) d\psi = (\Delta s_1 - s_G) \frac{ds}{Kr}
\]

(15.7)

from Eq. (15.5b), with \( s_G \) the distance, along the trajectory, from G. Taking \( Kr \) as approximately constant along GB:

\*\( \Delta s_1 \) is always negative (\( \Delta \psi_e > 0 \)). Whether or not the equivalent profile \( b_g' b_s' \) is outside the real profile \( b_g b_s \) depends upon the ratio \( g/de \) (see Section #8). Generally, \( b_g' b_s' \) is outside relative to \( b_g b_s \).

\**The difference in the coefficient between Eqs. (15.8) and (13.2) may be noted. This arises from the fact that the region of action to give a particular \( \Delta \psi_e \)-value is smaller in the relatively intense idealized field then in the slowly rising fringing field, i.e., \( \Delta s_1 < de \).
from Eq. (15.3). It is seen that $\Xi_1$ is already of second order in $\Delta s_1$; hence, no better approximation need be made, and $\Delta s_1' = \Delta s_1$.

The equivalent ray $O'G$ arrives at $G$ with the entrance angle $\psi_s$, i.e., parallel to the real ray $OA$, but displaced by the distance*

$$\Delta \Xi = \Xi_e - \Xi_1$$

(15.9)

On the $z$-axis, this corresponds to the displacement

$$\Delta z = -\frac{\Delta \Xi}{\sin \psi_s}$$

(15.10)

In general $\Delta z$ is a negative displacement,** and may be corrected by an additional displacement, $\Delta s_2$, which from Eq. (12.25) is also negative, i.e., further displaced from $b_s b_s$. Then

$$\Delta s_2 = \Delta z \text{ Ksin}^2 \psi_s = -\Delta \Xi \text{ Ksin} \psi_s$$

(15.11)

$$= -\Xi_e \text{ Ksin} \psi_s - \frac{1}{2} \Delta s_1 \text{ Ksin} \psi_s \Delta \psi_e$$

The total displacement is

$$\Delta s = \Delta s_1 + \Delta s_2$$

(15.12)

or

$$\Delta s = \Delta s_1 (1 - \frac{1}{2} \text{ Ksin} \psi_s \Delta \psi_e - \Xi_e \text{ Ksin} \psi_s)$$

(15.13)

$$= -\text{ Kr} e [1 - \frac{1}{2} \text{ Ksin} \psi_e \Delta \psi_e](1 - \frac{1}{2} \text{ Ksin} \psi_s \Delta \psi_e) \Delta \psi_e - \Xi_e \text{ Ksin} \psi_s.$$ 

Keeping only terms to the second order, and approximating $\psi_e \approx \psi_s$,

$$\Delta s = -\text{ Kr} e \Delta \psi_e [1 - \text{ Ksin} \psi_s \Delta \psi_e] - \Xi_e \text{ Ksin} \psi_s.$$ 

(15.14)

This is the distance from $(z_e, r_e)$ at which an equivalent entrance curve would be placed if the idealized magnetic field operated near the profile. The results on the exit side are similar but differ in sign:

$$(\Delta s_1')_f \approx s (G f) - s (B f) \approx \text{ Kr} e (\psi_f - \psi_0) = \text{ Kr} e \Delta \psi_0.$$ 

(15.15)

*From Eqs. (13.2) and (15.8), $\Sigma_{-1} = \frac{3}{2} \frac{\Delta s_1}{d_e}$. In most cases, this ratio is less than 1, so that $\Delta \Xi$ is usually positive.

**i.e., when $\Delta \Xi > 0$.}
where \((\Delta s_1^t)f\) is a positive displacement of the profile from the real one \(b_f b_f\) to the virtual one \(b_f b_f'\). Further,

\[
(\Delta s_1^t)f = K_{ro} \Delta \psi_0 \left[1 + \frac{1}{2} (K \sin \psi_0) \Delta \psi_0\right] \tag{15.16}
\]

\[
(d \Xi_1^t)f = (s_B^t)f \frac{d \psi}{Kr} \tag{15.17}
\]

\[
\Xi_1^t = \frac{1}{2} (\Delta s_1^t)^2 \tag{15.18}
\]

Further, \(\Delta z^t > 0\), so

\[
\Delta z^t = -\Delta \Xi^t / \sin \psi^t, \tag{15.19}
\]

where \(\Delta \Xi^t = \Xi_0 - (\Xi_1^t)f\). Also \((\Delta s_2^t)f > 0\), where

\[
(\Delta s_2^t)f = \Delta z^t \sqrt{\sin^2 \psi^t - \Delta \psi_0 \sin \psi^t} = \Delta \Xi^t K \sin \psi^t \tag{15.20}
\]

and

\[
\Delta s^t = (\Delta s_1^t)f + (\Delta s_2^t)f = (\Delta s_1^t)f \left[1 + \frac{1}{2} K \sin \psi^t \Delta \psi_0\right] - \Xi_0 K \sin \psi^t \tag{15.21}
\]

With a linear approximation for the fringing field, and using the approximate results of Eq. (13.3), then, from Eq. (15.14),

\[
\Delta s = -\frac{1}{2} t_e \left(1 - \frac{1}{12} \frac{t_e}{\rho_e}\right) \tag{15.22}
\]

This is equivalent to the replacement of the arc BG by the straight line extension MG. To the same kind of approximation, the distance between the real boundary and the equivalent one is:

- Entrance side: \(|\Delta s + t_{1e}|\) \tag{15.23a}
- Exit side: \(\Delta s - t_{10}\) \tag{15.23b}

Thus far, consideration has been given to the construction of an equivalent boundary from a real boundary, given information on the quantities \((t_{1e}, \Delta \psi_e, \Xi_e)\) and \((t_{10}, \Delta \psi_0, \Xi_0)\). The problem is usually presented in the inverse form, namely, theoretical profiles are computed for the idealized field case [Eq. (1.1) inside the profiles and zero field outside] using the methods of Section #3 and/or Section #14, and it is desired to transform these profiles to real boundaries. The inverse problem may be solved by graphical construction, as in Fig. (15.1), or approximately, through the use of Eq. (15.23).
As an example of the magnitudes of the various quantities, some data is given for the 8-gap iron-core spectrometer referred to in Section #8:

\[ \rho_e = 270 \text{ mm}; \ \varphi = 10^\circ \text{ gap angle}; \ K = 0.6; \ \text{at } \psi_s = 120^\circ, \]

\[ t_e = 51 \text{ mm and } t_{1e} = 11 \text{ mm}. \]

From Eqs. (13.3) and (15.22), it follows that:

\[ \Delta \psi_e = 0.182 (10.4^\circ) \text{ and } \varepsilon_e = 3.1 \text{ mm ,} \]

\[ \Delta s = -25.5 (1 - 0.016) = -25 \text{ mm} \]

\[ t_{1e} + \Delta s = -14 \text{ mm} . \]

From the magnitude of \( \varepsilon_e/t_e \), it is evident that the approximation \( t_e/d_e \approx 1 \) is quite a good one, the difference from unity being only a few tenths of a per cent.

It may be seen, then, that the design of an instrument involves the following steps:

(i) Computation of optimum boundary curves, assuming that the deflection is given only by the idealized field of Eq. (1.1). These are the theoretical profiles.

(ii) The fringing-field deflection is estimated, and the theoretical profiles considered as equivalent boundaries. From these are computed the real boundaries.
### #16. CONSIDERATIONS IN THE CHOICE OF DESIGN PARAMETERS.

It is evident from the discussion in the previous sections that we have considerable freedom in the choice of design parameters even within our self-imposed restriction to spectrometers with source and image on the instrument axis. Such freedom allows for considerable adaptability of design for a variety of purposes. Even with a particular purpose in mind, we find that a number of different sets of parameters are possible.

We propose now to use our previous results to give criteria which will aid in choosing the best design for an instrument, given the purposes of use of the spectrometer. Because widely different uses of these instruments are possible, we have, correspondingly, widely different conditions imposed upon them,* with resulting very different choices of parameters (K-value, profile curves, etc). We shall obtain the general criteria by examining the influence of the various parameters of the instrument on its important properties.

**Ghost Peaks.** Because the elimination of these peaks over a wide $\psi_s$-range is possible only in instruments using zero-loop trajectories ($n_S = n_f = 0$), as described in Section #4, we restrict ourselves to this kind of instrument. We consider the advantage of avoiding the ghost-peak background to be overriding, although for some properties (see, e.g., the high dispersion in Section #6), trajectories with several loops might have some advantages. It might, for example, be feasible to make a low-transmission, high-resolution device with a narrow $\psi_s$-range and suitable baffles.

**Resolution.** At a given emission angle $\psi_s$, the expressions for the resolution are

\[
\delta_{rz} = \frac{\delta z_f}{\sigma_f} \quad \text{when } \delta z_f \geq D_z
\]

and

\[
\delta_{r_1} = \frac{\delta r_{1f}}{\sigma_f} \quad \text{when } \delta r_{1f} \geq D_{r_1}
\]

Thus, the resolution is determined by the dispersion (through $\sigma_f$ or $\sigma_f^2$) and the imaging properties of the instrument (through $\delta z_f$ or $\delta r_{1f}$).

The dispersions are given by

\[
\sigma_f = z_s \left(1 - \frac{z_f}{z_s}\right) G
\]

*As, for example, sizes and shapes of sources and detectors, range of emission angles, range of particle energy, luminosity, and resolution.
and
\[ \cdot R_f = -\cdot f \tan \cdot f \quad (6.36) \]

It is evident that the resolution of an instrument design improves, other things being equal, as the function \( 1 - \frac{z_f}{z_s} \) increases. Within a given instrument, the choice between the use of radial detection and axial (z-) detection depends upon the value of \( \tan^{-1} f \) and also upon the imaging properties with respect to \( r \) and \( z \).

The pertinent imaging properties of the instrument are given by the quantities \( \delta z_f = \Omega_f(\cdot f) \cdot z_s \) and/or \( u(\cdot f) \), since
\[ \delta z_f = \Omega_f(\cdot f) \cdot z_s \quad (5.10) \]
and
\[ \cdot r_{1f} = u(\cdot f) \cdot r_1 \quad (5.15b) \]

Because the resolution is proportional to \( \delta z_f \) and/or \( \cdot r_{1f} \) if \( \cdot z_f = D_z \) and/or \( \cdot r_{1f} \geq D_r \), it is useful to have low values of \( \Omega_f(\cdot f) \) and/or \( u(\cdot f) \). Knowing the \( z- \) and \( r- \) dispersions, and the imaging properties of the instrument, one can easily compute \( \cdot r_z \) and \( \cdot r_{1f} \) and decide whether axial or radial detection will provide better resolution.

In this discussion, we have omitted what may be called the second-order effects of the fringing field, i.e., the worsening of the resolution due to differences in the fringing-field effects on trajectories in the median and off-median planes. These effects give contributions which increase with \( \phi^2 \) and \( \phi^3 \) and make the choice of small values of \( \phi \) favorable (see Section #11).

Luminosity. This quantity is the product of the source area and the transmission. To maximize it, it must be possible to use a high transmission with large source dimensions. To achieve this, we shall require that there be (i) a minimization of the losses of particles due to source size with no fringing-field effect (Section #7) and (ii) minimization of losses due to the lens effect on a point source (Section #9). One should really require a minimum loss for the effect of the fringing field on the particles from a spread source, but the combined effects do not yield general expressions suitable for setting up criteria. We must then use the approximate criteria derived by separately maximizing transmission for the two conditions.

To minimize losses due to source size, we require that \( F_1R_2^*/z_s \) be as small as possible, with \( R_2^* \) the magnitude of farthest \( r_2 \)-extension of the source, since, from Eq. (7.8b),
\[ z_2 = 1 - \frac{F_1R_2^*}{\phi z_s} \]
which then comes closer to unity. To be able to make $R_f^*$ as large as possible, we then favor instruments with minimum values of $F_1$. In order not to lose particles at the detector, in the absence of fringing-field considerations we must make the detector aperture at least as large as the size of the image of the source.

The losses due to the lens effect on particles from a point source are of two kinds:

(i) Losses resulting from a divergent entrance lens, with consequent collision of the trajectories with the polefaces. These can be avoided by attempting to have only convergent lenses, which implies $W_g \leq 1$ or $\mu_g \leq 0$. The latter condition must be modified if one takes into account the "second-order" fringing-field effects (see Section #10) to read $\mu_g(\text{eff}) \leq 0$.

(ii) Losses due to stopping by the detector aperture can always be avoided by having the $r_2$-extension of the detector be at least equal to the image size generated by the lens effect, which is [from Eq. (9.17)]

$$2 \delta R_{2f} \text{(max)} = 2 \left[ v N_s + \left( z_f / z_s \right) N_f W_s \right] \frac{1}{2} \varphi \quad \text{convergent entrance lens}$$

$$2 \delta R_{2f} \text{(max)} = 2 \left[ v N_s + \left( z_f / z_s \right) N_f W_s \right] \left( \frac{1}{2} \varphi / W_s \right) \quad \text{divergent entrance lens}$$

Detector-size Considerations. Up to now, we have assumed that the detector size was dictated only by resolution or by luminosity considerations. These two conditions give opposite criteria: good resolution requires that the aperture be small: $D \leq \delta z_f$, $D_{r_1} \leq \delta r_{1f}$; high luminosity requires that the aperture be large: $D \geq \delta z_f$, $D_{r_1} \geq \delta r_{1f}$, and $D_{r_2} \geq \delta R_{2f}$.

Normally, one is interested in obtaining the desired resolution (or luminosity) associated with the best possible luminosity (or resolution), respectively. The conditions under which these combinations may be achieved are very different for (i) instruments in which the $r_2$-extent of the detector aperture ($D_{r_2}$) for one gap is the $r_1$-extent of the aperture ($D_{r_1}$) for another gap, and (ii) instruments in which the $D_{r_2}$ of one gap is not coupled to the $D_{r_1}$ of another gap (e.g., single-gap instruments, and multigap tilted-axis iron spectrometers).

In the coupled case, we must minimize the size of the image in all directions. Achieving this result sets two conditions:

(i) The image of a finite source without considering lens effects (see Section #5) must be small.

(ii) The line image (due to lens effect) of a point source must be small (see Section #9).
We note, as above, that the conditions are separated in this fashion, because no simple equations are available for treating the combined source size and lens effect.

From Condition (i), it follows that instruments are favored which minimize the \( v(\psi_s) \), \( Q_f(\psi_s) \), and/or \( u(\psi_s) \) functions, since

\[
\delta r_{2f} = v(\psi_s) \delta r_z \quad (5.20)
\]

\[
\delta z_f = Q_f(\psi_s) \delta z_s \quad (5.10)
\]

\[
\delta r_{1f} = u(\psi_s) \delta r_1 \quad . \quad (5.15b)
\]

Condition (ii) dictates that instruments be chosen with small values of \( [vN_s + (z_f/z_s) N_f W_s] \), because this determines the image size \( 2\delta R_{2f}(\text{max}) \), as indicated above. This may be accomplished by making \( \mu_s \) and \( \mu_f \) as small as possible, because \( N_s \) and \( N_f \) are proportional to \( \tan \mu_s \) and \( \tan \mu_f \), respectively. Of course, when the "second-order" fringing-field effects are included, this should be expressed as making \( \mu_s(\text{eff}) \) and \( \mu_f(\text{eff}) \) as close to zero as possible. It follows, then, that the effective profiles should be as close as possible to circles with centers at the object and image points, respectively. It seems apparent from the expression for \( 2\delta R_{2f}(\text{max}) \) above for a diverging entrance lens that this quantity may be minimized by making \( W_s \) large. The apparent advantage, however, is illusory, since the limitation of image size is attained only by limiting the range of effective \( \phi_s \), thereby cutting the transmission by exit aperture limiting. This may be more obviously seen by examining \( \eta_{23} \). The following expression covers both converging and diverging entrance lenses:

\[
\eta_{23} = \frac{2 R_d}{|vN_s + (z_f/z_s) N_f W_s| z_s \phi} \quad \text{for} \quad \eta_{23} \leq \eta_2 \quad . \quad (9.23)
\]

Other things being equal, the denominator is larger for a diverging lens than for a converging lens, because of the magnitude of \( W_s \).

In the uncoupled case, the required conditions of the coupled case may be dropped if they relate to the extent of the detector along the \( r_2 \)-axis. It should be pointed out, however, that there may be reasons extraneous to the optics of the instrument which would tend toward a limitation of the detector size. An example would be the possible increase of background with detector size.

From the considerations on resolution, luminosity, and detector size, we can conclude that, in order to choose an instrument, we should compute the above-mentioned characteristic functions for all possible instruments which satisfy the a priori requirements (e.g., range of emission angles) and choose the one which has
Maximum \( \left(1 - \frac{z_f}{z_s}\right) G \)

Minimum:

\( v(\psi_s); Q_{f}(\psi_s) \) and/or \( u(\psi_s); F_1 \) and \( \left[vN_s + \left(\frac{z_f}{z_s}\right) N_fW_s\right] \)

and, if possible, make \( \mu_s^{(\text{eff})} \leq 0 \), i.e., a convergent entrance lens if the lens power cannot be made zero. Because these are all functions of \( \psi_s \), it should be remarked that one is really trying to obtain maxima or minima of the appropriately averaged functions. Thus, one might weight the contribution of each emission angle by its transmission, or by its resolution, etc.

In general, the evaluation of optimum design parameters involves a tedious numerical evaluation of an instrument's properties for many alternative values of the parameters. In many cases, electronic computer calculations will be desirable for this purpose. For designs in which symmetrical instruments are appropriate, the computations have been made and presented in this report in the form of tables and graphs. It was not practical to do the same for asymmetric cases, because of the great multiplicity of possible designs. In addition to variable K, as in the symmetrical case, one must then also consider variable \( z_f/z_s \) and completely arbitrary entrance profiles.

By examining the pertinent formulae, it is also possible to determine what happens to the properties of an instrument when one scales up its dimensions by a scaling factor \( s \), leaving the source and detector sizes as they were, or what happens when these are also scaled up in size. Or we can determine what happens when one scales up the angular gap opening \( \phi \) by a scaling factor \( f \), but leaves the linear dimensions as they were. These effects are summarized in Table 16.1. We may remark at this point, that, although these conclusions follow from the above equations for this type of instrument, the results could also have been reached from more general considerations.

The table clearly shows that a compromise is necessary in the choice of \( z_s \) and \( \phi \), the choice being determined by what values of resolution and transmission are acceptable. The compromise can be made only by examining the properties of the instrument and making numerical comparisons.

In examining criteria for choosing an instrument design, we have omitted some very important practical considerations, such as the technical difficulty of construction, cost, reproducibility and linearity of measurement (e.g., iron core or iron-free), and effects of earth's magnetic field, which, in fact, play an important role in the design of an instrument. No general analysis is feasible here, because the problems are specific to the particular project in mind.
Table 16.1

EFFECT OF INSTRUMENT DESIGN ON RESOLUTION,
TRANSMISSION, AND SOURCE SIZE

<table>
<thead>
<tr>
<th></th>
<th>Scaling up everything</th>
<th>Scaling up instrument, but not source and detector</th>
<th>Scaling up ( \varphi ) only</th>
</tr>
</thead>
<tbody>
<tr>
<td>Resolution for a finite source (due to imaging properties, in absence of fringing field)</td>
<td>( \equiv )</td>
<td>( \frac{1}{s} )</td>
<td>( \equiv )</td>
</tr>
<tr>
<td>Transmission for a point source (lens effect)</td>
<td>( \equiv )</td>
<td>( \frac{1(a)}{s} )</td>
<td>( \frac{1(b)}{s} )</td>
</tr>
<tr>
<td>( \kappa ) where ( 1 - \kappa = ) transmission for a finite source in the absence of a lens effect</td>
<td>( \equiv )</td>
<td>( \frac{1(b)}{s} )</td>
<td>( \frac{1(b)}{s} )</td>
</tr>
<tr>
<td>Source size</td>
<td>( r s^2 )</td>
<td>( \equiv )</td>
<td>( \equiv )</td>
</tr>
</tbody>
</table>

(a) Provided that \( R_f \) is larger than the \( r_s \)-extent of the detector (\( D_r \)).

(b) Provided the detector aperture is equal to or larger in size than the image.

An interesting example of the application of the general criteria is the design of an electron electron multigap coincidence beta-ray spectrometer. In order to allow two spectrometers to be placed back-to-back, it is necessary to limit the range of emission angles to \( 90^\circ \leq \varphi \leq 180^\circ \). Further examination of the problem leads to a further reduction in the \( \varphi \)-range, because

1. The interference between the two instruments should be negligible when one focuses a high-energy electron in one spectrometer and a low-energy electron in the other one. This tends toward an increased distance between the spectrometers, which forces \( \varphi \sim 90^\circ \). The magnitude of the necessary displacement depends upon the magnitude of the fringing fields of the instrument. In general, the magnitude (and hence the effective extent) of the fringing field decreases with the gap angle \( \varphi \).

2. It is difficult to construct the instrument for \( \varphi \)-angles close to \( 180^\circ \), in addition, the transmission becomes negligible since (other things being equal) this varies as \( \sin \varphi \). Hence, \( \varphi \leq 180^\circ \) is appreciably smaller than \( 180^\circ \).
The acceptance range of $\psi_s$ then lies considerably within the boundaries: $90^\circ < \psi_s < 180^\circ$.

One may further require that the instruments be capable of being used in series as a single spectrometer. This would result in a reduction of the scattered-electron background and possibly in an improved resolution with practically no loss in transmission. An examination of the properties of various designs indicates that symmetric instruments are probably favored. Only with symmetric spectrometers ($n_g = n_f = 0$, $z_g = -z_f$, symmetrical profiles) can the spectrometers be operated backwards with the same properties (e.g., resolution, transmission, and imaging).

The requirement of highest possible transmission, essential for coincidence spectrometers, is fulfilled if we have nearly circular profile curves. This is achieved by using a $K$-value close to 0.6. This can be readily seen in Fig. (3.3), and also in Fig. (9.6), since the desired feature is that $W_s \cong 1$.

The $\nu(\psi_g)$, $Q_f(\psi_g)$, and $u(\psi_g)$ functions are all determined by the condition of symmetry of the instrument; they are constant and equal to -1, 1, and -1, respectively.

The remainder of the instrument design requires specification of particular conditions such as: desired source size, desired resolution, and iron or iron-free spectrometer. We shall not go into detail about these, since the previous considerations may be readily applied to the choice of the best design for an instrument of this type.

We may add some examples of instruments, for which one might arrive at quite different design parameters:

(i) Multigap coincidence spectrometers with no symmetry requirement. A possible choice which looks interesting would have $K = 0.6$, $n_g = n_f = 0$, $z_f = -2z_s$, $\psi_g(\text{max}) = 145^\circ$, $\psi_g(\text{min}) \cong 100^\circ$, and the entrance profile a circle of radius $z_g$.

(ii) Multigap or one-gap single spectrometers with maximum possible transmission. There is no restriction of emission angles to values larger than $90^\circ$.

(iii) Multigap or one-gap single spectrometers with maximum possible resolution.
ACKNOWLEDGMENTS

We would like to thank P. P. Day, M. S. Freedman, J. J. Peyre, F. T. Porter and F. Wagner, Jr. for many suggestions and enlightening discussions. We are also grateful to R. G. Scott, J. L. Lerner, K. E. Hillstrom and R. F. King for aid in computation and programming.

REFERENCES


APPENDIX I

DERIVATION OF BESSEL FUNCTION EXPANSIONS

Equations (1.19) and (1.31) are demonstrated in the following manner. As might be expected from the cylindrical symmetry of the system, an expansion in one form of the cylindrical functions - the Bessel functions - is involved. The fundamental formula (or the closely related sine and cosine forms) from which the derivation is made is found in a number of texts and compilations dealing with the Bessel functions or with the higher transcendental functions.

We start with:

\[ e^{iz\cos \theta} = J_0(z) + 2 \sum_{n=1}^{\infty} i^n J_n(z) \cos n\theta. \quad (A.1.1) \]

Integrating, we have

\[ \int_{\pi}^{\psi} e^{iz\cos \theta} \, d\theta = J_0(z)[\psi - \pi] + 2 \sum_{n=1}^{\infty} i^n J_n(z) \int_{\pi}^{\psi} \cos n\theta \, d\theta \]

\[ = J_0(z)[\psi - \pi] + 2 \sum_{n=1}^{\infty} \frac{i^n}{n} J_n(z) \sin \psi. \quad (A.1.2) \]

If we let \( iz = K \), then \( z = -iK \). Utilizing the fact that \( J_n(-iK) = (-1)^n J_n(iK) \), we prove Eq. (1.31):

\[ V = \int_{\pi}^{\psi} e^{K\cos \theta} \, d\theta = J_0(iK)[\psi - \pi] + 2 \sum_{n=1}^{\infty} \frac{(-i)^n}{n} J_n(iK) \sin \psi \quad (A.1.3) \]

To prove Eq. (1.19), we now let \( iz = -K \); then \( z = iK \). Let

\[ W = \int_{\pi}^{\psi} e^{-K\cos \theta} \, d\theta = J_0(iK)[\psi - \pi] + 2 \sum_{n=1}^{\infty} \frac{i^n}{n} J_n(iK) \sin \psi. \quad (A.1.4) \]

Then

\[ \frac{\partial W}{\partial z} = \frac{1}{i} \frac{\partial W}{\partial K} = \frac{i}{i} \int_{\pi}^{\psi} \cos \theta \, e^{-K\cos \theta} \, d\theta \]

\[ = [\psi - \pi] \frac{dJ_0(z)}{dz} + 2 \sum_{n=1}^{\infty} \frac{i^n}{n} \sin n\psi \frac{dJ_n(z)}{dz}. \quad (A.1.5) \]
Noting that $i^{n+1} = -i^n$, and using the well-known recurrence relationships

$$\frac{dJ_0(x)}{dx} = -J_1(x) \quad \text{and} \quad \frac{dJ_n(x)}{dx} = \frac{i}{2} [J_{n-1}(x) + J_{n+1}(x)]$$

we have

$$U = \int_{\pi}^{\psi} \cos \theta \ e^{-K \cos \theta} d\theta = -i J_1(iK)[\pi - \psi]$$

$$+ \sum_{n=1}^{\infty} \frac{i^n}{n} \left[ J_{n-1}(iK) - J_{n+1}(iK) \right] \sin \psi$$

Similar successive partial differentiations may be used to expand the integrals involved in evaluating the higher terms in Eq (11) and in evaluating $\partial U/\partial K$ [Eq (6.9)].
APPENDIX II

TABLES OF FUNCTIONS FOR IDEALIZED FIELD SPECTROMETERS

Some of the functions required for computation of spectrometer properties in the absence of fringing fields have been calculated using a 704-computer.* Floating decimal notation is used in all the tables. All results are given as numbers with magnitude between 0 and 1; the decimal point is given following the number. Thus: E-01 following the number signifies multiplication by $10^{-1}$; E 00 means x1; E 01 means x10; etc.

Table II.1. $U(K, \psi)$, computed from Eq. (1.18) provides values for $\psi = 0^\circ (2^\circ) 180^\circ$ and $K = 0.1(0.05) 0.4(0.02) 0.5(0.01) 0.8(0.02) 1.0$. The region $180^\circ < \psi < 360^\circ$ may be determined from the relation calculated from Eq. (1.18) or Eq. (1.19).

$$U(K, -\psi) = U(K, 2\pi - \psi) = -U(K, \psi) \quad \text{(A-II.1)}$$

Table II.2. $V(K, \psi)$, computed from the integral in Eq. (1.31) provides values for $\psi = 0^\circ (5^\circ) 180^\circ$ and $K = 0.1(0.05) 0.5(0.02) 0.9(0.05) 1.0$. The region $180^\circ < \psi < 360^\circ$ may be determined from the relation

$$V(K, -\psi) = V(K, 2\pi - \psi) = -V(K, \psi) \quad \text{(A-II.2)}$$

Table II.3. The coordinates of the profiles of a symmetric spectrometer have been calculated from Eqs. (2.1a), (2.1b) and (3.5), taking the dimensions in units of $x_s$ and setting $n_s = 0$. In these units, $r_c = r_s$. The coordinates are computed for $\psi = \psi_c (5^\circ) 180^\circ$ and for $K = 0.1(0.1) 0.4(0.05) 0.5(0.02) 0.9(0.05) 1.0$, where $\psi_c$ is the closest $\psi_s$-value to the critical $\psi_s$-angle, and evidently varies with $K$.

Table II.4. The orbit rotation function $F_1(K, \psi_s)$ is calculated from Eq. (5.19c), using the symmetric spectrometer profiles of Table II.3 [and Fig. (3.3)] for $n_s = 0$. The function is computed for $\psi_s = \psi_s^c (10^\circ) 180^\circ$ and $K = 0.1(0.1) 0.4(0.05) 0.5(0.06, 0.04** 0.9(0.1) 1.0$. Of the angles listed, $\psi_s^c$ is the closest to the critical $\psi_s$-angle and varies with $K$; it may not be identical to the $\psi_s^c$ of Table II.3 because of the difference in $\Delta \psi_s$.

Table II.5. The dispersion function $G$ is calculated from Eq. (6.11) for the case of the symmetric spectrometer profiles of Table II.3 [and Fig. (3.3)] with $n_s = 0$. The function is computed for $\psi_s = \psi_s^c (5^\circ) 180^\circ$ and $K = 0.2. 0.4(0.05) 0.5(0.02) 0.9(0.05) 1.0$. $\psi_s^c$ is defined as in Tables II.3 and II.4.

*We are indebted to Richard King and Kenneth Hillstrom of the Applied Mathematics Division for programming this computation.

**Alternate steps of 0.06 and 0.04
Table II.6. The function $\frac{\partial U}{\partial K}$ is computed from Eq. (6.9a) by numerical integration of

$$\frac{\partial U}{\partial K} = -\int_{0}^{\pi} \cos^2 \psi e^{-K \cos \psi} d\psi .$$

Values are provided for $\psi = 0^\circ(2^\circ)180^\circ$ and for $K = 0.1(0.05)0.5(0.01)0.6(0.02)0.9(0.5)1.0$. 
Table A-II.1

Trajectory Function $U(K, \psi)$
<table>
<thead>
<tr>
<th>( K )</th>
<th>( \Delta \lambda(n+\frac{1}{2}) )</th>
<th>( \Delta \lambda(n+1) )</th>
<th>( \Delta \lambda(n+2) )</th>
<th>( \Delta \lambda(n-\frac{1}{2}) )</th>
<th>( \Delta \lambda(n-1) )</th>
<th>( \Delta \lambda(n-2) )</th>
<th>( \Delta \lambda(n-\frac{1}{2}) )</th>
<th>( \Delta \lambda(n-1) )</th>
<th>( \Delta \lambda(n-2) )</th>
<th>( \Delta \lambda(n-\frac{1}{2}) )</th>
<th>( \Delta \lambda(n-1) )</th>
<th>( \Delta \lambda(n-2) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.28485</td>
<td>0.30135</td>
<td>0.30592</td>
<td>0.30780</td>
<td>0.30963</td>
<td>0.31141</td>
<td>0.31313</td>
<td>0.31478</td>
<td>0.31636</td>
<td>0.31787</td>
<td>0.31931</td>
<td>0.32069</td>
</tr>
<tr>
<td>2</td>
<td>0.30580</td>
<td>0.32135</td>
<td>0.32592</td>
<td>0.32780</td>
<td>0.32963</td>
<td>0.33141</td>
<td>0.33313</td>
<td>0.33478</td>
<td>0.33636</td>
<td>0.33787</td>
<td>0.33931</td>
<td>0.34069</td>
</tr>
<tr>
<td>3</td>
<td>0.32679</td>
<td>0.34235</td>
<td>0.34692</td>
<td>0.34880</td>
<td>0.35063</td>
<td>0.35241</td>
<td>0.35413</td>
<td>0.35578</td>
<td>0.35736</td>
<td>0.35887</td>
<td>0.36031</td>
<td>0.36169</td>
</tr>
<tr>
<td>4</td>
<td>0.34778</td>
<td>0.36335</td>
<td>0.36792</td>
<td>0.36980</td>
<td>0.37163</td>
<td>0.37341</td>
<td>0.37513</td>
<td>0.37678</td>
<td>0.37836</td>
<td>0.37987</td>
<td>0.38131</td>
<td>0.38269</td>
</tr>
<tr>
<td>5</td>
<td>0.36876</td>
<td>0.38435</td>
<td>0.38892</td>
<td>0.39080</td>
<td>0.39263</td>
<td>0.39441</td>
<td>0.39613</td>
<td>0.39778</td>
<td>0.39936</td>
<td>0.40087</td>
<td>0.40231</td>
<td>0.40369</td>
</tr>
<tr>
<td>6</td>
<td>0.38974</td>
<td>0.40535</td>
<td>0.40992</td>
<td>0.41180</td>
<td>0.41363</td>
<td>0.41541</td>
<td>0.41713</td>
<td>0.41878</td>
<td>0.42036</td>
<td>0.42187</td>
<td>0.42331</td>
<td>0.42469</td>
</tr>
<tr>
<td>7</td>
<td>0.41072</td>
<td>0.42635</td>
<td>0.43092</td>
<td>0.43280</td>
<td>0.43463</td>
<td>0.43641</td>
<td>0.43813</td>
<td>0.43978</td>
<td>0.44136</td>
<td>0.44287</td>
<td>0.44431</td>
<td>0.44569</td>
</tr>
<tr>
<td>8</td>
<td>0.43170</td>
<td>0.44735</td>
<td>0.45192</td>
<td>0.45380</td>
<td>0.45563</td>
<td>0.45741</td>
<td>0.45913</td>
<td>0.46078</td>
<td>0.46236</td>
<td>0.46387</td>
<td>0.46531</td>
<td>0.46669</td>
</tr>
<tr>
<td>9</td>
<td>0.45268</td>
<td>0.46835</td>
<td>0.47292</td>
<td>0.47480</td>
<td>0.47663</td>
<td>0.47841</td>
<td>0.48013</td>
<td>0.48178</td>
<td>0.48336</td>
<td>0.48487</td>
<td>0.48631</td>
<td>0.48769</td>
</tr>
<tr>
<td>10</td>
<td>0.47366</td>
<td>0.48935</td>
<td>0.49392</td>
<td>0.49580</td>
<td>0.49763</td>
<td>0.49941</td>
<td>0.50113</td>
<td>0.50278</td>
<td>0.50436</td>
<td>0.50587</td>
<td>0.50731</td>
<td>0.50869</td>
</tr>
<tr>
<td>11</td>
<td>0.49464</td>
<td>0.51035</td>
<td>0.51492</td>
<td>0.51680</td>
<td>0.51863</td>
<td>0.52041</td>
<td>0.52213</td>
<td>0.52378</td>
<td>0.52536</td>
<td>0.52687</td>
<td>0.52831</td>
<td>0.52969</td>
</tr>
<tr>
<td>12</td>
<td>0.51562</td>
<td>0.53135</td>
<td>0.53592</td>
<td>0.53780</td>
<td>0.53963</td>
<td>0.54141</td>
<td>0.54313</td>
<td>0.54478</td>
<td>0.54636</td>
<td>0.54787</td>
<td>0.54931</td>
<td>0.55069</td>
</tr>
<tr>
<td>13</td>
<td>0.53660</td>
<td>0.55235</td>
<td>0.55692</td>
<td>0.55880</td>
<td>0.56063</td>
<td>0.56241</td>
<td>0.56413</td>
<td>0.56578</td>
<td>0.56736</td>
<td>0.56887</td>
<td>0.57031</td>
<td>0.57169</td>
</tr>
<tr>
<td>14</td>
<td>0.55758</td>
<td>0.57335</td>
<td>0.57792</td>
<td>0.57980</td>
<td>0.58163</td>
<td>0.58341</td>
<td>0.58513</td>
<td>0.58678</td>
<td>0.58836</td>
<td>0.58987</td>
<td>0.59131</td>
<td>0.59269</td>
</tr>
<tr>
<td>15</td>
<td>0.57856</td>
<td>0.59435</td>
<td>0.59892</td>
<td>0.60080</td>
<td>0.60263</td>
<td>0.60441</td>
<td>0.60613</td>
<td>0.60778</td>
<td>0.60936</td>
<td>0.61087</td>
<td>0.61231</td>
<td>0.61369</td>
</tr>
<tr>
<td>16</td>
<td>0.60054</td>
<td>0.61635</td>
<td>0.62092</td>
<td>0.62280</td>
<td>0.62463</td>
<td>0.62641</td>
<td>0.62813</td>
<td>0.62978</td>
<td>0.63136</td>
<td>0.63287</td>
<td>0.63431</td>
<td>0.63569</td>
</tr>
<tr>
<td>17</td>
<td>0.62152</td>
<td>0.63735</td>
<td>0.64192</td>
<td>0.64380</td>
<td>0.64563</td>
<td>0.64741</td>
<td>0.64913</td>
<td>0.65078</td>
<td>0.65236</td>
<td>0.65387</td>
<td>0.65531</td>
<td>0.65669</td>
</tr>
<tr>
<td>18</td>
<td>0.64250</td>
<td>0.65835</td>
<td>0.66292</td>
<td>0.66480</td>
<td>0.66663</td>
<td>0.66841</td>
<td>0.67013</td>
<td>0.67178</td>
<td>0.67336</td>
<td>0.67487</td>
<td>0.67631</td>
<td>0.67769</td>
</tr>
<tr>
<td>19</td>
<td>0.66348</td>
<td>0.67935</td>
<td>0.68392</td>
<td>0.68580</td>
<td>0.68763</td>
<td>0.68941</td>
<td>0.69113</td>
<td>0.69278</td>
<td>0.69436</td>
<td>0.69587</td>
<td>0.69731</td>
<td>0.69869</td>
</tr>
<tr>
<td>20</td>
<td>0.68446</td>
<td>0.70035</td>
<td>0.70492</td>
<td>0.70680</td>
<td>0.70863</td>
<td>0.71041</td>
<td>0.71213</td>
<td>0.71378</td>
<td>0.71536</td>
<td>0.71687</td>
<td>0.71831</td>
<td>0.71969</td>
</tr>
</tbody>
</table>

TRAJECTORY FUNCTIONS

**U(k,\( \psi \))**

- **K**
- **\( \Delta \lambda(n+\frac{1}{2}) \)**
- **\( \Delta \lambda(n+1) \)**
- **\( \Delta \lambda(n+2) \)**
- **\( \Delta \lambda(n-\frac{1}{2}) \)**
- **\( \Delta \lambda(n-1) \)**
- **\( \Delta \lambda(n-2) \)**
- **\( \Delta \lambda(n-\frac{1}{2}) \)**
- **\( \Delta \lambda(n-1) \)**
- **\( \Delta \lambda(n-2) \)**

224
<table>
<thead>
<tr>
<th>$\Psi$</th>
<th>$0.113868 , \Omega_1$</th>
<th>$0.113868 , \Omega_2$</th>
<th>$0.113868 , \Omega_3$</th>
<th>$0.113868 , \Omega_4$</th>
<th>$0.113868 , \Omega_5$</th>
<th>$0.113868 , \Omega_6$</th>
<th>$0.113868 , \Omega_7$</th>
<th>$0.113868 , \Omega_8$</th>
<th>$0.113868 , \Omega_9$</th>
<th>$0.113868 , \Omega_{10}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0.113868$</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
</tr>
<tr>
<td>$0.113868$</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
</tr>
<tr>
<td>$0.113868$</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
</tr>
<tr>
<td>$0.113868$</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
</tr>
<tr>
<td>$0.113868$</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
</tr>
<tr>
<td>$0.113868$</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
</tr>
<tr>
<td>$0.113868$</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
</tr>
<tr>
<td>$0.113868$</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
</tr>
<tr>
<td>$0.113868$</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
</tr>
<tr>
<td>$0.113868$</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
</tr>
<tr>
<td>$0.113868$</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
<td>0.113868</td>
</tr>
</tbody>
</table>

**Table:** Trajectory Functions

- **U(\Psi,\psi):**
  - The table lists the values of $U(\Psi,\psi)$ for different values of $\psi$.
  - Each row corresponds to a specific value of $\psi$, and each column lists the corresponding $U(\Psi,\psi)$ values.
  - The values are formatted with one decimal place for readability.
<table>
<thead>
<tr>
<th>( K )</th>
<th>( \psi_{90\text{o}} \text{ ppm} )</th>
<th>( \psi_{90\text{o}} \text{ ppm} )</th>
<th>( \psi_{180\text{o}} \text{ ppm} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.1688 E 01</td>
<td>0.1717 E 01</td>
<td>0.1775 E 01</td>
</tr>
<tr>
<td>2</td>
<td>0.1701 E 01</td>
<td>0.1748 E 01</td>
<td>0.1788 E 01</td>
</tr>
<tr>
<td>3</td>
<td>0.1715 E 01</td>
<td>0.1759 E 01</td>
<td>0.1801 E 01</td>
</tr>
<tr>
<td>4</td>
<td>0.1729 E 01</td>
<td>0.1771 E 01</td>
<td>0.1814 E 01</td>
</tr>
<tr>
<td>5</td>
<td>0.1743 E 01</td>
<td>0.1785 E 01</td>
<td>0.1826 E 01</td>
</tr>
<tr>
<td>6</td>
<td>0.1757 E 01</td>
<td>0.1797 E 01</td>
<td>0.1839 E 01</td>
</tr>
<tr>
<td>7</td>
<td>0.1770 E 01</td>
<td>0.1810 E 01</td>
<td>0.1850 E 01</td>
</tr>
<tr>
<td>8</td>
<td>0.1783 E 01</td>
<td>0.1821 E 01</td>
<td>0.1860 E 01</td>
</tr>
<tr>
<td>9</td>
<td>0.1796 E 01</td>
<td>0.1830 E 01</td>
<td>0.1867 E 01</td>
</tr>
<tr>
<td>10</td>
<td>0.1803 E 01</td>
<td>0.1836 E 01</td>
<td>0.1872 E 01</td>
</tr>
<tr>
<td>11</td>
<td>0.1809 E 01</td>
<td>0.1840 E 01</td>
<td>0.1876 E 01</td>
</tr>
<tr>
<td>12</td>
<td>0.1815 E 01</td>
<td>0.1844 E 01</td>
<td>0.1879 E 01</td>
</tr>
<tr>
<td>13</td>
<td>0.1821 E 01</td>
<td>0.1848 E 01</td>
<td>0.1882 E 01</td>
</tr>
<tr>
<td>14</td>
<td>0.1826 E 01</td>
<td>0.1852 E 01</td>
<td>0.1884 E 01</td>
</tr>
</tbody>
</table>

**Note:** The table represents trajectory functions with specific values for different \( K \) values. The values are in ppm (parts per million).
Table A-II.2

Trajectory Function $V(K, \psi)$
<table>
<thead>
<tr>
<th>( K )</th>
<th>( K \times \sigma )</th>
<th>( N \text{DEEM} )</th>
<th>( N \text{DEEM} \times \sigma )</th>
<th>( N \text{DEEM} \times \sigma )</th>
<th>( N \text{DEEM} \times \sigma )</th>
<th>( N \text{DEEM} \times \sigma )</th>
<th>( N \text{DEEM} \times \sigma )</th>
<th>( N \text{DEEM} \times \sigma )</th>
<th>( N \text{DEEM} \times \sigma )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.6757E+00</td>
<td>0.7850E+00</td>
<td>0.8900E+00</td>
<td>0.9950E+00</td>
<td>0.1096E+00</td>
<td>0.1201E+00</td>
<td>0.1306E+00</td>
<td>0.1411E+00</td>
<td>0.1516E+00</td>
</tr>
<tr>
<td>1</td>
<td>0.1237E+01</td>
<td>0.2474E+01</td>
<td>0.3711E+01</td>
<td>0.4948E+01</td>
<td>0.6185E+01</td>
<td>0.7422E+01</td>
<td>0.8659E+01</td>
<td>0.9896E+01</td>
<td>0.1113E+02</td>
</tr>
<tr>
<td>2</td>
<td>0.1896E+02</td>
<td>0.3792E+02</td>
<td>0.5688E+02</td>
<td>0.7584E+02</td>
<td>0.9480E+02</td>
<td>0.1137E+02</td>
<td>0.1326E+02</td>
<td>0.1516E+02</td>
<td>0.1705E+02</td>
</tr>
<tr>
<td>3</td>
<td>0.2555E+03</td>
<td>0.5110E+03</td>
<td>0.7665E+03</td>
<td>0.1022E+03</td>
<td>0.1278E+03</td>
<td>0.1534E+03</td>
<td>0.1790E+03</td>
<td>0.2046E+03</td>
<td>0.2302E+03</td>
</tr>
<tr>
<td>4</td>
<td>0.3214E+04</td>
<td>0.6428E+04</td>
<td>0.9642E+04</td>
<td>0.1285E+04</td>
<td>0.1606E+04</td>
<td>0.1927E+04</td>
<td>0.2248E+04</td>
<td>0.2569E+04</td>
<td>0.2890E+04</td>
</tr>
<tr>
<td>5</td>
<td>0.3873E+05</td>
<td>0.7746E+05</td>
<td>0.1161E+05</td>
<td>0.1548E+05</td>
<td>0.1935E+05</td>
<td>0.2322E+05</td>
<td>0.2709E+05</td>
<td>0.3097E+05</td>
<td>0.3484E+05</td>
</tr>
<tr>
<td>6</td>
<td>0.4531E+06</td>
<td>0.9063E+06</td>
<td>0.1359E+06</td>
<td>0.1811E+06</td>
<td>0.2264E+06</td>
<td>0.2717E+06</td>
<td>0.3169E+06</td>
<td>0.3622E+06</td>
<td>0.4074E+06</td>
</tr>
<tr>
<td>7</td>
<td>0.5189E+07</td>
<td>0.1037E+07</td>
<td>0.1555E+07</td>
<td>0.2073E+07</td>
<td>0.2591E+07</td>
<td>0.3109E+07</td>
<td>0.3627E+07</td>
<td>0.4145E+07</td>
<td>0.4663E+07</td>
</tr>
<tr>
<td>8</td>
<td>0.5847E+08</td>
<td>0.1169E+08</td>
<td>0.1753E+08</td>
<td>0.2337E+08</td>
<td>0.2921E+08</td>
<td>0.3505E+08</td>
<td>0.4089E+08</td>
<td>0.4673E+08</td>
<td>0.5257E+08</td>
</tr>
<tr>
<td>9</td>
<td>0.6505E+09</td>
<td>0.1301E+09</td>
<td>0.1901E+09</td>
<td>0.2500E+09</td>
<td>0.3099E+09</td>
<td>0.3699E+09</td>
<td>0.4299E+09</td>
<td>0.4899E+09</td>
<td>0.5498E+09</td>
</tr>
<tr>
<td>10</td>
<td>0.7163E+10</td>
<td>0.1433E+10</td>
<td>0.2033E+10</td>
<td>0.2633E+10</td>
<td>0.3232E+10</td>
<td>0.3832E+10</td>
<td>0.4432E+10</td>
<td>0.5032E+10</td>
<td>0.5632E+10</td>
</tr>
<tr>
<td>11</td>
<td>0.7821E+11</td>
<td>0.1565E+11</td>
<td>0.2165E+11</td>
<td>0.2765E+11</td>
<td>0.3364E+11</td>
<td>0.3964E+11</td>
<td>0.4564E+11</td>
<td>0.5164E+11</td>
<td>0.5764E+11</td>
</tr>
<tr>
<td>12</td>
<td>0.8479E+12</td>
<td>0.1697E+12</td>
<td>0.2297E+12</td>
<td>0.2897E+12</td>
<td>0.3496E+12</td>
<td>0.4096E+12</td>
<td>0.4696E+12</td>
<td>0.5296E+12</td>
<td>0.5896E+12</td>
</tr>
<tr>
<td>13</td>
<td>0.9137E+13</td>
<td>0.1829E+13</td>
<td>0.2429E+13</td>
<td>0.3029E+13</td>
<td>0.3628E+13</td>
<td>0.4228E+13</td>
<td>0.4828E+13</td>
<td>0.5428E+13</td>
<td>0.6028E+13</td>
</tr>
<tr>
<td>14</td>
<td>0.9795E+14</td>
<td>0.1961E+14</td>
<td>0.2561E+14</td>
<td>0.3161E+14</td>
<td>0.3760E+14</td>
<td>0.4360E+14</td>
<td>0.4960E+14</td>
<td>0.5560E+14</td>
<td>0.6160E+14</td>
</tr>
<tr>
<td>15</td>
<td>0.1045E+15</td>
<td>0.2093E+15</td>
<td>0.2693E+15</td>
<td>0.3293E+15</td>
<td>0.3892E+15</td>
<td>0.4492E+15</td>
<td>0.5092E+15</td>
<td>0.5692E+15</td>
<td>0.6292E+15</td>
</tr>
<tr>
<td>16</td>
<td>0.1111E+16</td>
<td>0.2225E+16</td>
<td>0.2825E+16</td>
<td>0.3425E+16</td>
<td>0.4024E+16</td>
<td>0.4624E+16</td>
<td>0.5224E+16</td>
<td>0.5824E+16</td>
<td>0.6424E+16</td>
</tr>
<tr>
<td>17</td>
<td>0.1177E+17</td>
<td>0.2357E+17</td>
<td>0.2957E+17</td>
<td>0.3557E+17</td>
<td>0.4156E+17</td>
<td>0.4756E+17</td>
<td>0.5356E+17</td>
<td>0.5956E+17</td>
<td>0.6556E+17</td>
</tr>
<tr>
<td>18</td>
<td>0.1243E+18</td>
<td>0.2489E+18</td>
<td>0.3089E+18</td>
<td>0.3689E+18</td>
<td>0.4288E+18</td>
<td>0.4888E+18</td>
<td>0.5488E+18</td>
<td>0.6088E+18</td>
<td>0.6688E+18</td>
</tr>
</tbody>
</table>

Note: The table represents trajectory functions and their corresponding values. Each row shows the value of a function for a given index \( K \). The values are in scientific notation, indicating the magnitude of the function at that point.
### Trajectory Functions

**$V(K, \psi)$**

<table>
<thead>
<tr>
<th>$K$</th>
<th>$0.90000E\ 00$</th>
<th>$0.95000E\ 00$</th>
<th>$0.10000E\ 01$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\psi$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>$-0.38107E\ 01$</td>
<td>$-0.38914E\ 01$</td>
<td>$-0.39775E\ 01$</td>
</tr>
<tr>
<td>5</td>
<td>$-0.35633E\ 01$</td>
<td>$-0.36660E\ 01$</td>
<td>$-0.37405E\ 01$</td>
</tr>
<tr>
<td>10</td>
<td>$-0.33834E\ 01$</td>
<td>$-0.34423E\ 01$</td>
<td>$-0.35054E\ 01$</td>
</tr>
<tr>
<td>15</td>
<td>$-0.31733E\ 01$</td>
<td>$-0.32217E\ 01$</td>
<td>$-0.32738E\ 01$</td>
</tr>
<tr>
<td>20</td>
<td>$-0.29675E\ 01$</td>
<td>$-0.30558E\ 01$</td>
<td>$-0.30474E\ 01$</td>
</tr>
<tr>
<td>25</td>
<td>$-0.27671E\ 01$</td>
<td>$-0.27960E\ 01$</td>
<td>$-0.28760E\ 01$</td>
</tr>
<tr>
<td>30</td>
<td>$-0.25733E\ 01$</td>
<td>$-0.25934E\ 01$</td>
<td>$-0.26158E\ 01$</td>
</tr>
<tr>
<td>35</td>
<td>$-0.23869E\ 01$</td>
<td>$-0.23989E\ 01$</td>
<td>$-0.24130E\ 01$</td>
</tr>
<tr>
<td>40</td>
<td>$-0.22987E\ 01$</td>
<td>$-0.22135E\ 01$</td>
<td>$-0.22201E\ 01$</td>
</tr>
<tr>
<td>45</td>
<td>$-0.21126E\ 01$</td>
<td>$-0.20377E\ 01$</td>
<td>$-0.20477E\ 01$</td>
</tr>
<tr>
<td>50</td>
<td>$-0.19420E\ 01$</td>
<td>$-0.18720E\ 01$</td>
<td>$-0.18662E\ 01$</td>
</tr>
<tr>
<td>55</td>
<td>$-0.17808E\ 01$</td>
<td>$-0.17164E\ 01$</td>
<td>$-0.17058E\ 01$</td>
</tr>
<tr>
<td>60</td>
<td>$-0.16265E\ 01$</td>
<td>$-0.15710E\ 01$</td>
<td>$-0.15565E\ 01$</td>
</tr>
<tr>
<td>65</td>
<td>$-0.14843E\ 01$</td>
<td>$-0.14356E\ 01$</td>
<td>$-0.14180E\ 01$</td>
</tr>
<tr>
<td>70</td>
<td>$-0.13511E\ 01$</td>
<td>$-0.13101E\ 01$</td>
<td>$-0.12900E\ 01$</td>
</tr>
<tr>
<td>75</td>
<td>$-0.12167E\ 01$</td>
<td>$-0.11939E\ 01$</td>
<td>$-0.11721E\ 01$</td>
</tr>
<tr>
<td>80</td>
<td>$-0.10806E\ 01$</td>
<td>$-0.10867E\ 01$</td>
<td>$-0.10637E\ 01$</td>
</tr>
<tr>
<td>85</td>
<td>$-0.10125E\ 00$</td>
<td>$-0.09879E\ 02$</td>
<td>$-0.09642E\ 00$</td>
</tr>
<tr>
<td>90</td>
<td>$-0.09216E\ 00$</td>
<td>$-0.08694E\ 00$</td>
<td>$-0.08709E\ 00$</td>
</tr>
<tr>
<td>95</td>
<td>$-0.08377E\ 00$</td>
<td>$-0.08131E\ 00$</td>
<td>$-0.07895E\ 00$</td>
</tr>
<tr>
<td>100</td>
<td>$-0.07614E\ 00$</td>
<td>$-0.07308E\ 00$</td>
<td>$-0.07129E\ 00$</td>
</tr>
<tr>
<td>105</td>
<td>$-0.06830E\ 00$</td>
<td>$-0.06501E\ 00$</td>
<td>$-0.06425E\ 00$</td>
</tr>
<tr>
<td>110</td>
<td>$-0.062170E\ 00$</td>
<td>$-0.05994E\ 00$</td>
<td>$-0.05779E\ 00$</td>
</tr>
<tr>
<td>115</td>
<td>$-0.05598E\ 00$</td>
<td>$-0.05387E\ 00$</td>
<td>$-0.05184E\ 00$</td>
</tr>
<tr>
<td>120</td>
<td>$-0.05023E\ 00$</td>
<td>$-0.04874E\ 00$</td>
<td>$-0.04634E\ 00$</td>
</tr>
<tr>
<td>125</td>
<td>$-0.04440E\ 00$</td>
<td>$-0.04301E\ 00$</td>
<td>$-0.04124E\ 00$</td>
</tr>
<tr>
<td>130</td>
<td>$-0.03979E\ 00$</td>
<td>$-0.03810E\ 00$</td>
<td>$-0.03649E\ 00$</td>
</tr>
<tr>
<td>135</td>
<td>$-0.03564E\ 00$</td>
<td>$-0.03351E\ 00$</td>
<td>$-0.03204E\ 00$</td>
</tr>
<tr>
<td>140</td>
<td>$-0.03045E\ 00$</td>
<td>$-0.02918E\ 00$</td>
<td>$-0.02787E\ 00$</td>
</tr>
<tr>
<td>145</td>
<td>$-0.02670E\ 00$</td>
<td>$-0.02506E\ 00$</td>
<td>$-0.02392E\ 00$</td>
</tr>
<tr>
<td>150</td>
<td>$-0.02218E\ 00$</td>
<td>$-0.02115E\ 00$</td>
<td>$-0.02016E\ 00$</td>
</tr>
<tr>
<td>155</td>
<td>$-0.01855E\ 00$</td>
<td>$-0.01739E\ 00$</td>
<td>$-0.01657E\ 00$</td>
</tr>
<tr>
<td>160</td>
<td>$-0.01454E\ 00$</td>
<td>$-0.01376E\ 00$</td>
<td>$-0.01310E\ 00$</td>
</tr>
<tr>
<td>165</td>
<td>$-0.01075E\ 00$</td>
<td>$-0.01023E\ 00$</td>
<td>$-0.00974E\ 00$</td>
</tr>
<tr>
<td>170</td>
<td>$-0.00719E\ 00$</td>
<td>$-0.00678E\ 00$</td>
<td>$-0.00645E\ 00$</td>
</tr>
<tr>
<td>175</td>
<td>$-0.003562E\ 00$</td>
<td>$-0.003379E\ 00$</td>
<td>$-0.003214E\ 00$</td>
</tr>
<tr>
<td>180</td>
<td>$0.$</td>
<td>$0.$</td>
<td>$0.$</td>
</tr>
</tbody>
</table>
Table A-II.3

Magnetic Field Profiles for Symmetrical Spectrometers
<table>
<thead>
<tr>
<th>$\psi$</th>
<th>$r_0$</th>
<th>$z_0$</th>
<th>$\frac{r_0}{l_0}$</th>
<th>$\frac{z_0}{l_0}$</th>
<th>$\frac{r_0}{l_0}$</th>
<th>$\frac{z_0}{l_0}$</th>
<th>$\frac{r_0}{l_0}$</th>
<th>$\frac{z_0}{l_0}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>60</td>
<td>2.0746E+0</td>
<td>0.1665E+0</td>
<td>0.1558E+0</td>
<td>0.7603E+0</td>
<td>0.1764E+0</td>
<td>0.4360E+0</td>
<td>0.9979E+0</td>
<td>0.2339E+0</td>
</tr>
<tr>
<td>65</td>
<td>0.1054E+0</td>
<td>0.1544E+0</td>
<td>0.1544E+0</td>
<td>0.9180E+0</td>
<td>0.1764E+0</td>
<td>0.4360E+0</td>
<td>0.9979E+0</td>
<td>0.2339E+0</td>
</tr>
<tr>
<td>70</td>
<td>0.4947E+0</td>
<td>0.1665E+0</td>
<td>0.1558E+0</td>
<td>0.7603E+0</td>
<td>0.1764E+0</td>
<td>0.4360E+0</td>
<td>0.9979E+0</td>
<td>0.2339E+0</td>
</tr>
<tr>
<td>75</td>
<td>0.2557E+0</td>
<td>0.1665E+0</td>
<td>0.1558E+0</td>
<td>0.7603E+0</td>
<td>0.1764E+0</td>
<td>0.4360E+0</td>
<td>0.9979E+0</td>
<td>0.2339E+0</td>
</tr>
</tbody>
</table>

**MAGNETIC FIELD PROFILE**

- $\psi$: Magnetic potential
- $r_0$, $z_0$: Cartesian coordinates
- $\frac{r_0}{l_0}$, $\frac{z_0}{l_0}$: Normalized coordinates
<table>
<thead>
<tr>
<th>$\psi$</th>
<th>$\theta$</th>
<th>$Z_0$</th>
<th>$I_0$</th>
<th>$Z_0$</th>
<th>$I_0$</th>
<th>$Z_0$</th>
<th>$I_0$</th>
<th>$Z_0$</th>
<th>$I_0$</th>
<th>$Z_0$</th>
<th>$I_0$</th>
<th>$Z_0$</th>
<th>$I_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
<td>13</td>
<td>14</td>
</tr>
<tr>
<td>0.69629 F0</td>
<td>0.65636 F0</td>
<td>0.74563 F0</td>
<td>0.78754 F0</td>
<td>0.87642 F0</td>
<td>0.95834 F0</td>
<td>0.99563 F0</td>
<td>0.97423 F0</td>
<td>0.92674 F0</td>
<td>0.86456 F0</td>
<td>0.79324 F0</td>
<td>0.70987 F0</td>
<td>0.60876 F0</td>
<td></td>
</tr>
</tbody>
</table>

**Note:** The table contains data for various parameters, likely related to a scientific or technical context, but the exact nature of the data is not clear from the image alone.
<table>
<thead>
<tr>
<th>( V_s )</th>
<th>( f_0 )</th>
<th>( Z_0 )</th>
<th>( I_e )</th>
<th>( Z_0 )</th>
<th>( f_0 )</th>
<th>( Z_0 )</th>
<th>( f_0 )</th>
<th>( Z_0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>0.1</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>0.2</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>0.3</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>0.4</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>0.5</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>0.6</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>0.7</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>0.8</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>0.9</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

**Magnetic Field Profile**

- **Table:**
  - Columns: \( V_s \), \( f_0 \), \( Z_0 \), \( I_e \), \( Z_0 \), \( f_0 \), \( Z_0 \)
  - Rows: 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9

- **Note:** The table represents the magnetic field profile with specific values for each parameter given in the columns.
Table A-II.4

Orbit Rotation Function $F_1(K, \psi_s)$
Table A-II.5

Dispersion Function \( G(K, \psi_g) \)
Table A-II.6

Dispersion Function $\frac{\partial U(K,\psi)}{\partial K}$
APPENDIX III

EVALUATION OF $\psi_f$

From Eq. (3.2a) we have

\[ r_0 e^{K \cos \psi_f} = r_e e^{K \cos \psi_s}. \]  

(A-III.1)

Differentiating with respect to $\psi_s$, keeping $z_s, z_f, K, n_s$, and $n_f$ constant, we have

\[ r_0 e^{K \cos \psi_f} \frac{d \psi_f}{d \psi_s} = r_e e^{K \cos \psi_s} \left[ -K \sin \psi_s + \frac{1}{r_0} \frac{dr_0}{d \psi_s} \right]. \]

Equations (5.24) and (5.25) give

\[ \psi_f = \frac{-K \sin \psi_s + \frac{1}{r_e} \frac{dr_e}{d \psi_s}}{-K \sin \psi_f + \frac{1}{r_0} \frac{dr_0}{d \psi_f}} = \frac{\Lambda_s - K \sin \psi_s}{\Lambda_f - K \sin \psi_f}. \]

(A-III.3a)

with

\[ \frac{dr_e}{d \psi_s} = z_s \frac{df_s(\psi_s)}{d \psi_s} = \Lambda_s z_s f_s(\psi_s) = \Lambda_s r_e \]

(A-III.3b)

and

\[ \frac{dr_0}{d \psi_f} = z_f \frac{df_f(\psi_f)}{d \psi_f} = \Lambda_f z_f f_f(\psi_f) = \Lambda_f r_0. \]

(A-III.3c)

We take the $\rho$-value at the profile [distance from $(z_s,0)$ to the profile] to be $\rho_s$, while the $(z_f,0)$-to-profile distance to $\rho_f$. We have, then,

\[ r_e = \rho_s \sin \psi_s; \quad r_0 = -\rho_f \sin \psi_f. \]

(A-III.4)

Then,

\[ \Lambda_s = \frac{1}{r_e} \frac{dr_e}{d \psi_s} = \frac{1}{\rho_s \sin \psi_s} \left[ \rho_s \cos \psi_s + \sin \psi_s \frac{d \rho_s}{d \psi_s} \right] = \cot \psi_s + \frac{1}{\rho_s} \frac{d \rho_s}{d \psi_s}. \]

(A-III.5a)
\[
\Lambda_f = \frac{1}{\sin \psi_f} \frac{d \rho_f}{d \psi_f} = \frac{1}{-\rho_f \sin \psi_f} \left[ -\rho_f \cos \psi_f - \sin \psi_f \frac{d \rho_f}{d \psi_f} \right] \\
= \cot \psi_f + \frac{1}{\rho_f} \frac{d \rho_f}{d \psi_f} 
\]

Hence,

\[
\psi_f' = \frac{-K \sin \psi_s + \cot \psi_f + \frac{1}{\rho_s} \frac{d \rho_s}{d \psi_s}}{-K \sin \psi_f + \cot \psi_f + \frac{1}{\rho_f} \frac{d \rho_f}{d \psi_f}} 
\] (A-III.6)

The form in Eq. (A-III.6) has been introduced because the terms \( \frac{1}{\rho} \frac{d \rho}{d \psi} \) are important in considerations of transmission problems (see Section #9), and hence require computation for other reasons than for the determination of \( \psi_f \).

For a symmetrical instrument, it may be noted, that, since

\[
\sin \psi_f = -\sin \psi_s; \ \cot \psi_f = -\cot \psi_s; \frac{d \rho_f}{d \psi_f} = -\frac{d \rho_s}{d \psi_s} \left( \text{or} \frac{dr_0}{d \psi_f} = -\frac{dr_e}{d \psi_s} \right) 
\]

then \( \psi_f' = -1 \), which is also evident from the fact that \( \psi_s + \psi_f = 2\pi \).

Also \( \psi_f' \) may be computed if \( \psi_f \) has been evaluated, but not the exit profile [i.e., \( \ell_f(\psi_f) \)] or its derivative. Differentiating Eq. (3.3) with respect to \( \psi_s \), we have, since

\[
\frac{dU}{d \psi_s} = \cos \psi_s e^{-K \cos \psi_s} 
\]

that

\[
\left[ \frac{-r_0}{\sin^2 \psi_f} \right] e^{-K \cos \psi_f} \frac{r_0}{r_e} = e^{-K \cos \psi_s} \left\{ \frac{-r_e}{\sin^2 \psi_s} + (z_s - z_f) \left\{ K \sin \psi_s - \frac{1}{r_e} \frac{dr_e}{d \psi_s} \right\} \right\} 
\]

Then, from Eq. (5.24),

\[
\psi_f' = \left[ \frac{\sin^2 \psi_f}{\sin^2 \psi_s} + \frac{z_s - z_f}{z_s \ell_f(\psi_s)} \left\{ \Lambda_s - K \sin \psi_s \right\} \sin^2 \psi_f \right]\ e^{K(\cos \psi_f - \cos \psi_s)} 
\] (A-III.7)

which, it is evident, is closely related to Eq. (5.7b).
Because \( \frac{1}{\rho} \frac{d\rho}{d\psi} \) is used for the transmission computations of Section #9, we compute here an explicit formula for the symmetrical instrument of Eq. (3.5), in which we set \( b_1 = 0 \), since \( b_1 \) does not affect the derivative. From Eq. (3.5), since \( f_s(\psi_s) = \frac{r_e}{z_s} \), we have

\[
K[U(K, \psi_s) + 2\pi n_s iJ_1(iK)] - \cot \psi_s e^{-K\cos \psi_s} = \frac{z_s}{r_e} e^{-K\cos \psi_s}. \tag{A-III.8}
\]

Differentiating with respect to \( \psi_s \),

\[
\frac{e^{-K\cos \psi_s}}{\sin^2 \psi_s} = \frac{z_s}{r_e^2} e^{-K\cos \psi_s} \left[ r_e \sin \psi_s - \frac{d^2 e}{d\psi_s^2} \right] \]

or

\[
\frac{1}{r_e} \frac{d^2 e}{d\psi_s} = \frac{1}{f_s(\psi_s)} \frac{df_s}{d\psi_s} = K \sin \psi_s - \frac{r_e}{z_s} \frac{1}{\sin^2 \psi_s} = K \sin \psi_s - \frac{f_s(\psi_s)}{\sin^2 \psi_s} \tag{A-III.9}
\]

and

\[
\frac{d^2 e}{d\psi_s^2} = z_s \frac{df_s}{d\psi_s} = K r_e \sin \psi_s - \frac{r_e^2}{z_s} \frac{1}{\sin^2 \psi_s} = z_s \left[ K f_s \sin \psi_s - \frac{f_s^2}{\sin^2 \psi_s} \right]. \tag{A-III.10}
\]

From Eqs. (5.28) and (5.32), we have

\[
\frac{1}{\rho_s} \frac{d\rho_s}{d\psi_s} = K \sin \psi_s - \frac{r_e}{z_s} \frac{1}{\sin^2 \psi_s} - \cot \psi_s \tag{A-III.11}
\]

and

\[
\frac{d\rho_s}{d\psi_s} = K \rho_s \sin \psi_s - \frac{r_e}{z_s} \frac{\rho_s}{\sin^2 \psi_s} - \rho_s \cot \psi_s
\]

\[
= \frac{z_s f_s(\psi_s)}{\sin \psi_s} \left[ K \sin \psi_s - \frac{f_s(\psi_s)}{\sin^2 \psi_s} - \cot \psi_s \right]. \tag{A-III.12}
\]
APPENDIX IV

TABLE OF FUNCTIONS FOR SPECTROMETERS WITH FRINGING FIELDS

Table IV. Column 1. \( \frac{d\rho}{d\psi} \) was computed from Eq. (A-III.12).

Computation was performed setting \( z_s = 1 \); hence, the quantity computed is
\[
\frac{1}{z_s} \frac{d\rho_s}{d\psi_s},
\]
with \( \psi_s \) in radians. The magnetic profile considered was that of the symmetrical system of Fig. (3.3) and Table II.3. The derivative was computed for \( \psi_s = \psi_s^o(15^\circ)175^\circ \) and \( K = 0.2, 0.4(0.05)0.5(0.02)0.9(0.05)1.0. \)
Here \( \psi_s^o \) is the angle closest to the critical angle, but may not be the same as \( \psi_s^o \) in Tables II.3, II.4, or II.5, because of the difference in \( \Delta \psi_s \).

Table IV. Column 2. \( \frac{1}{\rho_s} \frac{d\rho_s}{d\psi_s} \) was computed from Eq. (A-III.11)
over the same range as Column 1. Also, \( \tan \mu_s \) is given by this quantity, as in Eq. (9.8a).

Table IV. Column 3. Here \( W_s \) is computed from Eq. (9.12b) over the same \( \psi_s \)- and \( K \)-ranges as in Column 1. Values for which \( W_s < 1 \) arise when \( \frac{d\rho_s}{d\psi_s} \) (and hence \( \tan \mu_s \)) is negative, i.e., when the entrance lens is convergent. In this case, values of \( \psi_s \) for which \( W_s > 0 \) arise when the lens is slightly or moderately convergent. Where the lens is strongly convergent, \( |\tan \mu_s| \) is large and hence \( W_s < 0 \). For a divergent lens, \( \frac{d\rho_s}{d\psi_s} \) and \( \tan \mu_s \) are positive, and \( W_s > 1 \).

Table IV. Column 4. Here \( \eta_2 \) is computed from Eqs (9.19) and (9.20), i.e., the absolute value of the inverse of \( W_s \). Now \( \eta_2 = 1/|W_s| \) is valid only for values where \( |W_s| > 1 \). When \( |W_s| < 1 \), then \( \eta_2 \) is set equal to one. The condition \( |W_s| > 1 \) can arise in two ways.

(i) Divergent entrance lens \( (W_s > 1) \). Some rays are always lost by being driven into the polefaces

(ii) Strongly convergent entrance lens \( (W_s < -1) \). Trajectories cross the median plane at such large angles that they are driven into the polefaces on the opposite sides to those in which they entered the field.
Table IV. Column 5. The quantity \( \frac{1}{2} \frac{\varphi}{R_d/z_s} \eta_{23} \) is computed from Eq. (9.28). Actual values of \( \eta_{23} \) require specification of \( R_d/z_s \) and \( \frac{1}{2} \varphi \); an example is shown in the text using \( R_d = 0.01 \) (in units of \( z_s \)) and \( \varphi = 10^\circ \).

For a given \( Y \)-value and \( R_d \), increase of \( \varphi \) tends to decrease \( \eta_{23} \) because of the larger \( \delta R_{2f} \) which can get through a wider aperture; increase of \( R_d \) tends to increase \( \eta_{23} \) because a larger fraction of the rays spread out into \( \delta R_{2f}(\text{max}) \) can be accepted by the detector. However, \( \eta_{23} \) has \( \eta_2 \) as an upper limit, and if \( Y \frac{R_d/z_s}{\frac{1}{2} \varphi} \) exceeds \( \eta_2 \), then \( \eta_{23} \) is set equal to \( \eta_2 \); in this case, the detector is accepting all rays which pass through the exit aperture.
Table A-IV

Table of Functions for Spectrometers with Fringing Fields
APPENDIX V

THE DISPERSION FORMULA

The identity of the two dispersion formulae [Eqs. (6.6), (6.7), and (6.10b)] and Eq. (12.52) may be readily shown. We write Eqs. (12.42) and (12.52) as

\[
\frac{d\xi}{dK} = (z_e - z_0) \left[ \frac{F'}{F} + \cos\psi_s \right] + r_0 \tan \gamma_f (\cos\psi_f - \cos\psi_s) \tag{A-V.1}
\]

\[
\sigma_f = \frac{A_f}{\sin\psi_f} \frac{d\xi}{dK} = -\frac{A_f}{\sin\psi_f} \frac{d\xi}{dK} \tag{A-V.2}
\]

From Eqs. (2.5b) and (3.2a), using the definition in Eq. (5.2c),

\[
z_e - z_0 = -F r_e e \cos\psi_s = -F r_0 e \cos\psi_f \tag{A-V.3}
\]

From Eq. (12.8b),

\[
\tan \gamma_f = \frac{1}{A_f \sin^2\psi_f} - \cot\psi_f \tag{A-V.4}
\]

From Eqs. (A-V.1), (A-V.3), and (A-V.4),

\[
\frac{d\xi}{dK} = r_0 \left[ -F e \cos\psi_f \left( \frac{F' + F \cos\psi_s}{F} \right) + (\cos\psi_f - \cos\psi_s) \left( \frac{1}{A_f \sin^2\psi_f} - \cot\psi_f \right) \right]
\]

\[
= \frac{r_0}{A_f \sin\psi_f} \left\{ -A_f \left[ F_3 \sin\psi_f e \cos\psi_f + \cos\psi_f (\cos\psi_f - \cos\psi_s) \right] + \frac{(\cos\psi_f - \cos\psi_s)}{\sin\psi_f} \right\} \tag{A-V.5}
\]

Since

\[
\frac{-r_0}{\sin^2\psi_f} = (z_s - z_f) \frac{e \cos\psi_f}{F_3 \sin^2\psi_f} \tag{A-V.6}
\]

we have from Eqs. (A-V.5) and (12.10a) that

\[
\sigma_f = \frac{-r_0}{\sin^2\psi_f} \left\{ A_f \cos\psi_f (\cos\psi_s - \cos\psi_f) - F_3 \sin\psi_f e \cos\psi_f \right\}
\]

\[
-\left[ \frac{\cos\psi_s - \cos\psi_f}{\sin\psi_f} \right] = (z_s - z_f) G \tag{A-V.7}
\]
For a symmetrical spectrometer, from Eq. (12.44),

\[
\frac{d\Sigma}{dK} = -F \, r_e \, e^{K\cos\psi_s} \left[ \frac{F' - F\cos\psi_s}{F} \right]
\]

\[
= -r_e \, e^{-K\cos\psi_s} \left[ F' - F\cos\psi_s \right].
\]  \hspace{1cm} \text{(A-V.8)}

We have, also,

\[
\sigma_f = -\frac{r_e^2}{z_s} \frac{e^{K\cos\psi_s}}{\sin^2\psi_s} = -2z_s \frac{2e^{-K\cos\psi_s}}{F_3^2 \sin^2\psi_s} = 2z_s G \hspace{1cm} \text{(A-V.9)}
\]

where G is given by Eq. (6.11).
## APPENDIX VI

### PUBLISHED DATA ON CONSTRUCTED SPECTROMETERS

<table>
<thead>
<tr>
<th>References</th>
<th>Iron (O. or free 1%)</th>
<th>( z ) cm</th>
<th>K</th>
<th>( \psi_0 ) mm</th>
<th>( \psi_0 ) cm</th>
<th>( \psi_0 ) um</th>
<th>Diameter of source mm</th>
<th>Diameter of detector mm</th>
<th>Resolution ( % )</th>
<th>Transmission % of 4 ( \theta )</th>
<th>Maximum energy of electrons (MeV)</th>
<th>Purpose</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>6</td>
<td>0.6</td>
<td>1</td>
<td>160</td>
<td>750</td>
<td>1250</td>
<td></td>
<td></td>
<td>4</td>
<td>3</td>
<td>5.0</td>
<td>B ray spectrometer and study of conversion electrons produced in Coulomb excitation</td>
</tr>
<tr>
<td>7</td>
<td></td>
<td></td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Electron spectrometer for photon monochromator</td>
</tr>
<tr>
<td>8</td>
<td>6</td>
<td>~6</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Electron spectrometer for photon monochromator</td>
</tr>
<tr>
<td>9</td>
<td>1</td>
<td>~15.6</td>
<td>1</td>
<td>160</td>
<td>800</td>
<td>1000</td>
<td></td>
<td></td>
<td>16</td>
<td>9</td>
<td>5.0</td>
<td>Electron spectrometer for photon monochromator used as spectrograph from K 0.6 to K 1.0</td>
</tr>
<tr>
<td>10</td>
<td>1</td>
<td>~25 ~9.6</td>
<td>6</td>
<td>200</td>
<td>~750 ~1300</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>B ray spectrometer</td>
</tr>
<tr>
<td>11</td>
<td>1</td>
<td>10.5 ~9.6</td>
<td>6</td>
<td>200</td>
<td>~750 ~1300</td>
<td>3 x 6</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>B ray spectrometer (Ring focus)</td>
</tr>
<tr>
<td>12</td>
<td>1</td>
<td>27</td>
<td>8</td>
<td>100</td>
<td>950</td>
<td>1450</td>
<td></td>
<td></td>
<td>4</td>
<td>18</td>
<td>3.3</td>
<td>B ray spectrometer Half of an ( e^+ e^- ) coincidence spectrometer</td>
</tr>
<tr>
<td>14</td>
<td>IF</td>
<td>25.6</td>
<td>36</td>
<td>1250</td>
<td>1500</td>
<td>15</td>
<td></td>
<td></td>
<td>15</td>
<td>38</td>
<td>7.3</td>
<td>B rays and recoil protons from the decay of the free neutron</td>
</tr>
<tr>
<td>15</td>
<td>IF</td>
<td>~35</td>
<td>100</td>
<td>100</td>
<td>1300</td>
<td>8</td>
<td></td>
<td></td>
<td>8</td>
<td>11</td>
<td>8.6</td>
<td>Conversion electrons ( \alpha ) and ( \gamma ) rays</td>
</tr>
<tr>
<td>16</td>
<td>IF</td>
<td>33.5</td>
<td>100</td>
<td>100</td>
<td>1300</td>
<td>8</td>
<td></td>
<td></td>
<td>5</td>
<td>10</td>
<td>5.5</td>
<td>B rays Half of an ( e^+ e^- ) coincidence spectrometer</td>
</tr>
<tr>
<td>18</td>
<td>IF</td>
<td>~35</td>
<td>50</td>
<td>100</td>
<td>1300</td>
<td>1500</td>
<td></td>
<td></td>
<td>4</td>
<td>14</td>
<td>1.5</td>
<td>B rays of short lived activities High energy ( \alpha ) rays</td>
</tr>
<tr>
<td>19</td>
<td>IF</td>
<td>30 &gt;1.0</td>
<td>30</td>
<td>1250</td>
<td>1500</td>
<td>2</td>
<td></td>
<td></td>
<td>14</td>
<td>15</td>
<td>10</td>
<td>B rays of short lived activities High energy ( \alpha ) rays</td>
</tr>
</tbody>
</table>