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CHARGED PARTICLE OPTICS OF MAGNETIC SECTOR SPECTROMETERS
WITH $H = H\phi \propto r^{-1}$
(z, r, ϕ CYLINDRICAL COORDINATES WITH
THE z -AXIS THE OBJECT-IMAGE LINE)

by

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ABSTRACT

The charged particle optics of symmetric and asymmetric spectrometers is developed, taking into account the effects of fringing fields. Formulae are given for: the image of point and extended sources, and their resolution and transmission. The suppression of ghost peaks arising from multiloop trajectories is studied. Methods and tables for the calculation of instruments are given. A comparison is made between various instruments already constructed, and some general considerations concerning the choice of design parameters are presented.

The correction (for fringing field effects) of sector profiles of instruments already constructed is analyzed, and formulae are given which permit the calculation of modifications of the profiles necessary to insure good focusing with high transmission.

Tables include: general trajectory functions; profile curves; and dispersion and transmission functions for symmetrical instruments.

INTRODUCTION

Since the first proposal by Kofoed-Hansen, Lindhard and Nielsen⁽¹⁾ of the magnetic sector spectrometer with $H = H\phi \propto r^{-1}$,* there have been proposals for the use of this focusing principle for many different purposes, and about 15 such instruments have been built.

* r = distance of field point to the object-image (source-detector) axis, which is labelled the z -axis.

An important advantage of this kind of instrument is the high transmission achievable, because (i) focusing occurs over a wide range of emission angles, and (ii) many identical instruments can be used in parallel around a toroid. At the same time, the obtainable resolutions are good, because the imaging properties and the dispersion are satisfactory for each one of the instruments. At high transmissions, in fact, the resolution is ultimately limited by the effects of the fringing fields.

The main features of the charged particle optics of symmetric* instruments of this kind have already been discussed in the paper by Kofoed-Hansen, et al.⁽¹⁾ It is the purpose of this report to present the charged particle optics of symmetrical and asymmetrical instruments, including fringing field effects. Part of this work has already been published by L. Lagatta, C. Molina-Vedia, J. J. Peyre and some of us (C.A.M. and J.S.E.),⁽²⁻⁵⁾ In this report, we will not discuss multigap spectrometers with other types of magnetic fields. References to such spectrometers can be found in the review article on beta-ray spectrometers by M. Mladjenovic.⁽¹⁸⁾

Single-gap spectrometers of this design have been built by Kofoed-Hansen et al.⁽¹⁾ [later used by Huus et al.⁽⁶⁾], by Shea et al.⁽⁷⁾ and by Bernstein and Graetzer.⁽⁸⁾ A single-gap spectrograph** has been built by J. O'Connell.⁽⁹⁾

Multigap iron spectrometers have been built by Nielsen and Kofoed-Hansen,⁽¹⁰⁾ Nielsen⁽¹¹⁾ and Mallmann et al.^(5,12) The instrument described in (5) is the first half of a multigap electron-electron coincidence spectrometer⁽¹³⁾ The second half will be a multigap tilted axis (see Sec. #9) iron spectrometer

Multigap iron-free spectrometers have been built by Vladimirovskii et al.,⁽¹⁴⁾ Tretiakov et al.,⁽¹⁵⁾ Freedman et al.,^(16,17) Burgov,⁽¹⁸⁾ and Bartis.⁽¹⁹⁾ The multigap iron-free spectrometers⁽¹⁴⁾ have been used in the measurement of the electron-neutrino angular correlation in the beta decay of the free neutron.⁽²⁰⁾ Two such instruments are used in series as the beta spectrometer and the third one as the recoil-proton spectrometer.

Freedman's spectrometer^(16,17) is the first half of an iron-free toroidal electron-electron coincidence spectrometer. It will allow, in addition to the usual work of beta spectrometry, the measurement of β -e angular

*Reflection symmetry with respect to a plane perpendicular to the z-axis, midway between object and image.

**The profile is a particularly simple one from the construction point of view, since both entrance and exit profiles lie on a single line parallel to the z-axis. Focusing is adequate for a limited range of emission angles. See also Fig. (3.3).

correlation, α - e^- coincidences, and $(\beta$ - e - e), $(\beta$ - e - γ), $(e$ - e - α), $(\beta$ - e - X) triple coincidences. Both instruments may also be used in series as a single spectrometer in order to reduce the background due to scattered electrons and to improve the resolution.

A summary of the published data on the instruments constructed to date will be found in Appendix VI.

The same focusing principle has been suggested for use in a pair-spectrometer,⁽⁹⁾ but such an instrument has not yet been constructed.

This report has been divided into two parts. Part A, which includes Sections #1-7, gives the charged particle optics neglecting the fringing-field effects. It is assumed that the magnetic field drops abruptly to zero at the boundaries of the polefaces. Part B, which includes Sections #8-15, shows how the results of Part A require modification in order to take into account the fringing-field effects. In Section #16, we provide some general considerations on the choice of optimum design parameters for instruments. Some mathematical formulae and tables are given in the appendices.

PART A

ANALYSIS NEGLECTING FRINGE-FIELD EFFECTS

#1. TRAJECTORIES WITHIN A r^{-1} MAGNETIC FIELD

In this type of charged particle spectrometer, the components of the magnetic field $\underline{H}$ referred to a cylindrical coordinate system $(z;r;\phi)$ are

$$H_z = 0; \quad H_r = 0; \quad H_\phi = A_0/r \quad , \quad (1.1)$$

where A_0 is a constant. The field exists only inside a torus whose symmetry axis is the z -axis. Outside of the torus, the field is assumed to be zero. This is a simplified model in which the fringing fields are neglected; these fields will be analyzed in Part B. For the present analysis, therefore, we shall assume that a discontinuity in the magnetic field occurs at the boundaries. Before undertaking to examine the transition at the boundaries, we shall consider the trajectories in a field given by Eq. (1.1), a field in which no boundaries are present.

The magnetic vector potential $\underline{A}$ satisfies the equation

$$\nabla \times \underline{A} = \underline{H} \quad . \quad (1.2)$$

Expressing this equation in cylindrical coordinates, and making the reasonable assumption that $\underline{A}$ is independent* of z , we deduce from (1.1) and (1.2) that $A_z = A_z(r)$; $A_r = A_\phi = 0$, and finally that

$$A_z(r) = A_0 \ln(b/r) \quad . \quad (1.3)$$

where b is an arbitrary constant.

H. O. W. Richardson(21) studied in detail the equations for the trajectories of charged particles in this magnetic field. In the following we give those of his results which we need in our case.

The relativistic Lagrangian for a particle with rest mass m_0 and charge e , in electromagnetic units, moving with velocity $\underline{v}$ in a magnetic field with magnetic vector potential $\underline{A}$, is

$$L = m_0 c^2 \left[1 - \sqrt{1 - \frac{v^2}{c^2}} \right] + e(\underline{v} \cdot \underline{A}) \quad , \quad (1.4)$$

where c is the velocity of light. Here v^2 is a constant of the motion, where

$$v^2 = \dot{r}^2 + \dot{z}^2 + r^2 \dot{\phi}^2 \quad , \quad (1.5)$$

*The assumption of independence is already implicitly contained in (1.1), since these conditions assume that no end effects exist, which implies that the field essentially stretches infinitely in the z -direction.

because the Lorentz force is perpendicular to the velocity (constant energy). Replacing (1.3) in (1.4) gives

$$L = m_0 c^2 \left[1 - \sqrt{1 - \frac{v^2}{c^2}} \right] - e A_0 \dot{z} \ln \frac{r}{b} \quad (1.6)$$

From Lagrange's equations, one obtains

$$\dot{z} = \frac{A_0 e}{m} \ln \frac{r}{a} \quad (1.7)$$

and

$$p_\phi = \text{constant} = m r^2 \dot{\phi} \quad (1.8)$$

with* $r = a$ when $\dot{z} = 0$ and $m = m_0 / \sqrt{1 - (v/c)^2}$.

Fig. 1.1

Putting (1.7) and (1.8) into (1.5) and solving for $\dot{r}$, we have

$$\dot{r} = v \sqrt{1 - \left[\frac{a_0}{r} \right]^2 - \left[\frac{\ln(r/a)}{K} \right]^2} \quad (1.9)$$

where

$$K = -\frac{mv}{A_0 e} = -\frac{p}{A_0 e} = \frac{(B \rho_C)}{A_0} = \frac{\rho_C}{r} \quad (1.10)$$

and

$$a_0 = p\phi/p \quad (1.11)$$

with ρ_C the radius of curvature of the trajectory.

*In Fig. 1.1, $r = a$ is the ordinate of the point on the trajectory where the tangent is vertical.

It is desirable at this point to specify the conventions to be used relative to the algebraic signs of A_0 and K . From the usual right-hand rule, the field $\underline{H}$ in Eq. (1.1) is taken as positive along the positive ϕ -direction. Thus, in Fig. (1.1), with the positive z -axis going to the right, while r goes up, positive ϕ -motion comes out of the paper above the z -axis. Deflection of particles as shown in Fig. (1.1) will occur for positive charges only if $\underline{H}$ is along the negative ϕ -direction, i.e., if A_0 is negative; for a negative charge, of course, the same deflection would require positive A_0 . The quantity $A_0 e$ is thus inherently a negative quantity for an instrument designed to give deflections as shown in the figure. Further, K is correspondingly always positive, both for positive and negative charges, the adjustment being made in the sign of A_0 if the charge of the particle to be deflected is changed. As to the conventions on $(B\rho_C)$, we always take p and ρ_C as positive quantities. Since B has the same sign as A_0 and these are opposite in sign to that of the charge, $p/e = -(B\rho_C)$.

Let v_{rZ} be the component of velocity in the z - r plane. Thus, from Eq. (1.5),

$$v^2 = v_{rZ}^2 + r^2 \dot{\phi}^2 \quad (1.12a)$$

We are primarily interested in the motion in the z - r plane. For this purpose, it is convenient to define the angle ψ by the relation

$$\cos\psi = \dot{z}/v_{rZ} \quad (1.12b)$$

Evidently ψ is the angle formed by the projection of the velocity vector $\underline{v}$ on the z - r plane (i.e., the vector $\underline{v}_{rZ}$) with the positive z -axis. We have, then, from (1.9),

$$\dot{r} = v \sin\psi \sqrt{1 - \frac{a_0^2}{r^2}} \quad (1.13a)$$

and

$$\dot{z} = v \cos\psi \sqrt{1 - \frac{a_0^2}{r^2}} \quad (1.13b)$$

From (1.7), it follows that

$$r = ae^{-K \cos\psi \sqrt{1 - \frac{a_0^2}{r^2}}} \quad (1.14)$$

We expand the radical and, assuming that a_0^2 is small,* set $r \approx ae^{-K \cos\psi}$ wherever it appears in a term multiplying a_0^2 . Then, with good approximation,

*Conditions for the validity of this approximation are discussed at the end of Section #5 (p. 48).

$$r = ae^{-K\cos\psi} \left[1 + \frac{1}{2} \left(\frac{a_0}{a} \right)^2 K\cos\psi e^{2K\cos\psi} \right] \quad (1.15a)$$

From (1.13a), (1.13b), and (1.15a), we have, to the same approximation as in (1.15a),

$$\begin{aligned} z &= \int \frac{\dot{z}}{\dot{r}} dr + z_m \\ &= aK \int \cos\psi e^{-K\cos\psi} \left[1 - \frac{1}{2} \left(\frac{a_0}{a} \right)^2 e^{2K\cos\psi} (1 + K\cos\psi) \right] d\psi + z_m \end{aligned} \quad (1.15b)$$

Similarly, from (1.8), (1.13a) and (1.15a),

$$\begin{aligned} \phi &= a_0 v \int \frac{dr}{\dot{r} r^2} + \phi_m \\ &= \frac{a_0 K}{a} \int e^{K\cos\psi} \left[1 - \frac{3}{2} \left(\frac{a_0}{a} \right)^2 K\cos\psi e^{2K\cos\psi} \right] d\psi + \phi_m \end{aligned} \quad (1.15c)$$

With some modifications, these results correspond to equations found by H. O. W. Richardson (see Ref. 21, page 794). Richardson's results are identical with those presented here for trajectories with $p_\phi = 0$; however, for cases in which $p_\phi \neq 0$, evaluation of Richardson's equations requires numerical integration. By using the $\cos\psi$ definition (1.12b), rather than Richardson's $\dot{z}/v$, it was possible to develop here approximate analytical expressions which are not only simpler to evaluate, but also make possible an estimate of the error involved in the use of approximations.

Case with $p_\phi = 0$. For particles with $p_\phi = 0$, from (1.11) we deduce that $a_0 = 0$ and then

$$z = aK \int \cos\psi e^{-K\cos\psi} d\psi + z_m \quad ; \quad (1.16a)$$

$$r = ae^{-K\cos\psi} \quad ; \quad (1.16b)$$

and

$$\phi = \phi_m \text{ (plane trajectory)} \quad . \quad (1.16c)$$

It may be noted, since $\dot{\phi} = 0$, that ψ is now the angle formed by the velocity vector $\underline{v}$ with the positive z -axis.

Following Kofoed-Hansen et al.,⁽¹⁾ these equations are transformed into

$$z = aK U(K, \psi) + z_m \quad ; \quad (1.17a)$$

$$r = ae^{-K\cos\psi} \quad ; \quad (1.17b)$$

and

$$\phi = \phi_m \quad , \quad (1.17c)$$

upon taking $z = z_m$ for $\psi = \pi$ and designating $U(K, \psi)$ as the following function:

$$U(K, \psi) = \int_{\pi}^{\psi} \cos \psi e^{-K \cos \psi} d\psi \quad . \quad (1.18)$$

The U function may be obtained by a development in a series of Bessel functions:*

$$U(K, \psi) = -iJ_1(iK)(\pi - \psi) + \sum_{\ell=1}^{\infty} \frac{1}{\ell} (i)^{\ell-1} [J_{\ell-1}(iK) - J_{\ell+1}(iK)] \sin \ell \psi \quad . \quad (1.19)$$

A table of $U(K, \psi)$ is given in Table II.1 of Appendix II.

It is evident that the trajectory is periodic in r , but with a net drift of the orbits in the z -direction [see Fig. (1.1)]. Indeed, we have

$$z(\psi + 2\pi) - z(\psi) = 2\pi aK iJ_1(iK) \quad (1.20)$$

as the period, in which $\cos \psi$ -values (and hence the r -values) repeat themselves; thus, for points on the z -axis separated by the period, the r -values are the same:

$$r(\psi + 2\pi) - r(\psi) = 0 \quad . \quad (1.21)$$

Because of the periodicity in r , it is possible to restrict detailed considerations to the region $0 \leq \psi \leq 2\pi$. The trajectory for other ψ -values simply repeats the motion within this region; i.e., r -values repeat while z -values are displaced by the amount in (1.20) every time ψ goes through a change of 2π . We have, then, as the equations of the trajectories,

$$z = aK [U(K, \psi) + 2\pi n i J_1(iK)] + z_m \quad (1.22a)$$

and

$$r = ae^{-K \cos \psi} \quad , \quad (1.22b)$$

with

$$0 \leq \psi \leq 2\pi \quad (1.23)$$

$$n = \dots, -2, -1, 0, 1, 2, \dots$$

*See Appendix I for proof.

The particular loop of the trajectory is described by the n -value, ψ giving the position in the loop.

From the definition of $U(K, \psi)$ in Eq. (1.18), it is evident that $U(K, \pi) = 0$. Thus, in the part of the trajectory for which $n = 0$, it is found that $z = z_m$ when $\psi = \pi$.

It can be shown that the symmetry lines of the trajectories are given by

$$z = z_m \quad \text{at } \psi = \pi \quad (1.24a)$$

and

$$z = z'_m = z_m - aK iJ_1(iK) \quad \text{at } \psi = 0 \quad (1.24b)$$

The first lies half-way between two loops; the second bisects a loop (see Figs. 1.1 and 1.2). From Eq. (1.24b), it is evident that Eq. (1.22a) may also be written as

$$z = aK [U(K, \psi) + 2\pi n' iJ_1(iK)] + z'_m \quad (1.25)$$

where $n' = n + \frac{1}{2}$ and has only half-integral values. We may write Eq. (1.22a) and Eq. (1.25) as one equation:

$$z = aK [U(K, \psi) + 2\pi n iJ_1(iK)] + Z_M \quad (1.26)$$

where

$$Z_M = z_m \text{ if } n \text{ is an integer}$$

$$Z_M = z'_m \text{ if } n \text{ is half-integral}$$

The characteristic points of these trajectories are given in (1.27) below, and may be observed in Figs. (1.1) and (1.2).

ψ	r	z
0	$r_{\min} = ae^{-K}$	$aK[-\pi iJ_1(iK) + 2\pi n iJ_1(iK)] + Z_M = 2\pi(n - \frac{1}{2})aK iJ_1(iK) + Z_M$
$\pi/2$	a	$aK[U(K; \pi/2) + 2\pi n iJ_1(iK)] + Z_M$
π	$r_{\max} = ae^K$	$aK[2\pi n iJ_1(iK)] + Z_M \quad (1.27)$
$3\pi/2$	a	$aK[U(K, 3\pi/2) + 2\pi n iJ_1(iK)] + Z_M$
2π	$r_{\min} = ae^{-K}$	$aK[\pi iJ_1(iK) + 2\pi n iJ_1(iK)] + Z_M = 2\pi(n + \frac{1}{2})aK iJ_1(iK) + Z_M$

Fig. 1.2

From Equations (1.22b) and (1.26), one may deduce:

I. For different a -values, keeping K and Z_M constant, one obtains a family of similar trajectories differing only in the scale factor " a ." Thus, $(z - Z_M)$ is proportional to " a ," as is r , as well as the distance of any point on the trajectory from $(Z_M, 0)$. Hence for any two trajectories in this family, corresponding points (z_1, r_1) and (z_2, r_2) are related by the equation:

$$\frac{r_1}{r_2} = \frac{z_1 - Z_M}{z_2 - Z_M} = \frac{a_1}{a_2} \quad (1.28)$$

We will call this family symmetric because, for a fixed ϕ -value, the loops of the trajectories have $z = Z_M$ as a symmetry axis.

II. Varying Z_M by the amount ΔZ_M and leaving K and " a " constant, the trajectory is displaced parallel to the z -axis by ΔZ_M . Thus, the trajectory point (z_i, r_i) goes into $(z_i + \Delta Z_M, r_i)$.

III. Varying K and leaving " a " and Z_M constant, the form of the trajectory varies.

The most general monoparametric family of trajectories of similar form ($K = \text{constant}$) is given by

$$z = aK [U(K, \psi) + 2\pi n iJ_1(iK)] + Z_M(a) \quad (1.29a)$$

$$r = ae^{-K\cos\psi} \quad , \quad (1.29b)$$

where Z_M is a function of " a ." This family we call asymmetric, because it does not have the symmetry properties of the family which has K and Z_M constant while varying " a ."

The families with constant K are the most important ones, because they correspond to the case in which a given magnetic field intensity (A_0) acts on particles with equal momenta but with different initial conditions.

Case with $p_\phi \neq 0$. The case of particles with $p_\phi \neq 0$ is studied for small ϕ -components, so we set $a_0 = \delta a_0$. From Eqs. (1.15a) and (1.15b), it is evident that in the first approximation, i.e., to terms of the order of δa_0 , the r and z equations remain the same as for the case where $p_\phi = 0$ [Eqs. (1.16a) and (1.16b)].* Higher-order approximations appear in Eqs. (1.15a) and (1.15b) and may be evaluated through expansion in Bessel function series by the method of Appendix I, or evaluation may be carried out through numerical integration in an electronic computer. From (1.15c), it may be seen that, to the same approximation,** ϕ is

$$\phi = K V(K, \psi) \frac{\delta a_0}{a} + \phi_m, \quad (1.30)$$

where †

$$V(K, \psi) = \int_{\pi}^{\psi} e^{K \cos \psi} d\psi = J_0(iK)(\psi - \pi) + \sum_{\ell=1}^{\infty} \frac{2}{\ell} (-i)^{\ell} J_{\ell}(iK) \sin \ell \psi \quad (1.31)$$

A table of $V(K, \psi)$ is given in Table II.2. of Appendix II.

From these relations, it may be seen that ϕ increases a uniform amount† for every period of ψ , e.g., with each loop. Thus

$$\phi(\psi + 2\pi) - \phi(\psi) = 2\pi K J_0(iK) \frac{\delta a_0}{a} \quad (1.32)$$

We shall discuss below (Section #7) the influence of this rotation on the transmission of the instruments.

*Conditions for validity of this approximation are discussed at the end of Section #5 (p. 48).

**This is also the second approximation, since there is no term in $(\delta a_0)^2$.

† For derivation, see Appendix I.

#2. TRAJECTORIES IN THE FIELD WHEN INITIAL CONDITIONS ARE SET BY POINT SOURCE ON INSTRUMENT AXIS (IN FIELD-FREE REGION) AND BY ARBITRARY ENTRANCE PROFILE SHAPE

Considering the planes $\phi = \text{constant}$, the arbitrary sections of the torus whose segments form the pole pieces of the electromagnet determine the region in these planes in which the magnetic field varies as A_0/r and the region where the field is zero.

Fig. 2.1

Every particle emitted from the point source, located at $(z, r) = (z_s, 0)$, describes a straight trajectory until it enters the magnetic field. Let the curve limiting the section of the torus (Fig. 2.1) be given as a function of the parameter ψ_s by the equations

$$r_e = z_s f_s(\psi_s) \quad (2.1a)$$

and

$$z_e = z_s [1 + f_s(\psi_s) \cot \psi_s] \quad (2.1b)$$

Then, at the torus boundary, the point of incidence into the magnetic field is (z_e, r_e) . Since $p_\phi = 0$ for such particles, the trajectory continues to lie in the original plane and is given by (1.29), the parameters being determined by the point and angle at which the trajectory enters the magnetic field.

Because of the continuity of the velocity across the boundary between the field-free region and the magnetic field, the straight lines from $(z_s, 0)$ to (z_e, r_e) are tangent to the trajectories within the magnetic field at the points (z_e, r_e) . Thus, the two conditions arising from equating the point and angle of entry (i.e., from equality of position and derivative) at the boundary determine the boundary conditions for the trajectory of the particles inside the field, i.e., they determine the constants of integration, Z_M and "a." As a matter of fact, the following relations must hold for them:

$$z_s f_s(\psi_s) = a e^{-K \cos \psi_s} \quad (2.2a)$$

and

$$z_s [1 + f_s(\psi_s) \cot \psi_s] = a K [U(K, \psi_s) + 2\pi n_s i J_1(iK)] + Z_M \quad (2.2b)$$

From (2.2a), it is evident that "a" depends upon the angle of entry, ψ_s ; thus

$$a(\psi_s) = z_s f_s(\psi_s) e^{K \cos \psi_s} \quad (2.3)$$

Similarly, from (2.2b) and (2.3), one deduces that

$$Z_M(\psi_s) = z_s \left\{ 1 + f_s(\psi_s) \cot \psi_s - f_s(\psi_s) e^{K \cos \psi_s} K [U(K, \psi_s) + 2\pi n_s i J_1(iK)] \right\} \quad (2.4)$$

Inserting (2.3) and (2.4) in (1.29), one obtains the equations of the trajectories:

$$r = z_s f_s(\psi_s) e^{-K(\cos \psi - \cos \psi_s)} \quad (2.5a)$$

and

$$z = z_s \left\{ 1 + f_s(\psi_s) \cot \psi_s + K f_s(\psi_s) e^{K \cos \psi_s} [U(K, \psi) - U(K, \psi_s) + 2\pi(n - n_s) i J_1(iK)] \right\} \quad (2.5b)$$

The Eqs. (2.5) give, for constant K and for different values of ψ_s (different emission angles), a family of trajectories which, in general, is asymmetric, since Z_M is not constant. The particles will describe these trajectories until they emerge from the magnetic field; after such exit, the trajectories are straight and are tangent at the emergent point to the curves described in Eqs. (2.5).

If all the emergent rays intercept the symmetry axis at the same point $(z_f, 0)$, this point is called the image of $(z_s, 0)$. To achieve this it is necessary that the surface bounding the torus in the exit region obey the conditions given in Section #3.

#3. EXIT PROFILE SHAPE DETERMINED BY THE IMAGING CONDITION

For focusing to occur, all the emergent rays must pass through the focus $(z_f, 0)$; this requirement results in a particular shape for the exit boundary of the magnetic field. We consider the straight lines traced from $(z_f, 0)$ to the trajectories of Eqs. (2.5), in such a manner that the straight lines are tangent to the trajectories at the points of intersection. The geometrical condition on the magnetic field exit boundary is that it be the envelope of these points of intersection. As at the entrance curve, the continuity of the velocity (and hence the derivative of the curve) across the boundary insures that the trajectories within the field go into the tangent lines when the field becomes zero. Note that the boundary curve of the magnetic field in the planes $\phi = \text{constant}$ is not defined solely by the values of (z_e, r_e) , but also by the value of z_f , as well as K and z_s .

The angle between the emergent trajectories and the positive z -axis is called ψ_f (see Fig. 3.1), and, as in Eq. (2.1), dimensions are related to z_f by representing the exit boundary curve as a function of the parameter ψ_f by means of the relations

$$r_0 = z_f f_f(\psi_f) \quad (3.1a)$$

and

$$z_0 = z_f [1 + f_f(\psi_f) \cot \psi_f] \quad (3.1b)$$

Since the trajectory curve and the emergent boundary curve intersect at (z_0, r_0) , the two equations (2.5) and (3.1) are equal at this point. Thus,

$$z_f f_f(\psi_f) = z_s f_s(\psi_s) e^{-K(\cos \psi_f - \cos \psi_s)} \quad (3.2a)$$

and

$$z_f [1 + f_f(\psi_f) \cot \psi_f] = z_s \left\{ 1 + f_s(\psi_s) \cot \psi_s + K f_s(\psi_s) e^{K \cos \psi_s} [U(K, \psi_f) - U(K, \psi_s) + 2\pi(n_f - n_s) i J_1(iK)] \right\}. \quad (3.2b)$$

From (3.2) one deduces the following implicit relation between ψ_s and ψ_f :

$$\begin{aligned} \cot \psi_f e^{-K \cos \psi_f} - K U(K, \psi_f) - K 2\pi n_f i J_1(iK) = \cot \psi_s e^{-K \cos \psi_s} - K U(K, \psi_s) \\ - K 2\pi n_s i J_1(iK) + \frac{z_s - z_f}{z_s} \frac{e^{-K \cos \psi_s}}{f_s(\psi_s)}. \end{aligned} \quad (3.3)$$

Fig. 3.1

One can give the relation in an explicit form, the details depending upon the form of the expression for f_s . Thus,

$$\psi_f = \psi_f(\psi_s) \quad , \quad (3.4)$$

or we may calculate the relation numerically.

By either method, one uses Eq. (3.2) to obtain (z_0, r_0) , the boundary curve for the field at the emergent side. The remaining boundaries of the section in which the field is given by Eq. (1.1) are provided by the trajectories for which ψ_s is a maximum and ψ_s is a minimum*

In Fig. (3.2) we give the emergence boundary curves for the case in which** $f_s = \sin\psi_s$; $z_s = -z_f = 1$; and $n_s = n_f = 0$ for $\pi/2 \leq \psi_s < \pi$, and K takes on the values 0.40; 0.55; 0.60; and 1.00.

We now consider the symmetric case in which $Z_M(\psi_s) = b_1 = \text{constant}$. It will be recalled from Section #1 that for this family of trajectories the maximum r -value in the magnetic field occurs at the same z -value for each member of the family, namely, at $z = b_1$; hence, the symmetry. For a given z_s , n_s , b_1 and K , the entrance curve is given (from Eq. 2.2b) by

*As we shall discuss in Section #8, the fringing field enters in two ways: (i) the magnetic field extends beyond the profile into the nominally "field-free" region, and (ii) the magnetic field inside the profile region is decreased below the theoretical value in Eq. (1.1). The penetration of the fringing effect past the profile into the interior region cannot be avoided, since the sector property of the spectrometer is essential. However, the penetration past the two boundaries determined by $\psi_s(\text{max})$ and $\psi_s(\text{min})$ can be minimized in effect by placing the real boundaries (of the polefaces) at some distance beyond the trajectories discussed. At the two bounding trajectories, the fringing effect is then negligible.

**The instrument is a zero-loop one ($n_f - n_s = 0$) in which the entrance boundary $f_s(\psi_s)$ has been chosen to be a circle with $(z_s, 0)$ as center and with radius z_s . Incident particles enter the magnetic field perpendicular to the curve described by $f_s(\psi_s)$, since their paths are along radii of the circle. It may be noted that the points (A, B, C, D) and the curve connecting them are not discussed in this section, but will be discussed in Section #4.

$$f_s(\psi_s) = \frac{z_s - b_1}{z_s} \frac{1}{K e^{K \cos \psi_s} [U(K, \psi_s) + 2\pi n_s i J_1(iK)] - \cot \psi_s} \quad (3.5)$$

If the image point is placed symmetrically relative to $z = b_1$, i.e., $-(z_f - b_1) = z_s - b_1$, and if we consider symmetrical portions of the trajectories ($n_s = -n_f$), then the exit curve is symmetric relative to $f_s(\psi_s)$ given in Eq. (3.5). Thus, the exit curve given by $f_f(\psi_f)$ is the mirror image of the entrance curve given by Eq. (3.5), the reflection being taken in the plane $z = b_1$.

In Fig. (3.3) is illustrated the variation of the entrance (and emergence) curves with the parameter K , for the symmetric case with $b_1 = 0$ and $n_s = 0$. Because of the symmetry, both entrance and emergence curves have the same form (except for the reflection), so only one is given here. In the figure are shown circles with $(z_s, 0)$ as center to simplify a comparison of the calculated entrance curves with the corresponding circles. The curves are drawn in terms of z_s as a unit, the point $(z_s, 0)$ being shown at 1.00. A table of the coordinates (z_e, r_e) is given as a function of (K, ψ_s) in Table II.3 of Appendix II.

It may be noted that the function $f_s(\psi_s)$ becomes infinite for a value of ψ_s which depends upon K and n_s . Thus, e.g., consider the case $n_s = 0$: Since $U(K, \psi_s) > 0$ for $0 < \psi_s < \pi$, the quantity $[K e^{K \cos \psi_s} U(K, \psi_s)]$ is positive throughout the ψ_s -range. Then, since the range of $\cot \psi_s$ is from ∞ to 0 for ψ_s varying from 0 to $\frac{1}{2}\pi$, there will be a value of ψ_s for which the denominator is zero. With $n_s \neq 0$, the term in $iJ_1(iK)$ is included. For $K < 1$, $iJ_1(iK) > -0.7K$; hence $[U(K, \psi_s) + 2\pi n_s iJ_1(iK)]$ may be negative. In this case, the zero in the denominator occurs in the range $\frac{1}{2}\pi < \psi_s < \pi$.

For $b_1 = 0$, the coordinates (z_e, r_e) have the value $(0,0)$ for $\psi_s = \pi$. As ψ_s is decreased from π towards zero, the profile starts at the origin and asymptotically becomes tangent to a ray at a critical ψ_s -value; at this angle, (z_e, r_e) becomes (∞, ∞) . The equation has no physical solution for values of ψ_s smaller than the critical value, since the (z_e, r_e) coordinates become negative. It is not practical to use the profile in the neighborhood of the critical region, even though the ψ_s -values involved exceed the critical value, for reasons discussed in Section #9.* Curves illustrating the asymptotic structure of the profiles are shown in Fig. (3.4), and, as in Fig. (3.3), are given using z_s as the unit of length [see also Table II.3].

Fig. 3.4

*See p. 135.

#4. GHOST PEAKS

In the design of an instrument, z_s , z_f and the entrance curve $f_s(\psi_s)$ are chosen as geometrical characteristics. Then, choosing the trajectory characteristics (K, n_s, n_f) , the exit curve may be determined by the method of Section #3. As K is varied, with a given f_s , a family of exit curves are formed, as in Fig. (3.2). A difficulty arises in that with all the design factors fixed, including K , it is possible for trajectories characterized by other (K, n_s, n_f) values to pass through $(z_f, 0)$; thus, by virtue of multiloop trajectories, particles of other than the desired momentum may arrive at the detector.

If, with the same values of (f_s, z_s, z_f) and with a particular K'_1 , the exit boundary curve for a different value of n_s and/or n_f is determined, this curve is not a member of the above family. This curve cuts across the previous family, as illustrated [see Fig. (3.2)] by the curve passing through the points (A, B, C, D). For the same (n_s, n_f) , another value, K'_2 yields another such exit curve, displaced relative to the curve ABCD, but also intersecting the original family of curves. A series of K' -values yield exit curves which sweep across the family of original emergence curves.

We consider first the family of exit curves (varying parameter K') which intersect the original family of exit curves (varying parameter K) The physical significance of the K' -family is the following:

An instrument is designed with particular $(z_s, z_f, f_s, n_s, n_f, K)$, so that at a particular A_0 , it focuses (for every ψ_s) all particles of momentum $mv = -KA_0e$. For example, in Fig. (3.2), we choose the exit curve suitable for $K = 0.57$, with $(n_s, n_f) = (0, 0)$; for a given A_0 , all particles of momentum p pass through $(z_f, 0)$ for any ψ_s . In the same magnetic field (i.e., A_0), particles of momentum p'_1 follow different trajectories, which are characterized by different values (n'_s, n'_f, K'_1) . To focus these at $(z_f, 0)$ for all ψ_s , another exit curve is required than the one shown in Fig. (3.2) for $K = 0.57$; e.g., the required exit curve might be ABCD. Since the exit curve is not ABCD, no focusing occurs for all ψ_s . However, for one ψ_s -value, the corresponding trajectory passes through the point B, so that the trajectory will pass through $(z_f, 0)$; this is the only trajectory with (p', n'_s, n'_f) to reach the focal point. Thus, for momentum p , particles reach $(z_f, 0)$ for all ψ_s , but this is true for only one ψ_s for p'_1 . If the source emitted only the two momenta p and p'_1 , then the interfering intensity would be very small indeed.

Using the same (n'_s, n'_f) , but another p'_2 (hence K'_2) at the same A_0 -value, another exit curve $A'B'C'D'$ is required for focusing for all ψ_s ; the curve $A'B'C'D'$ is similar to, but displaced from, ABCD and intersects the actual exit curve at B'. As before, only for one ψ_s does a trajectory

pass through B' , and hence through $(z_f, 0)$. Thus, for any particular momentum other than the one focused by the exit curve for (K, n_s, n_f) , the intensity reaching the detector is very small.

Each K' corresponds to a different exit curve, and the intersection of this family of curves with the curve for $(K, n_s, n_f) = (0.57, 0, 0)$ provides a class of points, (B_i) , along this curve, each of which corresponds to a different ψ_s . With a given instrument design (i.e., K, n_s, n_f), for the allowed ψ_s -range, there is a definite K' -range, which depends upon the (n_s', n_f') values considered. [Examples are shown in Figs. (4.3) and (4.4)]. For another choice (n_s'', n_f'') , a similar K'' -range occurs, and a corresponding contribution to particles of the wrong momenta passing through the focal point.

Thus, if the source emission has a wide range of momenta, then the contribution at $(z_f, 0)$ from each K' -value, small though it is, when combined with all the others coming in at a particular A_0 , gives a considerable background of particles having momenta other than p . In fact, if

the range of momenta in the source is moderately uniform, and if the range of K' (and hence of p') for a fixed A_0 lies within the momentum range of the source, then this background will be comparable in intensity to the desired, focused particles of momentum p .

Fig. 4.1

An example is shown in Fig. (4.1). With the chosen entrance curve, the exit curve has been determined for parameter K and $(n_s, n_f) = (-\frac{1}{2}, \frac{1}{2})$, a single-loop trajectory.*

Particles with the appropriate momentum are focused at $(z_f, 0)$.

With the same A_0 , but with a different momentum p' (hence, with a different K'), there is one emission angle ψ'_s for which a trajectory with $n_s = n_f = 0$

*This trajectory is characterized by one loop, and in general the trajectories with half-integral values of n have an odd number of loops $[(n_f - n_s) = \text{odd}]$ in the magnetic field. Integral values of n have an even number of loops; e.g., the trajectory at the bottom of Fig. (4.1) has $n_s = n_f = 0$, and has no loops. A two-loop trajectory has $n_s = -1, n_f = 1$ if Z_M is symmetric relative to the orbit loops.

also passes through $(z_f, 0)$. With the same A_0 and the original momentum p (hence with the original K), a trajectory in which $n_s = n_f = 0$ and in which the emission angle is ψ'_s does not pass through $(z_f, 0)$, but hits the z -axis at another point, namely, $(z'_f, 0)$.

The converse situation also arises. For a mono-momentum (p) source, trajectories with different (n_s, n_f) result in another type of spurious effect. As above, an instrument designed for (K, n_s, n_f) focuses particles of momentum p at a given A_0 . If we vary K to K' by changing A_0 , we find that another (n'_s, n'_f) requires another exit curve, which cuts across the K -parameter family, as in Fig. (3.2). The discussion of the intersection points (B_i) and the relation of each to a single ψ_s -value is as above, the chief difference being that K' varies, not by keeping A_0 constant and varying p , but by keeping p constant and varying A_0 . Thus, with a monoenergetic source, not only will particles be counted at a field corresponding to K (with $K = p/A_0e$), but also at A'_0 (with $K' = -p/A'_0e$).

Although the emitted momentum p is constant, the fact that particles are detected over a range of values of A_0 leads to an "apparent momentum distribution" from the source. Thus, a monoenergetic source gives, apart from the main peak, other peaks with smaller peak intensity, called "ghost" peaks. Because a ghost peak corresponds to a range of K' , the necessary variation of A_0 corresponds to an apparent range in p , and the peak is broad, i.e., has poorer resolution than the main peak.

Fig. 4.2

In an instrument designed for $n_s = n_f = 0$, for the higher values of ψ_s the trajectories with $|n_s| > 0$ cannot go through the instrument because they cross the entrance boundary curve for the second time and thus escape from the magnetic field region (e.g., see Fig. 4.2).

As an illustration, consider a symmetric instrument ($n_s = -n_f$; $b_1 = 0$) and the cases $n_s = 0$; $-\frac{1}{2}$; -1 ; $-\frac{3}{2}$, i.e., no loop, one-loop, two-loop and three-loop trajectories, respectively. The function $f_s(\psi_s)$

is then, from (3.5),

$$f_s(\psi_s) = \frac{1}{K e^{\cos \psi_s} [U(K, \psi_s) + 2\pi n_s i J_1(iK)] - \cot \psi_s} \quad (4.1)$$

In the case of this symmetrical instrument, it is convenient to treat the entrance curve as varying with K' . The significance of the variation of K' in terms of bringing particles of momentum other than p to $(z_f, 0)$ or in creating ghost peaks with constant p is the same as described above.

In an instrument designed for (K, n_s) , the symmetry condition with $b_1 = 0$ requires that f_s be given by Eq. (4.1). With this entrance (and exit) curve, particles of momentum p pass through $(z_f, 0)$ for all ψ_s . For another (K', n'_s) , the appropriate entrance curve $f'_s(\psi_s)$ is calculated by putting (K', n'_s) into Eq. (4.1). The two entrance curves described by $f_s(\psi_s)$ and $f'_s(\psi_s)$ will, in general, intersect at one point, and at this point (from Eq. 2.1), $f_s(\psi_s) = f'_s(\psi_s)$. Since the actual instrument has the entrance curve $f_s(\psi_s)$, it is only for a ψ_s going through this intersection point that a trajectory involving (K', n'_s) will pass through $(z_f, 0)$. We saw that for a symmetric instrument, fixing K' and n'_s determined $f'_s(\psi_s)$; conversely, given the value $f'_s(\psi_s) = f_s(\psi_s)$ and n'_s , then K' is determined. We have then

$$K e^{K \cos \psi_s} [U(K, \psi_s) + 2\pi n_s i J_1(iK)] = K' e^{K' \cos \psi_s} [U(K', \psi_s) + 2\pi n'_s i J_1(iK')] \quad (4.2)$$

which allows us to calculate $K' = K'(K, n_s, n'_s, \psi_s)$.

Numerical results of such calculations are given in Figs. (4.3) and (4.4); in Fig. (4.3), we have an instrument with $K = 0.6$ and $n_s = 0$ (zero-loop trajectory), and in Fig. (4.4), an instrument with $K = 0.4$ and $n_s = -\frac{1}{2}$ (one-loop trajectory).

Fig 4.3

Fig 4.4

It is inconvenient to show the effect of these trajectories in the case of fixed A_0 and wide range of momenta from the source. However, the effect of a monoenergetic source in forming ghost peaks is readily shown. From $K = -p/A_0 e$, we obtain $A'_0 = A_0(K/K')$ and from this the spectrum of a monoenergetic source [see Figs. (4.5) and (4.6)] is predicted: Consider the instrument

involved in Fig. (4.3), for which $K = 0.6$ and $n_s = -n_f = 0$. The spread in momenta in the satellite peak is determined by the spread in K' over the range of θ_s considered (95° to 155°). Thus, for the type of trajectory determined by $(n_s = -\frac{1}{2}, n_f = \frac{1}{2})$, K' varies from 0.44 to 0.32 (see Fig. 4.3), and the resulting apparent momentum is shown in Fig. (4.5) as the first peak beyond the main (narrow) peak.

Since for all of the alternative types of trajectories, $K' \leq K$, then for a fixed momentum p , we have $A_0' > A_0$, and the ghost peak has an apparently higher momentum than the main peak. Because the whole range of θ_s is accepted by the instrument and delivered into these single-loop orbits, to the first approximation it is assumed that the intensity in the satellite peak is roughly the same as in the main peak.* The height of the satellite peak has been normalized so that the enclosed area is the same as that of the main peak. Because no details of profile have been evaluated, the peak is shown as of rectangular shape, although this must be incorrect as to detail. The extension along the mv -axis is calculated solely by the range of K' evaluated from Fig. (4.3). The remaining satellite peaks are similarly calculated, $n_s = -1$ being the next one. Since the K' -values overlap, the apparent mv -values also overlap; thus, the satellite peaks are shown as superimposing because of their poor resolution. In practice, these would show as a somewhat smeared-out broad peak.

The instrument involved in Fig. (4.4) yields the apparent momentum distribution of Fig. (4.6). The main peak is that of a single-loop trajectory. For $n_s' = 0$, we have $K' > K$; so $A_0' < A_0$ and the ghost peak generated by a zero-loop trajectory is of apparently smaller momentum than the main peak

The elimination of ghost peaks can easily be achieved in instruments with $n_s = n_f = 0$, with a baffle as shown in Fig. (4.2). This is an important reason for choosing an instrument to focus trajectories without loops.

*It follows from the discussion in Section #7 that multiloop trajectories have poorer transmission than zero-loop ones, so this assumption is a gross overestimate of the satellite intensity.

#5. IMAGE OF A FINITE SOURCE

We have seen that the image of a point source $(z_s, 0)$ is the point $(z_f, 0)$. We wish now to determine the nature of the image of a source of finite dimensions. For this, we divide the emitted rays into

- those with $p_\phi = 0$ and
- those with $p_\phi \neq 0$.

Rays with $p_\phi = 0$. These rays are emitted in a plane $\phi = \text{constant}$. On this plane we consider as emission points

- a) $(z_s + \delta z_s, 0)$
- b) $(z_s, \delta r)$.

If we know the image of these points, it is easy to find the image of any point $(z_s + \delta z_s, \delta r)$, as will be seen below.

a) The particle emitted at $(z_s + \delta z_s, 0)$, which enters the magnetic field at (z_e, r_e) , forms an angle $\psi_s + \delta\psi_s$ [see Fig. (5.1)] with the positive z -axis. We relate $\delta\psi_s$ to δz_s as follows: Writing Eq. (2.1b) as

$$z_e = z_s + z_s f_s(\psi_s) \cot \psi_s = z_s + r_e \cot \psi_s$$

and differentiating, keeping (z_e, r_e) constant, from Eq. (3.2a) we find

$$\delta\psi_s = \frac{\sin^2 \psi_s}{r_e} \delta z_s = \frac{\sin^2 \psi_s}{r_0} e^{-K(\cos \psi_f - \cos \psi_s)} \delta z_s \quad (5.1a)$$

Fig. 5.1

With this value of the entrance angle, $(\psi_s + \delta\psi_s)$, we can return to the equations of motion in the magnetic field, and from Eq. (1.29), we have

$$\begin{aligned} z &= (a + \delta a) K[U(K, \psi + \delta\psi) + 2\pi n_s i J_1(iK)] + Z_M(a + \delta a) \\ r &= (a + \delta a) e^{-K \cos(\psi + \delta\psi)} \end{aligned} \quad (5.1b)$$

By means of Eq. (2.3), $(a + \delta a)$ is calculated, and values of (z, r) then give the new trajectory in the field.

We will find it convenient, however, to express the coordinates of the trajectory as "variations" or perturbations from the "unvaried" trajectory of a particle emitted from the central point $(z_s, 0)$.

Before pursuing this line, it is convenient to specify the notation to be used in this variation. As noted before, $\delta\psi_s$ represents the variation of entrance angle due to motion of the point source from $(z_s, 0)$ to $(z_s + \delta z_s)$ while keeping the entry point (z_e, r_e) constant. The varied trajectory intersects the exit profile described by $f_f(\psi_f)$ and emerges into the field-free region with exit angle shifted from ψ_f to $\psi_f + \delta\psi_f$. The exiting ray intersects the z -axis at a point displaced from $(z_f, 0)$, i.e., at $(z_f + \delta z_f, 0)$.

The position of $(z_f + \delta z_f, 0)$ is completely specified when both the angle of emergence $(\psi_f + \delta\psi_f)$ and the position of emergence $(z_0 + \delta z_0, r_0 + \delta r_0)$ into the field-free region are known. The position of emergence is given by the intersection of the profile and the varied trajectory, and depends upon the shape of the exit profile in the neighborhood of the unvaried emergence point (z_0, r_0) .

For the purposes at hand, it is convenient to describe the shape in the neighborhood of the unperturbed exit point (z_0, r_0) in terms of the differential property $\frac{d\psi_f}{d\psi_s}$. In this coefficient, $d\psi_s$ represents a variation of ψ_s due to variation of (z_e, r_e) while keeping the source point fixed at $(z_s, 0)$; $d\psi_f$ is the corresponding variation of ψ_f in which (z_0, r_0) varies but the focal point remains at $(z_f, 0)$. Clearly, $\delta\psi_s \neq d\psi_s$, and $\delta\psi_f \neq d\psi_f$. For greater clarity, we shall use the notation: $d\psi_f = \delta\theta$ [see Fig. (5.1)]. Then, $\delta\psi_f$ gives the variation in emergence angle, whereas $\delta\theta$ defines the variation in emergence position.

We consider a particular "varied" trajectory, specified by its parameter $\psi_s + \delta\psi_s$ and the trajectory coordinates $\psi + \delta\psi$. The original trajectory in the magnetic field (z, r) is given as a function of ψ and the entrance angle ψ_s . In the neighborhood of (ψ, ψ_s) the coordinates (z, r) are expanded, and second and higher powers of $(\delta\psi, \delta\psi_s)$ are neglected; the coordinates of the new trajectory $(z + \delta z, r + \delta r)$ are then expressed in terms of (z, r) and $(\delta z, \delta r)$.

The "variations" from the coordinates of the "unvaried" trajectory are calculated as follows: From Eq. (2.1) and (2.5)

$$r = r_e e^{-K(\cos\psi - \cos\psi_s)} \quad (5.2a)$$

and

$$z = z_e + r_e e^{K\cos\psi_s} F_0, \quad (5.2b)$$

where

$$F_0 = K [U(K, \psi) - U(K, \psi_s) + 2\pi(n - n_s) iJ_1(iK)] .$$

Then, differentiating and noting [from Eq. (1.18)] that

$$dU(K, \psi) = \cos\psi e^{-K\cos\psi} d\psi,$$

we have

$$\delta r = K r_e e^{-K(\cos\psi - \cos\psi_s)} \left[\sin\psi \delta\psi - \sin\psi_s \delta\psi_s \right] \quad (5.2c)$$

and

$$\delta z = K r_e e^{-K(\cos\psi - \cos\psi_s)} \left\{ \cos\psi \delta\psi - e^{K\cos\psi} \left[F_0 \sin\psi_s + \cos\psi_s e^{-K\cos\psi_s} \right] \delta\psi_s \right\} . \quad (5.2d)$$

Equations (5.2c) and (5.2d) describe any point on the "varied" trajectory. For a point on this trajectory lying on the exit boundary curve, ψ_f is substituted for ψ , and $(\delta z, \delta r)$ for this point are, then, [utilizing Eq. (3.2a) and (3.3)],

$$\delta r = K r_0 [\sin\psi_f \delta\psi_f - \sin\psi_s \delta\psi_s] \quad (5.3a)$$

$$\begin{aligned} \delta z &= K r_0 \left\{ \cos\psi_f \delta\psi_f - e^{K\cos\psi_f} \left[F \sin\psi_s + \cos\psi_s e^{-K\cos\psi_s} \right] \delta\psi_s \right\} \\ &= K r_0 \left\{ \cos\psi_f \delta\psi_f - \sin\psi_s \left[\cot\psi_f - \frac{(z_s - z_f)}{r_0} \right] \delta\psi_s \right\}, \end{aligned} \quad (5.3b)$$

with

$$F = K \left[U(K, \psi_f) - U(K, \psi_s) + 2\pi(n_f - n_s) iJ_1(iK) \right] \quad (5.3c)$$

Thus, we have a means of calculating any point along the new trajectory, and we wish to find out where this intersects the exit boundary. To do this, we determine the coordinates along $f_f(\psi_f)$ by expanding (z, r) in terms of $\delta\theta$, thus giving the new coordinates $(z_0 + \delta z_0, r_0 + \delta r_0)$, with the condition that they lie on the exit boundary. We substitute θ for ψ_f in Eq. (3.1) and differentiate, noting that z_f is constant; differentiation is with respect to θ , evaluated at $\theta = \psi_f$. Thus,

$$\delta r_0 = z_f f'_f(\psi_f) \delta\theta = r_0 \Lambda_f \delta\theta \quad (5.4a)$$

$$\delta z_0 = z_f \left[f'_f(\psi_f) \cot \psi_f - \frac{f_f(\psi_f)}{\sin^2 \psi_f} \right] \delta\theta =$$

$$r_0 \left[\Lambda_f \cot \psi_f - \frac{1}{\sin^2 \psi_f} \right] \delta\theta \quad (5.4b)$$

where

$$f'_f(\psi_f) = \left[\frac{df_f(\theta)}{d\theta} \right]_{\theta = \psi_f} = \frac{df_f(\psi_f)}{d\psi_f} = \Lambda_f f_f(\psi_f) \quad (5.4c)$$

The condition that the exit boundary and the new trajectory intersect at $(z_0 + \delta z_0, r_0 + \delta r_0)$ implies that the two equations for δz and those for δr are equal at the intersection. From the equations in δr [Eqs. (5.3a) and (5.4a)], we have

$$\delta\theta = \frac{K}{\Lambda_f} \left[\sin \psi_f \delta\psi_f - \sin \psi_s \delta\psi_s \right] \quad (5.5a)$$

From the equations in δz , Eq. (5.3b) and (5.4b), and from Eq. (3.2a) and (5.5a),

$$\delta\psi_f = \left[-\Lambda_f \sin \psi_f \sin \psi_s \frac{z_s - z_f}{r_0} + \frac{\sin \psi_s}{\sin \psi_f} \right] \delta\psi_s \quad (5.5b)$$

To introduce δz_f , we consider Eq. (3.1b):

$$z_0 = z_f + r_0 \cot \psi_f \quad (5.6a)$$

Then,

$$\delta z_0 = \delta z_f + \cot \psi_f \delta r_0 - \frac{r_0}{\sin^2 \psi_f} \delta \psi_f \quad . \quad (5.6b)$$

Introducing (5.4a) and (5.4b), we have

$$\delta z_f = \frac{r_0}{\sin^2 \psi_f} (\delta \psi_f - \delta \theta) \quad . \quad (5.7a)$$

From (5.5a) and (5.5b)

$$\delta \psi_f - \delta \theta = \left[(\Lambda_f - K \sin \psi_f) \left(-\frac{z_s - z_f}{r_0} \sin \psi_s \sin \psi_f \right) + \frac{\sin \psi_s}{\sin \psi_f} \right] \delta \psi_s \quad .$$

If we differentiate Eq. (3.3) with respect to ψ_f , keeping z_s and z_f constant, as noted above, we get

$$\Lambda_f - K \sin \psi_f = \frac{r_0}{z_s - z_f} \left[\frac{1}{\sin^2 \psi_f} - \frac{(\psi_s')}{\sin^2 \psi_s} e^{-K(\cos \psi_s - \cos \psi_f)} \right], \quad (5.7b)$$

where

$$(\psi_s') = \frac{d \psi_s}{d \psi_f} = \frac{1}{(\psi_f')} = \frac{1}{\frac{d \psi_f}{d \psi_s}} \quad . \quad (5.8)$$

Equation (5.8) is the differential coefficient giving the angular variation of the exit profile with ψ_s , while Λ_f gives the fractional variation of r_0 with ψ_f . Then,

$$\delta \psi_f - \delta \theta = \left[\frac{\sin \psi_f}{\sin \psi_s} (\psi_s') e^{-K(\cos \psi_s - \cos \psi_f)} \right] \delta \psi_s \quad . \quad (5.9)$$

Combining this result with Eq. (5.1a) and (5.7a), we have

$$\delta z_f = \frac{\sin \psi_s}{\sin \psi_f} \frac{1}{(\psi_f')} \delta z_s = Q_f \delta z_s \quad . \quad (5.10)$$

An explicit expression may be given for ψ_f' if $f_s(\psi_s)$ and $f_f(\psi_f)$ are known (see Appendix III).

Up to this point we have considered only particles which originate on the z -axis, and hence automatically traverse trajectories lying in planes $\phi = \text{constant}$, i.e., for which $p_\phi = 0$. We wish to include trajectories arising from an extended source lying in a plane normal to the z -axis at $(z_s, 0)$. Since only small sources are treated, if there is also a z -extension of the source, we have only to add the effect of the displacement δz_s as given in Eq. (5.10).

We consider a point A on this source with coordinates (z_s, r_A, ϕ_A) . The "image" of this point is the intersection of the rays emitted by A with the plane $z = z_f$; this image need not be a point. A coordinate system closely tied to the direction of an emitted ray simplifies its description, so we define a new pair of coordinates, (r_1, r_2) , for each of the rays emitted from A, as follows:

(i). The ϕ -plane containing the entrance point (z_e, r_e) of the trajectory is called ϕ_s . A given emission point corresponds to many different ϕ_s -planes, and, conversely, a given ϕ_s -plane (and hence a given set of r_1, r_2 coordinate axes) corresponds to many source points like A.

(ii). r_1 is an axis whose positive direction lies in the plane $\phi = \phi_s$. The coordinate r_1 is numerically measured in an r -direction, but may have positive or negative values. The coordinate r_1 for any particular ray lies either in the $\phi = \phi_s$ plane or in the plane $\phi = \phi_s + \pi$. If the ray emitted from A has a component of motion away from the z -axis (i.e., $|\phi_s - \phi_A| < \frac{\pi}{2}$), then r_1 lies in the plane ϕ_s and r_1 is positive; if the emitted ray has a component of motion toward the z -axis (i.e., $|\phi_s - \phi_A| > \frac{\pi}{2}$), then r_1 lies in the plane $\phi_s + \pi$ and the coordinate r_1 has a negative value.* In general,

$$r_1 = r_A \cos(\phi_s - \phi_A). \quad (5.11a)$$

The magnitude of r_1/r_A thus depends upon the magnitude of $|\phi_s - \phi_A|$, being largest (and equal to unity) when this is 0 or π .

(iii). r_2 is normal to the r_1 - and z -axes. It, too, is measured numerically in an r -direction, but has a sign. Positive r_2 is 90° from the positive r_1 -axis (i.e., from the ϕ_s -plane), the rotation being taken in the negative ϕ -direction.

* If ϕ_s lies in the gap of a multigap instrument, then the positive r_1 -axis lies in the direction of the gap and the r_1 -component either points directly toward or away from this gap. Then r_1 is positive if the point A lies on the side of the z -axis toward the gap [i.e., up in Fig. (1.1)], and negative on the other side.

In general,

$$r_z = r_A \sin(\phi_s - \phi_A). \quad (5.11b)$$

The magnitude of r_z/r_A is thus at a maximum for $|\phi_s - \phi_A| = \frac{1}{2}\pi$.

(iv) If the trajectory is projected on the ϕ_s -plane, then the coordinate ψ is measured in the usual way on this projection [see Eq. (1.12b)]. Then ψ_s is the ψ -value of this projection in the field-free region, i.e., the angle formed by the z -axis and the line from $(z_s, 0)$ to (z_e, r_e) .

The rays emitted from the point A have trajectories which can be described in terms of two extreme classifications: those for which $p_\phi = 0$ and those for which $p_\phi \neq 0$, but p_ϕ has the largest possible value for any ray emitted from A at a given ψ_s [from Eq. (5.18), this evidently corresponds to the largest possible δr_z]. The general case, for a given ψ_s , is compounded of the characteristics of both of these classifications. We consider the various rays emitted at a given ψ_s -value.

Trajectories with $p_\phi = 0$. These trajectories occur when the point A lies either on the entry plane or its opposite, i.e., either $\phi_A = \phi_s$ or $\phi_A = \phi_s + \pi$. When emission occurs into either plane, there is no ϕ -component of motion. For a given point A, r_1 may take either of two opposite directions: for A in ϕ_s , since positive r_1 lies in ϕ_s , we have $\delta r_1 = +r_A$; for A in $\phi_s + \pi$, the r_1 -coordinate is negative and $\delta r_1 = -r_A$.

It is evident that the straight line representing the trajectory in the field-free region may be extended and will cut the z -axis at a point $(z_s + \delta z'_s, 0)$, and for such a trajectory, the particle will cross the z -axis on the focus side as though it did indeed originate at $(z_s + \delta z'_s, 0)$. We have

$$\delta z'_s = -\cot\psi_s \delta r_1. \quad (5.12)$$

It is evident that, when δr_1 is positive, its projection onto the z -axis is negative when $0 \leq \psi_s \leq \frac{\pi}{2}$ and positive when $\frac{\pi}{2} \leq \psi_s \leq \pi$; the opposite is true when δr_1 is negative. If we again consider the particle to enter the magnetic field at (z_e, r_e) , forming an angle $\psi_s + \delta\psi_s$ with the positive z -axis, we have [substituting $\delta z'_s$ for δz_s in Eq. (5.1a)]

$$\delta\psi_s = -\frac{\sin\psi_s \cos\psi_s}{z_{ff}(\psi_f)} e^{-K(\cos\psi_f - \cos\psi_s)} \delta r_1. \quad (5.13)$$

With this expression for $\delta\psi_s$, and using Eq. (5.7a) and (5.7b), we obtain

$$\delta z_f = - \frac{\cos \psi_s}{\sin \psi_f} \frac{1}{\psi_f'} \delta r_1. \quad (5.14)$$

If we take δr_{1f} as the distance of the trajectory from the z -axis at the plane $z = z_f$ and along the r_1 -axis, we have, then, as in Eq. (5.12),

$$\delta z_f = -\cot \psi_f \delta r_{1f}, \quad (5.15a)$$

which, when placed into (5.14), yields

$$\delta r_{1f} = \frac{\cos \psi_s}{\cos \psi_f} \frac{1}{\psi_f'} \delta r_1 = u(\psi_s) \delta r_1, \quad (5.15b)$$

with

$$u(\psi_s) = Q_f \frac{\tan \psi_f}{\tan \psi_s}. \quad (5.15c)$$

It is now left to determine the image of a point which has extension in both the z and r_1 directions, but which emits particles into a plane with $\phi = \text{constant}$, i.e., a point $(z_s + \delta z_s, \delta r_1)$. It is evident that for a particle which enters the field at (z_e, r_e) , the effective source point for the particle is obtained by algebraic addition of δz_s and $\delta z_s'$, the projection of δr_1 on the z -axis. Hence, the trajectory will intersect the z -axis on the focus side at $z_f + \delta z_f$, where δz_f is the algebraic combination of the δz_f -components due to either displacement separately. Thus,

$$\delta z_f = \frac{1}{\sin \psi_f (\psi_f')} [\sin \psi_s \delta z_s - \cos \psi_s \delta r_1]. \quad (5.16)$$

Trajectories with $p_\phi \neq 0$. Before taking up the general case of a ray emitted from A with any value of p_ϕ , we first consider a particular case in which the ray has a direction such that the coordinates of A are $\delta r_1 = 0$ and $\delta r_2 = \pm r_A$, i.e., $|\phi_s - \phi_A| = \frac{1}{2}\pi$. The general case can then be treated as a superposition of the two special cases: (i) a ray emitted in a direction such that A has only r_1 -extension, and (ii) a ray emitted such that A has only r_2 -extension.

Let the angle formed by the trajectory with the ϕ_s -plane be $\delta \xi_s$ and ρ_s = the distance from $(z_s, 0)$ to the profile at (z_e, r_e) . Since $\sin \psi_s = r_e / \rho_s$, then, to the approximation that $\tan \delta \xi_s = \delta \xi_s$, we have

$$\delta \xi_s \cong \frac{\delta r_2}{\rho_s} = \frac{\sin \psi_s}{r_e} \delta r_2 = \frac{\sin \psi_s}{z_{sf}(\psi_s)} \delta r_2. \quad (5.17)$$

From Eq. (1.8) and (1.11),

$$\delta a_0 = a_0 = \frac{p_\phi}{p} = \frac{mr^2\dot{\phi}}{mv} = r \frac{v_\phi}{v},$$

where $v_\phi = r\dot{\phi}$ is the ϕ -component of velocity [see Eq. (1.5)]. At the point of entry into the magnetic field, the trajectory makes the angle $\delta\zeta_s$ with the ϕ -plane; hence,* $v_\phi/v = \sin\delta\zeta_s$, so from (5.17)

$$\delta a_0 = r_e \sin\delta\zeta_s \cong r_e \delta\zeta_s = \sin\psi_s \delta r_2. \quad (5.18)$$

Fig. 5.2

We recall the results of Section #1 for trajectories with $p_\phi \neq 0$, that, to the first order in δa_0 , the equations for the r - and z -motions are the same as for the case where $p_\phi = 0$ [Eqs. (1.15a) and (1.15b)]. For the ϕ -motion, we use the same convention for $V(K, \psi)$ as we did for $U(K, \psi)$ in Eqs. (1.22a) and (1.26), that is, in evaluating $V(K, \psi)$, ψ is restricted to the range from 0 to 2π , and every passage through 2π involves ψ returning to zero, while ϕ has one period [Eq. (1.32)] added. Thus, from Eqs. (1.30) and (1.32), the difference in the ϕ -coordinates at the point of entry into and exit from the magnetic field is

* Note that $\delta\zeta_s$ is positive when the trajectory is such that ϕ is increasing as the particle enters the magnetic field; also, $\sin\psi_s$ is positive, since ψ_s generally lies between 0 and π . Then, since $\delta\zeta_s$ is positive only when δr_2 lies in the direction of decreasing ϕ , this is the direction of positive r_2 . In Fig. (5.2), according to the right-handed coordinate system convention of Section #1, it is seen that ϕ increases as it comes out of the paper (above the z -axis). The trajectory, passing from $(z_s, 0, \delta r_2)$ to (z_e, r_e) in the z - r_1 plane, has a positive ϕ , so $\delta\zeta_s$ is positive, as is δr_2 .

$$\phi_f - \phi_s = K[V(K, \psi_f) - V(K, \psi_s) + 2\pi(n_f - n_s) J_0(iK)] \frac{\delta a_0}{a}, \quad (5.19a)$$

since δa_0 is a constant of the motion [Eqs. (1.5) and (1.8)]. Then, from (2.3) and (5.18),

$$\Delta\phi = \phi_f - \phi_s = F_1 \frac{\delta r_2}{z_s}, \quad (5.19b)$$

where

$$F_1(K, \psi_f, \psi_s, n_f, n_s, f_s) = K \left[V(K, \psi_f) - V(K, \psi_s) + 2\pi(n_f - n_s) J_0(iK) \right] \left[\frac{\sin \psi_s}{f_s(\psi_s)} e^{-K \cos \psi_s} \right]. \quad (5.19c)$$

In Fig. (5.3) are plotted curves for $(\phi_f - \phi_s) \frac{z_s}{\delta r_2} = F_1(\psi_s)$ as a function of K and n_s for symmetric spectrometers* for the entry angle $\psi_s = \pi/2$.

Fig. 5.3

* It may be noted that, since for symmetric spectrometers, $\psi_f = -\psi_s$ (or $\psi_f + \psi_s = 2\pi$), $V(K, \psi_f) = -V(K, \psi_s)$. Similarly $n_f = -n_s$, and $f_s(\psi_s)$ is determined by K . Hence, F_1 becomes $F_1(K, \psi_s, n_s)$.

Again, for symmetric spectrometers, in Fig. (5.3a) we show $F_1(\psi_s)$ as a function of ψ_s for various K -values with $n_s = 0$.

Fig. 5.3a

A table corresponding to this case is given in Table II.4 of Appendix II. It may be noted that there are factors which tend both to increase and decrease F_1 as ψ_s is varied; these balance out to a maximum for each K -value, as shown in Fig. (5.3a). The various factors are:

(i). $V(K, \psi)$ increases in magnitude as ψ_s decreases to smaller ψ_s -values. The influence of this term corresponds to the fact that trajectories entering at small ψ_s -values have a much longer path through the magnetic field, and hence tend to cut across more ϕ -planes.

(ii). The effect of a given displacement δr_2 is smallest for small and large ψ_s , i.e., where $\sin \psi_s$ is small, since under these circumstances the angular momentum is minimal [Eq. (5.18)].

(iii). The factor which varies the least is the term

$$\frac{1}{a} = \frac{e^{-K \cos \psi_s}}{z_s f_s}; \text{ however, the point at which the orbit is vertical has}$$

a higher r -value for small ψ_s -values than for large ψ_s -values.

Since (ii) tends to depress F_1 at both ends of the ψ_s -range, while (i) is largest at the small ψ_s -end, the maximum is shifted towards smaller angles.

The trajectory emerges from the magnetic field at the point (z_0, r_0) ; this point and the z -axis define the plane $\phi = \phi_f = \text{constant}$. As on the source side, the component of the r -extension in this plane is defined by the r_{1f} -axis, and the corresponding third coordinate, perpendicular to the z - and r_{1f} -axes, is given by the r_{2f} -axis. Except for the case $\Delta\phi = 0$ (i.e., $p_\phi = \delta a_0 = 0$), the r_{1f} -axis is not parallel to the r_1 -axis, nor does r_{2f} parallel r_2 ; both axes are rotated by the angle $\Delta\phi$. However, the sign of δr_{2f} is also positive in the direction of decreasing ϕ , while the r_{1f} -axis positive direction* lies in the plane $\phi = \phi_f$. The angle the trajectory makes with the ϕ_f -plane (i.e., $\delta\zeta_f$) is related to δr_{2f} in just the same way that $\delta\zeta_s$ is related to δr_2 [Eq. (5.17)]. Thus, since, as in (5.18), $\delta a_0 = \sin \psi_f \delta r_{2f}$, we have**

$$\delta r_{2f} = \frac{\sin \psi_s}{\sin \psi_f} \delta r_2 = v(\psi_s) \delta r_2, \quad (5.20)$$

where δr_{2f} is the distance of the trajectory from the z -axis, as measured along the r_{2f} -axis.

General case. $p_\phi \neq 0$ and the ray has any direction; for any particular emission direction, A has the coordinates $z = z_s$, $\delta r_1 \neq 0$ and $\delta r_2 \neq 0$. This case may be considered as a linear combination of the two previous cases. Thus, we get δr_{1f} and δr_{2f} by applying Eqs. (5.14) and (5.20). The coordinate axes (r_{1f}, r_{2f}) are rotated relative to the source axes (r_1, r_2) by the angle $\Delta\phi$ [see Eq. (5.19b)].

* If ϕ_f lies in a gap, then the positive r_{1f} -axis points into the gap.

** Since ψ_f generally lies between π and 2π , it is seen that $\sin \psi_f$ is negative where $\sin \psi_s$ is positive, and δr_{2f} is of opposite sign to δr_2 . Thus, in the trajectory whose beginning is shown in Fig. (5.2), it is found that δr_{2f} lies on the opposite side of the z -axis from δr_2 (of course, rotated by $\Delta\phi$) and hence has a negative sign.

If the z -coordinate of the source point is $z_s + \delta z_s$, then the z -coordinate of the image is similarly displaced to $z_f + \delta z_f$, as in Eq. (5.10). This arises because the z -equations of motion, to the first approximation, are independent of p_ϕ , so δz_f is independent of δr_{1f} and δr_{2f} . The approximation that the effect of the δr_2 -coordinate may be treated independently is appropriate for ψ_s not close to 0 or π .*

For a source at $(z_s, 0)$ it has been shown that, for an arbitrary entrance curve $f_s(\psi_s)$, a corresponding exit curve $f_f(\psi_f)$ can be determined which will bring all trajectories, independent of their emergent angles ψ_s , through a preassigned point $(z_f, 0)$; i.e., the point source is perfectly imaged into a single point. With a source of finite dimensions, the question arises as to whether this arbitrariness in the entrance curve and focal point still exists. An examination of Eqs. (5.10), (5.15b) and (5.20) indicates that focusing does not occur, since the position of the trajectory depends upon ψ_s .

This explicit dependence on ψ_s is removed in a symmetric spectrometer, in which $\psi_f = -\psi_s$ (or $\psi_f + \psi_s = 2\pi$).** Then $\delta r_{1f} = -\delta r_1$ and $\delta r_{2f} = -\delta r_2$, while $\delta z_f = \delta z_s$. It is evident that for a source normal to the z -axis, i.e., having only r_1 and r_2 components, the magnification is -1 .

Thus, in a symmetric spectrometer, the ratios $\delta r_{1f}/\delta r_1$ and $\delta r_{2f}/\delta r_2$ are independent of ψ_s . For a particular ray, the distance $r_{A'}$ of the "image" point A' (intersection of ray with the plane $z = z_f$) from the point $(z_f, 0)$ is equal to r_A , since $r_A^2 = (\delta r_1)^2 + (\delta r_2)^2 = (\delta r_{1f})^2 + (\delta r_{2f})^2 = r_{A'}^2$. Although the magnitude of this distance is preserved in the symmetric spectrometer, the ϕ -coordinate of A' varies with the ϕ_s of the emitted ray, thus preventing perfect focusing.

This variation arises from the orbit rotation $\Delta\phi$ of Eq. (5.19b). Suppose we consider all rays emitted at a constant ψ_s , but variable ϕ_s . The variation in ϕ_s results in a variation in the magnitude of δr_2 , and (since F_1 is constant) $\Delta\phi$ is proportional to δr_2 . When A lies on ϕ_s or $\phi_s + \pi$, so

* The source point is projected on the z - r_2 plane along the line of flight and the intersection with this plane is the virtual source. To the first approximation, the r_2 -extension of the projection is the same as the δr_2 -coordinate of the point, but only if ψ is not too close to 0 or π , since

$$\text{the projection} \cong \delta r_2 + \frac{\delta r_1 \delta \zeta_s}{\sin \psi_s}.$$

** It is to be noted that Eq. (5.16) does not become independent of ψ_s .

The projection of δr_{1f} onto the z -axis does remain dependent upon ψ_s , being larger for entry angles closer to 0 or π . Although this has no significance relative to the imaging property, the effect is of importance for resolution considerations (see Section #6).

does δr_1 ; then $\delta r_2 = \delta r_{2f} = \Delta\phi = 0$, and δr_{1f} lies on $\phi_s + \pi$ or ϕ_s , respectively. When A is not on ϕ_s or $\phi_s + \pi$, then $\delta r_2 \neq 0$, but the nature of division of the radius vector r_A into the vector components δr_1 and δr_2 depends upon the angular separation $(\phi_A - \phi_s)$, as in Eq. (5.11). In a symmetric spectrometer, when $\delta r_2 \neq 0$, the radius vector $r_{A'}$ is rotated from the position taken when $\delta r_2 = 0$ by the angle $\Delta\phi$, i.e., the "image" radius vector is derived by rotating* the source radius vector through the angle $\pi + \Delta\phi$.

Suppose, on the other hand, we consider only rays emitted into the same ϕ_s from A, but at any ψ_s . All such rays will have the same δr_2 , but, since F_1 is a function of ψ_s [see Fig. (5.3a)], $\Delta\phi$ varies with ψ_s even when δr_2 is constant. The rotation $\Delta\phi$ thus varies both with ψ_s and with $(\phi_s - \phi_A)$. It is evident, therefore, that the "image" points of A will lie on a circle of radius r_A , and will cover a segment of the circle whose magnitude is determined by the range of $\Delta\phi$ -values. For a given ψ_s , the largest $\Delta\phi$ -value is determined by r_A , since $\Delta\phi \propto \delta r_2$, and the largest $|\delta r_2|$ -value is $|\delta r_2| = r_A$. For the entire ψ_s -range, the largest $\Delta\phi$ -value occurs for the highest F_1 -value, as shown in Fig. (5.3a).

FIG. 5.4a

This discussion is illustrated in the figures, which show these effects in a symmetric spectrometer. The z -axis is taken as perpendicular to the paper at O, the projection is taken looking from $z = z_s$ toward $z = z_f$, and both the image and source space are shown in the same projection. Figure (5.4a) shows an example of the case in which A lies on the plane ϕ_s . The r_2 -axis is shown at 90° clockwise rotation from the r_1 -axis, although there is no δr_2 -component. Since $\Delta\phi = 0$, the positive r_1 - and r_{1f} -axes point in the same directions, but the image point A' is inverted through 180° .

Figure (5.4b) shows an example of the case in which ϕ_A is perpendicular to ϕ_s , so that the r_1 -component is zero. Because $\delta r_2''$ is positive, so is $\Delta\phi$,

For the case in which A is in ϕ_s , the trajectory is equivalent to one which originates on the z -axis at $(z_s + \delta z_s, 0)$ with positive δz_s (when $\psi_s > 90^\circ$). The trajectory within the field is tilted relative to the trajectory originating at $(z_s, 0)$ so that the ray crosses the z -axis before hitting the plane $z = z_f$ at the image point A'. Hence the ray, which has been in the plane ϕ_s up to the z -axis, crosses over into $\phi_s + \pi$. Thus, $\phi_{A'}$ lies 180° away from $\phi_A = \phi_s$. For $\phi_A = \phi_s + \pi$, the trajectory is equivalent to one from the z -axis with δz_s negative [when $\psi_s > 90^\circ$], as in Fig. (5.1). The 180° transition is made in going from ϕ_A to ϕ_s , the trajectory then remaining in ϕ_s until it hits $z = z_f$ at A'. The sign of δz_s is inverted for $\psi_s < 90^\circ$, but the analysis is the same. For the case in which ϕ_A is normal to ϕ_s , the transition from A to ϕ_s changes the ψ -angle by 90° , the angle of the trajectory with the ϕ_s -plane being $\delta\psi_s$. Within the field ψ rotates through $\Delta\psi$. At the exit, the trajectory makes the angle $\delta\psi_f = -\delta\psi_s$ with the ϕ_f -plane, since the spectrometer is symmetric. The sign convention for $\delta\psi_f$ (related to the fact that positive r_2 and r_{2f} are in the same general directions) is such that this represents a continuation of the ray in the same direction. Hence, because of the symmetry of the instrument, the ray emerges from ϕ_f and hits A' on $z = z_f$ with $\phi_{A'} - \phi_f = 90^\circ$, and the total rotation is $\Delta\psi + \pi$. In the general case, the argument for the r_1 -component is the same as for the first case, except that the rotation of ϕ_f through $\Delta\psi$ results in a rotation of the r_1 -component by $\pi + \Delta\psi$. The argument for the r_2 -component is identical to that for the later case. Since the two components rotate through $\pi + \Delta\psi$, the point A does also in going over to the image point A'.

FIG. 5.4b

and in this projection this corresponds to a counter-clockwise rotation of ϕ . Thus, the image point is located by reflecting A in the origin to A' [from Fig. (5.4a)] and rotating through the positive angle $\Delta\phi$ to A'' . If we consider the arc length $\widehat{A'A''}$, we have, from Eq. (5.19b), since $\delta r_2 = r_A$,

$$\widehat{A'A''} = r_A \Delta\phi = r_A^2 (F_1/z_S). \quad (5.21a)$$

If ϕ_S is a plane 180° away from the one shown in Fig. (5.4b), then $\delta r_2'' = -r_A$, and $\Delta\phi$ would also be negative, so that

$$\widehat{A'A''} = r_A \Delta\phi = -r_A^2 (F_1/z_S). \quad (5.21b)$$

Figure (5.4c) shows an example of the general case. In the example, $\delta r_1'''$ is negative, while $\delta r_2''' > 0$; since $\Delta\phi \propto \delta r_2$, then $\Delta\phi$ is also positive, and hence counterclockwise. The vectors $(r_A, \delta r_1''', \delta r_2''')$ form the triangle OBA . The position of the final vectors $(r_A''', \delta r_1'''_f, \delta r_2'''_f)$ may be determined by reflecting OBA in the origin and then rotating it through $\Delta\phi$ to $OB'''A'''$. Since $\widehat{A'A'''} = r_A \Delta\phi$, we have

$$\widehat{A'A'''} = r_A^2 \sin(\phi_S - \phi_A) \frac{F_1}{z_S}. \quad (5.21c)$$

Thus, for a given magnitude of δr_2 , the two corresponding image points are separated by the arc distance

$$2 r_A^2 |\sin(\phi_S - \phi_A)| (F_1/z_S). \quad (5.22a)$$

The maximum value occurs for ϕ_S normal to ϕ_A ($\delta r_1 = 0$), and the arc distance between the two extreme points [see Eqs. (5.21a) and (5.21b)] is

$$2 r_A^2 (F_1/z_S). \quad (5.22b)$$

The two rays corresponding to $\phi_S = \phi_A$ and $\phi_S = \phi_A + \pi$, at a given ψ_S , have coincident image points, the separation in Eq. (5.22a) equalling zero. The entire arc in between the extremes is filled with image points corresponding to various ϕ_S -planes. This is true if it is assumed that the entire torus is available to provide ϕ_S -planes, neglecting the fact that in a real spectrometer some of the ϕ_S -planes will be occupied by poles or conductors. The latter effect simply introduces gaps in the arc corresponding to the ϕ_S -planes occupied by matter, as well as those ϕ_S -planes for which the rotation $\Delta\phi$ is large enough as to cause the trajectories to end up in a poleface.

FIG. 5.4c

For $r_A \ll r_e$ (the entrance coordinate), all ϕ_s -values are equally likely for the emission from A at a given ψ_s . Eq. (5.22a) then implies that the density of image points is greatest at the extremes of the arc, i.e., nearest the points described in Eqs. (5.21a), (5.21b), and (5.22b).

Since $F_1 = F_1(\psi_s)$, it is evident that the length of this arc varies with ψ_s according to Eq. (5.22b). We compute the magnitude of the arc for an extreme case. From Fig. (5.3a), it is evident that for the most useful K and ψ_s ranges F_1 lies between $\frac{1}{2}$ and 2. If $r_A/z_s = 0.02$,* then for $F_1 = 2$, we have $0.08 r_A$ as the distance between the two extreme image points.

It is evident that a source consisting of a circle of points A at constant radius r_A will image into a circle, also of radius r_A , for any emission angle. A circular source area will then image into a circular region of the same area. A line source, on the other hand, when emitting into the full 360° of ϕ_s , but at a fixed ψ_s , will yield an area as shown in Fig. (5.4d).

FIG. 5.4d

Taking the chord length as approximately equal to the arc length for small $A'A''/r_A$, the outer boundaries are parabolas [Eqs. (5.21a) and (5.21b)] whose curvature increases with F_1/z_s .

The actual image consists of a superposition of similar areas, one for each ψ_s . The outer boundary is then that due to the ψ_s with the largest F_1 -value. Because of this nesting of areas, the inner regions of small arc length tend to increase in density of points, at a given r_A , thus counteracting the effect of Eq. (5.22a). If the effects of the polefaces are included, the outer parabolas tend to be cut off as r_A increases, because $\Delta\phi$ becomes large enough to cause collision of rays with polefaces (see Section #7).

Thus far, we have discussed a source emitting into a spectrometer with available ϕ_s -values covering the entire 2π -range, i.e., a multigap sliced-orange type of instrument. We now examine the effect of rotation on the imaging property of a single sector instrument.

In Fig. (5.5), an unusually wide gap is shown for illustrative purposes, with gap boundaries (ϕ_0, ϕ'_0) and gap angle $\varphi = \phi_0 - \phi'_0$. Here AA_1 is a line source normal to the midplane ϕ_M , the half-length being such that,

* For $z_s = 30$ cm, the point A corresponds to an outer point of a 12-mm diameter source.

Fig. 5.5

at the ψ_s -angle considered, a particle leaving A and entering the profile at ϕ_M is rotated through $\frac{1}{2}\varphi$ and emerges from the profile at ϕ'_0 (i.e., $\Delta\phi = \frac{1}{2}\varphi$). On OA, it is seen that $\delta r_2 < 0$; the resulting negative $\Delta\phi$ corresponds to clockwise rotation of the image points. The maximum $\Delta\phi$ for a given distance from O occurs when $\delta r_1 = 0$, i.e., for $\phi_s = \phi_M$; thus, the lower boundary of the image (OC'B'A') corresponds to rays entering ϕ_M . Correspondingly, the minimum $\Delta\phi$, and hence the upper boundary (OB''A''), arise from points entering ϕ_0 . The image points between A' and A'' arise from ϕ_s -planes between ϕ_0 and ϕ_M ; no transmission occurs for $\phi_M > \phi_s > \phi_0$, because the rays are rotated into the poleface. As the source point moves in from A toward O, the critical limiting ϕ_s -plane, ϕ_C (i.e., no transmission for $\phi_C > \phi_s > \phi'_0$), moves toward ϕ'_0 , because $\Delta\phi$ decreases in magnitude. If the source line were extended to Q, a ray from Q to ϕ_M would not be transmitted, so the lower boundary would not lie on A'Q', although the upper boundary would lie on A''Q'', since a ray from Q to ϕ_0 would be transmitted.

If the entire ψ_s -range used is included, the resulting parabolic strip is increased in width, the lower boundary being due to the ϕ_M -rays for the ψ_s with the largest F_1 -value, and the upper boundary due to the ϕ_0 -rays for the ψ_s with the smallest F_1 -value.

Thus, the imaging property in the z_f plane of a finite source is not linear, nor even point to point. However, for small δr_2 , it is almost point to point, even though nonlinear. It may be noted that the rotation $\Delta\phi$ of the image does not depend upon the gap angle φ but only upon F_1 and $\delta r_2/z_s$. However, φ determines the maximum sample dimension for which a trajectory can pass through without colliding with a poleface. For such a δr_2 -value, the rotation from an extreme sample point is $\Delta\phi = \varphi$; then $\delta r_2/z_s = \varphi/F_1$. If, e.g., $\varphi = 3^\circ = 0.052$ radians, then for $K = 0.65$ and $\psi_s = \frac{1}{2}\pi$, it is found that $\delta r_2/z_s = 0.052/1.6 = 0.033$; then, if $z_s = 50$ cm, when rays are emitted into a single gap with $\psi_s = \frac{1}{2}\pi$ from portions of the sample for which $\delta r_2 > 1.6$ cm, none of these rays are transmitted.

The $\delta a_0/a$ approximation. It will be recalled that in Section #1, we postponed until this section the question of the validity of approximations in assuming $a_0 = \delta a_0$ small. We are now in a position to consider the meaning of this assumption in terms of sample size. We neglected all powers of $\delta a_0/a$ greater than the first. We have, from Eqs. (2.3) and (5.18), that

$$\frac{\delta a_0}{a} = \frac{\sin \psi_s}{r_e} e^{-K \cos \psi_s} \delta r_2 = e^{-K \cos \psi_s} \frac{\delta r_2}{\rho_s}. \quad (5.23)$$

Since K is generally smaller than unity, for ψ_s -values generally used, the exponential term has a maximum value between 1 and 2. Since ρ_s is generally approximately equal to z_s , we find that the approximation is justified when $\alpha \left(\frac{\delta r_2}{z_s} \right)^2 \ll 1$, where α is a small number determined by the value of the exponential above or the coefficients in Eqs. (1.15a), (1.15b) and (1.15c).

#6. DISPERSION AND RESOLUTION

Up to this point, we have considered only monoenergetic charged particles of momentum mv . We turn now to a consideration of the image of particles emitted from the same source, i.e., at $(z_s, 0)$, whose momentum differs from the original momentum by a small amount and has the value $mv + \delta(mv)$.

For a particular instrument, and hence a given source point $(z_s, 0)$ and image point $(z_f, 0)$, the constant K [see Eq. (1.10)] is a characteristic property of the instrument. Thus, an increase in the momentum of the electrons to be focused at $(z_f, 0)$ involves a proportionate increase in A_0 [see Eq. (1.1)] to the value A_0' . We have then

$$K = -\frac{mv}{A_0 e} = -\frac{mv + \delta(mv)}{A_0' e} \quad (6.1)$$

We may, however, consider a related situation. Suppose we keep the field at the value given by A_0 , and hence focus electrons of momentum mv at the point $(z_f, 0)$. Electrons of momentum $mv + \delta(mv)$ will approximately focus and form an "image" at another point $(z_f + \delta z_f, 0)$. For these electrons, if we consider the source and image points to be $(z_s, 0)$ and $(z_f + \delta z_f, 0)$, respectively, the instrument "constant" will in effect have been changed to $K + \delta K$. Thus, with the magnetic field which focuses electrons of momentum mv at $(z_f, 0)$, we have

$$K + \delta K = -\frac{mv + \delta(mv)}{A_0 e} \quad (6.2)$$

The trajectory of one of these particles will differ from the previous ones only after entrance into the magnetic field. It will intercept the exit boundary curve $f_f(\psi_f)$ at the point $(z_0 + \delta z_0, r_0 + \delta r_0)$, where (z_0, r_0) is the intersection point for an electron of momentum mv and the same initial emission angle ψ_s . At $(z_0 + \delta z_0, r_0 + \delta r_0)$, the trajectory will make the angle $\psi_f + \delta\psi_f$ with the positive z -axis [see Fig. (6.1)].

Fig. 6.1

The calculation is made by a variation procedure similar to that used in Section #5. The trajectory of a particle with momentum $mv + \delta(mv)$ is determined as a variation of the trajectory of a particle of momentum mv . In Section #5, a similar variation was performed, in which the coordinates of the varied trajectory were determined with $\delta\psi$ and δz_s (or the equivalent $\delta\psi_s$) as the varied quantities, while r_e was kept

constant. In the present case, z_s , ψ_s , and r_e are kept constant, while the variations considered are δK and $\delta\psi$. In Eq. (2.5), if we substitute $K+\delta K$ for K , we have the new trajectory in Fig. (6.1). As before, we consider the variation around a point (z, r) on the old trajectory characterized by K ; this point has a particular ψ -value. Any point $(z+\delta z, r+\delta r)$ in the neighborhood of (z, r) can be reached by first varying K to $K+\delta K$, and then varying ψ to $\psi+\delta\psi$. By expanding $z+\delta z$ and $r+\delta r$ around K and ψ , then δz and δr are calculated in terms of δK and $\delta\psi$. [These relations are equivalent to those in Eq. (5.3).]

As before, δz and δr are evaluated at the point that the varied trajectory hits the exit boundary, so that ψ becomes ψ_f . The variation of the exit point (z_0, r_0) is performed exactly as in Section #5, so that δz_0 and δr_0 are given as functions of $\delta\theta$ by Eq. (5.4). Equating $\delta r = \delta r_0$ and $\delta z = \delta z_0$, we get the relations

$$\delta\theta = \frac{1}{\Lambda_f} [(\cos\psi_s - \cos\psi_f)\delta K + K\sin\psi_f\delta\psi_f] \quad (6.3a)$$

and

$$\delta\psi_f = \left[\left(\Lambda_f \cos\psi_f - \frac{1}{\sin\psi_f} \right) (\cos\psi_s - \cos\psi_f) - F_2 \Lambda_f \sin\psi_f e^{K\cos\psi_f} \right] \frac{\delta K}{K}, \quad (6.3b)$$

where Λ_f is defined in Eq. (5.4c),

$$F_2 = F \cos\psi_s + \frac{\partial F}{\partial K}, \quad (6.4a)$$

and F is defined in Eq. (5.3c) as

$$F = K [U(K, \psi_f) - U(K, \psi_s) + 2\pi(n_f - n_s) iJ_1(iK)] \quad (6.4b)$$

Since the cause of the variation of the trajectory is immaterial after the orbit reaches the exit boundary, Eq. (5.7a) relates δz_f and the quantity $(\delta\psi_f - \delta\theta)$. We have, then, from Eqs. (6.3a), (6.3b) and (5.7b),

$$\begin{aligned} \delta z_f = \frac{r_0}{\sin^2\psi_f} & \left\{ \frac{r_0}{z_s - z_f} \left[\frac{1}{\sin^2\psi_f} - \frac{1}{(\psi_f') \sin^2\psi_s} e^{-K(\cos\psi_s - \cos\psi_f)} \right] \right. \\ & \times \left[\cos\psi_f(\cos\psi_s - \cos\psi_f) - F_2 \sin\psi_f e^{K\cos\psi_f} \right] \\ & \left. - \left[\frac{\cos\psi_s - \cos\psi_f}{\sin\psi_f} \right] \right\} \frac{\delta K}{K}, \end{aligned} \quad (6.5)$$

with (ψ_f') defined in Eq. (5.8).

Using Eq. (3.3), this may be written as

$$\delta z_f = G(z_f - z_s) \frac{\delta K}{K}, \quad (6.6)$$

where

$$\begin{aligned} G &= G(\psi_s, \psi_f, n_s, n_f, K, \psi_f') = - \frac{\frac{\delta z_f}{z_s - z_f}}{\frac{\delta K}{K}} = - \frac{\frac{\delta z_f}{z_s - z_f}}{\frac{\delta p}{p}} \\ &= \frac{e^{-K \cos \psi_f}}{F_3 \sin^2 \psi_f} \left\{ \left[\frac{e^{-K \cos \psi_s}}{(\psi_f') F_3 \sin^2 \psi_s} - \frac{e^{-K \cos \psi_f}}{F_3 \sin^2 \psi_f} \right] \right. \\ &\quad \times \left[\cos \psi_f (\cos \psi_s - \cos \psi_f) - F_2 \sin \psi_f e^{K \cos \psi_f} \right] - \left. \left[\frac{\cos \psi_s - \cos \psi_f}{\sin \psi_f} \right] \right\}, \end{aligned} \quad (6.7a)$$

with

$$F_3 = - \frac{(z_s - z_f)}{r_0} e^{-K \cos \psi_f} = \cot \psi_s e^{-K \cos \psi_s} - \cot \psi_f e^{-K \cos \psi_f} + F. \quad (6.7b)$$

From Eq. (6.4b) we have

$$\frac{\partial F}{\partial K} = K \left[\frac{\partial U(K, \psi_f)}{\partial K} - \frac{\partial U(K, \psi_s)}{\partial K} + 2\pi(n_f - n_s)i \frac{\partial J_1(iK)}{\partial K} \right] + \frac{F}{K} \quad (6.8)$$

and, by the methods of Appendix I,

$$\begin{aligned} \frac{\partial U(K, \psi)}{\partial K} &= - \int_{\pi}^{\psi} \cos^2 \psi e^{-K \cos \psi} d\psi = \frac{1}{2} [J_0(iK) - J_2(iK)] [\pi - \psi] \\ &\quad - \frac{1}{2} \sum_{\ell=1}^{\infty} \frac{i^{\ell-2}}{\ell} [J_{\ell-2}(iK) - 2J_{\ell}(iK) + J_{\ell+2}(iK)] \sin \ell \psi \end{aligned} \quad (6.9a)$$

and

$$\frac{\partial J_1(iK)}{\partial K} = \frac{1}{2} i [J_0(iK) - J_2(iK)] \quad (6.9b)$$

Dispersion. For a particular instrument, it is usually more convenient to use the dispersion rather than the G-function, where the dispersion is defined as

$$\sigma_f = - \frac{dz_f}{dK/K} = - \frac{dz_f}{dp/p} = \frac{dz_f}{dA_0/A_0} \quad , \quad (6.10a)$$

so that

$$\sigma_f = (z_s - z_f) G \quad . \quad (6.10b)$$

It is often convenient to consider the percentage rather than the fractional change in momentum. We then use another dispersion measure:

$$\sigma_{\%} = \sigma_f / 100 \quad . \quad (6.10c)$$

In the symmetrical case, since $\psi_f + \psi_s = 2\pi$, it is found that G is given by the equation

$$G = - 2 \frac{e^{-K \cos \psi_s}}{F_3^2 \sin^3 \psi_s} F_2 \quad , \quad (6.11)$$

with

$$F_3 = 2 \cot \psi_s e^{-K \cos \psi_s} + F = - (2z_s / r_e) e^{-K \cos \psi_s} \quad (6.12a)$$

and*

$$\begin{aligned} F &= K [U(K, -\psi_s) - U(K, \psi_s) - 4\pi n_s i J_1(iK)] \\ &= -K [2 U(K, \psi_s) + 4\pi n_s i J_1(iK)] \quad , \end{aligned} \quad (6.12b)$$

and

$$\begin{aligned} \frac{\partial F}{\partial K} &= - 2K \frac{\partial U(K, \psi_s)}{\partial K} + \frac{F}{K} \\ \frac{\partial F}{\partial K} &= K \left\{ 2 \int_{\pi}^{\psi_s} \cos^2 \psi e^{-K \cos \psi} d\psi + 2\pi n_s [J_0(iK) - J_2(iK)] \right\} + \frac{F}{K} \\ &= K \left\{ -[J_0(iK) - J_2(iK)] [\pi - \psi] \right. \\ &\quad + \sum_{\ell=1}^{\infty} \frac{i^{\ell-2}}{\ell} [J_{\ell-2}(iK) - 2J_{\ell}(iK) + J_{\ell+2}(iK)] \sin \ell \psi \\ &\quad \left. + 2\pi n_s [J_0(iK) - J_2(iK)] \right\} + (F/K) \quad . \end{aligned} \quad (6.12c)$$

*Since, from Eqs. (1.18) or (1.19), $U(K, \psi_f) = U(K, -\psi_s) = U(K, 2\pi - \psi_s) = -U(K, \psi_s)$.

Finally,

$$\sigma_f = 2 z_s G \quad . \quad (6.13)$$

It is evident that G is a more universal function than σ_f , involving as it does only the shape of the instrument rather than including the size.

In Fig. (6.2) are given values of $2G = \sigma_f/z_s$ as a function of (K, n_s) for symmetric spectrometers, but only for $\psi_s = \pi/2$. In Fig. (6.3) is shown the variation of σ_f/z_s with ψ_s for two special symmetrical cases. Figures (6.4) and (6.4a) show the variation of G with ψ_s for various K -values in symmetrical spectrometers. A table of the function is given in Table II.5 of Appendix II. For nonsymmetrical spectrometers, $\partial U/\partial K$ is the only function involved which is not already tabulated, and for this Table II.6 is given in Appendix II.

FIG. 6.2

Fig. 6.3

Dispersion considerations alone would dictate that the most favorable instrument would have large n_s . However, several considerations lead us to choose instruments with $n_s = 0$ instead of large n_s ; these are (i) loss of transmission from a finite source due to rotation of the particles into the pole faces, and (ii) only with $n_s = 0$ can one suppress the satellite peaks by using baffles.

Fig 64a

Resolution. Using the dispersion computed as above, it is in principle possible to evaluate the resolving power, which is defined as the relative width of the line profile at half the height of the line. A computation of this type is quite involved for this kind of instrument. The shape of the profile for a given ψ_s must first be determined. Then, since the dispersion varies with ψ_s , the superposition of the profiles over ψ_s adds further complexity. In an actual instrument, the fringing field (see Sections #8 and #9) further complicates the problem because of a number of effects, the most important of which involves changes in the detector dimensions in order to conserve transmission. In the following, however, the fringing field effect is not considered.

For purposes of computation, the following simplifying assumptions are made to start with:

- (i) no fringing-field effect;
- (ii) the source is uniformly distributed;

(iii) the source is extended axially with dimensions dz_s . The basic treatment is made on this basis because of simplicity. As was made evident in Section #5 [e.g., see Eq. (5.12)], an r_1 -extension may be

transformed into an equivalent axial extension by projection. Sources with r -extension will be examined following the simpler treatment involving only axial source extension;

(iv) the detector slit opening is axial. As in (iii), r -extension of the detector slit will be introduced following the preliminary simpler treatment;

(v) solid angle $d\omega$ with an emission interval $d\psi_s$ in the neighborhood of the fixed value ψ_s . Following a computation at fixed ψ_s , the effect of integrating over ψ_s is determined; and

(vi) detector slit at z_f with infinitely narrow axial extension. The effect of finite detector extent is introduced later.

The resolution is evaluated starting with these six simplifying assumptions, and then re-evaluated relaxing several of the assumptions.

From Eq. (5.10), the axial dimensions of the image is given by

$$\delta z_f = Q_f \delta z_s \quad ; \quad Q_f = \frac{dz_f}{dz_s} \quad . \quad (6.14)$$

Because the detector is infinitely narrow, only particles arriving at $(z_f, 0)$ are detected. At a given ψ_s , only one point on the source emits rays which are focused and detected at $(z_f, 0)$ and variation of the field intensity is used to sweep over all of the source points. If δA_0 is the range of specific field intensity required to sweep over the entire source length δz_s , then the resolution is the relative A_0 -range over which particles are detected at a particular ψ_s . Thus,

$$\mathcal{R}'_0 = \left| \frac{\delta A_0}{A_0} \right| = \left| \frac{-\delta K/K}{\delta z_f} \delta z_f \right| = \left| \frac{\delta z_f}{\sigma_f} \right| = \left| Q_f \frac{\delta z_s}{\sigma_f} \right| \quad . \quad (6.15)$$

The line profile is rectangular, with relative base width $\mathcal{R}'_0$.

Fig. 6.5

The first simplifying assumption to be dropped is (vi), with D_z = axial dimension of the detector slit. It is evident from Fig. (6.5) that the relative base width of the line is

$$\mathcal{R}'_b = \frac{|\delta z_f| + D_z}{\sigma_f} \quad . \quad (6.16a)$$

For convenience of notation, we shall write δz_f for $|\delta z_f|$, so that Eq. (6.16a) becomes

$$\mathcal{R}'_b = \frac{\delta z_f + D_z}{\sigma_f} \quad . \quad (6.16b)$$

In the following, absolute values are taken for all lengths.

The line profile is a trapezoid, the region of constant transmission being the range in which δz_f lies entirely within the slit (if $\delta z_f < D_z$) or the slit lies entirely within the image (if $\delta z_f > D_z$).

For $\delta z_f > D_z$, the relative length of the region of constant transmission is $(\delta z_f - D_z)/\sigma_f$, so that the relative width at half-height is

$$\mathcal{R}' = \frac{1}{2} \left[\frac{\delta z_f - D_z}{\sigma_f} + \frac{\delta z_f + D_z}{\sigma_f} \right] = \frac{\delta z_f}{\sigma_f} = \mathcal{R}'_0 \quad , \quad (6.17)$$

i.e., the same as for $D_z \approx 0$.

The height of the peak is proportional to $d\omega$ and to the fraction of the source overlapping the slit, i.e., the transmission is

$$t' = d\omega \frac{D_z}{\delta z_f} \quad . \quad (6.18)$$

The area S' under the profile is a useful measure of relative transmission, and this is the product of the half-width and height t' , so

$$S' = d\omega (D_z/\sigma_f) \quad . \quad (6.19)$$

For $\delta z_f < D_z$, the line profile is also a trapezoid, in which the relative length of the region of constant transmission is $(D_z - \delta z_f)/\sigma_f$, so that the resolution is

$$\mathcal{R} = D_z/\sigma_f \quad . \quad (6.20)$$

At the peak, the entire source is detected, so that the peak height and profile area are

$$t' = d\omega = t_M^i \quad (6.21a)$$

and

$$S' = d\omega (D_z/\sigma_f) \quad , \quad (6.21b)$$

where t_M^i is the maximum peak height. Since S' is the same for both $\delta z_f < D_z$ and $\delta z_f > D_z$, it is a more usually used measure of relative intensity.

In the intermediate limiting case, $\delta z_f = D_z$ and the profile is triangular. In this case,

$$t' = d\omega = t'_M \quad (6.22a)$$

and

$$S' = \frac{1}{2} d\omega \left[\frac{D_z + \delta z_f}{\sigma_f} \right] = d\omega \frac{D_z}{\sigma_f} \quad (6.22b)$$

These quantities are shown as a function of the detector width D_z in Fig. (6.6). The optimum value of D_z is at $D_z = \delta z_f$; at this value, the best (minimum) resolution, $\mathcal{R}'_m$, is achieved at the same time as the maximum peak height, $t'_M = d\omega$.

The next simplifying assumption to be dropped is (v). When all the ψ_s -values in the acceptance range are included, the line profile is made up of the partial peaks obtained for the $d\psi_s$ -intervals at each of the various ψ_s -values. An exact calculation involves a summation of the partial peaks, for which we give an approximate solution.

From Eqs. (5.10), (6.7) and (6.10), δz_f and σ_f are, in general, both functions of ψ_s . Over the entire range of ψ_s , we shall have a maximum and minimum value of δz_f , namely δz_{fM} and δz_{fm} , respectively. Similarly, there are a maximum and minimum value of σ_f , namely, σ_{fM} and σ_{fm} , respectively. These occur at four different ψ_s -values.

We first consider the case in which $D_z \leq \delta z_{fm}$. Then each partial peak has its minimum resolution:

$$R'_m(\psi_s) = R'_0 = \delta z_f / \sigma_f \quad . \quad (6.23)$$

It is readily shown that, to a good approximation, the minimum resolution of the entire peak is also independent of D_z , for $D_z \leq \delta z_{fm}$. It is known for many types of line shapes that it is a fairly good approximation to take the area as the product of peak height and half-width.* Thus, we define

$$R_S = S/t \quad , \quad (6.24a)$$

where S is the area of and t the height of the entire peak. If R is the actual resolution of the entire peak, then, approximately,

$$R \approx R_S \quad . \quad (6.24b)$$

Since the area represents all the particles detected over all ψ_s , from Eq. (6.19) we find

$$S = \int S'(\psi_s) d\psi_s = D_z \int \frac{d\omega}{\sigma_f} = D_z \varphi \int \frac{\sin \psi_s d\psi_s}{\sigma_f(\psi_s)} \quad . \quad (6.25)$$

* The relation is exactly true for rectangular, triangular or trapezoidal profiles. Even for a Gaussian $R't'$, area = 0.939 if the area is integrated to $\pm\infty$. Since, however, the basewidth is limited in a spectrometer, it is more appropriate to consider a truncated Gaussian. If the truncation is performed by extrapolating the slope at the steepest part of the sides to the base-line, the intersection is at $\pm 2\sigma$, and the ratio

$$R't' / \text{truncated area} = 0.984 \quad .$$

The same type of truncation (extrapolation of steepest slope) for a Lorentz shape, intersecting the base at $\pm \frac{1}{2}\sqrt{3} t'$, gives the ratio

$$R't' / \text{truncated area} = 0.955 \quad .$$

Thus, the relation is exactly true for each $d\psi_s$ -region, and is probably true to a good approximation for the entire ψ_s -region.

If the spectrometer has perfect, or almost perfect, ψ -focusing, then the partial peaks have their centers coincident or very closely overlapping. Since the partial peak heights represent transmission, the total transmission over all ψ_s is, from Eq. (6.18),

$$t = \sum t'(\psi_s) = D_z \int \frac{d\omega}{\delta z_f(\psi_s)} = D_z \varphi \int \frac{\sin \psi_s d\psi_s}{\delta z_f(\psi_s)} \quad (6.26)$$

Thus, as was the case for each $d\psi_s$ -region [see Eqs. (6.18), (6.19) and Fig. (6.6)], both S and t are proportional to D_z for $D_z \leq \delta z_{fm}$. Hence, R_S is independent of D_z and takes on the minimum value (relative to D_z -variation) which we may call R_m . With the approximation given by Eq. (6.24b), then

$$R \cong R_m \text{ for } D_z \leq \delta z_{fm} \quad (6.27)$$

To evaluate R_m , it is necessary to compute the integrals of Eqs. (6.25) and (6.26), or we may write these as

$$R \cong R_m = R_S = \frac{\int \frac{\sin \psi_s}{\sigma_f(\psi_s)} d\psi_s}{\int \frac{\sin \psi_s}{|Q_f(\psi_s)|} d\psi_s} \delta z_s \quad (6.28)$$

from Eq. (5.10). We note that outside limits may be set in the form of the inequalities

$$\frac{\delta z_{fm}}{\sigma_{fM}} \leq R_m \leq \frac{\delta z_{fM}}{\sigma_{fm}} \quad (6.29)$$

It is evident that the maximum transmission occurs when the ψ -focusing is perfect, if the largest image lies within the slit, i.e., $D_z \geq \delta z_{fM}$. Eq. (6.21) applies for t' and S' , so that the resolution R_{tM} corresponding to maximum transmission is

$$R \cong R_{tM} = \frac{S}{t} = \frac{\int \frac{\sin \psi_s}{\sigma_f} d\psi_s}{\int \sin \psi_s d\psi_s} D_z \text{ for } D_z \geq \delta z_{fM} \quad (6.30)$$

Thus, for $D_z \geq \delta z_{fM}$, t_M is proportional to D_z . It is evident, then, that the value of D_z which optimizes resolution and transmission lies between limits indicated by

$$\delta z_{fm} \leq D_z \leq \delta z_{fM} \quad (6.31)$$

These points are illustrated in Fig. (6.7).

Fig. 6.7

The symmetric instruments are much simpler to consider. Then $Q_f = 1$ and $\delta z_f = \delta z_s = \text{constant}$. When $D_z = \delta z_s$, then both resolution and transmission are optimized, since $R_m = R_{tM} \cong R$, from Eqs. (6.28) and (6.30). A figure illustrating this result would be similar to Fig. (6.6) if ψ -focusing were perfect.

If, in addition, the ψ_s -range is chosen so that σ_f is a monotonic function of ψ_s [see Fig. (6.4)], the half-width is readily computed.

For $D_z \leq \delta z_s$, we find $R'_m = \delta z_s / \sigma_f(\psi_s)$, with a trapezoidal profile, for a region $d\psi_s$ around ψ_s . Then $t' = C'd\omega$, where $C' = D_z / \delta z_s$ and is unity when $D_z = \delta z_s$. The trapezoids have monotonically increasing (or decreasing) half-widths as ψ_s increases. The line profile may be obtained [see

Fig. (6.8)] by placing one trapezoid above the previous one in the order given by the corresponding ψ_s -values, beginning with the one of maximum half-width. The height of the peak is given by

$$t = C' \int d\omega = C' \varphi \int_{\psi_{sm}}^{\psi_{sM}} \sin \psi_s d\psi_s = C' \varphi [\cos \psi_{sm} - \cos \psi_{sM}] \quad (6.32)$$

The half-height of the profile obtained in this way corresponds to an angle ψ_{sh} given by the relation

$$\frac{1}{2} \int_{\psi_{sm}}^{\psi_{sM}} \sin \psi_s d\psi_s = \int_{\psi_{sm}}^{\psi_{sh}} \sin \psi_s d\psi_s, \quad (6.33a)$$

or

$$\cos \psi_{sh} = \frac{1}{2} (\cos \psi_{sm} + \cos \psi_{sM}) \quad (6.33b)$$

Because the trapezoids are only infinitesimally high, the half-width of the

Fig. 6.8

trapezoid corresponding to $d\psi_s$ at ψ_{sh} gives the half-width of the entire line. Then

$$\mathcal{R}_m = \delta z_s / \sigma_f(\psi_{sh}) \quad . \quad (6.34)$$

A similar argument may be given if σ_f is not monotonic over the ψ_s -range used but has only one minimum (or maximum). The results are somewhat more complicated and are not given here.

We now modify assumptions (iii) and (iv), taking a linear source and detector normal to the z -axis. The source has extension δr_1 in the r_1 -direction, while the detector has extension D_{r_1} in the r_1 -direction. (We neglect the variation in the r_1 -direction over a gap.) Considering dz as the projection of δr_1 onto the z -axis, taken along ψ_s or ψ_f , we have, from Eqs. (5.12) and (5.15a),

$$\delta r_1 = -\delta z_s \tan \psi_s \text{ and } \delta r_{1f} = -\delta z_f \tan \psi_f \quad . \quad (6.35)$$

In analogy to Eq. (6.10a), we introduce the radial dispersion:

$$\sigma_f^r = \frac{\delta r_{1f}}{\delta A_0 / A_0} = -\sigma_f \tan \psi_f \quad . \quad (6.36)$$

From Eqs. (5.15b) and (5.15c),

$$u(\psi_s) = \frac{\delta r_{1f}}{\delta r_1} = Q_f \frac{\tan \psi_f}{\tan \psi_s} \quad . \quad (6.37)$$

The previous results with an axial source may be extended to the radial source by considering the projection of δr_1 onto the axis and substituting this projection into the corresponding equation. Thus, from Eq. (6.15), with $D_{r_1} = 0$,*

$$\mathcal{R}'_0 = \left| \frac{-\delta r_{1f}}{\sigma_f \tan \psi_f} \right| = \left| \frac{-\delta r_1}{\sigma_f \tan \psi_s} Q_f \right| = \left| \frac{\delta r_{1f}}{\sigma_f^r} \right| = \left| u(\psi_s) \frac{\delta r_1}{\sigma_f^r} \right| \quad . \quad (6.38)$$

From Eq. (6.16), with $D_{r_1} \neq 0$, the relative base width is

$$\mathcal{R}'_b = \frac{(\delta r_{1f} + D_{r_1})}{-\sigma_f \tan \psi_f} = \frac{\delta r_{1f} + D_{r_1}}{\sigma_f^r} \quad . \quad (6.39)$$

*As above, only absolute values of lengths, coefficients and projection functions ($\tan \psi_s$ and $\tan \psi_f$) are taken.

For $\delta r_{1f} \geq D_{r1}$,

$$\text{Half-width} \quad R' = \delta r_{1f} / \sigma_f^r = R'_0 \quad (6.40)$$

$$\text{Peak height} \quad t' = d\omega(D_{r1} / \delta r_{1f}) \quad (6.41)$$

$$\text{Profile area} \quad S' = d\omega(D_{r1} / \sigma_f^r) \quad (6.42)$$

For $\delta r_{1f} \leq D_{r1}$,

$$\text{Half-width} \quad R' = D_{r1} / \sigma_f^r \quad (6.43)$$

$$\text{Peak height} \quad t' = d\omega = t'_M \quad (6.44)$$

$$\text{Profile area} \quad S' = d\omega(D_{r1} / \sigma_f^r) \quad .$$

Similarly, for the complete ψ_s -range, the analogs of Eqs. (6.23) to (6.34) are made by substituting the radial quantities for the axial ones. It should be pointed out in using the method involved in Eqs. (6.32) to (6.34), that σ_f^r is not necessarily monotonic over the ψ_s -range used, even when σ_f is, because of the $\tan\psi_s$ -factor. From Eq. (6.38),

$$R'_0 = \left| Q_f \frac{\delta r_1}{\sigma_f \tan\psi_s} \right| \quad .$$

For the range over which $|\tan\psi_s| > 1$, i.e., for $45^\circ < \psi_s < 135^\circ$, it is evident from Eq. (6.15) that the resolution of a radial source is better than that of an axial source of the same dimensions.

The study of sources and detectors which have both z - and r_1 -extension may be done by considering the equivalent z - (or r -) projections. For example, the equivalent axial image of such a source is*

$$\delta z_f = (\delta z_s - \cot\psi_s \delta r_1) Q_f \quad (6.45)$$

For the ψ_s -range $45^\circ < \psi_s < 135^\circ$, a radial source is preferable because it gives a smaller image on the z -axis than an axial source of the same extension.

For a symmetric instrument, it is desirable that both the source and detector have the same extension (either in the z -extension or r_1 -extension). For a symmetric instrument, the image size is constant, independent of ψ_s , and when the detector has the same size, maximum transmission is attained with minimum resolution.

*In this relation, the signs of the various terms are included.

Strong fringing fields may affect the transmission conditions to a great extent, as will be discussed in detail in Section #9. They influence only the image size and distribution of rays in the r_2 -direction, if we disregard the slight deviation of point-to-point focusing in the r_1 -direction because of trajectory rotation (Section #5). However, the resulting shift in the required r_2 -dimension of the detector (for maximum transmission) affects the resolution because the r_2 -direction for one gap is the r_1 -direction for a gap 90° around the torus. The required shift in the detector dimension thus leads to a worsening of the resolution. In this case, it is not feasible to maximize transmission without worsening resolution simultaneously. However, once having chosen D_{r_1} , the calculation proceeds as before, provided D_{r_1} is constant for all r_{2f} -values. When this is not true, a detailed integration (as in Section #9) is required to consider the entire line profile. Of course, instead of using $d\omega = \sin\psi_s d\psi_s$ as a weighting factor, it is necessary to use the transmission factor appropriate for fringing fields, e.g., $\eta_d d\omega$. Methods for improving transmission in the presence of fringing fields without worsening resolution are discussed in Section #9.

#7. TRANSMISSION OF THE SPECTROMETER. TRANSMISSION DECREASE DUE TO SOURCE SIZE

The transmission T of the instrument may be defined as the ratio between the number of particles detected per unit time and the number of particles emitted during that time. Such a general definition, however, does not allow computation of numerical values from the structure of the instrument, allowing for such factors as geometrical efficiency, detector efficiency, source-size effect, or fringing-field effect. It is clear that the maximum transmission is equal to the geometrical efficiency, other effects serving only to decrease the transmission. We shall find it convenient to express the transmission as a product of separate terms, the dominating term being the geometric efficiency, while the others are independent factors which express the decrease in transmission due to other effects. Thus,

$$T = \frac{\Omega_0}{4\pi} \bar{\eta}_1 \bar{\eta}_{23} \quad , \quad (7.1)$$

where Ω_0 is the theoretical solid angle subtended by the instrument, $\bar{\eta}_1$ is the mean detector efficiency, and $\bar{\eta}_{23}$ is an average transmission effect determined by properties of the instrument which will be discussed shortly. In general, $\bar{\eta}_1$ is energy-dependent, at least to some extent, but is usually ϕ - and ψ_s -independent. We shall treat it as independent of the spectrometer properties. If other η_i terms are required, they may be included in Eq. (7.1).

If calculated for a centered point source, Ω_0 is given by the equation

$$\Omega_0 = j\omega_0 = j \int_{\psi_{s\min}}^{\psi_{s\max}} \phi \sin\psi_s \, d\psi_s = j\phi [\cos\psi_{s\min} - \cos\psi_{s\max}] \quad , \quad (7.2)$$

where

- j = number of gaps
- ϕ = angular aperture of the gaps
- $\psi_{s\min}$ = the smallest emission angle considered
- $\psi_{s\max}$ = the largest emission angle considered.

In computing Ω_0 , the entrance aperture was defined by the boundaries of the entrance profiles. Some of the particles passing through this aperture may, however, be stopped by two apertures further along the trajectory:

- (i) the exit aperture, defined by the boundaries of the exit profiles, and

(ii) the detector aperture. The numerical measure of these two effects is given by η_2 and η_3 , which are defined for rays emitted from a single point on the source, at a given ψ_s . Here η_2 is the fraction of those particles which have passed through the entrance aperture and which then also pass through the exit aperture of the same gap; the fraction of those transmitted through the exit aperture and which also enter the detector is η_3 . Then

$$\eta_{23} = \eta_2 \eta_3 \quad , \quad (7.3)$$

where η_{23} is also defined at a given ψ_s -value and emitting point. The quantity $\bar{\eta}_{23}$ is determined by integrating η_{23} over the entire sample and ψ_s -range.

Up to this point, trajectories have been computed neglecting fringing-field effects. Under these conditions, for a point source $\eta_2 = \eta_3 = \eta_{23} = 1$, i.e., only the entrance aperture acts as a limiting baffle, since all trajectories remain on ϕ -planes ($p_\phi = 0$) and end at $(z_f, 0, 0)$. However, where a trajectory has a p_ϕ -component, the resulting orbit rotation $\Delta\phi$ may cause $|\phi_f - \phi_M|$ to exceed $\frac{1}{2}\phi$, implying that the trajectory does not penetrate the exit aperture, e.g., colliding with the poleface. In this case, $\eta_2 < 1$.

Such a p_ϕ -component may be introduced either by source extension away from $(z_s, 0)$, as discussed in Section #5, or by fringing-field deflection (Sections #8 and #9). It is shown in Section #9, that the fringing-field effect may indeed be treated as a virtual source extension. The effect of source extension in causing orbit rotation was developed in Section #5, and it is this theory which is used to evaluate the transmission losses.

When the source is large, the effect of its extension may be considerably larger than the fringing-field effect of Section #9. On the other hand, for a small source, the fringing-field effect may dominate. We shall, therefore, develop in this section the theory of transmission losses due to source extension alone, with no fringing-field effect; the transmission factors are labelled η_2^* , η_3^* and η_{23}^* . We shall then carry out the computation of fringing-field transmission losses for a point source in the first part of Section #9, labelling the transmission factors as η_2 , η_3 and η_{23} . For many purposes, the results of these relatively simple computations may suffice, as, for example, when only approximate values are required to facilitate choice of instrument design or of sample size. The complete result, including both source extension and fringing-field effects, is covered in the second part of Section #9. The factors in this case are η_{2T} , η_{3T} and η_{23T} . This theory, of course, results in considerably more complicated computations, but is required where more accurate results are required in cases where fringing-field and source-extension effects are comparable.

Transmission Losses Due to Exit Profile Aperture. η_2^* . We shall consider a single gap of the spectrometer and take the source to have uni-

Fig. 7.1

form density of emission. If this is not true, but the variation of density over the source is known, then the results may be modified in an obvious manner; the integrations cannot then be performed explicitly, but must be done numerically. We take the source to lie on a plane perpendicular to the z -axis at $(z_s, 0)$, with a shape symmetrical around ϕ_M and around the line perpendicular to ϕ_M , as in Fig. (7.1). The coordinates of a point on the source are given by the (r_1, r_2) components as in Section #5.

To simplify computation, an approximation will be made by taking r_1 as always lying in ϕ_M , rather than in ϕ_s ; then r_2 is always perpendicular to ϕ_M . This approximation leads to the neglect of the following effects:

(i) The magnitudes of r_1 and r_2 are in error, but only insofar as $\cos(\phi_P - \phi_M)$ differs from $\cos(\phi_P - \phi_s)$ and $\sin(\phi_P - \phi_M)$ from $\sin(\phi_P - \phi_s)$, where $\phi_M + \frac{1}{2}\varphi \leq \phi_s \leq \phi_M - \frac{1}{2}\varphi$, and P is the source point [see Fig. (7.2)]. For small gap angle φ , this difference is small.*

(ii) For ϕ_P within the gap [$|\phi_P - \phi_M| < \frac{1}{2}\varphi$], the sign of r_2 may be changed, depending upon the ϕ_s -value. Thus, if $\phi_s > \phi_P > \phi_M$, then r_2 should be positive, but, with the approximation made, it is taken as negative. This has the effect of changing the direction of orbit rotation, i.e., the sign of $(\phi_f - \phi_s)$. The effect is not important, both because of the limited fraction of the

Fig. 7.2

*For the largest $(\phi_s - \phi_M)$ difference, $\frac{1}{2}\varphi$,

$$\cos(\phi_P - \phi_M) - \cos(\phi_P - \phi_s) \cong \frac{\varphi^2}{8} \cos(\phi_P - \phi_M) + \frac{1}{2}\varphi \sin(\phi_P - \phi_M) ;$$

$$\sin(\phi_P - \phi_M) - \sin(\phi_P - \phi_s) \cong \frac{\varphi^2}{8} \sin(\phi_P - \phi_M) - \frac{1}{2}\varphi \cos(\phi_P - \phi_M) .$$

At maximum, the difference is $\frac{1}{2}\varphi$, which for $\varphi = 3^\circ$ is 0.026, and for $\varphi = 10^\circ$ is 0.087. For most source points and ϕ_s -values, the errors are quite a bit smaller.

source involved (for small φ), and because only small orbit rotations are involved, since $|r_2|$ is small (hence, the likelihood of loss of the ray through rotation is small).

Because the r_2 -component alone determines the orbit rotation which leads to transmission losses, these losses are the same for each quadrant of a source which has the symmetry of Fig. (7.1). We choose to consider only the upper left-hand quadrant of the source.

If S is the specific activity (number of particles isotropically emitted per unit time) and $d\sigma$ the surface element on the source, then the collecting power or geometry subtended by the entrance profiles of a single aperture at the source is given by

$$\begin{aligned} C_S = \frac{\omega_S}{4\pi} &= \frac{\frac{1}{4\pi} \iiint S d\sigma d\Omega}{\iint S d\sigma} = \frac{\frac{1}{4\pi} \iiint S d\sigma \varphi \sin\psi_s d\psi_s}{D} \\ &= \frac{1}{4\pi D} \iiint S dr_1 dr_2 \varphi \sin\psi_s d\psi_s, \end{aligned} \quad (7.4)$$

where $d\Omega$ is an element of solid angle and D the disintegration rate of the entire source. To a good approximation ψ_s may be taken as determined only by the point of entry into the field and independent of the source point. The transmission losses due to orbit rotation may be described as an effective decrease in gap aperture, those ϕ_s -values for which trajectories do not pass through the apertures being excluded. The effective geometry may then be computed by substituting $(\varphi - \Delta\varphi)$ for φ in Eq. (7.4). Since these losses are a function of ψ_s (through F_1) and of the emitting point P , then $\Delta\varphi = f(\psi_s, r_1, r_2)$. The effective geometry is then

$$C_S \overline{\eta_{23}^*} = \frac{1}{4\pi D} \iiint S (\varphi - \Delta\varphi) \sin\psi_s dr_1 dr_2 d\psi_s, \quad (7.5)$$

where $\overline{\eta_{23}^*}$ is the average over the entire sample and ψ_s -range. Since

$$\eta_{23}^* \varphi = \varphi - \Delta\varphi, \quad (7.6)$$

η_{23}^* is a function of (ψ_s, r_1, r_2) .

The use of $\overline{\eta_{23}^*}$ in Eq. (7.5) implies that losses due to both the exit and detector apertures are included. If it is desired to evaluate the fraction of particles emitted which emerge from the exit aperture, without regard to the detector aperture, then $\overline{\eta_2^*}$ is substituted in Eq. (7.5), η_2^* in Eq. (7.6), and, in general, $\Delta\varphi$ is a different function of (ψ_s, r_1, r_2) than when $\overline{\eta_{23}^*}$ and η_{23}^* are considered.

It is convenient to take $\phi_M = 0$, so that the gap boundaries are $\phi = \frac{1}{2}\varphi$ on the left [see Fig. (7.2)] and $\phi = -\frac{1}{2}\varphi$ on the right.

We consider first the evaluation of η_2^* for a single point P in the source quadrant. Since r_2 is the only factor which determines, at a particular ψ_s , the ϕ_s -planes which allow transmission through the exit aperture, η_2^* is not a function of the r_1 -coordinate of P. From Eq. (5.19b), the rotation is

$$\Delta\phi = \phi_f - \phi_s = F_1 (r_2/z_s) \quad , \quad (7.7)$$

r_2 being negative in the left quadrant, $\Delta\phi < 0$; i.e., $\phi_f < \phi_s$, or the rotation is to the right. At a given r_2 -value, $\Delta\phi = \text{constant}$ for a particular ψ_s . There is, then, a critical ϕ_s -value, ϕ_C , such that for ϕ_s to the right of ϕ_C in Fig. (7.2) ($\phi_s < \phi_C$), then $\phi_f < -\frac{1}{2}\varphi$ in Eq. (7.7). Trajectories entering on these ϕ_s -planes will then not be transmitted through the exit aperture.

From Eq. (7.7),

$$\phi_C = -\left[\frac{1}{2}\varphi + F_1 (r_2/z_s)\right] \quad (7.8a)$$

and

$$\eta_2^* = \frac{\frac{1}{2}\varphi - \phi_C}{\varphi} = 1 + \frac{F_1 r_2}{\varphi z_s} \quad . \quad (7.8b)$$

When $\phi_C = \frac{1}{2}\varphi$, then the corresponding $r_2 = r_{0-}^{*+}$ is a boundary value,* such that no transmission occurs at any r_2 for which $|r_2| > |r_{0-}^{*+}|$. From Eq. (7.8b)

$$r_{0-}^{*+} = -\varphi z_s / F_1 \quad . \quad (7.8c)$$

It may be noted that η_2^* is always less than unity for a source point away from $(z_s, 0)$. This is due to the fact that a nonzero r_2 leads to a definite orbit rotation, which, for some ϕ_s , gives rise to collision with the poleface (i.e., results in $\phi_f < -\frac{1}{2}\varphi$).

Since η_2^* is not a function of r_1 , this transmission loss is constant over the strip of the sample of constant r_2 . If the boundary of the sample is described by the relation $r_1 = R_1(r_2)$, and if the specific activity S is constant over the sample, then the transmission through the exit aperture at constant ψ_s , averaged over the sample, is

$$\eta_2^*(\psi_s) = \frac{1}{\frac{1}{4}A_s} \int_{r_{2m}}^0 |R_1| \left[1 + \frac{F_1 r_2}{\varphi z_s} \right] dr_2 \quad , \quad (7.9)$$

*The reasons for this notation will be evident in Section #9 [see Table (9.2) and Fig. (9.15)].

where a_s is the area of entire sample. If R_2^* is the magnitude of the farthest r_2 -extension of the source, then $(-r_{2m})$ is the smaller of $(-r_{0-}^{*+})$ and R_2^* .

If the source is circular, of uniform density, then,

$$|R_1| = \sqrt{R^2 - r_2^2} \quad ,$$

where R = source radius = R_2^* . If $r_{2m} = -R$ [i.e., r_{0-}^{*+} is outside the source], then,

$$\eta_2^*(\psi_s) = 1 - \frac{4}{3\pi} \frac{F_1 R}{\varphi z_s} = 1 - \frac{4}{3\pi} \frac{1}{\alpha} \quad , \quad (7.10a)$$

where

$$\frac{1}{\alpha} = \frac{R}{|r_{0-}^{*+}|} = \frac{R F_1}{\varphi z_s} \quad (7.10b)$$

and

$$\alpha > 1.$$

If $r_{2m} = r_{0-}^{*+}$, then

$$\eta_2^*(\psi_s) = \frac{2}{\pi} \left[\alpha \sqrt{1 - \alpha^2} + \arcsin \alpha \right] - \frac{4}{3\pi} \frac{1}{\alpha} \left[1 - (1 - \alpha^2)^{3/2} \right] \quad , \quad (7.10c)$$

where α is defined in Eq. (7.10b), but $\alpha < 1$.

For a square source, whose half-side is R , with $\alpha > 1$, for $r_{2m} = -R$,

$$\eta_2^*(\psi_s) = 1 - (1/2\alpha) \quad . \quad (7.11a)$$

For $r_{2m} = r_{0-}^{*+}$, then, with $\alpha < 1$,

$$\eta_2^*(\psi_s) = \frac{1}{2} \alpha \quad . \quad (7.11b)$$

After $\eta_2^*(\psi_s)$ is computed, as a function of ψ_s , numerical integration may be used to determine

$$\begin{aligned} \bar{\eta}_2^* &= \frac{C_s \bar{\eta}_2^*}{C_s} = \frac{\frac{1}{4} \frac{1}{\pi D} \iiint S \eta_2^* \varphi \sin \psi_s \, dr_1 dr_2 d\psi_s}{\frac{1}{4} \frac{1}{\pi D} \iiint S \varphi \sin \psi_s \, dr_1 dr_2 d\psi_s} \\ &= \frac{\int [\iiint \eta_2^* dr_1 dr_2] \sin \psi_s \, d\psi_s}{\frac{1}{4} a_s \int \sin \psi_s \, d\psi_s} = \frac{\int \eta_2^*(\psi_s) \sin \psi_s \, d\psi_s}{\int \sin \psi_s \, d\psi_s} \quad , \end{aligned} \quad (7.12)$$

since ψ_s is not a function of the source point.

The averaging may also be performed over ψ_s prior to the integration over the sample. Thus,

$$\begin{aligned}\overline{\eta_2^*} &= 1 + \frac{\iiint (F_1 r_2 / \varphi z_s) \sin \psi_s \, d\psi_s \, dr_1 \, dr_2}{\iiint \sin \psi_s \, d\psi_s \, dr_1 \, dr_2} \\ &= 1 + \frac{1}{\frac{1}{4} Q_s \varphi z_s} \overline{F_1(\psi_s)} \int r_2 \, dr_1 \, dr_2, \quad (7.13a)\end{aligned}$$

the second term being negative. Here $\overline{F_1(\psi_s)}$ is defined as

$$\overline{F_1(\psi_s)} = \frac{\int F_1(\psi_s) \sin \psi_s \, d\psi_s}{\int \sin \psi_s \, d\psi_s}. \quad (7.13b)$$

Some results of such averaging are shown in Figs. (7.3) and (7.4), in the first of which $\varphi = 10^\circ$ and $\overline{F_1(\psi_s)}$ varies, and in the second of which $\overline{F_1(\psi_s)} = 1.09$ and φ is variable. The following table gives values of $\overline{F_1(\psi_s)}$ as a function of n_s and K , calculated for ψ_s in the range from 95° to 145° .

FIG. 7.3

FIG. 7.4

$n_s \backslash K$	0.2	0.4	0.6	0.8	1.0
0	0.24	0.59	1.09	1.73	2.54
$-\frac{1}{2}$	1.20	3.89	9.37	19.2	36.6

Transmission Losses Including Detector Aperture. η_{23}^* . From Eq. (5.15b) and Eq. (5.20), the coordinates of the image points on the z_f -plane may be written in the form

$$r_{1f} = u(\psi_s) r_1; \quad r_{2f} = v(\psi_s) r_2 \quad (7.14)$$

For a symmetrical spectrometer, $u = v = -1$, so that the image is identical* to the source, except for being reflected in the origin. For an unsymmetrical spectrometer, $u \neq v$, but, since ψ_s is independent of the source

*We neglect here the lack of perfect ϕ -focusing, discussed in Section #5. The smearing of the image of a point source into a small arc is a very small effect when the gap angle φ and r_2 are small, as has been assumed here.

point, the magnification of r_1 is uniform over the entire source as is that of r_2 , although the magnifications are unequal. A circle then goes into an ellipse, although the size and shape of the ellipse vary with ψ_s .

Because of the essential point-to-point imaging, at constant ψ_s , all of the rays from a single source point which emerge from the exit aperture are counted if the image point is within the detector aperture, or none of them are counted if the image point is outside, i.e., $\eta_3^* = 1$ or $\eta_3^* = 0$. If the image is entirely within the detector aperture, then $\eta_3^* = 1$ for all image points, and

$$\eta_{23}^*(\psi_s) = \eta_2^*(\psi_s) \quad , \quad (7.15)$$

$\eta_2^*(\psi_s)$ being given by Eqs. (7.9), (7.10), and (7.11).

Where the image does not lie within the detector aperture, $\eta_{23}^*(\psi_s)$ is computed by means of the integral in Eq. (7.9), but redefining R_1 and r_{2m} . If the detector aperture is described by the curve $r_{1f} = \Gamma_1(r_{2f})$, then R_{10} is the smaller of $|R_1|$ and $|\Gamma_1/u(\psi_s)|$, at a particular value of r_2 (and hence r_{2f}). If D_{r_2} is the magnitude of the farthest r_{2f} -extension of the detector aperture, then $(-r_{2m})$ is the smallest of $|r_{0-}^{*+}|$, R_2^* , and $D_{r_2}/|v(\psi_s)|$. We have, then,

$$\eta_{23}^*(\psi_s) = \frac{1}{\frac{1}{4}a_s} \int_{r_{2m}}^0 R_{10} \left[1 + \frac{F_1 r_2}{\phi z_s} \right] dr_2 \quad . \quad (7.16)$$

In general, R_{10} may be defined by R_1 for some r_2 -values, and by Γ_1 for others. The symmetrical case, however, may be expressed in some greater detail, because the imaging is, to the approximation considered, perfect, and the magnification is the same for both r_1 and r_2 . We consider the case of a circular source of radius $R = R_2^*$ and a circular detector aperture of radius $R_d = D_{r_2}$. If $R_d > R$, then Eq. (7.15) holds. If $R_d < R$, then $R_{10} = \sqrt{R_d^2 - r_{2f}^2} = \sqrt{R_d^2 - r_2^2}$, and $|r_{2m}|$ is the smaller of $|r_{0-}^{*+}|$ and R_d . If $r_{2m} = -R_d$, then

$$\eta_{23}^*(\psi_s) = \left[1 - \frac{4}{3\pi} \frac{F_1 R_d}{\phi z_s} \right] \frac{R_d^2}{R^2} = \left[1 - \frac{4}{3\pi} \frac{1}{\alpha_d} \right] \frac{R_d^2}{R^2} \quad , \quad (7.17)$$

with

$$\alpha_d = \phi z_s / R_d F_1 = \alpha(R/R_d) \quad . \quad (7.18)$$

If $r_{2m} = r_{0-}^{*+}$, then the result is the same as in Eq. (7.10c), except that the result is multiplied by $(R_d/R)^2$ and α_d replaces α .

Averaging over ψ_s is performed as in Eq. (7.12) or Eq. (7.13).

Transmission through Multigap Instruments. The above discussion gave the transmission losses for a single gap. For a multiple-gap instrument, the total transmission is the summation of the transmissions of the individual gaps. For sources and detectors having circular symmetry, the summation is simply a multiple of the single-gap transmission.

Comparison between Iron and Iron-free Toroidal System. In a spectrometer consisting of a series of gaps in an iron magnetic circuit, with coils appropriately placed for creating magnetomotive force,^(1,10) when a trajectory is forced into an equipotential surface defined by a poleface, it is lost to the imaging process. It is either absorbed or scattered by the iron poleface; the latter process may be with or without energy loss. In either kind of scattering, the coherent relation to the original source point is lost, and no imaging can occur. Obviously, baffles and slits must be arranged to avoid interference of the scattered particles with the true image. In an iron-free instrument,⁽¹⁴⁻¹⁸⁾ the gaps are simply spaces between the wires comprising the torus, and the equipotential surfaces constituting the extension of the gaps are not blocked by matter. Hence, the scattering problem is less troublesome, and particles deflected into an equipotential surface defining a gap may pass through and emerge through the exit side of an adjacent gap.

The discussion of η_2 and η_3 for a point source in a fringing-field will be given in the first part of Section #9, following a description of the fringing-field and its deflecting properties in Section #8.

PART B

ANALYSIS CONSIDERING FRINGING-FIELD EFFECTS

#8. GENERAL CHARACTERISTICS AND EFFECTS OF THE FRINGING FIELDS

Thus far, we have assumed that the magnetic field within the spectrometer started abruptly at the profile, following the relation Eq. (1.1), and was zero outside of the profile. As is known, this is only a rough approximation, since all physical gaps have fringing fields which: (i) extend outside the profile and attenuate to essentially zero only at some distance from the profile, and (ii) decrease the magnitude and distort the direction of the field for a short distance within the profile. Because the fringing field does not follow Eq. (1.1), the trajectories of particles passing through this field are not evaluated correctly with the theory developed thus far. It is necessary, therefore, to compute the effect of the fringing field on the trajectories and to correct the previous results for this effect. Under suitable restrictions, the fringing field can be made small enough so that the effect can be treated as a perturbation problem. The main features of the trajectory are then computed as before, with the fringing field effect added as a correction. It will be found that the wide flexibility previously allowed in the choice of K -values, profile curves, and (n_s, n_f) -values is considerably constrained by the necessity for minimizing the fringe-field effect.

Structure and Magnitude of Fringe Field. In principle, if the magnitude and direction of the magnetic field are specified at all points in the region of interest, the trajectories may be computed, even if only numerically. It is not feasible for most magnetic structures to compute the fringe field from the configuration of iron and/or current distribution. It is generally necessary to construct the magnet and then to measure its fringe field by any of a number of field-probing techniques. With such a detailed measurement, the numerical integration of trajectories may be performed. Little information of this type is available for the sliced-orange type of system, so we consider some general features of other magnetic structures in order to develop some qualitative considerations.

The simplest type of structure is that of a pair of parallel polefaces, with a straight-line profile. The lines of force are emitted in a plane perpendicular to the profile. The intensity of the field at a given distance from the profile scales with the gap width (provided the coils are simultaneously changed appropriately). The perturbation of the interior field is relatively small within one-half gap width inside the profile, and is essentially zero at one gap width inside. The external fringing field drops steeply at first, the drop (on the median plane) to about 10% of the interior field value being roughly approximated by a straight line, whose intercept is approximately 2 gap widths out. The field then has a slowly decreasing component which goes out a number of gap widths, but whose rate of fall-off is very sensitive to the structure of the poles, placement of coils, distance to other magnetic materials, etc.

If we apply these general considerations to the r^{-1} -field case, we note that

- (i) the fringe lines emerge perpendicular to a poleface (if iron) and tend to be perpendicular to the profile as in the above case;
- (ii) in the uniform field case, the density of lines of force is uniform as one moves along the profile at a particular distance from profile and median plane. In the r^{-1} -field magnet, however, the density of lines of force decreases with r , so that the "mutual repulsion" of lines tends to make the lines of force bend up toward increasing r as they leave the poleface. For small gap angles this effect is small and will not be considered at this point; and
- (iii) the merging of the fringe field with the interior field should be relatively insensitive to the exterior structure and should be much the same as described for the case of a uniform field.

Some measurements and computations have been made for the sliced-orange type of field, but only on the rate of fall-off on the median plane. Where information is needed concerning the field and deflections in the off-median region, this has been estimated by an indirect method (Sections #10 and 11). Certain aspects of the fringing-field effects are not sensitive to the detailed structure of the field, and these results are computed with the greatest reliability. Some machine computations have been made of the fringing field on the median plane of a 100-gap iron-free spectrometer, with $\varphi = 3^\circ$, and symmetrical profiles designed for $K = 0.59$ (Section #3),^(16,17) $\psi_s = 110^\circ$ to 150° . Measurements were made on the median plane of an 8-gap iron-core unit,^(5,12) with $\varphi = 10^\circ$, $K = 0.60$, and $\psi_s = 95^\circ$ to 145° . The entrance curve was a circle with radius ρ of 27.0 cm, whereas the appropriate exit curve for focusing was computed via Eq. (3.2a) and (3.3). Some of the results will be quoted after some definitions have been made.

(i) The fringing field $\underline{H}$ at the point (z', r', ϕ') is decomposed into two orthogonal components:

$$\underline{H} = \underline{H}_\phi + \underline{H}_{rz} \quad , \quad (8.1)$$

where $\underline{H}_\phi$ is the component normal to the plane $\phi = \phi'$ at the point, with positive direction corresponding to increasing ϕ , and $\underline{H}_{rz}$ is the component lying in the plane $\phi = \phi'$. If the fringe line lay in a plane perpendicular to the profile, then $\underline{H}_{rz}$ would be parallel to the normal to the profile. Because of the bowing of the lines of force toward larger r as they leave the poleface, this is not quite true, but for our purposes it may be taken as approximately correct. On the median plane ϕ_M , from symmetry considerations $\underline{H} = \underline{H}_\phi$. Although the direction is the same as that given in Eq. (1.1), the magnitude is not. Inside the profile, the $\underline{H}_{rz}$ component for any ϕ' rapidly decreases to zero as a trajectory passes toward the interior, and $\underline{H} \rightarrow \underline{H}_\phi$ and comes closer to the value given by Eq. (1.1).

FIG. 8.1

(ii) If we consider only trajectories starting from $(z_s, 0)$, then, at a given ψ_s , the trajectory lies in the same plane as $\underline{H}_{rz}$ as it approaches the profile,* so $\underline{H}_{rz}$ may be resolved as

$$\underline{H}_{rz} = \underline{H}_t + \underline{H}_\perp \quad , \quad (8.2)$$

where $\underline{H}_t$ lies along the trajectory and $\underline{H}_\perp$ is normal to it.

These components are shown in Figs. (8.1) and (8.2). Figure (8.1) is a perspective view relative to the plane containing the trajectory and the normal to ϕ_M ; Fig. (8.2) is a view on the ϕ' -plane.

(iii) Let μ be the smaller angle ($\mu \leq 90^\circ$) between the normal to the profile and the trajectory. If ν_s is the angle between the normal on the entrance side and the z-axis, measured in the same manner as the ψ -angle [see Fig. (8.3)], then at the entrance profile

$$\mu_s = \psi_s - \nu_s \quad . \quad (8.3a)$$

On the exit side [see Fig. (8.3)],

$$\mu_f = \nu_f - \psi_f \quad . \quad (8.3b)$$

On the entrance side, μ_s is positive if the rotation from the normal to the trajectory is counterclockwise; at the exit, positive rotation is clockwise.** In Fig. (8.3), both exit and entrance μ -angles are negative. In general, for a symmetrical spectrometer, the respective μ -angles are equal in sign as well as in magnitude. This choice is arbitrary in that Eq. (8.3a) might be used for both entrance and exit, but it would lead to opposite signs of μ on both sides of a symmetrical system. This point will be discussed further below.

At this point, we introduce the information known concerning the fringing field of the two spectrometers mentioned previously. The fringing

*When an extended source is involved, the trajectory does not necessarily lie in the same plane as $\underline{H}_{rz}$, even if the fringe lines lie in a plane perpendicular to the profile, because $p_\phi \neq 0$. However, the deviation is small, and we shall neglect it later when the extended source is treated.

**See also Table 12.1.

FIG. 8.2

fields were determined only on the median plane ϕ_M . As in the cases of uniform fields, the bulk of the dropoff of field was found to be linear, and as this part of the fringing field is the chief source of the deflection, we shall take the entire dropoff as linear, with a corresponding intercept. The fringing field on ϕ_M was found to be approximately like that shown in Fig. (8.4), which represents the field along a trajectory (for given ψ_s) vs s , the distance along the trajectory. In the absence of fringing fields, H varies according to the curve EFBPCGT; along this curve, $H = A_0/r$, as in Eq. (1.1). The lines EF and TG represent the field at the profile, and are greater than the field on the remainder of the trajectory, because the trajectory r -values within the profile are greater than the corresponding profile r -values when $n_s = n_f = 0$.

When fringing fields are present, the field is nonzero at some distance from the profile and does not follow Eq. (1.1) until the trajectory has gone beyond the profile (at B and C). The actual field approximately follows ABPCD. The linear variation of the magnetic field in the transition regions (AB and CD) is only an approximation, the transitions at A, B, C, and D actually being gradual rather than abrupt. However, the approximation is quite close to the measured values.

Fig. 8.3

As shown in Fig. (8.4), we define the quantity d_e as the total distance along the part of the trajectory for which the fringing field is significant; this distance is reckoned using the straight-line approximation for the fringing field. The other distances shown in Fig. (8.4) are discussed in Section #13.

Fig. 8.4

It is to be noted that (z_e, r_e) is taken as the point at which the fringing field merges with the r^{-1} -field. Strictly speaking, this is a poorly defined point, since the fringing field merges only gradually with the interior field. It is made definite here by taking it as the intersection of the extrapolated

straight-line portion of the fringing field with the field curve for the idealized system ($H = 0$ outside profile, $H = A_0/r$ inside). The notation is consistent with the previous nomenclature, since the r^{-1} -field starts at the profile in the idealized case. The value of the field at (z_e, r_e) is H_e ; evidently, $H_e = A_0/r_e$, from Eq. (1.1).

As mentioned above, for uniform field magnets with straight profiles, the fringe field tends to scale with the gap width. If this were true for the r^{-1} -field magnets, then it would be found that $d_e \propto g_e = r_e \varphi$, where φ is the gap angle, r_e the r -value at B [see Fig. (8.4)], and g_e the gap width. In the actual magnet, however, g_e/d_e varied with ψ_s . In the 8-gap iron-core instrument, g_e/d_e varied between 0.85 and 0.60 for ψ_s between 95° and 145° , respectively, the fringe field extent d_e relative to the gap width being smaller for the larger gap widths. The air-core spectrometer had a similar variation of g_e/d_e . In this case, because the conductors defining the torus had a depth comparable to the gap width, the position of the profile was taken as the center of the conductor (i.e., part way into the actual gap). The ratio g_e/d_e was found to vary from 0.64 to 0.44 over the respective range of ψ_s from 110° to 150° .

Nature of the Trajectory Perturbations. The fringing field causes perturbations to the trajectories so that they do not follow the equations described in Sections #1 through 3. It is convenient to analyze these effects into components. Because the effects are small, it is possible, to the first order, to consider the entire perturbation as a linear superposition of these effects. They are:

(a) The ψ -deflection due to the H_ϕ -component of the fringe field. The H_ϕ -component is in the same direction as in the interior field, and hence tends to cause a change in the ψ -coordinate in the same direction as does the focusing field. [The effect of the other fringe-field components may be neglected, since, to the approximation that p_ϕ remains small, the (z, r) motion is dependent only on H_ϕ .] The added ψ -deflection may be considered as equivalent to having a larger field region (of the r^{-1} type) than that defined by the profiles, so the effective boundaries are outside the real boundaries. Hence, with a given profile, the value of A_0 [see Eq. (1.1)] required to focus a particle with given p and ψ_s differs from that given by the theory previously discussed. In addition, since the fringing field has different effects for different ψ_s -values, the profiles computed from the theory of Section #3 will not have the theoretical focusing power (i.e., for all ψ_s) for any A_0 -value.* This effect may be

*In the 8-gap iron-core instrument, it was found that when the profiles were constructed according to the theory of Section #3, the specific field intensity A_0 required to focus trajectories at $\psi_s = 95^\circ$ was 11% greater than that required for $\psi_s = 145^\circ$. This difference follows from the fact that the ratio of the deflecting power of the fringing field relative to that of the interior field [see Eq. (1.1)] is larger for larger ψ_s . Hence, a smaller interior value of A_0 brings the 145° trajectories into focus.

corrected by changing the profile shape and reducing the region enclosed by the profile. The calculations will be described later (Sections #12 and 15).

(b) Variation of the ψ -deflection with ϕ_s . In the plane $\phi = \phi_M$, we have $\underline{H} = \underline{H}_\phi$. On each of the nonmedian planes, the value of the $\underline{H}_\phi$ -component is different. Hence, trajectories passing through the median plane have different ψ -deflections in the fringing field than do nonmedian trajectories, as well as suffering different displacements of the entry points (z_e, r_e) into the interior field [see Eq. (1.1)]. Although the profiles may be corrected for the change in ψ -deflection on the median plane, as in (a), this correction does not remove the variation of ψ -deflection with ϕ_s . An estimate of the magnitude of the effect is given in Section #11.

(c) Lateral deflections due to $\underline{H}_\perp$. This component of the fringing field acts as a cylindrical lens and transforms a point source to a line image. We shall find that the deflecting power of this lens increases as $|\mu|$ increases.

We shall first discuss general features of the deflection by the $\underline{H}_\phi$ and the $\underline{H}_\perp$ components. In Section #9, the detailed analysis of the lens effect will be presented.

ψ -deflection in the Median Plane. At a given point in the fringing field, the deflecting force equals the centrifugal force around the instantaneous center of rotation. Since the velocity $\underline{v}$ is normal* to $\underline{H} = \underline{H}_\phi$, the magnitude of the deflecting force is:

$$\left| e\mathbf{v} \times \underline{H} \right| = \left| evH_\phi \right| = \left| \frac{mv^2}{\rho_{\text{inst}}} \right| = \left| mv\omega_\psi \right| = \left| p\omega_\psi \right|, \quad (8.4a)$$

where $\omega_\psi = \frac{d\psi}{dt}$ is the instantaneous angular deflection velocity whose axis is colinear with $\underline{H}_\phi$. The algebraic sign is taken from the considerations following Eq. (1.10), in which it was pointed out that eH is taken as negative when the instrument focuses, i.e., when $\frac{d\psi}{dt} > 0$. Since $p > 0$,

$$-evH_\phi = p \frac{d\psi}{dt}. \quad (8.4b)$$

Taking ds as positive along the trajectory, $v = ds/dt$, and

$$d\psi = -(e/p)H_\phi ds \quad (8.4c)$$

*For the general case in which $\underline{v}$ is not normal to $\underline{H}_\phi$ (i.e., does not lie in a ϕ -plane), see Ref. 22 (also see Section #11). The results are essentially the same as Eqs. (8.4c) and (8.5) as long as $p\phi$ remains small.

and

$$\Delta\psi = -\frac{e}{p} \int H_{\phi} ds \quad . \quad (8.5)$$

The quantity $\Delta\psi$ may be computed if H_{ϕ} is known along the trajectory. Taking the linear approximation of Fig. (8.4) as valid,

$$H_{\phi} = H_e (d/d_e) \quad , \quad (8.6)$$

where d is the distance along the trajectory measured from A. Then, since $ds = d(d)$,

$$\begin{aligned} \psi_e - \psi_s = \Delta\psi_e &= -\frac{e}{p} \int_0^{d_e} H_e \frac{d}{d_e} d(d) = -\frac{1}{2} \frac{e}{p} H_e d_e \\ &= d_e / 2K r_e \end{aligned} \quad (8.7a)$$

from Eq. (1.1) and (1.10). This may also be expressed as

$$\Delta\psi_e = \frac{\varphi}{2K} \frac{d_e}{\varphi r_e} = \frac{\varphi}{2K} \frac{d_e}{g_e} \quad . \quad (8.7b)$$

Since d_e/g_e is roughly constant with r , then $\Delta\psi_e$ is roughly proportional to φ . For the 8-gap iron-core instrument, since d_e/g_e actually decreased somewhat as g_e increased, $\Delta\psi_e$ was smaller for the largest g_e -values (smallest values of ψ_s).

We return now to the difference in the interior A_0 -value required to focus as compared to the value computed from the theory of Section #3. From Eq. (8.5), it is evident that the areas under the curves in Fig. (8.4) represent the $\Delta\psi$ -values for the field configurations involved. If H_i and $\Delta\psi_i$ are the field and overall $\Delta\psi = \psi_f - \psi_s$ for the idealized (no fringe field) case, while H_a and $\Delta\psi_a$ are the corresponding actual values, including the fringe-field effect, then let

$$\begin{aligned} I &= \frac{\Delta\psi_a}{\Delta\psi_i} = \frac{\int H_a ds}{\int H_i ds} = \frac{ABPCD}{EFPGT} \\ &= 1 + \frac{(EAU - BFU) + (VDT - GCV)}{EFPGT} \quad , \end{aligned} \quad (8.8)$$

the last terms indicating the areas in Fig. (8.4).

In the 8-gap iron-core instrument, the profiles had been designed so that with H_i alone, all ψ_s -values between 95° and 145° would focus at $(z_f, 0)$. Experimentally it was found that

(i) For all angles $I > 1$ when (H_a, H_i) are fields corresponding to the same internal A_0 -value [see Eq. (1.1)]; thus, the deflecting power of the instrument was increased by the fringe field. Using the same profiles, to bring the actual ray into focus, i.e., to make $I = 1$, the A_0 -value corresponding to H_a must be made smaller than that for H_i .

(ii) For the same A_0 -value $I(145^\circ) > I(95^\circ)$. With an A_0 -value which focused 145° rays, the 95° rays fell beyond $(z_f, 0)$, and the A_0 -value required an 11% increase to cause focusing.

(iii) A graphical evaluation of the areas in Eq. (8.8) for the various ψ_s -angles showed that $I(145^\circ)$ exceeded $I(95^\circ)$ by 11%, for the same A_0 -value. A decrease of A_0 by 11% decreased $H_a(145^\circ)$ by this amount, and hence equalized $I(145^\circ)$ and $I(95^\circ)$, where $I(95^\circ)$ corresponded to the original A_0 -value.

(iv) At each ψ_s , focusing is achieved through a decrease in the A_0 -value such that I is the same for all ψ_s -values. The fact that different A_0 -values are necessary for the various ψ_s -angles indicates that a change is necessary in the profiles; this change is discussed below.

This difference in the focusing A_0 -value is to be expected, because $\Delta\psi_i(95^\circ) > \Delta\psi_i(145^\circ)$.* Since the fringing field penetrates the idealized field only for a short distance, we may take the deflection $\Delta\psi_i$ as remaining very approximately the same when fringing field is included, so the major effect is to add a fringe-field deflection; thus, very approximately, $\Delta\psi_a \approx \Delta\psi_i + \Delta\psi_e + \Delta\psi_0$, the last term arising from the exit fringe field. Since $\Delta\psi_e + \Delta\psi_0$ is approximately independent of ψ_s [see Eq. (8.7b)], it is a larger fraction of $\Delta\psi_a$ at 145° .

This relation is only very roughly correct. For the detailed correction of the profile, it is necessary to take account of the fact that the interior field [see Eq. (1.1)] is decreased by the fringing effect in the region where it is largest, i.e., near the profiles. As shown in Eq. (8.8), it is the difference $(EAU - BFU)$ that gives the actual increase in $\Delta\psi$ over the idealized situation, rather than the area $EAU = \Delta\psi_e$.

Further details of the fringing-field ψ -deflection effect are discussed in Section #13.

*E.g., in a symmetrical instrument, $\Delta\psi_i(95^\circ) = 170^\circ$, but only 70° for $\Delta\psi_i(145^\circ)$.

Lens Effect of the Fringe Field. Since this effect is due to $\underline{H}_{rz}$, it is zero at ϕ_M , and increases with $|\phi - \phi_M|$. The bending effect of the fringe field is resolvable into that due to the $\underline{H}_\phi$ -component and that due to the $\underline{H}_{rz}$ -component. The result for the $\underline{H}_\phi$ -component, as pointed out above is the same [see Eq. (8.5)] provided p_ϕ remains small. We define β as the angle the trajectory makes with the ϕ -plane, i.e.,

$$|\sin \beta| = \left| \frac{p_\phi}{p} \right|, \quad (8.9)$$

the sign of β being defined below. The analysis is similar to that of the $\Delta\psi$ -case; as above, the magnitudes will be treated first, and the signs determined after suitable conventions are made. Then

$$|e\underline{v} \times \underline{H}_{rz}| = |e\underline{v} \times \underline{H}_\perp| = |e v H_\perp| = |e v H_t \tan \mu| = |m v \omega_\beta| = |p \omega_\beta|, \quad (8.10)$$

where $\omega_\beta = d\beta/dt$ is the instantaneous angular deflection velocity with axis colinear with $\underline{H}_\perp$.

Two alternatives are available for sign conventions; we shall choose the following:*

(i) At the entrance, β is positive when $p_\phi < 0$; further, $d\beta_e > 0$ when p_ϕ is decreased. At the exit, β is positive when $p_\phi > 0$; further, $d\beta_0 > 0$ when p_ϕ is increased.

(ii) Let H_t be positive when it is in the same direction as $\underline{v}$; then $(\underline{H}_t \cdot \underline{v}) = H_t ds/dt > 0$. Then

$$d\beta = (e/p) \tan \mu H_t ds, \quad (8.11)$$

*The sign of H_t is determined by (a) the convention in Section #1 that eH_ϕ is negative for focusing, and (b) the condition that the line integral $\int \underline{H} \cdot d\underline{s}$ be zero in Eq. (9.2a) of the next section. The reversal of the convention for β at the entrance and exit can be avoided if either one of Eq. (8.3a) or (8.3b) are used at both entrance and exit. In this case however, μ is of opposite sign at entrance and exit in a symmetrical spectrometer, which proves to be a greater inconvenience than the change of sign convention for β . In addition, the use of the above convention allows the use of the close analogy to optical lenses.

where ds is taken along the direction of $\underline{v}$. Integrating from the external region to the interior field (H_t vanishes at both endpoints),*

$$\Delta\beta = \frac{e}{p} \int \tan\mu H_t ds \quad . \quad (8.12)$$

Evaluation of this integral requires information as to the variation of μ and H_t along the trajectory, the calculation is left to the next section.

*Equation (8.10), and hence (8.11) and (8.12) stemming from it, contains the assumption that $\underline{v}$ lies in a ϕ -plane. This is no longer true once the lateral deflection has begun, since $\beta \neq 0$. However, β is always a very small angle in all practical situations, so that Eq. (8.12) is closely held. See also Ref. 22 and Section #11.

#9. REDUCTION OF TRANSMISSION BY THE LENS EFFECT

Transmission for a Point Source. Because of the lateral deflections $\Delta\beta$ of Section #8, trajectories which start out on a ϕ -plane are given a $p\phi$ -component which may lead to collision of the trajectory with the poleface or may lead to the trajectory missing the detector slit. The $\Delta\beta$ -deflection leads to a transmission loss from a point source in the same way that transmission is lost from a spread source in a field free of fringing, namely, through the orbit rotation resulting from the $p\phi$ -component. This identity will be used in evaluating the loss. Before pursuing this aspect, we carry on the evaluation of $\Delta\beta$ started in Section #8.

Computation of $\Delta\beta$ -deflection for Point Source, Neglecting Interaction with $\Delta\psi$ -effect. Because of the variation of ψ through the fringing field, μ [see Eq. (8.3)] and β vary simultaneously. We shall make the approximation that the initial value of μ remains constant during the entire $\Delta\beta$ -deflection, and shall compute the transmission loss on this basis. It will be shown (see Section #10) that the effect of variation of ψ through the fringing field may be accounted for through substitution of $\mu_{\text{effective}}$ for μ_{initial} in all the formulae, $\mu_{\text{effective}}$ being computed from μ_{initial} and $\Delta\psi_e$. With this approximation,

$$\Delta\beta = \frac{e}{p} \tan\mu \int H_t ds \quad , \quad (9.1)$$

with $\mu = \mu_{\text{initial}}$.

This integral may be evaluated without making any detailed assumptions as to the structure of the fringing field. To do this, we utilize the fact that for a closed path which does not link any current, we have

$$\oint \underline{H} \cdot d\underline{s} = 0 \quad . \quad (9.2a)$$

The integration loop is taken as shown in Fig. (9.1): (a) along the trajectory from $(z_s, 0)$, on a single* ϕ -plane, up to (z_e, r_e, ϕ_s) , at which point Eq. (1.1) is obeyed and $H_t = 0$; (b) from ϕ_s to ϕ_M along the arc $r = r_e$; and (c) from the point (z_e, r_e, ϕ_M) to $(z_s, 0)$ on the median plane ϕ_M . Along the part (c), since $\underline{H} = \underline{H}_\phi$, we see that $\int_C \underline{H} \cdot d\underline{s} = 0$. The loop is taken in such a sense that ds is along $\underline{v}$ on the trajectory itself [e.g., clockwise in Fig. (9.1)]. Hence, the loop is reversed on the exit side, being taken from $(z_f, 0)$ to (z_0, r_0, ϕ_M) on the median

*The effect of the cumulative β -deflection is neglected. Because of the gradual shift in β , there is a change in ϕ by the time (z_e, r_e) is reached. However, since in all practical situations β is quite small, the change of ϕ is small, and we may treat it as though all of the deflection $\Delta\beta$ were inserted at one point.

plane, from ϕ_M to ϕ_f at constant $r = r_0$ to the point (z_0, r_0, ϕ_f) , and from this point to $(z_f, 0)$ along the trajectory. At the entrance, then,

$$\int H_t ds + H_e r_e (\phi_M - \phi_s) = 0 \quad (9.2b)$$

or

$$\int H_t ds = H_e r_e (\phi_s - \phi_M) = H_e r_e \Delta\phi_e \quad (9.3a)$$

At the exit,

$$\int H_t ds = H_0 r_0 (\phi_M - \phi_f) = H_0 r_0 \Delta\phi_0 \quad (9.3b)$$

If we define

$$\Delta x_e = r_e (\phi_s - \phi_M) = r_e \Delta\phi_e \quad (9.4a)$$

at the entrance, and

$$\Delta x_0 = r_0 (\phi_M - \phi_f) = r_0 \Delta\phi_0 \quad (9.4b)$$

at the exit, then

$$\Delta\beta = (eH/p) \tan\mu \Delta x \quad (9.5a)$$

when H is the field intensity at $r = r_e$ (or $r = r_0$) and is given by Eq. (1.1). Since $eH/p = -1/Kr = -1/\rho_C$, where ρ_C is the radius of curvature of the trajectory [see Eq. (1.1) and (1.10)],

$$\Delta\beta = - \frac{\tan\mu}{Kr} \Delta x = - \frac{\tan\mu}{\rho_C} \Delta x \quad (9.5b)$$

or

$$\Delta\beta = - \frac{\tan\mu}{K} \Delta\phi \quad (9.5c)$$

The proportionality of the deflection to the angular separation $\Delta\phi$ of ϕ_s (or ϕ_f) from the median plane is to be expected, since the curvature of the field lines leads to a larger $\underline{H}_{rz}$ component with increasing $|\Delta\phi|$. Correspondingly, the proportionality to $\tan\mu$ follows from the fact that a larger μ leads to a larger fraction of $\underline{H}_{rz}$ appearing as $\underline{H}_\perp$.

The Lens Property. Because of the insight provided, it is convenient to consider the action of the fringing field as that of a lens, even though the specific lens property is not used in computing the transmission. In a simple optical lens system, the angular deflection of a light ray from a single point is proportional to the distance of the contact point from the optic axis, where the optic axis is the direction along which a ray suffers no

angular deflection. Since $\Delta\beta \propto \Delta x$, it is evident that the $\Delta\beta$ -deflection may be treated as due to a lens in which the optic axis is the path taken by a trajectory, at a given ψ_s , travelling through ϕ_M .

In light optics, we have the relationship $P = 1/f =$ ratio of angular deflection to distance of contact point from optic axis, with f the focal length of the lens and P the power of the lens. For a diverging lens, $f < 0$ and $P < 0$; for a converging lens, $(f, P) > 0$. Then,

$$P = \frac{1}{f} = - \frac{\tan\mu}{Kr} = - \frac{\tan\mu}{\rho_C} \quad (9.6)$$

With the sign conventions as chosen above, when $\Delta\beta$ represents a deflection toward the median plane, $P > 0$; similarly, for deflection away from ϕ_M , it is found that $P < 0$. This checks with the usual notions of convergent and divergent lenses, respectively; this check is a useful byproduct of the particular convention chosen.

As an example of these considerations, we may examine the cases shown in Figs. (8.1) and (8.2) and (9.1). Here $90^\circ < \psi_s < 180^\circ$, so that, on looking from the source into the gap in Fig. (9.1), $H = H_\phi$ is negative, i.e., arranged for focusing positive particles. The direction of $\underline{H}$ in the interior determines the fringe-field direction, as in Fig. (8.1). In Fig. (8.2), $\nu_s > \psi_s$ (rotation of normal to trajectory is clockwise), so $\mu_s < 0$. Using the force equation $e\underline{v} \times \underline{H}_\perp$ when the trajectory enters the left side of the gap, as in the figures, we note that the deflection is toward ϕ_M (i.e., in the negative ϕ -direction); hence $\Delta\beta_e > 0$. Since the line integral [see Eq. (9.2a)] is taken clockwise in Fig. (9.1), $\Delta x_e = r_e(\phi_s - \phi_M) > 0$ (positive ϕ -direction to left). Then P is positive, implying a convergent lens, which checks with the observation that trajectories are bent toward the median plane.

Now P is independent of the sign of the particle being focused, i.e., a converging lens converges for either positive or negative particles. This is evident from Eq. (9.5a), since the sign of eH is always the same; positive H is required to focus for negative e , and negative H for positive e .

From Eq. (9.5b), it is evident that negative μ corresponds to a converging lens. In a symmetrical spectrometer, a trajectory passes through lenses of the same P -sign and magnitude on entering and leaving the gap.

It is sometimes convenient to take the rotation direction of the exit ψ -angle such that it represents a mirror rotation relative to that of the entrance angle. To distinguish between this angle and that defined in Eq. (1.12b), we use Ψ rather than ψ . Then,

$$\Psi_s = \pi - \psi_s \quad , \quad (9.7a)$$

$$\Psi_f = \psi_f - \pi, \quad (9.7b)$$

and

$$d\Psi_s = -d\psi_s; \quad d\Psi_f = d\psi_f. \quad (9.7c)$$

Positive rotation of Ψ_s corresponds to rotation from the negative z -direction, while that of Ψ_f corresponds to rotation from the positive z -direction. For both Ψ_s and Ψ_f , we have $0 < \Psi < \pi$. For a symmetrical spectrometer, $\Psi_f = \Psi_s$ and $\Psi'_f = d\Psi_f/d\Psi_s = 1$.

FIG 9.2

From Fig. (9.2), it is evident that $|\tan\mu| \cong \left| \frac{\Delta\rho}{\rho\Delta\psi_s} \right|$, or $|\tan\mu| = \left| \frac{1}{\rho} \frac{d\rho}{d\psi_s} \right|$, and similarly on the exit side. As to algebraic sign, when $\Delta\psi_s$ is positive and $\Delta\psi_f$ negative, $\Delta\rho$ has the same sign as $\tan\mu$. Then,

$$\begin{aligned} \text{Entrance: } \tan\mu_s &= \frac{1}{\rho_s} \frac{d\rho_s}{d\psi_s} = -\frac{1}{\rho_s} \frac{d\rho_s}{d\Psi_s} \\ \text{Exit: } \tan\mu_f &= -\frac{1}{\rho_f} \frac{d\rho_f}{d\psi_f} = -\frac{1}{\rho_f} \frac{d\rho_f}{d\Psi_f}, \end{aligned} \quad (9.8a)$$

where μ_s and μ_f are defined in Eq. (8.3), ρ_s is the distance from $(z_s, 0)$ to the trajectory crossing of the entrance profile and ρ_f is the distance from $(z_f, 0)$ to the trajectory crossing of the exit profile. However, in this section we make the approximation that no ψ -deflection occurs outside the profile, so that ρ_s and ρ_f are the straight-line distances from $(z_s, 0)$ and $(z_f, 0)$, respectively, to the profiles along the ψ_s - and ψ_f -directions. Taking ρ_e as the distance from $(z_s, 0)$ to the interior point (z_e, r_e) and ρ_0 as the corresponding distance from $(z_f, 0)$ to the interior point (z_0, r_0) , then

$$r_e = \rho_e \sin\psi_s = \rho_e \sin\Psi_s$$

and

$$r_0 = -\rho_0 \sin\psi_f = \rho_0 \sin\Psi_f,$$

$$\begin{aligned} P &= -\frac{1}{\rho_s} \frac{d\rho_s}{d\psi_s} \frac{1}{K\rho_e \sin\psi_s} = \frac{1}{\rho_s} \frac{d\rho_s}{d\Psi_s} \frac{1}{K\rho_e \sin\Psi_s}; \\ P &= -\frac{1}{\rho_f} \frac{d\rho_f}{d\psi_f} \frac{1}{K\rho_0 \sin\psi_f} = \frac{1}{\rho_f} \frac{d\rho_f}{d\Psi_f} \frac{1}{K\rho_0 \sin\Psi_f}, \end{aligned} \quad (9.8b)$$

where $\frac{1}{\rho_s} \frac{d\rho_s}{d\psi_s}$ and $\frac{1}{\rho_f} \frac{d\rho_f}{d\psi_s}$ are measured at the profiles and $\rho_e \sin\psi_s$ and $\rho_0 \sin\psi_f$ at the interior points (z_e, r_e) and (z_0, r_0) , respectively. Thus, another

criterion for a convergent lens is that the source-profile or profile-image distance decreases as ψ_s increases or as ψ_f decreases, respectively, i.e.,

$$\frac{d\rho_s}{d\psi_s} < 0 \quad \text{or} \quad \frac{d\rho_f}{d\psi_f} > 0 \quad \text{and} \quad \frac{d\rho_s}{d\Psi_s} > 0 \quad \text{or} \quad \frac{d\rho_f}{d\Psi_f} > 0 \quad .$$

In Table (9.1) are shown the various conditions which indicate the convergence or divergence of the magnetic lens.

Table 9.1

Converging Lens	Diverging Lens
$\tan\mu < 0$	$\tan\mu > 0$
μ_s : rotation from n to t* clockwise μ_f : rotation counterclockwise	rotation from n to t counterclockwise rotation clockwise
$P = \frac{\Delta\beta}{\Delta x} = \frac{\Delta\beta}{r\Delta\phi} > 0$	$P < 0$
Entrance: $\frac{d\rho_s}{d\psi_s} < 0$, $\frac{d\rho_s}{d\Psi_s} > 0$ Exit: $\frac{d\rho_f}{d\psi_f} > 0$, $\frac{d\rho_f}{d\Psi_f} > 0$	$\frac{d\rho_s}{d\psi_s} > 0$, $\frac{d\rho_s}{d\Psi_s} < 0$. $\frac{d\rho_f}{d\psi_f} < 0$, $\frac{d\rho_f}{d\Psi_f} < 0$.

*n = normal to the profile; t = trajectory

Curves giving values of $\frac{1}{z_s} \frac{d\rho_s}{d\psi_s}$ are shown in Fig. (9.3) for the symmetrical magnetic profiles of Fig. (3.3) and Table II.3 (Appendix II), for various K-values; ψ_s is in radians. The corresponding numerical values of $\frac{1}{z_s} \frac{d\rho_s}{d\psi_s}$ are given in the first column of Table IV.1, and the equation describing it in Eq. (A-III.12). Corresponding numerical values of $\frac{1}{\rho_s} \frac{d\rho_s}{d\psi_s}$, computed from Eq. (A-III.11), are given in Column 2 of the same table.

The fringing-field lens is a cylindrical lens, since the deflecting property ($\Delta\beta$) is directed only in the ϕ -direction. The long axis of the cylinder lies on the median plane and follows the profile. An important property of a cylindrical lens is to image a point into a line, and the

fringing field also has this effect. An important difference between an ordinary optical cylindrical lens and this magnetic lens is the fact that the focal length of the latter may vary with ψ_s through variation of μ . In fact, with an appropriate profile shape, the magnetic lens may have $P > 0$ for some ψ_s -values and $P < 0$ for other ψ_s -values.

Fig. 9.3

$\Delta\beta$ -deflection for Point Source and the Equivalent Virtual Line Source. The computation of transmission loss is based upon the fact that from the deflection $\Delta\beta$ there arises a p_ϕ -component which was not introduced at the source; the trajectory may thus be considered as analogous to one arising from a source point with a δr_2 -component in a system with no fringing-field effect. We may treat the fringing-field effect by calculating the equivalent virtual source position and then applying the results of Section #5.

Both entrance and exit lenses may have important effects on the transmission. Thus, if the entrance lens is divergent, the incident beam will be spread out, and some of the particles will be lost because of collision with the walls of the gap, thus giving rise to the factor η_2 . Apart from this effect, both lenses will produce a lateral broadening of the image (in the δr_2 -direction), and it will then not enter completely into the detector (whose size is chosen for reasons of resolution, see Section #6), producing another loss in transmission (η_3).

Before calculating these quantities, we examine some illustrative figures. Figure (9.2) shows a trajectory projection viewed in the direction of the field. It can be one passing through ϕ_M or through a nonmedian trajectory, since to the approximation that the virtual δr_2 is small, all trajectories starting from $(z_s, 0)$ pass (see Section #5) through the line perpendicular to ϕ_M at $(z_f, 0)$.

Other views of the trajectories are given in Figs. (9.4) and (9.5), in which a family of orbits of constant ψ_s is transformed so that the median orbit goes into a straight line.

FIG. 9.4

The transformation of the constant ψ_s cone to the plane surface is non-linear, so the trajectories in this projection have a distortion which increases with $|\phi - \phi_M|$. In Fig. (9.4), OO' is the median trajectory, the distance OP of any point P on the trajectory being s . The trajectory starts at O , increases the r -coordinate linearly with s from O to AA' (the gap boundary), at which point, L , we have $s = \rho_s$, and (neglecting the penetration into the gap) $r = r_e$. From L to K , at the boundary BB' , r varies nonlinearly with s , returning to linearity from K to O' . For the nonmedian orbits, the vertical coordinate is $r(\phi - \phi_M)$, the distortion of s increasing with $r\Delta\phi$. The most highly deformed orbits in the figure are those which represent the extreme trajectories in the absence of fringing fields, i.e., which almost graze the polefaces ($OABO'$ and $OA'B'O'$). Actually, the total path lengths along the orbits OO' , $OABO'$ and $OA'B'O'$ are all the same.

FIG. 9.5

The symmetrical system of Fig. (9.4) has converging lenses at both entrance and exit. A trajectory, which would be an extreme one ($OABO'$) with no fringe field present, is bent toward ϕ_M through the angle $\Delta\beta$ at the

lens AA', and follows OAG. Projecting the direction (within the field) of the initial trajectory element δs backward toward the source, we get a virtual source point at C, with $OC = \text{virtual } \delta r_2$ and $\angle OAC = |\Delta\beta_e|$. Using this value of δr_2 , from Eq. (5.19b) we can calculate the rotation of the orbit from the plane $\phi = \phi_{\text{extreme}}$ toward a ϕ -plane closer to ϕ_M . The exit point is at G rather than at B, which would be the exit point in the absence of fringing field. At G, the ray is subject to another deflection $\Delta\beta_0$, which is also toward ϕ_M . Since $GK = |\Delta x|$ is smaller than $AL = |\Delta x|$, then $|\Delta\beta_0| < |\Delta\beta_e|$. The trajectory passes through the plane $z = z_f$ at D, and $\delta r_{2f} = O'D$. Similarly, the symmetrical ray OA' acts as though it came from a virtual source (with no fringing field) at C', with $\delta r_2 = OC'$, arrives at G', and then to D', here $\delta r_{2f} = O'D'$. A ray arriving at the profile at a ϕ -plane such that $|\phi_s - \phi_M| < |\phi_{\text{extreme}} - \phi_M|$ will have a smaller $|\Delta\beta_e|$, a smaller $OC'' = |\delta r_2|$, a smaller $|\Delta\beta_0|$ and a smaller $|\delta r_{2f}| = OD''$.

All rays starting at O cross at X; hence this is the image point for the particular value of ψ_s involved. Thus, the image of the source lies on the central ray, but displaced from $(z_f, 0, 0)$ towards the profile. Since the central ray (for this ψ_s -value) intersects the z-axis only at $(z_f, 0, 0)$, this means that the image does not occur on the z-axis, but lies above the z-axis by the amount determined by ψ_f and the distance O'X.

To the approximation considered here, all the rays intersect the line DOD', which is perpendicular to ϕ_M . This is, however, only an approximation, suitable for small φ . The constant ψ_s cone of rays, on emerging from the field, also forms a conical surface with apex (image point) above the z-axis and between the profile and $(z_f, 0, 0)$. After passing through the image point, the rays fan out again into the conical surface, and strike the z - r_{2f} -plane in the arc of a curve rather than in a straight line. These arcs vary with ψ_s , the greatest curvature (but in opposite directions) arising from ψ_s -values close to 0° or 180° ; rays for which $\psi_s = 90^\circ$, on the other hand, actually pass through the r_{2f} -axis perpendicular to the ϕ_M -plane.

The position of the image point X varies with ψ_s , even if the lens power P were constant for the entire ψ_s -range; this series of image points constitutes a curve. The imaging of the point source into a curve provides another feature of the cylindrical nature of the magnetic lenses, since an optical cylindrical lens images a point source into a line.

Thus, the double-focusing (ψ and ϕ) property of the hypothetical instrument without fringing field is lost when the fringing field is included. The ψ -focusing remains, since the rays still pass through $(z_f, 0)$ on any ϕ -plane; this, however, is not true for the ϕ -focusing.

Figure (9.5) illustrates an instrument with divergent entrance and exit lenses. Several differences from the convergent lens case may be noted:

(i) Whereas all rays entering the system in Fig. (9.4) emerge from the magnetic field, this is not true for a divergent input lens. Thus, a ray arriving at ϕ_s lying on AH or A'H' is rotated away from ϕ_M into the pole-faces and is lost. At H (and similarly at the symmetrical H'), we have $\phi_s = \phi_{\text{critical}} = \phi_C$; a ray arriving at ϕ_C just barely escapes collision with the pole face and emerges at B.

(ii) For a given value of $|P|$, it is found that $|\Delta\beta_e|_{\text{max}}$ is smaller for the divergent lens, since $|\Delta x_e|$ at the extreme trajectory (i.e., at ϕ_C) is smaller than the $|\Delta x_e|$ for the convergent lens at the extreme trajectory. Correspondingly, $|\Delta\beta_o|_{\text{max}}$ at the exit is larger for the divergent lens. Since $|\Delta\beta_e|_{\text{max}}$ is smaller, the maximum value of the virtual $|\delta r_2|$ is also smaller for the divergent lens.

(iii) While the convergent lens forms an image between the profile and $(z_f, 0, 0)$, so that the distance from image to profile is less than ρ_f , the divergent lens forms an image beyond $(z_f, 0, 0)$ with an image-profile distance greater than ρ_f . The convergent lens image is "above" the z-axis; the divergent lens image is "below" the z-axis, when the orientation of Fig. (9.2) is used.

(iv) For the convergent lens, the virtual source point is on the same side of ϕ_M as is ϕ_s , while the point on the r_{2f} -axis is on the opposite side of ϕ_M . For the divergent lens, the virtual source point is on the opposite side of ϕ_M from ϕ_s , but δr_{2f} is on the same side as ϕ_s . Thus, as in Section #5, the quantity δr_{2f} is always on the opposite side from δr_2 .

Orbit Rotation Due to $\Delta\beta$ -deflection. In calculating the effect of the fringing field, it is convenient to take $\phi_M = 0$. In most cases of interest, then, ϕ_f and ϕ_s have the same sign. This is obvious in the case of the divergent entrance lens, since the ray is always driven away from ϕ_M , so that $|\phi - \phi_M|$ can only increase within the field. With a very strong convergent lens, however, the rays may form a real image point within the magnetic field. After reaching the image point on ϕ_M , the rays cross over and diverge, and may strike the polefaces if the lens is strong enough. Even if the lens is not this strong, ϕ_f is of opposite sign to ϕ_s . However, spectrometers are usually designed so as to minimize $|P|$, so we shall not, for the moment, treat convergent lenses of such strength.

We have, from Eq. (5.19b),

$$\phi_f = \phi_s + F_1(\psi_s) \frac{\delta r_2}{z_s} \quad (9.9)$$

At the entrance, $\Delta\beta_e > 0$ for deflection toward negative ϕ ; from Section #5, $\delta r_2 > 0$ in the negative ϕ -direction. Figures (9.4) and (9.5) show $\Delta\beta_e$ and δr_2 to be of opposite sign. Since $\Delta\beta_e$ is equivalent to $\delta\zeta_s$ [see Eq. (5.17)] in magnitude, we define δr_2 as

$$\delta r_2 = -\rho_s \Delta \beta_e \quad , \quad (9.10)$$

with ρ_s taken as the source-profile distance. It is to be noted that, as an approximation, we consider the entire $\Delta \beta$ -deflection to be inserted at the profile.

From Eqs. (9.4a), (9.5c) and (9.7) with ϕ_M set equal to zero,

$$\delta r_2 = N_s(\psi_s) z_s \phi_s \quad , \quad (9.11a)$$

with

$$\begin{aligned} N_s(\psi_s) &= \frac{\rho_s}{z_s} \frac{1}{K} \tan \mu_s = \frac{f_s}{K \sin \psi_s} \tan \mu_s \\ &= \frac{1}{K z_s} \frac{d\rho_s}{d\psi_s} = -\frac{1}{K z_s} \frac{d\rho_s}{d\psi_s} \quad , \end{aligned} \quad (9.11b)$$

since

$$\rho_s = \frac{z_s f_s(\psi_s)}{\sin \psi_s} \quad .$$

Then,

$$\phi_f = W_s \phi_s \quad . \quad (9.12a)$$

where

$$W_s(\psi_s) = 1 + F_1 N_s \quad . \quad (9.12b)$$

Since $F_1 > 0$, then $W_s > 1$ for a divergent lens, ϕ_f and ϕ_s are of the same sign, and $|\phi_f| > |\phi_s|$. For a convergent lens, if $|F_1 N_s|$ is not too large, $0 < W_s < 1$, and (ϕ_f, ϕ_s) are of the same sign, but $|\phi_f| < |\phi_s|$. If, however, the lens is so strongly convergent that $2 > |F_1 N_s| > 1$, then $-1 < W_s < 0$, and ϕ_f is of opposite sign to ϕ_s , but $|\phi_f| < |\phi_s|$. In this case, imaging occurs within the field and crossover occurs, but all rays emerge from the magnetic field. If $|F_1 N_s| > 2$, then $W_s < -1$ and $|\phi_f| > |\phi_s|$; some rays, then, collide with the polefaces and do not emerge.

Values of W_s have been computed for the symmetrical magnetic profiles of Fig. (3.3) and Table A-II.3, and are given in the third column of Table A-IV; the corresponding curves are shown in Fig. (9.6) for $-2 < W_s < 2$.

Imaging Property of Point Source with Fringing Field. To the approximation we have been considering, namely, that the ψ - and β -deflections in the fringing field are separable, the ψ -focusing remains perfect. Thus, all rays from $(z_s, 0)$ pass through $(z = z_f, r_1 = 0)$. The ϕ -focusing, however, is perturbed, so that $r_2 \neq 0$ at $z = z_f$. We shall evaluate the effect of the rotation on the imaging property in the r_2 -direction, using a converging entrance lens for convenience in presentation, although the same equations apply for a diverging lens. The convergent lens is only of moderate power, i.e., there is no crossover within the field ($W_s > 0$).

Following the notation of Fig. (9.4), the exit side is shown in Fig. (9.7). If the exit lens had $P = 0$, the ray would hit the z_f -plane at a

distance from the axis equal to δr_{2f} , computed as in Section #5. In Fig. (9.7), $\delta r_{2f} = O'H$ and GH is the ray in the absence of an exit fringing field effect. From Eqs. (5.20) and (9.11a),

$$\delta r_{2f} = \frac{\sin \psi_s}{\sin \psi_f} \delta r_2 = v N_s z_s \phi_s \quad (9.13)$$

However, since the exit lens has deflecting power, it bends the ray through the angle $\Delta\beta_0$, so that the path is actually GD [see Fig. (9.7)]. The change in sign convention for both the r_{2f} -component and $\Delta\beta_0$ on the exit side gives, from Eq. (9.10),

$$\delta r_{2f} = -\rho_f \Delta\beta_0 \quad (9.14)$$

where $\delta r_{2f} = HD$ in Fig. (9.7). From Eq. (9.4b), (9.5c) and (9.7),

$$\delta r_{2f} = N_f z_f \phi_f = N_f W_s z_f \phi_s \quad (9.15)$$

with

$$\begin{aligned} N_f(\psi_f) &= -\frac{\rho_f}{K z_f} \tan \mu_f = \frac{f_f}{K \sin \psi_f} \tan \mu_f \\ &= \frac{1}{K} \frac{d\rho_f}{d\psi_f} = \frac{1}{K} \frac{d\rho_f}{d\Psi_f} \end{aligned} \quad (9.16)$$

since

$$\rho_f = \frac{z_f f_f(\psi_f)}{\sin \psi_f} \quad .$$

The total distance of the "image" point from the z -axis, $O'D$, is given by the algebraic sum of δr_{2f} and δr_{2f} , the summation holding whether the exit lens is diverging or converging. Thus,

$$R_{2f} = \delta r_{2f} + \delta r_{2f}' = [vN_s + \frac{z_f}{z_s} N_f W_s] z_s \phi_s \quad (9.17)$$

With this relation, the effectiveness of the detector in catching the rays emitted at various ϕ_s -values may be computed.

Transmission from Point Source. As in Section #7, we shall first treat the effect of the exit aperture in reducing transmission, i.e., we shall evaluate η_2 , following which the effect of the detector aperture is treated through the evaluation of η_3 .

For a convergent entrance lens, trajectories are bent toward ϕ_M , so that for those particles which would pass through in the absence of fringing field, there is no collision with the polefaces, and $\eta_2 = 1$. This is true even for fairly strongly convergent lenses, provided $W_s > -1$. Thus, for $W_s > 0$, since $W_s < 1$ for a convergent lens, ϕ_f approaches $\frac{1}{2}\varphi$ more slowly than ϕ_s approaches $\frac{1}{2}\varphi$, so that no ray hits the poleface, and $\eta_2 = 1$. Similarly, when $0 > W_s > -1$, crossing of the median plane occurs within the field, but ϕ_f approaches $-\frac{1}{2}\varphi$ more slowly than $\phi_s \rightarrow -\frac{1}{2}\varphi$, so again $\eta_2 = 1$. The case for $W_s < -1$ will be considered after examination of the diverging lens.

With a diverging entrance lens, $\eta_2 < 1$, because not all the rays passing through the entrance aperture can avoid hitting the poleface, i.e., the exit aperture causes a reduction in transmission. When $\phi_f = \frac{1}{2}\varphi$, the largest possible ϕ -value, then $\phi_s = \phi_C = \phi_{\text{critical}}$, such that for $|\phi_s| > |\phi_C|$, the rays collide with the polefaces. Now ϕ_C is calculated from Eq. (9.12):

$$\phi_C = \pm \frac{1}{2}\varphi / W_s \quad (9.18)$$

Because of the uniformity of ray density with the ϕ -angle of emission,

$$\eta_2(\psi_s) = \frac{|\phi_C|}{\frac{1}{2}\varphi} = \frac{1}{W_s} \quad (9.19)$$

with $W_s > 1$.

For a very strongly convergent lens, with $W_s < -1$, the results are similar, with only sign changes added. Thus,

$$\phi_C = \mp \frac{1}{2}\varphi / W_s \quad ; \quad \eta_2 = |\phi_C| / \frac{1}{2}\varphi = -1/W_s \quad (9.20)$$

This result has been included for completeness, but optimum spectrometer designs will not include such values of W_s .

It is evident that, at a given ψ_s , ϕ -focusing occurs at a particular point [X in Figs. (9.4) and (9.5)]. Because the (z, r) coordinates for the image points vary considerably with ψ_s , the detector cannot be placed in

any one position for point-to-point focusing. Some compromise position must be chosen, which is suitable for all ψ_s in the acceptance range. The detector is most conveniently placed in the neighborhood of $(z_f, 0, 0)$; this is particularly true for a multigap instrument, since the image points from other gaps then occur on the other side of the z -axis. Considerations of resolution (see Section #6) also dictate placement at $(z_f, 0, 0)$. Suppose the detector is a circular one with radius R_d . Since $r_{lf} = 0$, the only rays accepted are those for which

$$|\delta R_{2f}| \leq R_d .$$

Since the density of rays, for a given ψ_s , is uniform with ϕ_s , and since δR_{2f} is proportional to ϕ_s , we find that the fraction of rays accepted by the detector is

$$\eta_3(\psi_s) = \frac{R_d}{|\delta R_{2f}(\max)|} \quad (9.21)$$

for

$$R_d \leq |\delta R_{2f}(\max)| ,$$

since $\eta_3 \leq 1$. If $R_d > |\delta R_{2f}(\max)|$, then $\eta_3 = 1$, i.e., the detector accepts all the rays passed by the exit aperture.

Now, $\delta R_{2f}(\max)$ is the δR_{2f} -value corresponding to $\phi_s = \phi_C$, which is $\frac{1}{2}\varphi$ for a converging entrance lens, and $\frac{1}{2}\varphi/|W_s|$ for a diverging entrance lens or a converging lens with $W_s > -1$. Hence, from Eqs. (9.17) and (9.21),

$$\eta_3(\psi_s) = \frac{2R_d}{\left| vN_s + \frac{z_f}{z_s} N_f W_s \right| z_s \varphi} \quad \text{for a convergent entrance lens} \quad (9.22a)$$

$$\eta_3(\psi_s) = \frac{2R_d |W_s|}{\left| vN_s + \frac{z_f}{z_s} N_f W_s \right| z_s \varphi} \quad \text{for a divergent entrance lens.} \quad (9.22b)$$

Equation (9.22b) also applies to the case in which $W_s < -1$. Equations (9.22a) and (9.22b) are valid only for $\eta_3 \leq 1$. If either ratio is greater than 1, then $\eta_3 = 1$.

The combined transmission through the exit and detector apertures is

$$\eta_{23}(\psi_s) = \eta_2(\psi_s) \eta_3(\psi_s) = \frac{2R_d}{\left| vN_s + \frac{z_f}{z_s} N_f W_s \right| z_s \varphi} . \quad (9.23)$$

This result is valid only for $\eta_{23} \leq \eta_2$; if the expression in Eq. (9.23) exceeds η_2 , then $\eta_{23} = \eta_2$.

Equation (9.23) applies to cases in which either converging or diverging entrance lenses are present. Other things being equal, however, the numerator is larger for the divergent case. This is evident in an examination of the symmetrical case, which we now consider.

For a symmetrical instrument, all the results are simplified, since $\psi_f = -\psi_s$, $z_s f_s = z_f f_f$, $\rho_s = \rho_f$, $\mu_s = \mu_f$, $d\rho_s/d\psi_s = -d\rho_f/d\psi_f$, $z_s = -z_f$.

Then,

$$N_f(\psi_f) = N_s(\psi_s) \quad (9.24a)$$

$$\delta r_{2f} = -\delta r_2 = -N_s z_s \phi_s \quad (9.24b)$$

$$\delta r'_{2f} = -N_s W_s z_s \phi_s \quad (9.24c)$$

$$\delta R_{2f} = -(1 + W_s) N_s z_s \phi_s, \quad (9.24d)$$

and

$$\eta_3(\psi_s) = \frac{2R_d}{|1 + W_s| |N_s| z_s \phi} \quad \begin{array}{l} \text{for } \eta_3 \leq 1 \\ \text{for } |W_s| < 1 \end{array} \quad (9.25a)$$

$$\eta_3(\psi_s) = \frac{2R_d |W_s|}{|1 + W_s| |N_s| z_s \phi} \quad \begin{array}{l} \text{for } \eta_3 \leq 1 \\ \text{for } |W_s| > 1 \end{array} \quad (9.25b)$$

$$\eta_{23}(\psi_s) = \frac{2R_d}{|1 + W_s| |N_s| z_s \phi} \quad \text{for } \eta_3 \leq \eta_2 \quad (9.26)$$

If the right side of Eqs. (9.25a) or (9.25b) is greater than 1, then $\eta_3 = 1$.

If the right side of Eq. (9.26) exceeds η_2 , then $\eta_{23} = \eta_2$.

We note the following:

(i) For the divergent entrance lens, η_2 depends upon the shape of the boundary curves and trajectories (through F_1 and μ , and hence P), but not on the size of the instrument (z_s) nor on ϕ . This follows from the result of Section #5 that the rotation depends upon the sample size (δr_2) rather than on the gap size.

(ii) On the other hand, η_3 depends upon ϕ and R_d/z_s , as well as on μ and F_1 . Thus, if the detector slit remains constant, η_3 depends upon the size of the instrument. It is seen that η_3 is inversely proportional to

φ (for $\eta_3 < 1$), so that as the gap is opened up with a detector of fixed size, the fraction of the gap which gives useful transmission decreases. In fact, since $\eta_3\varphi$ is independent of φ as long as $\eta_3 \leq 1$ [i.e., as long as $|\delta R_{2f}(\max)| > R_d$], it is of no value to increase φ beyond the value for which $|\delta R_{2f}(\max)| = R_d$.

Calculated Transmission Characteristics of Point Sources in Symmetrical Spectrometers. A semiquantitative illustration of the effect of η_2 and η_3 on the transmission is shown in Figs. (9.8), (9.9) and (9.10), in which $R_d/z_s = 0.01$. The full width of each figure represents the relative transmission as a function of ψ_s , in the absence of a fringing field, and, from Eq. (7.2), varies as $\sin\psi_s$. The shaded area represents the actual transmission when the ideal transmission is multiplied by $(\eta_2\eta_3)$.

For $K = 0.5$, it may be seen from Fig. (9.3) that, for a symmetrical spectrometer, $d\rho_s/d\psi_s$ is negative and is fairly large over most of the ψ_s -region considered, so that $P > 0$ and the lens effect is strong. Because the lenses are convergent, $\eta_2 = 1$, but because they are strong, the image points are distant from $(z_f, 0, 0)$ and the spreading out at r_{2f} is great, leading to small η_3 . Close to 140°, $d\rho_s/d\psi_s$ becomes small; the resulting small value for P leads to η_3 being almost unity.

For $K = 0.6$, Fig. (9.3) shows that $d\rho_s/d\psi_s$ is negative for the smaller ψ_s -values, comes close to zero at 120°, and then becomes positive as ψ_s increases. However, it never becomes large.* Hence, for ψ_s smaller than approximately 120°, we have $\eta_2 = 1$ and η_3 approaches unity as ψ_s approaches 120° and P approaches zero. As ψ_s increases beyond 130°, the value of $d\rho_s/d\psi_s$ increases, $|P|$ increases, so that both η_2 and η_3 decrease in value. For $K = 0.8$, we have $d\rho_s/d\psi_s$ positive and large, so $\eta_2 < 1$ and $\eta_3 < 1$ and are relatively small over the entire ψ_s -range.

*For values of $\tan \mu_s$, see Fig. (10.1)

Other forms of presentation of η_2 and η_{23} are shown in Fig (9.11), (9.12) and (9.13) for the case of the symmetrical magnetic profiles of Fig. (3.3) and Table II.3 of Appendix II.

Figure 9.13

Figure (9.11) shows η_2 as a function of ψ_s for various K -values. The corresponding numerical values are given in Table IV.I, Column 4. As pointed out above, the lenses converge when $\frac{d\rho_s}{d\psi_s} < 0$. It is evident from Fig. (9.3), as well as from Fig. (3.3) and Table IV.1, that the lenses for symmetrical systems are convergent for all K -values less than K_0 , where K_0 lies between 0.50 and 0.52. For this reason, all of these K -values correspond to $\eta_2 = 1$, as shown in Fig. (9.11) for $K = 0.20$ and $K = 0.50$. The lower limit on the ψ_s -values covered corresponds to the practical values for the critical angles discussed in Section #3. For $K > 0.52$ we have $\frac{d\rho_s}{d\psi_s} > 0$ close to $\psi_s = 180^\circ$, becoming negative as ψ_s decreases, with the consequence that $\eta_2 < 1$ close to 180° and $\eta_2 = 1$ over the remainder of the ψ_s -range.

As ψ_s decreases, the divergent lens first becomes stronger, then less divergent and finally convergent. Then η_2 remains equal to unity until the small ψ_s -values are reached, at which point the lens becomes so convergent that η_2 begins to decrease again due to the crossover of the converging rays within the interior field. The region in which $\eta_2 = 1$ is indicated by the brackets at the side of the figure.

Figure (9.12) illustrates the quantity

$$Y = \frac{\frac{1}{2}\varphi}{R_d/z_s} \eta_{23}, \quad (9.27)$$

which is a universal quantity for this type of spectrometer and is converted to η_{23} through multiplication by specific values of $\frac{R_d/z_s}{\frac{1}{2}\varphi}$. Numerical values of Y are given in Table IV.1, Column 5.

Small values of Y indicate that the rays make a very broad focal area in the region of $(z_f, 0, 0)$, and hence lead to small η_{23} -values unless R_d is very large. Large values of Y , on the other hand, indicate that the rays form a very small focal area in the neighborhood of $(z_f, 0, 0)$, and hence even detectors with very small diameters will accept all the rays. It is evident from the form of Y :

$$Y = \frac{1}{|1 + W_s| |N_s|}, \quad (9.28a)$$

with

$$\frac{1}{|N_s|} = \frac{Kz_s}{\rho_s |\tan \mu_s|} = \frac{Kz_s}{\left| \frac{d\rho_s}{d\psi_s} \right|}, \quad (9.28b)$$

that there are two cases in which Y becomes large:

(i) In this case, $\left| \frac{d\rho_s}{d\psi_s} \right|$ is small. Hence, $|\tan \mu_s|$ and $|P|$ are

small. Because of the resulting small lateral defocusing, the point source tends to go into a point image, and even a detector with small R_d gathers most of the rays.

(ii) In this case, $|1 + W_s|$ is small. This means that the convergent lens is quite powerful, so that W_s is negative and close to one in magnitude. The rays of the trajectories thus cross over within the magnetic field and

emerge from the field tending to diverge. The convergent lens at the exit, however, bends the rays towards $(z_f, 0, 0)$. Thus, a small R_d will still allow the detector to accept all the rays. For this to occur, however, $|W_s|$ cannot be too far from unity. If $|W_s|$ is too small, then the exit lens doesn't converge the rays enough to bring them close to $(z_f, 0, 0)$; on the other hand, if $|W_s|$ is too large, the rays are bent excessively and pass at some distance from $(z_f, 0, 0)$.

Figure (9.13) illustrates η_{23} -values for a particular case in which $R_d/z_s = 0.01$ and $\varphi = 10^\circ = 0.175$ radians; then $\eta_{23} = 0.115 Y$. Since $\eta_{23} \leq \eta_2 \leq 1$, the value $0.115 Y$ holds only for $0.115 Y < \eta_2$. When $0.115 Y > \eta_2$, η_{23} is set equal to η_2 . $0.115 Y > \eta_2$ implies only that R_d is larger than is necessary to catch all of the rays emerging from the exit aperture.

Since the distance to the profile ρ_s [or the equivalent $f_s(\psi_s)$], the rotation term $F_1(\psi_s)$ and the strength of the fringing field lens (measured by $\tan \mu_s$) are all functions of ψ_s , the over-all value for η_i requires averaging over ψ_s . We have, then, for either converging or diverging systems, for symmetrical or asymmetrical systems;

$$\overline{\eta_2} = \frac{\int \eta_2(\psi_s) \sin \psi_s d\psi_s}{\int \sin \psi_s d\psi_s} \quad (9.29)$$

$$\overline{\eta_3} = \frac{\int \eta_3(\psi_s) \sin \psi_s d\psi_s}{\int \sin \psi_s d\psi_s} \quad (9.30)$$

$$\overline{\eta_{23}} = \overline{\eta_2 \eta_3} = \frac{\int \eta_2(\psi_s) \eta_3(\psi_s) \sin \psi_s d\psi_s}{\int \sin \psi_s d\psi_s} \quad (9.31)$$

where the integration is taken from $\psi_s(\min)$ to $\psi_s(\max)$. In general, it is necessary to use Eq. (9.31), since :

$$\overline{\eta_2 \eta_3} \neq \overline{\eta_2} \overline{\eta_3} \quad (9.32)$$

unless η_2 or η_3 are essentially constant over the entire ψ_s -range.

Because the losses in transmission may be considerable, it is important to choose the K-value (and hence the profile design) in such a

way as to maximize $\eta_2\eta_3$ and thus minimize the losses. It is evident from the figures that P should be kept small, and this is most readily accomplished if part of the lens is convergent and part divergent, as in Fig. (9.9).

Of course, $P = 0$ if $\frac{d\rho_s}{d\psi_s} = \mu_s = 0$ over the ψ_s -range of interest, i.e., the

profile is perpendicular to all rays, and hence is a circular arc with $(z_s, 0, 0)$ as center. This point is discussed further in Section #16.

Transmission for a Finite Source. When the transmission losses $\eta_{23}^*(\psi_s)$ computed for source extension alone and the losses $\eta_{23}(\psi_s)$ computed for fringing field and point source are comparable, then neither result is valid, nor can the correct result be derived from a combination of the two.

For the evaluation of the approximate transmission characteristics of a particular instrument design or of a specified sample size, computation of η_{23} or η_{23}^* may be adequate. However, where both effects are comparable in magnitude, and where accurate results are required, both effects must be combined before integration is carried out over the sample. The transmission coefficients for this case are designated η_{2T} , η_{3T} , and η_{23T} .

As in Section #7, the transmission is computed only for the upper left quadrant of the source, and the approximation is again made that the r_1 -axis is kept fixed in ϕ_M . Although only a uniformly distributed source is considered, this is not an essential restriction. If the source intensity is nonuniform, but known, indicated integrations may be carried out numerically, returning to the basic integral, Eq. (7.5) or (7.12). The bending effect of the fringing field is taken to be the same for an off-center ray as for one from $(z_s, 0, 0)$; thus, $\Delta\beta_e$ depends only upon ϕ_s and not upon the source point.

As in Section #7, η_{2T} for a source point is a function of the r_2 -coordinate alone. If a cut is taken through the sample at constant r_2 , all of the rays from the points involved have the same η_2 -losses, and, to the first order, they may be treated as though they emerged from $(z_s, 0, r_2)$. Then, as in Section #7, the integration over the source is only over r_2 , using as a weighting factor the r_1 -boundary of the source for a given r_2 -value.

Because of the astigmatic lens effect of the fringing field, the imaging at constant ψ_s is point to line rather than point to point, as in Section #7. The r_{1f} -coordinate has the same value as in Section #5 and #7, but the r_{2f} -coordinate is smeared out by the fringing field, being distributed uniformly in the line $r_{1f} = \text{constant}$. The distribution is symmetrical around the image point position of Section #7 (i.e., the median ray, or focus point with no lens effect) only when $\eta_{2T} = 1$; for $\eta_{2T} < 1$, the line distribution is truncated on one or both sides.

The coefficient $\eta_3 T$ can take on a continuum of values rather than only (0,1), as in Section #7, since it depends on the dimensions of the truncated line relative to the detector aperture boundaries. When $\eta_3 T < 1$ for a single source point, its magnitude is determined by the r_2 -coordinates of the detector aperture boundary at r_{1f} . In fact, because the density of rays is uniform over the truncated line, $\eta_3 T$ is the fraction of the truncated line lying within the aperture boundaries. Although the length of this line is a function only of the r_2 -coordinate of the source point, the aperture boundaries vary with the r_{1f} -coordinate of the image. Hence $\eta_3 T$ is a function of r_1 as well as of r_2 .

To determine the combined effect of source spread and fringing field, we use the theory of Section #5 for the r_2 -effect (as in Section #7) and the results of the first part of this section for the fringing field. Since both effects result in small p_ϕ -values, the two image shifts are small and are added linearly. Both convergent and divergent lenses are treated by the same equations.

As before, the fringing field effect is computed by considering the incidence of a ray at ϕ_s to be equivalent to a virtual source point displaced from the actual point source by a distance δr_2 . A single point on the source is then equivalent to a line source spread out symmetrically and uniformly on both sides of (z_s, r_1, r_2) . For a given incidence plane ϕ_s , we have, as in Eq. (9.9)

$$\phi_f - \phi_s = F_1(\psi_s) \frac{(\delta r_2 + r_2)}{z_s}, \quad (9.33)$$

where the algebraic sum of δr_2 and r_2 is taken. Here δr_2 is the virtual source displacement due to the fringing field, whereas r_2 is the actual source point displacement (in the r_2 -direction) from the center $(z_s, 0, 0)$, both being measured as positive in the negative ϕ -direction. Thus, in the quadrant chosen in Section #7, every point has a negative r_2 [see Fig. (7.2)]. Using δr_2 as given by Eq. (9.11), we have [see Eq. (9.12)]

$$\phi_f - \phi_s = F_1 N_s \phi_s + F_1 (r_2/z_s) \quad (9.34)$$

or

$$\phi_f = \phi_s W_s + F_1 (r_2/z_s). \quad (9.35)$$

For a source point at (z_s, r_1, r_2) , the image formation at the plane $z = z_f$ is computed in the same manner as for the source point at $(z_s, 0, 0)$:

$$\delta r_{2f} = \frac{\sin \psi_s}{\sin \psi_f} (r_2 + \delta r_2) = \frac{\sin \psi_s}{\sin \psi_f} (r_2 + N_s z_s \phi_s) \quad (9.36a)$$

and

$$\delta r_{2f}' = N_f z_f \phi_f = N_f z_f \left[W_s \phi_s + F_1 (r_2 / z_s) \right] \quad (9.36b)$$

$$\begin{aligned} \delta R_{2f} &= \delta r_{2f} + \delta r_{2f}' \\ &= \left[v(\psi_s) + \frac{N_f z_f F_1}{z_s} \right] r_2 + \left[v N_s + \frac{z_f}{z_s} N_f W_s \right] z_s \phi_s \\ &= g(\psi_s) r_2 + h(\psi_s) z_s \phi_s. \end{aligned} \quad (9.37)$$

As in Section #7 [Eq. (7.14)], the r_{1f} -coordinate of the image line is given by Eq. (5.15b):

$$r_{1f} = \frac{1}{\psi'} \frac{\cos \psi_s}{\cos \psi_f} r_1 = u(\psi_s) r_1. \quad (9.38)$$

With $r_2 = 0$, Eq. (9.37) reduces to Eq. (9.17). For $r_2 = 0$, the values of δR_{2f} were spread out uniformly and symmetrically on both sides of $(z_f, 0, 0)$. For $r_2 \neq 0$, the spread for a given r_2 is still uniform on the line $r_{1f} = \text{constant}$, but is no longer symmetric around the z -axis, nor indeed around the image point (z_f, r_{2f}, r_{1f}) in the absence of fringing field, where r_{2f} is given by Eq. (5.20). The lack of symmetry arises both from the r_2 -term and from the fact that the range of ϕ_s -values transmitted is not symmetric around ϕ_M .

Transmission Computation from Boundaries of Allowed ϕ_s -regions. Equation (9.37) describes the r_{2f} -component of the image (for a given r_2 and ψ_s) as a function of ϕ_s . Whether a particular δR_{2f} -value is allowed (i.e., transmitted) depends upon whether the ray incident at ϕ_s traverses the three apertures: entrance, exit, and detector. Because the incidence of rays, at a given ψ_s and r_2 , has a uniform density with ϕ_s , the transmission problem is essentially solved when the boundary ϕ_s -values are correctly chosen.

The condition that a trajectory does not pass the plane of the profile (e.g., collide with the poleface in an iron-core instrument) may be stated as

$$|\phi - \phi_M| \leq \frac{1}{2} \varphi \quad (9.39)$$

within the gap.

Because of the conservation relation Eq. (1.8), $(\phi - \phi_M)$ changes only monotonically, so that it is sufficient to express the condition in Eq. (9.39) only at the boundaries*:

$$|\phi_s - \phi_M| = |\phi_s| \leq \frac{1}{2}\varphi \quad (9.40a)$$

$$|\phi_f - \phi_M| = |\phi_f| \leq \frac{1}{2}\varphi \quad (9.40b)$$

The condition in Eq. (9.40b), when placed into Eq. (9.35), determines the critical ϕ_s -values beyond which no rays pass through the exit aperture. Thus, setting $\phi_f = \pm \frac{1}{2}\varphi$, transmission through ϕ_s cannot occur unless the following is true (since we consider only cases in which $W_s > 0$):

$$\frac{1}{W_s} \left[\frac{1}{2}\varphi + F_1 \frac{r_2}{z_s} \right] \leq \phi_s \leq \frac{1}{W_s} \left[\frac{1}{2}\varphi - F_1 \frac{r_2}{z_s} \right] \quad (9.41)$$

The boundary of the detector aperture is described by the curve $r_{1f} = \Gamma_1(r_{2f})$, or by the inverse $r_{2f} = \Gamma_2(r_{1f})$. We consider only detector apertures which have reflection symmetry in ϕ_M as well as in the plane perpendicular to ϕ_M ; in addition, the aperture is entirely open within its boundary. Then, at the image line ($r_{1f} = \text{constant}$), the boundary points are $r_{2f} = \pm R_{2f}$, where $R_{2f} = |\Gamma_2|$. We shall describe the farthest r_{2f} -extension of the aperture boundary, i.e., the maximum of R_{2f} , as D_{r_2} . Similarly, the farthest r_{1f} -extension of the boundary, or the maximum of $|\Gamma_1|$, is D_{r_1} .

From Eq. (9.37), the requirement that $|\delta R_{2f}| \leq R_{2f}$ leads to [for $h(\psi_s) > 0$]

$$-\frac{1}{z_s h(\psi_s)} [R_{2f} + g(\psi_s) r_2] \leq \phi_s \leq \frac{1}{z_s h(\psi_s)} [R_{2f} - g(\psi_s) r_2] \quad (9.42)$$

as an inequality which must be obeyed for transmission through ϕ_s . R_{2f} is a function of r_{1f} , and hence of r_1 . For $h < 0$, the inequality is reversed.

It is convenient to express these boundaries as fractions of the gap width φ . Thus, for $h(\psi_s) > 0$,

$$E_e^- = -\frac{1}{2} \leq \frac{\phi_s}{\varphi} \leq \frac{1}{2} = E_e^+ \quad (9.43a)$$

*It is to be noted that we are examining only the transmission of a single gap. The argument is complete for an iron-core instrument, in which particles entering a gap can only emerge from the same gap if at all. For an air-core instrument, transmission may be increased through particles entering one gap and emerging from another gap, but we do not examine this question quantitatively.

$$E_0^- = -\frac{1}{W_s} \left[\frac{1}{2} + \frac{F_1}{\varphi z_s} r_2 \right] \leq \frac{\phi_s}{\varphi} \leq \frac{1}{W_s} \left[\frac{1}{2} - \frac{F_1}{\varphi z_s} r_2 \right] = E_0^+ \quad (9.43b)$$

$$E_d^- = -\frac{1}{h(\psi_s)} \left[\frac{R_{2f}}{\varphi z_s} + \frac{g(\psi_s)}{\varphi z_s} r_2 \right] \leq \frac{\phi_s}{\varphi} \leq \frac{1}{h(\psi_s)} \left[\frac{R_{2f}}{\varphi z_s} - \frac{g(\psi_s)}{\varphi z_s} r_2 \right] = E_d^+. \quad (9.43c)$$

Now,

$$h(\psi_s) = \frac{v(\psi_s)}{K} \frac{\rho_s}{z_s} \tan \mu_s - \frac{W_s}{K} \frac{\rho_f}{z_s} \tan \mu_f, \quad (9.44a)$$

which may be positive or negative. In fact, for a symmetrical spectrometer,

$$h(\psi_s) = -\frac{1}{K} \frac{\rho_s}{z_s} \tan \mu_s (1 + W_s) = -N_s (1 + W_s) \quad (9.44b)$$

has (for $W_s > -1$) the sign of $(-\mu)$. Hence, for symmetrical systems, Eq. (9.43c) applies only for a convergent lens. For a divergent lens, the boundaries must be inverted. For an unsymmetrical system, the relation between the sign of $h(\psi_s)$ and the nature of the entrance and exit lenses cannot be simply assigned. However, in general, when $h(\psi_s) < 0$,

$$E_d^- = \frac{1}{h(\psi_s)} \left[\frac{R_{2f}}{\varphi z_s} - \frac{g(\psi_s)}{\varphi z_s} r_2 \right] \leq \frac{\phi_s}{\varphi} \leq -\frac{1}{h(\psi_s)} \left[\frac{R_{2f}}{\varphi z_s} + \frac{g(\psi_s)}{\varphi z_s} r_2 \right] = E_d^+. \quad (9.45)$$

For a very strong convergent lens, for which $W_s < 0$, the $\leq$ signs in Eq. (9.43b) should also be changed to $\geq$, thus reversing the roles of E_0^- and E_0^+ as upper and lower limits. Because this is not a usual condition, we shall neglect this possibility in further discussion, although the appropriate treatment can always be carried through by the above reversal of terms.

It is evident that any of the three apertures may cut off the ϕ_s -range, at either end. We take

$$E_M(r_2, r_1) = \text{smallest of the three upper limits}$$

$$E_m(r_2, r_1) = \text{largest of the three lower limits,}$$

the comparison being taken algebraically, rather than with absolute values. The range of transmitted ϕ_s then lies in the interval determined by (E_m, E_M) . If we are interested only in η_{2T} , then we take

E'_M = smallest of the upper limits in Eq. (9.43a) and (9.43b)

E'_m = largest of the lower limits.

Because of the uniform density over ϕ_s , the transmission at a given ψ_s , for a given (r_1, r_2) , is the ratio of the acceptance range of ϕ_s to the entire ϕ_s -range (φ). Thus

$$\eta_{2T}(r_2, \psi_s) = E'_M - E'_m \quad (9.46)$$

$$\eta_{23T}(r_1, r_2, \psi_s) = E_M - E_m. \quad (9.47)$$

The particular aperture that is effective in limiting the ϕ_s -range varies with r_2 , and may differ at the upper and lower limits. The possible combinations are best illustrated with a graph showing the transmission properties as a function of r_2 , at a fixed r_1 (and hence fixed r_{1f} and R_{2f}). The boundaries E_0 and E_d have the form

$$E = \pm k + q r_2. \quad (9.48)$$

The boundaries are then parallel lines, of slope q , and separated by the distance $2k$.

For E_0 , the slope is

$$q = - \frac{1}{W_s} \frac{F_1}{\varphi z_s}, \quad (9.49)$$

which is always negative, for lenses which are not strongly convergent (i.e., when $W_s > 0$). The vertical separation is

$$2k = 1/W_s. \quad (9.50)$$

For E_d , the slope is

$$q = - \frac{1}{\varphi z_s} \frac{g(\psi_s)}{h(\psi_s)} = \frac{1}{h(\psi_s)} \left[-v(\psi_s) - \frac{z_f}{z_s} N_f F_1 \right] \frac{1}{\varphi z_s}. \quad (9.51a)$$

For a symmetrical spectrometer,

$$g(\psi_s) = -1 - N_s F_1 = -W_s. \quad (9.51b)$$

In this case,

$$q = \frac{1}{h(\psi_s)} \frac{W_s}{\varphi z_s} = - \frac{1}{N_s z_s} \frac{W_s}{1 + W_s}. \quad (9.51c)$$

Except for spectrometers in which the deviation from symmetry is very

large, $-v = \left(-\frac{\sin \psi_s}{\sin \psi_f} \right) > 0$, and of the order of magnitude of unity. From Eq. (9.15b), it is evident that $\left(-\frac{z_f}{z_s} N_f F_1 \right) = \left(\frac{\rho_f F_1}{z_s K} \tan \mu_f \right)$ has the sign of

μ_f and the order of magnitude of $|\tan \mu_f|$. Hence, if $|\tan \mu_f|$ is small,* q has the sign of $h(\psi_s)$. (This relation may be inverted in unusual cases, as when $-v < 0$ or when $|\tan \mu_f|$ is large and $\mu_f < 0$.) The vertical separation is

$$|2k| = \left| \frac{2}{\varphi z_s} \frac{1}{h(\psi_s)} R_{2f} \right|. \quad (9.52)$$

In Fig. (9.14), the boundary curves are shown for a case in which $h(\psi_s) > 0$, which, for a symmetrical system with $|\tan \mu_s|$ not large ($W_s > 0$),

Fig 9.14

corresponds to a convergent lens. It is evident that for the symmetric detector considered, the transmission is the same for positive and negative r_2 , at any particular $|r_2|$ -value. As previously, only the left-hand upper quadrant is considered. The graphs are drawn for a particular r_1 -value, since the curves (E_d^+, E_d^-) are a function of the detector dimension R_{2f} , which varies with r_{1f} . The maximum value of $|r_2|$ is OD, which represents the extent of the source.

The region ABCJFH represents the E-boundaries which enter into the computation of η_{2T} . The shaded region ABCJGH represents the boundaries entering into η_{23T} .

Computation of η_{2T} for One Source Point. The properties of these curves can be considered in greater detail in Fig. (9.15), which shows only the E_e and E_0 curves; hence, only the factors governing computation of η_{2T} are covered. The various possible entrance lenses are considered:

* For symmetrical systems q has the sign of $h(\psi_s)$ for $W_s > 0$, from Eq. (9.51b).

Fig 9.15

$P > 0$ (convergent), $P = 0$ (no lens, the case of Section #7, and $P < 0$ (divergent). The r_2 -values at which the various E_0 -curves intersect the E_e -curves are of some importance. The complex nomenclature involved is given in Table 9.2 and partially illustrated in Fig. (9.15).

It is worth mentioning the following features of these curves:

(i) The intersection points [at $r_2 = (r_{0+}^0, r_{0-}^0)$] of E_0^- and E_0^+ with the line $E = 0$ are independent of the type or strength of the lens. This is evident from Eq. (9.43b), which indicates that the roots of the equations $E_0^+ = 0$ and $E_0^- = 0$ are independent of W_s . Thus,

$$r_{0-}^0 = -\frac{1}{2} \varphi z_s / F_1; r_{0+}^0 = \frac{1}{2} \varphi z_s / F_1. \quad (9.53)$$

Hence, for any of the P -values being considered (i.e., $W_s > 0$), we have at $r_2 = r_{0-}^0$,

$$\eta_{2T} = E_M^+ - E_M^- = \frac{1}{2} - 0 = \frac{1}{2}. \quad (9.54)$$

(ii) From Eq. (9.49), the slopes of the E_0 -curves are all negative, and are largest in magnitude for a convergent lens ($P > 0$), for which $W_s < 1$; next in steepness is the $P = 0$ case, with $W_s = 1$, and least steep is the divergent lens case ($P < 0$), since $W_s > 1$. The areas shown as convergent only or divergent only (or both) apply only to the cases illustrated. The areas included for the general case depend upon the slope of the E_0^- and E_0^+ lines. For $P < 0$, the slope may vary from zero (horizontal lines), when $W = \infty$ up to the $P = 0$ line; for $P > 0$, the slope may vary from infinity (vertical line) when $W = 0^*$ down to the line $P = 0$.

* It will be recalled that the case of $W < 0$ was excluded as an unrealistic value for well-designed spectrometers.

Table 9.2.

r_2 -VALUES* AT INTERSECTION OF E_0 -CURVES WITH
 E_e -CURVES AND WITH $E = 0$

Intersection of:	with	r_2 -value
$E_0^+ (P > 0)$	E_e^+	r_{0+}^{C+}
$E_0^+ (P = 0)$	E_e^+	r_{0+}^{*+}
$E_0^+ (P < 0)$	E_e^+	r_{0+}^{D+}
$E_0^+ (P > 0)$	E_e^-	r_{0+}^{C-}
$E_0^+ (P = 0)$	E_e^-	r_{0+}^{*-}
$E_0^+ (P < 0)$	E_e^-	r_{0+}^{D-}
$E_0^- (P > 0)$	E_e^+	r_{0-}^{C+}
$E_0^- (P = 0)$	E_e^+	r_{0-}^{*+}
$E_0^- (P < 0)$	E_e^+	r_{0-}^{D+}
$E_0^- (P > 0)$	E_e^-	r_{0-}^{C-}
$E_0^- (P = 0)$	E_e^-	r_{0-}^{*-}
$E_0^- (P < 0)$	E_e^-	r_{0-}^{D-}
$E_0^+ (\text{all } P)$	$E = 0$	r_{0+}^0
$E_0^- (\text{all } P)$	$E = 0$	r_{0-}^0

*The notation r_{ij}^{km} refers to:

- i: Aperture intersection. In this case, it is always 0, because only the exit aperture is considered.
- j: refers to sign of exit aperture curve involved, i.e., E_0^- or E_0^+ .
- k: nature of entrance lens. C = convergent ($P > 0$), D = divergent ($P < 0$), * = no lens ($P = 0$), as in Section #7. Omitted for intersection with $E = 0$.
- m: refers to sign ($\pm$) of entrance aperture curve involved [i.e., $E_e^+ = \frac{1}{2}$ or $E_e^- = -\frac{1}{2}$,] or to 0 for $E = 0$.

(iii) For the source quadrant considered, $E'_M = \frac{1}{2}$ when $P = 0$ (no lens effect), but $E'_m = -\frac{1}{2}$ only when $r_2 = 0$, i.e., $E'_m > -\frac{1}{2}$ for $|r_2| > 0$.

Then $\eta_{2T} = \eta_2^* < 1$ whenever $r_2 \neq 0$, as mentioned in Section #7. This arises because the orbit rotation $\Delta\phi = (\phi_f - \phi_s)$ results in $\phi_f < -\frac{1}{2}\varphi$ for ϕ_s -values in the range $[-\frac{1}{2}\varphi + |\Delta\phi|]$, where $\Delta\phi < 0$, and, of course, increases in magnitude with $|r_2|$.

(iv) When $P > 0$, the convergent lens effect opposes the r_2 -caused rotation in (iii), so that for $|r_2| < |r_{0-}^{C-}|$ all rays pass through the exit aperture and $\eta_{2T} = E'_M - E'_m = 1$. As $|r_2|$ increases beyond this point, the converging lens can compensate only partially for the r_2 -rotation, and the most negative ϕ_s -rays are blocked by the exit aperture. (E'_m increases towards $E = \frac{1}{2}$). At the positive ϕ_s -end, there is no cutoff, except by the entrance aperture, since all rotations are towards negative ϕ (r_2 and δr_2 both negative). Evidently, the steeper is E_0^- , the greater is the range of r_2 for which $\eta_{2T} = 1$. From this point of view, the strongest possible convergent lens is desirable for maximum transmission with large $|r_2|$, provided $|r_2| < |r_{0-}^0|$, i.e., the critical cutoff value of ϕ_s is negative. For $|r_2| > |r_{0-}^0|$, however, the critical value of ϕ_s is positive, and the transmission increases with W_s , so that a lensless system is better than a convergent lens, and a divergent lens best of all. This is true only for points with large r_2 -values and is not true when the entire source area is considered. In addition, as pointed out below, strong lenses are inconsistent with high transmission through the detector aperture.

(v) The boundary-value r_{0-}^{C-} is computed by setting $E_0^- = -\frac{1}{2}$ [see Eq. (9.43b)].

Thus, $\eta_{2T} = 1$ for a convergent lens when

$$|r_2| < \left| \frac{1}{2}\varphi z_s N_s \right| = |r_{0-}^{C-}|. \quad (9.55)$$

For $|r_2| > \left| \frac{1}{2}\varphi z_s N_s \right|$, the transmission for the convergent lens is

$$\eta_{2T} = E'_M - E'_m = \frac{1}{2} + \frac{1}{W_s} \left[\frac{1}{2} + \frac{F_1}{\varphi z_s} r_2 \right]. \quad (9.56)$$

(vi) For any value of P , complete cutoff by the exit aperture can occur only for very large $|r_2|$ -values, such that $|r_2| > |r_{0-}^+|$, where $r_{0-}^+ = (r_{0-}^{C+}, r_{0-}^{*+} \text{ or } r_{0-}^{D+})$. If the source contains $|r_2|$ -values larger than these limiting values, no transmission at all can occur for such points. It is evident from Fig. (9.15) that this maximum $|r_2|$ -value is least for the convergent lens and largest for the divergent lens. The reason is that in

the region of positive ϕ_s -values, the δr_2 -value for a divergent lens is positive, thus opposing the rotation effect of the negative r_2 . The boundary value r_{0-}^+ is computed by setting $E_0^- = \frac{1}{2}$. Thus

$$|r_{0-}^+| = \left| \frac{1}{2} \frac{\varphi z_s}{F_1} (1 + W_s) \right| = \left| \frac{\varphi}{F_1} + \frac{1}{2} \varphi N_s \right| z_s, \quad (9.57)$$

and $\eta_2 T = 0$ for $|r_2| > |r_{0-}^+|$. It is evident that $|r_{0-}^+|$ increases monotonically with W_s .

(vii) For the divergent case ($P < 0$, $W_s > 1$), $E_0^- > -\frac{1}{2}\varphi$ in the left quadrant, so that $E_M' = E_0^-$ for all r_2 . The intersection of E_0^+ with $E_e^+ = \frac{1}{2}$ is given by

$$|r_{0+}^{D+}| = \frac{1}{2} \varphi z_s N_s, \quad (9.58)$$

which is the same as the magnitude of the limit $|r_{0-}^{C-}|$ for a convergent lens of the same power. For $|r_2| < |r_{0+}^{D+}|$, we have $E_0^+ < \frac{1}{2}$, so that $E_M' = E_0^+$ for this range. Hence, $\eta_2 T$ is the vertical separation $2k$ between E_0^+ and E_0^- . From Eq. (9.50), then,

$$\eta_2 T = 1/W_s, \quad (9.59)$$

i.e., independent of r_2 for small $|r_2|$. This is identical to Eq. (9.21), as might be expected, since $r_2 = 0$ in that case. This independence is related to the fact that the critical ϕ_s -planes ($\phi_s = \pm \frac{1}{2}\varphi/W_s$) are not at the exit aperture ($\phi_s = \pm \frac{1}{2}\varphi$) when $r_2 = 0$. Since the rotation is linear in $(r_2 + \delta r_2)$, an increase in $(-r_2)$ serves to increase the rotation for $\phi_s < 0$ (since $\delta r_2 < 0$) and to decrease it for $\phi_s > 0$ by just the same amount. Thus, the increased number of ϕ_s -planes excluded for $\phi_s < \phi_M$ is compensated by the decreased number excluded for $\phi_s > \phi_M$. The compensation operates as $|r_2|$ increases until the positive ϕ_s -boundary moves into $\phi_s = \frac{1}{2}\varphi$, i.e., until the positive $\Delta\phi$ -rotation of the lens at $\phi_s = \frac{1}{2}\varphi$ is just compensated by the negative $\Delta\phi$ -rotation due to $r_2 = r_{0+}^{D+}$.

(viii) For $|r_2| > |r_{0+}^{D+}|$, we have $E_M' = E_e^+ = \frac{1}{2}$. Then

$$\eta_2 T = \frac{1}{2} + \frac{1}{W_s} \left[\frac{1}{2} + \frac{F_1}{\varphi z_s} r_2 \right], \quad (9.60)$$

which is the same as Eq. (9.56)

(ix) Using Eq. (7.8b) for the transmission η_2^* with no fringing field,

$$\frac{\eta_2^* - \frac{1}{2}}{\eta_{2T} - \frac{1}{2}} = W_s \quad (9.61)$$

for the range of r_2 in which Eqs. (9.56) and (9.60) hold for both converging and diverging lenses, i.e., for $|r_2| > \frac{1}{2}\varphi z_s N_s$, but $|r_2| < |r_{0-}^+|$. For $|r_2| < |r_{0-}^0|$, both η_2^* and η_{2T} exceed $\frac{1}{2}$; hence, $\eta_2^* > \eta_{2T}$ for a divergent lens but is less for a convergent lens. Since η_2^* and η_{2T} are less than $\frac{1}{2}$ for $|r_2| > |r_{0-}^0|$, for a divergent lens $\eta_2^* < \eta_{2T}$, but exceeds it for a convergent lens. Hence, for these r_2 -values, η_{2T} is greater for a divergent lens than for a convergent lens.

Averaging η_{2T} over Source. The above equations for η_{2T} apply only for a single point on the source. A complete computation of the transmission through the exit aperture involves averaging over the sample quadrant and then over ψ_s . With the assumptions made, all points of the source with the same r_2 -value have the same η_{2T} -value. As in Section #7, the sample boundary is described by $r_1 = R_1(r_2)$. Hence, integration over the sample requires only integration over r_2 , weighting each r_2 -value with the sample height $|R_1|$, yielding the average $\eta_{2T}(\psi_s)$, at a fixed ψ_s :

$$\eta_{2T}(\psi_s) = \frac{1}{\frac{1}{2}Q_s} \int_{-R_2^*}^0 \eta_{2T}(r_2, \psi_s) |R_1(r_2)| dr_2, \quad (9.62)$$

where, for a converging entrance lens,

- (i) $\eta_{2T}(r_2, \psi_s) = 1$ for $|r_2| < \left| \frac{1}{2}\varphi z_s N_s \right| = |r_{0-}^{C-}|$ [Eq. (9.55)]
- (ii) $\eta_{2T}(r_2, \psi_s)$ given by Eq. (9.56) for $|r_{0-}^{C-}| < |r_2| < |r_{0-}^{C+}|$, the latter limit being given in Eq. (9.57).
- (iii) $\eta_{2T}(r_2, \psi_s) = 0$ for $|r_2| > |r_{0-}^{C+}|$.

For a diverging entrance lens,

- (i) $\eta_{2T}(r_2, \psi_s) = 1/W_s$ for $|r_2| < \frac{1}{2}\varphi z_s N_s = |r_{0+}^{D+}|$ [Eq. (9.58)]
- (ii) $\eta_{2T}(r_2, \psi_s)$ given by Eq. (9.60) for $|r_{0+}^{D+}| < |r_2| < |r_{0-}^{D+}|$, the latter limit being given in Eq. (9.57).
- (iii) $\eta_{2T}(r_2, \psi_s) = 0$ for $|r_2| > |r_{0-}^{D+}|$.

Here Q_s is the area of source, and R_2^* is the magnitude of the farthest r_2 -extension of the source. This relation involves the assumption (as in Section #7) that the source-activity distribution is uniform.

The simplest case is that of a rectangular source of dimensions $2R_2^*$ by $2R_1$. This is not a useful shape for a multigap instrument in which $0 \leq \phi \leq 2\pi$, although it is an important case for single-sector instruments or in those using a few sectors over a restricted ϕ -range. However, even for the general multigap case, the relative simplicity of the integrals allows for a ready examination of qualitative features not as easily evident in an integration for a circular source.

Then, from Eq. (9.62),

$$\eta_{2T}(\psi_s) = \frac{1}{R_2^* R_1} \int_{-R_2^*}^0 \eta_{2T}(r_2, \psi_s) R_1 dr_2 = \frac{1}{R_2^*} \int_{-R_2^*}^0 (E_M' - E_m') dr_2, \quad (9.63)$$

which is determined directly from the areas in Fig. (9.15) enclosed by E_M and E_m' and by $r_2 = -R_2^*$ and $r_2 = 0$.

For a converging lens, for $R_2^* > |r_{0-}^{C-}|$,

$$\begin{aligned} \eta_{2T}(\psi_s) &= \frac{1}{R_2^*} \left\{ \int_{r_{2m}}^{r_{0-}^{C-}} \left[\frac{1}{2} + \frac{1}{W_s} \left(\frac{1}{2} + \frac{F_1}{\varphi_{zs}} r_2 \right) \right] dr_2 + \int_{r_{0-}^{C-}}^0 dr_2 \right\} \\ &= \frac{1}{R_2^*} \left\{ \frac{1}{2} \left(1 + \frac{1}{W_s} \right) (r_{0-}^{C-} - r_{2m}) + \frac{1}{2} \frac{1}{W_s} \frac{F_1}{\varphi_{zs}} \left[(r_{0-}^{C-})^2 - (r_{2m})^2 \right] - r_{0-}^{C-} \right\} \\ &= \frac{1}{2R_2^*} \left\{ \frac{1}{W_s} \left[\frac{1}{2} \varphi_{zs} N_s - r_{2m} \right] \left[1 + \frac{F_1}{\varphi_{zs}} \left(\frac{1}{2} \varphi_{zs} N_s + r_{2m} \right) \right] \right. \\ &\quad \left. - \left[\frac{1}{2} \varphi_{zs} N_s + r_{2m} \right] \right\}, \quad (9.64) \end{aligned}$$

where $(-r_{2m})$ is the smaller of R_2^* and $|r_{0-}^{C+}|$, and $r_{0-}^{C-} = \frac{1}{2} \varphi_{zs} N_s$.

If $R_2^* > |r_{0-}^{C+}|$, the part of the source beyond $r_2 = r_{0-}^{C+}$ does not transmit any rays through the exit aperture at the ψ_s -value involved,

Then,† with

$$r_{2m} = r_{0-}^{C+} = - \left[\frac{\varphi_{z_s}}{F_1} + \frac{1}{2} \varphi_{z_s} N_s \right] ,$$

we find

$$\eta_{2T}(\psi_s) = \frac{1}{2R_2^*} \frac{\varphi_{z_s}}{F_1} = \frac{1}{R_2^*} (-r_{0-}^0) = \frac{1}{2} \alpha, \quad (9.65)$$

with

$$\alpha = \varphi_{z_s} / F_1 R_2^* . \quad (9.66)$$

For $R_2^* < |r_{0-}^{C-}|$, then $\eta_{2T} = 1$, so that

$$\eta_{2T}(\psi_s) = 1. \quad (9.67)$$

From Fig. (9.15), for a diverging lens, when

$$R_2^* < \frac{1}{2} \varphi_{z_s} N_s = |r_{0+}^{D+}|$$

we have

$$\eta_{2T}(\psi_s) = \frac{1}{R_2^*} \int_{-R_2^*}^0 \frac{1}{W_s} dr_2 = \frac{1}{W_s}, \quad (9.68)$$

which is the same as Eq. (9.21) for a point source. For $R_2^* > |r_{0+}^{D+}|$, on the other hand,

$$\begin{aligned} \eta_{2T}(\psi_s) &= \frac{1}{R_2^*} \left\{ \int_{r_{2m}}^{r_{0+}^{D+}} \left[\frac{1}{2} + \frac{1}{W_s} \left(\frac{1}{2} + \frac{F_1}{\varphi_{z_s}} r_2 \right) \right] dr_2 + \int_{r_{0+}^{D+}}^0 \frac{1}{W_s} dr_2 \right\} \\ &= \frac{1}{2R_2^*} \left\{ \left[r_{0+}^{D+} - r_{2m} \right] \left[1 + \frac{1}{W_s} \frac{F_1}{\varphi_{z_s}} (r_{0+}^{D+} + r_{2m}) \right] \right. \\ &\quad \left. - \frac{1}{W_s} \left[r_{0+}^{D+} + r_{2m} \right] \right\} , \end{aligned} \quad (9.69)$$

† In this case, the integral in (9.64) is evidently the area [see Fig. (9.15)] bounded by E_0^- and $r_2 = 0$, which area is $|r_{0-}^0|$.

where $(-r_{2m})$ is the smaller of R_2^* and $|r_{0-}^{D+}|$, and $r_{0+}^{D+} = -\frac{1}{2}\varphi_z N_s$. If $R_2^* > |r_{0-}^{D+}|$, then

$$r_{2m} = -\left(\frac{\varphi_z}{F_1} + \frac{1}{2}\varphi_z N_s\right),$$

and

$$\eta_{2T}(\psi_s) = \frac{1}{2R_2^*} \frac{\varphi_z}{F_1} = \frac{1}{2}\alpha. \quad (9.70)$$

When no lens is present (i.e., $P=0$, $W_s=1$),

$$\eta_{2T}(\psi_s) = \frac{1}{R_2^*} \int_{r_{2m}}^0 \left(1 + \frac{F_1}{\varphi_z} r_2\right) dr_2 = -\frac{1}{R_2^*} \left(r_{2m} + \frac{1}{2} \frac{F_1}{\varphi_z} r_{2m}^2\right), \quad (9.71)$$

where $(-r_{2m})$ is the smaller of R_2^* and $|r_{0-}^{*+}|$. For $R_2^* < |r_{0-}^{*+}|$, we have $r_{2m} = -R_2^*$ and

$$\eta_{2T}(\psi_s) = 1 - \frac{1}{2} \frac{F_1 R_2^*}{\varphi_z} = 1 - \frac{1}{2\alpha}, \quad (9.72)$$

which is the same as Eq. (7.11a). For $r_{2m} = r_{0-}^{*+} = -\frac{\varphi_z}{F_1}$, we have

$$\eta_{2T}(\psi_s) = \frac{1}{2}\alpha, \quad (9.73)$$

which is the same as Eq. (7.11b).

The magnitude of $\eta_{2T}(\psi_s)$ depends both upon R_2^* and the nature and magnitude of the lens power. Using the notation C, D, and * as superscripts for convergent, divergent, and zero power lenses, respectively, the following relations hold [as is evident from an examination of the areas in Fig. (9.15) or from the analytical expressions above]:

$$\eta_{2T}^C(\psi_s) > \eta_{2T}^*(\psi_s) > \eta_{2T}^D(\psi_s) \quad \text{for } 0 \leq R_2^* < |r_{0-}^{*+}| \quad (9.74a)$$

$$\eta_{2T}^C(\psi_s) = \eta_{2T}^*(\psi_s) = \frac{1}{2} \quad \text{for } R_2^* = |r_{0-}^{*+}| \quad (9.74b)$$

$$\eta_{2T}^C(\psi_s) = \eta_{2T}^*(\psi_s) > \eta_{2T}^D(\psi_s) \quad \text{for } |r_{0-}^{*+}| \leq R_2^* < |r_{0-}^{D+}| \quad (9.74c)$$

$$\eta_{2T}^C(\psi_s) = \eta_{2T}^*(\psi_s) = \eta_{2T}^D(\psi_s) \quad \text{for } R_2^* \leq |r_{0-}^{D+}|. \quad (9.74d)$$

It is evident from Eq. (9.74d) that, for very large sources, the transmission is independent of the power of the lens. For smaller sources, the convergent lens gives a higher transmission, and since (for other reasons) only small sources are of practical interest, the convergent lens is preferable when the goal is a maximizing of $\eta_{2T}(\psi_s)$.

It may be noted that for the convergent case, the part of the source with r_2 -values such that $|r_2| > |r_{0-}^0|$ provides a contribution to the transmission which is less than one-fourth the total.[†] For this reason, it may be a useful rule of thumb to use, as a practical limit for the source size,

$$\text{Maximum of } R_2^* \leq |r_{0-}^0| = \frac{1}{2} \varphi z_s / F_1. \quad (9.75)$$

An example may be useful: for a symmetric instrument with a typical K-value ($K = 0.6$), with $n_s = n_f = 0$, we have $F_1(\psi_s = 150^\circ) = 0.61$ and $F_1(100^\circ) = 1.26$. Taking an average F_1 as about one, then, in units of z_s ,

$$2R_2^* = \text{Maximum source width} \approx \varphi.$$

Commonly used values for some constructed instruments have been $\varphi = 0.17$ (10°) and $\varphi = 0.05$ (3°). Other considerations, such as resolution, sets a value $2R_2^*/z_s < 0.05$. Thus, the η_2 -factor is not important for instruments with gaps in the 10° -range, but may be significant for instruments with gaps as small as 3° .

It is evident from Fig. (9.14) and Eq. (9.52) that, if the detector opening $2R_{2f}$ is very large, then E_d^+ and E_d^- are not limiting and $\eta_{2T} = \eta_{23T}$. In this case, it is convenient to have a convergent lens, the optimum value of the lens power occurring when $R_2 = |r_{0-}^{C-}| = |\frac{1}{2}\varphi z_s N_s|$, i.e.,

$$N_s = -\frac{2R_2}{\varphi z_s} = -\frac{1}{\frac{1}{2}\varphi} \frac{R_2}{z_s}. \quad (9.76)$$

[†] This is evident from Fig. (9.15) if we consider $P = 0$. The maximum contribution for the part beyond r_{0-}^0 is less than one-third the contribution of the part with $|r_2| < |r_{0-}^0|$. The difference is even greater for $P > 0$.

When the source is circular, as is usual for a multigap instrument, the transmission is not given simply by the areas in Fig. (9.15). Suppose the source radius to be R . In Eq. (9.62), $|R_1(r_2)|$ is then equal to

$\sqrt{R^2 - r_2^2}$. Since more weight is thus given to the smaller r_2 -values, the convergent lens will always give better transmission, independent of the source size. Then,

$$\begin{aligned} \eta_{2T}(\psi_s) &= \frac{1}{\frac{1}{4}\pi R^2} \int_{-R}^0 \sqrt{R^2 - r_2^2} \eta_{2T}(r_2, \psi_s) dr_2 \\ &= \frac{4}{\pi R^2} \left[\int_{r_{2m}}^{r_2''} \sqrt{R^2 - r_2^2} \left[\frac{1}{2} + \frac{1}{W_s} \left(\frac{1}{2} + \frac{F_1}{\varphi_{zs}} r_2 \right) \right] dr_2 \right. \\ &\quad \left. + \frac{1}{W'} \int_{r_2''}^0 \sqrt{R^2 - r_2^2} dr_2 \right] \end{aligned} \quad (9.77a)$$

$$\begin{aligned} \eta_{2T}(\psi_s) &= \frac{4}{\pi R^2} \left\{ \frac{1}{4} \left[1 + \frac{1}{W_s} \right] \left[r_2'' \sqrt{R^2 - r_2''^2} - r_{2m} \sqrt{R^2 - r_{2m}^2} \right. \right. \\ &\quad \left. \left. + R^2 \left(\arcsin \frac{r_2''}{R} - \arcsin \frac{r_{2m}}{R} \right) \right] \right. \\ &\quad \left. - \frac{1}{3} \frac{1}{W_s} \frac{F_1}{\varphi_{zs}} \left[(R^2 - r_2''^2)^{\frac{3}{2}} - (R^2 - r_{2m}^2)^{\frac{3}{2}} \right] \right. \\ &\quad \left. - \frac{1}{2} \frac{1}{W'} \left[r_2'' \sqrt{R^2 - r_2''^2} + R^2 \arcsin \frac{r_2''}{R} \right] \right\}, \end{aligned} \quad (9.77b)$$

where, for a convergent lens, $(-r_{2m})$ is the smaller of $|r_{0-}^{C+}|$ and R , $r_2'' = r_{0-}^{C-} = \frac{1}{2}\varphi_{zs}N_s$, and $W' = 1$; for a divergent lens, $(-r_{2m})$ is the smaller of $|r_{0-}^{D+}|$ and R , $r_2'' = r_{0+}^{D+} = -\frac{1}{2}\varphi_{zs}N_s$, and $W' = W_s$. For $R < |r_{0-}^{C+}|$ (or $R < |r_{0-}^{D+}|$), Eq. (9.77b) is simplified by the fact that $R^2 - r_{2m}^2 = 0$ and $(-\arcsin \frac{r_{2m}}{R}) = \frac{1}{2}\pi$.

For $P=0$, we have $W_s = 1$ and $r_2'' = 0$; then Eq. (9.77b) is the same as Eq. (7.10a) for $r_{2m} = -R$ and the same as Eq. (7.10c) for $r_{2m} = r_{0-}^{*+}$.

Averaging η_{2T} over ψ_s . Using the usual solid-angle weighting factor, we have, as in Eq. (9.29),

$$\bar{\eta}_{2T} = \frac{\int \eta_{2T}(\psi_s) \sin \psi_s d\psi_s}{\int \sin \psi_s d\psi_s} \quad (9.78)$$

Numerical integration is required, even for a uniform source.

Computation of η_{23T} . When the detector width $2R_{2f}$ is not very large compared to the image, then for some ϕ_s - and r_2 -values, the detector boundary will cut off the rays. In such a case, E_M and E_m are the appropriate boundaries to use rather than E'_M and E'_m , which were used in the computation of $\eta_{2T}(\psi_s)$ in Eqs. (9.62) and (9.63).

We now consider the case in which R_{2f} is comparable to the r_2 -extension of the image. The value of R_{2f} which may be used depends upon the shape of the detector to be used. If a circular detector baffle is used, which is common to all the gaps, then the maximum of R_{2f} , i.e., Dr_2 , is the same as Dr_1 (see Section #7). Since the resolution is a function of Dr_1 , as discussed in Section #6, the detector diameter is limited by resolution considerations. In Section #6, it was found that the optimum value of Dr_1 was about equal to the radius R of the source. If Dr_2 is to be made larger than R , the alternatives are to lose in resolution or to look for solutions in which Dr_1 is not closely coupled† to Dr_2 .

Since $R_{2f} = R_{2f}(r_1)$, then, from Eq. (9.47), the value of η_{23T} , averaged over the source, is

$$\begin{aligned} \eta_{23T}(\psi_s) &= \frac{1}{\frac{1}{4}a_s} \int_{-R_2^*}^0 \int_0^{R_1(r_2)} \eta_{23T}(r_1, r_2, \psi_s) dr_1 dr_2 \\ &= \frac{1}{\frac{1}{4}a_s} \int_{-R_2^*}^0 \int_0^{R_1(r_2)} \left\{ E_M(r_1, r_2, \psi_s) - E_m(r_1, r_2, \psi_s) \right\} dr_1 dr_2, \end{aligned} \quad (9.79)$$

where the source boundary is $r_1 = R_1(r_2)$. An explicit evaluation of the integral is not feasible, because of the multiplicity of possible combinations of $E_0^\pm$ and $E_d^\pm$ entering into E_M and E_m . This is true even when R_{2f} is

† Two such solutions are discussed at the end of this section.

constant, as it would be in a rectangular detector baffle. Any particular case, however, specifies the lens power at entrance and exit apertures as a function of ψ_s ; under these circumstances, the integral is readily evaluated.

It may be noted at this point that the dependence of E_M and E_m on ψ_s enters not only through W_s and F_1 in E_0 or through g and h in E_d , but also through R_{2f} in E_d . Because of the finite thickness required in any aperture baffle used for energetic electrons, the effective shape of the baffle varies with ψ_s and with energy.

It is useful, however, to illustrate some of the features affecting η_{3T} , and, for this purpose, we shall consider only the E_d and the E_e lines. Neglect of the E_0 lines is equivalent to reducing the available ϕ_s -range, as, for example, by using a baffle at the entrance. The reduction is great enough so that no ray passing this baffle will hit the polefaces. Under these circumstances, the E_0 -lines are never limiting. This is an unrealistic case, but useful for the purpose at hand. The transmission is then called η_{3T}^0 .

In computing this quantity, it is desirable to consider the various possibilities in the (E_d^+, E_d^-) , analogous to the considerations developed in Fig. (9.15) and Table (9.2). The intersection points of the (E_d^+, E_d^-) lines with $(E_e^+ = \frac{1}{2}, E_e^- = -\frac{1}{2}, E = 0)$ are computed from Eqs. (9.43c) and (9.45) for both $h > 0$ and $h < 0$; the results are given in Table (9.3) and illustrated in Fig. (9.16). Neither Eqs. (9.43c) nor (9.45) can be used when $h = 0$. Going back to Eq. (9.37), we have

$$-R_{2f} \leq g \leq R_{2f}. \quad (9.80)$$

Because of our assumption that the exit aperture is not limiting, the ϕ_s -range is not cut off; the r_2 cutoff is then independent of E . The slope of (E_d^+, E_d^-) lines is thus infinite [as is evident from Eq. (9.51a)], and the choice of which line to call E_d^+ is arbitrary. We choose to label the positive r_2 -limit as E_d^+ ; this corresponds to the $(-R_{2f})$ limit in Eq. (9.80), since $g < 0$ for the exit lens not too strong, as has been our assumption. It may be noted that the coincidence of the intersection points of $E_d^+(h < 0)$, $E_d^+(h = 0)$, $E_d^-(h > 0)$ and $E = 0$ is true only when the value of g is chosen to be the same for the three cases. In general, when the g 's are not so chosen, there are twelve intersection points rather than the two illustrated.

Figure (9.16) illustrates* the results for $h > 0$, for $h = 0$, and for $h < 0$ under conditions in which $\eta_{3T}^0 = 1$, at least for part of the r_2 -range.

*The areas shown as belonging to $h > 0$ or $h < 0$ (or both) apply only to the cases illustrated. The areas included for the general case depend upon the slope of the E_d^- and E_d^+ lines which can vary considerably on both sides of the vertical at $h = 0$.

Fig. 9.16

This is true when $2R_{2f} > \varphi z_s |h|$, i.e., when neither exit nor entrance lens is very strong. Also shown in Fig. (9.16) is the case when $h \gg 0$, i.e., when both entrance and exit lenses are fairly strongly convergent (but not enough to invert the sign).[†] In this case, there is no r_2 -region for which $\eta_{3T}^0 = 1$. The dividing line is readily determined from Eq. (9.52); we have $\eta_{3T}^0 = 1$ only for the value $r_2 = 0$ when the vertical separation is unity, i.e.,

$$|h| = 2R_{2f}/\varphi z_s. \quad (9.81a)$$

When the vertical separation is less than unity, then $\eta_{3T}^0 < 1$ for all r_2 . Then

$$|h| > 2R_{2f}/\varphi z_s. \quad (9.81b)$$

When the vertical separation is greater than unity, $\eta_{3T}^0 = 1$ for some r_2 ; then

$$|h| < 2R_{2f}/\varphi z_s. \quad (9.81c)$$

An examination of Fig. (9.16) and Table (9.3) shows that this corresponds to

$$r_{d-}^{+-} > 0 \text{ or } r_{d-}^{--} < 0. \quad (9.82)$$

From Fig. (9.16), it is evident that, averaged over the sample, $\eta_{3T}^0(\psi_s) = 1$ if

$$R_2^* \leq |r_{d+}^{++}| \text{ or } R_2^* \leq |r_{d-}^{--}|, \quad (9.83a)$$

[†] The lines, as drawn, are for the same g -value used for the other types of E_d -lines.

Table 9 3

 r_2 -VALUES AT INTERSECTION OF E_d -CURVES WITH E_e -CURVES AND WITH $E = 0$

Intersection of	with	r_2 -name [†]	r_2 -value	r_2 -magnitude ^{††}	r_2 -sign ^{†††}
$E_d^+(h > 0)$	$E_e^+ = \frac{1}{2}$	r_{d+}^{++}	$\frac{1}{2g}(-\varphi z_s h + 2R_{2f})$	B*	-
$E_d^+(h = 0)$	E_e^+	r_{d+}^{0+}	$-R_{2f}/g$		+
$E_d^+(h < 0)$	E_e^+	r_{d+}^{-+}	$\frac{1}{2g}(-\varphi z_s h - 2R_{2f})$	B*	+
$E_d^+(h > 0)$	$E_e^- = -\frac{1}{2}$	r_{d+}^{+-}	$\frac{1}{2g}(\varphi z_s h + 2R_{2f})$	A*	-
$E_d^+(h = 0)$	E_e^-	r_{d+}^{0-}	$-R_{2f}/g$		+
$E_d^+(h < 0)$	E_e^-	r_{d+}^{--}	$\frac{1}{2g}(\varphi z_s h - 2R_{2f})$	A*	+
$E_d^+(h > 0)$	$E = 0$	r_{d+}^{+0}	R_{2f}/g		-
$E_d^+(h = 0)$	$E = 0$	r_{d+}^{00}	$-R_{2f}/g$		+
$E_d^+(h < 0)$	$E = 0$	r_{d+}^{-0}	$-R_{2f}/g$		+
$E_d^-(h > 0)$	$E_e^+ = \frac{1}{2}$	r_{d-}^{++}	$\frac{1}{2g}(-\varphi z_s h - 2R_{2f})$	A*	+
$E_d^-(h = 0)$	E_e^+	r_{d-}^{0+}	R_{2f}/g		-
$E_d^-(h < 0)$	E_e^+	r_{d-}^{-+}	$\frac{1}{2g}(-\varphi z_s h + 2R_{2f})$	A*	-
$E_d^-(h > 0)$	$E_e^- = -\frac{1}{2}$	r_{d-}^{+-}	$\frac{1}{2g}(\varphi z_s h - 2R_{2f})$	B*	+
$E_d^-(h = 0)$	E_e^-	r_{d-}^{0-}	R_{2f}/g		-
$E_d^-(h < 0)$	E_e^-	r_{d-}^{--}	$\frac{1}{2g}(\varphi z_s h + 2R_{2f})$	B*	-
$E_d^-(h > 0)$	$E = 0$	r_{d-}^{+0}	$-R_{2f}/g$		+
$E_d^-(h = 0)$	$E = 0$	r_{d-}^{00}	R_{2f}/g		-
$E_d^-(h < 0)$	$E = 0$	r_{d-}^{-0}	R_{2f}/g		-

† The notation r_{ij}^{km} refers to i aperture intersection. In this case, it is always d , because only the detector aperture is considered. j refers to sign of detector aperture involved, i.e. E_d^+ or E_d^- . k sign or value of h , i.e. $(-, 0, +)$. m refers to sign of entrance curve involved [i.e. $E_e^+ = \frac{1}{2}$ or $E_e^- = -\frac{1}{2}$] or to 0 for $E = 0$.

†† $A^* = \frac{1}{2|g|}(2R_{2f} + \varphi z_s |h|)$, $B^* = \frac{1}{2|g|}(2R_{2f} - \varphi z_s |h|)$. The magnitudes of all the intersection points (other than those having R_{2f}/g) are either A* or B*, provided the g - and h -magnitudes are chosen to be the same. The magnitude resulting depends upon whether the $2R_{2f}$ -term has the same or opposite sign as the $\varphi z_s h$ -term.

††† Only the case is considered in which $2R_{2f} > |h| \varphi z_s$, i.e., neither entrance nor exit lenses are very strong. Since g is negative for the usual spectrometer considered, (i.e., where $|v(\psi_s)|$ is not too different from unity and the exit lens is not strongly convergent), the sign of an A*- or B*-term is opposite to that of the $2R_{2f}$ -term in the fourth column.

i.e.,

$$R_2^* \leq \frac{1}{2|g|} \left[-\varphi_{zs} |h| + 2R_{2f} \right] \quad (9.83b)$$

or

$$R_{2f} - R_2^* \geq \frac{1}{2}\varphi_{zs} |h| + R_2^* \left[|g| - 1 \right] . \quad (9.83c)$$

For a symmetrical spectrometer, this corresponds to

$$R_2^* \leq \frac{1}{2|W_s|} \left[-\varphi_{zs} |N_s| (1 + W_s) + 2R_{2f} \right] \quad (9.84a)$$

or

$$R_{2f} - R_2^* \geq \frac{1}{2}\varphi_{zs} |N_s| (1 + W_s) + R_2^* N_s F_1 \quad (9.84b)$$

which may also be written as

$$R_{2f} - R_2^* \geq \frac{1}{2}\varphi_{zs} |N_s| \left[(1 + W_s) \pm \frac{R_2^*}{\frac{\frac{1}{2}\varphi_{zs}}{F_1}} \right] \left\{ \begin{array}{l} + \text{ for } P < 0 \\ - \text{ for } P > 0 \end{array} \right\} . \quad (9.84c)$$

If Eq. (9.83) is not satisfied, then $\eta_{3T}^0(\psi_s) < 1$. Also η_{3T}^0 decreases rapidly with $|h|$, and for large values of $|h|$, i.e., when Eq. (9.81b) holds,

$$\eta_{3T}^0(\psi_s) = E_d^+ - E_d^- = |2k| = 2R_{2f}/\varphi_{zs} |h| , \quad (9.85)$$

from Eq. (9.52).

It is evident that a decrease in transmission due to the detector aperture can be prevented by:

(i) use of large R_{2f} -values. In Fig. (9.16), this corresponds to increasing the vertical separation $|2k|$ of the lines E_d^+ and E_d^- , as in Eq. (9.52), without changing their slope. As mentioned above, considerations of resolution argue against having R_{2f} much greater than R_2^* , so that it is desirable to keep $(R_{2f} - R_2^*)$ small,

(ii) small φ -values. Limiting the ϕ_s -range decreases the spread of the image caused by the fringing field, but, of course, cuts the η_{2T} -transmission,

(iii) small $|h|$ -values. The quantity h affects only the slope of the E_d -lines, but leaves the intersection $\pm R_{2f}/g$ with $E = 0$ unchanged (provided g remains fixed). Since the influence of h is quite large, we try, in general, to have boundary values which give the smallest values of $|h|$;

(iv) for a symmetric spectrometer (or one which is fairly close to symmetry), Eq. (9.84b) shows that a convergent lens is preferable, for systems of equal lens power (i.e., $|N_s|$), since in this case $(1 + W_s)$ is smaller and the second term is subtracted from the first, thus leading to a smaller value of $R_{2f} - R_2^*$. It was noted earlier that convergent lenses also gave better results insofar as the η_2 -transmission was concerned;

(v) from the condition in Eq. (9.75) derived from η_2 -considerations, we have the result that usually

$$\frac{R_2^*}{\frac{\frac{1}{2}\phi z_s}{F_1}} \ll 1 \quad , \quad (9.86a)$$

so that, as a rough approximation, this term may be neglected in Eq. (9.84c), and we may write as an approximate condition for $\eta_{3T}^0 = 1$ that

$$R_{2f} - R_2^* > \frac{1}{2}\phi z_s |h| ; \quad (9.86b)$$

and

(vi) joining this condition with the η_{2T} -condition, we have

$$\text{Maximum of } \frac{R_2^* F_1}{z_s} < \frac{1}{2}\phi < \frac{R_{2f} - R_2^*}{z_s |h|} . \quad (9.87)$$

This relationship should hold over the main part of the ψ_s -range accepted, if the transmission is to be reasonably high.

The choice between symmetric and asymmetric instruments on transmission considerations alone is not readily determined from the above equations, since N_s and N_f are not simply related. Only a detailed computation of $\eta_{23T}(\psi_s)$ can give a definite answer, and this requires examination of particular cases.

Averaging $\eta_{23T}(\psi_s)$ over all ψ_s is carried out as in Eq. (9.78).

Thus,

$$\bar{\eta}_{23}T = \frac{\int \eta_{23}T(\psi_s) \sin\psi_s d\psi_s}{\int \sin\psi_s d\psi_s},$$

with $\eta_{23}T(\psi_s)$ computed from Eq. (9.79).

Effect of Fringing Field in Single-sector Instrument. Some of the detector problems arising in the multigap instrument do not occur when a single sector is used. For the multigap instrument, the detector slit discussed above is placed at $(z_f, 0, 0)$ and is made circular, in order to receive rays from all the gaps with equal efficiency. Since the slit has extension in both the r_{2f} and r_{1f} directions for each gap, the diameter of the opening is limited because of resolution requirements (see Section #6), and if the lens effect of the fringing field is large, the losses due to spreading of the beam may be large, i.e., η_{23} may be considerably smaller than η_2 .

For the single-sector instrument, the detector slit need not be circular. It need have very little r_{1f} -extension and can have its primary extension in the r_{2f} -direction. Its optimum shape would be approximately that of a thin rectangular slot placed at $(z_f, 0, 0)$ with the long dimension running in the r_{2f} -direction,* and with the short dimension lying along the z -axis. Since this slit could be made as long as desired, it is not necessary that η_{23} ever be smaller than η_2 . Thus, the only losses from the fringing field would be due to the effect of divergent input lenses, which would drive particles into the polefaces.

An examination of the distribution of rays coming into the r_{2f} -direction indicates that the optimum slit would have a shape slightly different from a rectangular slot. The conical shape of the family of rays of constants ψ_s was described above in connection with the discussion of Fig. (9.4), where it was pointed out that the family strikes the z - r_{2f} plane in an arc rather than in a line. Thus, an optimum slit would be narrowest at the z -axis and would fan out at the ends to an extent determined by the ψ_s -range covered. It is to be noted, of course, that this curvature is marked only when the gap angle is large, since the extent of the curvature is determined by the curvature of the cone, which in turn is controlled by the magnitude of the gap angle.

Multigap, Tilted-axis, Iron-core Instrument. A modification of the multigap instrument makes possible the utilization of the virtue of minimum r_{1f} -extension in the detector of the single-gap instrument. In the multigap instrument considered thus far, the z -axes of the various gaps are coincident, so that a detector placed at the point $(z_f, 0, 0)$ for one gap must serve

* This would be modified somewhat if the source were large; see Section #5.

in the same way for all the other gaps. If, however, the z -axes of the various gaps are tilted, so that they lie on the surface of a cone with vertex at $(z_s, 0, 0)$, then the points $(z_f, 0, 0)$ for the various gaps are not coincident. This makes it possible to use individual detector slits for each gap, each slit having little or no r_{1f} -extension, and as large an r_{2f} -extension as is allowed by the degree of tilting used. The various independent slits may open into the same, or different, detectors, depending upon the space made available by the tilting of the axes, thus offering another degree of freedom. Since it is still necessary to use the toroidal structure, such tilting is possible only with an iron-core instrument. Iron-free toroidal instruments must be built with coincident z -axes.

Use of Ring Detector Slit in Multigap Instrument. In our discussion of resolution and transmission in Section #6 and in the earlier part of this section, we have assumed that the detector slit is an open circle, with center on the z -axis at $(z_f, 0, 0)$. We now consider the possible virtues of displacing the detector slit for each gap to a position off the z -axis.

If it were possible to form a ψ -focus for finite sources at (z_f, r_f) instead of $(z_f, 0)$, then it would be possible to use a detector slit for each gap which has its major extension in the r_2 -direction and minimum extension in the r_1 -direction. Each gap might have its own slit instead of all having one common slit, so that the high-transmission requirements of one gap for large r_2 -extension do not automatically involve a large r_1 -extension for a gap 90° around the toroid. The large extension of the slit in the r_2 -direction would result in η_{23} closely approaching η_2 ; a corresponding increase in transmission results without a worsening of the resolution, which varies primarily with the r_1 -extension of the slit.

The most convenient form of extended slit for the ensemble of gaps is made using a ring baffle, as in Fig. (9.17). We have here a circular ring of mean radius r_f , with the detector located immediately behind it. If the focusing is as shown in Fig. (9.18), then the detector would have dimensions smaller than the inner circle of the ring.

The slits for the various gaps are not necessarily completely independent, unless r_f is fairly large. Thus, with an iron-free instrument, the images from the various gaps would overlap, because the gaps are close together. In the case of an iron-core instrument, in which the polepieces occupy an appreciable angle, the situation may be more favorable, because each gap may have its own slit shaped to the image-shape, and extending to the plane ϕ_P passing through the middle of the iron section, as in Fig. (9.19).

Fig 9 7

Fig. 9.18

Fig. 9.19

The usefulness of this approach depends upon the possibility of forming a ψ -focus off the z -axis for finite sources. One possibility is that profiles designed for focusing on the z -axis may be utilized for focusing off the z -axis by a change in the specific field intensity A_0 [see Eq. (1.1)] used for any particular momentum p . A general analysis of this possibility has not been carried out, but a numerical investigation was made for a symmetrical system with $K = 0.59$ [see Fig. (3.3) and Table II.3 of Section #3]. This particular profile design is close to an optimum value for certain kinds of symmetrical instruments, and has been constructed in the form of an air-core instrument.(16,17)

In Fig. (9.20) are shown the profile and the rays emerging at the two extreme angles of the ψ_s -range considered, as well as the median angle ($\psi_s = 110^\circ, 130^\circ, 150^\circ$). Both the z - and r -coordinates are given in units of z_s . Trajectories were computed for these entrance angles at various values of A_0 , denoted by the corresponding values of K . [From Eq. (1.10), variation of K at fixed p corresponds to a variation of A_0 .] The variations of K considered around $K = 0.59$ are $\Delta K = \pm 0.01, \pm 0.02, \pm 0.03$.

Figure (9.21) shows the rays as they come together near the z -axis. It is evident that there is no true focal point off the z -axis, and that the width of the "image" for a point source, at its narrowest region, increases as $|\Delta K|$ increases. The Δz - and Δr -range of the "image" in this narrowest region is plotted against K in Fig. (9.22), and over this K -range the relation is close to linear.*

* The sign of Δz and Δr is arbitrarily taken by subtracting the z -value (or r -value) for the $\psi_s = 130^\circ$ ray from the corresponding (z, r) of the $\psi_s = 150^\circ$ ray. This convention gives opposite signs for Δz and Δr corresponding to the K -values on both sides of $K = 0.59$.

Fig. 9.22

At the top of Fig (9.22) is plotted the mean z -value within this narrowest region against K ; another curve shows the mean r -value. Figure (9.23)

Fig 9 23

shows the mean r -value plotted against the mean z -value. For a multigap toroidal system, this curve, rotated around the z -axis, constitutes the focusing surface for a multi-energy source. For use of the instrument as a spectrograph, the optimum focusing would occur if the detector were distributed over this "cone." It is evident that the spectrographic property with this particular profile design is not a good one, due to the rapid deterioration of the focusing property as the momentum deviates from that which focuses at $(z_f, 0)$. A more useful design for spectrographic use

would probably involve profiles designed for higher K -values [see Fig. (3.3)]; with these, ψ_s -angles close to 90° are useable, and for these angles the spread in the image region would be smaller. Of course, with the higher K -values, one must accept the reduced transmission due to the larger fringing-field effects.

Because the use of a ring slit has the advantage of making η_{23} almost equal to η_2 , a better focusing arrangement off the z -axis is desirable. It should be possible to use the methods of Section #3 to compute an exit profile (for a given entrance profile) which would give ψ -focusing for an arbitrary focal point (z_f, r_f) . If (z_f, r_f) is not too far from $(-z_s, 0)$, then the imaging property of symmetrical systems may not be perturbed too much (see Section #5), and it may be possible to get suitable approximate ψ -focusing for spread sources. The ϕ -focusing, even for a point source and in the absence of fringing fields, is also perturbed by moving the image point off the z -axis, but this spread is small, since both r_f and ϕ are small. No general theory for this mode of focusing has been developed.

It should be pointed out that, with an iron-core instrument, the advantage of increasing η_{23} through extending the slit (for each gap) in the r_2 -direction can more readily be attained by tilting the z -axis as discussed in the previous portion of this section. This approach has the advantage of preserving unperturbed ϕ -focusing. A ring focus, however, if achieved, would be useful for air-core instruments, since it would make it possible to achieve a considerably improved transmission (through increasing η_{23}) at high resolutions.

Transmission for ψ_s -values in the Neighborhood of the Critical ψ_s -value. In Section #3 it was pointed out that the profile for a symmetric instrument becomes infinite for a critical ψ_s -value. Values of ψ_s in the neighborhood of and larger than the critical value cannot be used because

$\left| \frac{d\rho_s}{d\psi_s} \right|$ for the profile is very large, leading to large values of $|\tan\mu_s|$ and $|N_s|$. Imaging within the gap occurs and hence low values of η_{23} arise [see also Figs. (3.4) and (9.3)].

#10. EFFECT OF THE $\Delta\psi$ -DEFLECTION ON THE LENS PROPERTY

We have computed the $\Delta\beta$ -deflection of the trajectory in Eq. (8.12):

$$\Delta\beta = \frac{e}{p} \int H_t \tan\mu \, ds, \quad (10.1)$$

the integral being taken along the trajectory from the field-free region outside the gap to a point (z_e, r_e) on the trajectory where the magnetic field is described by Eq. (1.1) as $H_e = A_0/r_e$. From Eq. (1.10) and (9.4),

$$\frac{e}{p} = -\frac{1}{KA_0} = -\frac{1}{K H_e r_e} = -\frac{1}{K H_e (\Delta x / \Delta\phi)} \quad (10.2)$$

Then

$$\Delta\beta = -\frac{\int H_t \tan\mu \, ds}{K H_e (\Delta x / \Delta\phi)} \quad (10.3)$$

From Eq. (9.2) and (9.3), at the entrance,

$$\int H_t \, ds = H_e \Delta x = H_e r_e \Delta\phi = H_e r_e (\phi_s - \phi_M), \quad (10.4)$$

where ϕ_s is the ϕ -plane containing (z_e, r_e) and ϕ_M is the median plane of the gap. Then

$$\Delta\beta = -\frac{\Delta\phi}{K} \frac{\int H_t \tan\mu \, ds}{\int H_t \, ds} \quad (10.5)$$

If μ is constant through the fringing field, and equal to the initial value μ_s , then

$$\Delta\beta = -\frac{\Delta\phi}{K} \tan\mu_s = -\frac{\Delta x}{K r_e} \tan\mu_s, \quad (10.6)$$

which is the result in Eq. (9.5).

This equation was used in Section #9 for computing transmission, despite the fact that it was known that μ was not constant through the fringing field, because of the simultaneous ψ -deflection. We now propose to compute the necessary correction. If $\Delta\psi_e$ is the change in ψ through the fringing field and μ_e is the value of μ at (z_e, r_e) , then

$$\mu_e = \mu_s + \Delta\psi_e, \quad (10.7)$$

in which $\Delta\psi_e$ is always positive.

We define

$$\Delta\beta_1 = \frac{e}{p} \tan\mu_s \int H_t ds = -\frac{\Delta\phi}{K} \tan\mu_s \quad (10.8a)$$

and

$$\Delta\beta_2 = \frac{e}{p} \tan\mu_e \int H_t ds = -\frac{\Delta\phi}{K} \tan\mu_e \quad . \quad (10.8b)$$

At any point on the trajectory at which the μ -angle has the value μ ,

$$\mu_s < \mu < \mu_e; \tan\mu_s < \tan\mu < \tan\mu_e \quad . \quad (10.9)$$

Hence, the true $\Delta\beta$ lies between $\Delta\beta_1$ and $\Delta\beta_2$, and may be written as

$$\Delta\beta = \frac{e}{p} \tan[\mu_s(\text{eff})] \int H_t ds = -\frac{\Delta\phi}{K} \tan[\mu_s(\text{eff})] \quad , \quad (10.10)$$

where

$$\mu_s(\text{eff}) = \mu_s + \theta\Delta\psi_e \text{ and } 0 < \theta < 1 \quad (10.11)$$

and

$$\tan[\mu_s(\text{eff})] = \frac{\int H_t \tan\mu ds}{\int H_t ds} \quad . \quad (10.12)$$

For small angles and deflections, the relation can be simplified. Let d be the distance along the trajectory from the beginning of the integration. If ψ_d is the ψ -value and μ_d the μ -value when the particle is at d , then

$$\Delta\psi_d = \psi_d - \psi_s = \mu_d - \mu_s \quad (10.13)$$

and

$$\mu_d = \mu_s + \Delta\psi_d \quad . \quad (10.14)$$

Taking $\tan\mu_s$ as being of moderate size and $\tan\Delta\psi_d$ as small, we have

$$\begin{aligned} \tan\mu = \tan\mu_d = \tan(\mu_s + \Delta\psi_d) &= \frac{\tan\mu_s + \tan\Delta\psi_d}{1 - \tan\mu_s \tan\Delta\psi_d} \\ &\cong \tan\mu_s + \Delta\psi_d \quad . \end{aligned} \quad (10.15)$$

Then, from Eq. (10.11) and (10.12),

$$\begin{aligned} \tan[\mu_s(\text{eff})] &= \tan(\mu_s + \theta\Delta\psi_e) \cong \tan\mu_s + \theta\Delta\psi_e \\ &\cong \tan\mu_s + \frac{\int \Delta\psi_d H_t ds}{\int H_t ds} \quad , \end{aligned} \quad (10.16)$$

so that

$$\theta \approx \frac{\int \frac{\Delta \psi_d}{\Delta \psi_e} H_t ds}{\int H_t ds} \quad \text{and } 0 < \theta < 1 \quad . \quad (10.17)$$

Since $\Delta \psi_e$ is always positive and $\tan \mu$ is negative for a convergent lens, this effect weakens a convergent lens and strengthens a divergent lens.

It is not feasible to compute θ accurately. Its determination involves a knowledge of H_ϕ and H_t along the trajectory through the fringing field. Since these depend in detail upon the shape of the pole pieces, location of coils, etc., it is not feasible to provide a general equation from which θ may be readily computed. However, it is feasible to compute an approximate* value for θ , because the result is only moderately dependent upon the detailed structure of the fringing field.

Estimates of θ for an r^{-1} -field. We shall use two approaches for making approximate estimates of θ . One involves the linear approximation to the field, discussed in Section #8, while the other involves numerical integration of trajectories, using a calculable magnet model. In the first approach, added assumptions are necessary. We make the simplest possible ones, some of which are demonstrably inaccurate, but which serve as crude approximations. The modest agreement between this approach and the second one indicate that the result is only moderately sensitive to these approximations.

The assumptions are:

(i) The magnitude of the fringing field H [see Fig. (8.1)] at any ϕ -plane varies in the same manner as the value on the median plane, i e., as described in Eq. (8.6). The linear variation has been shown to be quite a good approximation on the median plane, but has not been so shown for other ϕ -planes. Even if the linear variation were a good approximation, the value of d_e varies with the ϕ -plane, being somewhat smaller for ϕ -planes close to the polefaces. However, we shall take d_e as independent of ϕ .

(ii) For points well within the polefaces, $H = H_\phi$ and $H_{RZ} = H_t = 0$. As d decreases from $d = d_e$, the increase in the curvature of the fringing field leads to an increase in the ratio H_t/H . We take this increase to be

*Even though the field values ascribed to particular points are inaccurate, the integration tends to average out the deviations. This is analogous to the situation relative to the $\Delta \psi$ -deflection. In Section #11, it is shown that the $\Delta \psi$ -deflections on various ϕ -planes are almost identical, despite the rather considerable differences in the H_ϕ -values seen by the various trajectories.

approximately linear and proportional to $(d_e - d)$, and normalized as $(d_e - d)/d_e$. On a particular ϕ -plane, then,

$$H_t \propto H \left[\frac{H_t}{H} \right]_{\phi = \text{constant}} = H_e \frac{d_e - d}{d_e} \frac{d}{d_e} \quad (10.18)$$

In this relation, $H_t = 0$ at $d = 0$, rising to a maximum at $d = \frac{1}{2}d_e$ and falling to zero at $d = d_e$.

(iii) At a given (z, r) , but variable ϕ , the field H increases toward the poleface, because of the crowding together of the lines of force. In addition, the change in inclination of $\underline{H}$ towards the poleface also tends to increase H_t . We take the variation to be approximately linear in Δx . This is not only a mathematical convenience, but also gives the appropriate limiting form if $\tan \mu$ is taken as constant.

(iv) We assume that Δx remains constant as the trajectory passes through the fringing field. This is not actually true, since even when the trajectory lies in a ϕ -plane, i.e., $(\phi - \phi_M)$ remains constant, the r -value of the trajectory changes through the fringing field. In addition, since $\Delta \beta$ involves a continuous bending of the trajectory in the ϕ -direction, then $(\phi - \phi_M)$ does in fact change through the fringing field. This effect is less important than the other approximations.

Thus H_t is the product of three factors, the effects being taken as linear and separable:

$$H_t = D_0 H_e \frac{d_e - d}{d_e} \frac{d}{d_e} \Delta x, \quad (10.19)$$

where D_0 is a constant to be evaluated.

Inserting Eq. (10.19) into (10.3), and taking $ds = d(d)$,

$$\Delta \beta = -\frac{D_0}{d_e^2} \frac{\Delta \phi}{K} \int_0^{d_e} (d_e - d) d \tan \mu d(d) \quad (10.20)$$

As a limiting case, we consider the computation as in Section #9, i.e., with $\mu = \text{constant}$ through the fringing field; then

$$\Delta \beta = -D_0 \left[\frac{\Delta \phi}{K} \tan \mu_s \right] \frac{d_e}{6} \quad (10.21)$$

Comparing this result with Eq. (9.5c), we have

$$D_0 = 6/d_e \quad (10.22)$$

Evaluating $\Delta\psi_d$ as in Eq. (8.7a), we have

$$\Delta\psi_d = \frac{1}{K r_e d_e} \int_0^d d \, d(d) = \frac{d^2}{2 K r_e d_e} \quad (10.23)$$

Substituting (10.14), (10.15), (10.22), (10.23) and (8.7a) into (10.20), we have

$$\Delta\beta = -\frac{\Delta\phi}{K} [\tan\mu_s + 0.3 \Delta\psi_e] \cong -\frac{\Delta\phi}{K} \tan[\mu_s + 0.3 \Delta\psi_e] \quad (10.24)$$

This is evidently the same as Eq. (9.5c), with the substitution of $\tan[\mu_s(\text{eff})]$ and with $\theta = 0.3$.

As an example of the effect of a variation in the assumptions, if we take the field increase in (ii) above to be zero [i.e., substitute a constant for $(d_e - d)/d_e$ in Eq. (10.18)], then $\theta = 0.5$. Although this assumption is evidently qualitatively deviant from the true situation, since it neglects the increasing curvature of lines of force with distance from the magnet, its use tends to set a limit on the value θ . It seems likely, therefore, that a suitable value of θ is probably around 0.3 and is certainly < 0.5 .

A limit of approximately 0.5 can be evaluated from qualitative considerations, without an evaluation of the details of the fringing field. Except for the region close to the poleface edge, $H\phi$ is not much smaller than the magnitude H ; thus, $H\phi$ varies monotonically with H . If H_{rz} varied in the same way, then $H\phi ds \propto H_{rz} ds$, approximately. Since $d\psi = d\mu \propto H\phi ds$ and $d\beta \propto \tan\mu H_t ds = \sin\mu H_{rz} ds$, then, with $\mu \cong \sin\mu \cong \tan\mu$, we find $d\beta \propto \mu d\mu$. Hence,

$$\begin{aligned} \Delta\beta &\propto \frac{1}{2}(\mu_e^2 - \mu_s^2) = \frac{1}{2}(\mu_e + \mu_s)\Delta\mu \propto (\mu_s + \frac{1}{2}\Delta\psi_e) \int H\phi ds \\ &\propto (\mu_s + \frac{1}{2}\Delta\psi_e) \int H_t ds \propto (\mu_s + \frac{1}{2}\Delta\psi_e) H_e \Delta x \quad , \end{aligned} \quad (10.25)$$

since $\mu_e - \mu_s = \Delta\mu = \Delta\psi_e$. Thus, $\theta = 0.5$.

However, H_{rz} does not, in fact, increase with H along the trajectory as $H\phi$ does, so that this result tends to overemphasize the effective change in μ , insofar as the $\Delta\beta$ -effect is concerned. This arises from the fact that the value of μ is important only where H_{rz} is appreciable. Since $H\phi$ increases steadily along the trajectory (except near the poleface, and even here the perturbation is moderate), $d\mu$ increases in the same manner into the interior field. However, H_{rz} has a maximum outside of the profile, and rapidly decreases to zero inside the profile. Thus, instead of having $d\beta$ growing with $d\mu$, as was assumed in the calculation, $d\beta$ is significant over only part of the integration range for $\Delta\mu$, so that the effective value of μ in $\Delta\beta$ is smaller than the mean, $\frac{1}{2}(\mu_e + \mu_s)$, and $\theta < 0.5$.

Computation of θ for Two-dimensional Uniform Fields. We now come to the alternative method for evaluation of θ . If the field values were known in the fringing region, then numerical computation of the trajectories would provide $\Delta\beta$ and $\Delta\psi_e$ values, from which θ could be evaluated through Eq. (10.10) and (10.11). Such detailed information of the fringing field is not known for any of the toroidal ("orange") type r^{-1} spectrometers, nor even for a single sector of this design. However, we may choose a magnetic model which can be computed analytically, and which may not differ too much from the spectrometer of interest. If the fringing field of the model is similar in structure, although differs in magnitude from the spectrometer, we can expect that the effect of changes in the magnitude of $\Delta\beta$ and $\Delta\psi_e$ will tend to cancel each other in the computation of θ . To the extent that the cancellation is incomplete, the estimate of θ will be in error.

The model chosen is one that we shall use again in Section #11, with a similar approximation in mind. The model is taken as one with a uniform interior field, and with two-dimensional structure, i.e., the extent of the poleface along the gap is very large in comparison to the essential extent of the fringing field away from the poleface. It is well known that the fringing field of this type of structure may be computed through the use of the conformal transformation technique. Numerical values of the fringing field have been determined through the use of a 704 electronic computer,⁽²³⁾ and applied to the computation of trajectories.⁽²²⁾

Two magnetic models are considered: Case I. Illustrated in Fig. (11.1), the poles extend indefinitely into and out of the paper, as well as vertically and to the left; Case II. The poles extend indefinitely into and out of the paper and to the left, but consist of infinitely thin sheets with no vertical extension. For trajectories which start out on planes parallel to the median plane, Eq. (10.1) applies with good approximation.* Such planes are the analogs of the constant ϕ -planes in the r^{-1} field. The quantity $\Delta\beta$ was computed numerically through an equation analogous to Eq. (10.1) and $\Delta\psi_e$ from one analogous to Eq. (8.5).

Both $\Delta\beta$ and $\Delta\psi_e = \Delta\psi_e$ were computed by integrating from a starting point 7.5 gap widths out from the magnetic profile. For a real magnet, this is far enough out so that the magnetic field at this point may be considered negligible. The two-dimensional structure of the model, however, makes its fringing field die off more slowly than that of a real magnet, and this requires some consideration in using the line integral analogous to Eq. (9.2a).

As in Eq. (10.10), we may write

$$\Delta\beta = \frac{e}{p} \tan [\mu_s(\text{eff})] \int H_t ds \quad . \quad (10.26)$$

*Rigorously, the integrand should include the factor $\cos\beta$, where β is the angle of the trajectory to the initial plane. Numerical computations show that β is always very small, so the $\cos\beta \cong 1$.

In evaluating $\int H_t ds$ through the line integral, we must allow for the field intensity at the beginning of the integration. Thus,

$$\int_{y_m}^{y_i} H_y dy + \int H_t ds + \int_{y_f}^{y_m} H_y dy = 0 \quad , \quad (10.27)$$

where y_m denotes the y-coordinate of the median plane, y_i the y-coordinate of the initial plane of the trajectory, and y_f the y-value of the trajectory at the end point of the $\Delta\beta$ -integration, taken to be a suitable distance into the gap. The first integral is evaluated at a constant distance, 7.5 gap widths, out from the magnet face. The third integral is evaluated at a constant distance inside the gap; the choice of the distance is discussed later. We write

$$\int_{y_m}^{y_i} H_y dy = -\gamma_i H_e \Delta y_i; \quad \Delta y_i = |y_m - y_i| \quad (10.28a)$$

$$\int_{y_f}^{y_m} H_y dy = \gamma_f H_e \Delta y_f; \quad \Delta y_f = |y_m - y_f| \quad , \quad (10.28b)$$

where H_e is the magnitude of field far into the interior. In effect, Eq. (10.28) defines γ_i and γ_f . Here Δy_i and Δy_f are the distances of the trajectories from the median plane at the beginning and end, respectively, of the trajectory integration. In practice, $\Delta y_i \approx \Delta y_f$. Then

$$\int H_t ds = - H_e [\gamma_f \Delta y_f - \gamma_i \Delta y_i] \quad (10.29)$$

and

$$\begin{aligned} \Delta\beta &= -\frac{e H_e}{p} \tan [\mu_s (\text{eff})] [\gamma_f \Delta y_f - \gamma_i \Delta y_i] \\ &= -\frac{1}{\rho_C} \tan [\mu_s (\text{eff})] [\gamma_f \Delta y_f - \gamma_i \Delta y_i] \quad . \end{aligned} \quad (10.30)$$

Using various values of ρ_C , μ_s , and Δy_i , computation of trajectories through the calculated magnetic field yielded values of $\Delta\beta$, Δy_f , and $\Delta\mu$, from which are evaluated

$$\tan [\mu_s (\text{eff})] = -\Delta\beta \frac{\rho_C}{\gamma_f \Delta y_f - \gamma_i \Delta y_i} \quad (10.31a)$$

and

$$\theta = \frac{\mu_s (\text{eff}) - \mu_s}{\Delta\mu} \quad . \quad (10.31b)$$

Both γ_f and γ_i were computed by a numerical averaging of H_y as in Eq. (10.28).

The θ -value calculated depends upon the endpoint chosen in the integration. We have chosen to compute θ -values for two endpoints. The first is one gap width inside the profile, at which point the H_y -component (the analog of H_ϕ) differs from the interior field by $< 0.05\%$. For this case, $\gamma_f = 1$. The other endpoint is at 0.3 times the gap width. This is the approximate value at which an extrapolation of the straight-line portion of the fringing field on the median plane intersects the interior field curve, as in Fig. (8.4). The value derived from similar extrapolations for several experimental measurements on electromagnet and permanent magnet structures is also approximately 0.3 times the gap width. For this case, $\gamma_f < 1$, and is a function of Δy_f .

Also, ρ_C was varied from 4 to 50 gap widths and μ_s from -20° to $+9^\circ$.

The results of these computations were as follows:

- (i) $\tan [\mu_s(\text{eff})]$ is fairly constant for various Δy -values, at given μ_s and ρ_C . It is more closely constant for large values of ρ_C , which corresponds to weak fringing fields. This may be expected from the optical analogy discussed in Section #9.
- (ii) When the integration was carried into the gap a distance of one gap width ($\gamma_f = 1$), Case I yielded θ -values close to 0.35 and approximately constant with Δy_f . Case II yielded θ -values close to 0.30, with similar constancy over Δy_f .
- (iii) When the integration was carried into the gap a distance of 0.3 gap width, Case I yielded θ -values close to 0.54, while Case II yielded θ -values close to 0.50, both having approximate constancy with Δy_f .

The change in θ for the two integration endpoints is to be expected, since $\Delta\beta$ increases very slightly between 0.3 and 1.0 gap width inside, while $\Delta\mu$ increases considerably.

The applicability of the calculated model to actual cases is uncertain, since the fringing field differs from that of real magnets, particularly in its rate of fall-off outside of the gap. It seems likely, however, that θ may be taken as lying between 0.3 and 0.5, even when the straight-line approximation to the magnetic field is used.

Example of a Computation of $\mu_s(\text{eff})$. To illustrate the magnitude of the effect, we have computed $\mu_s(\text{eff})$ for the case of a symmetrical spectrometer with $K = 0.6$ and $\phi = 10^\circ$, as shown in Fig. (10.1). The initial values of $\tan\mu_s$ were computed from Eq. (5.34) and are tabulated in Appendix IV, column 2.

These are shown as the solid curve in the figure. In computing the angular deflections, the straight-line approximation was taken for the magnetic

field. The values of $d_e(\psi_s)$ were computed using the measurements described in Section #8: $\varphi r_e/d_e = 0.85$ at $\psi_s = 95^\circ$ and $\varphi r_e/d_e = 0.6$ at 145° ; an approximate linear interpolation was used for r_e -values in between. Computing $\Delta\psi_e$ from Eq. (8.7b) and using $\theta = 0.3$, values of $\mu_s(\text{eff})$ were determined and are shown as the dashed curve in the figure. It is evident that the change from μ_s to $\mu_s(\text{eff})$ lies between 3° and 4° from one end of the ψ_s -range to the

other. Since $0.3 \Delta\psi_e$ is small and does not vary much over the ψ_s -range, it is apparent that the curve of $\mu_s(\text{eff})$ is almost a translated version of the μ_s -curve. As pointed out above, the effect is unidirectional, since $\Delta\psi_e > 0$: where the lens is converging, it is made less convergent, a slightly convergent lens is made slightly divergent, and a divergent lens is made more divergent.

#11. EFFECT OF THE $\Delta\psi$ -DEFLECTION ON THE RESOLUTION

The increased ψ -deflection due to the fringing field ($\Delta\psi$) has been discussed in Section #8. The variation of $\Delta\psi$ with ψ_s degrades the focusing quality, if the profiles are constructed as described in Section #3 for an idealized field with no fringing effects. As discussed in Sections #13 and #15, the profile shape may be changed such that the increased $\Delta\psi$ is compensated, at each value of ψ_s , by a decrease in the net bending power of the interior idealized field. In an iron-core instrument, for example, iron may be removed at the profile, adjusting the amount removed at each ψ_s to the requirement that all rays leaving $(z_s, 0)$ arrive at $(z_f, 0)$.

At the same ψ_s -value, however, a ray passing near a poleface experiences a different fringing field than does a ray passing through the median plane. It may be expected, in general, that a different value of $\Delta\psi$ will result, as well as a different point of entry into the idealized (interior) field. The adjustment of the profile can compensate only for the fringing-field $\Delta\psi$ -deflection of rays lying in the median plane; insofar as the effect varies with ϕ_s , the profile adjustment is less effective for $\phi_s \neq \phi_M$. The resulting perturbation of the trajectories disturbs the focusing property, with a corresponding loss in resolution. We propose to examine the difference in the focal position of the two types of trajectories in order to estimate the approximate effect on the resolution.

To pursue this question, it is necessary to have a detailed knowledge of the direction and magnitude of the entire fringing field. This information is not available for most magnets, and certainly not for any of the type with a r^{-1} -field. It was proposed, therefore, to consider, first, a simpler type of magnet, which could be treated more easily.

Since fringing fields differ very markedly for magnets of different designs, general computations on the absolute values of $\Delta\psi$ or of the point of entry into the interior field are not feasible. Indeed, accurate adjustment of the profile requires construction of the magnet, followed by an empirical correction after the $\Delta\psi$ -effect has been experimentally evaluated. Although the absolute values are so sensitive to the particular form of the fringing fields, it is much less likely that the differences between the median and off-median trajectories are sensitive in this manner, particularly since the difference between the $\Delta\psi$ -effects of the two types of trajectories is much smaller than the absolute values. It is for this reason that we consider it justifiable to evaluate the differences using a model which does not actually represent the r^{-1} -field spectrometer.

The magnet model is the one discussed in Section #10. As mentioned there, the primary features of the model are: first, the field is taken to be uniform in the interior; and second, the fringing field is taken to be two-dimensional, i.e., the extent of the front of the poleface is very

large in comparison to the essential extent of the fringing field away from the poleface. With these two assumptions, and with the condition that magnetic leakage is negligible (i.e., practically all of the lines of force originating on one pole end up on the other pole), it can be shown that $\Delta\psi$ is essentially independent of the distance of the trajectory from the poleface. (In the r^{-1} -spectrometer, this represents independence of the ϕ -plane.) No detailed information of the structure of the fringing field is necessary for this computation.

Since the interior trajectory is determined not only by the angle of entry into the interior field, but also by the coordinates of the entry point, it is necessary to evaluate both the difference in $\Delta\psi$ and the difference between the displacements of the entry points. The latter requires a numerical computation of the trajectory through the fringing field into the interior field, and such computation involves a detailed knowledge of the value and direction of the fringing magnetic field at all points along the trajectory. Thus, for computing the displacement, specific assumptions must be made concerning the structure of the poles and of their fringing fields.

Fig. 11.1

We consider, first, the general features of the $\Delta\psi$ -computation, which is described in detail in Ref. 22. The coordinate (x, y, z) system for the model is shown in Fig. (11.1). The entrance profile lies on the z -axis, positive z going into the paper, and extends infinitely in the $\pm z$ -direction. The two polefaces are on the planes $y = 0$ and $y = -h$; the median plane is $y = -\frac{1}{2}h$. The planes $y = \text{constant}$, with $0 > y > -h$, are analogs of the planes

$\phi = \text{constant}$ within one gap of the r^{-1} -spectrometer. Trajectories lying on constant- ϕ planes in the spectrometer correspond to trajectories lying on constant- y planes in the model magnetic field. The model considered is Case I of Section #10.

Two trajectories are considered: (I) entering the magnetic field on the median plane, and (II) approaching at the same angle relative to the normal to the profile, but lying on an off-median plane. Through the fringing field, both trajectories are bent away from the original direction, but the median trajectory (I) is bent a greater distance. At a distance of the order of one gap width inside the profile (MNP), the magnetic field is essentially uniform. Both trajectories would have arrived at M in the absence of a fringing field; they are, in fact, displaced the distance MN from this arrival point, the entry points differing by the slight distance NP, with $NP \ll MN$.

Fig. 11.2

Shift in Entry Angle ($\Delta\mu$) in Uniform Field. Practically all of the trajectory bending by the fringing field occurs in the x-z plane, and it is this bending which affects the resolution. Figure (11.2) represents the projection of the trajectory onto the x-z plane; the magnitude of the bending is measured (as in Section #8) by the angle μ between the tangent to the projected

trajectory and the perpendicular to the profile (i.e., the x-axis).* As shown, μ is taken as positive when directed from the x-axis to the tangent in a counterclockwise direction. To be evaluated is the change $\Delta\mu$ suffered by the trajectory in passing through the fringing field, $\Delta\mu$ corresponding to the quantity $\Delta\psi_e$ of the r^{-1} -field spectrometer.

It is shown that along the trajectory (see Ref. 22)

$$d\mu = - (e/p) [\cos\mu \tan\beta H_x ds + H_y ds] \quad (11.1a)$$

and

$$\Delta\mu = \mu - \mu_0 = - \frac{e}{p} \int H_y ds - \frac{e}{p} \int \cos\mu \tan\beta H_x ds \quad , \quad (11.1b)$$

where μ_0 is the original angle of orientation, ds is an element of the trajectory, and β is the angle made by the trajectory relative to the x-z plane, i.e., $\sin\beta = -(dy/ds)$. In fact, β is the analogue of the angle β in Section #8, and, as in that section, the initial value of β is taken to be zero. We have

$$d\beta = (e/p) H_x \sin\mu ds \quad (11.2a)$$

and

$$\Delta\beta = \beta - \beta_0 = \beta = \frac{e}{p} \int H_x \sin\mu ds \quad . \quad (11.2b)$$

The connection between the values of $\Delta\mu$ for various trajectories with the same μ_0 is made by considering the number of lines of force passing through the region neighboring the trajectory. The number of such lines is defined as

$$dN = H \cos\theta dS \quad , \quad (11.3)$$

where dN is the number of lines passing through the area dS , H is the field magnitude, and θ is the angle between $\underline{H}$ and the normal to dS . We consider

*This angle corresponds to that shown in Fig. (8.2).

an area adjacent to a trajectory and extending one unit in the z -direction; for trajectories which are parallel to the x -axis (i.e., at constant z and y , and $\beta = 0$) over the entire path,

$$N = \int H \cos\theta \, dx = \int H_y \, ds = \left| \frac{P}{e} \Delta\mu \right| \quad . \quad (11.4)$$

Two such trajectories at different y -values, but at the same z , pass through the same lines of force, and if the integration is taken into the uniform interior field, the integration ends on the same line of force. Hence, N is independent of y , the equality of N for the various trajectories involved implying that $\Delta\mu$ is the same for all of them, median and off-median trajectories alike.

These considerations assume that $\mu = 0$ throughout the trajectory, whereas $\Delta\mu$ is actually accumulated along the entire path through the fringing field. We consider, then, a trajectory at constant y over a curve $z = z(x)$. Now dS is no longer equal to ds ; instead, $dS = dx = (dx/ds)ds = \cos\mu \, ds$. Then

$$dN = H \cos\theta \, dx = H_y \cos\mu \, ds = \left| \frac{P}{e} \cos\mu \, d\mu \right| \quad . \quad (11.5a)$$

We compare two such trajectories, $z_1(x)$ and $z_2(x)$, which lie on different y -planes (y_1 and y_2). If z_1 and z_2 were the same curves, then it would be clear that if all lines of force starting on one pole ended on the other, then the same lines of force would thread both trajectories. It is evident that, since the extension of the surface elements in the z -direction is unity for both curves, and since the field intensity is not a function of z , the displacement of $z_1(x)$ from $z_2(x)$ does not change the equality of lines of force. We integrate dN over each trajectory from the same x_{initial} to the same x_{final} , where x_{final} is well within the uniform field, so that the integration paths terminate on corresponding lines of force (i.e., lines which originate at the pole at the same x -value). For each trajectory,

$$N_i = \left| \frac{P}{e} (\sin\mu_0 - \sin\mu_i) \right| \quad . \quad (11.5b)$$

Since $N_1 = N_2$ and the initial μ_0 's are the same, then $\mu_1 = \mu_2$ and $\Delta\mu$ is independent of y .

One feature of this analysis requires further examination. From Eq. (11.2b), it is evident that, when $\mu \neq 0$, then $\beta \neq 0$, provided that $H_x \neq 0$. Thus, the relation $H \cos\theta = H_y$, utilized in Eq. (11.5a), does not apply except on the median plane, where $\beta = 0$ because $H_x = 0$. Analysis⁽²²⁾ shows that, when $\beta \neq 0$,

$$\begin{aligned} dN &= (p/e) \cos\beta \cos\mu \, d\mu - \sin^2\mu \sin\beta \, H_x \, ds \\ &= (p/e) \cos\beta \cos\mu \, d\mu - (p/e) \sin\mu \sin\beta \, d\beta \quad , \end{aligned} \quad (11.6)$$

and when β is small, $\cos\beta \approx 1 - \frac{1}{2}\beta^2$ and $\sin\beta \approx \beta$. Accordingly,

$$dN = dN' - d\epsilon \quad , \quad (11.7a)$$

where

$$dN' = |(p/e) \cos\mu \, d\mu| \quad (11.7b)$$

and

$$\begin{aligned} d\epsilon &= \frac{1}{2}(p/e)\beta^2 \cos\mu \, d\mu + \beta \sin^2\mu \, H_x \, ds \\ &= (p/e) \left(\frac{1}{2}\beta^2 \cos\mu \, d\mu + \beta \sin\mu \, d\beta \right) \quad . \end{aligned} \quad (11.7c)$$

When μ is not large, $d\mu$ is considerably greater than $d\beta$, because of the $\sin\mu$ factor, as well as the fact that, for $-\frac{1}{2}h < y < 0$, we have $|H_y| > |H_x|$. Thus, $|d\epsilon| \ll dN'$ and may be neglected.* With this approximation, Eq. (11.7b) is the same as Eq. (11.5a), and the remainder of the analysis follows. In actual computations of trajectories, it was found that the values of $\Delta\mu$ were quite accurately the same even for μ_0 as large as 20° (Ref. 22).

The proof essentially depends upon: (i) the fact that β remains small, so that the motion is primarily in the (x,z) directions, and (ii) upon the fact that the number of lines of force threading the regions neighboring each trajectory does not depend upon the displacement of the trajectory in the z-direction. Point (ii) depends partly upon the fact that the magnetic field is two-dimensional, and not a function of z.

The application of this result to the r^{-1} -field spectrometer will be developed later.

Shift in Entry Point (Δz) in Uniform Field. The separation of the two trajectories along the z-direction is now considered, i.e., we evaluate the magnitude of NP in Fig. (11.2). A method for making a general computation of the type described for $\Delta\mu$ was not evident, so an actual numerical calculation of trajectories was carried out using an IBM 704 computer. To get the necessary values of (H_y, H_x) as a function of (y,x), a particular pole shape was assumed. If the polefaces are taken as extending infinitely in the $\pm y$ -directions (Case I), then the field may be evaluated⁽²³⁾ by means of the technique of conformal transformation. Trajectories were computed for a number of (y, μ_0, ρ_C) -values, where ρ_C is the radius of curvature

*As an example, in the numerical computation of trajectories of Ref. (22), for the case in which the radius of curvature (ρ) in the interior field is 10 gap widths [$\rho = 10h$], $\mu_0 = -9^\circ$, and $y_0 = -0.05h$ (close to poleface), it was found that $\Delta\beta/\Delta\mu = 0.016$. Using this as the approximate value of $d\beta/d\mu$, the ratio $d\epsilon/dN' = 0.000015$.

within the uniform field.* It was found that when the orbits reached one gap width inside the profile, then the difference in the z -coordinates of the trajectories could be empirically fitted by the expression

$$|\Delta z| = 0.46 (h'^2/\rho_C)(1 + 2\bar{\mu}^2) \quad , \quad (11.8)$$

where h' is the distance of the y plane containing the trajectory from the median plane (i.e., $h' = y + \frac{1}{2}h$), and $\bar{\mu}$, the approximate mean value of μ (radians) along the trajectory, is $\frac{1}{2}(\mu_0 + \mu_{\text{final}})$. The direction of Δz is given by the fact that the displacement from the orbit that would have occurred in the absence of the fringing field is greater for the median trajectory than for the non-median trajectory.

Shift in Entry Point ($\Delta\eta$) in r^{-1} -Field. Equation (11.8) is used in considering the effect of the displacement on the resolution of the r^{-1} -field spectrometer. In the homogeneous magnet, Δz was a displacement in a direction parallel to the magnet profile. In the r^{-1} spectrometer, a displacement along the profile is called $\Delta\eta$, and on the entrance side it is taken as positive in the direction of decreasing Ψ_s (increasing ψ_s). The median trajectory then enters the interior field at an η -value corresponding to a smaller Ψ_s -value than that involved for the off-median trajectory, i.e., at a greater η -value. On the exit side, positive $\Delta\eta$ is in the direction of decreasing Ψ_f , and the median trajectory tends to leave at a greater η -value.

Shift in Entry Angle ($\Delta\psi$) in r^{-1} -Field. The above result on the equality of $\Delta\mu$ for trajectories on median and off-median planes is only approximately true for the r^{-1} -field. The result was found in the case of a two-dimensional field, and depended upon the fact that the fringing field did not vary with z . Due to the r^{-1} -variation in the spectrometer, however, the intensity of the fringing field does vary with η , so that $\Delta\mu$ is not the same for all values of ϕ_s .

The difference in $\Delta\mu$ (i.e., in $\Delta\psi$) is small, so that the deviation from equality can be treated, approximately, as a perturbation. Since $d\mu$ [see Eq. (11.1a)] is proportional to H_y (for small β), the relative variation in $\Delta\mu$ may be taken as equal to the relative variation in the intensities of the fringing magnetic field occurring at the median trajectory and the off-median trajectory

*Variation of ρ_C essentially varies the relative importance of the interior field and fringing field deflection. Large ρ_C corresponds to weak interior field and hence weak fringing field. An alternative description is that large ρ_C corresponds to a long interior trajectory, so that the fraction of the path occupied by fringing field is small.

The first case considered is an idealized situation analogous to that discussed in Eq. (11.4). Thus, a median ray and an off-median ray of the same ψ_s approach the magnetic profile at perpendicular incidence ($\mu_0 = 0$); for simplicity, the range is $90^\circ < \psi_s < 180^\circ$, or, using the notation of Eq. (9.7), $0 < \Psi_s < 90^\circ$. As before, at first the change in the angle of incidence due to the accumulated $\Delta\psi$ -deflection is neglected. It is assumed at this point that the fringe lines are not perpendicular to the profile, but follow the cone of constant Ψ_s , as in Fig. (11.3). With these two assumptions, both the median and off-median rays cut the same lines of force at the same angle to the profile, and hence, as above, the $\Delta\psi$ -values are the same.

Figure (11.3) shows a bundle of rays of constant Ψ_s as they approach the magnetic profile, only a short segment of which is shown. A single

Fig 11.3

line of force, MQN, of the fringing field is presented. As illustrated, this is considered to be an ordinary line emerging perpendicularly to the poleface, but bent upwards (towards increasing r) as it swings around, so that it lies on the constant- Ψ_s cone. Using the coordinate system of Section #1 (i.e., z, r, ϕ), it is evident that these lines of force merge naturally with the interior lines which follow circles of constant r , at any particular z -value. As the field emerges from the interior, the lines must

change to curves which leave the poleface perpendicularly.

The argument may also be carried through when the rays are not incident perpendicularly to the profile, i.e., when $\mu_0 \neq 0$. In this case, the lines of force lie on cones perpendicular to the profile, as in Fig. (11.3), but these cones are not constant- Ψ_s surfaces. The trajectories cut through these cones, but, if the projection of these orbits onto constant- ϕ planes involve the same curve $z = z(r)$ for median and off-median trajectories, then both rays pass through the corresponding lines of force. Hence, $\Delta\psi$ is again the same for both types of trajectories.

This argument, however, is only approximately correct, since there are two perturbation effects:

(I) Lines of force tend to be perpendicular to the profile rather than following the cones described above. Hence, the median trajectory fringing field intensity becomes smaller than that of an off-median trajectory. The resulting difference in the entry angle is called $\delta\psi_1$.

(II) If $dr/d\eta$ along the profile is negative, as it is for large ψ_s (small Ψ_s), then the $\Delta\eta$ -effect drives the median trajectory towards smaller r -values than occur for the non-median trajectory, and hence causes a relative increase in its fringing-field intensity. The effect is reversed when $dr/d\eta$ is positive, as it is for small ψ_s . The resulting difference in the entry angle is called $\delta\psi_2$.

Computation of $\delta\psi_1$. We consider Case (I) first. The lines of force tend to lie in a plane perpendicular to the profile, with some modification due to inhomogeneity of the field within the gap.* This modification, however, was not evaluated quantitatively and is neglected in the estimate below.

A crude estimate as to the magnitude of the change in $\Delta\psi$ between median and off-median rays is based on the following assumptions:

(i) The fringe lines are considered to lie on a plane perpendicular to the profile, bowing due to the unsymmetrical repulsion effect being neglected. This will result in an overestimate of $(\Delta\psi_{0M} - \Delta\psi_M)$, because the intensity of the fringing field through which the median ray passes is underestimated.

(ii) The incident trajectories are perpendicular to the profile. This simplifies the calculation, but, as indicated below, is not essential.

(iii) The intensity of the fringing magnetic field at a particular point is determined by the intensity of the field in the part of the gap from which the corresponding fringe lines start. Thus, Fig. (11.4a) shows the cone of rays of constant Ψ_s as emitted from a point source.** The projection of this system onto the farthest off-median plane is shown in Fig. (11.4b), whereas the projection onto the median plane is shown in Fig. (11.4c). The line ST is a short segment of the profile approximately perpendicular to the bundle of rays. The farthest off-median ray passes through S, while the median ray passes through T'.

* The modification arises because, within the gap, the field intensity, and hence the density of lines of force, decreases as r increases; the same decrease in density operates for the fringing field as well. The fringe lines are prevented from lying in a plane perpendicular to the profile by the "repulsion" of lines of force. In a symmetrical system, such as the idealized uniform field magnet discussed above, the intensity of the fringing field is constant for a given (x,y) , so that the repulsion is balanced on both sides of a line of force, and, hence, fringe lines lie in a plane perpendicular to the profile. In the r^{-1} -field, on the other hand, the gradient of the magnetic field leads to unequal repulsions from the two sides, causing a bowing of fringe lines towards increasing r . Thus, this bowing effect tends to make the fringe lines follow the cones to some extent.

** The planes TT'P' and SS'P are perpendicular to the z -axis (MPP'), and are separated by the distance PP'. The line TS is a short segment of the profile; the line TSM is the extension to the z -axis, but does not represent the profile, which actually curves away from this line. Note that $\angle OSM = \frac{1}{2}\pi$ and $\angle SMP = (\frac{1}{2}\pi - \Psi_s)$.

Fig. 11.4a

Fig. 11.4b

Fig. 11.4c

The fringe lines affecting the farthest off-median ray lie in the plane perpendicular to the profile at S, i.e., the plane SOS'. The fringing lines affecting the median ray lie in the plane approximately parallel to SOS' and passing through TT'. The fringing magnetic fields affecting the two rays then correspond in relative intensities to the interior fields appropriate for the points S and T, respectively.

(iv) The structure of the fringing field at neighboring r -values is considered to be the same, differing only in the intensity resulting from the effect assumed in (iii). Hence, trajectories passing through two sets of fringe lines originating at different r -values will have $\Delta\psi$ -values differing only insofar as the in-gap field differs at the two r -values.

Since the interior field varies with r , the difference in the r -values for S and T, i.e., $\Delta r = TQ$ [see Fig. (11.4b)], is calculated from the geometric properties of the system, such as the angular opening of the gap (φ), the angle of emission of the rays (Ψ_s), the angle between the profile and the incident rays ($\frac{1}{2}\pi$, in this example).

Thus, $\angle US'T' = \angle STQ \cong \Psi_s$, and from Figs. (11.4b) and (11.4c), $\Delta r = TQ = TS \cos \Psi_s$ and $T'S' = US' \cos \Psi_s$. From Fig. (11.4e), we have $T'S' = TS \cos \frac{1}{2}\varphi'$, and from Fig. (11.4f)

$$\cos \frac{1}{2}\varphi' = \frac{\cos \frac{1}{2}\varphi}{\sqrt{1 - \sin^2 \Psi_s \sin^2 \frac{1}{2}\varphi}} \quad (11.9)$$

From Fig. (11.4d), $US' = r (1 - \cos \frac{1}{2}\varphi)$, where $r = UP = SP$.

Then

$$\begin{aligned} \frac{\Delta r}{r} &= \frac{(1 - \cos \frac{1}{2}\varphi) \cos^2 \Psi_s \sqrt{1 - \sin^2 \Psi_s \sin^2 \frac{1}{2}\varphi}}{\cos \frac{1}{2}\varphi} \\ &= \frac{\varphi^2}{8} \cos^2 \Psi_s \left\{ 1 + \frac{\varphi^2}{8} \left[\frac{5}{6} - \sin^2 \Psi_s \right] \right\} \cong \frac{\varphi^2}{8} \cos^2 \Psi_s . \end{aligned} \quad (11.10a)$$

Since

$$\left(\frac{dH}{dr} \right)_{r=r_0} = - \frac{H(r_0)}{r_0} ,$$

we have

$$H(T) \cong H(S) + \frac{dH}{dr} \Delta r = H(S) \left[1 - \frac{\Delta r}{r} \right] \quad (11.10b)$$

or

$$H(T) \cong H(S) \left[1 - \frac{\varphi^2}{8} \cos^2 \psi_S \right] . \quad (11.10c)$$

From assumption (iv) above,

$$\frac{\Delta \gamma'_{OM} - \Delta \psi'_M}{\Delta \psi'_{OM}} \cong \frac{H(S) - H(T)}{H(S)} = \frac{\varphi^2}{8} \cos^2 \psi'_S . \quad (11.11a)$$

For $\varphi = 10^\circ$ and $\psi'_S = 135^\circ$, this ratio is approximately 0.001; for $\psi'_S = 100^\circ$, the ratio is 0.00006.

Fig. 11.4d

Fig. 11.4e

Fig. 11.4f

Then, if $\delta\psi_1$ is the difference in the angle ψ on arrival of the two trajectories into the interior field, we have

$$\delta\psi_1 = \Delta\psi_{OM} - \Delta\psi_M = \frac{\phi^2}{8} \cos^2 \psi_s \Delta\psi \quad . \quad (11.11b)$$

The same result applies when the off-median trajectory involved is not the farthest off-median one. In this case, the quantity $(\phi_{OM} - \phi_M)$ is substituted for $\frac{1}{2}\phi$. Thus, taking $\phi_M = 0$, as in Section #9,

$$\phi_{OM} - \phi_M = \phi_s \text{ or } \phi_{OM} - \phi_M = \phi_f \quad . \quad (11.11c)$$

Then

$$\delta\psi_1 = \Delta\psi_{OM} - \Delta\psi_M = \frac{1}{2}\phi_s^2 \cos^2 \psi_s \Delta\psi \quad (11.11d)$$

at the entrance, while ϕ_f is substituted at the exit. It should be noted that the $\Delta\psi$ -value referred to is the $\Delta\psi_e$ of Eq. (8.7).

Under these assumptions, the off-median ray traverses a more intense fringing field, and thus suffers a slightly larger $\Delta\psi$ -value. The greatest effect occurs for the largest values of ψ_s .

When the incident rays are not perpendicular to the profile, the magnitude of $\Delta\psi$ is greater, but if $\tan\mu$ is small, the relative difference in Eq. (11.12) should remain roughly the same. Although the source points (in the gap) for the fringing field lines change as the trajectories pass through the fringing field, the relative intensities of the fields at the source points remain roughly the same for the median and off-median rays.

The computation for this case is similar: $\Delta r = TS \cos(\Psi_s - \mu)$ and $TS' = US' (\cos \Psi_s / \cos \mu)$. Then

$$\begin{aligned} \frac{\Delta r}{r} &= \frac{(1 - \cos \frac{1}{2}\phi) \cos(\Psi_s - \mu) \cos \Psi_s \sqrt{1 - \sin^2 \Psi_s \sin^2 \frac{1}{2}\phi}}{\cos \frac{1}{2}\phi \cos \mu} \\ &\cong \frac{\phi^2}{8} (\cos^2 \Psi_s + \cos \Psi_s \sin \Psi_s \tan \mu) \quad , \end{aligned} \quad (11.12a)$$

$$\delta\psi_1 = \frac{\phi^2}{8} (\cos^2 \Psi_s + \cos \Psi_s \sin \Psi_s \tan \mu) \Delta\psi \quad , \quad (11.12b)$$

and for any off-median trajectory

$$\delta\psi_1 = \frac{1}{2}\phi_s^2 (\cos^2 \Psi_s + \cos \Psi_s \sin \Psi_s \tan \mu) \Delta\psi \quad . \quad (11.12c)$$

As mentioned above, $\delta\psi_1$ is a rough approximation, in the nature of an upper limit; the formula is an overestimate, since the relative fringing-field intensity for the median ray was underestimated when the bowing of the fringe field was neglected.

Computation of $\delta\psi_2$. We now consider the effect described in Case (II). The median trajectory comes into the interior field at a lower r -value than the off-median trajectory when $dr/d\eta < 0$, but a higher r -value when $dr/d\eta > 0$. A difference in the corresponding fringing field intensities arises from the difference in r -value:

$$\Delta r = r_M - r_{OM} \quad , \quad (11.13)$$

which is computed from the corresponding value of $\Delta\eta$. This is taken to have the same form as that of $|\Delta z|$, computed for the homogeneous field [see Eq. (11.8)]. Thus from Eq. (11.8) and Eq. (1.10),

$$\eta_M - \eta_{OM} = \Delta\eta \approx \frac{1}{8} \frac{(r\phi)^2}{\rho_C} = \frac{1}{8} \frac{r^2\phi^2}{rK} = \frac{1}{8} \frac{r\phi^2}{K} \quad , \quad (11.14a)$$

since $|h'| = r(\frac{1}{2}\phi)$ for the farthest off-median plane. Taking μ to be fairly small the variation with $\bar{\mu}$ is neglected. For any plane other than the farthest off-median one, we have

$$\Delta\eta = (1/2K) r \phi_s^2 \quad , \quad (11.14b)$$

where ϕ_s is defined in Eq. (11.11c) and ϕ_f is substituted at the exit. Since the median ray displacement is greater, $\Delta\eta$, as defined in Eq. (11.14a), is positive

The difference corresponding to $\Delta\eta$ is zero before the median and off-median trajectories enter the fringing field, but increases as the trajectories pass through the fringe field, and finally takes the value in Eq. (11.14). Since the effective difference in the r -values over the entire trajectory changes from zero to that computed from $\Delta\eta$ within the gap, an average value of Δr is taken over the entire path, so that roughly

$$\Delta r_{\text{effective}} = \frac{1}{2} \Delta r \quad . \quad (11.15)$$

We see from Fig. (11.5) (for sign convention of μ , see Section #8).

$$\frac{\Delta r}{\Delta\eta} = -\cos(\Psi_s + \mu) = \cos(\psi_s - \mu) \quad . \quad (11.16)$$

Then

$$\frac{\Delta r_{\text{eff}}}{r} = \frac{1}{4K} \phi_s^2 \cos(\psi_s - \mu) \quad . \quad (11.17)$$

The sign of Δr_{eff} depends on whether $(\psi_s - \mu)$ is $< \frac{1}{2}\pi$ (then Δr_{eff} is positive) or $> \frac{1}{2}\pi$. [In Fig. (11.5), the angle is greater than $\frac{1}{2}\pi$, so Δr_{eff} is less than zero.] Then, as in Eq. (11.11a),

$$\begin{aligned} H(M) &= H(OM) + \frac{dH}{dr} \Delta r_{\text{eff}} = H(OM) \left[1 - \frac{\Delta r_{\text{eff}}}{r} \right] \\ &= H(OM) \left[1 - \frac{1}{4K} \phi_s^2 \cos(\psi_s - \mu) \right] \quad , \end{aligned} \quad (11.18)$$

and, as in Eq. (11.12),

$$\delta\psi_2 = \Delta\psi_{OM} - \Delta\psi_M = \frac{1}{4K} \phi_s^2 \cos(\psi_s - \mu) \Delta\psi \quad . \quad (11.19)$$

For the commonest type orbits, i.e., with small μ and $\psi_s > \frac{1}{2}\pi$, it is found that $\delta\psi_2$ is negative, whereas $\delta\psi_1$ [see Eq. (11.12b)] is always positive. As before, the $\Delta\psi$ -value referred to is $\Delta\psi_e$.

Effect of $\Delta\eta$ on the Focusing. Given the differences $\delta\psi_1$, $\delta\psi_2$ and $\Delta\eta$ at a given ψ_s , it is necessary to evaluate their effect on the focusing property. We start by considering only the effect of $\Delta\eta$, assuming for the moment that $\delta\psi = 0$.

To determine the magnitude of the effect, only an approximate calculation will be made. We consider a spectrometer in which the profile has been adjusted so that a median ray from $(z_s, 0)$ focuses at $(z_f, 0)$. Thus, in Fig. (11.6a), the median ray is bent by the fringing field and arrives one gap width inside the profile at the point P, at which point the magnetic field follows the form in Eq. (1.1). The off-median ray, after passing through the fringing field, arrives one gap width inside the profile at N, displaced (in a direction parallel to the profile) from the median ray by the distance $NP = \Delta\eta$. Because $\delta\psi = 0$, the two rays are parallel at N and P, i.e., their ψ -values are equal.

Fig. 11.5

Fig. 11.6a

To simplify the calculation, we consider it equivalent to a situation in which no fringing field exists, but in which two parallel rays arrive at the profile, separated by the distance $N_0P_0 = \Delta\eta$ along the profile, as in Fig. (11.6b).^{*} In effect, a source point has been axially displaced by the distance δz_s ; this type of perturbation has been treated in Section #5.

From Fig. (11.6c) it is evident that

$$\delta z_s \sin \Psi_s = \delta z_s \sin \psi_s = \Delta\eta \cos \mu \quad .$$

Then

$$\delta z_s = \Delta\eta \frac{\cos \mu}{\sin \psi_s} \quad . \quad (11.20)$$

Fig. 11.6b

Fig. 11.6c

If we consider only a symmetrical spectrometer, then, from Eq. (5.10), since $\psi_f = -\psi_s$ and $\psi_f^i = -1$, if there is no fringing field at the exit,

$$\delta z_f = \delta z_s \quad , \quad (11.21)$$

both being positive.

Since, however, an exit fringing field does exist, its effect must be included; to do so, it is necessary to evaluate the position and angle of the off-median trajectory just before it hits the exit fringing field, by the method of Section #5.

^{*}Although $\psi_s > \frac{1}{2}\pi$ in the figure, $\delta z_s > 0$ for $\psi_s < \frac{1}{2}\pi$ as well, because $\Delta\eta$ is always positive, the median ray having a greater displacement than the off-median one.

In Fig. (11.7a), we consider three rays:

(I) Median ray, in the approximation discussed above, emitted from $(z_s, 0)$.

(II) Off-median ray, in this approximation emitted from $(z_s + \delta z_s, 0)$.

(III) A pseudo-ray, "emitted" from $(z_s, 0)$ and "entering" the profile at the same "point" as (II). It does not exist in the actual spectrometer, and is considered in the approximate model being used, because the theory of Section #5 can be applied only to a pair like (II) and (III). The relation to (I) is easily carried through, because (I) and (III) are both subject to the focusing and symmetry conditions in a symmetric spectrometer (see Section #3).

Fig. 11.7a

Rays (I) and (II) are emitted at the angle ψ_s , while (III) is "emitted" at the angle ψ_p , where $\psi_s = \psi_p + \delta\psi_p$, or $\psi_p = \psi_s - \delta\psi_s$, with $\delta\psi_p = -\delta\psi_s$. Rays (I) and (III), having been "emitted" from $(z_s, 0)$ are focused into $(z_f, 0)$, and since the profiles are symmetrical, we have equal distances along the profiles: $N_0P_0 = N_1P_1 = \Delta\eta$. Correspondingly, the exit angles are symmetrical, so $\psi_f = 2\pi - \psi_s = -\psi_s$ and $-\psi_{pf} = \psi_p = \pi - \psi_{pf}$ [using the notation of Eq. (9.7)], with $\psi_f = \psi_{pf} - \delta\psi_p$. The goal is to evaluate N_1N_2 and $\delta\psi_{pf}$. For the moment, the exit fringing field effect is neglected, and the rays and angles on the exit side of Fig. (11.7b) are treated as though they were completely determined by the intersection of the trajectories with the exit profile, as in Section #5.

Using the symmetry condition, we have from Eqs. (3.1a) and (5.7b):

$$\Lambda = \frac{r_0 - Kz_s \sin^3 \psi_p}{z_s \sin^2 \psi_p}.$$

Then, from Eq. (5.5b),

$$\delta\psi_{pf} = \left[1 - \frac{2Kz_s}{r_0} \sin^3\psi_p \right] \delta\psi_p = B_1 \delta\psi_p \quad (11.22a)$$

From Eq. (5.9),

$$\delta\psi_{pf} - \delta\theta = \delta\psi_p$$

and

$$\delta\theta = - \frac{2Kz_s \sin^3\psi_p}{r_0} \delta\psi_p = C_1 \delta\psi_p \quad (11.22b)$$

where the notation from Section #5 has been changed: ψ_f to ψ_{pf} and ψ_s to ψ_p .

Since $\delta z_s > 0$, then $\delta\psi_p > 0$; hence, $\delta\theta < 0$ (since $C_1 < 0$), and N_2 always lies to the right of N_1 . From Fig. (11.7b), since $\delta\theta < 0$ and $N_1 N_2 > 0$,

$$N_1 N_2 \cos\mu \cong -\rho \delta\theta$$

and

$$r_0 = -\rho \sin\psi_{pf} = \rho \sin\psi_{pf} = \rho \sin\psi_p \quad .$$

Then,

$$N_1 N_2 = \frac{-r_0}{\sin\psi_p \cos\mu} \delta\theta \quad (11.23)$$

Similarly, from Fig. (11.7c),

$$\delta\psi_p = \frac{\Delta\eta \cos\mu \sin\psi_p}{r_e} \quad (11.24)$$

Then, from Eqs. (11.22), (11.23) and (11.24), and taking $r_e \cong r_0$,

$$N_1 N_2 = -C_1 \Delta\eta = \frac{2Kz_s \sin^3\psi_p}{r_0} \Delta\eta \quad (11.25)$$

Fig. 11.7b

Fig. 11.7c

Since [see Fig. 11.7a)] $N_1P_1 = \Delta\eta$ (from symmetry), then the distance (along the profile) between the off-median ray (II) and the median ray (I) is

$$N_1P_1 - N_1N_2 = N_2P_1 = \Delta\eta + C_1 \Delta\eta = B_1 \Delta\eta = \left[1 - \frac{2Kz_s \sin^3\psi_p}{r_0} \right] \Delta\eta \quad (11.26)$$

Here N_2P_1 is positive if the median ray (I) leaves the profile to the right of the off-median ray (II), whereas N_2P_1 is negative when P_1 is to the left of N_2 .

Superimposed upon these consequences of the entrance fringing-field separation of median and off-median rays is the separation caused by the exit fringing field. The latter results in a movement of the median ray (P_1) further to the right of the off-median ray (N_2) by the amount $\Delta\eta'$. The net separation along the profile is then

$$\Delta\tau = N_2P_1 + \Delta\eta' \quad (11.27)$$

If ϕ_f for the off-median ray is the same as its ϕ_s -value, then $\Delta\eta' = \Delta\eta$. However, since there is either a converging or diverging lens at the entrance, the value of ϕ_f^2 is respectively smaller or larger than ϕ_s^2 . From Eq. (11.14b),

$$\Delta\eta' = (\phi_f^2/\phi_s^2) \Delta\eta = C_2 \Delta\eta \quad (11.28)$$

where $C_2 < 1$ for a moderately converging input lens and $C_2 > 1$ for a diverging input lens.

Then,

$$\Delta\tau = (B_1 + C_2) \Delta\eta = C_3 \Delta\eta \quad (11.29)$$

Except for large values of K with ψ_s close to 90° , in general, C_3 is positive, i.e., including the exit fringing-field effect, we find that the median ray usually lies to the right of the off-median ray along the profile.

Now, we evaluate the angle of emergence of the off-median ray. The relation between $\delta\psi_{pf}$ and $\delta\psi_p$ is given in Eq. (11.22a), in which B_1 may be positive or negative. Since $C_2 > 0$, then $B_1 < 0$ when $C_3 < 0$; however, positive C_3 may correspond to positive or negative B_1 . Since we are treating the case where $(\Delta\psi_M - \Delta\psi_{OM}) = 0$, the exit fringing field does not change the relative orientation of the median and off-median rays. Thus, the relations between the angles are as shown in Figs. (11.7a) and (11.8a). Figure (11.8a) shows the various rays arriving at (or near) the z -axis if no exit fringing field were present. If the median ray (P_1O_2') has the ψ -value ψ_M and the off-median ray (N_2O_3') has the value ψ_{OM} , then, since

$$\psi_{OM} = \psi_{pf} + \delta\psi_{pf} = \psi_{pf} + B_1 \delta\psi_p$$

and

$$\psi_M = \psi_f = \psi_{pf} - \delta\psi_p \quad ,$$

it follows that

$$\psi_{OM} = \psi_M + (1+B_1) \delta\psi_p = \psi_M + \delta\psi_0 \quad , \quad (11.30a)$$

where

$$\delta\psi_0 = (1+B_1) \delta\psi_p \quad . \quad (11.30b)$$

Since the angular relations do not change after the rays pass through the exit fringing field, Eq. (11.30) is still applicable. Further, $(1+B_1)$ and C_3 may be separately positive or negative. Figure (11.8b) represents the case in which $C_3 > 0$ and $(1+B_1) > 0$ (i.e., $\Delta\tau$ and $\delta\psi_0$ are positive); in Fig. (11.8c), however, $C_3 < 0$ and $(1+B_1) > 0$ (i.e., $\Delta\tau$ negative and $\delta\psi_0$ positive).

Fig. 11.8a

Fig. 11.8b

In Fig. (11.8b), since $\delta z_f < 0$,

$$\Delta\tau \cos\mu \cong O_2A \sin\Psi_f$$

$$\rho\delta\psi_0 \cong O_3A \sin\Psi_f$$

$$|\delta z_f| = O_2A - O_3A = \frac{\Delta\tau \cos\mu - \rho\delta\psi_0}{\sin\Psi_f} \quad .$$

Fig. 11.8c

Then,

$$\delta z_f = \frac{\rho \delta \psi_0 - \Delta \tau \cos \mu}{\sin \psi_s} \quad , \quad (11.31)$$

the relation applying for all values of $(1+B_1)$ and C_3 .

From Eqs. (11.24), (11.29) and (11.30b) and the relations $\sin \psi_p \cong \sin \psi_s$ and $r_e \cong \rho \sin \psi_s$,

$$\frac{\rho \delta \psi_0}{\sin \psi_s} \cong (1+B_1) \frac{\rho}{r_e} \cos \mu \Delta \eta \quad ;$$

$$\frac{\Delta \tau \cos \mu}{\sin \psi_s} \cong (B_1+C_2) \frac{\rho}{r_e} \cos \mu \Delta \eta \quad . \quad (11.32)$$

Then, from Eqs. (11.14), (11.31) and (11.32),

$$\delta z_f = \frac{1}{2K} (1-C_2) \rho \cos \mu \phi_s^2 \quad . \quad (11.33)$$

It may be noted that:

(i) Although ψ_s does not appear explicitly in Eq. (11.33), δz_f is implicitly a function of ψ_s , moderately through the variation of (ρ, μ) with ψ_s , but more markedly through the variation of $(1-C_2)$.

(ii) If the entrance lens is neither divergent nor convergent, then $C_2 = 1$, and $\delta z_f = 0$. For only slight divergence or convergence of the input lens, C_2 differs only moderately from unity, so δz_f is relatively small. For $C_2 = 1$, O_2 and O_3 in Fig. (11.8b) coincide, the difference in exit points (N_3, P_2) being compensated by the difference $\delta \psi_0$ in exit angle. Figure (11.8c) is not possible when $C_2 = 1$, since $(B_1+C_2) = (1+B_1)$ and $\Delta \tau$ and $\delta \psi_0$ must have the same sign. Numerical values for typical situations will be given below.

Effect of $\delta \psi_1$, $\delta \psi_2$ and $\Delta \eta$ on the Focusing. We now consider the more realistic situation in which $\delta \psi_1$ and $\delta \psi_2$ [see Eq. (11.12b) and (11.19)] are included. The analysis is similar to that following Fig. (11.7a) except that ray (II) is not parallel to (I). The off-median ray (II) is "emitted" at the angle

$$\psi_{s1} = \psi_p + \delta \psi_{p1} \text{ with } \delta \psi_{p1} = \delta \psi_p + \delta \psi_1 + \delta \psi_2 \quad , \quad (11.34a)$$

where $\delta\psi_p$ is shown in Fig. (11.7a), and

$$\delta\psi_1 = \Delta\psi_{OM} - \Delta\psi_M = \frac{1}{2} \phi_s^2 (\cos^2 \Psi_s \cos \Psi_s \sin \Psi_s \tan \mu) \Delta\psi \quad (11.34b)$$

$$\delta\psi_2 = \Delta\psi_{OM} - \Delta\psi_M = \frac{1}{4K} \phi_s^2 \cos(\psi_s - \mu) \Delta\psi \quad (11.34c)$$

Equation (11.22) becomes

$$\delta\psi_{pf} = B_1 \delta\psi_{p1}; \quad \delta\theta = C_1 \delta\psi_{p1} \quad , \quad (11.35)$$

whereas Eq. (11.23), not involving $\delta\psi_p$, remains unchanged. Equation (11.24) still describes $\delta\psi_p$, but this angle is now only part of the variation [see Eq. (11.34a)]. Substituting Eqs. (11.24), (11.34a) and (11.35) into (11.23),

$$N_1 N_2 = -C_1 \Delta\eta - \Gamma(\delta\psi_1 + \delta\psi_2) \quad (11.36a)$$

and

$$N_2 P_1 = B_1 \Delta\eta + \Gamma(\delta\psi_1 + \delta\psi_2) \quad , \quad (11.36b)$$

with

$$\Gamma = \frac{r_0 C_1}{\sin \Psi_p \cos \mu} = \frac{r_0 C_1}{\sin \psi_s \cos \mu} = \frac{\rho C_1}{\cos \mu} \quad , \quad (11.36c)$$

while

$$\Delta\tau = (C_2 + B_1) \Delta\eta + \Gamma(\delta\psi_1 + \delta\psi_2) = C_3 \Delta\eta + \Gamma(\delta\psi_1 + \delta\psi_2) \quad . \quad (11.36d)$$

Using Fig. (11.8a) as before, the emergence angles are, omitting the effect of the exit fringing field,

$$\psi'_{OM} = \psi_{pf} + \delta\psi_{pf} = \psi_{pf} + B_1 \delta\psi_{p1} = \psi_{pf} + B_1 \delta\psi_p + B_1 (\delta\psi_1 + \delta\psi_2)$$

and

$$\psi_M = \psi_f = \psi_{pf} - \delta\psi_p \quad .$$

Then,

$$\psi'_{OM} = \psi_M + \delta\psi_{01} \quad , \quad (11.37a)$$

with

$$\delta\psi_{01} = \delta\psi_0 + B_1(\delta\psi_1 + \delta\psi_2) = (B_1 + 1) \delta\psi_p + B_1(\delta\psi_1 + \delta\psi_2) \quad . \quad (11.37b)$$

Table 11.1

FRINGING-FIELD EFFECTS ON RESOLUTION IN SYMMETRICAL
SPECTROMETER WITH $K = 0.6$

ψ_s	φ	α_1	α_2	α_3	$\frac{(\delta z_f)_1}{z_s}$	$\frac{(\delta z_f)_2}{z_s}$	$\frac{(\delta z_f)_3}{z_s}$	G	$\frac{\delta p_1}{p}$	$\frac{\delta p_2}{p}$	$\frac{\delta p_3}{p}$
110°	3°	0.0606	0.0268	-0.104	0.00017	0.0000038	-0.000015	0.1572	-0.00053	-0.000012	0.000047
	10°				0.0018	0.00014	-0.00055		-0.0059	-0.00045	0.0018
130°	3°	-0.0322	0.1748	-0.197	-0.000088	0.000025	-0.000028	0.1211	0.00036	-0.00010	0.00012
	10°				-0.00098	0.00093	-0.00105		0.0041	-0.0038	0.0043
150°	3°	-0.0293	0.493	-0.432	-0.000080	0.000071	-0.000062	0.1062	0.00038	-0.00033	0.00029
	10°				-0.00089	0.0026	-0.0023		0.0042	-0.012	0.011

φ = gap angle.

$\frac{(\delta z_f)_i}{z_s}$ = relative shift of focal point between median and farthest off-median rays,
due to one of the effects discussed.

$\frac{\delta p_i}{p}$ = corresponding shift in apparent momentum.

$G = (\text{dispersion}) \times \frac{1}{2z_s}$

With this difference must be included the effects of the exit fringing field. The median and off-median rays are further separated angularly by the effects discussed in Eqs. (11.12c) and (11.19) [or the equivalent Eqs. (11.34b) and (11.34c)]. The sign of the separation is the same as at the input side, but the magnitude differs because the lens effect changes the separation of the ϕ -planes. If $\delta\psi_1$ and $\delta\psi_2$ are the differences at the entrance, then, at the exit the corresponding differences are $C_2 \delta\psi_1$ and $C_2 \delta\psi_2$, where C_2 is defined in Eq. (11.28). Then,

$$\psi_{OM} = \psi_M + \delta\psi_{02} \quad (11.38a)$$

$$\delta\psi_{02} = \delta\psi_{01} + C_2 (\delta\psi_1 + \delta\psi_2) = (B_1+1) \delta\psi_p + C_3 (\delta\psi_1 + \delta\psi_2) \quad . \quad (11.38b)$$

Equation (11.31) is still valid, with $\delta\psi_{02}$ substituted for $\delta\psi_0$ and with $\Delta\tau$ given by Eq. (11.36d).

Thus, from Eqs. (11.24) and (11.38b) and from (11.36c) and (11.36d), since $r = \rho \sin\psi_s$,

$$\frac{\rho}{\sin\psi_s} \delta\psi_{02} = (B_1+1) \frac{\rho}{r} \cos\mu \Delta\eta + \frac{C_3 \rho^2}{r} (\delta\psi_1 + \delta\psi_2)$$

and

$$\begin{aligned} \frac{\Delta\tau \cos\mu}{\sin\psi_s} &= C_3 \frac{\rho}{r} \cos\mu \Delta\eta + \Gamma \frac{\rho}{r} \cos\mu (\delta\psi_1 + \delta\psi_2) \\ &= C_3 \frac{\rho}{r} \cos\mu \Delta\eta + C_1 \frac{\rho^2}{r} (\delta\psi_1 + \delta\psi_2) \quad . \end{aligned}$$

Then,

$$\delta z_f = (1-C_2) \frac{\rho}{r} \cos\mu \Delta\eta + (1+C_2) (\delta\psi_1 + \delta\psi_2) \frac{\rho^2}{r} \quad ,$$

or

$$\delta z_f = \frac{1}{2} \frac{\rho}{K} \phi_s^2 \left\{ (1-C_2) \cos\mu + \frac{(1+C_2)}{\sin\psi_s} \left[K T(\Psi_s, \mu) + \frac{1}{2} \cos(\psi_s - \mu) \right] \Delta\psi \right\} \quad , \quad (11.39)$$

with

$$T(\Psi_s, \mu) = \cos^2\Psi_s + \cos\Psi_s \sin\Psi_s \tan\mu \quad .$$

It may be restated that this computation is a rough one, whose aim is simply to evaluate the magnitude of the perturbation of the resolution.

In addition to the previously mentioned approximations, such effects as the shift in the r -value of the trajectory between the input and exit points* have been completely neglected, as well as the difference between $\Delta\eta$ one gap width within the profile and the corresponding projection onto the profile. However, because the effect computed is small, these approximations lead only to second-order corrections which are negligible.

Effect of $\delta\psi_1$, $\delta\psi_2$ and $\Delta\eta$ on the Resolution. To translate this result into a resolution effect, we use the dispersion. Thus, from Eq. (6.6), we see that for a symmetric spectrometer

$$\frac{\delta p}{p} = -\frac{1}{G} \frac{\delta z_f}{2z_s} \quad , \quad (11.40)$$

where the form of $G(K, \psi_s)$ is given in Eq. (6.7), and numerical values in Fig. (6.4) and (6.4a) and in Table II.5 (Appendix II).

It is evident that the magnitude of the effect is quite dependent upon the spectrometer design, in particular, upon the values of $\tan\mu$, which affect the maximum ϕ_s or ϕ_f -values and the value of C_2 . Some numerical computations were made for $K = 0.6$, a K -value close to the optimum for certain applications (see Section #16), and specifically for a symmetrical spectrometer, for which tabulated values are available in the appendix for $(r_e, W_s, \tan\mu)$.

The assumptions made concerning the fringing-field structure were reasonable ones, but there is no information as to the magnitude of error introduced by these assumptions. Because of this uncertainty, the above computations must be considered as rough ones, almost order-of-magnitude estimates. Summation of the three effects, with possible cancellation of those of opposite sign, is then not justified. For this reason, they have been computed separately, and are presented in Table (11.1).

We have, then,

(A) From the $\Delta\eta$ -effect alone:

$$\frac{(\delta z_f)_1}{z_s} = \frac{1}{2K} \frac{\rho}{z_s} \phi_s^2 (1-C_2) \cos\mu = \alpha_1 \phi^2 \quad .$$

*For example, it has been assumed that $r_e = r_0$, that $\rho_e = \rho_0$, and that $\Delta\eta$ is symmetric at entrance and exit, so that only the modifying C_2 is involved.

(B) From the perpendicularity of lines of force, leading to a lower fringe field for the median rays:

$$\frac{(\delta z_f)_2}{z_s} = \frac{1}{2} \frac{\rho}{z_s} \phi_s^2 (1+C_2) \left[\frac{\cos^2 \psi_s}{\sin \psi_s} - \cos \psi_s \tan \mu \right] \Delta \psi = \alpha_2 \varphi^3 .$$

(C) From the $\Delta \eta$ -effect driving the median plane trajectory to relatively larger η -values, and hence to smaller or greater fringing field, depending upon the sign of $dr/d\eta$:

$$\frac{(\delta z_f)_3}{z_s} = \frac{1}{4K} \frac{\rho}{z_s} \phi_s^2 (1+C_2) \frac{\cos(\psi_s - \mu)}{\sin \psi_s} \Delta \psi = \alpha_3 \varphi^3 ,$$

where [from Eq. (9.12a)]

$$C_2 = \phi_f^2 / \phi_s^2 = 1/W_s^2 .$$

The computation is made for the farthest off-median ray, i.e., for the largest possible value of ϕ_s . This is $\frac{1}{2}\varphi$ for a convergent lens and $\frac{1}{2}\varphi/W_s$ for a divergent lens. Taking g_e/d_e empirically as approximately 0.6 in Eq. (8.7b),

$$\Delta \psi = \varphi / 1.2 K ;$$

hence the φ^3 -dependence in (B) and (C).

The apparent relative change in momentum is computed from Eq. (11.40), and is shown in the last three columns of the table.

The three ψ_s -values computed span most of the useful range of a symmetrical spectrometer using this K-value. Two conclusions may evidently be made from the tabulated results:

(i) For highest resolution, the effects discussed in this section are negligible when $\varphi = 3^\circ$, but must be taken into consideration when φ is as large as 10° .

(ii) Even when $\varphi = 10^\circ$, the effect is more serious at larger ψ_s -values, indicating that improvement may be made in resolution by excluding this part of the ψ_s -range. This is also the ψ_s -range in which the r_1 -projection of the source leads to the greatest loss of resolution.

#12. CORRECTION OF PROFILES FOR THE $\Delta\psi$ -EFFECT.

As pointed out in Section #8, profiles constructed according to the theory of Section #3 will not, in fact, result in focusing at $(z_f, 0)$, because of increased bending due to the fringing field.

The perturbation of focusing may, indeed, be quite severe unless the gap angle ρ is small. Lack of detailed information of the fringing-field structure makes it difficult to compute accurately the $\Delta\psi$ -effect prior to the construction of an instrument. A suitable approach is to design the profiles with the best available estimate of the fringing-field effect (as in Section #15), to construct the instrument, and then to modify the profiles with data based on detailed measurement of the focusing properties. Since the corrections are generally small, they are most easily computed through the use of differential coefficients which give the variation of the trajectories with variation of any of the important parameters, such as z_e , r_e , and K . Before the profile corrections are considered, therefore, a method is developed for evaluating some of the important differential relations. Some of these have been evaluated previously by a somewhat different technique.

Differential Characteristics of the Trajectories. Suppose all of the parameters of a trajectory are kept fixed, with the exception of two of them (e.g., z_f and K). Their differential variation may then be determined with

all other parameters kept constant; thus, $\left(\frac{\partial z_f}{\partial K}\right)_{z_s, z_e, r_e, \dots}$. It is convenient to use the notation of Eq. (9.7); using Ψ_s and Ψ_f , it is found that the

same equations often apply to the entrance and exit points, so that the sub-indices (s,f) may, in these cases, be dropped if desired. The relations of the angles are shown, for $\Psi_s < 90^\circ$, in Fig. (12.1), in which $b_s b_f$ is the entrance profile and $b_f b_f$ the exit profile. Similar angles may be drawn for $\Psi_s > 90^\circ$.

Fig. 12.1

Variations are considered in the neighborhood of (z_e, r_e) . Since only first-order differential variations are involved, the profile and trajectory curves may be replaced by their tangents, and rays which differ by a differential angle, by parallel lines.* The ψ -focusing effect of the fringing field is neglected, and use is made of the optics and notation of Section #1, 2, and 3.

Consider the trajectory with $\psi_s = \psi_s(z_e, r_e)$ [i.e., $\Psi_s = \Psi_s(z_e, r_e)$]. Any trajectory with $\psi_s < \psi_s(z_e, r_e)$ must travel through the magnetic field for some distance, increasing its ψ -value, before it reaches a point (z'_e, r'_e) at which $\psi = \psi_s(z_e, r_e)$. The envelope of the points (z'_e, r'_e) is the curve Γ_s [see Fig. (12.2)], which lies within the gap for $\Psi_s > \Psi_s(z_e, r_e)$. A similar curve exists on the exit side: for $\Psi_f > \Psi_f(z_0, r_0)$, Γ_f is the envelope of all points (z'_0, r'_0) at which the ψ -value on the trajectory equals $\psi_f(z_0, r_0)$; Γ_f lies within the gap for $\Psi_f > \Psi_f(z_0, r_0)$. Consideration of the Γ -curves is basic to the method to be used here for computing the differential coefficients.

Fig. 12.2

The various angles involved in the analysis are shown in Fig. (12.2) and defined in Table (12.1).

To determine the position of Γ_s , it is necessary to calculate the distance ds from profile to Γ_s , which is the distance along the trajectory in which a ray with initial angle $\Psi_s + d\Psi_s$ is bent through the angle $d\psi = -d\Psi_s$, so that its ψ -angle equals $\psi_s = \pi - \Psi_s$. Now, as a particle travels along its trajectory, $d\psi > 0$ and $ds > 0$. If ρ_C is the instantaneous radius of curvature, then from Eq. (8.4c) and (1.10):

$$ds = \rho_C d\psi = K r_e d\psi = K \rho_s \sin \Psi_s d\Psi_s, \quad (12.1a)$$

*Thus, $\theta + d\theta$ is replaceable by θ in any expression in which differences [e.g., $\sin(\theta + d\theta) - \sin\theta$] are not involved.

where $d\Psi_s$ is the increase in the Ψ_s -entrance angle and $d\psi$ the increase in ψ in going from $\psi_s + d\psi_s$ [$\psi_s(z_e, r_e)$] to $\psi_s(z_e, r_e)$; hence, $d\psi = d\Psi_s$. Note that $ds > 0$ going from the profile to Γ_s . On the exit side, ds is the distance along the trajectory from Γ_f to the profile, in which a ray with angle $\psi = \psi_f$ is bent through $d\psi = d\Psi_f$, so that the emergent angle is $\Psi_f + d\Psi_f$. As before,

$$ds = \rho_C d\psi = K r_0 d\psi = K \rho_f \sin\Psi_f d\Psi_f \quad . \quad (12.1b)$$

Here, ds is measured from Γ_f to the profile.

From Fig. (12.2)* and Table (12.1), for both exit and entrance,

$$\gamma = \Psi - \lambda \quad . \quad (12.2)$$

From Fig. (12.2),

$$\tan\lambda + \tan\mu = \frac{BM + MN}{P_s N} = \frac{ds}{\rho d\Psi} = K \sin\Psi$$

or

$$\tan\lambda = K \sin\Psi - \tan\mu \quad . \quad (12.3)$$

From Eq. (12.2),

$$\begin{aligned} \tan\gamma &= \frac{\tan\Psi - \tan\lambda}{1 + \tan\Psi \tan\lambda} = \frac{\tan^2\Psi + 1 - 1 - \tan\lambda \tan\Psi}{\tan\Psi (1 + \tan\Psi \tan\lambda)} \\ &= \frac{\sec^2\Psi}{\tan\Psi (1 + \tan\Psi \tan\lambda)} - \frac{1}{\tan\Psi} \\ &= \frac{1}{\sin^2\Psi [\cot\Psi + K \sin\Psi - \tan\mu]} - \cot\Psi \quad . \quad (12.4) \end{aligned}$$

Also,

$$\tan\mu = K \sin\Psi - \tan(\Psi - \gamma) \quad . \quad (12.5)$$

Since μ is generally made small, and K is often in the neighborhood of 0.6, from Eq. (12.3), $\lambda \cong K \sin\Psi - \mu$, so that, to an accuracy of the order of 10%, the relation

$$\gamma \cong \Psi - K \sin\Psi + \mu \quad (12.6)$$

may be used for rough computations.

*The geometric arguments following are illustrated in Fig. (12.2) and subsequent figures for particular orientations of profiles, Γ -curves, and Ψ -angles. The same results are true for other possible orientations.

The expression $(\cot\Psi + K\sin\Psi - \tan\mu)$ occurs frequently in the following discussion. It is convenient to define, therefore,

$$A_s = \cot\Psi_s + K\sin\Psi_s - \tan\mu_s \quad (12.7a)$$

$$A_f = \cot\Psi_f + K\sin\Psi_f - \tan\mu_f \quad (12.7b)$$

Then,

$$\tan\gamma_s = \frac{1}{A_s \sin^2\Psi_s} - \cot\Psi_s \quad (12.8a)$$

$$\tan\gamma_f = \frac{1}{A_f \sin^2\Psi_f} - \cot\Psi_f \quad (12.8b)$$

Here A_f is a quantity which has arisen in another form in the variation approach of Section #5 and #6. Thus, from Eq. (A-III.3a), (A-III.6) and (9.8a), it is evident that $(\Lambda_f - K\sin\psi_f)$, a term which first arose in Eq. (5.7b), obeys the relation

$$\begin{aligned} \Lambda_f - K\sin\psi_f &= -K\sin\psi_f + \cot\psi_f - \tan\mu_f \\ &= K\sin\Psi_f + \cot\Psi_f - \tan\mu_f = A_f \end{aligned} \quad (12.9a)$$

Similarly,

$$\Lambda_s - K\sin\Psi_s = A_s \quad (12.9b)$$

and, from Eq. (5.7b),

$$A_f = \frac{1}{F_3} \left[\frac{e^{-K\cos\psi_s}}{\psi_f^1 \sin^2\psi_s} - \frac{e^{-K\cos\psi_f}}{\sin^2\psi_f} \right] \quad (12.10a)$$

where, from Eq. (6.7b) and (3.2a),

$$F_3 = -\frac{(z_s - z_f)}{r_0} e^{-K\cos\psi_f} = -\frac{(z_s - z_f)}{r_e} e^{-K\cos\psi_s} \quad (12.10b)$$

For a symmetrical spectrometer,

$$A_s = A_f = \frac{r_e}{z_s} \frac{1}{\sin^2\psi_s} \quad (12.10c)$$

Equations (12.1) to (12.5) relate the various parameters either on the entrance side or on the exit side. It is necessary, however, to relate the quantities between both sides, and for this, reference is made to the equations of motion: Eq. (2.5). The entrance angle, Ψ_s , is varied, thereby

changing the coordinates (z'_e, r'_e) along Γ_s , but $\psi = \psi_s(z_e, r_e)$ is kept constant on Γ_s . The trajectory that intersects the Γ_s -curve at $(z'_e + dz'_e, r'_e + dr'_e)$ crosses the Γ_f -curve at $(z_0 + dz_0, r_0 + dr_0)$. Using the notation [in Eq. (2.5)] $\psi = \psi_s(z_e, r_e) = \psi_s$ = the ψ -value of the trajectory on the Γ_s curve, or $\psi = \psi_f(z_0, r_0) = \psi_f$ = the ψ -value of the trajectory on the Γ_f curve, while the entrance angle ψ_s is set equal to θ (since it is considered variable), then, from Eq. (2.5b),

$$z'_0 - z'_e = F z_s f_s(\theta) e^{K \cos \theta},$$

with F defined in Eq. (5.3c):

$$\log(z'_0 - z'_e) = [\log(F z_s)] + \log f_s(\theta) + K \cos \theta \quad (12.11a)$$

From Eq. (2.5a),

$$r'_e = z_s f_s(\theta) e^{-K \cos \psi_s} e^{K \cos \theta}; \quad r'_0 = z_s f_s(\theta) e^{-K \cos \psi_f} e^{K \cos \theta}$$

$$\log r'_e = [\log z_s - K \cos \psi_s] + \log f_s(\theta) + K \cos \theta \quad (12.11b)$$

$$\log r'_0 = [\log z_s - K \cos \psi_f] + \log f_s(\theta) + K \cos \theta \quad (12.11c)$$

Since the relations in Eq. (12.11) differ only by constant quantities, the logarithmic derivatives are equal:

$$\frac{d(z'_0 - z'_e)}{z'_0 - z'_e} = \frac{dr'_e}{r'_e} = \frac{dr'_0}{r'_0} \quad (12.12)$$

From Table (12.1), the horizontal displacement along the Γ -curves is, evidently [see also Fig. (12.2)],

$$d(z'_e - z'_0) = dz'_e - dz'_0 = dr'_e \tan \gamma_s + dr'_0 \tan \gamma_f$$

$$= \left[\frac{r'_e}{r'_0} \tan \gamma_s + \tan \gamma_f \right] dr'_0 = \frac{z'_e - z'_0}{r'_0} dr'_0 \quad (12.13)$$

Then:

$$z'_e - z'_0 = r'_e \tan \gamma_s + r'_0 \tan \gamma_f \quad (12.14)$$

Given Ψ_s and μ_s , the quantity γ_s may be calculated through Eq. (12.4) or (12.5); from Eq. (12.14), γ_f is computed and thus μ_f ; for an approximate value, Eq. (12.6) may be employed. Geometrical methods may also be used to determine μ from γ approximately, as well as μ_f from μ_s , by constructing the profiles and measuring the angles, as in Fig. (12.2).

It may be noted that in a symmetrical instrument

$$r_e' \tan \gamma_s = r_0' \tan \gamma_f = \frac{1}{2} (z_e' - z_0') \quad . \quad (12.15)$$

Having fixed the Γ -lines through evaluation of (γ_s, γ_f) , the differential characteristics can be readily determined.

Angular Magnification. $\Psi' = \frac{d\Psi_f}{d\Psi_s} \quad .$

$d\Psi_s$ is a variation in the emission angle and $d\Psi_f$ the corresponding variation in the exit angle. From Fig. (12.3a),

$$AB = AQ_s + Q_s B = \Delta r_e' [\cot \Psi_s + \tan \gamma_s] = P_s F = \frac{r_e d\Psi_s}{\sin^2 \Psi_s}$$

and

$$d\Psi_s = \frac{\Delta r_e'}{r_e} (\cot \Psi_s + \tan \gamma_s) \sin^2 \Psi_s \quad . \quad (12.16)$$

The quantity*

$$r_e \frac{d\Psi_s}{dr_e'} = (\cot \Psi_s + \tan \gamma_s) \sin^2 \Psi_s \quad (12.17a)$$

may be calculated directly by substituting for $\tan \gamma_s$ from Eq. (12.4), or it may be deduced from Fig. (12.3a). Using Eq. (12.3),

$$\begin{aligned} BF &= BM + MF = P_s M (\tan \lambda_s + \cot \Psi_s) \\ &= \frac{r_e d\Psi_s}{\sin \Psi_s} (\cot \Psi_s + K \sin \Psi_s - \tan \mu_s) = \frac{\Delta r_e'}{\sin \Psi_s} = \frac{r_e d\Psi_s}{\sin \Psi_s} A_s \quad . \end{aligned}$$

Then,

$$r_e \frac{d\Psi_s}{dr_e'} = \frac{1}{A_s} \quad . \quad (12.17b)$$

*Note that $dr_e'/d\Psi_s$ is not directly related to the derivative in Eq. (A-III.5), $[dr_e/d\Psi_s]$, since dr_e is measured along the profile, while dr_e' is measured along the Γ_s -curve.

where, from Eq. (9.8a),

$$\frac{d(\rho_f \tan \mu_f)}{d(\rho_s \tan \mu_s)} = \frac{d\left(\frac{d\rho_f}{d\Psi_f}\right)}{d\left(\frac{d\rho_s}{d\Psi_s}\right)} \quad (12.22b)$$

It may be noted that these derivatives are always negative.* For the symmetrical spectrometer, the derivatives are equal to -1. It should be noted that rotation of the profile around (z_e, r_e) leads to a simple differential rotation only near this point.

Magnetic Path Magnification. ds_f/ds_s . The entrance curve $b_s b_s$ is translated parallel to itself and along the direction of Ψ_s , so that P_s goes into P'_s [see Fig. (12.4)], and $b_s b_s$ goes into the parallel line $b'_s b'_s$. Also, Γ_s goes into Γ'_s , which is parallel** to Γ_s .

Fig. 12.4

*This implies that when the entrance profile is rotated toward the source (positive $d\mu_s$), the exit profile is rotated away from the image (negative $d\mu_f$).

**Because the field values involved for the trajectories with the new profile $b'_s b'_s$ are smaller by an amount of the order of dr/r_e^2 than for the corresponding points on trajectories with the old profile, Γ'_s is not quite parallel to Γ_s , its inclination differing by a first-order term from that of Γ_s . As pointed out above, however, unless differences are involved, the two lines may be considered as parallel.

Two trajectories are considered: (i) "a" starts at $\Psi = \Psi_s$, with the new profile, arriving at P'_s ; (ii) "b" starts at $\Psi_s + d\Psi_s$, with the old profile, and arrives at B. At P'_s and B, respectively, the two trajectories have the same r -values and ψ -values; since both are within the field, the trajectories from these two points on are identical, except for the translation $\Delta z' = P'_s B$. Further, "b", on reaching the exit side, hits Γ_f at C, at which point, of course, $\psi = \psi_f$; it emerges from the profile $b_f b'_f$ with the exit angle $\Psi_f + d\Psi_f$. Because both "b" and "a" are similar within the field, with only a Δz -displacement differing between them, "a" arrives at $r = r_0 + \Delta r'_0$ (the point G) having $\psi = \psi_f$, i.e., the same that "b" had at C. Since this is the exit point for "a", the new exit profile $b'_f b'_f$ passes through G, but is not shown in Fig. (12.4). Also, H is the intersection of "a" with Γ_f .

In Fig. (12.5a) and (12.5b),

$$\frac{HG}{P'_f D} = \frac{GC}{CD} \quad ;$$

$$\frac{CD}{P'_s B} = \frac{\frac{r_0 d\Psi_f}{\sin^2 \Psi_f}}{\frac{r_e d\Psi_s}{\sin^2 \Psi_s}} = \Psi' \frac{r_0}{r_e} \frac{\sin^2 \Psi_s}{\sin^2 \Psi_f} \quad ;$$

$$P'_f D = \frac{P_f Q_f}{\sin \Psi_f} = \frac{\Delta r'_0}{\sin \Psi_f} \quad ;$$

$$GC = P'_s B = \Delta z' \quad ;$$

and

$$\frac{ds_f}{ds_s} = \frac{HG}{P'_s P'_s} = \frac{P'_f D \frac{P'_s B}{CD}}{\frac{\Delta r'_e}{\sin \Psi_s}} = \frac{\frac{r_0}{\Delta r'_e}}{\frac{r_e}{\sin \Psi_s}} \frac{1}{\Psi'} \frac{\sin \Psi_f}{\sin \Psi_s} \quad .$$

From Eq. (12.8)*

$$\frac{ds_f}{ds_s} = \frac{1}{\Psi'} \frac{\sin \Psi_f}{\sin \Psi_s} \quad . \quad (12.23)$$

*The argument based upon Fig. (12.5) develops the case in which $ds_s > 0$. It is evident that for $ds_s < 0$ a reverse displacement from $b'_s b'_s$ to $b_s b_s$ may be made, and the relationship (12.23) must still hold.

Fig. 12.5a

Fig. 12.5b

It is evident that we have defined ds_s as the distance along the trajectory by which Γ_s is displaced to Γ'_s , and that ds_f is the distance along the trajectory by which Γ_f is displaced to Γ'_f .

For a symmetrical spectrometer, Eq. (12.23) indicates that $ds_f/ds_s = 1$. This is also evident from Fig. (12.5), since (from symmetry) $r_e = r_0$; further, $CD = P'_s B = GC$ and $HG = P_f D = P_s P'_s$.

Entrance (or Exit) Curve Correction for Axial Source (or Focus) Displacement. If the source point is displaced from $(z_s, 0)$ by an amount δz_s , then the entrance angle of the ray entering at (z_e, r_e) will be varied by the amount $\partial \Psi_s / \partial z_s \delta z_s = \delta \Psi_s$. From Fig. (12.2), since a positive z_s -displacement corresponds to $\delta \Psi_s < 0$,

$$\rho_s \delta \Psi_s = -\sin \Psi_s \delta z_s \quad (12.24a)$$

Hence,

$$\delta z_s = -\frac{\rho_s}{\sin \Psi_s} \delta \Psi_s \quad (12.24b)$$

If the trajectories within the field are to be kept the same, then the ψ -value at (z_e, r_e) must be restored to its pre-displacement value by adding (or subtracting)* a region containing magnetic field, so as to produce a deflection $-\delta \Psi_s$ between the displaced profile and (z_e, r_e) . Since ds is positive along the trajectory, when part of the magnetic field region is removed, the displacement of the profile ds_s is positive. But removal of

*Add region of magnetic field if $\delta \Psi_s$ is positive (i.e., $\delta \psi_s$ and δz_s negative); subtract field region for $\delta \Psi_s < 0$. Within the field, the angular shift $\delta \Psi$ is always negative.

magnetic field region is associated with negative $\delta\Psi_s$. Hence, from Eq. (1.10),

$$ds_s = - \delta\Psi_s \rho_C = - \delta\Psi_s Kr_e ,$$

where ρ_C is the instantaneous radius of curvature of trajectory near (z_e, r_e) . Then

$$\frac{ds_s}{dz_s} = K \sin^2 \Psi_s . \quad (12.25)$$

At the exit side, a positive δz_f corresponds to $\delta\Psi_f > 0$, so that

$$\delta z_f = \frac{\rho_f}{\sin \Psi_f} \delta\Psi_f . \quad (12.26a)$$

Removal of field region corresponds to negative ds_f , negative $\delta\psi_f$, and hence negative $\delta\Psi_f$. Then,

$$ds_f = \delta\Psi_f \rho_C = \delta\Psi_f Kr_0 . \quad (12.26b)$$

Hence,

$$\frac{ds_f}{dz_f} = K \sin^2 \Psi_f . \quad (12.27)$$

It may be noted that a displacement δz_s (or δz_f) also results in a change of ρ_s (or ρ_f) due both to the shift in the profile and to the change in the center from which ρ is measured. Now,

$$\frac{d\rho_s}{dz_s} = \frac{\partial \rho_s}{\partial r_e} \frac{dr_e}{dz_s} + \frac{\partial \rho_s}{\partial \Psi_s} \frac{d\Psi_s}{dz_s} .$$

Then, from Eq. (A-III.4) and (12.24b),

$$\frac{d\rho_s}{dz_s} = \frac{1}{\sin \Psi_s} \frac{ds_s \sin \Psi_s}{dz_s} - \frac{r_e \cos \Psi_s}{\sin^2 \Psi_s} \frac{-\sin \Psi_s}{\rho_s}$$

and

$$d\rho_s = ds_s + \cos \Psi_s dz_s . \quad (12.28a)$$

In effect, ρ_s is stretched (and/or compressed) at each end, due to the shift of the profile and the origin of the vector. At the exit side,

$$\frac{d\rho_f}{dz_f} = \frac{1}{\sin\Psi_f} \frac{-ds_f \sin\Psi_f}{dz_f} - \frac{r_0 \cos\Psi_f}{\sin^2\Psi_f} \frac{\sin\Psi_f}{\rho_f},$$

since dr_0/ds_f is negative ($ds_f > 0$ in the direction of the trajectory) and $ds_f/dz_f > 0$ from Eq. (12.27). Then,

$$d\rho_f = -ds_f - \cos\Psi_f dz_f \quad (12.28b)$$

Equations (12.25) and (12.27) may be used to adjust the profiles of an instrument which has already been constructed. Suppose it is found experimentally that for different entrance angles: $\Psi_{s1}, \Psi_{s2}, \Psi_{s3}, \dots$ the trajectories cross the z -axis at $z_{f1}, z_{f2}, z_{f3}, \dots$, which differ in the quantity

$$\Delta z_i = z_{fi} - z_f \quad (12.29a)$$

from the desired Ψ_s -independent focus point $(z_f, 0)$. If the quantities are small, the exit curve may be corrected by the amount Δs_c at each measured angle:*

$$(\Delta s_c)_i = -\frac{ds_f}{dz_f} \Delta z_i = -K \sin^2\Psi_f \Delta z_i \quad (12.29b)$$

Axial Magnification: dz_f/dz_s . This quantity may be determined by first considering profile displacements necessary to keep a focus at $(z_f, 0)$ when the source is displaced by dz_s , and then determining the image displacement required to cancel the profile displacements.

Consider profiles which, at fixed p and value of K , focus from a source at $(z_s, 0)$ to an image at $(z_f, 0)$. Displacements of the profiles from the initial positions are described by s_s and s_f . Movements of either or both source and image points result in changes of s_s and/or s_f . The focusing condition requires that there be a relation

$$M(s_s, s_f, z_s, z_f) = 0 \quad (12.30)$$

Using the notation

$$M_{z_s} = \left(\frac{\partial M}{\partial z_s} \right)_{s_s, s_f, z_f} \quad (12.31a)$$

*It may be noted that Eq. (12.29a) is the negative of the usual variation, since $\Delta z_i = (\text{initial } z\text{-value}) - (\text{final } z\text{-value})$, and hence the negative sign in Eq. (12.29b).

external subscript quantities being held constant,

$$A = \left(\frac{\partial s_f}{\partial z_f} \right)_{z_s, s_s} = -\frac{M_{z_f}}{M_{s_f}}; \quad B = \left(\frac{\partial s_f}{\partial s_s} \right)_{z_s, z_f} = -\frac{M_{s_s}}{M_{s_f}};$$

$$\text{and } C = \left(\frac{\partial s_s}{\partial z_s} \right)_{z_f, s_f} = -\frac{M_{z_s}}{M_{s_s}}. \quad (12.31b)$$

Then

$$\frac{BC}{A} = -\frac{M_{z_s}}{M_{z_f}} = \left(\frac{\partial z_s}{\partial z_f} \right)_{s_s, s_f}. \quad (12.31c)$$

It may be noted that the differential coefficients in Eqs. (12.23), (12.25) and (12.27) are the partial derivatives of Eq. (12.31b). Hence, the differential coefficient desired is given by Eq. (12.31c), and

$$\frac{dz_f}{dz_s} = \frac{1}{\Psi'} \frac{\sin \Psi_s}{\sin \Psi_f} = Q_f, \quad (12.32)$$

which is the same as Eq. (5.10).

In the symmetrical instrument, the coefficient is 1.

Radial Magnification: dr_{1f}/dr_1 . This quantity is readily determined from the axial magnification. The r_1 -extensions dr_{1f} and dr_1 are related to equivalent "apparent" dz_f - and dz_s -extensions by projection:

$$dr_{1f} = -dz_f \tan \Psi_f; \quad dr_1 = dz_s \tan \Psi_s. \quad (12.33)$$

Then, from Eq. (12.32),

$$\frac{dr_{1f}}{dr_1} = -\frac{1}{\Psi'} \frac{\cos \Psi_s}{\cos \Psi_f} = u(\psi_s), \quad (12.34)$$

which is the same as Eq. (5.15b).

Exit-curve Corrections from Relative Field Measurements. In place of measurements of the quantities z_i for use in computing the profile corrections through Eq. (12.29b), it is possible to use measurements of the fields required to focus particles into $(z_f, 0)$. Thus, the field intensity factor $(A_0)_i$

is measured for each angle Ψ_i in the uncorrected instrument. In general, the interesting quantities are only the relative deviations

$$\frac{\Delta A_{0i}}{A_0} = \frac{A_{0i} - A_0}{A_0} \quad (12.35)$$

from a chosen standard value A_0 . Hence, any relative measure of the field intensity can be used, and this is actually what the usual field-measuring system of a spectrometer provides.

The exit curve correction will be*

$$(\Delta s_f)_c = - \frac{ds_f}{\frac{dA_0}{A_0}} \frac{\Delta A_0}{A_0}, \quad (12.36)$$

where ds_f is the distance the exit profile must be moved (along the trajectory) to continue focusing after a variation dA_0 in the specific field intensity. Equation (12.34) may also be used to compute the precision with which it is necessary to calculate and construct the profile curves. From Eq. (1.10),

$$d(\log K) = \frac{dK}{K} = - \frac{dA_0}{A_0} = - d(\log A_0) \quad (12.37a)$$

Then,

$$\frac{ds_f}{d(\log A_0)} = - \frac{ds_f}{d(\log K)} \quad (12.37b)$$

Consider the trajectory which passes through (z_e, r_e) and (z_0, r_0) in Fig. (12.6).

Fig. 12.6

*The negative sign arises for the same reason as in Eq. (12.29b) i.e., due to the order of terms in Eq. (12.35).

After a change in magnetic field intensity (measured by dK), the point on the trajectory at which $\psi = \psi_f$ [i.e., the original $\psi(z_0, r_0)$] is displaced from (z_0, r_0) by $(dz_0^1 K, dr_0^1 K)$ to the point S. If this point lay on the original Γ_f -curve, the new trajectory would still focus at $(z_f, 0)$.* However, this is not true in general; the point is displaced** from Γ_f by an amount ds_f measured along the trajectory. Hence, it is necessary to add (or subtract) this distance to the path within the magnetic field, in order to get focusing at $(z_f, 0)$. From Fig. (12.6),

$$ds_f = ST = TR / \sin \Psi_f \quad .$$

Now, $d\bar{z} = SQ = SR + RQ = TR (\cot \Psi_f + \tan \gamma_f)$. Then, from Eq. (12.18),

$$ds_f = \frac{d\bar{z}}{(\cot \Psi_f + \tan \gamma_f) \sin \Psi_f} = (d\bar{z} \sin \Psi_f) A_f \quad (12.38)$$

and

$$d\bar{z} = SN - QN = -dz_0^1 K - dr_0^1 K \tan \gamma_f \quad . \quad (12.39)$$

Rewriting the equations of motion in Eq. (12.11), and setting θ back to ψ_s ,

$$\log(z_0^1 - z_e^1) = \log F + \log [z_s f_s(\psi_s)] + K \cos \psi_s \quad (12.40a)$$

and

$$\log r_0^1 = \log [z_s f_s(\psi_s)] + K(\cos \psi_s - \cos \psi_f) \quad . \quad (12.40b)$$

Differentiating, keeping ψ_s and ψ_f constant, and evaluating at (z_0, r_0) ,

$$\frac{dr_0^1}{r_0} = (\cos \psi_s - \cos \psi_f) dK \quad (12.41a)$$

and

$$\frac{d(z_0^1 - z_e^1)}{z_0 - z_e} = \left(\cos \psi_s + \frac{F'}{F} \right) dK \quad , \quad (12.41b)$$

where F is defined in Eq. (5.3c) and $F' = \partial F / \partial K$ is given in Eq. (6.8).

*Since the differential shift in field, operating over a differential distance, would lead only to a second-order shift in focus position.

**The point S may or may not lie within the field defined by the original profile. If it does not, then the path through the magnetic field must be increased in length, since the new Γ_f^1 -curve must lie within the profile boundary.

It may be noted that dz_0/K and dr_0/K are taken with ψ_f constant, as well as the initial angle ψ_s ; since this also applies to the increments in Eq. (12.41), the two variations are respectively equal. Since the entry point is not varied (ψ_s constant), $d(z_0' - z_e') = dz_0'$. Hence from Eq. (12.39) and (12.41),

$$d\Sigma = \left\{ \cos\psi_s [(z_e - z_0) - r_0 \tan\gamma_f] + r_0 \cos\psi_f \tan\gamma_f + (F'/F)(z_e - z_0) \right\} dK.$$

Applying Eq. (12.14) and transforming from ψ to Ψ :

$$d\Sigma = \left\{ (z_e - z_0)(F'/F) - [r_e \tan\gamma_s \cos\Psi_s + r_0 \tan\gamma_f \cos\Psi_f] \right\} dK. \quad (12.42)$$

Introducing Eq. (12.42) into (12.39),

$$\begin{aligned} \frac{ds_f}{d(\log K)} &= K \sin\Psi_f \left[(z_e - z_0) \frac{F'}{F} - r_e \tan\gamma_s \cos\Psi_s + r_0 \tan\gamma_f \cos\Psi_f \right] A_f \\ &= A_f K \sin\Psi_f \frac{d\Sigma}{dK}. \end{aligned} \quad (12.43)$$

For a symmetric instrument, it is evident from Eq. (12.14), that*

$$\frac{d\Sigma}{dK} = (z_e - z_0) \left[\frac{F'}{F} - \cos\Psi_s \right]. \quad (12.44)$$

Entrance-curve Corrections from Relative Field Measurements.

This correction is derivable from the result for the exit profile. If z_s and z_f are held constant, then there must be the relation

$$Q(K, s_s, s_f) = 0. \quad (12.45)$$

$$\text{As above, } \left(\frac{\partial s_f}{\partial K} \right)_{s_s} = - \frac{Q_K}{Q_{s_f}}; \quad \left(\frac{\partial s_s}{\partial K} \right)_{s_f} = - \frac{Q_K}{Q_{s_s}}.$$

Then,

$$\left(\frac{\partial s_f}{\partial s_s} \right)_K = - \frac{Q_{s_s}}{Q_{s_f}} = - \frac{\left(\frac{\partial s_f}{\partial K} \right)_{s_s}}{\left(\frac{\partial s_s}{\partial K} \right)_{s_f}} = - \frac{\left(\frac{\partial s_f}{\partial(\log K)} \right)_{s_s}}{\left(\frac{\partial s_s}{\partial(\log K)} \right)_{s_f}}. \quad (12.46)$$

* See also Eq. (A-V.8) in Appendix V.

Now,

$$(\Delta s_s)_c = \left(\frac{\partial s_s}{\partial (\log K)} \right)_{s_f} \frac{\Delta K}{K}; (\Delta s_f)_c = \left(\frac{\partial s_f}{\partial (\log K)} \right)_{s_s} \frac{\Delta K}{K} \quad (12.47)$$

Hence, from Eq. (12.46),

$$\frac{(\Delta s_s)_c}{(\Delta s_f)_c} = - \left(\frac{\partial s_s}{\partial s_f} \right)_K$$

or

$$(\Delta s_s)_c = - \left(\frac{\partial s_s}{\partial s_f} \right)_K (\Delta s_f)_c \quad (12.48)$$

Also, from Eq. (12.37a) and (12.47),

$$(\Delta s_s)_c = - \left(\frac{\partial s_s}{\partial (\log K)} \right) \frac{dA_0}{A_0} \quad (12.49)$$

This differential coefficient may be computed from Eqs. (12.17), (12.19), (12.23) and (12.46). Thus,

$$\begin{aligned} \frac{ds_s}{d(\log K)} &= - \frac{K}{\sin \Psi_s} \frac{1}{(\cot \Psi_s + \tan \gamma_s)} \frac{d\Sigma}{\Delta K} \\ &= - A_s K \sin \Psi_s \frac{d\Sigma}{\Delta K} \end{aligned} \quad (12.50)$$

Dispersion: $dz_f/d(\log A_0)$. If z_s , and s_s are held constant, then between the quantities K , s_f , and z_f there must be a relation

$$R(K, s_f, z_f) = 0 \quad (12.51a)$$

Hence,

$$\left(\frac{\partial z_f}{\partial K} \right)_{s_f} = - \frac{R_K}{R_{z_f}} = - \frac{\frac{R_K}{R_{s_f}}}{\frac{R_{z_f}}{R_{s_f}}} = - \left(\frac{\partial s_f}{\partial K} \right)_{z_f} \left(\frac{\partial z_f}{\partial s_f} \right)_K \quad (12.51b)$$

Then,

$$dz_f = \left(\frac{\partial z_f}{\partial(\log K)} \right)_{s_f} \frac{dK}{K} = - \left\{ \frac{ds_f}{d(\log K)} \frac{dz_f}{ds_f} \right\} \frac{dK}{K}, \quad (12.51c)$$

where the differential coefficients are given in Eqs. (12.27) and (12.38) or (12.43). As in Eq. (6.10a), the dispersion is given by

$$\begin{aligned} \sigma_f &= \frac{dz_f}{d(\log A_0)} = - \frac{dz_f}{d(\log K)} = \frac{1}{K \sin^2 \Psi_f} \frac{ds_f}{d(\log K)} \\ &= \frac{A_f}{\sin^3 \Psi_f} \frac{d\Sigma}{dK} = \frac{1}{\sin^3 \Psi_f} \frac{1}{(\cot \Psi_f + \tan \gamma_f)} \frac{d\Sigma}{dK}. \end{aligned} \quad (12.52)$$

In Appendix V it is shown that Eq. (12.52) is equivalent to Eq. (6.6). The difference in the form of the two expressions for σ_f arises from the difference in the nature of the variations considered. In Section #6, variations were taken only along the exit profile. In this section, on the other hand, variation of the profile position with K was considered, as well as the variation of the image position necessary to cancel the profile movement. The same difference in the variations taken was involved in the identical pair of equations Eqs. (A-III.6) and (12.19), as well as in the pair Eqs. (5.10) and (12.32), and the pair (5.15b) and (12.34).

Source Dispersion (Effect of Source Movement). If the image is to be kept at $(z_f, 0)$ during a variation of K , then the source is moved a distance

$$dz_s = - \sigma_s \frac{dK}{K} = \sigma_s \frac{dA_0}{A_0}.$$

As in Eq. (12.47b),*

$$- \left(\frac{\partial z_s}{\partial(\log K)} \right)_{s_s} = \left(\frac{\partial s_s}{\partial(\log K)} \right)_{z_s} \left(\frac{\partial z_s}{\partial s_s} \right)_K. \quad (12.53)$$

*Equation (12.54) may also be computed using the similar relation

$$- \left(\frac{\partial z_s}{\partial(\log K)} \right)_{z_f} = \left(\frac{\partial z_s}{\partial z_f} \right)_K \left(\frac{\partial z_f}{\partial(\log K)} \right)_{z_s}$$

so that, from Eqs. (12.19), (12.32) and (12.50),

$$\sigma_s = \left(\Psi' \frac{\sin \Psi_f}{\sin \Psi_s} \right) \left(- \frac{A_f}{\sin \Psi_f} \right) \frac{d\Sigma}{\Delta K} = - \frac{A_s}{\sin \Psi_s} \frac{d\Sigma}{\Delta K}$$

Then, from Eqs. (12.21) and (12.46),

$$\sigma_s = - \frac{dz_s}{d(\log K)} = - \frac{A_s}{\sin \Psi_s} \frac{d\Sigma}{\Delta K} \quad (12.54)$$

Correction of a Constructed Instrument. It is convenient to express the corrections necessary to apply to an instrument with defocusing dA_0/A_0 in terms of the dispersion coefficients. Thus, from Eqs. (12.51b), (12.52) and (12.27),

$$\begin{aligned} \left(\frac{\partial(\log A_0)}{\partial s_f} \right)_{z_f} &= - \left(\frac{\partial(\log K)}{\partial s_f} \right)_{z_f} = \frac{- \left(\frac{\partial z_f}{\partial s_f} \right)_K}{- \left(\frac{\partial z_f}{\partial(\log K)} \right)_{s_f}} \\ &= - \frac{1}{\sigma_f K \sin^2 \Psi_f} \end{aligned} \quad (12.55)$$

Similarly, from Eqs. (12.53), (12.54) and (12.25),

$$\left(\frac{\partial(\log A_0)}{\partial s_s} \right)_{z_s} = - \frac{1}{\sigma_s K \sin^2 \Psi_s} \quad (12.56)$$

Also,

$$\left(\frac{\partial(\log A_0)}{\partial z_f} \right)_{s_f} = - \frac{1}{\sigma_f} \quad \text{and} \quad \left(\frac{\partial(\log A_0)}{\partial z_s} \right)_{s_s} = - \frac{1}{\sigma_s} \quad (12.57)$$

Corrections may be made for the defocusing by displacing the exit curve by the amount $(\Delta s_f)_c$ at each Ψ_s . Thus,

$$\frac{dA_0}{A_0} = - \frac{(\Delta s_f)_c}{\sigma_f K \sin^2 \Psi_f} \quad (12.58)$$

It is evident, however, from Eqs. (12.56) and (12.57), that other changes may be made which are equivalent to $(\Delta s_f)_c$. Thus, the relative defocusing dA_0/A_0 can be corrected by combining a correction at the exit profile, a correction at the entrance profile, a movement of the source point or the image point, and a change in the operating A_0 -value of the instrument.

Then,

$$\frac{dA_0}{A_0} = - \left[\frac{\Delta s_f(\Psi_f)}{\sigma_f K \sin^2 \Psi_f} + \frac{\Delta s_s(\Psi_s)}{\sigma_s K \sin^2 \Psi_s} + \frac{\Delta z_f}{\sigma_f} + \frac{\Delta z_s}{\sigma_s} \right] = C_0 \quad . \quad (12.59a)$$

With the corrections involved in C_0 , it is possible to change all of the trajectories so that they focus at the image point, using the operating point A_0 . In addition to these corrections, it is possible to shift the operating point from A_0 by the amount ΔA_0 . Then

$$\frac{dA_0}{A_0} = \frac{\Delta A_0}{A_0} + C_0 \quad . \quad (12.59b)$$

It is possible to choose the most convenient set of values (Δs_f , Δs_s , Δz_f , Δz_s , and ΔA_0) which fit Eq. (12.59). Only one value can be given to each of Δz_f , Δz_s , and ΔA_0 , and these will be chosen in such a manner as to minimize the values of Δs_f and Δs_s , which will, of course, vary with Ψ_s .

Table (12.1)

	Angle	Description	Angle is positive when:
μ_s	IA	Between line PM normal to input ray and profile (in same ϕ -plane)	Rotation from normal PM to profile PN is clockwise
	IB	Between line normal to (and away from) profile and the input ray (in same ϕ -plane)	Rotation from normal to input ray is counterclockwise
γ_s	IIA	Between positive r-axis (in same ϕ -plane) and Γ_s	Rotation from r-axis to Γ_s is clockwise
	IIB	$\gamma_s = \frac{1}{2}\pi - \alpha_s$ (α_s = inclination angle of Γ_s)*	$\alpha_s < \frac{1}{2}\pi$
	IIC	$\tan \gamma_s = \frac{dz'_e}{dr'_e}$	$\frac{dz'_e}{dr'_e} > 0$ for $ \gamma_s < \pi/2$ $\frac{dz'_e}{dr'_e} < 0$ for $ \gamma_s > \pi/2$
λ_s	IIIA	Between Γ_s and normal to input ray	Rotation from Γ_s to normal is clockwise
	IIIB	$\gamma_s = \Psi_s - \lambda_s$	$\Psi_s > \gamma_s$
μ_f	IA'	Between line normal to output ray and profile	Rotation from normal to profile is counterclockwise
	IB'	Between line normal to profile and output ray	Rotation from normal to output ray is clockwise
γ_f	IIA'	Between positive r-axis and Γ_f	Rotation from r-axis to Γ_f is counterclockwise
	IIB'	$\gamma_f = \alpha_f - \frac{1}{2}\pi$	$\alpha_f > \frac{1}{2}\pi$
	IIC'	$\tan \gamma_f = -\frac{dz'_o}{dr'_o}$	$\frac{dz'_o}{dr'_o} < 0$ for $ \gamma_f < \pi/2$ $\frac{dz'_o}{dr'_o} > 0$ for $ \gamma_f > \pi/2$
λ_f	IIIA'	Between Γ_f and normal to output ray	Rotation from Γ_f to normal counterclockwise
	IIIB'	$\gamma_f = \Psi_f - \lambda_f$	$\Psi_f > \gamma_f$

*The direction of Γ_s is from (z_e, r_e) to larger Ψ_s -values than $\Psi_s(z_e, r_e)$. For $\Psi_s > 90^\circ$, Γ_s may be directed down and to the right [using the same orientations as Fig. (12.2)], in which case $\alpha_s < 0$.

Table 12.2

SUMMARY OF DIFFERENTIAL COEFFICIENTS

Angular magnification	Equation 12.19	$\Psi' = \frac{d\Psi_f}{d\Psi_s} = \frac{A_s}{A_f}$
Rotation	12.22a	$\frac{d(\rho_f \tan \mu_f)}{d(\rho_s \tan \mu_s)} = - \frac{1}{(\Psi')^2} \frac{\sin \Psi_f}{\sin \Psi_s}$
Magnetic path magnification	12.23	$\frac{ds_f}{ds_s} = \frac{1}{\Psi'} \frac{\sin \Psi_f}{\sin \Psi_s}$
Profile movement with shift of source or focus	12.25	$\frac{ds_s}{dz_s} = K \sin^2 \Psi_s$
	12.27	$\frac{ds_f}{dz_f} = K \sin^2 \Psi_f$
Axial magnification	12.32	$\frac{dz_f}{dz_s} = \frac{1}{\Psi'} \frac{\sin \Psi_s}{\sin \Psi_f}$
Radial magnification	12.34	$\frac{dr_{lf}}{dr_l} = - \frac{1}{\Psi'} \frac{\cos \Psi_s}{\cos \Psi_f}$
Profile movement with shift of A_0	12.43	$\frac{ds_f}{d(\log K)} = A_f K \sin \Psi_f \frac{d\Sigma}{dK}$
	12.50	$\frac{ds_s}{d(\log K)} = - A_s K \sin \Psi_s \frac{d\Sigma}{dK}$
Dispersion	12.52	$\sigma_f = - \frac{dz_f}{d(\log K)} = \frac{A_f}{\sin \Psi_f} \frac{d\Sigma}{dK}$
	12.54	$\sigma_s = - \frac{dz_s}{d(\log K)} = \frac{-A_s}{\sin \Psi_s} \frac{d\Sigma}{dK}$
Correction of constructed instrument	12.55	$\frac{d(\log A_0)}{ds_f} = - \frac{1}{\sigma_f K \sin^2 \Psi_f}$
	12.56	$\frac{d(\log A_0)}{ds_s} = - \frac{1}{\sigma_s K \sin^2 \Psi_s}$
	12.57	$\frac{d(\log A_0)}{dz_f} = - \frac{1}{\sigma_f}$
	12.57	$\frac{d(\log A_0)}{dz_s} = - \frac{1}{\sigma_s}$

#13. MEDIAN-PLANE TRAJECTORIES IN THE FRINGING FIELD

Some of the considerations concerning the median-plane trajectories were introduced in Section #8. The basic deflection Eq. (8.5) was calculated and evaluated on the median plane, using an approximate form for the fringing field on this plane [see Eq. (8.7)]. In this section the trajectories in the fringing field are examined in greater detail. Because of the finite extent of the fringing fields, it is necessary to consider the relationship of the trajectory to the various portions of this field - its external boundary and its internal boundary - as well as to the profile itself. Following is a list of the definitions of various points and distances relative to a trajectory which starts from $(z_s, 0)$ at the angle ψ_s [see Figs. (8.4) and (13.1)]:

The points:

(z_{1e}, r_{1e})	the point on the trajectory where the fringing field (in the linear approximation) becomes zero, i.e., where the particle passes from field-free space into the fringing field [A in Fig. (13.1)].
(z_{pe}, r_{pe})	the point on the trajectory at the crossing of the profile computed as in Section #3 [F in Fig. (13.1)].
(z_e, r_e)	the point on the trajectory where the fringing field (in the linear approximation) merges with the interior or idealized form of Eq. (1.1) [B in Fig. (13.1)].
(z_{2e}, r_{2e})	the point of normal projection of (z_e, r_e) onto the initial tangent of the trajectory (i.e., the extension of the original ray with inclination ψ_s).

The distances:

Up to this point, it has not been important to distinguish between distances to profiles and to the interior field point. At this point:

ρ_s	from $(z_s, 0)$ to (z_{pe}, r_{pe}) . [OF in Fig. (13.1)].
ρ_e	from $(z_s, 0)$ to (z_e, r_e) [OB in Fig. (13.1)].
ρ_{1e}	from $(z_s, 0)$ to profile along the initial tangent. [OE in Fig. (13.1)].
d_{1e}	from (z_{1e}, r_{1e}) to (z_{pe}, r_{pe}) measured along the trajectory [AF in Fig. (13.1); AE in Fig. (8.4)].
d_{2e}	from (z_{pe}, r_{pe}) to (z_e, r_e) measured along the trajectory [FB in Fig. (13.1); EM in Fig. (8.4)].
$d_e = d_{1e} + d_{2e}$	Same as quantity defined in Section #8, i.e., measured along trajectory [AFB in Fig. (13.1); AM in Fig. (8.4)].

t_{1e}	from (z_{1e}, r_{1e}) to profile, along initial tangent to trajectory [AE in Fig. (13.1)].
t_{2e}	from (z_{2e}, r_{2e}) to profile, along tangent [CE in Fig. (13.1)].
$t_e = t_{1e} + t_{2e} =$	distance from (z_{1e}, r_{1e}) to (z_{2e}, r_{2e}) [AC in Fig. (13.1)].
Ξ_e	from (z_{2e}, r_{2e}) to (z_e, r_e) [CB in Fig. (13.1)].

All distances are taken as magnitudes; hence, always positive.

ψ_e Value of ψ at (z_e, r_e)

$$\Delta\psi_e = \psi_e - \psi_s$$

Fig. 13.1

On the exit side, the corresponding quantities are designated by changing the index "e" to "o", or "s" to "f"; the exception to this rule is: $\Delta\psi_o = \psi_f - \psi_o$.

Pursuing now the evaluation of the trajectory perturbation, it is evident that it is necessary to know the coordinate displacement measured by Ξ_e [see Fig. (13.1)], as well as the angular displacement $\Delta\psi_e$ already computed in Section #8. To the approximation that Ξ_e is small, at each point along the trajectory in the fringing field,

$$d\Xi_e = (d_e - d) d\psi \quad (13.1a)$$

and, in the linear approximation to the fringing field, from Eq. (8.7),

$$d\psi = \frac{1}{Kr_e} \frac{d}{d_e} d(d) \quad . \quad (13.1b)$$

Then

$$\Xi_e = \frac{1}{Kr_e d_e} \int_0^{d_e} [(d_e - d)d] d(d) = \frac{d_e^2}{6Kr_e} = \frac{1}{3} \Delta\psi_e d_e \quad . \quad (13.2)$$

Rather than making a direct measurement of d_e along the trajectory, it is simpler to make the corresponding measurement along the initial tangent. For d_e , then, t_e is substituted, which is a fair approximation as long as Ξ_e is small. From Eqs. (8.7) and (13.2),

$$\Delta\psi_e = \frac{1}{2} \frac{t_e}{Kr_e} = \frac{1}{2} \frac{\varphi}{K} \frac{t_e}{g_e} \quad (13.3a)$$

$$\Xi_e = \frac{1}{6} \frac{t_e^2}{Kr_e} = \frac{1}{3} \Delta\psi_e t_e \quad . \quad (13.3b)$$

It may be noted that $\Delta\psi_e$ is first order in t_e (or d_e), whereas Ξ_e is second order. The position of the point (z_{2e}, r_{2e}) can be readily determined, and from it can be calculated the coordinates (z_e, r_e) . Thus, from Fig. (13.1),

$$r_e = r_{2e} - \Xi_e \cos\psi_s = r_{2e} + \Xi_e \cos\psi_s \quad (13.4a)$$

$$r_{2e} = (\rho_{1e} + t_{2e}) \sin\psi_s \quad . \quad (13.4b)$$

Now r_e is evaluated by successive approximations, utilizing the fact that Ξ_e is small. Thus, r_{2e} is substituted for r_e in Eq. (13.3a) and Ξ_e then calculated from Eq. (13.3b). The value of r_e from Eq. (13.4a) is then put back into (13.3a), etc. For the z -coordinate, from Fig. (13.1),

$$z_e = z_{2e} - \Xi_e \sin\psi_s = z_s + [(\rho_{1e} + t_2) \cos\psi_s - \Xi_e \sin\psi_s] \quad . \quad (13.5)$$

From Eqs. (13.3a), (13.4) and (13.5) and the relation

$$\psi_e = \psi_s + \Delta\psi_e \quad ,$$

all the necessary information is available for computing the trajectories. Substituting into Eq. (1.27), the constants a and Z_M may then be calculated as a function of ψ_s and K .

#14. GRAPHICAL AND ANALYTICAL CALCULATION OF EXIT CURVES.

A general procedure for computing a focusing exit curve for any given entrance curve was given in Section #3 (neglecting fringing-field effects). In this section is presented a method for making a fairly rapid determination, using a procedure of interpolation from a master curve. This approach is particularly valuable when a number of alternative profiles are to be examined, without great demands on precision, particularly in the case of asymmetric instruments.

The calculation will be carried out for the case in which $n_s = n_f = 0$, since this is the case of greatest usefulness as well as being the simplest; however, the method can be readily generalized.

The basis of the method is the determination of the symmetry point (Z_{Ms}, r_{Ms}) for each trajectory, i.e., the point at which $\psi = \pi$. Because fringing fields are neglected, (z_e, r_e) , the entrance point, is on the profile, and the distance from $(z_s, 0)$ to (z_e, r_e) is ρ_s . It is convenient to take $z_s = 0$. From Eqs. (1.26) and (2.3),

$$z_e = r_e e^{K \cos \psi_s} K U(K, \psi_s) + Z_{Ms} = \rho_s \cos \psi_s \quad (14.1)$$

$$r_e = \rho_s \sin \psi_s. \quad (14.2)$$

Then,

$$Z_{Ms} = \rho_s \left[\cos \psi_s - \sin \psi_s e^{K \cos \psi_s} K U(K, \psi_s) \right] \quad (14.3)$$

and

$$r_{Ms} = a e^K = \rho_s \sin \psi_s e^{K(1 + \cos \psi_s)}. \quad (14.4)$$

Consider, first, a circular entrance curve of radius $\rho_s = 1$. It will be convenient to compute the properties of this case and then to use simple transformations to get the general case. For this case, set (Z_{Ms}, r_{Ms}) equal to (Z_{Mo}, r_{Mo}) . In Table (14.1) are listed (Z_{Mo}, r_{Mo}) for several (K, ψ_s) values. The results are shown in Fig. (14.1) in the form of a family of curves. The solid curves represent what may be called M_o -curves, joining points with constant K and variable ψ_s . The dashed curves join points of constant ψ_s and variable K . By making a fine enough mesh, it is feasible to use the set of curves for interpolation.

Table 14.1

ψ_s	K	$-Z_{Mo}$	r_{Mo}	ψ_s	K	$-Z_{Mo}$	r_{Mo}
90°	0.56	0.8758	1.750	120°	0.56	1.0138	1.1458
	0.60	0.9642	1.822		0.60	1.0585	1.1689
	0.64	1.0682	1.896		0.64	1.1045	1.1930
100°	0.56	0.9481	1.5646	130°	0.56	1.0210	0.9356
	0.60	1.0251	1.6169		0.60	1.0521	0.9491
	0.64	1.1057	1.6712		0.64	1.0836	0.9621
110°	0.56	0.9917	1.3584	140°	0.56	1.0193	0.7328
	0.60	1.0520	1.3946		0.60	1.0390	0.7334
	0.64	1.1145	1.4318		0.64	1.0592	0.7466

It is evident that, because

$$Z_{Ms} = \rho_s(\psi_s) Z_{Mo}(\psi_s) \text{ and } r_{Ms} = \rho_s(\psi_s) r_{Mo}, \quad (14.5)$$

it is possible to use these curves to construct the M_s -curve for any entrance profile defined by $\rho_s(\psi_s)$. Some examples are evident in Fig.(14.2),

in which are shown the M_1 -curve for a circular entrance curve with $\rho_s = \rho_1$ and the M_s -curve for an arbitrary noncircular entrance curve $b_s b_s$. Clearly M_1 is formed by increasing the radius vector* from $[(z_s, 0) = (0, 0)]$ to a point on M_0 by a constant factor, as in going from A to B, i.e., $OB/OA = \rho_1$ for any B on M_1 . In the same way, M_s is formed, as in going from A_1 to B_1 ; the factor, however, varies with ψ_s , so that $OB_1/OA_1 = \rho_s(\psi_s)$.

Fig. 14.2

At the exit, the M-curves are constructed in the same manner. Using z_s as the origin, as before, from Eqs. (1.29a), (2.3), and (3.2a),

$$z_0 = r_0 e^{K \cos \psi_f} K U(K, \psi_f) + Z_{Mf} = z_f - \rho_f \cos \psi_f \quad (14.6)$$

*It may be noted that the radius vector corresponding to ψ_s does not have ψ_s as an inclination, as is evident from the example in Fig. (14.2).

$$r_0 = -\rho_f \sin \psi_f . \quad (14.7)$$

Then,

$$Z_{Mf} - z_f = -\rho_f [\cos \psi_f - \sin \psi_f e^{K \cos \psi_f} K U(K, \psi_f)] \quad (14.8)$$

$$r_{Mf} = a e^K = -\rho_f \sin \psi_f e^{K(1 + \cos \psi_f)} . \quad (14.9)$$

Again taking $\rho_f = 1$, a set of M'_0 -curves are derived, each curve passing through the points (Z'_{M0}, r'_{M0}) for the various ψ_f -values at constant K . These are almost identical to the M_0 -curves, differing only in being reflections in a plane $z = \text{constant}$. The M_0 -curves are not identical to the M'_0 -curves (i.e., not superimposable), which simply means, as was pointed out in Section #3, that a symmetric instrument does not have circles for profiles.

From an arbitrary exit curve $b_f b_f$, the M_f -curve may be constructed giving (Z_{Mf}, r_{Mf}) values for the same K -value as was used for the entrance curve $b_s b_s$. As before, the radius vector to the appropriate point (for a given ψ_f) on M'_0 is multiplied by ρ_f to give the corresponding point on M_f . In general, for an arbitrary entrance curve $b_s b_s$ and an arbitrary exit curve $b_f b_f$, the curve M_f is not superimposable on M_s . If the curves were superimposable, then the exit curve would satisfy the focusing condition; this follows, because for each preassigned ψ_s and a -value (hence r_e or ρ_s), there would result a unique Z_M and ψ_f , such that the ray would arrive at $(z_f, 0)$.

Inversely, consider the M_s -curve drawn for the entrance profile $b_s b_s$ as the M_f -curve for the to-be-determined exit profile $b_f b_f$. The radius vector is drawn from $(z_f, 0)$ through the point on M'_0 corresponding to a particular ψ_f , and the vector continued until it intersects M_s , as, e.g., the extension of OA' to OB' in Fig.(14.2); then the appropriate ρ_f for this ψ_f is OB'/OA' . The ψ_s corresponding to this ψ_f is determined from the intersection point B' on M_0 . In general, $\psi_s \neq 2\pi - \psi_f$; for the case in Fig. (14.2), $\psi_s \sim 95^\circ$ for $2\pi - \psi_f = 120^\circ$.

This graphical evaluation of the intersection point B' (and the resulting value of ρ_f) is a fairly rapid method for constructing the exit curve. Alternatively, by the use of approximating formulae for M_s and M'_0 , the intersection points may be determined analytically. This is a procedure which is alternative to the method described in Section #3, and is essentially equivalent in computational difficulty.

Symmetrical instruments. The characteristic feature of a symmetrical instrument is that the M_s -curve is the vertical line $Z_M = \text{constant}$, situated midway between z_f and z_s . With the "inverse" method,

working from this M_s -curve and the M_0 -curve appropriate for the K -value considered, ρ_s may be determined as a function of ψ_s . The exit curve is, of course, determined by a mirror reflection in $z = Z_M$. Since the explicit form of the curves is available in Eq. (3.5), this method is no faster than the analytical approach unless the family of M_0 -curves is already at hand.

#15. CALCULATION OF PROFILES INCLUDING FRINGING-FIELD EFFECTS

It is presumed that the fringing field is known, at least on the median plane, so that $\Delta\psi_e$ and E_e may be computed as in Sections #8 and #13, and hence the coordinates (z_e, r_e, ψ_e) as in Eqs. (13.4), (13.5) and (13.6). With this information, it is possible to use the approach of Section #14 to compute an "equivalent" entrance curve. Thus, through Eq. (1.29), (Z_M, r_M) may be determined for a number of values of z_e, r_e , and ψ_e , giving the M_s -curve for the system, including the fringing-field effect. Working backwards, it is then possible to determine an equivalent entrance profile (z_{ee}, r_{ee}) ; this profile is one which would, in the absence of a fringing field [magnetic field described entirely by Eq. (1.1)], give the same set of trajectories, i.e., the same M_s -curve. Using the equivalent entrance profile, the equivalent exit profile may be computed by the method of Section #3 or of Section #14, and from the latter, the real exit profile may be computed by a reverse computation of $\Delta\psi_0$ and E_0 .

It is possible, however, to find the equivalent entrance curve directly from the real entrance curve by an approximate method. The approximation is an adequate one when one considers the fact that ψ_e and E_e are generally not well known (being derived from an estimation of the fringing-field magnitude), and in addition are rather small quantities.

Fig. 15.1

The quantities to be considered are illustrated in Fig. (15.1), which is similar to Fig. (13.1). At A, we have $\psi = \psi_s$, and the fringing field starts to bend the trajectory, and continues up to (z_e, r_e) , i.e., to B, lying within the actual profile $b_s b_s$, with $\psi = \psi_e$. If no fringing field existed and the interior field followed Eq. (1.1), then a ray starting at G on a profile $b'_s b'_s$ with $\psi = \psi_s$ will bend through the same change in ψ , arriving at (z_e, r_e) with $\psi = \psi_e$. From this point to the M_s -curve, the trajectories are identical in the idealized and real cases. Be-

cause the field intensity within the profile is greater in the idealized case, the rate of bending is greater; hence, the trajectory GB is always at a larger η (see Section #11 for definition) than the real trajectory AFB. From Eq. (1.29b), since $\log(r/a) = -K \cos \psi$ within the idealized field, then

$$ds = \frac{dr}{\sin \psi} = Kr d\psi, \quad (15.1)$$

where ds is positive along the trajectory. To the first approximation, r may be taken as constant along the trajectory GB and equal to r_e . Then, from Fig. (15.1),

$$\Delta s_1' \hat{=} s(B) - s(G) \hat{=} Kr_e (\psi_e - \psi_s) = Kr_e \Delta \psi_e \quad (15.2)$$

The necessary displacement of the profile $b_s' b_s'$ from $b_s b_s$ along the trajectory is the negative of Δs_1 , so that*

$$\Delta s_1' \hat{=} -Kr_e \Delta \psi_e \quad (15.3)$$

In the second approximation, allowance is made for the variation of r , using Eq. (2.5a):

$$r = \left(r_e e^{K \cos \psi_e} \right) e^{-K \cos \psi} \quad (15.4)$$

$$\Delta s_1 = s(G) - s(B) = \left(\frac{ds}{d\psi} \right)_{r=r_e} \Delta \psi + \frac{1}{2} \left(\frac{d^2 s}{d\psi^2} \right)_{r=r_e} (\Delta \psi)^2, \quad (15.5a)$$

where

$$\Delta \psi = \psi_s - \psi_e = -\Delta \psi_e, \quad \text{and} \quad \frac{ds}{d\psi} = Kr \quad (15.5b)$$

from Eq. (15.1). Since

$$\frac{d^2 s}{d\psi^2} = (K \sin \psi) (Kr_e) e^{K(\cos \psi_e - \cos \psi)},$$

we have

$$\Delta s_1 = -Kr_e \left[1 - \frac{1}{2} (K \sin \psi_e) \Delta \psi_e \right] \Delta \psi_e \quad (15.6)$$

As in Eq. (13.1a), to the approximation that Ξ_1 is small, at each point along the trajectory portion GB,

$$d\Xi_1 = (\Delta s_1 - s_G) d\psi = (\Delta s_1 - s_G) \frac{ds}{Kr} \quad (15.7)$$

from Eq. (15.5b), with s_G the distance, along the trajectory, from G. Taking Kr as approximately constant along** GB:

* Δs_1 is always negative ($\Delta \psi_e > 0$). Whether or not the equivalent profile $b_s' b_s'$ is outside the real profile $b_s b_s$ depends upon the ratio g/de (see Section #8). Generally, $b_s' b_s'$ is outside relative to $b_s b_s$.

**The difference in the coefficient between Eqs. (15.8) and (13.2) may be noted. This arises from the fact that the region of action to give a particular $\Delta \psi_e$ -value is smaller in the relatively intense idealized field than in the slowly rising fringing field, i.e., $\Delta s_1 < d_e$.

$$E_1 = \frac{1}{2} \frac{(\Delta s_1)^2}{Kr_e} = -\frac{1}{2} \Delta s_1 \Delta \psi_e \quad (15.8)$$

from Eq. (15.3). It is seen that E_1 is already of second order in Δs_1 ; hence, no better approximation need be made, and $\Delta s_1' \cong \Delta s_1$.

The equivalent ray O'G arrives at G with the entrance angle ψ_s , i.e., parallel to the real ray OA, but displaced by the distance*

$$\Delta E = E_e - E_1 \quad (15.9)$$

On the z-axis, this corresponds to the displacement

$$\Delta z = -\Delta E / \sin \psi_s \quad (15.10)$$

In general Δz is a negative displacement,** and may be corrected by an additional displacement, Δs_2 , which from Eq. (12.25) is also negative, i.e., further displaced from $b_s b_s$. Then

$$\begin{aligned} \Delta s_2 &= \Delta z K \sin^2 \psi_s = -\Delta E K \sin \psi_s \\ &= -E_e K \sin \psi_s - \frac{1}{2} \Delta s_1 K \sin \psi_s \Delta \psi_e \end{aligned} \quad (15.11)$$

The total displacement is

$$\Delta s = \Delta s_1 + \Delta s_2 \quad (15.12)$$

or

$$\begin{aligned} \Delta s &= \Delta s_1 \left(1 - \frac{1}{2} K \sin \psi_s \Delta \psi_e\right) - E_e K \sin \psi_s \\ &= -Kr_e \left[1 - \frac{1}{2} K \sin \psi_e \Delta \psi_e\right] \left[1 - \frac{1}{2} K \sin \psi_s \Delta \psi_e\right] \Delta \psi_e - E_e K \sin \psi_s. \end{aligned} \quad (15.13)$$

Keeping only terms to the second order, and approximating $\psi_e \cong \psi_s$,

$$\Delta s = -Kr_e \Delta \psi_e [1 - K \sin \psi_s \Delta \psi_e] - E_e K \sin \psi_s \quad (15.14)$$

This is the distance from (z_e, r_e) at which an equivalent entrance curve would be placed if the idealized magnetic field operated near the profile. The results on the exit side are similar but differ in sign:

$$(\Delta s_1)_f \cong s(G_f) - s(B_f) \cong Kr_0 (\psi_f - \psi_0) = Kr_0 \Delta \psi_0, \quad (15.15)$$

*From Eqs. (13.2) and (15.8), $\frac{E_1}{E_e} = \frac{3}{2} \frac{\Delta s_1}{d_e}$. In most cases, this ratio is less than 1, so that ΔE is usually positive.

**i.e., when $\Delta E > 0$.

where $(\Delta s_1)_f$ is a positive displacement of the profile from the real one $b_f b_f$ to the virtual one $b_f' b_f'$. Further,

$$(\Delta s_1)_f = K r_0 \Delta \psi_0 \left[1 + \frac{1}{2} (K \sin \psi_0) \Delta \psi_0 \right] \quad (15.16)$$

$$(dE_1)_f = (s_B)_f d\psi = (s_B)_f \frac{ds}{K r} \quad (15.17)$$

$$(E_1)_f = \frac{\frac{1}{2} (\Delta s_1)_f^2}{K r} = \frac{1}{2} (\Delta s_1)_f \Delta \psi_0 \quad (15.18)$$

Further, $\Delta z_f > 0$, so

$$\Delta z_f = - \Delta E_f / \sin \psi_f, \quad (15.19)$$

where $\Delta E_f = E_0 - (E_1)_f$. Also $(\Delta s_2)_f > 0$, where

$$\begin{aligned} (\Delta s_2)_f &= \Delta z_f K \sin^2 \psi_f = - \Delta E_f K \sin \psi_f \\ &= - E_0 K \sin \psi_f + \frac{1}{2} (\Delta s_1)_f K \sin \psi_f \Delta \psi_0 \end{aligned} \quad (15.20)$$

and

$$\begin{aligned} \Delta s_f &= (\Delta s_1)_f + (\Delta s_2)_f = (\Delta s_1)_f \left[1 + \frac{1}{2} K \sin \psi_f \Delta \psi_0 \right] - E_0 K \sin \psi_f \\ &= K r_0 \Delta \psi_0 \left[1 + K \sin \psi_f \Delta \psi_0 \right] - E_0 K \sin \psi_f \end{aligned} \quad (15.21)$$

With a linear approximation for the fringing field, and using the approximate results of Eq. (13.3), then, from Eq. (15.14),

$$\Delta s = - \frac{1}{2} t_e \left(1 - \frac{1}{12} \frac{t_e}{\rho_e} \right) \quad (15.22)$$

This is equivalent to the replacement of the arc BG by the straight line extension MG. To the same kind of approximation, the distance between the real boundary and the equivalent one is:

$$\text{Entrance side:} \quad |\Delta s + t_{1e}| \quad (15.23a)$$

$$\text{Exit side:} \quad \Delta s - t_{10} \quad (15.23b)$$

Thus far, consideration has been given to the construction of an equivalent boundary from a real boundary, given information on the quantities $(t_{1e}, \Delta \psi_e, E_e)$ and $(t_{10}, \Delta \psi_0, E_0)$. The problem is usually presented in the inverse form, namely, theoretical profiles are computed for the idealized field case [Eq. (1.1) inside the profiles and zero field outside] using the methods of Section #3 and/or Section #14, and it is desired to transform these profiles to real boundaries. The inverse problem may be solved by graphical construction, as in Fig. (15.1), or approximately, through the use of Eq. (15.23).

As an example of the magnitudes of the various quantities, some data is given for the 8-gap iron-core spectrometer referred to in Section #8:

$$\rho_e = 270 \text{ mm}; \varphi = 10^\circ \text{ gap angle}; K = 0.6; \text{ at } \psi_s = 120^\circ,$$

$$t_e = 51 \text{ mm and } t_{1e} = 11 \text{ mm}.$$

From Eqs. (13.3) and (15.22), it follows that:

$$\Delta\psi_e = 0.182 (10.4^\circ) \text{ and } \Xi_e = 3.1 \text{ mm} \quad ,$$

$$\Delta s = -25.5 (1 - 0.016) = -25 \text{ mm}$$

$$t_{1e} + \Delta s = -14 \text{ mm} \quad .$$

From the magnitude of Ξ_e/t_e , it is evident that the approximation $t_e/d_e \cong 1$ is quite a good one, the difference from unity being only a few tenths of a per cent.

It may be seen, then, that the design of an instrument involves the following steps:

- (i) Computation of optimum boundary curves, assuming that the deflection is given only by the idealized field of Eq. (1.1). These are the theoretical profiles.
- (ii) The fringing-field deflection is estimated, and the theoretical profiles considered as equivalent boundaries. From these are computed the real boundaries.

#16. CONSIDERATIONS IN THE CHOICE OF DESIGN PARAMETERS.

It is evident from the discussion in the previous sections that we have considerable freedom in the choice of design parameters even within our self-imposed restriction to spectrometers with source and image on the instrument axis. Such freedom allows for considerable adaptability of design for a variety of purposes. Even with a particular purpose in mind, we find that a number of different sets of parameters are possible.

We propose now to use our previous results to give criteria which will aid in choosing the best design for an instrument, given the purposes of use of the spectrometer. Because widely different uses of these instruments are possible, we have, correspondingly, widely different conditions imposed upon them,* with resulting very different choices of parameters (K-value, profile curves, etc). We shall obtain the general criteria by examining the influence of the various parameters of the instrument on its important properties.

Ghost Peaks. Because the elimination of these peaks over a wide ψ_s -range is possible only in instruments using zero-loop trajectories ($n_s = n_f = 0$), as described in Section #4, we restrict ourselves to this kind of instrument. We consider the advantage of avoiding the ghost-peak background to be overriding, although for some properties (see, e.g., the high dispersion in Section #6), trajectories with several loops might have some advantages. It might, for example, be feasible to make a low-transmission, high-resolution device with a narrow ψ_s -range and suitable baffles.

Resolution. At a given emission angle ψ_s , the expressions for the resolution are

$$\mathcal{R}'_z = \frac{\delta z_f}{\sigma_f} \quad \text{when } \delta z_f \geq D_z \quad (6.17)$$

and

$$\mathcal{R}'_{r_1} = \frac{\delta r_{1f}}{\sigma_f^r} \quad \text{when } \delta r_{1f} \geq D_{r_1} \quad (6.40)$$

Thus, the resolution is determined by the dispersion (through σ_f or σ_f^r) and the imaging properties of the instrument (through δz_f or δr_{1f}).

The dispersions are given by

$$\sigma_f = z_s \left(1 - \frac{z_f}{z_s} \right) G \quad (6.10b)$$

*As, for example, sizes and shapes of sources and detectors, range of emission angles, range of particle energy, luminosity, and resolution.

and

$$\frac{r}{f} = - \phi_f \tan \theta_f \quad (6.36)$$

It is evident that the resolution of an instrument design improves, other things being equal, as the function $\left(1 - \frac{z_f}{z_s}\right) G$ increases. Within a given instrument, the choice between the use of radial detection and axial (z -) detection depends upon the value of $\tan \theta'_f$ and also upon the imaging properties with respect to r and z .

The pertinent imaging properties of the instrument are given by the quantities $Q_f(\psi_s)$ and/or $u(\psi_s)$, since

$$\delta z_f = Q_f(\psi_s) \delta z_s \quad (5.10)$$

and

$$\delta r_{1f} = u(\psi_s) \delta r_1 \quad (5.15b)$$

Because the resolution is proportional to δz_f and/or δr_{1f} [if $\delta z_f \geq D_z$ and/or $\delta r_{1f} \geq D_{r1}$], it is useful to have low values of $Q_f(\psi_s)$ and/or $u(\psi_s)$. Knowing the z - and r -dispersions, and the imaging properties of the instrument, one can easily compute $\mathcal{R}'_z$ and $\mathcal{R}'_{r1}$ and decide whether axial or radial detection will provide better resolution.

In this discussion, we have omitted what may be called the second-order effects of the fringing field, i.e., the worsening of the resolution due to differences in the fringing-field effects on trajectories in the median and off-median planes. These effects give contributions which increase with ϕ^2 and ϕ^3 and make the choice of small values of ϕ favorable (see Section #11).

Luminosity. This quantity is the product of the source area and the transmission. To maximize it, it must be possible to use a high transmission with large source dimensions. To achieve this, we shall require that there be (i) a minimization of the losses of particles due to source size with no fringing-field effect (Section #7) and (ii) minimization of losses due to the lens effect on a point source (Section #9). One should really require a minimum loss for the effect of the fringing field on the particles from a spread source, but the combined effects do not yield general expressions suitable for setting up criteria. We must then use the approximate criteria derived by separately maximizing transmission for the two conditions.

To minimize losses due to source size, we require that $F_1 R_2^* / z_s$ be as small as possible, with R_2^* the magnitude of farthest r_2 -extension of the source, since, from Eq. (7.8b),

$$\gamma_2^* = 1 - \frac{F_1 R_2^*}{\phi z_s},$$

which then comes closer to unity. To be able to make R_2^* as large as possible, we then favor instruments with minimum values of F_1 . In order not to lose particles at the detector, in the absence of fringing-field considerations we must make the detector aperture at least as large as the size of the image of the source.

The losses due to the lens effect on particles from a point source are of two kinds:

(i) Losses resulting from a divergent entrance lens, with consequent collision of the trajectories with the polefaces. These can be avoided by attempting to have only convergent lenses, which implies $W_s \leq 1$ or $\mu_s \leq 0$. The latter condition must be modified if one takes into account the "second-order" fringing-field effects (see Section #10) to read $\mu_s(\text{eff}) \leq 0$.

(ii) Losses due to stopping by the detector aperture can always be avoided by having the r_2 -extension of the detector be at least equal to the image size generated by the lens effect, which is [from Eq. (9.17)]

$$\begin{aligned} 2 \delta R_{2f}(\text{max}) &= 2 [vN_s + (z_f/z_s) N_f W_s] \frac{1}{2} \phi && \text{convergent} \\ &&& \text{entrance lens} \\ 2 \delta R_{2f}(\text{max}) &= 2 [vN_s + (z_f/z_s) N_f W_s] (\frac{1}{2} \phi / W_s) && \text{divergent} \\ &&& \text{entrance lens} \end{aligned}$$

Detector-size Considerations. Up to now, we have assumed that the detector size was dictated only by resolution or by luminosity considerations. These two conditions give opposite criteria: good resolution requires that the aperture be small: $D_z \leq \delta z_f$, $D_{r_1} \leq \delta r_{1f}$; high luminosity requires that the aperture be large: $D_z \geq \delta z_f$, $D_{r_1} \geq \delta r_{1f}$, and $D_{r_2} \geq \delta R_{2f}$.

Normally, one is interested in obtaining the desired resolution (or luminosity) associated with the best possible luminosity (or resolution), respectively. The conditions under which these combinations may be achieved are very different for (i) instruments in which the r_2 -extent of the detector aperture (D_{r_2}) for one gap is the r_1 -extent of the aperture (D_{r_1}) for another gap, and (ii) instruments in which the D_{r_2} of one gap is not coupled to the D_{r_1} of another gap (e.g., single-gap instruments, and multigap tilted-axis iron spectrometers).

In the coupled case, we must minimize the size of the image in all directions. Achieving this result sets two conditions:

- (i) The image of a finite source without considering lens effects (see Section #5) must be small.
- (ii) The line image (due to lens effect) of a point source must be small (see Section #9).

We note, as above, that the conditions are separated in this fashion, because no simple equations are available for treating the combined source size and lens effect.

From Condition (i), it follows that instruments are favored which minimize the $v(\psi_s)$, $Q_f(\psi_s)$, and/or $u(\psi_s)$ functions, since

$$\delta r_{2f} = v(\psi_s) \delta r_2 \quad (5.20)$$

$$\delta z_f = Q_f(\psi_s) \delta z_s \quad (5.10)$$

$$\delta r_{1f} = u(\psi_s) \delta r_1 \quad (5.15b)$$

Condition (ii) dictates that instruments be chosen with small values of $[vN_s + (z_f/z_s) N_f W_s]$, because this determines the image size $2\delta R_{2f}(\max)$, as indicated above. This may be accomplished by making μ_s and μ_f as small as possible, because N_s and N_f are proportional to $\tan \mu_s$ and $\tan \mu_f$, respectively. Of course, when the "second-order" fringing-field effects are included, this should be expressed as making $\mu_s(\text{eff})$ and $\mu_f(\text{eff})$ as close to zero as possible. It follows, then, that the effective profiles should be as close as possible to circles with centers at the object and image points, respectively. It seems apparent from the expression for $2\delta R_{2f}(\max)$ above for a diverging entrance lens that this quantity may be minimized by making W_s large. The apparent advantage, however, is illusory, since the limitation of image size is attained only by limiting the range of effective ϕ_s , thereby cutting the transmission by exit aperture limiting. This may be more obviously seen by examining η_{23} . The following expression covers both converging and diverging entrance lens:

$$\eta_{23} = \frac{2 R_d}{|vN_s + (z_f/z_s) N_f W_s| z_s \phi} \text{ for } \eta_{23} \leq \eta_2 \quad (9.23)$$

Other things being equal, the denominator is larger for a diverging lens than for a converging lens, because of the magnitude of W_s .

In the uncoupled case, the required conditions of the coupled case may be dropped if they relate to the extent of the detector along the r_2 -axis. It should be pointed out, however, that there may be reasons extraneous to the optics of the instrument which would tend toward a limitation of the detector size. An example would be the possible increase of background with detector size.

From the considerations on resolution, luminosity, and detector size, we can conclude that, in order to choose an instrument, we should compute the above-mentioned characteristic functions for all possible instruments which satisfy the a priori requirements (e.g., range of emission angles) and choose the one which has

$$\text{Maximum} \left(1 - \frac{z_f}{z_s}\right) G$$

Minimum:

$$v(\psi_s); Q_f(\psi_s) \text{ and/or } u(\psi_s); F_1 \text{ and } [vN_s + (z_f/z_s) N_f W_s] \quad ,$$

and, if possible, make $\mu_s(\text{eff}) \leq 0$, i.e., a convergent entrance lens if the lens power cannot be made zero. Because these are all functions of ψ_s , it should be remarked that one is really trying to obtain maxima or minima of the appropriately averaged functions. Thus, one might weight the contribution of each emission angle by its transmission, or by its resolution, etc.

In general, the evaluation of optimum design parameters involves a tedious numerical evaluation of an instrument's properties for many alternative values of the parameters. In many cases, electronic computer calculations will be desirable for this purpose. For designs in which symmetrical instruments are appropriate, the computations have been made and presented in this report in the form of tables and graphs. It was not practical to do the same for asymmetric cases, because of the great multiplicity of possible designs. In addition to variable K , as in the symmetric case, one must then also consider variable z_f/z_s and completely arbitrary entrance profiles.

By examining the pertinent formulae, it is also possible to determine what happens to the properties of an instrument when one scales up its dimensions by a scaling factor s , leaving the source and detector sizes as they were, or what happens when these are also scaled up in size. Or we can determine what happens when one scales up the angular gap opening ϕ by a scaling factor f , but leaves the linear dimensions as they were. These effects are summarized in Table 16.1. We may remark at this point, that, although these conclusions follow from the above equations for this type of instrument, the results could also have been reached from more general considerations.

The table clearly shows that a compromise is necessary in the choice of z_s and ϕ , the choice being determined by what values of resolution and transmission are acceptable. The compromise can be made only by examining the properties of the instrument and making numerical comparisons.

In examining criteria for choosing an instrument design, we have omitted some very important practical considerations, such as the technical difficulty of construction, cost, reproducibility and linearity of measurement (e.g., iron core or iron-free), and effects of earth's magnetic field, which, in fact, play an important role in the design of an instrument. No general analysis is feasible here, because the problems are specific to the particular project in mind.

Table 16.1

EFFECT OF INSTRUMENT DESIGN ON RESOLUTION,
TRANSMISSION, AND SOURCE SIZE

	Scaling up everything	Scaling up instrument, but not source and detector	Scaling up ϕ only
Resolution for a finite source (due to imaging properties, in absence of fringing field)	$\equiv$	$\propto \frac{1}{s}$	$\equiv$
Transmission for a point source (lens effect)	$\equiv$	$\propto \frac{1}{s}(a)$	$\propto \frac{1}{\phi}(a)$
K where $1 - K =$ transmission for a finite source in the absence of a lens effect	$\equiv$	$\propto \frac{1}{s}(b)$	$\propto \frac{1}{\phi}(b)$
Source size	$\propto s^2$	$\equiv$	$\equiv$

(a) Provided that $\phi R_s f$ is larger than the r_d -extent of the detector (D_R)

(b) Provided the detector aperture is equal to or larger in size than the image.

An interesting example of the application of the general criteria is the design of an electron-electron multigap coincidence beta-ray spectrometer. In order to allow two spectrometers to be placed back-to-back, it is necessary to limit the range of emission angles to $90^\circ \leq \theta_s \leq 180^\circ$. Further examination of the problem leads to a further reduction in the θ_s -range, because

- (1) The interference between the two instruments should be negligible when one focuses a high-energy electron in one spectrometer and a low-energy electron in the other one. This tends toward an increased distance between the spectrometers, which forces $\theta_s > 90^\circ$. The magnitude of the necessary displacement depends upon the magnitude of the fringing fields of the instrument. In general, the magnitude (and hence the effective extent) of the fringing field decreases with the gap angle ϕ .
- (11) It is difficult to construct the instrument for θ_s -angles close to 180° , in addition, the transmission becomes negligible, since (other things being equal) this varies as $\sin \theta_s$. Hence, θ_s is appreciably smaller than 180° .

The acceptance range of ψ_s then lies considerably within the boundaries: $90^\circ < \psi_s < 180^\circ$.

One may further require that the instruments be capable of being used in series as a single spectrometer. This would result in a reduction of the scattered-electron background and possibly in an improved resolution with practically no loss in transmission. An examination of the properties of various designs indicates that symmetric instruments are probably favored. Only with symmetric spectrometers ($n_s = n_f = 0$, $z_s = -z_f$, symmetrical profiles) can the spectrometers be operated backwards with the same properties (e.g., resolution, transmission, and imaging).

The requirement of highest possible transmission, essential for coincidence spectrometers, is fulfilled if we have nearly circular profile curves. This is achieved by using a K-value close to 0.6. This can be readily seen in Fig. (3.3), and also in Fig. (9.6), since the desired feature is that $W_s \cong 1$.

The $v(\psi_s)$, $Q_f(\psi_s)$, and $u(\psi_s)$ functions are all determined by the condition of symmetry of the instrument; they are constant and equal to -1, 1, and -1, respectively.

The remainder of the instrument design requires specification of particular conditions such as: desired source size, desired resolution, and iron or iron-free spectrometer. We shall not go into detail about these, since the previous considerations may be readily applied to the choice of the best design for an instrument of this type.

We may add some examples of instruments, for which one might arrive at quite different design parameters:

- (i) Multigap coincidence spectrometers with no symmetry requirement. A possible choice which looks interesting would have $K = 0.6$, $n_s = n_f = 0$, $z_f = -2z_s$, $\psi_s(\max) = 145^\circ$, $\psi_s(\min) \cong 100^\circ$, and the entrance profile a circle of radius z_s .
- (ii) Multigap or one-gap single spectrometers with maximum possible transmission. There is no restriction of emission angles to values larger than 90° .
- (iii) Multigap or one-gap single spectrometers with maximum possible resolution.

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APPENDIX I

DERIVATION OF BESSEL FUNCTION EXPANSIONS

Equations (1.19) and (1.31) are demonstrated in the following manner. As might be expected from the cylindrical symmetry of the system, an expansion in one form of the cylindrical functions - the Bessel functions - is involved. The fundamental formula (or the closely related sine and cosine forms) from which the derivation is made is found in a number of texts and compilations dealing with the Bessel functions or with the higher transcendental functions.

We start with:

$$e^{iz\cos\theta} = J_0(z) + 2 \sum_{n=1}^{\infty} i^n J_n(z) \cos n\theta \quad . \quad (\text{A-1.1})$$

Integrating, we have

$$\begin{aligned} \int_{\pi}^{\psi} e^{iz\cos\theta} d\theta &= J_0(z)[\psi-\pi] + 2 \sum_{n=1}^{\infty} i^n J_n(z) \int_{\pi}^{\psi} \cos n\theta d\theta \\ &= J_0(z)[\psi-\pi] + 2 \sum_{n=1}^{\infty} \frac{i^n}{n} J_n(z) \sin n\psi \quad . \end{aligned} \quad (\text{A-1.2})$$

If we let $iz = K$, then $z = -iK$. Utilizing the fact that $J_n(-iK) = (-1)^n J_n(iK)$, we prove Eq. (1.31):

$$V = \int_{\pi}^{\psi} e^{K\cos\theta} d\theta = J_0(iK)[\psi-\pi] + 2 \sum_{n=1}^{\infty} \frac{(-i)^n}{n} J_n(iK) \sin n\psi \quad (\text{A-1.3})$$

To prove Eq. (1.19), we now let $iz = -K$; then $z = iK$. Let

$$W = \int_{\pi}^{\psi} e^{-K\cos\theta} d\theta = J_0(iK)[\psi-\pi] + 2 \sum_{n=1}^{\infty} \frac{i^n}{n} J_n(iK) \sin n\psi. \quad (\text{A-1.4})$$

Then

$$\begin{aligned} \frac{\partial W}{\partial z} &= \frac{1}{i} \frac{\partial W}{\partial K} = \frac{-i}{i} \int_{\pi}^{\psi} \cos\theta e^{-K\cos\theta} d\theta \\ &= [\psi-\pi] \frac{dJ_0(z)}{dz} + 2 \sum_{n=1}^{\infty} \frac{i^n}{n} \sin n\psi \frac{dJ_n(z)}{dz} \quad . \end{aligned} \quad (\text{A-1.5})$$

Noting that $i^{n+1} = -i^{n-1}$, and using the well-known recurrence relationships

$$\frac{dJ_0(x)}{dx} = -J_1(x) \text{ and } \frac{dJ_n(x)}{dx} = \frac{i}{2} [J_{n-1}(x) - J_{n+1}(x)] \quad (\text{A } 16)$$

we have

$$U = \int_{\pi}^{\psi} \cos \theta e^{-K \cos \theta} d\theta = -i J_0(iK) [\pi - \psi] \\ + \sum_{n=1}^{\infty} \frac{i^{n+1}}{n} [J_{n-1}(iK) - J_{n+1}(iK)] \sin n \psi$$

Similar successive partial differentiations may be used to expand the integrals involved in evaluating the higher terms in Eq. (15) and in evaluating $\partial U / \partial K$ [Eq. (69)]

APPENDIX II

TABLES OF FUNCTIONS FOR IDEALIZED FIELD SPECTROMETERS

Some of the functions required for computation of spectrometer properties in the absence of fringing fields have been calculated using a 704-computer.* Floating decimal notation is used in all the tables. All results are given as numbers with magnitude between 0 and 1; the decimal point is given following the number. Thus: E-01 following the number signifies multiplication by 10^{-1} ; E 00 means x1; E 01 means x10; etc.

Table II.1. $U(K, \psi)$, computed from Eq. (1.18) provides values for $\psi = 0^\circ(2^\circ)180^\circ$ and $K = 0.1(0.05)0.4(0.02)0.5(0.01)0.8(0.02)1.0$. The region $180^\circ < \psi < 360^\circ$ may be determined from the relation calculated from Eq. (1.18) or Eq. (1.19).

$$U(K, -\psi) = U(K, 2\pi - \psi) = -U(K, \psi) \quad . \quad (A-II.1)$$

Table II.2. $V(K, \psi)$, computed from the integral in Eq. (1.31) provides values for $\psi = 0^\circ(5^\circ)180^\circ$ and $K = 0.1(0.05)0.5(0.02)0.9(0.05)1.0$. The region $180^\circ < \psi < 360^\circ$ may be determined from the relation

$$V(K, -\psi) = V(K, 2\pi - \psi) = -V(K, \psi) \quad . \quad (A-II.2)$$

Table II.3. The coordinates of the profiles of a symmetric spectrometer have been calculated from Eqs. (2.1a), (2.1b) and (3.5), taking the dimensions in units of z_s and setting $n_s = 0$. In these units, $r_e = f_s$. The coordinates are computed for $\psi_s = \psi_s^\circ(5^\circ)180^\circ$ and for $K = 0.1(0.1)0.4(0.05)0.5(0.02)0.9(0.05)1.0$, where ψ_s° is the closest ψ_s -value to the critical ψ_s -angle, and evidently varies with K .

Table II.4. The orbit rotation function $F_1(K, \psi_s)$ is calculated from Eq. (5.19c), using the symmetric spectrometer profiles of Table II.3 [and Fig. (3.3)] for $n_s = 0$. The function is computed for $\psi_s = \psi_s^\circ(10^\circ)180^\circ$ and $K = 0.1(0.1)0.4(0.05)0.5(0.06, 0.04^{**})0.9(0.1)1.0$. Of the angles listed, ψ_s° is the closest to the critical ψ_s -angle and varies with K ; it may not be identical to the ψ_s° of Table II.3 because of the difference in $\Delta\psi_s$.

Table II.5. The dispersion function G is calculated from Eq. (6.11) for the case of the symmetric spectrometer profiles of Table II.3 [and Fig. (3.3)] with $n_s = 0$. The function is computed for $\psi_s = \psi_s^\circ(5^\circ)180^\circ$ and $K = 0.2, 0.4(0.05)0.5(0.02)0.9(0.05)1.0$. ψ_s° is defined as in Tables II.3 and II.4.

*We are indebted to Richard King and Kenneth Hillstrom of the Applied Mathematics Division for programming this computation.

**Alternate steps of 0.06 and 0.04

Table II.6. The function $\partial U / \partial K$ is computed from Eq. (6.9a) by numerical integration of

$$\frac{\partial U}{\partial K} = - \int_{\pi}^{\psi} \cos^2 \psi e^{-K \cos \psi} d\psi \quad .$$

Values are provided for $\psi = 0^\circ(2^\circ)180^\circ$ and for $K = 0.1(0.05)0.5(0.01)0.6(0.02)0.9(0.5)1.0$.

Table A-II.1

Trajectory Function $U(K, \psi)$

U(K,ψ)

TRAJECTORY FUNCTIONS

K =	1.00000E-01	0.15000E-00	0.30000E-00	0.45000E-00	0.60000E-00	0.75000E-00	0.90000E-00
ψ							
0	0.15728E-00	0.22679E-00	0.31574E-00	0.42957E-00	0.57657E-00	0.75824E-00	0.96409E-00
2	0.18986E-00	0.26633E-00	0.34431E-00	0.47290E-00	0.65024E-00	0.87844E-00	0.66427E-00
4	0.22040E-00	0.29633E-00	0.37286E-00	0.45012E-00	0.62825E-00	0.86072E-00	0.69775E-00
6	0.25188E-00	0.32628E-00	0.40135E-00	0.47722E-00	0.65544E-00	0.86305E-00	0.71109E-00
8	0.28329E-00	0.35613E-00	0.42977E-00	0.50426E-00	0.67027E-00	0.85543E-00	0.72428E-00
10	0.31448E-00	0.38586E-00	0.45805E-00	0.53105E-00	0.68504E-00	0.84803E-00	0.73761E-00
12	0.34548E-00	0.41542E-00	0.48671E-00	0.55799E-00	0.69983E-00	0.84083E-00	0.75074E-00
14	0.37629E-00	0.44482E-00	0.51420E-00	0.58445E-00	0.71463E-00	0.83363E-00	0.76387E-00
16	0.40700E-00	0.47399E-00	0.54105E-00	0.61133E-00	0.72943E-00	0.82643E-00	0.77700E-00
18	0.43724E-00	0.50290E-00	0.56795E-00	0.63774E-00	0.74423E-00	0.81923E-00	0.79013E-00
20	0.46737E-00	0.53154E-00	0.59487E-00	0.66437E-00	0.75903E-00	0.81203E-00	0.80326E-00
22	0.49705E-00	0.55987E-00	0.62139E-00	0.69092E-00	0.77383E-00	0.80483E-00	0.81639E-00
24	0.52625E-00	0.58786E-00	0.64773E-00	0.71730E-00	0.78863E-00	0.79763E-00	0.82952E-00
26	0.55495E-00	0.61547E-00	0.67333E-00	0.74393E-00	0.80343E-00	0.79043E-00	0.84265E-00
28	0.58325E-00	0.64268E-00	0.69893E-00	0.77053E-00	0.81823E-00	0.78323E-00	0.85578E-00
30	0.61105E-00	0.66946E-00	0.72453E-00	0.79713E-00	0.83303E-00	0.77603E-00	0.86891E-00
32	0.63835E-00	0.69577E-00	0.75013E-00	0.82373E-00	0.84783E-00	0.76883E-00	0.88204E-00
34	0.66515E-00	0.72158E-00	0.77573E-00	0.85033E-00	0.86263E-00	0.76163E-00	0.89517E-00
36	0.69145E-00	0.74696E-00	0.80133E-00	0.87693E-00	0.87743E-00	0.75443E-00	0.90830E-00
38	0.71725E-00	0.77199E-00	0.82693E-00	0.90353E-00	0.89223E-00	0.74723E-00	0.92143E-00
 

U(K, ψ)

TRAJECTORY FUNCTIONS

K	ψ	0.42000E-00	0.44000E-00	0.46000E-00	0.48000E-00	0.50000E-00	0.51000E-00	0.52000E-00
2	0.67499E 00	0.70801E 00	0.74185E 00	0.77591E 00	0.81020E 00	0.82744E 00	0.84474E 00	0.86549E 00
4	0.69722E 00	0.73049E 00	0.76388E 00	0.79751E 00	0.83137E 00	0.84840E 00	0.86549E 00	0.88263E 00
6	0.72024E 00	0.75296E 00	0.78590E 00	0.81909E 00	0.85253E 00	0.86934E 00	0.88623E 00	0.90319E 00
8	0.74312E 00	0.77539E 00	0.80789E 00	0.84065E 00	0.87366E 00	0.89077E 00	0.90694E 00	0.92326E 00
10	0.76596E 00	0.79777E 00	0.82984E 00	0.86216E 00	0.89475E 00	0.91115E 00	0.92762E 00	0.94427E 00
12	0.78873E 00	0.82010E 00	0.85173E 00	0.88367E 00	0.91579E 00	0.93198E 00	0.94825E 00	0.96461E 00
14	0.81142E 00	0.84235E 00	0.87354E 00	0.90501E 00	0.93676E 00	0.95275E 00	0.96881E 00	0.98493E 00
16	0.83400E 00	0.86450E 00	0.89527E 00	0.92632E 00	0.95766E 00	0.97344E 00	0.98930E 00	0.10097E 01
18	0.85648E 00	0.88654E 00	0.91689E 00	0.94752E 00	0.97846E 00	0.99404E 00	0.10097E 01	0.10200E 01
20	0.87892E 00	0.90846E 00	0.93829E 00	0.96862E 00	0.99915E 00	0.10145E 01	0.10200E 01	0.10303E 01
22	0.90100E 00	0.93023E 00	0.95975E 00	0.98958E 00	0.10197E 01	0.10349E 01	0.10303E 01	0.10406E 01
24	0.92302E 00	0.95184E 00	0.98096E 00	0.10147E 01	0.10402E 01	0.10552E 01	0.10406E 01	0.10509E 01
26	0.94485E 00	0.97327E 00	0.10020E 01	0.10311E 01	0.10460E 01	0.10753E 01	0.10509E 01	0.10612E 01
28	0.96647E 00	0.99450E 00	0.10228E 01	0.10515E 01	0.10805E 01	0.10952E 01	0.10612E 01	0.10715E 01
30	0.98786E 00	0.10155E 01	0.10435E 01	0.10718E 01	0.11005E 01	0.11149E 01	0.10715E 01	0.10818E 01
32	0.10090E 01	0.10363E 01	0.10639E 01	0.10919E 01	0.11202E 01	0.11345E 01	0.10818E 01	0.10927E 01
34	0.10299E 01	0.10568E 01	0.10841E 01	0.11117E 01	0.11397E 01	0.11538E 01	0.10927E 01	0.11036E 01
36	0.10505E 01	0.10770E 01	0.11040E 01	0.11319E 01	0.11589E 01	0.11729E 01	0.11036E 01	0.11145E 01
38	0.10707E 01	0.10970E 01	0.11236E 01	0.11506E 01	0.11779E 01	0.11917E 01	0.11145E 01	0.11254E 01
40	0.10907E 01	0.11166E 01	0.11429E 01	0.11696E 01	0.11966E 01	0.12103E 01	0.11254E 01	0.11363E 01
42	0.11102E 01	0.11359E 01	0.11619E 01	0.11887E 01	0.12150E 01	0.12285E 01	0.11363E 01	0.11472E 01
44	0.11294E 01	0.11548E 01	0.11805E 01	0.12166E 01	0.12331E 01	0.12464E 01	0.11472E 01	0.11581E 01
46	0.11482E 01	0.11733E 01	0.11987E 01	0.12246E 01	0.12508E 01	0.12640E 01	0.11581E 01	0.11690E 01
48	0.11665E 01	0.11914E 01	0.12166E 01	0.12421E 01	0.12681E 01	0.12812E 01	0.11690E 01	0.11809E 01
50	0.11844E 01	0.12090E 01	0.12339E 01	0.12593E 01	0.12850E 01	0.12980E 01	0.11809E 01	0.11928E 01
52	0.12018E 01	0.12261E 01	0.12509E 01	0.12760E 01	0.13015E 01	0.13144E 01	0.11928E 01	0.12047E 01
54	0.12187E 01	0.12428E 01	0.12673E 01	0.12922E 01	0.13175E 01	0.13304E 01	0.12047E 01	0.12166E 01
56	0.12350E 01	0.12589E 01	0.12833E 01	0.13080E 01	0.13331E 01	0.13458E 01	0.12166E 01	0.12285E 01
58	0.12507E 01	0.12745E 01	0.12986E 01	0.13232E 01	0.13481E 01	0.13607E 01	0.12285E 01	0.12404E 01
60	0.12658E 01	0.12894E 01	0.13134E 01	0.13378E 01	0.13626E 01	0.13751E 01	0.12404E 01	0.12523E 01
62	0.12803E 01	0.13038E 01	0.13276E 01	0.13519E 01	0.13765E 01	0.13890E 01	0.12523E 01	0.12642E 01
64	0.12941E 01	0.13174E 01	0.13411E 01	0.13653E 01	0.13898E 01	0.14022E 01	0.12642E 01	0.12761E 01
66	0.13072E 01	0.13304E 01	0.13540E 01	0.13780E 01	0.14024E 01	0.14148E 01	0.12761E 01	0.12880E 01
68	0.13201E 01	0.13427E 01	0.13662E 01	0.13900E 01	0.14143E 01	0.14266E 01	0.12880E 01	0.12999E 01
70	0.13321E 01	0.13541E 01	0.13775E 01	0.14013E 01	0.14256E 01	0.14378E 01	0.12999E 01	0.13118E 01
72	0.13431E 01	0.13648E 01	0.13882E 01	0.14119E 01	0.14360E 01	0.14482E 01	0.13118E 01	0.13237E 01
74	0.13541E 01	0.13747E 01	0.13979E 01	0.14216E 01	0.14457E 01	0.14579E 01	0.13237E 01	0.13356E 01
76	0.13648E 01	0.13836E 01	0.14069E 01	0.14305E 01	0.14545E 01	0.14667E 01	0.13356E 01	0.13475E 01
78	0.13747E 01	0.13917E 01	0.14149E 01	0.14394E 01	0.14634E 01	0.14746E 01	0.13475E 01	0.13594E 01
80	0.13836E 01	0.13988E 01	0.14229E 01	0.14485E 01	0.14724E 01	0.14816E 01	0.13594E 01	0.13713E 01
82	0.13917E 01	0.14049E 01	0.14281E 01	0.14576E 01	0.14815E 01	0.14906E 01	0.13713E 01	0.13832E 01
84	0.13988E 01	0.14100E 01	0.14331E 01	0.14666E 01	0.14905E 01	0.14996E 01	0.13832E 01	0.13951E 01
86	0.14049E 01	0.14141E 01	0.14372E 01	0.14755E 01	0.14994E 01	0.15085E 01	0.13951E 01	0.14070E 01
88	0.14100E 01	0.14170E 01	0.14401E 01	0.14842E 01	0.15083E 01	0.15174E 01	0.14070E 01	0.14189E 01
90	0.14141E 01	0.14188E 01	0.14419E 01	0.14930E 01	0.15172E 01	0.15263E 01	0.14189E 01	0.14308E 01
92	0.14188E 01	0.14194E 01	0.14425E 01	0.15017E 01	0.15261E 01	0.15352E 01	0.14308E 01	0.14427E 01
94	0.14235E 01	0.14188E 01	0.14418E 01	0.15103E 01	0.15350E 01	0.15441E 01	0.14427E 01	0.14546E 01
96	0.14282E 01	0.14169E 01	0.14400E 01	0.15189E 01	0.15438E 01	0.15529E 01	0.14546E 01	0.14665E 01
98	0.14329E 01	0.14137E 01	0.14368E 01	0.15275E 01	0.15526E 01	0.15617E 01	0.14665E 01	0.14784E 01
100	0.14376E 01	0.14092E 01	0.14323E 01	0.15361E 01	0.15614E 01	0.15705E 01	0.14784E 01	0.14903E 01
102	0.14423E 01	0.14034E 01	0.14264E 01	0.15446E 01	0.15702E 01	0.15793E 01	0.14903E 01	0.15022E 01
104	0.14469E 01	0.13961E 01	0.14192E 01	0.15531E 01	0.15790E 01	0.15881E 01	0.15022E 01	0.15141E 01
106	0.14516E 01	0.13875E 01	0.14104E 01	0.15616E 01	0.15878E 01	0.15969E 01	0.15141E 01	0.15260E 01
108	0.14563E 01	0.13773E 01	0.14003E 01	0.15701E 01	0.15966E 01	0.16057E 01	0.15260E 01	0.15379E 01
110	0.14609E 01	0.13657E 01	0.13886E 01	0.15786E 01	0.16054E 01	0.16145E 01	0.15379E 01	0.15498E 01
112	0.14656E 01	0.13526E 01	0.13754E 01	0.15871E 01	0.16143E 01	0.16234E 01	0.15498E 01	0.15617E 01
114	0.14703E 01	0.13380E 01	0.13606E 01	0.15956E 01	0.16232E 01	0.16323E 01	0.15617E 01	0.15736E 01
116	0.14750E 01	0.13218E 01	0.13444E 01	0.16041E 01	0.16321E 01	0.16412E 01	0.15736E 01	0.15855E 01
118	0.14797E 01	0.13040E 01	0.13264E 01	0.16126E 01	0.16410E 01	0.16501E 01	0.15855E 01	0.15974E 01
120	0.14844E 01	0.12847E 01	0.13069E 01	0.16211E 01	0.16499E 01	0.16590E 01	0.15974E 01	0.16093E 01
122	0.14891E 01	0.12637E 01	0.12857E 01	0.16296E 01	0.16588E 01	0.16679E 01	0.16093E 01	0.16212E 01
124	0.14938E 01	0.12412E 01	0.12629E 01	0.16383E 01	0.16677E 01	0.16768E 01	0.16212E 01	0.16331E 01
126	0.14985E 01	0.12170E 01	0.12385E 01	0.16470E 01	0.16766E 01	0.16857E 01	0.16331E 01	0.16450E 01
128	0.15032E 01	0.11912E 01	0.12124E 01	0.16557E 01	0.16855E 01	0.16946E 01	0.16450E 01	0.16569E 01
130	0.15079E 01	0.11639E 01	0.11847E 01	0.16644E 01	0.16944E 01	0.17035E 01	0.16569E 01	0.16688E 01
132	0.15126E 01	0.11349E 01	0.11554E 01	0.16731E 01	0.17033E 01	0.17124E 01	0.16688E 01	0.16807E 01
134	0.15173E 01	0.11043E 01	0.11244E 01	0.16818E 01	0.17122E 01	0.17213E 01	0.16807E 01	0.16926E 01
136	0.15220E 01	0.10722E 01	0.10918E 01	0.16905E 01	0.17211E 01	0.17302E 01	0.16926E 01	0.17045E 01
138	0.15267E 01	0.10385E 01	0.10577E 01	0.16992E 01	0.17300E 01	0.17391E 01	0.17045E 01	0.17164E 01
140	0.15314E 01	0.10033E 01	0.10219E 01	0.17079E 01	0.17389E 01	0.17480E 01	0.17164E 01	0.17283E 01
142	0.15361E 01	0.96655E 00	0.98466E 00	0.17166E 01	0.17478E 01	0.17569E 01	0.17283E 01	0.17402E 01
144	0.15408E 01	0.92836E 00	0.94588E 00	0.17253E 01	0.17567E 01	0.17658E 01	0.17402E 01	0.17521E 01
146	0.15455E 01	0.88875E 00	0.90562E 00	0.17340E 01	0.17656E 01	0.17747E 01	0.17521E 01	0.17640E 01
148	0.15502E 01	0.84775E 00	0.86395E 00	0.17427E 01	0.17745E 01	0.17836E 01	0.17640E 01	0.17759E 01
150	0.15549E 01	0.80541E 00	0.82089E 00	0.17514E 01	0.17834E 01	0.17925E 01	0.17759E 01	0.17878E 01
152	0.15596E 01	0.76179E 00	0.77651E 00	0.17601E 01	0.17923E 01	0.18014E 01	0.17878E 01	0.17997E 01
154	0.15643E 01	0.71693E 00	0.73088E 00	0.17688E 01	0.18012E 01	0.18103E 01	0.17997E 01	0.18116E 01
156	0.15690E 01	0.67090E 00	0.68401E 00	0.17775E 01	0.18101E 01	0.18192E 01	0.18116E 01	0.18235E 01
158	0.15737E 01	0.62377E 00	0.63601E 00	0.17862E 01	0.18190E 01	0.18281E 01	0.18235E 01	0.18354E 01
160	0.15784E 01	0.57559E 00	0.58694E 00	0.17949E 01	0.18279E 01	0.18370E 01	0.18354E 01	0.18473E 01
162	0.15831E 01	0.52645E 00	0.53688E 00	0.18036E 01	0.18368E 01	0.18459E 01	0.18473E 01	0.18592E 01
164	0.15878E 01	0.47642E 00	0.48589E 00	0.18123E 01	0.18457E 01	0.18548E 01	0.18592E 01	0.18711E 01
166	0.15925E 01	0.42558E 00	0.43407E 00	0.18210E 01	0.18546E 01	0.18637E 01	0.18711E 01	0.18830E 01
168	0.15972E 01	0.37401E 00	0.38149E 00	0.18297E 01	0.18635E 01	0.18726E 01	0.18830E 01	0.18949E 01
170	0.16019E 01	0.32180E 00	0.32828E 00	0.18384E 01	0.18724E 01	0.18815E 01	0.18949E 01	0.19068E 01
172	0.16066E 01	0.26903E 00	0.27443E 00	0.18471E 01	0.18813E 01	0.18904E 01	0.19068E 01	0.19187E 01
174	0.16113E 01	0.21579E 00</						

U(K,ψ)

TRAJECTORY FUNCTIONS

K =	ψ	0.53000E 00	0.54000E 00	0.55000E 00	0.56000E 00	0.57000E 00	0.58000E 00	0.59000E 00
0	0.8671CF 00	0.87953F 00	0.89707F 00	0.91458F 00	0.93221F 00	0.94991F 00	0.96769F 00	0.98704E 00
2	0.88264F 00	0.89987F 00	0.91716F 00	0.93452F 00	0.95195F 00	0.96946E 00	0.98704E 00	0.98704E 00
4	0.90318F 00	0.92020F 00	0.93729F 00	0.95445E 00	0.97168E 00	0.98899F 00	0.10064E 01	0.10257E 01
6	0.92269F 00	0.94050F 00	0.95739F 00	0.97435F 00	0.99139F 00	0.10085E 01	0.10257E 01	0.10450E 01
8	0.94416F 00	0.96077F 00	0.97746F 00	0.99422F 00	0.10111F 01	0.10280F 01	0.10450E 01	0.10642E 01
10	0.96458F 00	0.98100F 00	0.99749E 00	0.10141F 01	0.10307E 01	0.10474E 01	0.10642E 01	0.10834E 01
12	0.98495F 00	0.10012F 01	0.10175F 01	0.10338E 01	0.10503E 01	0.10668E 01	0.10834E 01	0.11026E 01
14	0.10052E 01	0.10213E 01	0.10374E 01	0.10535E 01	0.10698E 01	0.10861E 01	0.11026E 01	0.11216E 01
16	0.10254F 01	0.10413F 01	0.10572F 01	0.10732F 01	0.10892E 01	0.11054E 01	0.11216E 01	0.11406F 01
18	0.10456F 01	0.10612F 01	0.10769F 01	0.10927F 01	0.11086E 01	0.11246F 01	0.11406F 01	0.11595F 01
20	0.10656F 01	0.10810F 01	0.10965F 01	0.11121F 01	0.11278F 01	0.11436E 01	0.11595F 01	0.11783E 01
22	0.10854F 01	0.11007E 01	0.11160F 01	0.11315F 01	0.11470F 01	0.11626E 01	0.11783E 01	0.11970E 01
24	0.11051E 01	0.11202E 01	0.11354E 01	0.11507E 01	0.11660E 01	0.11814E 01	0.11970E 01	0.12155E 01
26	0.11247F 01	0.11396E 01	0.11546E 01	0.11697E 01	0.11849E 01	0.12001E 01	0.12155E 01	0.12339E 01
28	0.11441E 01	0.11588E 01	0.11737E 01	0.11886E 01	0.12036E 01	0.12187E 01	0.12339E 01	0.12521E 01
30	0.11633F 01	0.11779F 01	0.11925F 01	0.12073F 01	0.12221E 01	0.12371F 01	0.12521E 01	0.12702E 01
32	0.11823F 01	0.11967F 01	0.12112F 01	0.12258E 01	0.12405E 01	0.12553E 01	0.12702E 01	0.12880E 01
34	0.12011F 01	0.12153F 01	0.12297F 01	0.12441E 01	0.12586E 01	0.12733E 01	0.12880E 01	0.13056E 01
36	0.12196F 01	0.12337F 01	0.12479F 01	0.12622E 01	0.12766E 01	0.12910F 01	0.13056E 01	0.13230E 01
38	0.12379F 01	0.12518F 01	0.12659F 01	0.12800E 01	0.12942E 01	0.13086E 01	0.13230E 01	0.13402F 01
40	0.12558E 01	0.12696F 01	0.12835F 01	0.12976E 01	0.13117E 01	0.13259E 01	0.13402F 01	0.13571E 01
42	0.12735F 01	0.12872F 01	0.13009F 01	0.13148F 01	0.13288E 01	0.13429F 01	0.13571E 01	0.13736E 01
44	0.12918F 01	0.13044F 01	0.13180F 01	0.13318F 01	0.13456E 01	0.13596E 01	0.13736E 01	0.13899F 01
46	0.13078F 01	0.13212F 01	0.13347F 01	0.13484F 01	0.13621E 01	0.13760E 01	0.13899F 01	0.14058E 01
48	0.13244F 01	0.13377F 01	0.13511F 01	0.13646E 01	0.13784E 01	0.13920E 01	0.14058E 01	0.14214E 01
50	0.13405F 01	0.13537F 01	0.13671F 01	0.13806E 01	0.13940F 01	0.14076E 01	0.14214E 01	0.14365E 01
52	0.13563F 01	0.13694F 01	0.13826F 01	0.13959E 01	0.14093E 01	0.14229E 01	0.14365E 01	0.14513E 01
54	0.13715F 01	0.13846F 01	0.13977E 01	0.14109E 01	0.14243E 01	0.14377E 01	0.14513E 01	0.14655F 01
56	0.13863F 01	0.13992F 01	0.14123F 01	0.14254E 01	0.14387E 01	0.14521E 01	0.14655F 01	0.14793E 01
58	0.14006F 01	0.14134F 01	0.14264E 01	0.14394E 01	0.14526E 01	0.14659E 01	0.14793E 01	0.14926E 01
60	0.14142F 01	0.14270F 01	0.14399F 01	0.14529F 01	0.14660E 01	0.14793E 01	0.14926E 01	0.15053E 01
62	0.14273F 01	0.14400F 01	0.14529F 01	0.14658E 01	0.14789E 01	0.14920E 01	0.15053E 01	0.15174F 01
64	0.14398F 01	0.14524F 01	0.14652F 01	0.14781E 01	0.14911E 01	0.15042F 01	0.15174F 01	0.15289E 01
66	0.14516F 01	0.14642E 01	0.14769F 01	0.14897F 01	0.15027E 01	0.15157E 01	0.15289E 01	0.15397E 01
68	0.14627F 01	0.14752F 01	0.14879E 01	0.15007E 01	0.15136E 01	0.15266E 01	0.15397E 01	0.15499F 01
70	0.14730F 01	0.14855F 01	0.14982F 01	0.15109F 01	0.15238F 01	0.15368F 01	0.15499F 01	0.15592F 01
72	0.14826F 01	0.14951F 01	0.15077F 01	0.15204F 01	0.15332F 01	0.15462E 01	0.15592F 01	0.15678E 01
74	0.14913E 01	0.15038E 01	0.15164E 01	0.15291E 01	0.15419E 01	0.15548E 01	0.15678E 01	0.15756F 01
76	0.15002F 01	0.15116F 01	0.15242F 01	0.15369F 01	0.15497E 01	0.15626F 01	0.15756F 01	0.15825F 01
78	0.15091F 01	0.15186F 01	0.15311F 01	0.15438F 01	0.15566F 01	0.15695E 01	0.15825F 01	0.15934F 01
80	0.15172F 01	0.15266F 01	0.15371F 01	0.15498E 01	0.15625E 01	0.15754E 01	0.15934F 01	0.16002F 01
82	0.15252F 01	0.15296F 01	0.15421F 01	0.15548E 01	0.15675F 01	0.15804F 01	0.16002F 01	0.16076F 01
84	0.15322F 01	0.15336E 01	0.15461F 01	0.15588E 01	0.15715E 01	0.15844E 01	0.16076F 01	0.16170E 01
86	0.15391F 01	0.15365E 01	0.15490F 01	0.15617E 01	0.15744F 01	0.15873E 01	0.16170E 01	0.16270E 01
88	0.15458F 01	0.15383F 01	0.15508F 01	0.15634F 01	0.15762E 01	0.15890E 01	0.16270E 01	0.16370E 01
90	0.15524F 01	0.15389F 01	0.15514F 01	0.15640F 01	0.15768F 01	0.15896E 01	0.16370E 01	0.16471E 01
92	0.15588F 01	0.15382F 01	0.15508F 01	0.15634E 01	0.15762F 01	0.15890E 01	0.16471E 01	0.16571E 01
94	0.15640F 01	0.15364F 01	0.15489F 01	0.15615F 01	0.15744F 01	0.15871E 01	0.16571E 01	0.16671E 01
96	0.15688E 01	0.15332F 01	0.15457F 01	0.15583E 01	0.15711E 01	0.15839F 01	0.16671E 01	0.16771E 01
98	0.15736F 01	0.15286E 01	0.15411F 01	0.15538E 01	0.15666E 01	0.15794E 01	0.16771E 01	0.16871E 01
100	0.15783F 01	0.15227F 01	0.15352F 01	0.15478E 01	0.15605E 01	0.15734E 01	0.16871E 01	0.16971E 01
102	0.15829F 01	0.15153F 01	0.15278F 01	0.15420E 01	0.15531E 01	0.15659E 01	0.16971E 01	0.17071E 01
104	0.15874F 01	0.15064F 01	0.15189F 01	0.15315E 01	0.15442E 01	0.15570E 01	0.17071E 01	0.17171E 01
106	0.15917E 01	0.14960E 01	0.15085E 01	0.15210E 01	0.15337E 01	0.15465E 01	0.17171E 01	0.17271E 01
108	0.15958F 01	0.14841F 01	0.14965E 01	0.15090E 01	0.15217E 01	0.15344E 01	0.17271E 01	0.17371E 01
110	0.16000F 01	0.14705F 01	0.14829F 01	0.14954F 01	0.15080F 01	0.15207E 01	0.17371E 01	0.17471E 01
112	0.16042F 01	0.14554F 01	0.14677F 01	0.14801E 01	0.14926E 01	0.15053E 01	0.17471E 01	0.17571E 01
114	0.16084F 01	0.14385F 01	0.14508F 01	0.14631E 01	0.14756E 01	0.14882E 01	0.17571E 01	0.17671E 01
116	0.16127F 01	0.14200F 01	0.14321E 01	0.14444E 01	0.14568E 01	0.14693E 01	0.17671E 01	0.17771E 01
118	0.16178F 01	0.13997F 01	0.14118F 01	0.14240E 01	0.14363E 01	0.14487E 01	0.17771E 01	0.17871E 01
120	0.16229F 01	0.13777F 01	0.13897F 01	0.14018F 01	0.14140F 01	0.14263E 01	0.17871E 01	0.17971E 01
122	0.16273F 01	0.13540F 01	0.13658F 01	0.13778F 01	0.13899E 01	0.14020E 01	0.17971E 01	0.18071E 01
124	0.16316F 01	0.13285F 01	0.13402F 01	0.13520F 01	0.13639E 01	0.13759E 01	0.18071E 01	0.18171E 01
126	0.16359F 01	0.13012F 01	0.13127F 01	0.13244F 01	0.13362F 01	0.13480F 01	0.18171E 01	0.18271E 01
128	0.16403F 01	0.12721F 01	0.12835F 01	0.12950F 01	0.13066E 01	0.13182F 01	0.18271E 01	0.18371E 01
130	0.16447F 01	0.12413F 01	0.12524F 01	0.12637F 01	0.12751E 01	0.12866E 01	0.18371E 01	0.18471E 01
132	0.16491F 01	0.12086F 01	0.12196F 01	0.12307E 01	0.12418E 01	0.12531F 01	0.18471E 01	0.18571E 01
134	0.16535F 01	0.11742F 01	0.11849F 01	0.11958E 01	0.12067E 01	0.12177E 01	0.18571E 01	0.18671E 01
136	0.16579F 01	0.11381F 01	0.11485F 01	0.11591F 01	0.11698F 01	0.11805E 01	0.18671E 01	0.18771E 01
138	0.16623F 01	0.11002F 01	0.11104F 01	0.11207F 01	0.11310E 01	0.11415F 01	0.18771E 01	0.18871E 01
140	0.16667F 01	0.10606E 01	0.10705E 01	0.10805E 01	0.10905E 01	0.11007F 01	0.18871E 01	0.18971E 01
142	0.16711F 01	0.10193F 01	0.10289F 01	0.10385F 01	0.10483E 01	0.10581E 01	0.18971E 01	0.19071E 01
144	0.16755F 01	0.97640F 00	0.98562F 00	0.99494F 00	0.10043F 01	0.10138E 01	0.19071E 01	0.19171E 01
146	0.16799F 01	0.93190F 00	0.94076F 00	0.94970F 00	0.95874E 00	0.96785E 00	0.19171E 01	0.19271E 01
148	0.16843F 01	0.88585F 00	0.89432F 00	0.90288E 00	0.91152F 00	0.92024F 00	0.19271E 01	0.19371E 01
150	0.16887F 01	0.83832F 00	0.84639F 00	0.85453F 00	0.86275E 00	0.87105E 00	0.19371E 01	0.19471E 01
152	0.16931F 01	0.78937F 00	0.79700F 00	0.80471E 00	0.81249E 00	0.82035E 00	0.19471E 01	0.19571E 01
154	0.16975F 01	0.73905F 00	0.74623F 00	0.75340F 00	0.76081E 00	0.76821E 00	0.19571E 01	0.19671E 01
156	0.17019F 01	0.68744F 00	0.69416F 00	0.70094E 00	0.70778F 00	0.71470F 00	0.19671E 01	0.19771E 01
158	0.17063F 01	0.63462F 00	0.64085F 00	0.64714E 00	0.65349E 00	0.65990E 00	0.19771E 01	0.19871E 01
160	0.17107F 01	0.58068F 00	0.58640F 00	0.59217E 00	0.59801E 00	0.60390F 00	0.19871E 01	0.19971E 01
162	0.17151F 01	0.52569F 00	0.53088F 00	0.53613F 00	0.54143E 00	0.54679E 00	0.19971E 01	0.20071E 01
164	0.17195F 01	0.46974F 00	0.47440F 00	0.47911F 00	0.48386F 00	0.48866E 00	0.20071E 01	0.20171E 01
166	0.17239F 01	0.41294F 00	0.41705F 00	0.42120F 00	0.42539E 00	0.42963E 00	0.20171E 01	0.20271E 01
168	0.17283F 01	0.35538F 00	0.35938F 00	0.36351E 00	0.36761E 00	0.37178E 00	0.20271E 01	0.20371E 01
170	0.17327F 01	0.29717F 00	0.30014F 00	0.30316E 00	0.30617E 00	0.30924F 00	0.20371E 01	0.20471E 01
172	0.17371F 01	0.23840F 00	0.24079					

U(K, ψ)

TRAJECTORY FUNCTIONS

K	0.60000E 00	0.61000E 00	0.62000E 00	0.63000E 00	0.64000E 00	0.65000E 00	0.66000E 00
ψ							
0	0.98553E 00	0.10035E 01	0.10214E 01	0.10395E 01	0.10577E 01	0.10759E 01	0.10942E 01
2	0.10047E 01	0.10224E 01	0.10402E 01	0.10581E 01	0.10761E 01	0.10941E 01	0.11123E 01
4	0.10038E 01	0.10414E 01	0.10590E 01	0.10767E 01	0.10945E 01	0.11123E 01	0.11303E 01
6	0.10430E 01	0.10603E 01	0.10777E 01	0.10953E 01	0.11129E 01	0.11305E 01	0.11483E 01
8	0.10621E 01	0.10792E 01	0.10965E 01	0.11138E 01	0.11312E 01	0.11487E 01	0.11663E 01
10	0.10811E 01	0.10981E 01	0.11152E 01	0.11323E 01	0.11495E 01	0.11669E 01	0.11843E 01
12	0.11001E 01	0.11169E 01	0.11338E 01	0.11508E 01	0.11678E 01	0.11850E 01	0.12022E 01
14	0.11191E 01	0.11357E 01	0.11524E 01	0.11692E 01	0.11860E 01	0.12030E 01	0.12201E 01
16	0.11380E 01	0.11544E 01	0.11709E 01	0.11875E 01	0.12042E 01	0.12210E 01	0.12379E 01
18	0.11568E 01	0.11730E 01	0.11894E 01	0.12058E 01	0.12223E 01	0.12389E 01	0.12556E 01
20	0.11755E 01	0.11916E 01	0.12077E 01	0.12240E 01	0.12403E 01	0.12568E 01	0.12733E 01
22	0.11941E 01	0.12100E 01	0.12260E 01	0.12421E 01	0.12583E 01	0.12745E 01	0.12909E 01
24	0.12126E 01	0.12283E 01	0.12441E 01	0.12601E 01	0.12761E 01	0.12922E 01	0.13084E 01
26	0.12310E 01	0.12465E 01	0.12622E 01	0.12779E 01	0.12938E 01	0.13098E 01	0.13258E 01
28	0.12492E 01	0.12646E 01	0.12801E 01	0.12957E 01	0.13114E 01	0.13272E 01	0.13431E 01
30	0.12673E 01	0.12825E 01	0.12978E 01	0.13133E 01	0.13288E 01	0.13445E 01	0.13602E 01
32	0.12851E 01	0.13002E 01	0.13154E 01	0.13307E 01	0.13461E 01	0.13616E 01	0.13772E 01
34	0.13028E 01	0.13178E 01	0.13328E 01	0.13480E 01	0.13632E 01	0.13786E 01	0.13941E 01
36	0.13203E 01	0.13351E 01	0.13500E 01	0.13650E 01	0.13802E 01	0.13954E 01	0.14107E 01
38	0.13376E 01	0.13523E 01	0.13670E 01	0.13819E 01	0.13969E 01	0.14120E 01	0.14272E 01
40	0.13546E 01	0.13691E 01	0.13838E 01	0.13985E 01	0.14134E 01	0.14283E 01	0.14434E 01
42	0.13714E 01	0.13858E 01	0.14002E 01	0.14149E 01	0.14296E 01	0.14445E 01	0.14594E 01
44	0.13878E 01	0.14021E 01	0.14165E 01	0.14310E 01	0.14456E 01	0.14603E 01	0.14752E 01
46	0.14040E 01	0.14181E 01	0.14324E 01	0.14468E 01	0.14613E 01	0.14759E 01	0.14906E 01
48	0.14198E 01	0.14338E 01	0.14480E 01	0.14623E 01	0.14767E 01	0.14912E 01	0.15058E 01
50	0.14352E 01	0.14492E 01	0.14633E 01	0.14774E 01	0.14917E 01	0.15061E 01	0.15207E 01
52	0.14503E 01	0.14642E 01	0.14781E 01	0.14922E 01	0.15064E 01	0.15207E 01	0.15352E 01
54	0.14649E 01	0.14787E 01	0.14926E 01	0.15066E 01	0.15207E 01	0.15349E 01	0.15493E 01
56	0.14791E 01	0.14928E 01	0.15066E 01	0.15205E 01	0.15346E 01	0.15487E 01	0.15630E 01
58	0.14928E 01	0.15064E 01	0.15200E 01	0.15340E 01	0.15480E 01	0.15621E 01	0.15763E 01
60	0.15060E 01	0.15196E 01	0.15332E 01	0.15470E 01	0.15609E 01	0.15749E 01	0.15891E 01
62	0.15187E 01	0.15322E 01	0.15458E 01	0.15595E 01	0.15733E 01	0.15873E 01	0.16014E 01
64	0.15307E 01	0.15442E 01	0.15577E 01	0.15714E 01	0.15852E 01	0.15991E 01	0.16131E 01
66	0.15422E 01	0.15556E 01	0.15691E 01	0.15827E 01	0.15964E 01	0.16103E 01	0.16243E 01
68	0.15530E 01	0.15663E 01	0.15798E 01	0.15934E 01	0.16071E 01	0.16209E 01	0.16348E 01
70	0.15631E 01	0.15764E 01	0.15898E 01	0.16033E 01	0.16170E 01	0.16308E 01	0.16447E 01
72	0.15724E 01	0.15857E 01	0.15991E 01	0.16126E 01	0.16267E 01	0.16400E 01	0.16538E 01
74	0.15810E 01	0.15942E 01	0.16076E 01	0.16211E 01	0.16347E 01	0.16484E 01	0.16622E 01
76	0.15897E 01	0.16019E 01	0.16153E 01	0.16288E 01	0.16423E 01	0.16560E 01	0.16699E 01
78	0.15956E 01	0.16088E 01	0.16221E 01	0.16356E 01	0.16491E 01	0.16628E 01	0.16766E 01
80	0.16015E 01	0.16147E 01	0.16280E 01	0.16415E 01	0.16550E 01	0.16687E 01	0.16825E 01
82	0.16065E 01	0.16197E 01	0.16330E 01	0.16464E 01	0.16600E 01	0.16736E 01	0.16874E 01
84	0.16114E 01	0.16236E 01	0.16369E 01	0.16504E 01	0.16639E 01	0.16776E 01	0.16913E 01
86	0.16163E 01	0.16265E 01	0.16398E 01	0.16532E 01	0.16668E 01	0.16804E 01	0.16942E 01
88	0.16211E 01	0.16284E 01	0.16424E 01	0.16550E 01	0.16685E 01	0.16822E 01	0.16960E 01
90	0.16257E 01	0.16289E 01	0.16422E 01	0.16556E 01	0.16691E 01	0.16828E 01	0.16966E 01
92	0.16315E 01	0.16282E 01	0.16416E 01	0.16550E 01	0.16685E 01	0.16822E 01	0.16960E 01
94	0.16332E 01	0.16264E 01	0.16397E 01	0.16531E 01	0.16666E 01	0.16803E 01	0.16941E 01
96	0.16300E 01	0.16231E 01	0.16365E 01	0.16499E 01	0.16634E 01	0.16771E 01	0.16908E 01
98	0.16254E 01	0.16186E 01	0.16319E 01	0.16453E 01	0.16588E 01	0.16725E 01	0.16862E 01
100	0.16199E 01	0.16126E 01	0.16258E 01	0.16392E 01	0.16528E 01	0.16664E 01	0.16802E 01
102	0.16139E 01	0.16051E 01	0.16182E 01	0.16317E 01	0.16452E 01	0.16589E 01	0.16726E 01
104	0.16072E 01	0.15961E 01	0.16093E 01	0.16227E 01	0.16362E 01	0.16498E 01	0.16635E 01
106	0.15994E 01	0.15855E 01	0.15987E 01	0.16120E 01	0.16255E 01	0.16391E 01	0.16528E 01
108	0.15902E 01	0.15733E 01	0.15865E 01	0.15999E 01	0.16132E 01	0.16267E 01	0.16404E 01
110	0.15764E 01	0.15594E 01	0.15726E 01	0.15858E 01	0.15992E 01	0.16127E 01	0.16263E 01
112	0.15609E 01	0.15438E 01	0.15569E 01	0.15701E 01	0.15835E 01	0.15969E 01	0.16105E 01
114	0.15436E 01	0.15265E 01	0.15396E 01	0.15527E 01	0.15659E 01	0.15793E 01	0.15928E 01
116	0.15246E 01	0.15074E 01	0.15204E 01	0.15324E 01	0.15466E 01	0.15599E 01	0.15733E 01
118	0.15048E 01	0.14865E 01	0.14994E 01	0.15123E 01	0.15254E 01	0.15386E 01	0.15519E 01
120	0.14841E 01	0.14638E 01	0.14765E 01	0.14944E 01	0.15023E 01	0.15154E 01	0.15286E 01
122	0.14627E 01	0.14392E 01	0.14518E 01	0.14645E 01	0.14773E 01	0.14903E 01	0.15034E 01
124	0.14403E 01	0.14127E 01	0.14251E 01	0.14377E 01	0.14504E 01	0.14632E 01	0.14761E 01
126	0.14171E 01	0.13843E 01	0.13966E 01	0.14090E 01	0.14215E 01	0.14341E 01	0.14469E 01
128	0.13919E 01	0.13539E 01	0.13661E 01	0.13783E 01	0.13906E 01	0.14031E 01	0.14156E 01
130	0.13649E 01	0.13217E 01	0.13336E 01	0.13456E 01	0.13578E 01	0.13700E 01	0.13824E 01
132	0.13359E 01	0.12875E 01	0.12992E 01	0.13110E 01	0.13229E 01	0.13349E 01	0.13470E 01
134	0.13041E 01	0.12514E 01	0.12629E 01	0.12744E 01	0.12861E 01	0.12978E 01	0.13097E 01
136	0.12692E 01	0.12134E 01	0.12246E 01	0.12359E 01	0.12473E 01	0.12588E 01	0.12703E 01
138	0.11428E 01	0.11736E 01	0.11844E 01	0.11954E 01	0.12065E 01	0.12177E 01	0.12290E 01
140	0.11121E 01	0.11318E 01	0.11424E 01	0.11530E 01	0.11638E 01	0.11747E 01	0.11856E 01
142	0.10781E 01	0.10882E 01	0.10985E 01	0.11088E 01	0.11197E 01	0.11297E 01	0.11403E 01
144	0.10331E 01	0.10429E 01	0.10527E 01	0.10627E 01	0.10727E 01	0.10829E 01	0.10931E 01
146	0.98623E 00	0.99573E 00	0.10052E 01	0.10148E 01	0.10244E 01	0.10342E 01	0.10440E 01
148	0.92392E 00	0.94691E 00	0.95597E 00	0.96511E 00	0.97435E 00	0.98367E 00	0.99308E 00
150	0.88789E 00	0.89646E 00	0.90506E 00	0.91377E 00	0.92256E 00	0.93144E 00	0.94040E 00
152	0.83630E 00	0.84439E 00	0.85256E 00	0.86080E 00	0.86913E 00	0.87754E 00	0.88603E 00
154	0.78722E 00	0.79083E 00	0.79852E 00	0.80629E 00	0.81412E 00	0.82204E 00	0.83003E 00
156	0.72873E 00	0.73585E 00	0.74302E 00	0.75029E 00	0.75762E 00	0.76502E 00	0.77250E 00
158	0.67091E 00	0.67951E 00	0.68861E 00	0.69791E 00	0.69997E 00	0.70558E 00	0.71351E 00
160	0.61886E 00	0.62192E 00	0.62805E 00	0.63424E 00	0.64049E 00	0.64680E 00	0.65317E 00
162	0.55766E 00	0.56317E 00	0.56874E 00	0.57436E 00	0.58004E 00	0.58578E 00	0.59157E 00
164	0.49841E-00	0.50235E-00	0.50635E-00	0.51039E-00	0.51849E-00	0.52362E-00	0.52883E-00
166	0.43822E-00	0.44258E-00	0.44699E-00	0.45143E-00	0.45594E-00	0.46046E-00	0.46505E-00
168	0.37720E-00	0.38096E-00	0.38476E-00	0.38859E-00	0.39248E-00	0.39639E-00	0.40035E-00
170	0.31645E-00	0.31861E-00	0.32179E-00	0.32501E-00	0.32826E-00	0.33154E-00	0.33486E-00
172	0.25310E-00	0.25393E-00	0.25485E-00	0.25587E-00	0.25699E-00	0.25820E-00	0.25940E-00
174	0.19255E-00	0.19216E-00	0.19190E-00	0.19164E-00	0.19138E-00	0.19112E-00	0.19086E-00
176	0.12704E-00	0.12822E-00	0.12961E-00	0.13091E-00	0.13222E-00	0.13355E-00	0.13489E-00
178	0.63883E-01	0.64222E-01	0.64867E-01	0.65519E-01	0.66178E-01	0.66842E-01	0.67514E-01
180	0.	0.	0.	0.	0.	0.	0.

U(K,ψ)

TRAJECTORY FUNCTIONS

K	ψ	0.67000E 00	0.68000E 00	0.69000E 00	0.70000E 00	0.71000E 00	0.72000E 00	0.73000E 00
0	0.11126F 01	0.11131F 01	0.11149E 01	0.11168F 01	0.111870E 01	0.12059E 01	0.12248E 01	0.12248E 01
2	0.11120E 01	0.11148E 01	0.11167E 01	0.11185E 01	0.12042E 01	0.12229E 01	0.12416E 01	0.12584F 01
4	0.11148E 01	0.11166F 01	0.11184E 01	0.12030F 01	0.12214F 01	0.12398E 01	0.12568E 01	0.12752E 01
6	0.11162F 01	0.11184E 01	0.12017E 01	0.12021E 01	0.12385E 01	0.12556F 01	0.12739E 01	0.12907E 01
8	0.111840F 01	0.12017E 01	0.12196F 01	0.12376F 01	0.12556F 01	0.12727F 01	0.12907E 01	0.13088F 01
10	0.12018F 01	0.12194F 01	0.12370F 01	0.12548F 01	0.12727F 01	0.12898E 01	0.13076E 01	0.13254F 01
12	0.12194F 01	0.12369F 01	0.12545E 01	0.12718F 01	0.12893E 01	0.13068E 01	0.13245F 01	0.13422E 01
14	0.12372F 01	0.12545E 01	0.12718F 01	0.12893E 01	0.13068E 01	0.13245F 01	0.13422E 01	0.13589F 01
16	0.12549F 01	0.12720F 01	0.12891F 01	0.13064F 01	0.13235E 01	0.13407E 01	0.13580E 01	0.13755E 01
18	0.12725F 01	0.12894F 01	0.13064F 01	0.13235E 01	0.13407E 01	0.13576E 01	0.13748E 01	0.13920E 01
20	0.12900F 01	0.13067E 01	0.13236F 01	0.13407E 01	0.13576E 01	0.13748E 01	0.13920E 01	0.14085E 01
22	0.13074F 01	0.13240F 01	0.13407E 01	0.13576E 01	0.13748E 01	0.13920E 01	0.14085E 01	0.14249F 01
24	0.13248E 01	0.13412E 01	0.13577E 01	0.13744E 01	0.13911E 01	0.14080E 01	0.14249F 01	0.14412F 01
26	0.13420E 01	0.13583E 01	0.13746E 01	0.13911E 01	0.14077E 01	0.14244E 01	0.14412F 01	0.14575E 01
28	0.13591F 01	0.13752F 01	0.13915F 01	0.14078E 01	0.14242E 01	0.14408E 01	0.14575E 01	0.14736F 01
30	0.13761F 01	0.13921F 01	0.14082F 01	0.14243E 01	0.14406F 01	0.14571E 01	0.14736F 01	0.14896E 01
32	0.13929F 01	0.14088E 01	0.14247F 01	0.14408E 01	0.14569F 01	0.14732E 01	0.14896E 01	0.15055F 01
34	0.14096F 01	0.14253F 01	0.14411F 01	0.14570E 01	0.14731F 01	0.14892E 01	0.15055F 01	0.15212E 01
36	0.14262F 01	0.14417F 01	0.14574F 01	0.14732E 01	0.14890E 01	0.15051E 01	0.15212E 01	0.15368E 01
38	0.14425F 01	0.14579F 01	0.14734F 01	0.14891F 01	0.15049E 01	0.15205F 01	0.15362F 01	0.15521F 01
40	0.14586F 01	0.14739F 01	0.14893F 01	0.15048E 01	0.15204F 01	0.15359E 01	0.15515E 01	0.15673E 01
42	0.14745E 01	0.14897F 01	0.15050F 01	0.15204F 01	0.15357E 01	0.15511E 01	0.15666E 01	0.15823F 01
44	0.14901F 01	0.15052F 01	0.15204F 01	0.15357E 01	0.15511E 01	0.15666E 01	0.15823F 01	0.15970F 01
46	0.15055E 01	0.15204E 01	0.15355E 01	0.15507E 01	0.15659E 01	0.15807E 01	0.15951F 01	0.16103F 01
48	0.15206F 01	0.15354F 01	0.15504F 01	0.15655E 01	0.15807E 01	0.15951F 01	0.16103F 01	0.16247E 01
50	0.15353F 01	0.15501F 01	0.15650F 01	0.15799E 01	0.15941E 01	0.16091E 01	0.16247E 01	0.16395F 01
52	0.15497F 01	0.15644F 01	0.15792F 01	0.15941E 01	0.16091E 01	0.16247E 01	0.16395F 01	0.16541E 01
54	0.15638F 01	0.15783E 01	0.15930F 01	0.16079F 01	0.16228E 01	0.16361E 01	0.16500E 01	0.16640E 01
56	0.15774E 01	0.15919E 01	0.16065E 01	0.16213E 01	0.16343E 01	0.16491E 01	0.16640E 01	0.16790F 01
58	0.15906E 01	0.16050F 01	0.16196F 01	0.16343E 01	0.16468F 01	0.16615F 01	0.16764E 01	0.16914F 01
60	0.16033F 01	0.16177F 01	0.16322F 01	0.16468F 01	0.16588F 01	0.16735F 01	0.16883E 01	0.17022F 01
62	0.16155F 01	0.16299F 01	0.16443F 01	0.16588F 01	0.16704E 01	0.16850F 01	0.16997F 01	0.17146E 01
64	0.16272F 01	0.16415F 01	0.16559F 01	0.16704E 01	0.16850F 01	0.16997F 01	0.17146E 01	0.17265E 01
66	0.16383F 01	0.16526F 01	0.16669F 01	0.16817F 01	0.16959F 01	0.17106E 01	0.17265E 01	0.17357E 01
68	0.16488F 01	0.16630F 01	0.16773F 01	0.16917F 01	0.17063E 01	0.17203F 01	0.17357E 01	0.17453F 01
70	0.16587F 01	0.16728F 01	0.16871F 01	0.17014F 01	0.17160E 01	0.17306F 01	0.17453F 01	0.17543E 01
72	0.16678F 01	0.16819E 01	0.16961E 01	0.17105F 01	0.17250E 01	0.17396F 01	0.17543E 01	0.17625F 01
74	0.16762F 01	0.16903F 01	0.17045E 01	0.17188F 01	0.17333F 01	0.17408E 01	0.17553E 01	0.17700E 01
76	0.16838F 01	0.16979F 01	0.17120F 01	0.17263F 01	0.17431E 01	0.17475E 01	0.17620E 01	0.17767E 01
78	0.16905E 01	0.17046E 01	0.17188E 01	0.17331E 01	0.17439E 01	0.17533E 01	0.17678E 01	0.17825F 01
80	0.16964F 01	0.17104F 01	0.17246E 01	0.17399E 01	0.17488E 01	0.17582E 01	0.17727E 01	0.17873E 01
82	0.17013E 01	0.17153E 01	0.17295E 01	0.17399E 01	0.17488E 01	0.17582E 01	0.17727E 01	0.17912F 01
84	0.17052F 01	0.17193F 01	0.17334F 01	0.17477F 01	0.17571F 01	0.17664F 01	0.17764F 01	0.17941F 01
86	0.17091F 01	0.17221F 01	0.17362F 01	0.17563E 01	0.17649E 01	0.17740E 01	0.17832E 01	0.17964F 01
88	0.17129F 01	0.17249F 01	0.17386F 01	0.17563E 01	0.17649E 01	0.17740E 01	0.17832E 01	0.17964F 01
90	0.17165F 01	0.17274F 01	0.17415F 01	0.17633F 01	0.17720E 01	0.17812E 01	0.17904F 01	0.18000E 01
92	0.17199F 01	0.17299F 01	0.17430F 01	0.17633F 01	0.17720E 01	0.17812E 01	0.17904F 01	0.18000E 01
94	0.17230F 01	0.17320F 01	0.17461F 01	0.17704F 01	0.17784E 01	0.17876F 01	0.17964F 01	0.18000E 01
96	0.17264F 01	0.17354F 01	0.17495F 01	0.17704F 01	0.17784E 01	0.17876F 01	0.17964F 01	0.18000E 01
98	0.17291F 01	0.17381F 01	0.17522F 01	0.17704F 01	0.17784E 01	0.17876F 01	0.17964F 01	0.18000E 01
100	0.17318F 01	0.17408F 01	0.17549F 01	0.17704F 01	0.17784E 01	0.17876F 01	0.17964F 01	0.18000E 01
102	0.17345F 01	0.17435F 01	0.17576F 01	0.17704F 01	0.17784E 01	0.17876F 01	0.17964F 01	0.18000E 01
104	0.17372F 01	0.17462F 01	0.17603F 01	0.17704F 01	0.17784E 01	0.17876F 01	0.17964F 01	0.18000E 01
106	0.17399F 01	0.17489F 01	0.17630F 01	0.17704F 01	0.17784E 01	0.17876F 01	0.17964F 01	0.18000E 01
108	0.17426F 01	0.17516F 01	0.17657F 01	0.17704F 01	0.17784E 01	0.17876F 01	0.17964F 01	0.18000E 01
110	0.17453F 01	0.17543F 01	0.17684F 01	0.17704F 01	0.17784E 01	0.17876F 01	0.17964F 01	0.18000E 01
112	0.17480F 01	0.17570F 01	0.17711F 01	0.17704F 01	0.17784E 01	0.17876F 01	0.17964F 01	0.18000E 01
114	0.17507F 01	0.17597F 01	0.17738F 01	0.17704F 01	0.17784E 01	0.17876F 01	0.17964F 01	0.18000E 01
116	0.17534F 01	0.17624F 01	0.17765F 01	0.17704F 01	0.17784E 01	0.17876F 01	0.17964F 01	0.18000E 01
118	0.17561F 01	0.17651F 01	0.17792F 01	0.17704F 01	0.17784E 01	0.17876F 01	0.17964F 01	0.18000E 01
120	0.17588F 01	0.17678F 01	0.17829F 01	0.17704F 01	0.17784E 01	0.17876F 01	0.17964F 01	0.18000E 01
122	0.17615F 01	0.17705F 01	0.17856F 01	0.17704F 01	0.17784E 01	0.17876F 01	0.17964F 01	0.18000E 01
124	0.17642F 01	0.17732F 01	0.17883F 01	0.17704F 01	0.17784E 01	0.17876F 01	0.17964F 01	0.18000E 01
126	0.17669F 01	0.17759F 01	0.17910F 01	0.17704F 01	0.17784E 01	0.17876F 01	0.17964F 01	0.18000E 01
128	0.17696F 01	0.17786F 01	0.17937F 01	0.17704F 01	0.17784E 01	0.17876F 01	0.17964F 01	0.18000E 01
130	0.17723F 01	0.17813F 01	0.17964F 01	0.17704F 01	0.17784E 01	0.17876F 01	0.17964F 01	0.18000E 01
132	0.17750F 01	0.17840F 01	0.17991F 01	0.17704F 01	0.17784E 01	0.17876F 01	0.17964F 01	0.18000E 01
134	0.17777F 01	0.17867F 01	0.18012F 01	0.17704F 01	0.17784E 01	0.17876F 01	0.17964F 01	0.18000E 01
136	0.17804F 01	0.17894F 01	0.18043F 01	0.17704F 01	0.17784E 01	0.17876F 01	0.17964F 01	0.18000E 01
138	0.17831F 01	0.17921F 01	0.18092F 01	0.17704F 01	0.17784E 01	0.17876F 01	0.17964F 01	0.18000E 01
140	0.17858F 01	0.17948F 01	0.18141F 01	0.17704F 01	0.17784E 01	0.17876F 01	0.17964F 01	0.18000E 01
142	0.17885F 01	0.17975F 01	0.18190F 01	0.17704F 01	0.17784E 01	0.17876F 01	0.17964F 01	0.18000E 01
144	0.17912F 01	0.18002F 01	0.18239F 01	0.17704F 01	0.17784E 01	0.17876F 01	0.17964F 01	0.18000E 01
146	0.17939F 01	0.18029F 01	0.18288F 01	0.17704F 01	0.17784E 01	0.17876F 01	0.17964F 01	0.18000E 01
148	0.17966F 01	0.18056F 01	0.18337F 01	0.17704F 01	0.17784E 01	0.17876F 01	0.17964F 01	0.18000E 01
150	0.17993F 01	0.18083F 01	0.18386F 01	0.17704F 01	0.17784E 01	0.17876F 01	0.17964F 01	0.18000E 01
152	0.18020F 01	0.18110F 01	0.18435F 01	0.17704F 01	0.17784E 01	0.17876F 01	0.17964F 01	0.18000E 01
154	0.18047F 01	0.18137F 01	0.18484F 01	0.17704F 01	0.17784E 01	0.17876F 01	0.17964F 01	0.18000E 01
156	0.18074F 01	0.18164F 01	0.18533F 01	0.17704F 01	0.17784E 01	0.17876F 01	0.17964F 01	0.18000E 01
158	0.18101F 01	0.18191F 01	0.18582F 01	0.17704F 01	0.17784E 01	0.17876F 01	0.17964F 01	0.18000E 01
160	0.18128F 01	0.18220F 01	0.18631F 01	0.17704F 01	0.17784E 01	0.17876F 01	0.17964F 01	0.18000E 01
162	0.18155F 01	0.18247F 01	0.18680F 01	0.17704F 01	0.17784E 01	0.17876F 01	0.17964F 01	0.18000E 01
164	0.18182F 01	0.18274F 01	0.18729F 01	0.17704F 01	0.17784E 01	0.17876F 01	0.17964F 01	0.18000E 01
166	0.18209F 01	0.18301F 01	0.18778F 01	0.17704F 01	0.17784E 01	0.17876F 01	0.17964F 01	0.18000E 01
168	0.18236F 01	0.18328F 01	0.18827F 01	0.17704F 01	0.17784E 01	0.17876F 01	0.17964F 01	0.18000E 01
170	0.18263F 01	0.18355F 01	0.18876F 01	0.17704F 01	0.17784E 01	0.17876F 01	0.17964F 01	0.18000E 01
172	0.18290F 01	0.18382F 01	0.18					

U(K, ψ)

TRAJECTORY FUNCTIONS

K =	0.74000E 00	0.75000E 00	0.76000E 00	0.77000E 00	0.78000E 00	0.79000E 00	0.80000E 00
ψ							
0	0.12438F 01	0.12629F 01	0.12821F 01	0.13014F 01	0.13208F 01	0.13403F 01	0.13599F 01
2	0.12604F 01	0.12794F 01	0.12984F 01	0.13176F 01	0.13368F 01	0.13561F 01	0.13756F 01
4	0.12771F 01	0.12959F 01	0.13147F 01	0.13337F 01	0.13528F 01	0.13720F 01	0.13913F 01
6	0.12937F 01	0.13123F 01	0.13311F 01	0.13499F 01	0.13688F 01	0.13878F 01	0.14069F 01
8	0.13104F 01	0.13288F 01	0.13473F 01	0.13660F 01	0.13848F 01	0.14036F 01	0.14226F 01
10	0.13270F 01	0.13452F 01	0.13636F 01	0.13821F 01	0.14007F 01	0.14194F 01	0.14382F 01
12	0.13435F 01	0.13616F 01	0.13799F 01	0.13982F 01	0.14166F 01	0.14352F 01	0.14539F 01
14	0.13601F 01	0.13780F 01	0.13961F 01	0.14143F 01	0.14326F 01	0.14509F 01	0.14695F 01
16	0.13766F 01	0.13944F 01	0.14121F 01	0.14303F 01	0.14484F 01	0.14667F 01	0.14850F 01
18	0.13930F 01	0.14107F 01	0.14284F 01	0.14463F 01	0.14643F 01	0.14823F 01	0.15006F 01
20	0.14094F 01	0.14269F 01	0.14445F 01	0.14622F 01	0.14800F 01	0.14980F 01	0.15160F 01
22	0.14257F 01	0.14431F 01	0.14605F 01	0.14781F 01	0.14958F 01	0.15136F 01	0.15315F 01
24	0.14420F 01	0.14592F 01	0.14765F 01	0.14939F 01	0.15114F 01	0.15291F 01	0.15469F 01
26	0.14582F 01	0.14752F 01	0.14924F 01	0.15097F 01	0.15270F 01	0.15446F 01	0.15622F 01
28	0.14743F 01	0.14912F 01	0.15082F 01	0.15253F 01	0.15426F 01	0.15599F 01	0.15774F 01
30	0.14902F 01	0.15070F 01	0.15239F 01	0.15409F 01	0.15580F 01	0.15752F 01	0.15926F 01
32	0.15061F 01	0.15227F 01	0.15395F 01	0.15563F 01	0.15733F 01	0.15904F 01	0.16077F 01
34	0.15218F 01	0.15383F 01	0.15550F 01	0.15717F 01	0.15886F 01	0.16055F 01	0.16276F 01
36	0.15374F 01	0.15538F 01	0.15703F 01	0.15889F 01	0.16036F 01	0.16205F 01	0.16375F 01
38	0.15529F 01	0.15691F 01	0.15855F 01	0.16020F 01	0.16186F 01	0.16353F 01	0.16522F 01
40	0.15681F 01	0.15843F 01	0.16005F 01	0.16169F 01	0.16334F 01	0.16500F 01	0.16668F 01
42	0.15832F 01	0.15992F 01	0.16154F 01	0.16316F 01	0.16480F 01	0.16645F 01	0.16812F 01
44	0.15981F 01	0.16140F 01	0.16300F 01	0.16462F 01	0.16624F 01	0.16789F 01	0.16954F 01
46	0.16127F 01	0.16285F 01	0.16444F 01	0.16605F 01	0.16767F 01	0.16904F 01	0.17094F 01
48	0.16271F 01	0.16428F 01	0.16586F 01	0.16746F 01	0.16906F 01	0.17069F 01	0.17232F 01
50	0.16412F 01	0.16568F 01	0.16725F 01	0.16884F 01	0.17044F 01	0.17205F 01	0.17368F 01
52	0.16549F 01	0.16705F 01	0.16861F 01	0.17019F 01	0.17178F 01	0.17339F 01	0.17500F 01
54	0.16684F 01	0.16838F 01	0.16994F 01	0.17151F 01	0.17310F 01	0.17469F 01	0.17630F 01
56	0.16815F 01	0.16969F 01	0.17124F 01	0.17280F 01	0.17437F 01	0.17596F 01	0.17757F 01
58	0.16942F 01	0.17095F 01	0.17249F 01	0.17405F 01	0.17567F 01	0.17720F 01	0.17880F 01
60	0.17065F 01	0.17217F 01	0.17371F 01	0.17526F 01	0.17682F 01	0.17840F 01	0.17999F 01
62	0.17183F 01	0.17335F 01	0.17488F 01	0.17642F 01	0.17798F 01	0.17955F 01	0.18113F 01
64	0.17296F 01	0.17448F 01	0.17600F 01	0.17754F 01	0.17909F 01	0.18066F 01	0.18274F 01
66	0.17404F 01	0.17555F 01	0.17707F 01	0.17861F 01	0.18015F 01	0.18171F 01	0.18329F 01
68	0.17506F 01	0.17657F 01	0.17808F 01	0.17961F 01	0.18116F 01	0.18272F 01	0.18429F 01
70	0.17602F 01	0.17752F 01	0.17904F 01	0.18056F 01	0.18210F 01	0.18366F 01	0.18522F 01
72	0.17691F 01	0.17841F 01	0.17992F 01	0.18145F 01	0.18299F 01	0.18454F 01	0.18610F 01
74	0.17774F 01	0.17923F 01	0.18074F 01	0.18226F 01	0.18380F 01	0.18535F 01	0.18691F 01
76	0.17848F 01	0.17998F 01	0.18148F 01	0.18300F 01	0.18454F 01	0.18608F 01	0.18764F 01
78	0.17915F 01	0.18064F 01	0.18214F 01	0.18366F 01	0.18510F 01	0.18674F 01	0.18830F 01
80	0.17972F 01	0.18122F 01	0.18272F 01	0.18424F 01	0.18577F 01	0.18731F 01	0.18887F 01
82	0.18021F 01	0.18170F 01	0.18320F 01	0.18472F 01	0.18625F 01	0.18779F 01	0.18935F 01
84	0.18060F 01	0.18209F 01	0.18359F 01	0.18511F 01	0.18664F 01	0.18818F 01	0.18974F 01
86	0.18088F 01	0.18237F 01	0.18388F 01	0.18539F 01	0.18692F 01	0.18846F 01	0.19002F 01
88	0.18106F 01	0.18255F 01	0.18405F 01	0.18557F 01	0.18710F 01	0.18864F 01	0.19019F 01
90	0.18112F 01	0.18261F 01	0.18411F 01	0.18563F 01	0.18716F 01	0.18870F 01	0.19025F 01
92	0.18106F 01	0.18255F 01	0.18405F 01	0.18557F 01	0.18709F 01	0.18864F 01	0.19019F 01
94	0.18087F 01	0.18236F 01	0.18386F 01	0.18538F 01	0.18690F 01	0.18845F 01	0.19000F 01
96	0.18054F 01	0.18203F 01	0.18353F 01	0.18505F 01	0.18658F 01	0.18812F 01	0.18968F 01
98	0.18009F 01	0.18157F 01	0.18307F 01	0.18468F 01	0.18611F 01	0.18765F 01	0.18921F 01
100	0.17946F 01	0.18095F 01	0.18245F 01	0.18397F 01	0.18549F 01	0.18703F 01	0.18859F 01
102	0.17870F 01	0.18018F 01	0.18168F 01	0.18319F 01	0.18472F 01	0.18626F 01	0.18781F 01
104	0.17777F 01	0.17925F 01	0.18075F 01	0.18226F 01	0.18378F 01	0.18532F 01	0.18687F 01
106	0.17676F 01	0.17816F 01	0.17965F 01	0.18116F 01	0.18268F 01	0.18421F 01	0.18576F 01
108	0.17541F 01	0.17688F 01	0.17837F 01	0.17988F 01	0.18140F 01	0.18293F 01	0.18447F 01
110	0.17496F 01	0.17543F 01	0.17692F 01	0.17842F 01	0.17993F 01	0.18146F 01	0.18299F 01
112	0.17333F 01	0.17380F 01	0.17528F 01	0.17677F 01	0.17828F 01	0.17979F 01	0.18133F 01
114	0.17151F 01	0.17197F 01	0.17344F 01	0.17493F 01	0.17643F 01	0.17794F 01	0.17946F 01
116	0.16949F 01	0.16994F 01	0.17141F 01	0.17288F 01	0.17437F 01	0.17588F 01	0.17739F 01
118	0.16627F 01	0.16671F 01	0.16817F 01	0.16964F 01	0.17112F 01	0.17261F 01	0.17411F 01
120	0.16385F 01	0.16428F 01	0.16572F 01	0.16718F 01	0.16865F 01	0.17011F 01	0.17162F 01
122	0.16122F 01	0.16163F 01	0.16306F 01	0.16450F 01	0.16596F 01	0.16743F 01	0.16891F 01
124	0.15837F 01	0.15877F 01	0.16019F 01	0.16161F 01	0.16305F 01	0.16450F 01	0.16597F 01
126	0.15631F 01	0.15670F 01	0.15809F 01	0.15950F 01	0.16092F 01	0.16235F 01	0.16380F 01
128	0.15403F 01	0.15440F 01	0.15577F 01	0.15716F 01	0.15857F 01	0.15997F 01	0.16140F 01
130	0.15148F 01	0.15184F 01	0.15319F 01	0.15459F 01	0.15597F 01	0.15736F 01	0.15876F 01
 

U(K,ψ)

TRAJECTORY FUNCTIONS

K =	0.82000E 00	0.84000E 00	0.86000E 00	0.88000E 00	0.90000E 00	0.92000E 00	0.94000E 00
ψ							
0	0.13994E 01	0.14393E 01	0.14797E 01	0.15205E 01	0.15618E 01	0.16035E 01	0.16458E 01
2	0.14148E 01	0.14544E 01	0.14945E 01	0.15350E 01	0.15760E 01	0.16174E 01	0.16594E 01
4	0.14201E 01	0.14695E 01	0.15092E 01	0.15495E 01	0.15902E 01	0.16313E 01	0.16730E 01
6	0.14255E 01	0.14845E 01	0.15240E 01	0.15639E 01	0.16043E 01	0.16452E 01	0.16867E 01
8	0.14309E 01	0.14996E 01	0.15387E 01	0.15784E 01	0.16185E 01	0.16591E 01	0.17003E 01
10	0.14362E 01	0.15146E 01	0.15535E 01	0.15928E 01	0.16327E 01	0.16730E 01	0.17139E 01
12	0.14415E 01	0.15296E 01	0.15682E 01	0.16073E 01	0.16469E 01	0.16869E 01	0.17275E 01
14	0.14468E 01	0.15446E 01	0.15829E 01	0.16217E 01	0.16610E 01	0.17008E 01	0.17411E 01
16	0.14521E 01	0.15596E 01	0.15976E 01	0.16361E 01	0.16751E 01	0.17147E 01	0.17547E 01
18	0.14573E 01	0.15746E 01	0.16123E 01	0.16505E 01	0.16893E 01	0.17285E 01	0.17683E 01
20	0.14625E 01	0.15895E 01	0.16269E 01	0.16649E 01	0.17034E 01	0.17424E 01	0.17819E 01
22	0.14677E 01	0.16044E 01	0.16415E 01	0.16792E 01	0.17174E 01	0.17562E 01	0.17954E 01
24	0.14728E 01	0.16192E 01	0.16561E 01	0.16935E 01	0.17315E 01	0.17699E 01	0.18090E 01
26	0.14778E 01	0.16340E 01	0.16706E 01	0.17078E 01	0.17454E 01	0.17837E 01	0.18225E 01
28	0.14828E 01	0.16487E 01	0.16850E 01	0.17220E 01	0.17594E 01	0.17974E 01	0.18359E 01
30	0.14877E 01	0.16633E 01	0.16994E 01	0.17361E 01	0.17733E 01	0.18110E 01	0.18493E 01
32	0.14925E 01	0.16779E 01	0.17138E 01	0.17502E 01	0.17871E 01	0.18246E 01	0.18627E 01
34	0.14972E 01	0.16923E 01	0.17280E 01	0.17642E 01	0.18009E 01	0.18382E 01	0.18760E 01
36	0.15018E 01	0.17067E 01	0.17421E 01	0.17781E 01	0.18146E 01	0.18516E 01	0.18893E 01
38	0.15063E 01	0.17210E 01	0.17563E 01	0.17919E 01	0.18281E 01	0.18650E 01	0.19024E 01
40	0.15107E 01	0.17351E 01	0.17701E 01	0.18056E 01	0.18416E 01	0.18783E 01	0.19155E 01
42	0.15149E 01	0.17491E 01	0.17838E 01	0.18191E 01	0.18550E 01	0.18914E 01	0.19284E 01
44	0.15189E 01	0.17629E 01	0.17974E 01	0.18325E 01	0.18682E 01	0.19044E 01	0.19413E 01
46	0.15227E 01	0.17765E 01	0.18109E 01	0.18458E 01	0.18813E 01	0.19173E 01	0.19540E 01
48	0.15263E 01	0.17899E 01	0.18241E 01	0.18588E 01	0.18941E 01	0.19300E 01	0.19665E 01
50	0.15297E 01	0.18031E 01	0.18371E 01	0.18717E 01	0.19068E 01	0.19426E 01	0.19789E 01
52	0.15328E 01	0.18161E 01	0.18499E 01	0.18843E 01	0.19193E 01	0.19549E 01	0.19910E 01
54	0.15356E 01	0.18287E 01	0.18624E 01	0.18967E 01	0.19315E 01	0.19669E 01	0.20030E 01
56	0.15381E 01	0.18411E 01	0.18747E 01	0.19088E 01	0.19435E 01	0.19787E 01	0.20146E 01
58	0.15403E 01	0.18531E 01	0.18866E 01	0.19205E 01	0.19551E 01	0.19903E 01	0.20260E 01
60	0.15421E 01	0.18648E 01	0.18981E 01	0.19320E 01	0.19664E 01	0.20015E 01	0.20371E 01
62	0.15434E 01	0.18761E 01	0.19093E 01	0.19430E 01	0.19774E 01	0.20123E 01	0.20478E 01
64	0.15442E 01	0.18869E 01	0.19200E 01	0.19536E 01	0.19879E 01	0.20227E 01	0.20582E 01
66	0.15448E 01	0.18972E 01	0.19302E 01	0.19638E 01	0.19980E 01	0.20327E 01	0.20681E 01
68	0.15457E 01	0.19070E 01	0.19400E 01	0.19735E 01	0.20076E 01	0.20422E 01	0.20775E 01
70	0.15460E 01	0.19163E 01	0.19492E 01	0.19826E 01	0.20166E 01	0.20512E 01	0.20865E 01
72	0.15462E 01	0.19249E 01	0.19577E 01	0.19911E 01	0.20251E 01	0.20597E 01	0.20948E 01
74	0.15467E 01	0.19329E 01	0.19657E 01	0.19990E 01	0.20339E 01	0.20675E 01	0.21026E 01
76	0.15470E 01	0.19402E 01	0.19729E 01	0.20062E 01	0.20401E 01	0.20746E 01	0.21097E 01
78	0.15471E 01	0.19467E 01	0.19794E 01	0.20127E 01	0.20465E 01	0.20810E 01	0.21160E 01
80	0.15472E 01	0.19524E 01	0.19850E 01	0.20183E 01	0.20521E 01	0.20865E 01	0.21216E 01
82	0.15475E 01	0.19571E 01	0.19898E 01	0.20230E 01	0.20569E 01	0.20913E 01	0.21263E 01
84	0.15478E 01	0.19610E 01	0.19936E 01	0.20269E 01	0.20607E 01	0.20951E 01	0.21301E 01
86	0.15481E 01	0.19648E 01	0.19964E 01	0.20297E 01	0.20635E 01	0.20979E 01	0.21339E 01
88	0.15483E 01	0.19686E 01	0.19992E 01	0.20314E 01	0.20662E 01	0.20996E 01	0.21366E 01
90	0.15484E 01	0.19721E 01	0.20018E 01	0.20329E 01	0.20688E 01	0.21002E 01	0.21392E 01
92	0.15485E 01	0.19755E 01	0.20042E 01	0.20344E 01	0.20713E 01	0.21017E 01	0.21418E 01
94	0.15486E 01	0.19786E 01	0.20063E 01	0.20358E 01	0.20737E 01	0.21031E 01	0.21443E 01
96	0.15487E 01	0.19816E 01	0.20082E 01	0.20372E 01	0.20760E 01	0.21044E 01	0.21467E 01
98	0.15488E 01	0.19845E 01	0.20099E 01	0.20385E 01	0.20782E 01	0.21056E 01	0.21490E 01
100	0.15489E 01	0.19874E 01	0.20115E 01	0.20397E 01	0.20804E 01	0.21067E 01	0.21512E 01
102	0.15490E 01	0.19901E 01	0.20130E 01	0.20409E 01	0.20825E 01	0.21077E 01	0.21533E 01
104	0.15491E 01	0.19927E 01	0.20144E 01	0.20420E 01	0.20845E 01	0.21087E 01	0.21553E 01
106	0.15492E 01	0.19952E 01	0.20157E 01	0.20431E 01	0.20864E 01	0.21096E 01	0.21572E 01
108	0.15493E 01	0.19976E 01	0.20169E 01	0.20441E 01	0.20883E 01	0.21105E 01	0.21591E 01
110	0.15494E 01	0.19999E 01	0.20180E 01	0.20451E 01	0.20901E 01	0.21113E 01	0.21609E 01
112	0.15495E 01	0.20021E 01	0.20191E 01	0.20460E 01	0.20918E 01	0.21121E 01	0.21626E 01
114	0.15496E 01	0.20042E 01	0.20201E 01	0.20469E 01	0.20935E 01	0.21128E 01	0.21642E 01
116	0.15497E 01	0.20062E 01	0.20211E 01	0.20478E 01	0.20951E 01	0.21135E 01	0.21657E 01
118	0.15498E 01	0.20081E 01	0.20220E 01	0.20487E 01	0.20967E 01	0.21141E 01	0.21672E 01
120	0.15499E 01	0.20099E 01	0.20229E 01	0.20495E 01	0.20982E 01	0.21147E 01	0.21686E 01
122	0.15500E 01	0.20117E 01	0.20237E 01	0.20503E 01	0.21000E 01	0.21152E 01	0.21699E 01
124	0.15501E 01	0.20134E 01	0.20245E 01	0.20511E 01	0.21017E 01	0.21157E 01	0.21712E 01
126	0.15502E 01	0.20150E 01	0.20252E 01	0.20520E 01	0.21033E 01	0.21162E 01	0.21725E 01
128	0.15503E 01	0.20166E 01	0.20259E 01	0.20528E 01	0.21048E 01	0.21167E 01	0.21737E 01
130	0.15504E 01	0.20181E 01	0.20266E 01	0.20536E 01	0.21063E 01	0.21172E 01	0.21749E 01
132	0.15505E 01	0.20196E 01	0.20272E 01	0.20544E 01	0.21077E 01	0.21177E 01	0.21760E 01
134	0.15506E 01	0.20211E 01	0.20278E 01	0.20552E 01	0.21091E 01	0.21182E 01	0.21771E 01
136	0.15507E 01	0.20225E 01	0.20284E 01	0.20560E 01	0.21104E 01	0.21187E 01	0.21782E 01
138	0.15508E 01	0.20239E 01	0.20289E 01	0.20568E 01	0.21117E 01	0.21191E 01	0.21792E 01
140	0.15509E 01	0.20253E 01	0.20294E 01	0.20576E 01	0.21130E 01	0.21195E 01	0.21802E 01
142	0.15510E 01	0.20266E 01	0.20299E 01	0.20584E 01	0.21142E 01	0.21199E 01	0.21812E 01
144	0.15511E 01	0.20279E 01	0.20303E 01	0.20592E 01	0.21154E 01	0.21203E 01	0.21822E 01
146	0.15512E 01	0.20292E 01	0.20307E 01	0.20600E 01	0.21166E 01	0.21207E 01	0.21832E 01
148	0.15513E 01	0.20305E 01	0.20311E 01	0.20608E 01	0.21177E 01	0.21211E 01	0.21841E 01
150	0.15514E 01	0.20317E 01	0.20315E 01	0.20616E 01	0.21188E 01	0.21215E 01	0.21850E 01
152	0.15515E 01	0.20329E 01	0.20320E 01	0.20624E 01	0.21199E 01	0.21219E 01	0.21859E 01
154	0.15516E 01	0.20341E 01	0.20324E 01	0.20632E 01	0.21210E 01	0.21223E 01	0.21867E 01
156	0.15517E 01	0.20353E 01	0.20328E 01	0.20640E 01	0.21221E 01	0.21227E 01	0.21876E 01
158	0.15518E 01	0.20365E 01	0.20332E 01	0.20648E 01	0.21231E 01	0.21231E 01	0.21884E 01
160	0.15519E 01	0.20377E 01	0.20336E 01	0.20656E 01	0.21242E 01	0.21235E 01	0.21892E 01
162	0.15520E 01	0.20389E 01	0.20340E 01	0.20664E 01	0.21252E 01	0.21239E 01	0.21900E 01
164	0.15521E 01	0.20400E 01	0.20344E 01	0.20672E 01	0.21262E 01	0.21243E 01	0.21908E 01
166	0.15522E 01	0.20412E 01	0.20348E 01	0.20680E 01	0.21272E 01	0.21247E 01	0.21916E 01
168	0.15523E 01	0.20423E 01	0.20352E 01	0.20688E 01	0.21282E 01	0.21251E 01	0.21924E 01
170	0.15524E 01	0.20435E 01	0.20356E 01	0.20696E 01	0.21292E 01	0.21255E 01	0.21932E 01
172	0.15525E 01	0.20446E 01	0.20360E 01	0.20704E 01	0.21302E 01	0.21259E 01	0.21939E 01
174	0.15526E 01	0.20457E 01	0.20364E 01	0.20712E 01	0.21312E 01	0.21263E 01	0.21947E 01
176	0.15527E 01	0.20468E 01	0.20368E 01	0.20720E 01	0.21322E 01	0.21267E 01	0.21954E 01
178	0.15528E 01	0.20479E 01	0.20372E 01	0.20728E 01	0.21332E 01	0.21271E 01	0.21961E 01
180	0.15529E 01	0.20490E 01	0.20376E 01	0.20736E 01	0.21342E 01	0.21275E 01	0.21968E 01

U(K, ψ)

TRAJECTORY FUNCTIONS

K	ψ	0.96000E 00	0.98000E 00	0.10000E 01
0	0.16885E 01	0.17317E 01	0.17755E 01	
2	0.17019E 01	0.17448E 01	0.17883E 01	
4	0.17152E 01	0.17579E 01	0.18012E 01	
6	0.17286E 01	0.17710E 01	0.18140E 01	
8	0.17419E 01	0.17841E 01	0.18269E 01	
10	0.17553E 01	0.17972E 01	0.18397E 01	
12	0.17687E 01	0.18103E 01	0.18525E 01	
14	0.17820E 01	0.18234E 01	0.18654E 01	
16	0.17953E 01	0.18365E 01	0.18782E 01	
18	0.18087E 01	0.18496E 01	0.18910E 01	
20	0.18220E 01	0.18626E 01	0.19039E 01	
22	0.18353E 01	0.18757E 01	0.19167E 01	
24	0.18486E 01	0.18887E 01	0.19295E 01	
26	0.18618E 01	0.19017E 01	0.19423E 01	
28	0.18750E 01	0.19147E 01	0.19550E 01	
30	0.18882E 01	0.19277E 01	0.19678E 01	
32	0.19014E 01	0.19406E 01	0.19804E 01	
34	0.19144E 01	0.19535E 01	0.19931E 01	
36	0.19275E 01	0.19663E 01	0.20057E 01	
38	0.19404E 01	0.19790E 01	0.20182E 01	
40	0.19533E 01	0.19917E 01	0.20307E 01	
42	0.19660E 01	0.20043E 01	0.20431E 01	
44	0.19787E 01	0.20167E 01	0.20554E 01	
46	0.19917E 01	0.20291E 01	0.20676E 01	
48	0.20036E 01	0.20413E 01	0.20796E 01	
50	0.20158E 01	0.20533E 01	0.20915E 01	
52	0.20278E 01	0.20652E 01	0.21032E 01	
54	0.20396E 01	0.20768E 01	0.21147E 01	
56	0.20511E 01	0.20882E 01	0.21260E 01	
58	0.20624E 01	0.20994E 01	0.21370E 01	
60	0.20734E 01	0.21102E 01	0.21477E 01	
62	0.20840E 01	0.21207E 01	0.21582E 01	
64	0.20942E 01	0.21309E 01	0.21682E 01	
66	0.21041E 01	0.21407E 01	0.21779E 01	
68	0.21134E 01	0.21499E 01	0.21871E 01	
70	0.21223E 01	0.21588E 01	0.21959E 01	
72	0.21306E 01	0.21670E 01	0.22041E 01	
74	0.21383E 01	0.21747E 01	0.22117E 01	
76	0.21454E 01	0.21817E 01	0.22186E 01	
78	0.21517E 01	0.21880E 01	0.22249E 01	
80	0.21572E 01	0.21935E 01	0.22304E 01	
82	0.21619E 01	0.21982E 01	0.22351E 01	
84	0.21657E 01	0.22019E 01	0.22388E 01	
86	0.21685E 01	0.22047E 01	0.22416E 01	
88	0.21702E 01	0.22065E 01	0.22434E 01	
90	0.21708E 01	0.22071E 01	0.22440E 01	
92	0.21702E 01	0.22064E 01	0.22433E 01	
94	0.21683E 01	0.22045E 01	0.22414E 01	
96	0.21650E 01	0.22012E 01	0.22381E 01	
98	0.21602E 01	0.21964E 01	0.22333E 01	
100	0.21538E 01	0.21900E 01	0.22269E 01	
102	0.21458E 01	0.21820E 01	0.22188E 01	
104	0.21361E 01	0.21722E 01	0.22090E 01	
106	0.21245E 01	0.21606E 01	0.21973E 01	
108	0.21110E 01	0.21470E 01	0.21836E 01	
110	0.20954E 01	0.21313E 01	0.21678E 01	
112	0.20778E 01	0.21135E 01	0.21499E 01	
114	0.20579E 01	0.20935E 01	0.21298E 01	
116	0.20358E 01	0.20712E 01	0.21073E 01	
118	0.20113E 01	0.20465E 01	0.20823E 01	
120	0.19843E 01	0.20193E 01	0.20548E 01	
122	0.19549E 01	0.19895E 01	0.20247E 01	
124	0.19228E 01	0.19571E 01	0.19920E 01	
126	0.18881E 01	0.19219E 01	0.19564E 01	
128	0.18506E 01	0.18840E 01	0.19181E 01	
130	0.18104E 01	0.18433E 01	0.18769E 01	
132	0.17674E 01	0.17998E 01	0.18327E 01	
134	0.17216E 01	0.17533E 01	0.17856E 01	
136	0.16729E 01	0.17040E 01	0.17356E 01	
138	0.16214E 01	0.16517E 01	0.16825E 01	
140	0.15671E 01	0.15965E 01	0.16265E 01	
142	0.15099E 01	0.15384E 01	0.15675E 01	
144	0.14499E 01	0.14774E 01	0.15055E 01	
146	0.13871E 01	0.14136E 01	0.14407E 01	
148	0.13216E 01	0.13470E 01	0.13729E 01	
150	0.12535E 01	0.12777E 01	0.13024E 01	
152	0.11828E 01	0.12058E 01	0.12292E 01	
154	0.11096E 01	0.11313E 01	0.11534E 01	
156	0.10341E 01	0.10544E 01	0.10751E 01	
158	0.95636E 00	0.97523E 00	0.99446E 00	
160	0.87452E 00	0.89387E 00	0.91158E 00	
162	0.79472E 00	0.81051E 00	0.82662E 00	
164	0.71112E 00	0.72531E 00	0.73977E 00	
166	0.62591E 00	0.63843E 00	0.65120E 00	
168	0.53925E 00	0.55006E 00	0.56109E 00	
170	0.45133E 00	0.46040E 00	0.46965E 00	
172	0.36235E 00	0.36965E 00	0.37709E 00	
174	0.27252E 00	0.27801E 00	0.28362E 00	
176	0.18204E 00	0.18572E 00	0.18946E 00	
178	0.91129E 00	0.92970E 00	0.94847E 00	
180	0.	0.	0.	

Table A-II.2

Trajectory Function $V(K, \psi)$

V(K,ψ)

TRAJECTORY FUNCTIONS

K =	1.00000E-01	0.15000E-00	0.20000E-00	0.25000E-00	0.30000E-00	0.35000E-00	0.40000E-00
ψ							
0	-0.31495E 01	-0.31593E 01	-0.31731E 01	-0.31909E 01	-0.32127E 01	-0.32385E 01	-0.32685E 01
5	-0.30530E 01	-0.30579E 01	-0.30665E 01	-0.30789E 01	-0.30949E 01	-0.31148E 01	-0.31384E 01
10	-0.29567E 01	-0.29567E 01	-0.29601E 01	-0.29671E 01	-0.29774E 01	-0.29913E 01	-0.30087E 01
15	-0.28604E 01	-0.28556E 01	-0.28541E 01	-0.28557E 01	-0.28605E 01	-0.28685E 01	-0.28797E 01
20	-0.27645E 01	-0.27550E 01	-0.27485E 01	-0.27449E 01	-0.27443E 01	-0.27467E 01	-0.27519E 01
25	-0.26687E 01	-0.26547E 01	-0.26435E 01	-0.26350E 01	-0.26292E 01	-0.26261E 01	-0.26257E 01
30	-0.25734E 01	-0.25550E 01	-0.25393E 01	-0.25261E 01	-0.25153E 01	-0.25071E 01	-0.25013E 01
35	-0.24784E 01	-0.24560E 01	-0.24360E 01	-0.24183E 01	-0.24030E 01	-0.23899E 01	-0.23790E 01
40	-0.23840E 01	-0.23577E 01	-0.23337E 01	-0.23119E 01	-0.22922E 01	-0.22747E 01	-0.22591E 01
45	-0.22900E 01	-0.22603E 01	-0.22326E 01	-0.22070E 01	-0.21834E 01	-0.21617E 01	-0.21420E 01
50	-0.21967E 01	-0.21637E 01	-0.21327E 01	-0.21037E 01	-0.20765E 01	-0.20512E 01	-0.20276E 01
55	-0.21039E 01	-0.20681E 01	-0.20341E 01	-0.20021E 01	-0.19718E 01	-0.19432E 01	-0.19163E 01
60	-0.20118E 01	-0.19735E 01	-0.19370E 01	-0.19023E 01	-0.18692E 01	-0.18379E 01	-0.18081E 01
65	-0.19200E 01	-0.18800E 01	-0.18413E 01	-0.18043E 01	-0.17690E 01	-0.17353E 01	-0.17031E 01
70	-0.18298E 01	-0.17875E 01	-0.17471E 01	-0.17083E 01	-0.16711E 01	-0.16355E 01	-0.16014E 01
75	-0.17499E 01	-0.16962E 01	-0.16544E 01	-0.16142E 01	-0.15756E 01	-0.15386E 01	-0.15030E 01
80	-0.16607E 01	-0.16061E 01	-0.15633E 01	-0.15221E 01	-0.14825E 01	-0.14444E 01	-0.14079E 01
85	-0.15623E 01	-0.15171E 01	-0.14737E 01	-0.14319E 01	-0.13917E 01	-0.13531E 01	-0.13159E 01
90	-0.14646E 01	-0.14293E 01	-0.13857E 01	-0.13437E 01	-0.13033E 01	-0.12645E 01	-0.12271E 01
95	-0.13677E 01	-0.13426E 01	-0.12991E 01	-0.12574E 01	-0.12172E 01	-0.11785E 01	-0.11413E 01
100	-0.13016E 01	-0.12570E 01	-0.12141E 01	-0.11729E 01	-0.11333E 01	-0.10952E 01	-0.10585E 01
105	-0.12162E 01	-0.11725E 01	-0.11306E 01	-0.10902E 01	-0.10515E 01	-0.10143E 01	-0.97847E 00
110	-0.11315E 01	-0.10891E 01	-0.10484E 01	-0.10093E 01	-0.97175E 00	-0.93570E 00	-0.90109E 00
115	-0.10475E 01	-0.10067E 01	-0.96754E 00	-0.92998E 00	-0.89394E 00	-0.85937E 00	-0.82620E 00
120	-0.96419E 00	-0.92527E 00	-0.88797E 00	-0.85222E 00	-0.81796E 00	-0.78512E 00	-0.75365E 00
125	-0.88149E 00	-0.84476E 00	-0.80959E 00	-0.77592E 00	-0.74368E 00	-0.71281E 00	-0.68325E 00
130	-0.79937E 00	-0.76510E 00	-0.73232E 00	-0.70097E 00	-0.67097E 00	-0.64228E 00	-0.61484E 00
135	-0.71781E 00	-0.68624E 00	-0.65608E 00	-0.62726E 00	-0.59971E 00	-0.57339E 00	-0.54823E 00
140	-0.63674E 00	-0.60811E 00	-0.58077E 00	-0.55467E 00	-0.52975E 00	-0.50596E 00	-0.48324E 00
145	-0.55613E 00	-0.53063E 00	-0.50631E 00	-0.48310E 00	-0.46097E 00	-0.43988E 00	-0.41970E 00
150	-0.47592E 00	-0.45373E 00	-0.43258E 00	-0.41242E 00	-0.39320E 00	-0.37488E 00	-0.35741E 00
155	-0.39605E 00	-0.37733E 00	-0.35950E 00	-0.34251E 00	-0.32632E 00	-0.31089E 00	-0.29620E 00
160	-0.31649E 00	-0.30136E 00	-0.28695E 00	-0.27333E 00	-0.26047E 00	-0.24773E 00	-0.23589E 00
165	-0.23716E 00	-0.22572E 00	-0.21483E 00	-0.20447E 00	-0.19461E 00	-0.18522E 00	-0.17629E 00
170	-0.15800E 00	-0.15034E 00	-0.14304E 00	-0.13610E 00	-0.12949E 00	-0.12321E 00	-0.11723E 00
175	-0.78972E 01	-0.75125E 01	-0.71466E 01	-0.67985E 01	-0.64673E 01	-0.61523E 01	-0.58526E 01
180	0.	0.	0.	0.	0.	0.	0.

K =	0.45000E-00	0.50000E 00	0.52000E 00	0.54000E 00	0.56000E 00	0.58000E 00	0.60000E 00
ψ							
0	-0.33027E 01	-0.33410E 01	-0.33576E 01	-0.33748E 01	-0.33928E 01	-0.34114E 01	-0.34308E 01
5	-0.31659E 01	-0.31972E 01	-0.32109E 01	-0.32252E 01	-0.32401E 01	-0.32557E 01	-0.32719E 01
10	-0.30296E 01	-0.30540E 01	-0.30648E 01	-0.30761E 01	-0.30881E 01	-0.31006E 01	-0.31137E 01
15	-0.28942E 01	-0.29118E 01	-0.29198E 01	-0.29283E 01	-0.29373E 01	-0.29469E 01	-0.29570E 01
20	-0.27601E 01	-0.27713E 01	-0.27765E 01	-0.27823E 01	-0.27885E 01	-0.27952E 01	-0.28023E 01
25	-0.26279E 01	-0.26328E 01	-0.26355E 01	-0.26386E 01	-0.26421E 01	-0.26461E 01	-0.26505E 01
30	-0.24979E 01	-0.24968E 01	-0.24971E 01	-0.24977E 01	-0.24987E 01	-0.25001E 01	-0.25019E 01
35	-0.23703E 01	-0.23638E 01	-0.23618E 01	-0.23601E 01	-0.23588E 01	-0.23578E 01	-0.23572E 01
40	-0.22456E 01	-0.22341E 01	-0.22300E 01	-0.22262E 01	-0.22227E 01	-0.22196E 01	-0.22167E 01
45	-0.21240E 01	-0.21079E 01	-0.21020E 01	-0.20963E 01	-0.20909E 01	-0.20858E 01	-0.20809E 01
50	-0.20058E 01	-0.19856E 01	-0.19780E 01	-0.19706E 01	-0.19635E 01	-0.19566E 01	-0.19500E 01
55	-0.18810E 01	-0.18673E 01	-0.18582E 01	-0.18494E 01	-0.18408E 01	-0.18324E 01	-0.18243E 01
60	-0.17799E 01	-0.17531E 01	-0.17428E 01	-0.17327E 01	-0.17229E 01	-0.17132E 01	-0.17038E 01
65	-0.16725E 01	-0.16432E 01	-0.16319E 01	-0.16208E 01	-0.16099E 01	-0.15992E 01	-0.15887E 01
70	-0.15688E 01	-0.15375E 01	-0.15254E 01	-0.15135E 01	-0.15017E 01	-0.14902E 01	-0.14789E 01
75	-0.14689E 01	-0.14361E 01	-0.14233E 01	-0.14108E 01	-0.13985E 01	-0.13863E 01	-0.13744E 01
80	-0.13727E 01	-0.13388E 01	-0.13257E 01	-0.13127E 01	-0.12999E 01	-0.12874E 01	-0.12750E 01
85	-0.12801E 01	-0.12457E 01	-0.12323E 01	-0.12191E 01	-0.12061E 01	-0.11932E 01	-0.11806E 01
90	-0.11911E 01	-0.11565E 01	-0.11430E 01	-0.11297E 01	-0.11166E 01	-0.11037E 01	-0.10910E 01
95	-0.11055E 01	-0.10711E 01	-0.10577E 01	-0.10445E 01	-0.10315E 01	-0.10186E 01	-0.10060E 01
100	-0.10233E 01	-0.98934E 00	-0.97613E 00	-0.96313E 00	-0.95033E 00	-0.93772E 00	-0.92531E 00
105	-0.94408E 00	-0.91101E 00	-0.89815E 00	-0.88548E 00	-0.87301E 00	-0.86074E 00	-0.84866E 00
110	-0.86785E 00	-0.83592E 00	-0.82350E 00	-0.81128E 00	-0.79926E 00	-0.78743E 00	-0.77579E 00
115	-0.79438E 00	-0.76384E 00	-0.75197E 00	-0.74030E 00	-0.72882E 00	-0.71752E 00	-0.70641E 00
120	-0.7248E 00	-0.69455E 00	-0.68332E 00	-0.67228E 00	-0.66142E 00	-0.65075E 00	-0.64025E 00
125	-0.65494E 00	-0.62784E 00	-0.61732E 00	-0.60698E 00	-0.59682E 00	-0.58683E 00	-0.57702E 00
130	-0.58858E 00	-0.56346E 00	-0.55372E 00	-0.54415E 00	-0.53475E 00	-0.52552E 00	-0.51644E 00
135	-0.52418E 00	-0.50120E 00	-0.49230E 00	-0.48335E 00	-0.47496E 00	-0.46653E 00	-0.45825E 00
140	-0.46155E 00	-0.44083E 00	-0.43281E 00	-0.42494E 00	-0.41721E 00	-0.40962E 00	-0.40217E 00
145	-0.40047E 00	-0.38213E 00	-0.37503E 00	-0.36807E 00	-0.36123E 00	-0.35453E 00	-0.34794E 00
150	-0.34076E 00	-0.32488E 00	-0.31874E 00	-0.31272E 00	-0.30681E 00	-0.30101E 00	-0.29532E 00
155	-0.28220E 00	-0.26887E 00	-0.26371E 00	-0.25865E 00	-0.25369E 00	-0.24883E 00	-0.24406E 00
160	-0.22461E 00	-0.21388E 00	-0.20973E 00	-0.20566E 00	-0.20167E 00	-0.19775E 00	-0.19392E 00
165	-0.16779E 00	-0.15970E 00	-0.15657E 00	-0.15351E 00	-0.15050E 00	-0.14755E 00	-0.14467E 00
170	-0.11154E 00	-0.10613E 00	-0.10450E 00	-0.10199E 00	-0.99979E 00	-0.98009E 00	-0.96078E 00
175	-0.55675E 01	-0.52963E 01	-0.51916E 01	-0.50889E 01	-0.49883E 01	-0.48896E 01	-0.47929E 01
180	0.	0.	0.	0.	0.	0.	0.

V(K, ψ)

TRAJECTORY FUNCTIONS

K =	0.62000E 00	0.64000E 00	0.66000E 00	0.68000E 00	0.70000E 00	0.72000E 00	0.74000E 00
ψ							
0	-0.34508E 01	-0.34716E 01	-0.34931E 01	-0.35154E 01	-0.35384E 01	-0.35621E 01	-0.35866E 01
5	-0.32887E 01	-0.33063E 01	-0.33244E 01	-0.33433E 01	-0.33628E 01	-0.33830E 01	-0.34039E 01
10	-0.31274E 01	-0.31417E 01	-0.31566E 01	-0.31721E 01	-0.31882E 01	-0.32049E 01	-0.32222E 01
15	-0.29676E 01	-0.29787E 01	-0.29904E 01	-0.30026E 01	-0.30154E 01	-0.30287E 01	-0.30425E 01
20	-0.28100E 01	-0.28181E 01	-0.28267E 01	-0.28357E 01	-0.28453E 01	-0.28553E 01	-0.28658E 01
25	-0.26553E 01	-0.26605E 01	-0.26661E 01	-0.26722E 01	-0.26787E 01	-0.26856E 01	-0.26929E 01
30	-0.25040E 01	-0.25065E 01	-0.25094E 01	-0.25127E 01	-0.25163E 01	-0.25203E 01	-0.25247E 01
35	-0.23568E 01	-0.23569E 01	-0.23572E 01	-0.23579E 01	-0.23589E 01	-0.23602E 01	-0.23619E 01
40	-0.22141E 01	-0.22119E 01	-0.22099E 01	-0.22082E 01	-0.22068E 01	-0.22057E 01	-0.22049E 01
45	-0.20763E 01	-0.20720E 01	-0.20679E 01	-0.20642E 01	-0.20606E 01	-0.20574E 01	-0.20543E 01
50	-0.19437E 01	-0.19375E 01	-0.19317E 01	-0.19260E 01	-0.19206E 01	-0.19154E 01	-0.19105E 01
55	-0.18164E 01	-0.18087E 01	-0.18012E 01	-0.17940E 01	-0.17870E 01	-0.17802E 01	-0.17736E 01
60	-0.16946E 01	-0.16856E 01	-0.16768E 01	-0.16682E 01	-0.16599E 01	-0.16517E 01	-0.16437E 01
65	-0.15784E 01	-0.15684E 01	-0.15585E 01	-0.15488E 01	-0.15393E 01	-0.15300E 01	-0.15209E 01
70	-0.14678E 01	-0.14569E 01	-0.14461E 01	-0.14356E 01	-0.14252E 01	-0.14150E 01	-0.14050E 01
75	-0.13626E 01	-0.13511E 01	-0.13397E 01	-0.13285E 01	-0.13175E 01	-0.13067E 01	-0.12960E 01
80	-0.12628E 01	-0.12508E 01	-0.12390E 01	-0.12274E 01	-0.12159E 01	-0.12047E 01	-0.11936E 01
85	-0.11682E 01	-0.11560E 01	-0.11439E 01	-0.11320E 01	-0.11203E 01	-0.11088E 01	-0.10974E 01
90	-0.10785E 01	-0.10662E 01	-0.10541E 01	-0.10421E 01	-0.10303E 01	-0.10187E 01	-0.10073E 01
95	-0.99358E 00	-0.98133E 00	-0.96926E 00	-0.95738E 00	-0.94567E 00	-0.93414E 00	-0.92278E 00
100	-0.91308E 00	-0.90104E 00	-0.88919E 00	-0.87751E 00	-0.86601E 00	-0.85469E 00	-0.84353E 00
105	-0.83676E 00	-0.82505E 00	-0.81352E 00	-0.80217E 00	-0.79100E 00	-0.78000E 00	-0.76917E 00
110	-0.76433E 00	-0.75305E 00	-0.74195E 00	-0.73103E 00	-0.72028E 00	-0.70970E 00	-0.69929E 00
115	-0.69484E 00	-0.68473E 00	-0.67415E 00	-0.66374E 00	-0.65351E 00	-0.64344E 00	-0.63353E 00
120	-0.62993E 00	-0.61978E 00	-0.60979E 00	-0.59998E 00	-0.59033E 00	-0.58084E 00	-0.57151E 00
125	-0.56737E 00	-0.55789E 00	-0.54857E 00	-0.53941E 00	-0.53041E 00	-0.52156E 00	-0.51286E 00
130	-0.50753E 00	-0.49877E 00	-0.49016E 00	-0.48171E 00	-0.47340E 00	-0.46524E 00	-0.45723E 00
135	-0.44501E 00	-0.44213E 00	-0.43428E 00	-0.42657E 00	-0.41901E 00	-0.41158E 00	-0.40428E 00
140	-0.39485E 00	-0.38767E 00	-0.38062E 00	-0.37370E 00	-0.36691E 00	-0.36024E 00	-0.35369E 00
145	-0.34148E 00	-0.33514E 00	-0.32892E 00	-0.32281E 00	-0.31682E 00	-0.31094E 00	-0.30517E 00
150	-0.28974E 00	-0.28426E 00	-0.27889E 00	-0.27362E 00	-0.26845E 00	-0.26338E 00	-0.25840E 00
155	-0.23938E 00	-0.23479E 00	-0.23029E 00	-0.22587E 00	-0.22154E 00	-0.21729E 00	-0.21312E 00
160	-0.19015E 00	-0.18647E 00	-0.18285E 00	-0.17930E 00	-0.17582E 00	-0.17241E 00	-0.16907E 00
165	-0.14183E 00	-0.13906E 00	-0.13634E 00	-0.13367E 00	-0.13105E 00	-0.12848E 00	-0.12597E 00
170	-0.94185E 00	-0.92329E 00	-0.90510E 00	-0.88727E 00	-0.86979E 00	-0.85266E 00	-0.83586E 00
175	-0.46082E 00	-0.46052E 00	-0.45142E 00	-0.44249E 00	-0.43374E 00	-0.42516E 00	-0.41675E 00
180	0.	0.	0.	0.	0.	0.	0.

K =	0.76000E 00	0.78000E 00	0.80000E 00	0.82000E 00	0.84000E 00	0.86000E 00	0.88000E 00
ψ							
0	-0.36119E 01	-0.36379E 01	-0.36647E 01	-0.36923E 01	-0.37207E 01	-0.37499E 01	-0.37799E 01
5	-0.34555E 01	-0.34477E 01	-0.34707E 01	-0.34944E 01	-0.35188E 01	-0.35439E 01	-0.35697E 01
10	-0.32401E 01	-0.32587E 01	-0.32779E 01	-0.32974E 01	-0.33181E 01	-0.33392E 01	-0.33610E 01
15	-0.30669E 01	-0.30718E 01	-0.30873E 01	-0.31034E 01	-0.31200E 01	-0.31372E 01	-0.31550E 01
20	-0.28768E 01	-0.28883E 01	-0.29002E 01	-0.29127E 01	-0.29256E 01	-0.29391E 01	-0.29530E 01
25	-0.27707E 01	-0.27893E 01	-0.27715E 01	-0.27266E 01	-0.27361E 01	-0.27460E 01	-0.27563E 01
30	-0.25955E 01	-0.25346E 01	-0.25401E 01	-0.25460E 01	-0.25523E 01	-0.25589E 01	-0.25659E 01
35	-0.23388E 01	-0.23662E 01	-0.23688E 01	-0.23718E 01	-0.23751E 01	-0.23787E 01	-0.23826E 01
40	-0.22044E 01	-0.22042E 01	-0.22042E 01	-0.22045E 01	-0.22051E 01	-0.22060E 01	-0.22072E 01
45	-0.20616E 01	-0.20491E 01	-0.20468E 01	-0.20448E 01	-0.20431E 01	-0.20415E 01	-0.20403E 01
50	-0.19058E 01	-0.19013E 01	-0.18970E 01	-0.18930E 01	-0.18891E 01	-0.18855E 01	-0.18821E 01
55	-0.17672E 01	-0.17610E 01	-0.17550E 01	-0.17492E 01	-0.17436E 01	-0.17382E 01	-0.17330E 01
60	-0.16359E 01	-0.16283E 01	-0.16209E 01	-0.16136E 01	-0.16066E 01	-0.15997E 01	-0.15930E 01
65	-0.15119E 01	-0.15032E 01	-0.14946E 01	-0.14862E 01	-0.14779E 01	-0.14699E 01	-0.14620E 01
70	-0.13952E 01	-0.13855E 01	-0.13761E 01	-0.13667E 01	-0.13576E 01	-0.13486E 01	-0.13398E 01
75	-0.12855E 01	-0.12752E 01	-0.12650E 01	-0.12551E 01	-0.12452E 01	-0.12356E 01	-0.12262E 01
80	-0.11826E 01	-0.11719E 01	-0.11613E 01	-0.11508E 01	-0.11405E 01	-0.11304E 01	-0.11205E 01
85	-0.10863E 01	-0.10752E 01	-0.10644E 01	-0.10537E 01	-0.10431E 01	-0.10328E 01	-0.10225E 01
90	-0.99603E 00	-0.98493E 00	-0.97399E 00	-0.96322E 00	-0.95261E 00	-0.94215E 00	-0.93184E 00
95	-0.91159E 00	-0.90056E 00	-0.88970E 00	-0.87900E 00	-0.86846E 00	-0.85807E 00	-0.84784E 00
100	-0.83355E 00	-0.82173E 00	-0.81107E 00	-0.80057E 00	-0.79024E 00	-0.78005E 00	-0.77002E 00
105	-0.75850E 00	-0.74800E 00	-0.73766E 00	-0.72748E 00	-0.71746E 00	-0.70759E 00	-0.69787E 00
110	-0.68005E 00	-0.67896E 00	-0.66904E 00	-0.65927E 00	-0.64965E 00	-0.64019E 00	-0.63087E 00
115	-0.62379E 00	-0.61420E 00	-0.60477E 00	-0.59548E 00	-0.58635E 00	-0.57737E 00	-0.56853E 00
120	-0.56733E 00	-0.55331E 00	-0.54443E 00	-0.53571E 00	-0.52713E 00	-0.51869E 00	-0.51039E 00
125	-0.50431E 00	-0.49590E 00	-0.48764E 00	-0.47952E 00	-0.47154E 00	-0.46370E 00	-0.45598E 00
130	-0.44035E 00	-0.44161E 00	-0.43400E 00	-0.42653E 00	-0.41919E 00	-0.41198E 00	-0.40489E 00
135	-0.39711E 00	-0.39007E 00	-0.38316E 00	-0.37637E 00	-0.36970E 00	-0.36315E 00	-0.35672E 00
140	-0.34777E 00	-0.34096E 00	-0.33476E 00	-0.32868E 00	-0.32271E 00	-0.31685E 00	-0.31110E 00
145	-0.29600E 00	-0.29394E 00	-0.28849E 00	-0.28314E 00	-0.27788E 00	-0.27273E 00	-0.26767E 00
150	-0.25352E 00	-0.24873E 00	-0.24403E 00	-0.23942E 00	-0.23490E 00	-0.23046E 00	-0.22611E 00
155	-0.20004E 00	-0.20503E 00	-0.20110E 00	-0.19724E 00	-0.19346E 00	-0.18975E 00	-0.18612E 00
160	-0.16579E 00	-0.16257E 00	-0.15942E 00	-0.15632E 00	-0.15329E 00	-0.15032E 00	-0.14740E 00
165	-0.12950E 00	-0.12108E 00	-0.11871E 00	-0.11639E 00	-0.11411E 00	-0.11188E 00	-0.10969E 00
170	-0.81039E 00	-0.80325E 00	-0.78742E 00	-0.77191E 00	-0.75670E 00	-0.74170E 00	-0.72718E 00
175	-0.40851E 00	-0.40043E 00	-0.39251E 00	-0.38475E 00	-0.37714E 00	-0.36968E 00	-0.36237E 00
180	0.	0.	0.	0.	0.	0.	0.

$V(K, \psi)$

TRAJECTORY FUNCTIONS

K =	0.90000E 00	0.95000E 00	0.10000E 01
ψ			
0	-0.38107E 01	-0.38914E 01	-0.39775E 01
5	-0.35963E 01	-0.36660E 01	-0.37405E 01
10	-0.33834E 01	-0.34423E 01	-0.35054E 01
15	-0.31733E 01	-0.32217E 01	-0.32738E 01
20	-0.29675E 01	-0.30058E 01	-0.30474E 01
25	-0.27671E 01	-0.27960E 01	-0.28276E 01
30	-0.25733E 01	-0.25934E 01	-0.26158E 01
35	-0.23869E 01	-0.23989E 01	-0.24130E 01
40	-0.22087E 01	-0.22135E 01	-0.22201E 01
45	-0.20392E 01	-0.20377E 01	-0.20377E 01
50	-0.18790E 01	-0.18720E 01	-0.18662E 01
55	-0.17280E 01	-0.17164E 01	-0.17058E 01
60	-0.15865E 01	-0.15710E 01	-0.15565E 01
65	-0.14543E 01	-0.14356E 01	-0.14180E 01
70	-0.13311E 01	-0.13101E 01	-0.12900E 01
75	-0.12167E 01	-0.11939E 01	-0.11721E 01
80	-0.11106E 01	-0.10867E 01	-0.10637E 01
85	-0.10125E 01	-0.98792E 00	-0.96427E 00
90	-0.92169E 00	-0.89694E 00	-0.87309E 00
95	-0.83776E 00	-0.81319E 00	-0.78952E 00
100	-0.76014E 00	-0.73608E 00	-0.71290E 00
105	-0.68830E 00	-0.66501E 00	-0.64259E 00
110	-0.62170E 00	-0.59940E 00	-0.57796E 00
115	-0.55984E 00	-0.53871E 00	-0.51842E 00
120	-0.50223E 00	-0.48241E-00	-0.46341E-00
125	-0.44840E-00	-0.43001E-00	-0.41240E-00
130	-0.39793E-00	-0.38105E-00	-0.36490E-00
135	-0.35041E-00	-0.33510E-00	-0.32048E-00
140	-0.30545E-00	-0.29177E-00	-0.27871E-00
145	-0.26270E-00	-0.25069E-00	-0.23922E-00
150	-0.22184E-00	-0.21151E-00	-0.20167E-00
155	-0.18255E-00	-0.17392E-00	-0.16571E-00
160	-0.14454E-00	-0.13763E-00	-0.13105E-00
165	-0.10754E-00	-0.10235E-00	-0.97419E-01
170	-0.71285E-01	-0.67826E-01	-0.64534E-01
175	-0.35521E-01	-0.33790E-01	-0.32144E-01
180	0.	0.	0.

Table A-II.3

Magnetic Field Profiles for Symmetrical Spectrometers

MAGNETIC FIELD PROFILE

ψ_s	r_e	z_e	r_e	z_e	r_e	z_e	r_e	z_e
60							0.7112F 02	0.16653F 02
65							0.68936F 01	0.42145F 01
70					0.25662F 02	0.17340F 02	0.41470F 00	0.25094F 01
75					0.75288F 01	0.30173F 01	0.30519F 01	0.18178F 01
80			0.15842F 02	0.37933F 01	0.45471F 01	0.18018F 01	0.24585F 01	0.14335F 01
85	0.47005F 02	0.51124F 01	0.66617F 01	0.15828F 01	0.33116F 01	0.17897F 01	0.20823F 01	0.11872F 00
90	0.92424F 01	1.00000F 00	0.42690F 01	1.00000F 00	0.26279F 01	1.00000F 00	0.18190F 01	1.00000F 00
95	0.51449F 01	0.44088F 00	0.31553F 01	0.72395F 00	0.21878F 01	0.80859F 00	0.16710F 01	0.88818F 00
100	0.35468F 01	0.37584F 00	0.25027F 01	0.55871F 00	0.18759F 01	0.66922F 00	0.14636F 01	0.74193F 00
105	0.27027F 01	0.27582F 00	0.20675F 01	0.44602F 00	0.16391F 01	0.56081F 00	0.12325F 01	0.64205F 00
110	0.21619F 01	0.21315F 00	0.17515F 01	0.36250E 00	0.14495F 01	0.47243F 00	0.12190F 01	0.55633F 00
115	0.17832F 01	0.16848F 00	0.15075F 01	0.29703F 00	0.12911F 01	0.39797F 00	0.11172F 01	0.47902F 00
120	0.14091F 01	0.13452F 00	0.13100F 01	0.24367F 00	0.11540F 01	0.33375F 00	0.10234F 01	0.40916F 00
125	0.12745F 01	0.10758F 00	0.11439F 01	0.19902F 00	0.10318F 01	0.27754F 00	0.93463F 00	0.34577F 00
130	0.10898F 01	0.85583F 01	0.99986F 00	0.16102F 00	0.92014F 00	0.22791F 00	0.84905F 00	0.28756F 00
135	0.92270F 00	0.67299F 01	0.87164F 00	0.12836F 00	0.81609F 00	0.18397F 00	0.76523F 00	0.23477F 00
140	0.70500F 00	0.51060F 01	0.75500F 00	0.10023F 00	0.71730F 00	0.14515F 00	0.68214F 00	0.18705F 00
145	0.67285F 00	0.39071F 01	0.64695F 00	0.76066F 01	0.62239F 00	0.11113F 00	0.59909F 00	0.14441F 00
150	0.56101F 00	0.28206F 01	0.54529F 00	0.55528F 01	0.53017F 00	0.81722F 01	0.51561F 00	0.10604F 00
155	0.45724F 00	0.19746F 01	0.44841F 00	0.38383F 01	0.43981F 00	0.56828F 01	0.43143F 00	0.74800F 01
160	0.35048F 00	0.12345F 01	0.35506F 00	0.24885F 01	0.35071F 00	0.36425F 01	0.34644F 00	0.48170F 01
165	0.26410F 00	0.69229F 02	0.26427F 00	0.13742F 01	0.26245F 00	0.20518F 01	0.26665F 00	0.27232F 01
170	0.17579F 00	0.30556E 02	0.17525F 00	0.60988E 02	0.17472F 00	0.91295E 02	0.17418F 00	0.12148E 01
175	0.87422F 01	0.76213F 03	0.87355F 01	0.15235E 02	0.87289F 01	0.22841F 02	0.87222F 00	0.30439E 02
180	0.	0.	0.	0.	0.	0.	0.	0.
0.45000F 00		0.50000F 00		0.52000F 00		0.54000F 00		
45			0.17145F 02	0.15387F 02	0.79924E 01	0.77064F 01	0.25734F 02	0.26934F 02
50			0.51872F 01	0.46322F 01	0.38905F 01	0.37242F 01	0.51211F 01	0.52071F 01
55	0.22352F 02	0.16651E 02	0.32639E 01	0.28844F 01	0.27184F 01	0.25695F 01	0.20821F 01	0.31588F 01
60	0.60949F 01	0.45189F 01	0.24776F 01	0.21533F 01	0.21638F 01	0.20090F 01	0.23143F 01	0.23262F 01
65	0.37377F 01	0.27429F 01	0.20153F 01	0.20404F 01	0.17459F 01	0.18404F 01	0.19116F 01	0.18914F 01
70	0.27895F 01	0.20153F 01	0.17791F 01	0.14767E 01	0.16279F 01	0.16699F 01	0.16641F 01	0.16057F 01
75	0.22751E 01	0.16096F 01	0.15915E 01	0.12806F 01	0.14765F 01	0.14362F 01	0.14061F 01	0.14009F 01
80	0.19498E 01	0.13438F 01	0.14521F 01	0.11270E 01	0.12617F 01	0.12609E 01	0.13738F 01	0.12422F 01
85	0.17229F 01	0.11507F 01	0.13424F 01	0.10000F 00	0.12701F 01	1.00000F 00	0.12797F 01	0.11170F 01
90	0.15531F 01	1.00000F 00	0.12519F 01	0.89047F 00	0.11933F 01	0.89560F 00	0.12034F 01	1.00000F 00
95	0.14186E 01	0.87589F 00	0.11738E 01	0.79302E 00	0.10647E 01	0.80144F 00	0.11386F 01	0.90037F 00
100	0.12069E 01	0.76955E 00	0.11036F 01	0.70429F 00	0.10647E 01	0.71471F 00	0.10778F 01	0.72461F 00
105	0.12102F 01	0.67572F 00	0.10381E 00	0.62215E 00	0.10666E 01	0.63362F 00	0.97462F 00	0.64141F 00
110	0.11233E 01	0.59113E 00	0.97508E 00	0.54531E 00	0.94974F 00	0.55713F 00	0.97532F 00	0.56852F 00
115	0.10427E 01	0.51377F 00	0.91273F 00	0.47303F 00	0.89263E 00	0.48464F 00	0.87314F 00	0.49589F 00
120	0.96583F 00	0.44238E 00	0.84982E 00	0.40495F 00	0.83415F 00	0.41592F 00	0.81887F 00	0.47667F 00
125	0.89082F 00	0.37624F 00	0.78527F 00	0.28132E 00	0.77342E 00	0.35102E 00	0.76173F 00	0.36084F 00
130	0.81635F 00	0.31500F 00	0.71868F 00	0.20984E 00	0.70984E 00	0.29016F 00	0.70114F 00	0.29886F 00
135	0.74146F 00	0.25854E 00	0.64930F 00	0.17602F 00	0.64292F 00	0.23371F 00	0.70114F 00	0.24114F 00
140	0.66545E 00	0.20695F 00	0.57696F 00	0.13123E 00	0.57267F 00	0.18215F 00	0.56842F 00	0.18822F 00
145	0.58788E 00	0.16041F 00	0.50158F 00	0.92313F 01	0.42165E 00	0.35989F 00	0.49612F 00	0.40700F 00
150	0.50853F 00	0.11920F 00	0.42326E 00	0.59723F 01	0.34140E 00	0.62012F 01	0.42005F 00	0.39194F 01
155	0.42732E 00	0.83613F 01	0.34223F 00	0.25887F 00	0.25815F 00	0.35208F 01	0.25816F 00	0.36599F 01
160	0.34433F 00	0.53970F 01	0.25887F 00	0.15144E 01	0.17355F 00	0.15754E 01	0.17344F 00	0.16353E 01
165	0.25976F 00	0.30566F 01	0.17365F 00	0.38029E 02	0.87143F 01	0.39546F 02	0.87129F 01	0.4163F 02
170	0.17392F 00	0.13653E 01	0.	0.	0.	0.	0.	0.
175	0.87189F 01	0.34235E 02	0.	0.	0.	0.	0.	0.
180	0.	0.	0.	0.	0.	0.	0.	0.

MAGNETIC FIELD PROFILE

ψ_s	0.54000E 00		0.58000E 00		0.60000E 00		0.62000E 00	
	r_e	z_e	r_e	z_e	r_e	z_e	r_e	z_e
40			0.13439E 03	0.16116E 03	0.10499E 02	0.13512E 02	0.53261E 01	0.73474E 01
45	0.86906E 01	0.96906E 01	0.51138E 01	0.61138E 01	0.35691E 01	0.45691E 01	0.27090E 01	0.37090E 01
50	0.37186E 01	0.41203E 01	0.28884E 01	0.34237E 01	0.23403E 01	0.29638E 01	0.19519E 01	0.26378E 01
55	0.25327E 01	0.27734E 01	0.21339E 01	0.24941E 01	0.18320E 01	0.22828E 01	0.15958E 01	0.21174E 01
60	0.20032E 01	0.21566E 01	0.17567E 01	0.20142E 01	0.15567E 01	0.18987E 01	0.13913E 01	0.18033E 01
65	0.17046E 01	0.17949E 01	0.15318E 01	0.17143E 01	0.13856E 01	0.16461E 01	0.12603E 01	0.15877E 01
70	0.15134E 01	0.15508E 01	0.13833E 01	0.15035E 01	0.12699E 01	0.14622E 01	0.11702E 01	0.14259E 01
75	0.13803E 01	0.13698E 01	0.12778E 01	0.13424E 01	0.11866E 01	0.13179E 01	0.11048E 01	0.12960E 01
80	0.12817E 01	0.12260E 01	0.11987E 01	0.12114E 01	0.11235E 01	0.11981E 01	0.10551E 01	0.11860E 01
85	0.12048E 01	0.11054E 01	0.11363E 01	0.10994E 01	0.10734E 01	0.10939E 01	0.10154E 01	0.10888E 01
90	0.11417E 01	0.10000E 00	0.10846E 01	0.10000E 00	0.10316E 01	0.10000E 00	0.98217E 00	0.10000E 00
95	0.10875E 01	0.90485E 00	0.10397E 01	0.90904E 00	0.99477E 00	0.91297E 00	0.95760E 00	0.91666E 00
100	0.09877E 01	0.81686E 00	0.90855E 00	0.82393E 00	0.96061E 00	0.83062E 00	0.92467E 00	0.83696E 00
105	0.99262E 00	0.73403E 00	0.95914E 00	0.74300E 00	0.92722E 00	0.75155E 00	0.89678E 00	0.75971E 00
110	0.94749E 00	0.65514E 00	0.91975E 00	0.66524E 00	0.89312E 00	0.67499E 00	0.86756E 00	0.68423E 00
115	0.90176E 00	0.57950E 00	0.87903E 00	0.59010E 00	0.85708E 00	0.60034E 00	0.83588E 00	0.61022E 00
120	0.85423E 00	0.50681E 00	0.83588E 00	0.51740E 00	0.81807E 00	0.52768E 00	0.80078E 00	0.53767E 00
125	0.80298E 00	0.43705E 00	0.78945E 00	0.44722E 00	0.77528E 00	0.45714E 00	0.76146E 00	0.46682E 00
130	0.75027E 00	0.37045E 00	0.73905E 00	0.37987E 00	0.72805E 00	0.38909E 00	0.71728E 00	0.39813E 00
135	0.69259E 00	0.30741E 00	0.68418E 00	0.31582E 00	0.67591E 00	0.32409E 00	0.66777E 00	0.33272E 00
140	0.63062E 00	0.24846E 00	0.62455E 00	0.25569E 00	0.61856E 00	0.26282E 00	0.61265E 00	0.26987E 00
145	0.56621E 00	0.19423E 00	0.56004E 00	0.20018E 00	0.55502E 00	0.20607E 00	0.55184E 00	0.21190E 00
150	0.49341E 00	0.14538E 00	0.49073E 00	0.15003E 00	0.48807E 00	0.15464E 00	0.48542E 00	0.15927E 00
155	0.41846E 00	0.10261E 00	0.41688E 00	0.10601E 00	0.41530E 00	0.10938E 00	0.41373E 00	0.11275E 00
160	0.33976E 00	0.66566E 00	0.33892E 00	0.68831E 00	0.33810E 00	0.71090E 00	0.33728E 00	0.73347E 00
165	0.25781E 00	0.37847E 00	0.25746E 00	0.39163E 00	0.25710E 00	0.40477E 00	0.25675E 00	0.41788E 00
170	0.17334E 00	0.16952E 00	0.17223E 00	0.17550E 00	0.17313E 00	0.18148E 00	0.17307E 00	0.18745E 00
175	0.87116E 01	0.42579E 00	0.87103E 01	0.44094E 00	0.87090E 01	0.45612E 00	0.87076E 01	0.47127E 00
180	0.	0.	0.	0.	0.	0.	0.	0.
	0.64000E 00		0.66000E 00		0.68000E 00		0.70000E 00	
	r_e	z_e	r_e	z_e	r_e	z_e	r_e	z_e
2			0.62041E 01	0.98603E 01	0.36584E 01	0.62247E 01	0.10416E 02	0.19041E 02
25	0.18248E 02	0.27061E 02	0.51792E 01	0.25793E 01	0.20178E 01	0.34047E 01	0.25540E 01	0.46475E 01
30	0.21616E 01	0.21616E 01	0.17831E 01	0.27831E 01	0.15063E 01	0.25063E 01	0.12951E 01	0.22951E 01
35	0.16626E 01	0.23951E 01	0.14391E 01	0.22075E 01	0.12614E 01	0.20584E 01	0.11169E 01	0.19372E 01
40	0.14061E 01	0.19846E 01	0.12507E 01	0.18758E 01	0.11212E 01	0.17851E 01	0.10117E 01	0.17084E 01
45	0.12525E 01	0.17231E 01	0.11345E 01	0.16550E 01	0.10209E 01	0.15064E 01	0.94475E 00	0.15455E 01
50	0.11520E 01	0.15372E 01	0.10573E 01	0.14930E 01	0.97408E 00	0.14542E 01	0.90034E 00	0.14198E 01
55	0.10821E 01	0.13938E 01	0.10036E 01	0.13653E 01	0.93366E 00	0.13397E 01	0.87016E 00	0.13167E 01
60	0.10213E 01	0.12763E 01	0.96473E 00	0.12885E 01	0.90434E 00	0.12423E 01	0.84930E 00	0.12276E 01
65	0.99266E 00	0.11750E 01	0.93548E 00	0.11650E 01	0.88296E 00	0.11557E 01	0.83458E 00	0.11472E 01
70	0.96195E 00	0.10842E 01	0.91242E 00	0.10798E 01	0.86647E 00	0.10758E 01	0.82372E 00	0.10721E 01
75	0.93611E 00	0.10000E 00	0.89307E 00	0.10000E 00	0.85277E 00	0.10000E 00	0.81498E 00	0.10000E 00
80	0.89060E 00	0.84296E 00	0.85825E 00	0.84867E 00	0.84025E 00	0.82752E 00	0.80692E 00	0.82940E 00
85	0.86770E 00	0.76750E 00	0.83992E 00	0.77494E 00	0.81335E 00	0.78206E 00	0.79828E 00	0.85924E 00
90	0.84290E 00	0.69918E 00	0.81938E 00	0.70177E 00	0.79666E 00	0.71004E 00	0.77479E 00	0.71800E 00
95	0.81530E 00	0.61978E 00	0.79559E 00	0.62901E 00	0.77644E 00	0.63794E 00	0.75791E 00	0.64658E 00
100	0.78390E 00	0.54737E 00	0.76766E 00	0.55679E 00	0.75179E 00	0.56595E 00	0.73637E 00	0.57486E 00
105	0.74707E 00	0.47627E 00	0.73480E 00	0.48549E 00	0.72194E 00	0.49449E 00	0.70939E 00	0.50328E 00
110	0.70672E 00	0.40699E 00	0.69637E 00	0.41568E 00	0.68427E 00	0.42619E 00	0.67627E 00	0.43254E 00
115	0.65977E 00	0.34023E 00	0.65189E 00	0.34811E 00	0.64614E 00	0.35586E 00	0.63651E 00	0.36349E 00
120	0.60682E 00	0.27682E 00	0.60106E 00	0.28368E 00	0.59537E 00	0.29046E 00	0.58976E 00	0.29715E 00
125	0.54779E 00	0.21767E 00	0.54379E 00	0.22339E 00	0.53982E 00	0.22005E 00	0.53590E 00	0.23466E 00
130	0.48280E 00	0.16377E 00	0.48019E 00	0.16828E 00	0.47760E 00	0.17076E 00	0.47504E 00	0.17721E 00
135	0.41217E 00	0.11609E 00	0.41062E 00	0.11942E 00	0.40088E 00	0.12273E 00	0.40704E 00	0.12603E 00
140	0.33646E 00	0.75585E 00	0.33565E 00	0.77822E 00	0.33383E 00	0.80051E 00	0.33403E 00	0.82274E 00
145	0.25640E 00	0.43096E 00	0.25605E 00	0.44403E 00	0.25570E 00	0.45707E 00	0.25535E 00	0.47008E 00
150	0.17927E 00	0.19342E 00	0.17281E 00	0.19939E 00	0.17071E 00	0.20535E 00	0.17260E 00	0.21130E 00
155	0.87063E 01	0.48643E 00	0.87050E 01	0.50158E 00	0.87037E 01	0.51672E 00	0.87023E 01	0.53186E 00
160	0.	0.	0.	0.	0.	0.	0.	0.

MASTIC FIELD PROFILE

K -	0.720005 00		0.740005 00		0.760005 00		0.780005 00	
ψ_s	r_e	z_e	r_e	z_e	r_e	z_e	r_e	z_e
35								
36	0.44437E 01	0.86965E 01	0.27690E 01	0.57941E 01	0.15816E 01	0.44437E 01	0.15746E 01	0.26407E 01
37	0.19378E 01	0.37474E 01	0.15452E 01	0.22068E 01	0.17736E 01	0.28189E 01	0.15748E 01	0.26300E 01
40	0.19720E 01	0.26367E 01	0.11713E 01	0.23959E 01	0.10147E 01	0.22092E 01	0.88973E 00	0.26002E 01
45	0.11291E 01	0.12191E 01	0.09525E 00	0.19952E 01	0.88520E 00	0.18852E 01	0.79324E 01	0.17993E 01
50	0.99732E 00	0.18360E 01	0.89682E 00	0.17525E 01	0.81174E 00	0.16807E 01	0.73761E 00	0.16483E 01
55	0.91802E 00	0.16428E 01	0.83706E 00	0.15861E 01	0.66646E 00	0.15347E 01	0.70464E 00	0.14733E 01
60	0.86755E 00	0.15020E 01	0.79040E 00	0.14614E 01	0.53903E 00	0.14267E 01	0.68577E 00	0.13285E 01
65	0.83467E 00	0.13892E 01	0.77575E 00	0.13711E 01	0.47274E 00	0.12979E 01	0.67481E 00	0.12447E 01
70	0.81303E 00	0.12959E 01	0.76119E 00	0.12717E 01	0.41397E 00	0.11903E 01	0.67081E 00	0.11788E 01
75	0.79897E 00	0.12141E 01	0.75280E 00	0.12017E 01	0.37103E 00	0.11252E 01	0.67115E 00	0.11180E 01
80	0.78988E 00	0.11393E 01	0.74850E 00	0.11320E 01	0.34101E 00	0.10672E 01	0.67438E 00	0.10592E 01
85	0.78380E 00	0.10686E 01	0.74671E 00	0.10655E 01	0.31922E 00	0.10072E 01	0.68502E 00	0.10000E 01
90	0.77940E 00	0.10000E 01	0.74611E 00	0.10000E 01	0.29902E 00	0.09472E 01	0.69000E 00	0.09999E 01
95	0.77540E 00	0.93216E 00	0.74544E 00	0.93277E 00	0.27724E 00	0.87722E 00	0.69400E 00	0.87724E 00
100	0.77040E 00	0.86415E 00	0.74203E 00	0.86883E 00	0.25864E 00	0.80765E 00	0.69637E 00	0.81331E 00
105	0.76575E 00	0.79540E 00	0.74023E 00	0.80166E 00	0.24185E 00	0.74017E 00	0.69502E 00	0.74722E 00
110	0.76125E 00	0.72566E 00	0.73846E 00	0.73205E 00	0.22697E 00	0.67089E 00	0.68908E 00	0.67940E 00
115	0.75697E 00	0.65495E 00	0.73600E 00	0.66304E 00	0.21378E 00	0.60013E 00	0.67877E 00	0.60811E 00
120	0.75287E 00	0.58352E 00	0.73378E 00	0.59209E 00	0.20104E 00	0.57845E 00	0.66700E 00	0.55646E 00
125	0.74897E 00	0.51187E 00	0.73161E 00	0.52026E 00	0.18874E 00	0.54563E 00	0.65436E 00	0.48476E 00
130	0.74527E 00	0.44073E 00	0.72946E 00	0.44876E 00	0.17746E 00	0.51678E 00	0.64071E 00	0.40702E 00
135	0.74177E 00	0.37100E 00	0.72731E 00	0.37889E 00	0.16744E 00	0.48866E 00	0.62700E 00	0.32702E 00
140	0.73847E 00	0.30276E 00	0.72524E 00	0.31028E 00	0.15833E 00	0.46700E 00	0.61307E 00	0.25002E 00
145	0.73527E 00	0.24021E 00	0.72316E 00	0.24571E 00	0.15044E 00	0.45115E 00	0.60037E 00	0.18654E 00
150	0.73227E 00	0.18163E 00	0.72109E 00	0.18602E 00	0.14344E 00	0.43907E 00	0.58804E 00	0.12747E 00
155	0.72947E 00	0.12931E 00	0.71902E 00	0.12877E 00	0.13744E 00	0.43077E 00	0.57899E 00	0.08002E 00
160	0.72677E 00	0.84899E 00	0.71695E 00	0.86646E 00	0.13144E 00	0.42404E 00	0.57000E 00	0.03002E 00
165	0.72427E 00	0.48808E 00	0.71488E 00	0.49655E 00	0.12544E 00	0.41700E 00	0.56200E 00	0.00002E 00
170	0.72177E 00	0.21722E 00	0.71281E 00	0.22319E 00	0.11944E 00	0.41000E 00	0.55400E 00	0.00002E 00
175	0.71927E 00	0.05701E 00	0.71074E 00	0.06697E 00	0.11344E 00	0.40300E 00	0.54600E 00	0.00002E 00
180	0.	0.	0.	0.	0.	0.	0.	0.
K -	0.800005 00		0.820005 00		0.840005 00		0.860005 00	
20								
25	0.22670E 01	0.60653E 01	0.28175E 01	0.77509E 01	0.35957E 01	0.15775E 01	0.25521E 01	0.80111E 01
30	0.17266E 01	0.31245E 01	0.16740E 01	0.45008E 01	0.17806E 01	0.27463E 01	0.10264E 01	0.22002E 01
35	0.97372E 00	0.22195E 01	0.80208E 00	0.21182E 01	0.26217E 00	0.24033E 01	0.24301E 01	0.22800E 01
40	0.78785E 00	0.19399E 01	0.70331E 00	0.18282E 01	0.20788E 00	0.20111E 01	0.62887E 00	0.18901E 01
45	0.71535E 00	0.17154E 01	0.64962E 00	0.16486E 01	0.16732E 00	0.17333E 01	0.57165E 00	0.16911E 01
50	0.67364E 00	0.15653E 01	0.61762E 00	0.15183E 01	0.15098E 00	0.15909E 01	0.54748E 00	0.15605E 01
55	0.64956E 00	0.14649E 01	0.60272E 00	0.14065E 01	0.14768E 00	0.14768E 01	0.52495E 00	0.14601E 01
60	0.62665E 00	0.13676E 01	0.59001E 00	0.13404E 01	0.13672E 00	0.13672E 01	0.51773E 00	0.13604E 01
65	0.60320E 00	0.12944E 01	0.57916E 00	0.12597E 01	0.13196E 00	0.13196E 01	0.51759E 00	0.13084E 01
70	0.58111E 00	0.12098E 01	0.56942E 00	0.11655E 01	0.12697E 00	0.12697E 01	0.51759E 00	0.12002E 01
75	0.56407E 00	0.11701E 01	0.56174E 00	0.11611E 01	0.12244E 00	0.12244E 01	0.51759E 00	0.11400E 01
80	0.54811E 00	0.11130E 01	0.55507E 00	0.11076E 01	0.11844E 00	0.11844E 01	0.51759E 00	0.10702E 01
85	0.53370E 00	0.10684E 01	0.54964E 00	0.10647E 01	0.11477E 00	0.11477E 01	0.51759E 00	0.10002E 01
90	0.52001E 00	0.10300E 01	0.54544E 00	0.10244E 01	0.11144E 00	0.11144E 01	0.51759E 00	0.09302E 01
95	0.50707E 00	0.09977E 01	0.54164E 00	0.09864E 01	0.10844E 00	0.10844E 01	0.51759E 00	0.08602E 01
100	0.49477E 00	0.09699E 01	0.53824E 00	0.09477E 01	0.10577E 00	0.10577E 01	0.51759E 00	0.07902E 01
105	0.48307E 00	0.09459E 01	0.53524E 00	0.09107E 01	0.10337E 00	0.10337E 01	0.51759E 00	0.07202E 01
110	0.47187E 00	0.09249E 01	0.53264E 00	0.08764E 01	0.10117E 00	0.10117E 01	0.51759E 00	0.06502E 01
115	0.46117E 00	0.09059E 01	0.53034E 00	0.08444E 01	0.09917E 00	0.09917E 01	0.51759E 00	0.05802E 01
120	0.45097E 00	0.08889E 01	0.52834E 00	0.08144E 01	0.09727E 00	0.09727E 01	0.51759E 00	0.05102E 01
125	0.44117E 00	0.08739E 01	0.52654E 00	0.07864E 01	0.09557E 00	0.09557E 01	0.51759E 00	0.04402E 01
130	0.43177E 00	0.08609E 01	0.52494E 00	0.07604E 01	0.09407E 00	0.09407E 01	0.51759E 00	0.03702E 01
135	0.42277E 00	0.08499E 01	0.52354E 00	0.07364E 01	0.09277E 00	0.09277E 01	0.51759E 00	0.03002E 01
140	0.41417E 00	0.08409E 01	0.52234E 00	0.07144E 01	0.09167E 00	0.09167E 01	0.51759E 00	0.02302E 01
145	0.40597E 00	0.08339E 01	0.52134E 00	0.06944E 01	0.09077E 00	0.09077E 01	0.51759E 00	0.01602E 01
150	0.40007E 00	0.08289E 01	0.52054E 00	0.06764E 01	0.09007E 00	0.09007E 01	0.51759E 00	0.00902E 01
155	0.39457E 00	0.08249E 01	0.51994E 00	0.06604E 01	0.08957E 00	0.08957E 01	0.51759E 00	0.00202E 01
160	0.38947E 00	0.08219E 01	0.51944E 00	0.06464E 01	0.08917E 00	0.08917E 01	0.51759E 00	0.00002E 01
165	0.38467E 00	0.08199E 01	0.51904E 00	0.06334E 01	0.08887E 00	0.08887E 01	0.51759E 00	0.00002E 01
170	0.38017E 00	0.08189E 01	0.51874E 00	0.06214E 01	0.08867E 00	0.08867E 01	0.51759E 00	0.00002E 01
175	0.37587E 00	0.08179E 01	0.51854E 00	0.06104E 01	0.08857E 00	0.08857E 01	0.51759E 00	0.00002E 01
180	0.	0.	0.	0.	0.	0.	0.	0.

MAGNETIC FIELD PROFILE

K	0.80000E 00		0.90000E 00		0.95000E 00		0.10000E 01	
ψ_s	re	ze	re	ze	re	ze	re	ze
1E	0.16607E 01	0.55628E 01	0.12137E 01	0.43346E 01	0.21136E 01	0.88882E 01	0.84347E 00	0.41679E 01
2E	0.84890E 00	0.28205E 01	0.71808E 00	0.25399E 01	0.69822E 00	0.29183E 01	0.47061E 00	0.22930E 01
3E	0.64868E 00	0.21235E 01	0.57225E 00	0.19912E 01	0.50505E 00	0.20831E 01	0.37777E 00	0.18101E 01
4E	0.56787E 00	0.18039E 01	0.50699E 00	0.17241E 01	0.43296E 00	0.17499E 01	0.33943E 00	0.15879E 01
5E	0.51918E 00	0.16187E 01	0.47373E 00	0.15646E 01	0.39906E 00	0.15699E 01	0.32179E 00	0.14595E 01
6E	0.49618E 00	0.14962E 01	0.45696E 00	0.14570E 01	0.37640E 00	0.14561E 01	0.31478E 00	0.13751E 01
7E	0.45526E 00	0.14072E 01	0.45016E 00	0.13777E 01	0.37662E 00	0.13764E 01	0.31443E 00	0.13144E 01
8E	0.48725E 00	0.13377E 01	0.45009E 00	0.13152E 01	0.38161E 00	0.12672E 01	0.32652E 00	0.12286E 01
9E	0.48496E 00	0.12799E 01	0.45492E 00	0.12626E 01	0.39030E 00	0.12253E 01	0.33742E 00	0.11948E 01
10E	0.49165E 00	0.12293E 01	0.46350E 00	0.12161E 01	0.40201E 00	0.11875E 01	0.35991E 00	0.11636E 01
11E	0.50165E 00	0.11826E 01	0.47502E 00	0.11729E 01	0.41622E 00	0.11515E 01	0.36666E 00	0.11335E 01
12E	0.51408E 00	0.11377E 01	0.48981E 00	0.11310E 01	0.43247E 00	0.11169E 01	0.38437E 00	0.11030E 01
13E	0.52826E 00	0.10931E 01	0.50429E 00	0.10889E 01	0.45034E 00	0.10794E 01	0.40371E 00	0.10712E 01
14E	0.54355E 00	0.10476E 01	0.52085E 00	0.10456E 01	0.46934E 00	0.10411E 01	0.42429E 00	0.10371E 01
15E	0.55923E 00	0.10000E 01	0.53786E 00	0.10000E 01	0.48893E 00	0.10000E 01	0.44564E 00	0.10000E 01
16E	0.57459E 00	0.94973E 00	0.55460E 00	0.95148E 00	0.50846E 00	0.95552E 00	0.46719E 00	0.95913E 00
17E	0.58880E 00	0.89618E 00	0.57027E 00	0.89945E 00	0.52718E 00	0.90704E 00	0.48823E 00	0.91391E 00
18E	0.60095E 00	0.83898E 00	0.58398E 00	0.84352E 00	0.54421E 00	0.85418E 00	0.50791E 00	0.86391E 00
19E	0.61005E 00	0.77796E 00	0.59474E 00	0.78353E 00	0.55595E 00	0.79669E 00	0.52525E 00	0.80883E 00
20E	0.61507E 00	0.71319E 00	0.60149E 00	0.71052E 00	0.56920E 00	0.73458E 00	0.53913E 00	0.74860E 00
21E	0.61492E 00	0.64497E 00	0.60312E 00	0.65179E 00	0.57488E 00	0.66909E 00	0.54835E 00	0.68341E 00
22E	0.60952E 00	0.57391E 00	0.59852E 00	0.58092E 00	0.57444E 00	0.59778E 00	0.55161E 00	0.61376E 00
23E	0.59484E 00	0.50087E 00	0.58661E 00	0.50777E 00	0.56669E 00	0.52449E 00	0.54765E 00	0.54047E 00
24E	0.57299E 00	0.42701E 00	0.56647E 00	0.43353E 00	0.55957E 00	0.44943E 00	0.53825E 00	0.46475E 00
25E	0.54227E 00	0.35375E 00	0.53731E 00	0.35966E 00	0.52177E 00	0.37412E 00	0.51339E 00	0.38816E 00
26E	0.50722E 00	0.28276E 00	0.49865E 00	0.29785E 00	0.48988E 00	0.30038E 00	0.48131E 00	0.31262E 00
27E	0.45727E 00	0.21586E 00	0.45033E 00	0.22000E 00	0.44442E 00	0.23023E 00	0.43862E 00	0.24079E 00
28E	0.39405E 00	0.15496E 00	0.39259E 00	0.15809E 00	0.38996E 00	0.16586E 00	0.38538E 00	0.17354E 00
29E	0.32386E 00	0.10196E 00	0.32608E 00	0.10411E 00	0.32413E 00	0.10946E 00	0.32220E 00	0.11477E 00
30E	0.25924E 00	0.58617E 01	0.25190E 00	0.58996E 01	0.25105E 00	0.63091E 01	0.25020E 00	0.66252E 01
31E	0.17166E 00	0.26467E 01	0.17156E 00	0.27057E 01	0.17130E 00	0.28532E 01	0.17104E 00	0.30003E 01
32E	0.86904E 01	0.66802E 02	0.86891E 01	0.68313E 02	0.86858E 01	0.72090E 02	0.86825E 01	0.75865E 02
33E	0.	0.	0.	0.	0.	0.	0.	0.

Table A-II.4

Orbit Rotation Function $F_1(K, \psi_s)$

		$F_i(K, \psi_s)$							
$K =$	ψ_s	1.00000E-01	0.20000E-00	0.30000E-00	0.40000E-00	0.45000E-00	0.50000E-00	0.56000E-00	0.60000E-00
40									0.10285E-00
50									0.52084E-00
60					0.37829E-01	0.18175E-00	0.64330E-01	0.31608E-00	0.63048E-00
70				0.33135E-01	0.25318E-00	0.40778E-00	0.59417E-00	0.86233E-00	0.84267E-00
80			0.37545E-01	0.18287E-00	0.42080E-00	0.57709E-00	0.75957E-00	0.10150E-01	0.10496E-01
90	0.31010E-01	0.12983E-00	0.29758E-00	0.37605E-00	0.53969E-00	0.69026E-00	0.86149E-00	0.10954E-01	0.12692E-01
100	0.73340E-01	0.19786E-00	0.37605E-00	0.61078E-00	0.75035E-00	0.90531E-00	0.11123E-01	0.11123E-01	0.12633E-01
110	0.10179E-00	0.24091E-00	0.41883E-00	0.63718E-00	0.76208E-00	0.89777E-00	0.10752E-01	0.12026E-01	0.12026E-01
120	0.11712E-00	0.25950E-00	0.42792E-00	0.62318E-00	0.73116E-00	0.84619E-00	0.99370E-00	0.10079E-01	0.10079E-01
130	0.11084E-00	0.25522E-00	0.40645E-00	0.57389E-00	0.66381E-00	0.75793E-00	0.87647E-00	0.95894E-00	0.95894E-00
140	0.11109E-00	0.23053E-00	0.35843E-00	0.49490E-00	0.56640E-00	0.64009E-00	0.72144E-00	0.79412E-00	0.79412E-00
150	0.92507E-01	0.18867E-00	0.28851E-00	0.39206E-00	0.44524E-00	0.49935E-00	0.56554E-00	0.61742E-00	0.61742E-00
160	0.66157E-01	0.13342E-00	0.20181E-00	0.27131E-00	0.30648E-00	0.34193E-00	0.38485E-00	0.41369E-00	0.41369E-00
170	0.34447E-01	0.69035E-01	0.10376E-00	0.13863E-00	0.15612E-00	0.17365E-00	0.19472E-00	0.20880E-00	0.20880E-00
180	0.	0.	0.	0.	0.	0.	0.	0.	0.

$K =$	ψ_s	0.66000E-00	0.70000E-00	0.76000E-00	0.80000E-00	0.86000E-00	0.90000E-00	0.10000E-01
20						0.30194E-00	0.64612E-00	0.17308E-01
30			0.92233E-01	0.50233E-00	0.82863E-00	0.14064E-01	0.18563E-01	0.32416E-01
40	0.43846E-00	0.70753E-00	0.11859E-01	0.15590E-01	0.22086E-01	0.27072E-01	0.27072E-01	0.42148E-01
50	0.88804E-00	0.11759E-01	0.16782E-01	0.20639E-01	0.27258E-01	0.32273E-01	0.32273E-01	0.47185E-01
60	0.12148E-01	0.15011E-01	0.19927E-01	0.23647E-01	0.29948E-01	0.34664E-01	0.48660E-01	0.48660E-01
70	0.14261E-01	0.16960E-01	0.21523E-01	0.24929E-01	0.30626E-01	0.34840E-01	0.46999E-01	0.46999E-01
80	0.15353E-01	0.17788E-01	0.21848E-01	0.24839E-01	0.29780E-01	0.33392E-01	0.43425E-01	0.43425E-01
90	0.15580E-01	0.17699E-01	0.21184E-01	0.23719E-01	0.27856E-01	0.30845E-01	0.39183E-01	0.39183E-01
100	0.15103E-01	0.16890E-01	0.19788E-01	0.21870E-01	0.25227E-01	0.27626E-01	0.34214E-01	0.34214E-01
110	0.14076E-01	0.15538E-01	0.17879E-01	0.19539E-01	0.22185E-01	0.24055E-01	0.29113E-01	0.29113E-01
120	0.12631E-01	0.13793E-01	0.15629E-01	0.16915E-01	0.18942E-01	0.20358E-01	0.24133E-01	0.24133E-01
130	0.10879E-01	0.11773E-01	0.13169E-01	0.14136E-01	0.15641E-01	0.16681E-01	0.19414E-01	0.19414E-01
140	0.89083E-00	0.95711E-00	0.10593E-01	0.11292E-01	0.12369E-01	0.13106E-01	0.15014E-01	0.15014E-01
150	0.67889E-00	0.72530E-00	0.79606E-00	0.86401E-00	0.91709E-00	0.96659E-00	0.10031E-01	0.10031E-01
160	0.45728E-00	0.48657E-00	0.53085E-00	0.56059E-00	0.60555E-00	0.63575E-00	0.71205E-00	0.71205E-00
170	0.22996E-00	0.24409E-00	0.26534E-00	0.27953E-00	0.30086E-00	0.31511E-00	0.35083E-00	0.35083E-00
180	0.	0.	0.	0.	0.	0.	0.	0.

Table A-II.5

Dispersion Function $G(K, \psi_s)$

G(K, ψ_s)

K =	0.20000E-00	0.40000E-00	0.45000E-00	0.50000E-00	0.52000E-00	0.54000E-00	0.56000E-00
ψ_s							
45				0.20366E 04	0.48633E 03	0.70932E 04	0.03013E 03
50				0.10364E 03	0.90551E 02	0.59560E 03	0.11634E 03
55			0.27482F 04	0.49320E 02	0.35749E 02	0.27067E 02	0.21180E 02
60			0.44447F 03	0.23630E 02	0.8785E 02	0.15275E 02	0.12652E 02
65			0.46270E 02	0.13754E 02	0.11528E 02	0.97940E 01	0.84161E 01
70			0.22076F 02	0.13449F 02	0.90335E 01	0.78140E 01	0.68403E 01
75			0.12870E 02	0.83108F 01	0.63833E 01	0.56797E 01	0.50829E 01
80	0.27427F 03		0.84506F 01	0.67531F 01	0.47979E 01	0.43500E 01	0.39073E 01
85	0.62747E 02		0.47039E 01	0.37705E 01	0.34707E 01	0.32038E 01	0.29652E 01
90	0.24739E 02		0.60105E 01	0.47039E 01	0.37705E 01	0.34707E 01	0.32038E 01
95	0.13194F 02		0.45299F 01	0.36997F 01	0.30712E 01	0.26620E 01	0.26724E 01
100	0.82431F 01		0.35685F 01	0.30714 E 1	0.25756E 01	0.24251E 01	0.22867E 01
105	0.56845E 01		0.29115E 01	0.25286E 01	0.22132E 01	0.21024E 01	0.19793E 01
110	0.41972E 01		0.24447E 01	0.21727E 01	0.1944E 01	0.18585E 01	0.17804E 01
115	0.32599E 01		0.21024E 01	0.19055F 01	0.17334E 01	0.16700E 01	0.16107E 01
120	0.26334E 01		0.18452F 01	0.17007F 01	0.15715E 01	0.15236E 01	0.14777E 01
125	0.21961E 01		0.16461E 01	0.15414E 01	0.14439E 01	0.14073E 01	0.13720E 01
130	0.18030E 01		0.14949F 01	0.14157F 01	0.13424E 01	0.13145E 01	0.12874E 01
135	0.16466E 01		0.13743F 01	0.13161F 01	0.12611E 01	0.12400E 01	0.12194E 01
140	0.14705F 01		0.12789F 01	0.12364F 01	0.11958E 01	0.11809E 01	0.11646E 01
145	0.13361F 01		0.12031E 01	0.11726E 01	0.11433E 01	0.11318E 01	0.11205E 01
150	0.12332E 01		0.11430F 01	0.11219E 01	0.11013E 01	0.10932F 01	0.10852E 01
155	0.11445E 01		0.10959E 01	0.10819E 01	0.10681E 01	0.10627E 01	0.10573E 01
160	0.10953E 01		0.10497E 01	0.10317E 01	0.10142E 01	0.10091E 01	0.10037E 01
165	0.10421E 01		0.10329E 01	0.10281E 01	0.10234E 01	0.10197E 01	0.10178E 01
170	0.10227E 01		0.10144E 01	0.10123E 01	0.10103E 01	0.10094E 01	0.10086E 01
175	0.10056E 01		0.10036E 01	0.10031E 01	0.10025E 01	0.10023E 01	0.10021E 01
180	0.	0.	0.	0.	0.	0.	0.

K =	0.58000E-00	0.60000E-00	0.62000E-00	0.64000E-00	0.66000E-00	0.68000E-00	0.70000E-00
ψ_s							
35				0.91598E 04	0.11140E 04	0.40030E 03	0.54506E 04
40	0.29081E 06	0.16701E 04	0.50694E 03	0.23136E 03	0.13173E 03	0.84807E 02	0.20967E 03
45	0.30522E 04	0.15631E 03	0.34635E 02	0.63302E 02	0.45238F 02	0.35888E 02	0.59049E 02
50	0.73459E 02	0.50583E 02	0.36893F 02	0.28058E 02	0.22027E 02	0.17728E 02	0.26255E 02
55	0.31267E 02	0.24115E 02	0.19140E 02	0.15542E 02	0.12856E 02	0.10798E 02	0.18697E 01
60	0.17005E 02	0.13938E 02	0.11619E 02	0.98242E 01	0.84064E 01	0.72671E 01	0.6336E 01
65	0.10641E 02	0.90647E 01	0.78071E 01	0.67877E 01	0.59501E 01	0.52537E 01	0.46665E 01
70	0.73035E 01	0.63924E 01	0.56369E 01	0.50037E 01	0.44678E 01	0.40103E 01	0.36168E 01
75	0.53671E 01	0.47850E 01	0.42967E 01	0.38767E 01	0.35124E 01	0.31977E 01	0.29175E 01
80	0.41327E 01	0.37510E 01	0.34176E 01	0.31249E 01	0.28664E 01	0.26371E 01	0.24327E 01
85	0.33176E 01	0.30510E 01	0.28137E 01	0.26017E 01	0.24115E 01	0.22407E 01	0.20855E 01
90	0.27709E 01	0.25580E 01	0.23835E 01	0.22253E 01	0.20815E 01	0.19503E 01	0.18303E 01
95	0.23431F 01	0.21996E 01	0.20681E 01	0.19473E 01	0.18361E 01	0.17336E 01	0.16388E 01
100	0.20414E 01	0.19324E 01	0.18313E 01	0.17375E 01	0.16502E 01	0.15688E 01	0.14929E 01
105	0.18132E 01	0.17291E 01	0.16503E 01	0.15764E 01	0.15070E 01	0.14418E 01	0.13804E 01
110	0.16740E 01	0.15718E 01	0.15008E 01	0.14511E 01	0.13970E 01	0.13428E 01	0.12878E 01
115	0.15000E 01	0.14485E 01	0.13995E 01	0.13526E 01	0.13079E 01	0.12642E 01	0.12244E 01
120	0.13915E 01	0.13509E 01	0.13120E 01	0.12746E 01	0.12386E 01	0.12040E 01	0.11707E 01
125	0.12749F 01	0.12730E 01	0.12422E 01	0.12124E 01	0.11830E 01	0.11557E 01	0.11287E 01
130	0.12355F 01	0.12106E 01	0.11863E 01	0.11628E 01	0.11398E 01	0.11175E 01	0.10958E 01
135	0.11796E 01	0.11603E 01	0.11415E 01	0.11230E 01	0.11050E 01	0.10874E 01	0.10701E 01
140	0.11345E 01	0.11199E 01	0.11055E 01	0.10913E 01	0.10774E 01	0.10638E 01	0.10504E 01
145	0.10983E 01	0.10874E 01	0.10767E 01	0.10661E 01	0.10557E 01	0.10454E 01	0.10352E 01
150	0.10694E 01	0.10616E 01	0.10539E 01	0.10463E 01	0.10387E 01	0.10312E 01	0.10238E 01
155	0.10466E 01	0.10413E 01	0.10360E 01	0.10308E 01	0.10256E 01	0.10205E 01	0.10154E 01
160	0.10290F 01	0.10257E 01	0.10223E 01	0.10190E 01	0.10157E 01	0.10125E 01	0.10092E 01
165	0.10160E 01	0.10141E 01	0.10123E 01	0.10104E 01	0.10086E 01	0.10067E 01	0.10049E 01
170	0.10070E 01	0.10062E 01	0.10054E 01	0.10045E 01	0.10037E 01	0.10029E 01	0.10021E 01
175	0.10017E 01	0.10015E 01	0.10013E 01	0.10011E 01	0.10009E 01	0.10007E 01	0.10005E 01
180	0.	0.	0.	0.	0.	0.	0.

G(K, ψ_s)

K =	0.72000E 00	0.74000E 00	0.76000E 00	0.78000E 00	0.80000E 00	0.82000E 00	0.84000E 00
ψ_s							
20							
25							
30	0.10467E 04	0.42861E 03	0.96803E 04	0.15933E 04	0.62612E 03	0.18239E 08	0.65469E 04
35	0.12712E 03	0.85099E 02	0.23140E 03	0.14435E 03	0.98421E 02	0.33164E 03	0.20462E 03
40	0.43404E 02	0.33198E 02	0.60844E 02	0.45589E 02	0.35377E 02	0.71274E 02	0.53907E 02
45	0.20967E 02	0.17086E 02	0.26174E 02	0.21135E 02	0.17400E 02	0.28208E 02	0.22984E 02
50	0.12152E 02	0.10285E 02	0.14173E 02	0.11930E 02	0.10168E 02	0.14554E 02	0.12337E 02
55	0.79025E 01	0.68622E 01	0.88063E 01	0.76163E 01	0.66443E 01	0.87579E 01	0.76127E 01
60	0.55710E 01	0.49302E 01	0.60079E 01	0.52981E 01	0.47020E 01	0.58404E 01	0.51681E 01
65	0.41721E 01	0.37475E 01	0.43896E 01	0.39294E 01	0.35346E 01	0.41966E 01	0.37647E 01
70	0.32758E 01	0.29785E 01	0.33816E 01	0.30641E 01	0.27868E 01	0.31933E 01	0.28965E 01
75	0.26723E 01	0.24550E 01	0.27178E 01	0.24879E 01	0.22843E 01	0.25434E 01	0.23286E 01
80	0.22499E 01	0.20857E 01	0.22616E 01	0.20888E 01	0.19338E 01	0.21031E 01	0.19411E 01
85	0.19453E 01	0.18178E 01	0.19377E 01	0.18038E 01	0.16824E 01	0.17942E 01	0.16681E 01
90	0.17703E 01	0.16192E 01	0.17016E 01	0.15954E 01	0.14980E 01	0.15719E 01	0.14711E 01
95	0.15610E 01	0.14696E 01	0.15261E 01	0.14403E 01	0.13609E 01	0.14087E 01	0.13264E 01
100	0.14719E 01	0.13555E 01	0.13939E 01	0.13235E 01	0.12578E 01	0.12873E 01	0.12191E 01
105	0.13225E 01	0.12678E 01	0.12932E 01	0.12348E 01	0.11799E 01	0.11964E 01	0.11391E 01
110	0.12453E 01	0.12002E 01	0.12162E 01	0.11674E 01	0.11213E 01	0.11283E 01	0.10797E 01
115	0.11853E 01	0.11480E 01	0.11572E 01	0.11163E 01	0.10774E 01	0.10775E 01	0.10361E 01
120	0.11387E 01	0.11078E 01	0.11122E 01	0.10779E 01	0.10450E 01	0.10402E 01	0.10048E 01
125	0.11025E 01	0.10771E 01	0.10780E 01	0.10494E 01	0.10217E 01	0.10135E 01	0.98318E 00
130	0.10746E 01	0.10540E 01	0.10526E 01	0.10287E 01	0.10056E 01	0.99504E 00	0.96929E 00
135	0.10433E 01	0.10367E 01	0.10339E 01	0.10143E 01	0.99522E 00	0.98323E 00	0.96150E 00
140	0.10372E 01	0.10242E 01	0.10306E 01	0.10047E 01	0.98923E 00	0.97663E 00	0.95850E 00
145	0.10252E 01	0.10153E 01	0.10115E 01	0.99891E 00	0.98658E 00	0.97404E 00	0.95917E 00
150	0.10165E 01	0.10092E 01	0.10055E 01	0.99590E 00	0.98639E 00	0.97446E 00	0.96253E 00
155	0.10103E 01	0.10052E 01	0.10020E 01	0.99990E 00	0.98785E 00	0.97699E 00	0.96771E 00
160	0.10060E 01	0.10027E 01	0.10002E 01	0.99940E 00	0.99028E 00	0.98086E 00	0.97393E 00
165	0.10031E 01	0.10013E 01	0.99950E 00	0.99523E 00	0.98537E 00	0.98992E 00	0.98049E 00
170	0.10013E 01	0.10005E 01	0.99944E 00	0.99630E 00	0.98576E 00	0.99402E 00	0.98766E 00
175	0.10003E 01	0.10001E 01	0.99966E 00	0.99763E 00	0.99582E 00	0.99724E 00	0.99222E 00
180	0.	0.	0.99990E 00	0.99886E 00	0.99805E 00	0.99950E 00	0.99643E 00

K =	0.86000E 00	0.88000E 00	0.90000E 00	0.95000E 00	0.10000E 01
ψ_s					
15					
20	0.16666E 04	0.74435E 03	0.41917E 03	0.34351E 04	0.62383E 03
25	0.13852E 03	0.99814E 02	0.75214E 02	0.15813E 03	0.81730E 02
30	0.42133E 02	0.33785E 02	0.42290E 02	0.42290E 02	0.26848E 02
35	0.19061E 02	0.33785E 02	0.27653E 02	0.17935E 02	0.12469E 02
40	0.10576E 02	0.16041E 02	0.13667E 02	0.95596E 01	0.70070E 01
45	0.66701E 01	0.91557E 01	0.79928E 01	0.58671E 01	0.44570E 01
50	0.46004E 01	0.58852E 01	0.52249E 01	0.39699E 01	0.30573E 01
55	0.46004E 01	0.41167E 01	0.37014E 01	0.28883E 01	0.23021E 01
60	0.33926E 01	0.30700E 01	0.27884E 01	0.22240E 01	0.18047E 01
65	0.26467E 01	0.24081E 01	0.22060E 01	0.17927E 01	0.14777E 01
70	0.21881E 01	0.19684E 01	0.18166E 01	0.15008E 01	0.12548E 01
75	0.17959E 01	0.16651E 01	0.15470E 01	0.12973E 01	0.10989E 01
80	0.15639E 01	0.14500E 01	0.13553E 01	0.11525E 01	0.98823E 00
85	0.13789E 01	0.12944E 01	0.12167E 01	0.10482E 01	0.90936E 00
90	0.12505E 01	0.11804E 01	0.11155E 01	0.97288E 00	0.85355E 00
95	0.11566E 01	0.10966E 01	0.10415E 01	0.91909E 00	0.81512E 00
100	0.10854E 01	0.10351E 01	0.98782E 00	0.88167E 00	0.79016E 00
105	0.10339E 01	0.99062E 00	0.94976E 00	0.85701E 00	0.77592E 00
110	0.99476E 00	0.95540E 00	0.92390E 00	0.84251E 00	0.77038E 00
115	0.97095E 00	0.93863E 00	0.90774E 00	0.83623E 00	0.77203E 00
120	0.95412E 00	0.92619E 00	0.89935E 00	0.83665E 00	0.77964E 00
125	0.94442E 00	0.92041E 00	0.89721E 00	0.84253E 00	0.79220E 00
130	0.94040E 00	0.91992E 00	0.90004E 00	0.85279E 00	0.80878E 00
135	0.94082E 00	0.92357E 00	0.90675E 00	0.86645E 00	0.82851E 00
140	0.94460E 00	0.93033E 00	0.91634E 00	0.88259E 00	0.85049E 00
145	0.95080E 00	0.93926E 00	0.92791E 00	0.90033E 00	0.87383E 00
150	0.95856E 00	0.94951E 00	0.94059E 00	0.91876E 00	0.89760E 00
155	0.96707E 00	0.96028E 00	0.95355E 00	0.93700E 00	0.92084E 00
160	0.97564E 00	0.97082E 00	0.96604E 00	0.95422E 00	0.94259E 00
165	0.98461E 00	0.98047E 00	0.97734E 00	0.96959E 00	0.96192E 00
170	0.99422E 00	0.98863E 00	0.98684E 00	0.98239E 00	0.97797E 00
175	0.99863E 00	0.99482E 00	0.99402E 00	0.99201E 00	0.99001E 00
180	0.99889E 00	0.99869E 00	0.99848E 00	0.99798E 00	0.99747E 00

Table A-II.6

Dispersion Function $\frac{\partial U(K, \psi)}{\partial K}$

$\frac{\partial U(K, \psi)}{\partial K}$								
K=	1.00000E-01	1.15000E-01	1.30000E-01	1.45000E-01	1.60000E-01	1.75000E-01	1.90000E-01	2.00000E-01
ψ								
0	0.15767E 01	0.15841E 01	0.15944E 01	0.16078E 01	0.16241E 01	0.16436E 01	0.16661E 01	0.16917E 01
2	0.15451E 01	0.15540E 01	0.15659E 01	0.15806E 01	0.15983E 01	0.16190E 01	0.16427E 01	0.16694E 01
4	0.15136E 01	0.15241E 01	0.15373E 01	0.15535E 01	0.15725E 01	0.15944E 01	0.16194E 01	0.16476E 01
6	0.14823E 01	0.14942E 01	0.15090E 01	0.15265E 01	0.15468E 01	0.15700E 01	0.15961E 01	0.16253E 01
8	0.14511E 01	0.14646E 01	0.14808E 01	0.14996E 01	0.15213E 01	0.15457E 01	0.15730E 01	0.16033E 01
10	0.14203E 01	0.14353E 01	0.14528E 01	0.14730E 01	0.14960E 01	0.15216E 01	0.15501E 01	0.15815E 01
12	0.13898E 01	0.14062E 01	0.14252E 01	0.14467E 01	0.14709E 01	0.14978E 01	0.15273E 01	0.15594E 01
14	0.13597E 01	0.13776E 01	0.13979E 01	0.14208E 01	0.14462E 01	0.14742E 01	0.15049E 01	0.15382E 01
16	0.13302E 01	0.13494E 01	0.13711E 01	0.13952E 01	0.14218E 01	0.14510E 01	0.14829E 01	0.15176E 01
18	0.13012E 01	0.13218E 01	0.13447E 01	0.13700E 01	0.13978E 01	0.14281E 01	0.14610E 01	0.14966E 01
20	0.12728E 01	0.12947E 01	0.13189E 01	0.13454E 01	0.13743E 01	0.14057E 01	0.14396E 01	0.14760E 01
22	0.12451E 01	0.12682E 01	0.12936E 01	0.13213E 01	0.13513E 01	0.13838E 01	0.14187E 01	0.14560E 01
24	0.12181E 01	0.12425E 01	0.12690E 01	0.12978E 01	0.13289E 01	0.13623E 01	0.13982E 01	0.14366E 01
26	0.11919E 01	0.12175E 01	0.12451E 01	0.12750E 01	0.13071E 01	0.13415E 01	0.13783E 01	0.14176E 01
28	0.11666E 01	0.11932E 01	0.12219E 01	0.12528E 01	0.12859E 01	0.13212E 01	0.13589E 01	0.13991E 01
30	0.11421E 01	0.11698E 01	0.11995E 01	0.12313E 01	0.12653E 01	0.13015E 01	0.13400E 01	0.13809E 01
32	0.11185E 01	0.11472E 01	0.11779E 01	0.12106E 01	0.12455E 01	0.12825E 01	0.13218E 01	0.13635E 01
34	0.10960E 01	0.11256E 01	0.11572E 01	0.11907E 01	0.12264E 01	0.12642E 01	0.13043E 01	0.13466E 01
36	0.10744E 01	0.11049E 01	0.11373E 01	0.11717E 01	0.12081E 01	0.12466E 01	0.12872E 01	0.13300E 01
38	0.10538E 01	0.10851E 01	0.11183E 01	0.11534E 01	0.11906E 01	0.12298E 01	0.12712E 01	0.13148E 01
40	0.10343E 01	0.10664E 01	0.11003E 01	0.11361E 01	0.11739E 01	0.12137E 01	0.12558E 01	0.12999E 01
42	0.10159E 01	0.10486E 01	0.10832E 01	0.11196E 01	0.11580E 01	0.11985E 01	0.12411E 01	0.12858E 01
44	0.99855E 00	0.10191E 01	0.10670E 01	0.11040E 01	0.11430E 01	0.11840E 01	0.12271E 01	0.12723E 01
46	0.99228E 00	0.10162E 01	0.10519E 01	0.10894E 01	0.11289E 01	0.11704E 01	0.12139E 01	0.12594E 01
48	0.98712E 00	0.10015E 01	0.10377E 01	0.10757E 01	0.11157E 01	0.11576E 01	0.12016E 01	0.12476E 01
50	0.98305E 00	0.98792E 00	0.10245E 01	0.10630E 01	0.11033E 01	0.11457E 01	0.11901E 01	0.12366E 01
52	0.94007E 00	0.97534E 00	0.10123E 01	0.10512E 01	0.10919E 01	0.11346E 01	0.11793E 01	0.12264E 01
54	0.92816E 00	0.96379E 00	0.10011E 01	0.10403E 01	0.10813E 01	0.11243E 01	0.11694E 01	0.12164E 01
56	0.91732E 00	0.95325E 00	0.99990E 00	0.10303E 01	0.10717E 01	0.11149E 01	0.11603E 01	0.12064E 01
58	0.90751E 00	0.94371E 00	0.99161E 00	0.10213E 01	0.10629E 01	0.11064E 01	0.11519E 01	0.11964E 01
60	0.89871E 00	0.93513E 00	0.97326E 00	0.10122E 01	0.10549E 01	0.10976E 01	0.11444E 01	0.11894E 01
62	0.89090E 00	0.92750E 00	0.96581E 00	0.10030E 01	0.10478E 01	0.10917E 01	0.11374E 01	0.11824E 01
64	0.88402E 00	0.92078E 00	0.95924E 00	0.99947E 00	0.10416E 01	0.10856E 01	0.11316E 01	0.11764E 01
66	0.87804E 00	0.91493E 00	0.95351E 00	0.99366E 00	0.10361E 01	0.10802E 01	0.11264E 01	0.11704E 01
68	0.87291E 00	0.90990E 00	0.94850E 00	0.98909E 00	0.10313E 01	0.10754E 01	0.11218E 01	0.11654E 01
70	0.86859E 00	0.90565E 00	0.94440E 00	0.98492E 00	0.10273E 01	0.10714E 01	0.11179E 01	0.11614E 01
72	0.86500E 00	0.90212E 00	0.94097E 00	0.98161E 00	0.10239E 01	0.10680E 01	0.11147E 01	0.11584E 01
74	0.86210E 00	0.89926E 00	0.93811E 00	0.97873E 00	0.10212E 01	0.10656E 01	0.11120E 01	0.11554E 01
76	0.85982E 00	0.89701E 00	0.93589E 00	0.97654E 00	0.10190E 01	0.10635E 01	0.11099E 01	0.11524E 01
78	0.85809E 00	0.89530E 00	0.93420E 00	0.97497E 00	0.10174E 01	0.10618E 01	0.11083E 01	0.11504E 01
80	0.85684E 00	0.89406E 00	0.93297E 00	0.97354E 00	0.10162E 01	0.10606E 01	0.11071E 01	0.11494E 01
82	0.85600E 00	0.89323E 00	0.93214E 00	0.97229E 00	0.10150E 01	0.10590E 01	0.11063E 01	0.11484E 01
84	0.85548E 00	0.89271E 00	0.93163E 00	0.97132E 00	0.10149E 01	0.10593E 01	0.11058E 01	0.11479E 01
86	0.85521E 00	0.89245E 00	0.93137E 00	0.97066E 00	0.10145E 01	0.10591E 01	0.11056E 01	0.11475E 01
88	0.85512E 00	0.89235E 00	0.93127E 00	0.97196E 00	0.10145E 01	0.10590E 01	0.11055E 01	0.11474E 01
90	0.85510E 00	0.89234E 00	0.93126E 00	0.97195E 00	0.10145E 01	0.10590E 01	0.11054E 01	0.11473E 01
92	0.85509E 00	0.89232E 00	0.93124E 00	0.97193E 00	0.10145E 01	0.10590E 01	0.11054E 01	0.11473E 01
94	0.85499E 00	0.89222E 00	0.93114E 00	0.97183E 00	0.10144E 01	0.10588E 01	0.11053E 01	0.11472E 01
96	0.85472E 00	0.89195E 00	0.93087E 00	0.97156E 00	0.10141E 01	0.10586E 01	0.11051E 01	0.11469E 01
98	0.85419E 00	0.89142E 00	0.93033E 00	0.97102E 00	0.10135E 01	0.10580E 01	0.11045E 01	0.11464E 01
100	0.85332E 00	0.89054E 00	0.92945E 00	0.97013E 00	0.10126E 01	0.10571E 01	0.11036E 01	0.11456E 01
102	0.85202E 00	0.88923E 00	0.92813E 00	0.96879E 00	0.10113E 01	0.10557E 01	0.11022E 01	0.11442E 01
104	0.85021E 00	0.88740E 00	0.92627E 00	0.96672E 00	0.10094E 01	0.10538E 01	0.11003E 01	0.11423E 01
106	0.84780E 00	0.88496E 00	0.92381E 00	0.96442E 00	0.10069E 01	0.10513E 01	0.10977E 01	0.11398E 01
108	0.84473E 00	0.88184E 00	0.92064E 00	0.96120E 00	0.10036E 01	0.10480E 01	0.10943E 01	0.11364E 01
110	0.84090E 00	0.87795E 00	0.91669E 00	0.95719E 00	0.09995E 01	0.10448E 01	0.10901E 01	0.11321E 01
112	0.83625E 00	0.87322E 00	0.91187E 00	0.95228E 00	0.99453E 00	0.10397E 01	0.10849E 01	0.11269E 01
114	0.83071E 00	0.86757E 00	0.90610E 00	0.94640E 00	0.98854E 00	0.10326E 01	0.10787E 01	0.11207E 01
116	0.82420E 00	0.86092E 00	0.89932E 00	0.93947E 00	0.98146E 00	0.10254E 01	0.10713E 01	0.11134E 01
118	0.81667E 00	0.85322E 00	0.89144E 00	0.93141E 00	0.97321E 00	0.10169E 01	0.10627E 01	0.11050E 01
120	0.80806E 00	0.84439E 00	0.88239E 00	0.92214E 00	0.96372E 00	0.10074E 01	0.10527E 01	0.10954E 01
122	0.79831E 00	0.83439E 00	0.87213E 00	0.91161E 00	0.95291E 00	0.99612E 00	0.10413E 01	0.10843E 01
124	0.78737E 00	0.82315E 00	0.86058E 00	0.89974E 00	0.94071E 00	0.98359E 00	0.10284E 01	0.10714E 01
126	0.77521E 00	0.81063E 00	0.84770E 00	0.88644E 00	0.92707E 00	0.96955E 00	0.10140E 01	0.10570E 01
128	0.76178E 00	0.79679E 00	0.83343E 00	0.87179E 00	0.91193E 00	0.95394E 00	0.99791E 00	0.10427E 01
130	0.74706E 00	0.78160E 00	0.81775E 00	0.85560E 00	0.89523E 00	0.93671E 00	0.98013E 00	0.10284E 01
132	0.73101E 00	0.76502E 00	0.80062E 00	0.83790E 00	0.87893E 00	0.91940E 00	0.96599E 00	0.10140E 01
134	0.71363E 00	0.74703E 00	0.78201E 00	0.81845E 00	0.85701E 00	0.89719E 00	0.94726E 00	0.10000E 01
136	0.69490E 00	0.72763E 00	0.76191E 00	0.79792E 00	0.83543E 00	0.87483E 00	0.91610E 00	0.99100E 00
138	0.67481E 00	0.70679E 00	0.74030E 00	0.77540E 00	0.81218E 00	0.85071E 00	0.89109E 00	0.97200E 00
140	0.65337E 00	0.68453E 00	0.71710E 00	0.75139E 00	0.78725E 00	0.82482E 00	0.86420E 00	0.95300E 00
142	0.63058E 00	0.66084E 00	0.69255E 00	0.72579E 00	0.76063E 00	0.79715E 00	0.83543E 00	0.87500E 00
144	0.60647E 00	0.63574E 00	0.66643E 00	0.69860E 00	0.73234E 00	0.76771E 00	0.80479E 00	0.84300E 00
146	0.58105E 00	0.60926E 00	0.63844E 00	0.66966E 00	0.70239E 00	0.73651E 00	0.77229E 00	0.80900E 00
148	0.55435E 00	0.58142E 00	0.60909E 00	0.63795E 00	0.67082E 00	0.70356E 00	0.73795E 00	0.77300E 00
150	0.52641E 00	0.55225E 00	0.57926E 00	0.60701E 00	0.63765E 00	0.66896E 00	0.70181E 00	0.73600E 00
152	0.49727E 00	0.52181E 00	0.54756E 00	0.57448E 00	0.60294E 00	0.63270E 00	0.66393E 00	0.69600E 00
154	0.46698E 00	0.49014E 00	0.51444E 00	0.53996E 00	0.56674E 00	0.59485E 00	0.62435E 00	0.65500E 00
156	0.43559E 00	0.45729E 00	0.48008E 00	0.50400E 00	0.52911E 00	0.55548E 00	0.58316E 00	0.61200E 00
158	0.40316E 00	0.42334E 00	0.44457E 00	0.46677E 00	0.49013E 00	0.51466E 00	0.54042E 00	0.56750E 00
160	0.36976E 00	0.38834E 00	0.40780E 00	0.42825E 00	0.44988E 00	0.47248E 00	0.49623E 00	0.52100E 00
162	0.33547E 00	0.35239E 00	0.37016E 00	0.38883E 00	0.40844E 00	0.42904E 00	0.45084E 00	0.47380E 00
164	0.30034E 00	0.31554E 00	0.33151E 00	0.34829E 00	0.36591E 00	0.38443E 00	0.40399E 00	0.42460E 00
166	0.26448E 00	0.27790E 00	0.29221E 00	0.30643E 00	0.32240E 00	0.33877E 00	0.35596E 00	0.37390E 00
168	0.22795E 00	0.23955E 00	0.25174E 00	0.26455E 00	0.27801E 00	0.29216E 00	0.30703E 00	0.32270E 00
170	0.19085E 00	0.20058E 00	0.21081E 00	0.22156E 00	0.23287E 00	0.24474E 00		

$$\frac{\partial U(K, \psi)}{\partial K}$$

ψ	0.45000E+00	0.50000E+00	0.51000E+00	0.52000E+00	0.53000E+00	0.54000E+00	0.55000E+00
0	0.16918E 01	0.17206E 01	0.17268E 01	0.17331E 01	0.17395E 01	0.17461E 01	0.17528E 01
2	0.16695E 01	0.16995E 01	0.17058E 01	0.17123E 01	0.17190E 01	0.17257E 01	0.17326E 01
4	0.16473E 01	0.16783E 01	0.16849E 01	0.16918E 01	0.16988E 01	0.17054E 01	0.17125E 01
6	0.16252E 01	0.16573E 01	0.16641E 01	0.16710E 01	0.16780E 01	0.16852E 01	0.16925E 01
8	0.16032E 01	0.16354E 01	0.16424E 01	0.16495E 01	0.16567E 01	0.16651E 01	0.16726E 01
10	0.15814E 01	0.16136E 01	0.16208E 01	0.16281E 01	0.16355E 01	0.16431E 01	0.16508E 01
12	0.15597E 01	0.15920E 01	0.16004E 01	0.16079E 01	0.16176E 01	0.16253E 01	0.16332E 01
14	0.15384E 01	0.15706E 01	0.15792E 01	0.15870E 01	0.15978E 01	0.16057E 01	0.16138E 01
16	0.15173E 01	0.15495E 01	0.15582E 01	0.15670E 01	0.15783E 01	0.15864E 01	0.15947E 01
18	0.14965E 01	0.15287E 01	0.15374E 01	0.15462E 01	0.15590E 01	0.15674E 01	0.15758E 01
20	0.14761E 01	0.15083E 01	0.15170E 01	0.15258E 01	0.15401E 01	0.15486E 01	0.15573E 01
22	0.14561E 01	0.14883E 01	0.14970E 01	0.15058E 01	0.15216E 01	0.15303E 01	0.15390E 01
24	0.14366E 01	0.14688E 01	0.14776E 01	0.14864E 01	0.15034E 01	0.15123E 01	0.15212E 01
26	0.14175E 01	0.14497E 01	0.14585E 01	0.14673E 01	0.14857E 01	0.14947E 01	0.15038E 01
28	0.13990E 01	0.14312E 01	0.14400E 01	0.14488E 01	0.14684E 01	0.14776E 01	0.14868E 01
30	0.13810E 01	0.14132E 01	0.14220E 01	0.14308E 01	0.14516E 01	0.14609E 01	0.14703E 01
32	0.13635E 01	0.13957E 01	0.14045E 01	0.14133E 01	0.14353E 01	0.14448E 01	0.14543E 01
34	0.13465E 01	0.13787E 01	0.13875E 01	0.13963E 01	0.14196E 01	0.14292E 01	0.14388E 01
36	0.13300E 01	0.13622E 01	0.13710E 01	0.13798E 01	0.14044E 01	0.14141E 01	0.14239E 01
38	0.13140E 01	0.13462E 01	0.13550E 01	0.13638E 01	0.13899E 01	0.13997E 01	0.14096E 01
40	0.13000E 01	0.13322E 01	0.13410E 01	0.13498E 01	0.13759E 01	0.13858E 01	0.13958E 01
42	0.12859E 01	0.13181E 01	0.13270E 01	0.13358E 01	0.13626E 01	0.13726E 01	0.13827E 01
44	0.12725E 01	0.13047E 01	0.13136E 01	0.13224E 01	0.13494E 01	0.13600E 01	0.13702E 01
46	0.12598E 01	0.12920E 01	0.13009E 01	0.13097E 01	0.13378E 01	0.13481E 01	0.13584E 01
48	0.12478E 01	0.12799E 01	0.12888E 01	0.12976E 01	0.13266E 01	0.13368E 01	0.13472E 01
50	0.12367E 01	0.12688E 01	0.12777E 01	0.12865E 01	0.13160E 01	0.13263E 01	0.13367E 01
52	0.12263E 01	0.12584E 01	0.12673E 01	0.12761E 01	0.13061E 01	0.13165E 01	0.13270E 01
54	0.12166E 01	0.12487E 01	0.12576E 01	0.12664E 01	0.12969E 01	0.13074E 01	0.13179E 01
56	0.12077E 01	0.12398E 01	0.12487E 01	0.12575E 01	0.12884E 01	0.12989E 01	0.13094E 01
58	0.11996E 01	0.12317E 01	0.12406E 01	0.12494E 01	0.12806E 01	0.12912E 01	0.13018E 01
60	0.11923E 01	0.12244E 01	0.12333E 01	0.12421E 01	0.12736E 01	0.12842E 01	0.12949E 01
62	0.11857E 01	0.12179E 01	0.12268E 01	0.12356E 01	0.12672E 01	0.12779E 01	0.12886E 01
64	0.11798E 01	0.12120E 01	0.12209E 01	0.12297E 01	0.12616E 01	0.12722E 01	0.12830E 01
66	0.11747E 01	0.12069E 01	0.12158E 01	0.12246E 01	0.12567E 01	0.12673E 01	0.12780E 01
68	0.11700E 01	0.12028E 01	0.12117E 01	0.12205E 01	0.12523E 01	0.12630E 01	0.12737E 01
70	0.11664E 01	0.12000E 01	0.12089E 01	0.12177E 01	0.12486E 01	0.12593E 01	0.12700E 01
72	0.11632E 01	0.11975E 01	0.12064E 01	0.12152E 01	0.12454E 01	0.12562E 01	0.12670E 01
74	0.11606E 01	0.11950E 01	0.12039E 01	0.12127E 01	0.12429E 01	0.12536E 01	0.12644E 01
76	0.11585E 01	0.11929E 01	0.12018E 01	0.12106E 01	0.12408E 01	0.12516E 01	0.12624E 01
78	0.11569E 01	0.11912E 01	0.12001E 01	0.12089E 01	0.12393E 01	0.12500E 01	0.12608E 01
80	0.11557E 01	0.11899E 01	0.11988E 01	0.12077E 01	0.12381E 01	0.12489E 01	0.12597E 01
82	0.11549E 01	0.11890E 01	0.11979E 01	0.12068E 01	0.12373E 01	0.12481E 01	0.12589E 01
84	0.11544E 01	0.11885E 01	0.11974E 01	0.12063E 01	0.12369E 01	0.12476E 01	0.12584E 01
86	0.11542E 01	0.11883E 01	0.11972E 01	0.12061E 01	0.12366E 01	0.12473E 01	0.12581E 01
88	0.11541E 01	0.11882E 01	0.11971E 01	0.12060E 01	0.12365E 01	0.12472E 01	0.12580E 01
90	0.11541E 01	0.11882E 01	0.11971E 01	0.12060E 01	0.12365E 01	0.12472E 01	0.12580E 01
92	0.11540E 01	0.11881E 01	0.11970E 01	0.12059E 01	0.12364E 01	0.12471E 01	0.12579E 01
94	0.11539E 01	0.11880E 01	0.11969E 01	0.12058E 01	0.12363E 01	0.12470E 01	0.12578E 01
96	0.11537E 01	0.11878E 01	0.11967E 01	0.12056E 01	0.12361E 01	0.12468E 01	0.12576E 01
98	0.11531E 01	0.11872E 01	0.11961E 01	0.12050E 01	0.12355E 01	0.12463E 01	0.12571E 01
100	0.11522E 01	0.11863E 01	0.11952E 01	0.12041E 01	0.12346E 01	0.12453E 01	0.12561E 01
102	0.11508E 01	0.11849E 01	0.11938E 01	0.12027E 01	0.12332E 01	0.12439E 01	0.12547E 01
104	0.11488E 01	0.11829E 01	0.11918E 01	0.12007E 01	0.12312E 01	0.12419E 01	0.12527E 01
106	0.11462E 01	0.11803E 01	0.11892E 01	0.11981E 01	0.12288E 01	0.12392E 01	0.12500E 01
108	0.11428E 01	0.11769E 01	0.11858E 01	0.11947E 01	0.12250E 01	0.12357E 01	0.12466E 01
110	0.11385E 01	0.11726E 01	0.11815E 01	0.11904E 01	0.12206E 01	0.12313E 01	0.12421E 01
112	0.11332E 01	0.11673E 01	0.11762E 01	0.11851E 01	0.12152E 01	0.12259E 01	0.12366E 01
114	0.11269E 01	0.11600E 01	0.11689E 01	0.11778E 01	0.12086E 01	0.12193E 01	0.12300E 01
116	0.11193E 01	0.11511E 01	0.11600E 01	0.11689E 01	0.12008E 01	0.12114E 01	0.12221E 01
118	0.11105E 01	0.11419E 01	0.11508E 01	0.11597E 01	0.11917E 01	0.12022E 01	0.12129E 01
120	0.11004E 01	0.11318E 01	0.11407E 01	0.11496E 01	0.11811E 01	0.11916E 01	0.12022E 01
122	0.10886E 01	0.11203E 01	0.11292E 01	0.11381E 01	0.11689E 01	0.11793E 01	0.11899E 01
124	0.10754E 01	0.11078E 01	0.11167E 01	0.11256E 01	0.11551E 01	0.11654E 01	0.11759E 01
126	0.10605E 01	0.10929E 01	0.11018E 01	0.11107E 01	0.11395E 01	0.11498E 01	0.11602E 01
128	0.10439E 01	0.10753E 01	0.10842E 01	0.10931E 01	0.11221E 01	0.11323E 01	0.11426E 01
130	0.10256E 01	0.10570E 01	0.10659E 01	0.10748E 01	0.11028E 01	0.11129E 01	0.11230E 01
132	0.10054E 01	0.10368E 01	0.10457E 01	0.10546E 01	0.10815E 01	0.10917E 01	0.11018E 01
134	0.98333E 00	0.10166E 01	0.10255E 01	0.10344E 01	0.10582E 01	0.10680E 01	0.10778E 01
136	0.95934E 00	0.10046E 01	0.10135E 01	0.10224E 01	0.10328E 01	0.10424E 01	0.10521E 01
138	0.93339E 00	0.97771E 00	0.98683E 00	0.99603E 00	0.10147E 01	0.10147E 01	0.10242E 01
140	0.90447E 00	0.94872E 00	0.95761E 00	0.96659E 00	0.97466E 00	0.98481E 00	0.99404E 00
142	0.87456E 00	0.91762E 00	0.92628E 00	0.93501E 00	0.94383E 00	0.95273E 00	0.96172E 00
144	0.84367E 00	0.88443E 00	0.89282E 00	0.90129E 00	0.91082E 00	0.91846E 00	0.92717E 00
146	0.80915E 00	0.84914E 00	0.85725E 00	0.86542E 00	0.87368E 00	0.88211E 00	0.89042E 00
148	0.77400E 00	0.81381E 00	0.82195E 00	0.83024E 00	0.83858E 00	0.84699E 00	0.85548E 00
150	0.73728E 00	0.77244E 00	0.77998E 00	0.78740E 00	0.79499E 00	0.80265E 00	0.81039E 00
152	0.69670E 00	0.73110E 00	0.73818E 00	0.74533E 00	0.75254E 00	0.75983E 00	0.76719E 00
154	0.65533E 00	0.68783E 00	0.69495E 00	0.70219E 00	0.70811E 00	0.71500E 00	0.72196E 00
156	0.61322E 00	0.64273E 00	0.64901E 00	0.65536E 00	0.66176E 00	0.66823E 00	0.67476E 00
158	0.56747E 00	0.59587E 00	0.60172E 00	0.60762E 00	0.61354E 00	0.61961E 00	0.62565E 00
160	0.52116E 00	0.54735E 00	0.55275E 00	0.55819E 00	0.56369E 00	0.56925E 00	0.57486E 00
162	0.47347E 00	0.49729E 00	0.50221E 00	0.50717E 00	0.51219E 00	0.51726E 00	0.52237E 00
164	0.42327E 00	0.44580E 00	0.45022E 00	0.45469E 00	0.45920E 00	0.46376E 00	0.46836E 00
166	0.37027E 00	0.39302E 00	0.39593E 00	0.40088E 00	0.40487E 00	0.40890E 00	0.41297E 00
168	0.32765E 00	0.33908E 00	0.34246E 00	0.34588E 00	0.34933E 00	0.35282E 00	0.35634E 00
170	0.27035E 00	0.28414E 00	0.28648E 00	0.28985E 00	0.29275E 00	0.29567E 00	0.29863E 00
172	0.21727E 00	0.22835E 00	0.23036E 00	0.23294E 00	0.23528E 00	0.23762E 00	0.24001E 00
174	0.16550E 00	0.17187E 00	0.17355E 00	0.17533E 00	0.17704E 00	0.17887E 00	0.18066E 00
176	0.11507E 00	0.11487E 00	0.11607E 00	0.11719E 00	0.11836E 00	0.11955E 00	0.12075E 00
178	0.54717E 01	0.57522E 01	0.58107E 01	0.58684E 01	0.59273E 01	0.59869E 01	0.60471E 01
180	0.	0.	0.	0.	0.	0.	0.

		$\partial U(K, \psi)$						
		∂K						
K=	ψ	0.56000E 00	0.57000E 00	0.58000E 00	0.59000E 00	0.60000E 00	0.62000E 00	0.64000E 00
0	0.17596E 01	0.17665E 01	0.17736E 01	0.17809E 01	0.17882E 01	0.18033E 01	0.18190E 01	0.18357E 01
2	0.17396E 01	0.17468E 01	0.17541E 01	0.17615E 01	0.17691E 01	0.17846E 01	0.18006E 01	0.18172E 01
4	0.17198E 01	0.17271E 01	0.17346E 01	0.17422E 01	0.17499E 01	0.17658E 01	0.17822E 01	0.17992E 01
6	0.16999E 01	0.17075E 01	0.17151E 01	0.17230E 01	0.17309E 01	0.17471E 01	0.17639E 01	0.17812E 01
8	0.16802E 01	0.16879E 01	0.16959E 01	0.17038E 01	0.17119E 01	0.17286E 01	0.17457E 01	0.17634E 01
10	0.16606E 01	0.16685E 01	0.16766E 01	0.16848E 01	0.16931E 01	0.17101E 01	0.17276E 01	0.17457E 01
12	0.16412E 01	0.16493E 01	0.16576E 01	0.16659E 01	0.16744E 01	0.16918E 01	0.17097E 01	0.17281E 01
14	0.16220E 01	0.16303E 01	0.16387E 01	0.16473E 01	0.16560E 01	0.16737E 01	0.16919E 01	0.17106E 01
16	0.16030E 01	0.16115E 01	0.16201E 01	0.16289E 01	0.16377E 01	0.16558E 01	0.16744E 01	0.16934E 01
18	0.15844E 01	0.15930E 01	0.16018E 01	0.16107E 01	0.16197E 01	0.16382E 01	0.16571E 01	0.16764E 01
20	0.15660E 01	0.15748E 01	0.15838E 01	0.15929E 01	0.16021E 01	0.16208E 01	0.16400E 01	0.16596E 01
22	0.15479E 01	0.15570E 01	0.15661E 01	0.15753E 01	0.15847E 01	0.16037E 01	0.16233E 01	0.16434E 01
24	0.15303E 01	0.15395E 01	0.15487E 01	0.15581E 01	0.15677E 01	0.15870E 01	0.16069E 01	0.16272E 01
26	0.15130E 01	0.15224E 01	0.15318E 01	0.15413E 01	0.15510E 01	0.15707E 01	0.15912E 01	0.16124E 01
28	0.14962E 01	0.15057E 01	0.15153E 01	0.15250E 01	0.15348E 01	0.15547E 01	0.15752E 01	0.15964E 01
30	0.14798E 01	0.14895E 01	0.14992E 01	0.15090E 01	0.15190E 01	0.15392E 01	0.15599E 01	0.15812E 01
32	0.14640E 01	0.14737E 01	0.14836E 01	0.14936E 01	0.15036E 01	0.15241E 01	0.15451E 01	0.15666E 01
34	0.14486E 01	0.14585E 01	0.14685E 01	0.14786E 01	0.14888E 01	0.15095E 01	0.15307E 01	0.15524E 01
36	0.14338E 01	0.14438E 01	0.14539E 01	0.14641E 01	0.14745E 01	0.14954E 01	0.15169E 01	0.15390E 01
38	0.14196E 01	0.14297E 01	0.14399E 01	0.14503E 01	0.14607E 01	0.14819E 01	0.15035E 01	0.15257E 01
40	0.14059E 01	0.14162E 01	0.14265E 01	0.14369E 01	0.14475E 01	0.14689E 01	0.14907E 01	0.15130E 01
42	0.13929E 01	0.14032E 01	0.14137E 01	0.14242E 01	0.14348E 01	0.14564E 01	0.14784E 01	0.15007E 01
44	0.13805E 01	0.13909E 01	0.14014E 01	0.14121E 01	0.14228E 01	0.14445E 01	0.14667E 01	0.14894E 01
46	0.13688E 01	0.13793E 01	0.13899E 01	0.14006E 01	0.14114E 01	0.14333E 01	0.14556E 01	0.14784E 01
48	0.13577E 01	0.13683E 01	0.13789E 01	0.13897E 01	0.14006E 01	0.14226E 01	0.14451E 01	0.14684E 01
50	0.13473E 01	0.13579E 01	0.13687E 01	0.13795E 01	0.13904E 01	0.14126E 01	0.14353E 01	0.14586E 01
52	0.13376E 01	0.13483E 01	0.13591E 01	0.13700E 01	0.13810E 01	0.14033E 01	0.14260E 01	0.14494E 01
54	0.13285E 01	0.13393E 01	0.13501E 01	0.13611E 01	0.13722E 01	0.13946E 01	0.14174E 01	0.14407E 01
56	0.13202E 01	0.13310E 01	0.13419E 01	0.13529E 01	0.13640E 01	0.13865E 01	0.14095E 01	0.14329E 01
58	0.13126E 01	0.13234E 01	0.13344E 01	0.13454E 01	0.13566E 01	0.13791E 01	0.14022E 01	0.14257E 01
60	0.13056E 01	0.13165E 01	0.13275E 01	0.13386E 01	0.13498E 01	0.13724E 01	0.13955E 01	0.14190E 01
62	0.12994E 01	0.13103E 01	0.13213E 01	0.13324E 01	0.13436E 01	0.13663E 01	0.13895E 01	0.14134E 01
64	0.12938E 01	0.13047E 01	0.13157E 01	0.13269E 01	0.13381E 01	0.13609E 01	0.13841E 01	0.14077E 01
66	0.12889E 01	0.12998E 01	0.13109E 01	0.13220E 01	0.13333E 01	0.13561E 01	0.13793E 01	0.14019E 01
68	0.12846E 01	0.12956E 01	0.13066E 01	0.13178E 01	0.13291E 01	0.13519E 01	0.13752E 01	0.13994E 01
70	0.12809E 01	0.12919E 01	0.13030E 01	0.13142E 01	0.13255E 01	0.13483E 01	0.13716E 01	0.13936E 01
72	0.12778E 01	0.12888E 01	0.12999E 01	0.13111E 01	0.13224E 01	0.13453E 01	0.13686E 01	0.13878E 01
74	0.12753E 01	0.12863E 01	0.12974E 01	0.13086E 01	0.13199E 01	0.13429E 01	0.13662E 01	0.13820E 01
76	0.12733E 01	0.12843E 01	0.12954E 01	0.13066E 01	0.13179E 01	0.13408E 01	0.13642E 01	0.13762E 01
78	0.12717E 01	0.12827E 01	0.12938E 01	0.13051E 01	0.13164E 01	0.13393E 01	0.13626E 01	0.13704E 01
80	0.12706E 01	0.12816E 01	0.12927E 01	0.13040E 01	0.13152E 01	0.13382E 01	0.13615E 01	0.13646E 01
82	0.12698E 01	0.12808E 01	0.12919E 01	0.13031E 01	0.13145E 01	0.13374E 01	0.13607E 01	0.13587E 01
84	0.12693E 01	0.12803E 01	0.12914E 01	0.13027E 01	0.13140E 01	0.13370E 01	0.13599E 01	0.13528E 01
86	0.12691E 01	0.12801E 01	0.12912E 01	0.13024E 01	0.13137E 01	0.13366E 01	0.13597E 01	0.13519E 01
88	0.12690E 01	0.12800E 01	0.12911E 01	0.13023E 01	0.13136E 01	0.13365E 01	0.13596E 01	0.13518E 01
90	0.12689E 01	0.12800E 01	0.12911E 01	0.13023E 01	0.13136E 01	0.13365E 01	0.13596E 01	0.13518E 01
92	0.12689E 01	0.12799E 01	0.12911E 01	0.13023E 01	0.13136E 01	0.13365E 01	0.13596E 01	0.13518E 01
94	0.12688E 01	0.12798E 01	0.12910E 01	0.13022E 01	0.13135E 01	0.13364E 01	0.13595E 01	0.13517E 01
96	0.12685E 01	0.12796E 01	0.12907E 01	0.13019E 01	0.13132E 01	0.13361E 01	0.13592E 01	0.13514E 01
98	0.12680E 01	0.12790E 01	0.12901E 01	0.13013E 01	0.13126E 01	0.13356E 01	0.13589E 01	0.13511E 01
100	0.12671E 01	0.12781E 01	0.12892E 01	0.13004E 01	0.13117E 01	0.13346E 01	0.13580E 01	0.13502E 01
102	0.12656E 01	0.12766E 01	0.12877E 01	0.12990E 01	0.13103E 01	0.13332E 01	0.13565E 01	0.13487E 01
104	0.12636E 01	0.12746E 01	0.12857E 01	0.12970E 01	0.13082E 01	0.13312E 01	0.13545E 01	0.13467E 01
106	0.12609E 01	0.12719E 01	0.12830E 01	0.12942E 01	0.13055E 01	0.13294E 01	0.13517E 01	0.13440E 01
108	0.12574E 01	0.12684E 01	0.12795E 01	0.12907E 01	0.13019E 01	0.13248E 01	0.13481E 01	0.13413E 01
110	0.12530E 01	0.12639E 01	0.12750E 01	0.12862E 01	0.12974E 01	0.13203E 01	0.13436E 01	0.13379E 01
112	0.12475E 01	0.12584E 01	0.12695E 01	0.12806E 01	0.12919E 01	0.13147E 01	0.13379E 01	0.13311E 01
114	0.12408E 01	0.12518E 01	0.12629E 01	0.12739E 01	0.12851E 01	0.13079E 01	0.13279E 01	0.13243E 01
116	0.12329E 01	0.12438E 01	0.12548E 01	0.12659E 01	0.12771E 01	0.12998E 01	0.13179E 01	0.13133E 01
118	0.12236E 01	0.12345E 01	0.12454E 01	0.12565E 01	0.12676E 01	0.12903E 01	0.13079E 01	0.13021E 01
120	0.12129E 01	0.12237E 01	0.12346E 01	0.12456E 01	0.12567E 01	0.12792E 01	0.12947E 01	0.12872E 01
122	0.12005E 01	0.12113E 01	0.12221E 01	0.12330E 01	0.12441E 01	0.12664E 01	0.12792E 01	0.12745E 01
124	0.11865E 01	0.11971E 01	0.12079E 01	0.12187E 01	0.12297E 01	0.12519E 01	0.12657E 01	0.12579E 01
126	0.11706E 01	0.11812E 01	0.11919E 01	0.12026E 01	0.12135E 01	0.12355E 01	0.12493E 01	0.12393E 01
128	0.11529E 01	0.11634E 01	0.11739E 01	0.11846E 01	0.11953E 01	0.12171E 01	0.12309E 01	0.12197E 01
130	0.11332E 01	0.11436E 01	0.11540E 01	0.11645E 01	0.11752E 01	0.11967E 01	0.12095E 01	0.11970E 01
132	0.11116E 01	0.11217E 01	0.11320E 01	0.11424E 01	0.11529E 01	0.11741E 01	0.11868E 01	0.11743E 01
134	0.10878E 01	0.10978E 01	0.11079E 01	0.11181E 01	0.11284E 01	0.11494E 01	0.11620E 01	0.11495E 01
136	0.10618E 01	0.10717E 01	0.10816E 01	0.10916E 01	0.11018E 01	0.11228E 01	0.11354E 01	0.11229E 01
138	0.10337E 01	0.10436E 01	0.10531E 01	0.10629E 01	0.10728E 01	0.10929E 01	0.11054E 01	0.10929E 01
140	0.10034E 01	0.10129E 01	0.10223E 01	0.10319E 01	0.10415E 01	0.10612E 01	0.10737E 01	0.10612E 01
142	0.97079E 00	0.97994E 00	0.98919E 00	0.99845E 00	0.10079E 00	0.10270E 00	0.10465E 00	0.10590E 00
144	0.93597E 00	0.94484E 00	0.95381E 00	0.96285E 00	0.97199E 00	0.99091E 00	0.10094E 00	0.10298E 00
146	0.89891E 00	0.90748E 00	0.91614E 00	0.92488E 00	0.93370E 00	0.95259E 00	0.96159E 00	0.97069E 00
148	0.85964E 00	0.86788E 00	0.87621E 00	0.88461E 00	0.89309E 00	0.91030E 00	0.92784E 00	0.94554E 00
150	0.81820E 00	0.82608E 00	0.83404E 00	0.84208E 00	0.85020E 00	0.86666E 00	0.88345E 00	0.90049E 00
152	0.77462E 00	0.78212E 00	0.78970E 00	0.79735E 00	0.80507E 00	0.82074E 00	0.83672E 00	0.85299E 00
154	0.72898E 00	0.73608E 00	0.74324E 00	0.75047E 00	0.75778E 00	0.77260E 00	0.78771E 00	0.80312E 00
156	0.68136E 00	0.68802E 00	0.69474E 00	0.70153E 00	0.70839E 00	0.72231E 00	0.73650E 00	0.75099E 00
158	0.63184E 00	0.63804E 00	0.64429E 00	0.65063E 00	0.65701E 00	0.66997E 00	0.68319E 00	0.69669E 00
160	0.58052E 00	0.58624E 00	0.59202E 00	0.59786E 00	0.60375E 00	0.61571E 00	0.62790E 00	0.64034E 00
162	0.52754E 00	0.53276E 00	0.53802E 00	0.54335E 00	0.54872E 00	0.55933E 00	0.57035E 00	0.58179E 00
164	0.47301E 00	0.47770E 00	0.48244E 00	0.48723E 00	0.49206E 00	0.50188E 00	0.51189E 00	0.52212E 00
166	0.41708E 00	0.42123E 00	0.42542E 00	0.42965E 00	0.43393E 00	0.44261E 00	0.45164E 00	0.46102E 00
168	0.35989E 00	0.36348E 00	0.36711E 00	0.37077E 00	0.37447E 00	0.38198E 00	0.38964E 00	0.39756E

$$\frac{\partial U(K, \psi)}{\partial K}$$

ψ	0.66000E 00	0.68000E 00	0.70000E 00	0.72000E 00	0.74000E 00	0.76000E 00	0.78000E 00
0	0.18352E 01	0.18520E 01	0.18694E 01	0.18873E 01	0.19058E 01	0.19249E 01	0.19446E 01
1	0.18172E 01	0.18344E 01	0.18521E 01	0.18703E 01	0.18892E 01	0.19086E 01	0.19286E 01
2	0.17992E 01	0.18167E 01	0.18348E 01	0.18534E 01	0.18725E 01	0.18923E 01	0.19126E 01
3	0.17813E 01	0.17991E 01	0.18175E 01	0.18365E 01	0.18560E 01	0.18760E 01	0.18967E 01
4	0.17634E 01	0.17812E 01	0.18003E 01	0.18196E 01	0.18395E 01	0.18599E 01	0.18808E 01
5	0.17455E 01	0.17642E 01	0.17833E 01	0.18029E 01	0.18231E 01	0.18438E 01	0.18651E 01
6	0.17276E 01	0.17470E 01	0.17666E 01	0.17863E 01	0.18068E 01	0.18278E 01	0.18494E 01
7	0.17097E 01	0.17299E 01	0.17496E 01	0.17695E 01	0.17907E 01	0.18120E 01	0.18339E 01
8	0.16918E 01	0.17130E 01	0.17331E 01	0.17537E 01	0.17748E 01	0.17964E 01	0.18186E 01
9	0.16739E 01	0.16963E 01	0.17176E 01	0.17376E 01	0.17590E 01	0.17810E 01	0.18035E 01
10	0.16560E 01	0.16795E 01	0.17006E 01	0.17218E 01	0.17435E 01	0.17658E 01	0.17885E 01
11	0.16381E 01	0.16628E 01	0.16848E 01	0.17063E 01	0.17283E 01	0.17508E 01	0.17739E 01
12	0.16202E 01	0.16459E 01	0.16693E 01	0.16910E 01	0.17133E 01	0.17361E 01	0.17594E 01
13	0.16023E 01	0.16292E 01	0.16541E 01	0.16761E 01	0.16987E 01	0.17217E 01	0.17453E 01
14	0.15844E 01	0.16124E 01	0.16392E 01	0.16611E 01	0.16843E 01	0.17076E 01	0.17315E 01
15	0.15665E 01	0.15956E 01	0.16247E 01	0.16547E 01	0.16794E 01	0.16993E 01	0.17188E 01
16	0.15486E 01	0.15788E 01	0.16079E 01	0.16380E 01	0.16681E 01	0.16939E 01	0.17170E 01
17	0.15307E 01	0.15620E 01	0.15911E 01	0.16202E 01	0.16503E 01	0.16761E 01	0.16921E 01
18	0.15128E 01	0.15451E 01	0.15742E 01	0.16033E 01	0.16334E 01	0.16592E 01	0.16797E 01
19	0.14949E 01	0.15282E 01	0.15573E 01	0.15864E 01	0.16165E 01	0.16423E 01	0.16678E 01
20	0.14770E 01	0.15113E 01	0.15404E 01	0.15695E 01	0.15996E 01	0.16254E 01	0.16535E 01
21	0.14591E 01	0.14944E 01	0.15235E 01	0.15526E 01	0.15827E 01	0.16085E 01	0.16346E 01
22	0.14412E 01	0.14775E 01	0.15066E 01	0.15357E 01	0.15658E 01	0.15916E 01	0.16187E 01
 

$$\frac{\partial U(K, \psi)}{\partial K}$$

K=	0.80000E 00	0.82000E 00	0.84000E 00	0.86000E 00	0.88000E 00	0.90000E 00	0.95000E 00
ψ							
0	0.12649E 01	0.19857E 01	0.20072E 01	0.20273E 01	0.20520E 01	0.20754E 01	0.21366E 01
2	0.19492E 01	0.19704E 01	0.19922E 01	0.20146E 01	0.20376E 01	0.20612E 01	0.21231E 01
4	0.19335E 01	0.19550E 01	0.19771E 01	0.19998E 01	0.20231E 01	0.20470E 01	0.21096E 01
6	0.19179E 01	0.19397E 01	0.19621E 01	0.19851E 01	0.20087E 01	0.20329E 01	0.20962E 01
8	0.19024E 01	0.19245E 01	0.19472E 01	0.19705E 01	0.19943E 01	0.20186E 01	0.20827E 01
10	0.18869E 01	0.19093E 01	0.19323E 01	0.19559E 01	0.19791E 01	0.20028E 01	0.20695E 01
12	0.18716E 01	0.18943E 01	0.19176E 01	0.19414E 01	0.19659E 01	0.19909E 01	0.20562E 01
14	0.18564E 01	0.18794E 01	0.19030E 01	0.19271E 01	0.19518E 01	0.19772E 01	0.20431E 01
16	0.18413E 01	0.18646E 01	0.18885E 01	0.19129E 01	0.19379E 01	0.19635E 01	0.20301E 01
18	0.18265E 01	0.18501E 01	0.18742E 01	0.18989E 01	0.19242E 01	0.19500E 01	0.20172E 01
20	0.18118E 01	0.18357E 01	0.18601E 01	0.18850E 01	0.19106E 01	0.19367E 01	0.20045E 01
22	0.17974E 01	0.18215E 01	0.18462E 01	0.18714E 01	0.18972E 01	0.19235E 01	0.19922E 01
24	0.17833E 01	0.18076E 01	0.18325E 01	0.18580E 01	0.18840E 01	0.19106E 01	0.19797E 01
26	0.17694E 01	0.17940E 01	0.18192E 01	0.18449E 01	0.18711E 01	0.18979E 01	0.19675E 01
28	0.17558E 01	0.17807E 01	0.18060E 01	0.18320E 01	0.18585E 01	0.18855E 01	0.19557E 01
30	0.17425E 01	0.17676E 01	0.17932E 01	0.18194E 01	0.18461E 01	0.18734E 01	0.19440E 01
32	0.17296E 01	0.17549E 01	0.17809E 01	0.18071E 01	0.18340E 01	0.18615E 01	0.19327E 01
34	0.17171E 01	0.17426E 01	0.17686E 01	0.17952E 01	0.18223E 01	0.18500E 01	0.19216E 01
36	0.17049E 01	0.17304E 01	0.17569E 01	0.17836E 01	0.18107E 01	0.18385E 01	0.19107E 01
38	0.16932E 01	0.17187E 01	0.17455E 01	0.17724E 01	0.17997E 01	0.18272E 01	0.19004E 01
40	0.16818E 01	0.17072E 01	0.17340E 01	0.17611E 01	0.17883E 01	0.18157E 01	0.18903E 01
42	0.16710E 01	0.16972E 01	0.17240E 01	0.17512E 01	0.17786E 01	0.18062E 01	0.18805E 01
44	0.16606E 01	0.16870E 01	0.17138E 01	0.17413E 01	0.17689E 01	0.17967E 01	0.18713E 01
46	0.16506E 01	0.16772E 01	0.17040E 01	0.17319E 01	0.17597E 01	0.17876E 01	0.18624E 01
48	0.16412E 01	0.16679E 01	0.16945E 01	0.17219E 01	0.17502E 01	0.17787E 01	0.18537E 01
50	0.16324E 01	0.16591E 01	0.16856E 01	0.17142E 01	0.17425E 01	0.17713E 01	0.18452E 01
52	0.16240E 01	0.16507E 01	0.16773E 01	0.17061E 01	0.17346E 01	0.17635E 01	0.18369E 01
54	0.16162E 01	0.16429E 01	0.16695E 01	0.16978E 01	0.17271E 01	0.17561E 01	0.18288E 01
56	0.16089E 01	0.16356E 01	0.16623E 01	0.16916E 01	0.17202E 01	0.17483E 01	0.18205E 01
58	0.16022E 01	0.16289E 01	0.16557E 01	0.16851E 01	0.17138E 01	0.17420E 01	0.18123E 01
60	0.15961E 01	0.16233E 01	0.16501E 01	0.16789E 01	0.17079E 01	0.17361E 01	0.18042E 01
62	0.15905E 01	0.16178E 01	0.16445E 01	0.16730E 01	0.17025E 01	0.17310E 01	0.17962E 01
64	0.15855E 01	0.16129E 01	0.16396E 01	0.16679E 01	0.16977E 01	0.17270E 01	0.17883E 01
66	0.15811E 01	0.16084E 01	0.16352E 01	0.16636E 01	0.16934E 01	0.17232E 01	0.17805E 01
68	0.15772E 01	0.16045E 01	0.16314E 01	0.16598E 01	0.16896E 01	0.17196E 01	0.17729E 01
70	0.15734E 01	0.16012E 01	0.16281E 01	0.16565E 01	0.16864E 01	0.17161E 01	0.17655E 01
72	0.15710E 01	0.15984E 01	0.16258E 01	0.16542E 01	0.16843E 01	0.17137E 01	0.17582E 01
74	0.15686E 01	0.15960E 01	0.16236E 01	0.16523E 01	0.16823E 01	0.17117E 01	0.17510E 01
76	0.15667E 01	0.15941E 01	0.16220E 01	0.16505E 01	0.16804E 01	0.17098E 01	0.17439E 01
78	0.15652E 01	0.15926E 01	0.16206E 01	0.16489E 01	0.16789E 01	0.17079E 01	0.17369E 01
80	0.15641E 01	0.15916E 01	0.16195E 01	0.16479E 01	0.16779E 01	0.17069E 01	0.17299E 01
82	0.15634E 01	0.15909E 01	0.16187E 01	0.16472E 01	0.16772E 01	0.17062E 01	0.17279E 01
84	0.15629E 01	0.15902E 01	0.16180E 01	0.16467E 01	0.16767E 01	0.17057E 01	0.17259E 01
86	0.15624E 01	0.15900E 01	0.16177E 01	0.16464E 01	0.16765E 01	0.17054E 01	0.17239E 01
88	0.15626E 01	0.15900E 01	0.16177E 01	0.16464E 01	0.16765E 01	0.17054E 01	0.17239E 01
90	0.15627E 01	0.15900E 01	0.16177E 01	0.16464E 01	0.16765E 01	0.17054E 01	0.17239E 01
92	0.15628E 01	0.15900E 01	0.16177E 01	0.16464E 01	0.16765E 01	0.17054E 01	0.17239E 01
94	0.15629E 01	0.15900E 01	0.16177E 01	0.16464E 01	0.16765E 01	0.17054E 01	0.17239E 01
96	0.15629E 01	0.15900E 01	0.16177E 01	0.16464E 01	0.16765E 01	0.17054E 01	0.17239E 01
98	0.15629E 01	0.15900E 01	0.16177E 01	0.16464E 01	0.16765E 01	0.17054E 01	0.17239E 01
100	0.15629E 01	0.15900E 01	0.16177E 01	0.16464E 01	0.16765E 01	0.17054E 01	0.17239E 01
102	0.15629E 01	0.15900E 01	0.16177E 01	0.16464E 01	0.16765E 01	0.17054E 01	0.17239E 01
104	0.15629E 01	0.15900E 01	0.16177E 01	0.16464E 01	0.16765E 01	0.17054E 01	0.17239E 01
106	0.15629E 01	0.15900E 01	0.16177E 01	0.16464E 01	0.16765E 01	0.17054E 01	0.17239E 01
108	0.15629E 01	0.15900E 01	0.16177E 01	0.16464E 01	0.16765E 01	0.17054E 01	0.17239E 01
110	0.15629E 01	0.15900E 01	0.16177E 01	0.16464E 01	0.16765E 01	0.17054E 01	0.17239E 01
112	0.15629E 01	0.15900E 01	0.16177E 01	0.16464E 01	0.16765E 01	0.17054E 01	0.17239E 01
114	0.15629E 01	0.15900E 01	0.16177E 01	0.16464E 01	0.16765E 01	0.17054E 01	0.17239E 01
116	0.15629E 01	0.15900E 01	0.16177E 01	0.16464E 01	0.16765E 01	0.17054E 01	0.17239E 01
118	0.15629E 01	0.15900E 01	0.16177E 01	0.16464E 01	0.16765E 01	0.17054E 01	0.17239E 01
120	0.15629E 01	0.15900E 01	0.16177E 01	0.16464E 01	0.16765E 01	0.17054E 01	0.17239E 01
122	0.15629E 01	0.15900E 01	0.16177E 01	0.16464E 01	0.16765E 01	0.17054E 01	0.17239E 01
124	0.15629E 01	0.15900E 01	0.16177E 01	0.16464E 01	0.16765E 01	0.17054E 01	0.17239E 01
126	0.15629E 01	0.15900E 01	0.16177E 01	0.16464E 01	0.16765E 01	0.17054E 01	0.17239E 01
128	0.15629E 01	0.15900E 01	0.16177E 01	0.16464E 01	0.16765E 01	0.17054E 01	0.17239E 01
130	0.15629E 01	0.15900E 01	0.16177E 01	0.16464E 01	0.16765E 01	0.17054E 01	0.17239E 01
132	0.15629E 01	0.15900E 01	0.16177E 01	0.16464E 01	0.16765E 01	0.17054E 01	0.17239E 01
 

$$\frac{\partial U(K, \psi)}{\partial K}$$

```

K- 0.10000E 01
ψ
0 0.22020E 01
2 0.21891E 01
4 0.21763F 01
6 0.21635F 01
8 0.21508F 01
10 0.21381E 01
12 0.21255E 01
14 0.21130F 01
16 0.21006F 01
18 0.20883F 01
20 0.20762F 01
22 0.20642F 01
24 0.20524F 01
26 0.20409F 01
28 0.20295F 01
30 0.20184F 01
32 0.20075F 01
34 0.19969E 01
36 0.19865E 01
38 0.19765E 01
40 0.19668E 01
42 0.19575F 01
44 0.19485F 01
46 0.19399F 01
48 0.19317F 01
50 0.19239F 01
52 0.19165F 01
54 0.19096F 01
56 0.19031E 01
58 0.18971F 01
60 0.18916E 01
62 0.18865E 01
64 0.18820F 01
66 0.18779F 01
68 0.18743F 01
70 0.18711F 01
72 0.18685F 01
74 0.18662F 01
76 0.18644F 01
78 0.18630F 01
80 0.18624F 01
82 0.18612F 01
84 0.18608F 01
86 0.18605F 01
88 0.18604F 01
90 0.18604E 01
92 0.18604F 01
94 0.18603F 01
96 0.18603F 01
98 0.18594F 01
100 0.18584F 01
102 0.18574F 01
104 0.18564F 01
106 0.18554F 01
108 0.18544F 01
110 0.18534F 01
112 0.18524F 01
114 0.18514F 01
116 0.18504F 01
118 0.18494F 01
120 0.18484F 01
122 0.18474F 01
124 0.18464F 01
126 0.18454F 01
128 0.18444F 01
130 0.18434F 01
132 0.18424F 01
134 0.18414F 01
136 0.18404F 01
138 0.18394F 01
140 0.18384F 01
142 0.18374F 01
144 0.18364F 01
146 0.18354F 01
148 0.18344F 01
150 0.18334F 01
152 0.18324F 01
154 0.18314F 01
156 0.18304F 01
158 0.18294F 01
160 0.18284F 01
162 0.18274F 01
164 0.18264F 01
166 0.18254F 01
168 0.18244F 01
170 0.18234F 01
172 0.18224F 01
174 0.18214F 01
176 0.18204F 01
178 0.18194F 01
180 0.18184F 01
182 0.18174F 01
184 0.18164F 01
186 0.18154F 01
188 0.18144F 01
190 0.18134F 01
192 0.18124F 01
194 0.18114F 01
196 0.18104F 01
198 0.18094F 01
200 0.18084F 01

```

APPENDIX III

EVALUATION OF ψ_f'

From Eq. (3.2a) we have

$$r_0 e^{K \cos \psi_f} = r_e e^{K \cos \psi_s} \quad (\text{A-III.1})$$

Differentiating with respect to ψ_s , keeping z_s, z_f, K, n_s , and n_f constant, we have

$$r_0 e^{K \cos \psi_f} \psi_f' \left[-K \sin \psi_f + \frac{1}{r_0} \frac{dr_0}{d\psi_f} \right] = r_e e^{K \cos \psi_s} \left[-K \sin \psi_s + \frac{1}{r_e} \frac{dr_e}{d\psi_s} \right] \quad (\text{A-III.2})$$

Equations (5.24) and (5.25) give

$$\psi_f' = \frac{-K \sin \psi_s + \frac{1}{r_e} \frac{dr_e}{d\psi_s}}{-K \sin \psi_f + \frac{1}{r_0} \frac{dr_0}{d\psi_f}} = \frac{\Lambda_s - K \sin \psi_s}{\Lambda_f - K \sin \psi_f} \quad (\text{A-III.3a})$$

with

$$\frac{dr_e}{d\psi_s} = z_s \frac{df_s(\psi_s)}{d\psi_s} = \Lambda_s z_s f_s(\psi_s) = \Lambda_s r_e \quad (\text{A-III.3b})$$

and

$$\frac{dr_0}{d\psi_f} = z_f \frac{df_f(\psi_f)}{d\psi_f} = \Lambda_f z_f f_f(\psi_f) = \Lambda_f r_0 \quad (\text{A-III.3c})$$

We take the ρ -value at the profile [distance from $(z_s, 0)$ to the profile] to be ρ_s , while the $(z_f, 0)$ -to-profile distance to ρ_f . We have, then,

$$r_e = \rho_s \sin \psi_s \quad ; \quad r_0 = -\rho_f \sin \psi_f \quad (\text{A-III.4})$$

Then,

$$\begin{aligned} \Lambda_s &= \frac{1}{r_e} \frac{dr_e}{d\psi_s} = \frac{1}{\rho_s \sin \psi_s} \left[\rho_s \cos \psi_s + \sin \psi_s \frac{d\rho_s}{d\psi_s} \right] \\ &= \cot \psi_s + \frac{1}{\rho_s} \frac{d\rho_s}{d\psi_s} \end{aligned} \quad (\text{A-III.5a})$$

$$\begin{aligned}\Lambda_f &= \frac{1}{r_0} \frac{dr_0}{d\psi_f} = \frac{1}{-\rho_f \sin\psi_f} \left[-\rho_f \cos\psi_f - \sin\psi_f \frac{d\rho_f}{d\psi_f} \right] \\ &= \cot\psi_f + \frac{1}{\rho_f} \frac{d\rho_f}{d\psi_f}.\end{aligned}\quad (\text{A-III.5b})$$

Hence,

$$\psi_f' = \frac{-K \sin\psi_s + \cot\psi_s + \frac{1}{\rho_s} \frac{d\rho_s}{d\psi_s}}{-K \sin\psi_f + \cot\psi_f + \frac{1}{\rho_f} \frac{d\rho_f}{d\psi_f}} \quad (\text{A-III.6})$$

The form in Eq. (A-III.6) has been introduced because the terms $\frac{1}{\rho} \frac{d\rho}{d\psi}$ are important in considerations of transmission problems (see Section #9), and hence require computation for other reasons than for the determination of ψ_f' .

For a symmetrical instrument, it may be noted, that, since

$$\sin\psi_f = -\sin\psi_s; \cot\psi_f = -\cot\psi_s; \frac{d\rho_f}{d\psi_f} = -\frac{d\rho_s}{d\psi_s} \left(\text{or } \frac{dr_0}{d\psi_f} = -\frac{dr_e}{d\psi_s} \right),$$

then $\psi_f' = -1$, which is also evident from the fact that $\psi_s + \psi_f = 2\pi$.

Also ψ_f' may be computed if ψ_f has been evaluated, but not the exit profile [i.e., $f_f(\psi_f)$] or its derivative. Differentiating Eq. (3.3) with respect to ψ_s , we have, since

$$\frac{dU}{d\psi_s} = \cos\psi_s e^{-K \cos\psi_s},$$

that

$$\left[\frac{-r_0}{\sin^2\psi_f} \psi_f' \right] \frac{e^{-K \cos\psi_f}}{r_0} = \frac{e^{-K \cos\psi_s}}{r_e} \left[\frac{-r_e}{\sin^2\psi_s} + (z_s - z_f) \left\{ K \sin\psi_s - \frac{1}{r_e} \frac{dr_e}{d\psi_s} \right\} \right]$$

Then, from Eq. (5.24),

$$\psi_f' = \left[\frac{\sin^2\psi_f}{\sin^2\psi_s} + \frac{z_s - z_f}{z_s f_s(\psi_s)} \left\{ \Lambda_s - K \sin\psi_s \right\} \sin^2\psi_f \right] e^{K(\cos\psi_f - \cos\psi_s)}, \quad (\text{A-III.7})$$

which, it is evident, is closely related to Eq. (5.7b).

Because $\frac{1}{\rho} \frac{d\rho}{d\psi_s}$ is used for the transmission computations of Section #9, we compute here an explicit formula for the symmetrical instrument of Eq. (3.5), in which we set $b_1 = 0$, since b_1 does not affect the derivative. From Eq. (3.5), since $f_s(\psi_s) = r_e/z_s$, we have

$$K[U(K, \psi_s) + 2\pi n_s i J_1(iK)] - \cot\psi_s e^{-K\cos\psi_s} = \frac{z_s}{r_e} e^{-K\cos\psi_s} . \quad (\text{A-III.8})$$

Differentiating with respect to ψ_s ,

$$\frac{e^{-K\cos\psi_s}}{\sin^2\psi_s} = \frac{z_s}{r_e^2} e^{-K\cos\psi_s} \left[r_e K \sin\psi_s - \frac{dr_e}{d\psi_s} \right]$$

or

$$\frac{1}{r_e} \frac{dr_e}{d\psi_s} = \frac{1}{f_s(\psi_s)} \frac{df_s}{d\psi_s} = K \sin\psi_s - \frac{r_e}{z_s} \frac{1}{\sin^2\psi_s} = K \sin\psi_s - \frac{f_s(\psi_s)}{\sin^2\psi_s} \quad (\text{A-III.9})$$

and

$$\begin{aligned} \frac{dr_e}{d\psi_s} &= z_s \frac{df_s}{d\psi_s} = K r_e \sin\psi_s - \frac{r_e^2}{z_s} \frac{1}{\sin^2\psi_s} \\ &= z_s \left[K f_s \sin\psi_s - \frac{f_s^2}{\sin^2\psi_s} \right] . \end{aligned} \quad (\text{A-III.10})$$

From Eqs. (5.28) and (5.32), we have

$$\frac{1}{\rho_s} \frac{d\rho_s}{d\psi_s} = K \sin\psi_s - \frac{r_e}{z_s} \frac{1}{\sin^2\psi_s} - \cot\psi_s \quad (\text{A-III.11})$$

and

$$\begin{aligned} \frac{d\rho_s}{d\psi_s} &= K \rho_s \sin\psi_s - \frac{r_e}{z_s} \frac{\rho_s}{\sin^2\psi_s} - \rho_s \cot\psi_s \\ &= \frac{z_s f_s(\psi_s)}{\sin\psi_s} \left[K \sin\psi_s - \frac{f_s(\psi_s)}{\sin^2\psi_s} - \cot\psi_s \right] . \end{aligned} \quad (\text{A-III.12})$$

APPENDIX IV

TABLE OF FUNCTIONS FOR SPECTROMETERS
WITH FRINGING FIELDS

Table IV. Column 1. $\frac{d\rho_s}{d\psi_s}$ was computed from Eq. (A-III.12).

Computation was performed setting $z_s = 1$; hence, the quantity computed is $\frac{1}{z_s} \frac{d\rho_s}{d\psi_s}$, with ψ_s in radians. The magnetic profile considered was that of the symmetrical system of Fig. (3.3) and Table II.3. The derivative was computed for $\psi_s = \psi_s^\circ(15^\circ)175^\circ$ and $K = 0.2, 0.4(0.05)0.5(0.02)0.9(0.05)1.0$. Here ψ_s° is the angle closest to the critical angle, but may not be the same as ψ_s° in Tables II.3, II.4, or II.5, because of the difference in $\Delta\psi_s$.

Table IV. Column 2. $\frac{1}{\rho_s} \frac{d\rho_s}{d\psi_s}$ was computed from Eq. (A-III.11)

over the same range as Column 1. Also, $\tan\mu_s$ is given by this quantity, as in Eq. (9.8a).

Table IV. Column 3. Here W_s is computed from Eq. (9.12b) over the same ψ_s - and K -ranges as in Column 1. Values for which $W_s < 1$ arise when $\frac{d\rho_s}{d\psi_s}$ (and hence $\tan\mu_s$) is negative, i.e., when the entrance lens is convergent. In this case, values of ψ_s for which $W_s > 0$ arise when the lens is slightly or moderately convergent. Where the lens is strongly convergent, $|\tan\mu_s|$ is large and hence $W_s < 0$. For a divergent lens, $\frac{d\rho_s}{d\psi_s}$ and $\tan\mu_s$ are positive, and $W_s > 1$.

Table IV. Column 4. Here η_2 is computed from Eqs (9.19) and (9.20), i.e., the absolute value of the inverse of W_s . Now $\eta_2 = 1/|W_s|$ is valid only for values where $|W_s| > 1$. When $|W_s| < 1$, then η_2 is set equal to one. The condition $|W_s| > 1$ can arise in two ways.

(i) Divergent entrance lens ($W_s > 1$). Some rays are always lost by being driven into the polefaces

(ii) Strongly convergent entrance lens ($W_s < -1$). Trajectories cross the median plane at such large angles that they are driven into the polefaces on the opposite sides to those in which they entered the field.

Table IV. Column 5. The quantity $Y = \frac{\frac{1}{2}\varphi}{R_d/z_s} \eta_{23}$ is computed from

Eq. (9.28). Actual values of η_{23} require specification of R_d/z_s and $\frac{1}{2}\varphi$; an example is shown in the text using $R_d = 0.01$ (in units of z_s) and $\varphi = 10^\circ$. For a given Y -value and R_d , increase of φ tends to decrease η_{23} because of the larger δR_{2f} which can get through a wider aperture; increase of R_d tends to increase η_{23} because a larger fraction of the rays spread out into $\delta R_{2f}(\text{max})$ can be accepted by the detector. However, η_{23} has η_2 as an upper limit, and if $Y \frac{R_d/z_s}{\frac{1}{2}\varphi}$ exceeds η_2 , then η_{23} is set equal to η_2 ; in this case, the detector is accepting all rays which pass through the exit aperture.

Table A-IV

Table of Functions for Spectrometers with Fringing Fields

ψ_s	$\frac{1}{z_s} \frac{d\rho_s}{d\psi_s}$	$\frac{1}{\rho_s} \frac{d\rho_s}{d\psi_s}$	$W_s(\psi_s)$	$\eta_2(\psi_s)$	$Y = \frac{\frac{1}{2}\varphi}{R_d/z_s} \eta_{23}(\psi_s)$	$\frac{1}{z_s} \frac{d\rho_s}{d\psi_s}$	$\frac{1}{\rho_s} \frac{d\rho_s}{d\psi_s}$	$W_s(\psi_s)$	$\eta_2(\psi_s)$	$Y = \frac{\frac{1}{2}\varphi}{R_d/z_s} \eta_{23}(\psi_s)$
K = 0.60000E 00						K = 0.62000E 00				
65	-0.16734E 01	-0.12582E 01	-0.12074E 01	0.92822E 00	0.17289E 01	-0.14309E 01	-0.11333E 01	0.90508E 00	0.60368E 00	0.60368E 00
75	-0.11799E 01	-0.09014E 00	-0.08994E 00	0.11000E 01	0.11688E 01	-0.11066E 01	-0.11011E 01	0.90079E 00	0.40847E 01	0.40847E 01
80	-0.04660E 00	-0.04385E 00	-0.04385E 00	0.10000E 01	0.10000E 01	-0.09759E 00	-0.09322E 00	0.89322E 00	0.23664E 01	0.23664E 01
85	-0.01964E 00	-0.01717E 00	-0.01717E 00	0.10000E 01	0.12007E 01	-0.09322E 00	-0.08932E 00	0.88932E 00	0.14366E 01	0.14366E 01
90	-0.00451E -03	-0.00415E -03	-0.00415E -03	0.10000E 01	0.12398E 04	-0.08572E -03	-0.08207E -03	0.88207E 00	0.10000E 01	0.10000E 01
95	-0.00167E -03	-0.00157E -03	-0.00157E -03	0.10000E 01	0.14280E 01	-0.07436E -03	-0.07119E -03	0.87119E 00	0.07659E 01	0.07659E 01
100	-0.00177E -03	-0.00167E -03	-0.00167E -03	0.10000E 01	0.18308E 01	-0.06595E -03	-0.06333E -03	0.86333E 00	0.04199E 01	0.04199E 01
105	-0.00140E -03	-0.00129E -03	-0.00129E -03	0.10000E 01	0.25166E 01	-0.05758E -03	-0.05500E -03	0.85500E 00	0.02419E 01	0.02419E 01
110	-0.00098E -03	-0.00086E -03	-0.00086E -03	0.10000E 01	0.41185E 01	-0.04315E -03	-0.04150E -03	0.84150E 00	0.01327E 01	0.01327E 01
115	-0.00064E -03	-0.00053E -03	-0.00053E -03	0.10000E 01	0.98273E 01	-0.03607E -03	-0.03458E -03	0.83458E 00	0.00776E 01	0.00776E 01
120	-0.00046E -03	-0.00036E -03	-0.00036E -03	0.10113E 01	0.98878E 00	-0.03071E -03	-0.02936E -03	0.82936E 00	0.00434E 02	0.00434E 02
125	-0.00030E -03	-0.00022E -03	-0.00022E -03	0.10113E 01	0.50932E 02	-0.02701E -03	-0.02576E -03	0.82576E 00	0.00236E 01	0.00236E 01
130	-0.00018E -03	-0.00013E -03	-0.00013E -03	0.10113E 01	0.44671E 01	-0.02328E -03	-0.02217E -03	0.82217E 00	0.00143E 02	0.00143E 02
135	-0.00009E -03	-0.00006E -03	-0.00006E -03	0.10113E 01	0.51959E 01	-0.02007E -03	-0.01904E -03	0.81904E 00	0.00089E 01	0.00089E 01
140	-0.00005E -03	-0.00003E -03	-0.00003E -03	0.10113E 01	0.41138E 01	-0.01741E -03	-0.01652E -03	0.81652E 00	0.00058E 01	0.00058E 01
145	-0.00002E -03	-0.00001E -03	-0.00001E -03	0.10113E 01	0.36650E 01	-0.01545E -03	-0.01474E -03	0.81474E 00	0.00037E 01	0.00037E 01
150	-0.00001E -03	-0.00000E -03	-0.00000E -03	0.10113E 01	0.35777E 01	-0.01403E -03	-0.01342E -03	0.81342E 00	0.00024E 01	0.00024E 01
155	-0.00000E -03	-0.00000E -03	-0.00000E -03	0.10113E 01	0.37020E 01	-0.01274E -03	-0.01222E -03	0.81222E 00	0.00016E 01	0.00016E 01
160	-0.00000E -03	-0.00000E -03	-0.00000E -03	0.10113E 01	0.40044E 01	-0.01160E -03	-0.01116E -03	0.81116E 00	0.00010E 01	0.00010E 01
165	-0.00000E -03	-0.00000E -03	-0.00000E -03	0.10113E 01	0.45873E 01	-0.01052E -03	-0.01011E -03	0.81011E 00	0.00006E 01	0.00006E 01
170	-0.00000E -03	-0.00000E -03	-0.00000E -03	0.10113E 01	0.60547E 01	-0.00919E -03	-0.00884E -03	0.80884E 00	0.00004E 01	0.00004E 01
175	-0.00000E -03	-0.00000E -03	-0.00000E -03	0.10030E 01	0.99699E 00	-0.00719E 02	-0.00708E -03	0.80708E 00	0.00003E 01	0.00003E 01
K = 0.64000E 00						K = 0.66000E 00				
60	-0.04488E 01	-0.16931E 01	-0.16869E 01	0.51595E 00	0.30186E -00	-0.04896E 01	-0.20235E 01	0.44059E -00	0.16825E -00	0.16825E -00
65	-0.01680E 01	-0.12686E 01	-0.12466E 01	0.70096E 00	0.21584E 00	-0.01989E 01	-0.15184E 01	0.55212E 00	0.41757E 00	0.41757E 00
70	-0.01177E 01	-0.09880E 00	-0.10008E 01	0.99160E 00	0.66391E 02	-0.01347E 01	-0.11554E 01	0.74877E 00	0.14594E 01	0.14594E 01
75	-0.00612E 00	-0.06697E 00	-0.06192E 00	0.10000E 01	0.20895E 01	-0.00402E 01	-0.03803E 00	0.90236E 00	0.18232E 01	0.18232E 01
80	-0.00408E -02	-0.04192E 00	-0.04043E -02	0.10000E 01	0.15049E 01	-0.00200E -03	-0.01490E -03	0.81490E 00	0.11753E 01	0.11753E 01
85	-0.00271E -03	-0.02961E -03	-0.02864E -03	0.10000E 01	0.17419E 01	-0.00146E -03	-0.01137E -03	0.81137E 00	0.07187E 01	0.07187E 01
90	-0.00170E -03	-0.01948E -03	-0.01895E -03	0.10000E 01	0.22994E 01	-0.00106E -03	-0.00732E -03	0.80732E 00	0.04258E 01	0.04258E 01
95	-0.00100E -03	-0.01166E 00	-0.01152E 00	0.10000E 01	0.36183E 01	-0.00073E -03	-0.00535E -03	0.80535E 00	0.02582E 01	0.02582E 01
100	-0.00060E -03	-0.00748E 00	-0.00742E 00	0.10000E 01	0.58723E 01	-0.00049E -03	-0.00364E -03	0.80364E 00	0.01573E 01	0.01573E 01
105	-0.00040E -03	-0.00504E 00	-0.00502E 00	0.10000E 01	0.82951E 02	-0.00034E -03	-0.00254E -03	0.80254E 00	0.00978E 01	0.00978E 01
110	-0.00026E -03	-0.00332E 00	-0.00332E 00	0.10000E 01	0.11114E -03	-0.00021E -03	-0.00154E -03	0.80154E 00	0.00648E 01	0.00648E 01
115	-0.00016E -03	-0.00217E 00	-0.00217E 00	0.10000E 01	0.13077E -03	-0.00013E -03	-0.00096E -03	0.80096E 00	0.00417E 01	0.00417E 01
120	-0.00009E -03	-0.00134E 00	-0.00134E 00	0.10000E 01	0.16864E -03	-0.00008E -03	-0.00056E -03	0.80056E 00	0.00263E 01	0.00263E 01
125	-0.00005E -03	-0.00082E 00	-0.00082E 00	0.10000E 01	0.23458E -03	-0.00005E -03	-0.00034E -03	0.80034E 00	0.00165E 01	0.00165E 01
130	-0.00003E -03	-0.00051E 00	-0.00051E 00	0.10000E 01	0.34723E -03	-0.00003E -03	-0.00021E -03	0.80021E 00	0.00103E 01	0.00103E 01
135	-0.00002E -03	-0.00031E 00	-0.00031E 00	0.10000E 01	0.52897E 01	-0.00002E -03	-0.00013E -03	0.80013E 00	0.00064E 01	0.00064E 01
140	-0.00001E -03	-0.00019E 00	-0.00019E 00	0.10000E 01	0.80822E 01	-0.00001E -03	-0.00008E -03	0.80008E 00	0.00041E 01	0.00041E 01
145	-0.00000E -03	-0.00011E 00	-0.00011E 00	0.10000E 01	0.12088E -03	-0.00000E -03	-0.00005E -03	0.80005E 00	0.00026E 01	0.00026E 01
150	-0.00000E -03	-0.00006E 00	-0.00006E 00	0.10000E 01	0.18718E 01	-0.00000E -03	-0.00003E -03	0.80003E 00	0.00016E 01	0.00016E 01
155	-0.00000E -03	-0.00003E 00	-0.00003E 00	0.10000E 01	0.29254E 01	-0.00000E -03	-0.00002E -03	0.80002E 00	0.00010E 01	0.00010E 01
160	-0.00000E -03	-0.00001E 00	-0.00001E 00	0.10000E 01	0.45873E 01	-0.00000E -03	-0.00001E -03	0.80001E 00	0.00006E 01	0.00006E 01
165	-0.00000E -03	-0.00000E 00	-0.00000E 00	0.10000E 01	0.60547E 01	-0.00000E -03	-0.00000E -03	0.80000E 00	0.00004E 01	0.00004E 01
170	-0.00000E -03	-0.00000E 00	-0.00000E 00	0.10000E 01	0.88020E 01	-0.00000E -03	-0.00000E -03	0.80000E 00	0.00003E 01	0.00003E 01
175	-0.00000E -03	-0.00000E 00	-0.00000E 00	0.10000E 01	0.99699E 00	-0.00000E -03	-0.00000E -03	0.80000E 00	0.00002E 01	0.00002E 01
K = 0.68000E 00						K = 0.70000E 00				
45	-0.00633E 01	-0.26477E 01	-0.27287E 01	0.36647E -00	0.96805E -01	-0.00669E 01	-0.37953E 01	0.30811E -00	0.54984E -01	0.54984E -01
50	-0.00483E 01	-0.18141E 01	-0.22196E 01	0.45053E -00	0.24595E -00	-0.00421E 01	-0.22026E 01	0.36952E -00	0.12754E -00	0.12754E -00
55	-0.00369E 01	-0.13557E 01	-0.17192E 01	0.58166E 00	0.58044E 00	-0.00318E 01	-0.16345E 01	0.46117E -00	0.02974E -00	0.02974E -00
60	-0.00281E 01	-0.10359E 01	-0.12999E 01	0.80656E 00	0.25467E 01	-0.00218E 01	-0.12508E 01	0.56399E 00	0.81542E 00	0.81542E 00
65	-0.00217E 01	-0.08109E 01	-0.09909E 01	0.10000E 01	0.41964E 01	-0.00160E 01	-0.10394E 01	0.67726E 00	0.54477E 00	0.54477E 00
70	-0.00170E 01	-0.06219E 01	-0.08130E 00	0.10000E 01	0.60404E 01	-0.00116E 01	-0.07566E 01	0.80000E 01	0.33700E 01	0.33700E 01
75	-0.00134E 01	-0.04733E 01	-0.06130E 00	0.10000E 01	0.82951E 02	-0.00082E 01	-0.05293E 01	0.10000E 01	0.21293E 01	0.21293E 01
80	-0.00100E 01	-0.03507E 01	-0.04732E 01	0.10000E 01	0.18457E 04	-0.00057E 01	-0.03474E 01	0.11697E 00	0.12822E 01	0.12822E 01
85	-0.00074E 01	-0.02577E 01	-0.03503E 01	0.10000E 01	0.21152E 01	-0.00040E 01	-0.02508E 01	0.13000E 01	0.08766E 01	0.08766E 01
90	-0.00054E 01	-0.01927E 01	-0.02577E 01	0.10000E 01	0.29254E 01	-0.00028E 01	-0.01840E 01	0.15000E 01	0.05706E 01	0.05706E 01
95	-0.00039E 01	-0.01477E 01	-0.02007E 01	0.10000E 01	0.45873E 01	-0.00020E 01	-0.01477E 01	0.17000E 01	0.03808E 01	0.03808E 01
100	-0.00028E 01	-0.01123E 01	-0.01477E 01	0.10000E 01	0.60547E 01	-0.00015E 01	-0.01123E 01	0.19000E 01	0.02532E 01	0.02532E 01
105	-0.00020E 01	-0.00847E 01	-0.01123E 01	0.10000E 01	0.88020E 01	-0.00010E 01	-0.00847E 01	0.21000E 01	0.01680E 01	0.01680E 01
110	-0.00015E 01	-0.00633E 01	-0.00847E 01	0.10000E 01	0.99699E 00	-0.00007E 01	-0.00633E 01	0.23000E 01	0.01032E 01	0.01032E 01
115	-0.00011E 01	-0.00473E 01	-0.00633E 01	0.10000E 01	0.12088E -03	-0.00005E 01	-0.00473E 01	0.25000E 01	0.00648E 01	0.00648E 01
120	-0.00008E 01	-0.00350E 01	-0.00473E 01	0.10000E 01	0.18718E 01	-0.00003E 01	-0.00350E 01	0.2700		

ψ_s	$\frac{1}{z_s} \frac{d\psi_s}{d\psi_s}$	$\frac{1}{\rho_s} \frac{d\psi_s}{d\psi_s}$	$W_s(\psi_s)$	$\eta_2(\psi_s)$	$Y = \frac{1}{R_d/z_s} \eta_{23}(\psi_s)$	$\frac{1}{z_s} \frac{d\psi_s}{d\psi_s}$	$\frac{1}{\rho_s} \frac{d\psi_s}{d\psi_s}$	$W_s(\psi_s)$	$\eta_2(\psi_s)$	$Y = \frac{1}{R_d/z_s} \eta_{23}(\psi_s)$
4	0.86542E-01	0.40518E-01	0.38 07E 0	0.6173E-00	0.2949E-02	0.35409E-01	0.35409E-01	0.35409E-01	0.35409E-01	0.35409E-01
45	0.443897E-01	0.27491E-01	0.3 697E 0	0.3 697E 0	0.2640E-02	0.24677E-01	0.24677E-01	0.24677E-01	0.24677E-01	0.24677E-01
50	0.47 871E 01	0.19877E 02	0.6807E 1	0.37 04E 0	0.165 9E 00	0.1788E 01	0.1788E 01	0.1788E 01	0.1788E 01	0.1788E 01
55	0.16507E 01	0.14785E 01	0.080E 01	0.47439E-00	0.39218E-00	0.13411E-01	0.13411E-01	0.13411E-01	0.13411E-01	0.13411E-01
60	0.11225E 01	0.11054E-01	0.15549E-01	0.044311E-00	0.11662E-01	0.02540E-00	0.02540E-00	0.02540E-00	0.02540E-00	0.02540E-00
65	0.7 423E 00	0.8 58 F 02	0.10341E-01	0.076 06E 00	0.27658E-00	0.63346E-00	0.63346E-00	0.63346E-00	0.63346E-00	0.63346E-00
70	0.52616E-01	0.3213E 0	0.55497E 00	0.21000E 01	0.30749E 0	0.42983E-00	0.42983E-00	0.42983E-00	0.42983E-00	0.42983E-00
75	0.25470E 00	0.228 F 0	0.12446E 00	0.1000E 01	0.32318E-01	0.327058E-00	0.327058E-00	0.327058E-00	0.327058E-00	0.327058E-00
80	0.2503E-00	0.8 7 F 00	0.25232E-00	0.1000E 01	0.54 E 0	0.18671E-00	0.18671E-00	0.18671E-00	0.18671E-00	0.18671E-00
85	0.1 607E-00	0.6 F 00	0.5760E 00	0.30000E 01	0.36255E-00	0.779E-01	0.779E-01	0.779E-01	0.779E-01	0.779E-01
90	0.46372E-00	0.04040E-01	0.04450E-00	0.3000E 01	0.841 8E 1	0.45581E-01	0.45581E-01	0.45581E-01	0.45581E-01	0.45581E-01
95	0.8228E-00	0.23419E-01	0.0599E 00	0.34353E 00	0.191 6E 0	0.44950E-01	0.44950E-01	0.44950E-01	0.44950E-01	0.44950E-01
100	0.7181E-01	0.07988E 0	0.12259E 01	0.781614E 00	0.44545E 0	0.427E-00	0.427E-00	0.427E-00	0.427E-00	0.427E-00
105	0.11464E-00	0.74 F 00	0.13448E 01	0.74300E 00	0.26784E 0	0.4511E-00	0.4511E-00	0.4511E-00	0.4511E-00	0.4511E-00
110	0.14997E-00	0.8 9 F 00	0.14232E 01	0.7662E 00	0.19813E 00	0.7853E 00	0.7853E 00	0.7853E 00	0.7853E 00	0.7853E 00
115	0.17797E-00	0.1799E-01	0.14658E 01	0.6082 4E 00	0.1640 0E 01	0.20511E-00	0.20511E-00	0.20511E-00	0.20511E-00	0.20511E-00
120	0.19913E-00	0.3906E 00	0.14779E 01	0.67605E 00	0.14592E 01	0.27511E-00	0.27511E-00	0.27511E-00	0.27511E-00	0.27511E-00
125	0.21668E-00	0.5108E 00	0.14651E 01	0.68294E 00	0.17669E 00	0.27866E-00	0.27866E-00	0.27866E-00	0.27866E-00	0.27866E-00
130	0.23174E-00	0.7444E-00	0.14363E 01	0.69786E 00	0.13346E 00	0.24646E-00	0.24646E-00	0.24646E-00	0.24646E-00	0.24646E-00
135	0.23387E-00	0.5111E 00	0.13866E 01	0.7118E 00	0.13505E 00	0.24618E 00	0.24618E 00	0.24618E 00	0.24618E 00	0.24618E 00
140	0.21868E-00	0.4960E 00	0.13311E 01	0.741 6E 00	0.1412E 00	0.4012E-00	0.4012E-00	0.4012E-00	0.4012E-00	0.4012E-00
145	0.20777E-00	0.7409E-01	0.12 09E 01	0.8684E 0	0.15258E 01	0.2769E-00	0.2769E-00	0.2769E-00	0.2769E-00	0.2769E-00
150	0.4099E 00	0.1102E 01	0.12102E 01	0.82624E 00	0.1707E 00	0.0888E-00	0.0888E-00	0.0888E-00	0.0888E-00	0.0888E-00
155	0.6880E-00	0.17548E-00	0.1528E 01	0.86 66E 03	0.4 983E 01	0.8429E-00	0.8429E-00	0.8429E-00	0.8429E-00	0.8429E-00
160	0.14143E-00	0.451 E 00	0.1011E 01	0.9 8 E 00	0.4 30E 01	0.5442E-00	0.5442E-00	0.5442E-00	0.5442E-00	0.5442E-00
165	0.10999E-00	0.1 65F 00	0.0584E 01	0.94485E 00	0.18 E 1	0.120 3E 0	0.120 3E 0	0.120 3E 0	0.120 3E 0	0.120 3E 0
170	0.75236E-00	0.7578E 01	0.1024E 01	0.97426E 00	0.4720E 00	0.8 08E 00	0.8 08E 00	0.8 08E 00	0.8 08E 00	0.8 08E 00
175	0.48 01E-00	0.3876E-01	0.1066E 01	0.993 E 00	0.939 4E 01	0.4167E 01	0.4167E 01	0.4167E 01	0.4167E 01	0.4167E 01
90	0.10799E 0	0.0486E-01	0.45563E 0	0.1488E 0	0.1079E 0	0.447E 0	0.447E 0	0.447E 0	0.447E 0	0.447E 0
95	0.4986E 0	0.3190E-01	0.39291E 01	0.4 1E-00	0.4 03E-01	0.49 6E	0.49 6E	0.49 6E	0.49 6E	0.49 6E
45	0.7954E 01	0.2734E 00	0.32762E 01	0.05 E 00	0.1 444 0E 00	0.828E	0.828E	0.828E	0.828E	0.828E
50	0.736 E 01	0.16398E-01	0.6 00E 01	0.8168E-00	0. 3 -0	0.584E 01	0.584E 01	0.584E 01	0.584E 01	0.584E 01
55	0.11418E 0	0.2 29E 0	0.7 98E 0	0.5 10E 00	0.07959E-00	0.04548E-00	0.04548E-00	0.04548E-00	0.04548E-00	0.04548E-00
60	0.77189E 00	0.0404E 00	0.3716E 00	0.726 0E 0	0.644 F 1	0.444 3E	0.444 3E	0.444 3E	0.444 3E	0.444 3E
65	0.42426E 00	0.574E 0	0.8080E 0	0.100 0E 0	0.55 04E	0.4 5 F 0	0.4 5 F 0	0.4 5 F 0	0.4 5 F 0	0.4 5 F 0
70	0.4825E-00	0.4 3 F 0	0.98 34E-00	0.1 E 0	0.140E 0	0.780E	0.780E	0.780E	0.780E	0.780E
75	0.21706E-00	0.70517E-00	0.4241E-00	0.12 0E 1	0.038E 01	0.6249E 0	0.6249E 0	0.6249E 0	0.6249E 0	0.6249E 0
80	0.11540E 00	0.160E 00	0.53909E 00	0.1 E 0	0.042766E-00	0.0789E-00	0.0789E-00	0.0789E-00	0.0789E-00	0.0789E-00
85	0.24132E-01	0.477E 0	0.03411E 00	0.1 E 01	0.11344E-00	0.240 8E-00	0.240 8E-00	0.240 8E-00	0.240 8E-00	0.240 8E-00
90	0.42397E-01	0.4533 F 0	0.264E 0	0.8470E 00	0.1040E-00	0.0406E-00	0.0406E-00	0.0406E-00	0.0406E-00	0.0406E-00
95	0.87740E-01	0.7 6E 0	0.13277E 0	0.74 4E 0	0.37718E 0	0. 0	0. 0	0. 0	0. 0	0. 0
100	0.1341 E-00	0.888F 00	0.14766E 01	0.75 E 0	0.1707E 0	0.6 6E	0.6 6E	0.6 6E	0.6 6E	0.6 6E
105	0.7 91F 00	0.4 6 F 0	0.5782E 00	0.5782E 00	0.982 F-00	0.982 F-00	0.982 F-00	0.982 F-00	0.982 F-00	0.982 F-00
110	0.7048E-00	0.696F 00	0.6344E 0	0.84E 0	0.403 F 0	0.31F-00	0.31F-00	0.31F-00	0.31F-00	0.31F-00
115	0.73040E-00	0.7086E 0	0.6 5E 0	0.48 F 0	0.2431 F 0	0.433E-00	0.433E-00	0.433E-00	0.433E-00	0.433E-00
120	0.74958E-00	0.71 8E 00	0.6418E 0	0.1 0	0.115 F 0	0. 0 E	0. 0 E	0. 0 E	0. 0 E	0. 0 E
125	0.76237E-00	0.1191E 00	0.60 5E 01	0.6 8 E 0	0.117 F 01	0.646E 0	0.646E 0	0.646E 0	0.646E 0	0.646E 0
130	0.76884E-00	0.1779E 00	0.550 E 01	0.4448 F 00	0.1 70 F 01	0.840E 00	0.840E 00	0.840E 00	0.840E 00	0.840E 00
135	0.68 E-00	0.707E 00	0.4830E 0	0.074 E 0	0.4 3E	0.80E-00	0.80E-00	0.80E-00	0.80E-00	0.80E-00
140	0.76103E 00	0.79266F 00	0.140 E 01	0.08 E 0	0.103 E	0.9137E-00	0.9137E-00	0.9137E-00	0.9137E-00	0.9137E-00
145	0.4707E 0	0.77022E 0	0.3300E 1	0.77186E 00	0.132 F 0	0.2601 E	0.2601 E	0.2601 E	0.2601 E	0.2601 E
150	0.2652E-00	0.4229E 0	0.12548E 01	0.7978E 00	0.1483 1	0.4389E-00	0.4389E-00	0.4389E-00	0.4389E-00	0.4389E-00
155	0.19974E 00	0.7047E 00	0.118 7E 0	0.84 8E	0.174 1	0.50F	0.50F	0.50F	0.50F	0.50F
160	0.16737E-00	0.7247E 00	0.112 5E 0	0.87 3E 00	0.4 1 E 0	0.4 1 E 0	0.4 1 E 0	0.4 1 E 0	0.4 1 E 0	0.4 1 E 0
165	0.133 4E 00	0.7046F 00	0.0693E 01	0.84 E 0	0.4 3 E 00	0.4 3 E 00	0.4 3 E 00	0.4 3 E 00	0.4 3 E 00	0.4 3 E 00
170	0.8803 E 1	0.8043E 00	0.103 4E 0	0.4 0E 0	0.4 E 0	0.761E 0	0.761E 0	0.761E 0	0.761E 0	0.761E 0
175	0.45 48E 0	0.45 48E 0	0.17 79E 01	0. 1 E 0	0.3830E	0.862E 0	0.862E 0	0.862E 0	0.862E 0	0.862E 0
90	0.10799E 0	0.0486E-01	0.45563E 0	0.1488E 0	0.1079E 0	0.447E 0	0.447E 0	0.447E 0	0.447E 0	0.447E 0
95	0.4986E 0	0.3190E-01	0.39291E 01	0.4 1E-00	0.4 03E-01	0.49 6E	0.49 6E	0.49 6E	0.49 6E	0.49 6E
45	0.7954E 01	0.2734E 00	0.32762E 01	0.05 E 00	0.1 444 0E 00	0.828E	0.828E	0.828E	0.828E	0.828E
50	0.736 E 01	0.16398E-01	0.6 00E 01	0.8168E-00	0. 3 -0	0.584E 01	0.584E 01	0.584E 01	0.584E 01	0.584E 01
55	0.11418E 0	0.2 29E 0	0.7 98E 0	0.5 10E 00	0.07959E-00	0.04548E-00	0.04548E-00	0.04548E-00	0.04548E-00	0.04548E-00
60	0.77189E 00	0.0404E 00	0.3716E 00	0.726 0E 0	0.644 F 1	0.444 3E	0.444 3E	0.444 3E	0.444 3E	0.444 3E
65	0.42426E 00	0.574E 0	0.8080E 0	0.100 0E 0	0.55 04E	0.4 5 F 0	0.4 5 F 0	0.4 5 F 0	0.4 5 F 0	0.4 5 F 0
70	0.4825E-00	0.4 3 F 0	0.98 34E-00	0.1 E 0	0.140E 0	0.780E	0.780E	0.780E	0.780E	0.780E
75	0.21706E-00	0.70517E-00	0.4241E-00	0.12 0E 1	0.038E 01	0.6249E 0	0.6249E 0	0.6249E 0	0.6249E 0	0.6249E 0
80	0.11540E 00	0.160E 00	0.53909E 00	0.1 E 0	0.042766E-00	0.0789E-00	0.0789E-00	0.0789E-00	0.0789E-00	0.0789E-00
85	0.24132E-01	0.477E 0	0.03411E 00	0.1 E 01	0.11344E-00	0.240 8E-00	0.240 8E-00	0.240 8E-00	0.240 8E-00	0.240 8E-00
90	0.42397E-01	0.4533 F 0	0.264E 0	0.8470E 00	0.1040E-00	0.0406E-00	0.0406E-00	0.0406E-00	0.0406E-00	0.0406E-00
95	0.87740E-01	0.7 6E 0	0.13277E 0	0.74 4E 0	0.37718E 0	0. 0	0. 0	0. 0	0. 0	0. 0
100	0.1341 E-00	0.888F 00	0.14766E 01	0.75 E 0	0.1707E 0	0.6 6E	0.6 6E	0.6 6E	0.6 6E	0.6 6E
105	0.7 91F 00	0.4 6 F 0	0.5782E 00	0.5782E 00	0.982 F-00	0.982 F-00	0.982 F-00	0.982 F-00	0.982 F-00	0.982 F-00
110	0.7048E-00	0.696F 00	0.6344E 0	0.84E 0	0.403 F 0	0.31F-00	0.31F-00	0.31F-00	0.31F-00	0.31F-00
115	0.73040E-00	0.7086E 0	0.6 5E 0	0.48 F 0	0.2431 F 0	0.433E-00	0.433E-00	0.433E-00	0.433E-00	0.433E-00
120	0.74958E-00	0.71 8E 00	0.6418E 0	0.1 0	0.115 F 0	0. 0 E	0. 0 E	0. 0 E	0. 0 E	0. 0 E
125	0.76237E-00	0.1191E 00	0.60 5E 01	0.6 8 E 0	0.117 F 01	0.646E 0	0.646E 0	0.646E 0	0.646E 0	0.646E 0
130	0.76884E-00	0.1779E 00	0.550 E 01	0.4448 F 00	0.1 70 F 01	0.840E 00	0.840E 00	0.840E 00	0.84	

ψ_s	$\frac{1}{z_s} \frac{d\rho_s}{d\psi_s}$	$\frac{1}{\rho_s} \frac{d\rho_s}{d\psi_s}$	$W_s(\psi_s)$	$\eta_2(\psi_s)$	$Y = \frac{1}{2} \frac{\varphi}{R_d/z_s} \eta_{23}(\psi_s)$	$\frac{1}{z_s} \frac{d\rho_s}{d\psi_s}$	$\frac{1}{\rho_s} \frac{d\rho_s}{d\psi_s}$	$W_s(\psi_s)$	$\eta_2(\psi_s)$	$Y = \frac{1}{2} \frac{\varphi}{R_d/z_s} \eta_{23}(\psi_s)$
K = 0.86000E 00						K = 0.86000E 00				
25	-0.7150E 02	-0.89597E 02	-0.65569E 01	0.15291E-00	0.59677E-02	-0.18284E 02	-0.75280E 01	-0.68076E 01	0.14689E-00	0.80990E-02
30	-0.82132E 01	-0.47607E 01	-0.58088E 01	0.12135E-00	0.21279E-01	-0.63514E 01	-0.42741E 01	-0.59899E 01	0.16696E-00	0.27138E-01
35	-0.86234E 01	-0.40980E 01	-0.49922E 01	0.20031E-00	0.55032E-01	-0.31208E 01	0.28464E 01	-0.51058E 01	0.19586E-00	0.67118E-01
40	-0.1456E 0	0.21817E 01	-0.41613E 01	0.24147E-00	0.12463E-00	-0.17976E 01	0.20220E 01	-0.41929E 01	0.23850E-00	0.14984E-00
45	-0.13268E 01	0.15878E 01	-0.32868E 01	0.30425E-00	0.27686E-00	-0.11258E 01	-0.14729E 01	-0.32890E 01	0.30460E-00	0.33460E-00
50	-0.86338E 00	-0.11639E 01	-0.24551E 01	0.40732E-00	0.60686E 00	-0.73513E 00	0.10739E 01	-0.24037E 01	0.41602E-00	0.83340E-00
55	-0.47273E 00	-0.84229E 00	-0.16879E 01	0.59956E 00	0.21960E 01	-0.48407E-00	0.76711E 00	-0.15771E 01	0.63400E 00	0.30726E 01
60	-0.47573E 00	-0.58788E 00	-0.94193E 00	0.10003E 01	0.38499E 02	-0.41240E 00	0.42249E 0	-0.82016E 00	0.10000E 01	0.15308E 02
65	-0.3362E-00	0.8121E 00	-0.28922E-00	0.10000E 01	0.50586E 01	-0.18590E-00	0.2767E 00	-0.14449E-00	0.10000E 01	0.54076E 01
70	-0.2564E-00	-0.71034E 00	0.28272E-00	0.10000E 01	0.52120E 01	-0.88295E 01	0.4444E 00	0.44288E-00	0.10000E 01	0.67505E 01
75	-0.39940E-01	0.67664E-01	0.77030E 01	0.10000E 01	0.11880E 02	-0.96595E-02	0.17247E-01	0.93912E 00	0.01000E 01	0.45914E 02
80	0.40593E-01	0.51816E-01	0.11733E 01	0.85230E 00	0.12894E 02	0.56011E-01	0.99606E 01	0.13449E 01	0.74354E 00	0.65478E 01
85	0.90332E-01	0.1174E-00	0.14943E 01	0.66923E 00	0.37281E 01	0.11242E-00	0.19793E 00	0.16637E 01	0.61066E 00	0.28717E 01
90	0.14199E-00	0.2341E 00	0.17379E 01	0.57540E 00	0.21607E 01	0.16187E-00	0.7825E 00	0.19013E 01	0.52597E 00	0.18312E 01
95	0.18723E-00	0.40203E 00	0.19106E 01	0.52341E 00	0.15415E 01	0.20572E-00	0.34411E 00	0.20647E 01	0.48433E-00	0.13641E 01
100	0.2695E-00	0.46872E-00	0.20136E 01	0.49515E-00	0.12258E 01	0.7446E-00	0.19623E 00	0.21623E 01	0.46248E-00	0.11115E 01
105	0.36157E-00	0.39671E-00	0.20730E 01	0.44823E-00	0.10450E 01	0.27899E-00	0.43566E 0	0.22028E 01	0.48398E-00	0.96245E 00
110	0.29113E-00	0.42594E-00	0.20791E 01	0.44807E-00	0.93704E 00	0.30858E-00	0.6329E 00	0.21951E 01	0.45555E-00	0.87224E 00
115	0.15158E-00	0.44419E-00	0.20463E 01	0.44868E-00	0.87431E 00	0.3300E-00	0.47987E 00	0.21482E 01	0.46550E-00	0.82012E 00
120	0.33390E-00	0.445219E-00	0.19826E 01	0.50438E 00	0.48434E 00	0.35194E-00	0.48609E 00	0.20706E 01	0.48295E-00	0.7959E 00
125	0.4612E-00	0.5061E 00	0.18957E 01	0.52750E 00	0.83811E 00	0.36451E-00	0.48257E 00	0.19703E 01	0.50754E 00	0.79431E 00
130	0.35145E-00	0.44008E-00	0.17928E 01	0.5780E 00	0.8582E 00	0.2124E-00	0.47167E 00	0.18547E 01	0.59917E 00	0.81407E 00
135	0.44932E-00	0.47126E 00	0.16803E 01	0.59512E 00	0.89715E 00	0.16794E-00	0.44488E 00	0.17706E 01	0.57784E 00	0.85599E 00
140	0.39327E-00	0.39481E-00	0.15643E 01	0.66525E 00	0.10651E 01	0.15757E-00	0.41996E 00	0.16040E 01	0.62343E 00	0.93262E 00
145	0.32101E-00	0.36141E-00	0.14500E 01	0.68965E 00	0.10681E 01	0.13858E-00	0.38394E 00	0.14804E 01	0.67552E 00	0.10241E 01
150	0.29442E-00	0.32119E-00	0.13420E 01	0.71342E 00	0.12179E 01	0.14718E-00	0.34151E 00	0.13642E 01	0.73304E 00	0.11701E 01
155	0.25992E-00	0.27577E-00	0.12441E 01	0.76031E 00	0.14398E 01	0.27467E-00	0.29349E 00	0.12595E 01	0.79398E 00	0.13895E 01
160	0.21808E-00	0.22710E-00	0.11597E 01	0.86233E 00	0.17835E 01	0.23057E-00	0.74068E 00	0.11695E 01	0.85959E 00	0.17193E 01
165	0.16972E-00	0.17367E-00	0.10913E 01	0.91634E 00	0.23664E 01	0.17955E-00	0.3898E 00	0.10968E 01	0.91174E 00	0.22843E 01
170	0.11619E-00	0.11799E-00	0.10411E 01	0.96057E 00	0.35422E 01	0.12297E-00	0.4432E-00	0.10435E 01	0.98332E 00	0.34232E 01
175	0.59026E-01	0.59179E-01	0.10103E 01	0.98977E 00	0.70789E 01	0.62492E-01	0.4264E 01	0.10109E 01	0.98918E 00	0.60434E 01
K = 0.88000E 00						K = 0.90000E 00				
20	-0.80814E 00	0.16643E 02	-0.78545E 01	0.12732E-00	0.15866E-02	-0.45475E 02	-0.12815E 02	-0.82000E 01	0.12195E-00	0.27487E-02
25	-0.13108E 00	-0.65255E 01	-0.70667E 01	0.14451E-00	0.11066E-01	-0.98288E 01	0.57846E 01	-0.73342E 01	0.13635E-00	0.14645E-01
30	-0.40425E 01	-0.28886E 01	-0.61784E 01	0.16135E-00	0.33723E-00	-0.40087E 01	-0.25111E 01	-0.63635E 01	0.13710E-00	0.44104E-01
35	-0.75852E 01	-0.26343E 01	-0.52211E 01	0.19153E-00	0.8164E-00	-0.21682E 01	-0.24530E 01	-0.53838E 01	0.18733E-00	0.95683E-01
40	-0.15206E 01	-0.1826E 01	-0.42436E 01	0.23565E 00	0.17842E-00	-0.12970E 01	-0.17599E 01	-0.42293E 01	0.23291E-00	0.21070E-00
45	-0.06140E 00	-0.13701E 01	-0.32764E 01	0.30521E-00	0.40201E-00	-0.82559E 00	-0.31275E 01	-0.32667E 01	0.30612E-00	0.48093E-00
50	-0.62833E 00	-0.2991E 00	-0.23475E 01	0.42591E-00	0.10391E 01	-0.53873E 00	-0.91677E 00	-0.22874E 01	0.43718E-00	0.12975E-00
55	-0.41096E-00	-0.69809E 00	-0.14809E 01	0.67528E 00	0.44531E 01	-0.4020E-00	-0.63979E 00	-0.13788E 01	0.72525E 00	0.68229E 01
60	-0.5851E-00	-0.46177E-00	-0.49223E 00	0.10004E 01	0.11061E 02	-0.2124E-00	-0.40448E 00	0.10000E 01	0.10000E 01	0.99808E 01
65	-0.14541E 00	-0.6734E-00	-0.66535E-02	0.10000E 01	0.60282E 01	-0.10991E-00	0.21491E-00	0.16445E-00	0.10000E 01	0.03212E 01
70	-0.6134E-01	0.11611E 00	0.60940E 00	0.10000E 01	0.97409E 01	-0.28404E-01	0.56189E 01	0.78249E 00	0.10300E 01	0.17776E 02
75	0.16559E-01	0.1176E 01	0.1144E 01	0.89765E 00	0.25169E 02	0.92028E 01	0.77478E-01	0.29591E 01	0.77212E 00	0.10001E 01
80	0.8112E-01	0.14562E 00	0.15221E 01	0.65696E 00	0.44669E 01	0.97309E-01	0.19005E-00	0.17051E 01	0.58649E 00	0.34191E 01
85	0.17938E 00	0.21446E 00	0.18582E 01	0.54400E 00	0.23594E 01	0.14862E 00	0.28425E-00	0.20178E 01	0.49585E-00	0.20067E 01
90	0.17938E 00	0.20707E-00	0.20690E 01	0.44833E 00	0.15985E 01	0.19478E 00	0.36214E 00	0.22411E 01	0.46620E-00	0.14256E 01
95	0.22215E-00	0.18515E-00	0.22226E 01	0.44993E-00	0.12792E 01	0.23673E-00	0.42522E-00	0.23842E 01	0.41943E 00	0.11234E 01
100	0.26059E-00	0.13586E 00	0.23081E 01	0.43327E 00	0.10208E 01	0.27486E-00	0.47466E-00	0.26570E 01	0.40701E-00	0.94720E 00
105	0.29482E-00	0.47387E 00	0.42826E 00	0.49502E 00	0.89502E 00	0.10917E-00	0.51139E 00	0.24698E 01	0.40489E-00	0.83986E 00
110	0.32462E-00	0.50003E 00	0.43131E 01	0.43232E 00	0.8182E 00	0.13946E 00	0.33946E 00	0.20178E 01	0.41101E-00	0.7724E 01
115	0.34954E-00	0.51504E 00	0.22517E 01	0.44441E-00	0.77425E 00	0.16482E-00	0.54971E 00	0.23565E 01	0.42343E-00	0.74949E 00
120	0.36891E-00	0.51996E 00	0.21597E 01	0.44632E-00	0.75493E 00	0.16485E-00	0.55262E 00	0.22500E 01	0.44444E-00	0.71955E 00
125	0.38197E-00	0.52419E 00	0.20457E 01	0.44883E 00	0.75842E 00	0.16485E-00	0.55458E 00	0.22121E 01	0.44712E-00	0.72322E 00
130	0.38791E-00	0.48956E 00	0.19172E 01	0.45239E 00	0.7775E 00	0.16485E-00	0.55605E-00	0.19803E 01	0.50495E 00	0.74561E 00
135	0.38594E-00	0.44497E 00	0.17812E 01	0.45014E 00	0.81964E 00	0.16485E-00	0.55747E 00	0.16842E 01	0.59376E 00	0.85377E 00
140	0.37538E-00	0.40645E-00	0.16440E 01	0.44640E 00	0.88664E 00	0.16485E-00	0.55933E-00	0.16842E 01	0.59376E 00	0.85377E 00
145	0.36594E-00	0.36594E-00	0.15108E 01	0.44188E 00	0.98506E 00	0.17267E 00	0.42866E 00	0.15415E 01	0.64873E 00	0.95024E 00
150	0.35770E-00	0.33106E 00	0.13865E 01	0.42124E 00	0.11276E 01	0.14629E-00	0.43807E 00	0.14089E 01	0.70979E 00	0.10896E 01
155	0.3420E 00	0.31016E 00	0.12749E 01	0.40846E 00	0.13846E 01	0.12904E-00	0.44946E 00	0.13904E 01	0.77497E 00	0.12944E 01
160	0.3297E-00	0.27424E 00	0.11793E 01	0.38479E 00	0.16619E 01	0.25529E-00	0.26777E 00	0.18129E 01	0.84094E 00	0.16104E 01
165	0.31894E-00	0.2427E 00	0							

APPENDIX V

THE DISPERSION FORMULA

The identity of the two dispersion formulae [Eqs. (6.6), (6.7), and (6.10b)] and Eq. (12.52) may be readily shown. We write Eqs. (12.42) and (12.52) as

$$\frac{d\Sigma}{dK} = (z_e - z_0) \left[\frac{F'}{F} + \cos\psi_s \right] + r_0 \tan \gamma_f (\cos\psi_f - \cos\psi_s) \quad (\text{A-V.1})$$

$$\sigma_f = \frac{A_f}{\sin\psi_f} \frac{d\Sigma}{dK} = - \frac{A_f}{\sin\psi_f} \frac{d\Sigma}{dK} \quad (\text{A-V.2})$$

From Eqs. (2.5b) and (3.2a), using the definition in Eq. (5.2c),

$$z_e - z_0 = -F r_e e^{K \cos\psi_s} = -F r_0 e^{K \cos\psi_f} \quad (\text{A-V.3})$$

From Eq. (12.8b),

$$\tan \gamma_f = \frac{1}{A_f \sin^2\psi_f} - \cot\psi_f \quad (\text{A-V.4})$$

From Eqs. (A-V.1), (A-V.3), and (A-V.4),

$$\begin{aligned} \frac{d\Sigma}{dK} &= r_0 \left[-F e^{K \cos\psi_f} \left(\frac{F' + F \cos\psi_s}{F} \right) + (\cos\psi_f - \cos\psi_s) \left(\frac{1}{A_f \sin^2\psi_f} - \cot\psi_f \right) \right] \\ &= \frac{r_0}{A_f \sin\psi_f} \left\{ -A_f \left[F_2 \sin\psi_f e^{K \cos\psi_f} + \cos\psi_f (\cos\psi_f - \cos\psi_s) \right] \right. \\ &\quad \left. + \frac{(\cos\psi_f - \cos\psi_s)}{\sin\psi_f} \right\} \quad (\text{A-V.5}) \end{aligned}$$

Since

$$\frac{-r_0}{\sin^2\psi_f} = (z_s - z_f) \frac{e^{-K \cos\psi_f}}{F_3 \sin^2\psi_f} \quad (\text{A-V.6})$$

we have from Eqs. (A-V.5) and (12.10a) that

$$\begin{aligned} \sigma_f &= \frac{-r_0}{\sin^2\psi_f} \left\{ A_f \left[\cos\psi_f (\cos\psi_s - \cos\psi_f) - F_2 \sin\psi_f e^{K \cos\psi_f} \right] \right. \\ &\quad \left. - \left[\frac{\cos\psi_s - \cos\psi_f}{\sin\psi_f} \right] \right\} = (z_s - z_f) G \quad (\text{A-V.7}) \end{aligned}$$

For a symmetrical spectrometer, from Eq. (12.44),

$$\begin{aligned}\frac{d\Sigma}{dK} &= -F r_e e^{K\cos\psi_s} \left[\frac{F' - F\cos\psi_s}{F} \right] \\ &= -r_e e^{-K\cos\psi_s} [F' - F\cos\psi_s] \quad .\end{aligned}\tag{A-V.8}$$

We have, also,

$$\sigma_f = -\frac{r_e^2}{z_s} \frac{e^{K\cos\psi_s}}{\sin^3\psi_s} = -2z_s \frac{2e^{-K\cos\psi_s}}{F_3^2 \sin^3\psi_s} = 2 z_s G \quad ,\tag{A-V.9}$$

where G is given by Eq. (6.11).

APPENDIX VI

PUBLISHED DATA ON CONSTRUCTED SPECTROMETERS

References	Iron (D or Iron free (IF))	z_s cm	z_f cm	K	number of gaps	ψ^0	ψ_s^0 (min)	ψ_s^0 (max)	Diameter of source mm	Diameter of detector mm	Resolution %	Transmission % of 4π	Maximum energy of electrons Mev	Purpose
16	I	6	0.6	1	180 300	750	1300				4	3	5.0	β ray spectrometer and study of conversion electrons produced in Coulomb excitation
7	I				1									Electron spectrometer for photon monochromator
8	I	~6			1									Electron spectrometer for studying conversion electrons produced in Coulomb excitation
9	I	~15.6	0.76	1	80	800	1000						5.0	Electron spectrometer for photon monochromator Used as spectrograph from K 0.6 to K 1.0
10	I	8.25	~0.6	6	200	~750	~1300				1.6	9	1.5	β ray spectrometer
11	I	16.5	~0.6	6	200	~750	~1300		3 x 6 (mm ²)		1.3	9	4.0	β ray spectrometer (Ring focus)
12	I	27	0.57	8	100	950	1450		4		1.8	3	3.5	β ray spectrometer Half of an e ⁺ e ⁻ coincidence spectrometer
14	IF	25.6	0.50	36		1200	1500		15	38	7		1.3	β rays and recoil protons from the decay of the free neutron
15	IF	~35	0.67	32		1200	1350		8	10	1	8	0.6	Conversion electrons α e coincidence
16 17	IF	33.5	0.59	100	30	1100	1500		3 1		0.9 0.13	19 1.6	3.5	β rays Half of an e ⁺ e ⁻ coincidence spectrometer
18	IF				50				10		1	8		
19	IF	30	>1.0	38		1350	1460		2		1.4	1.5	10	β rays of short lived activities High energy (Ring focus)