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LOW-ENERGY PION-PHOTON INTERACTION: THE $(2\pi, 2\gamma)$ VERTEX

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Contents

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ABSTRACT

In the $(2\pi, 2\gamma)$ problem, the Mandelstam representation is written for the two independent gauge-invariant amplitudes. On the basis of unitarity limitations on the asymptotic behavior of these amplitudes, only a $j = 1$ subtraction in the $\gamma + \pi \to \gamma + \pi$ channel and a $j = 0$ subtraction in the $\gamma + \gamma \to \pi + \pi$ channel are allowed. No over-all subtraction constants are required and the Thomson limit is automatically maintained. Only the effect of 2π intermediate states is considered. The odd-j $\pi\pi$ contribution involves the amplitude for the process $\gamma + \pi \to 2\pi$ analyzed by Wong and shown to be proportional to a psuedo-elementary constant, Λ. Even with a $\pi\pi$ P resonance, the correction is negligible $(\lesssim 1\%)$ if we use the value of Λ estimated by Wong on the basis of π^0 decay and confirmed by Ball in connection with photopion production on nucleons. A moderately important contribution comes from the S-wave interaction. For the pion-pion coupling constant λ of order -0.20 (see below), this effect is $\sim 10\%$ in $\gamma + \pi \to \gamma + \pi$ scattering. For $\gamma + \gamma \to \pi + \pi$, the correction for the $I = 0$ state at threshold is positive and $\sim 100\%$ of the Born approximation. However, as the energy is increased, the correction quickly changes sign.

The $\pi\pi$ S-wave phase shifts needed in the above calculations are obtained by using crossing symmetry relations given by Chew and

Mandelstam. For a $\pi\pi$ P-resonance position $\simeq 20\mu^2$ (μ being the pion mass), we find that λ in the interval (-0.20, -0.15) is in good agreement with the S-wave enhancement observed by Abashian et al. in $\bar{p} + d$ collisions. The S-wave interaction is found to be much stronger in the $I = 0$ state than in the $I = 2$ state.

I. INTRODUCTION

In the $(2\pi, 2\gamma)$ problem, both strong and electromagnetic interactions are involved. In principle, one can calculate electromagnetic interactions on the basis of perturbation theory. Our purpose here is to understand the effects of strong pion interactions on the $(2\pi, 2\gamma)$ vertex.[1]

Attempts have been made in recent years to understand strong pion interactions at low energies by using the Mandelstam representation.[2,3] In particular, a P-wave pion-pion resonance has been conjectured in connection with the nucleon electromagnetic structure.[4] If such a resonance exists, one might expect its effects to be appreciable in Compton scattering on pions (e.g., $\gamma + \pi \rightarrow \gamma + \pi$). One may recall in this connection Compton scattering on protons (e.g., $\gamma + p \rightarrow \gamma + p$), where the 3-3 resonance causes a large increase in the cross section above the value given by the Klein-Nishina type formula.[5] Pion-pion forces may also be manifested in the final-state interactions of pion pairs produced by photons (e.g. $\gamma + \gamma \rightarrow \pi + \pi$). Such final-state interactions, if they are substantial, may be observed experimentally by producing a pion pair from a high-energy photon in the Coulomb field of a nucleus.

Further, an understanding of the $(2\pi, 2\gamma)$ vertex is a prerequisite for a theory of nucleon-photon scattering and in fact for most problems where a vertex connecting strongly interacting particles with two photons is involved. For example, in the calculation of the electromagnetic mass of charged pions, one needs the pion Compton scattering amplitude for virtual photons. The information obtained here may, therefore, be helpful in understanding the mass difference between charged and neutral pions.

We shall investigate the $(2\pi, 2\gamma)$ problem within the framework of double-dispersion relations proposed by Mandelstam.[2] We do not think it pertinent to go into the principles and conjectures underlying the Mandelstam representation, since we have nothing new to contribute to these general questions, which have been the subject of so many papers. Following the effective-range approximation given by Chew and Mandelstam we assume the behavior of the amplitudes to be dominated by nearby singularities.[3] Moreover, the contribution of intermediate states containing one or more photons will be neglected since, even-though they correspond to near singularities, powers higher than e^2 are involved.

In the next section, we shall go into the kinematics of the problem and show that because of Lorentz and gauge invariance only two invariant amplitudes are involved. The Mandelstam representation for these amplitudes is then written in Section III, and the question of subtractions discussed. In Section IV, the helicity amplitudes of Jacob and Wick are introduced.[6] In Section V, we consider Compton scattering, $\gamma + \pi \rightarrow \gamma + \pi$, and discuss the effect of the $\pi\pi$ interactions. In Section VI, pion-pair production, $\gamma + \gamma \rightarrow \pi + \pi$, is considered and the effect of final-state $\pi\pi$ S-wave interactions discussed. In the Appendix we give the calculations needed to obtain $\pi\pi$ S phase shifts.

One of our main results is negative and very surprising, in view of the large enhancement of nucleon Compton scattering by the 33 resonance.[5] We find that the effect of the 2π P resonance on pion Compton scattering is negligibly small. The important matrix element here is that for $\gamma + \pi \to \pi + \pi$ and has been estimated by Wong on the basis of the π^0 lifetime, where this amplitude also plays a role.[7] Wong's estimate, confirmed in order of magnitude by Ball in connection with photopion production from nucleons,[8] is smaller by about a factor of 10 than one might naively guess. Since this matrix element appears squared in the Compton amplitude, the 2π resonance turns out to make a contribution only of the order of 1%. In Section V, we shall discuss the probable reason for the smallness of Wong's amplitude. We do not here consider a 3π bound state or resonance, which may play a large role in pion Compton scattering.

In the $\gamma + \gamma \to \pi + \pi$ channel only even angular-momentum states are involved because of charge-conjugation invariance. By a reasonable choice of $\pi\pi$ S-phase shifts, we find in Section IV that the contribution of the final-state interaction is large. For the $I = 0$ state, where the interaction is strongest, the contribution at low energies is found to be positive corresponding to attraction and is of the order of 100% of the Born amplitude at threshold. As the energy is increased, however, it quickly changes sign. Such a circumstance corresponds to the fact that the pions are produced with a large relative separation (~ one pion Compton wave length) and have, therefore, a fairly small probability of interacting with each other.

II. KINEMATICS AND INVARIANCE CONSIDERATIONS

Figure 1 describes the $(2\pi, 2\gamma)$ vertex under consideration, where the wavy lines indicate photons and solid lines indicate pions. For the sake of symmetry, we shall take all the lines as incoming. Let p_1, p_2 be the four momenta of the pions and α, β the corresponding charge indices, while k_1, k_2 are the four momenta of the photons and e_1, e_2 the corresponding polarization vectors. We then define the three Lorentz invariants s, $\bar{s}$, and t as follows:

$$s = (k_1 + p_1)^2 = (k_2 + p_2)^2 \tag{2.1a}$$

$$\bar{s} = (k_1 + p_2)^2 = (k_2 + p_1)^2 \tag{2.1b}$$

$$t = (k_1 + k_2)^2 = (p_1 + p_2)^2 . \tag{2.1c}$$

From energy-momentum conservation, we have

$$s + \bar{s} + t = 2.$$

Notice that s, $\bar{s}$, and t are the squares of the energies of the following three reactions in the barycentric system:

$$k_1 + p_1 \rightarrow -k_2 - p_2 \qquad (\gamma + \pi \rightarrow \gamma + \pi) \tag{2.2a}$$

$$k_1 + p_2 \rightarrow -k_2 - p_1 \qquad (\gamma + \pi \rightarrow \gamma + \pi) \tag{2.2b}$$

$$k_1 + k_2 \rightarrow -p_1 - p_2 \qquad (\gamma + \gamma \rightarrow \pi + \pi) . \tag{2.2c}$$

The S matrix is defined as

$$S_{fi} = \delta_{fi} - i\,(2\pi)^4\,(16\,\omega_{k_1}\,\omega_{p_1}\,\omega_{k_2}\,\omega_{p_2})^{-1/2}\,\delta(k_1 + p_1 + k_2 + p_2)T_{fi},$$

where f and i indicate final and initial states, respectively, and the ω's indicate the energies of the different particles. For the given charge indices α and β we have for the T matrix

$$T_{\alpha\beta} = (\delta_{\alpha\beta} - \delta_{\alpha 3}\,\delta_{\beta 3})\,T^c + \delta_{\alpha 3}\,\delta_{\beta 3}\,T^n$$

where T^c and T^n denote the T matrices corresponding to charged and neutral pions, respectively. Henceforth we shall suppress the charged and neutral indices. We shall concentrate our attention mainly on the charged case and only comment on any alterations needed in the neutral case.

We may further write

$$T = e_{2\mu}\,T^{\mu\nu}\,e_{1\nu},$$

where $T^{\mu\nu}$ is a tensor of second rank which can be expressed in the most general form as:

$$T^{\mu\nu} = A\,k_1^{\mu}\,k_2^{\nu} + B\,\Delta^{\mu}\,k_2^{\nu} + C\,k_2^{\mu}\,k_2^{\nu} + D\,k_1^{\mu}\,k_1^{\nu} + E\,\Delta^{\mu}\,k_1^{\nu}$$

$$+ F\,k_2^{\mu}\,k_1^{\nu} + G\,k_1^{\mu}\,\Delta^{\nu} + H\,\Delta^{\mu}\,\Delta^{\nu} + I\,k_2^{\mu}\,\Delta^{\nu} + J\,g^{\mu\nu},$$

where $\Delta = p_1 - p_2$, and $g^{\mu\nu}$ is the conventional metric tensor.[1] The amplitudes $A \cdots J$ are functions of the invariants s, $\bar{s}$, and t. Gauge invariance requires that (a) $k_{2\mu}\,T^{\mu\nu} = 0$ and (b) $T^{\mu\nu}\,k_{1\nu} = 0$.

With the above conditions and the requirement of zero photon mass, $k_1^2 = 0 = k_2^2$, we obtain

$$T(s,\bar{s},t) = (e_2\cdot k_1\ e_1\cdot k_2 - k_2\cdot k_1\ e_2\cdot e_1)\ A(s,\bar{s},t)$$

$$+ \left(-e_1\cdot e_2\ k_2\cdot\Delta + \frac{k_2\cdot k_1}{k_2\cdot\Delta}\ e_2\cdot\Delta\ e_1\cdot\Delta + e_2\cdot\Delta\ e_1\cdot k_2 - e_2\cdot k_1\ e_1\cdot\Delta\right)$$

$$\times\ B(s,\bar{s},t). \qquad (2.3)$$

Crossing symmetry requires

$$A(s,\bar{s},t) = A(\bar{s},s,t), \text{ and } B(s,\bar{s},t) = -B(\bar{s},s,t). \qquad (2.4)$$

The foregoing results have been obtained independently by Gourdin and Martin.[9]

III. MANDELSTAM REPRESENTATION

The Mandelstam representation for A and B can be written for charged pions as

$$A(s,\bar{s},t) = \frac{4\pi e^2}{1-s} + \frac{4\pi e^2}{1-\bar{s}}$$

$$+ \frac{1}{\pi^2}\int_4^\infty ds' \int_4^\infty dt' \, \frac{\alpha_1(s',t')}{t'-t}\left(\frac{1}{s'-s} + \frac{1}{s'-\bar{s}}\right)$$

$$+ \frac{1}{\pi^2}\int_4^\infty ds' \int_4^\infty d\bar{s}' \, \frac{\alpha_2(s',\bar{s}')}{(s'-s)(\bar{s}'-\bar{s})} \qquad (3.1)$$

$$B(s,\bar{s},t) = \frac{4\pi e^2}{1-s} - \frac{4\pi e^2}{1-\bar{s}}$$

$$+ \frac{1}{\pi^2}\int_4^\infty ds' \int_4^\infty dt' \, \frac{\beta_1(s',t')}{t'-t}\left(\frac{1}{s'-s} - \frac{1}{s'-\bar{s}}\right)$$

$$+ \frac{1}{\pi^2}\int_4^\infty ds' \int_4^\infty d\bar{s}' \, \frac{\beta_2(s',\bar{s}')}{(s'-s)(\bar{s}'-\bar{s})} \, . \qquad (3.2)$$

Here α_1, α_2, β_1, and β_2 are the double spectral functions. Notice that the crossing condition (2.4) is explicitly contained in Eqs. (3.1) and (3.2) for $\alpha_2(s,\bar{s}) = \alpha_2(\bar{s},s)$ and $\beta_2(s,\bar{s}) = -\beta_2(\bar{s},s)$.

The poles at $s = 1$ and $\bar{s} = 1$ correspond to single-pion intermediate states in reactions (2.2a and b) respectively. The lower limits on the above integrals correspond to the fact that the least massive intermediate states in the three channels given in reactions (2.2a, b, and c) are the two-pion states. For neutral pions, the only difference is that the poles are absent. Subtractions are perhaps necessary in the above dispersions relations and we shall discuss them later on.

The region in which the double spectral functions α_1, α_2, β_1, and β_2 are nonzero are given as follows: For both $\alpha_1(s,t)$ and $\beta_1(s,t)$ the region is defined by the curves

$$t = \frac{4(2s+1)^2}{s(s-4)} \tag{3.3a}$$

and

$$t = \frac{4(s-1)}{s-9} \quad . \tag{3.3b}$$

For $\alpha_2(s,\bar{s})$ and $\beta_2(s,\bar{s})$, the curves are

$$(s-4)(\bar{s}-16) - 81 = 0 \tag{3.4a}$$

and

$$(s-16)(\bar{s}-4) - 81 = 0. \tag{3.4b}$$

Notice that there are no anomalous thresholds involved.

By a proper choice of amplitudes, the pole terms correspond in the $\gamma + \pi \rightarrow \gamma + \pi$ channel to the Thomson amplitude, which A and B should approach in the zero-energy limit. Hence on the basis of zero-energy-limit theorems, subtractions are unnecessary. We thus differ from the observations of Gourdin and Martin,[9] who use a different set of amplitudes and are uncertain, therefore, about the number of possible subtractions. We may go farther and discuss possible subtractions on the basis of unitarity limitations on the asymptotic behavior of the A and B amplitudes. Such an analysis was first carried out by Froissart in the case of scalar particles[10] and was applied by Singh and Udgaonkar to the pion-nucleon problem.[11] We give below the results for the A and B amplitudes which are derived in Sections Vb and VIb.

For the $\gamma + \pi \to \gamma + \pi$ channel as s approaches infinity, we have

$$|A| \lesssim s, \quad |B| \lesssim \text{constant}$$

for fixed t (i.e. for $\cos\theta = 1$), (3.5a)

$$|A| \lesssim s, \quad |B| \lesssim s$$

for fixed $\bar{s}$ (i.e. for $\cos\theta = -1$), and (3.5b)

$$|A| \lesssim s^{-1/4}, \quad |B| \lesssim s^{-1/4}$$

for any other value of $\cos\theta$, (3.5c)

where θ is the scattering angle in this channel. For the $\gamma + \gamma \to \pi + \pi$ channel as t approaches infinity, we have

$$|A| \lesssim t, \quad |B| \lesssim t$$

for fixed s or $\bar{s}$ (i.e. for $\cos\phi = \pm 1$) and (3.6a)

$$|A| \lesssim t^{-1/4}, \quad |B| \lesssim t^{-1/4}$$

for any other value of $\cos\phi$, (3.6b)

where ϕ is this scattering angle in the channel. Since $\gamma + \gamma \to \pi + \bar{\pi}$ is an inelastic channel, we may assume that the A and B amplitudes do not attain their maximum values given by expression (3.6a) in the forward or backward direction. For $\cos\phi = \pm 1$ we then have

$$|A| \lesssim t^{1-\epsilon}, \quad |B| \lesssim t^{1-\epsilon}, \qquad (3.6c)$$

where ϵ is any small positive number.

From the above asymptotic conditions, we observe that no arbitrary over-all subtraction constants are allowed in the A and B amplitudes since their presence violates conditions (3.5c) and (3.6b). Thus we do not anticipate that any new parameters will appear in our problem. One subtraction in t, corresponding to $j = 1$ in the $\gamma + \pi \to \gamma + \pi$ channel, is allowed for both A and B amplitudes. However, further subtractions bring in powers of t larger than or equal to unity and are incompatible with the asymptotic behavior of expression (3.6c). One subtraction in s (and $\bar{s}$) is allowed for the A amplitude, corresponding to $j = 0$ for the $\gamma + \gamma \to \pi + \pi$ channel, but subtractions for $j > 0$, where j is even, are incompatible with expression (3.5a) since they bring in powers of s (or $\bar{s}$) larger than or equal to two. For the B amplitude, the first subtraction involves $(s - \bar{s})$ and is incompatible with relation (3.5a).

IV. HELICITY AMPLITUDES

In the present problem, we shall use the helicity amplitudes given by Jacob and Wick.[6] Thus we have a simpler connection between unitarity and analyticity than when the conventional electric- and magnetic-multipole amplitudes are employed.

In a two-body collision, we denote the helicities of the initial particles by λ_a and λ_b and of the final particles by λ_c and λ_d, respectively. The corresponding scattering amplitude is given by

$$g_{\lambda_c, \lambda_d, \lambda_a, \lambda_b}(\theta) =$$

$$\frac{1}{p} \sum_j (j + \tfrac{1}{2}) \langle \lambda_c \lambda_d | T^j(E) | \lambda_a \lambda_b \rangle \, d_{\lambda\mu}^{\;j}(\theta) , \qquad (4.1)$$

while the differential cross section is

$$\frac{d\sigma}{d\Omega} = | g_{\lambda_c, \lambda_d, \lambda_a, \lambda_b}(\theta) |^2 . \qquad (4.2)$$

Here we have $\lambda = \lambda_a - \lambda_b$ and $\mu = \lambda_c - \lambda_d$; j is the total angular momentum, p, E, and Θ are the barycentric momentum, energy, and scattering angle, respectively; $\langle \lambda_c \lambda_d | T^j(E) | \lambda_a \lambda_b \rangle$ is the corresponding T matrix; and $d_{\lambda\mu}^{\;j}(\theta)$ is the function given by Jacob and Wick.[6]

In the $(2\pi, 2\gamma)$ problem, the pions have zero spin, and therefore zero helicity while the photons have helicity $+1$ or -1 depending on whether they are right or left circularly polarized.

V. COMPTON SCATTERING CHANNEL

In the barycentric system, we can write

$$k_1 = (k, \vec{k}_1),\ k_2 = (-k, \vec{k}_2);\ p_1 = (\sqrt{k^2+1}, -\vec{k}_1),\ p_2 = (-\sqrt{k^2+1}, -\vec{k}_2);$$

and $\vec{k}_1 \cdot \vec{k}_2 = -k^2 \cos\theta$, where θ is the scattering angle and we define

$$s = (k + \sqrt{k^2+1})^2\,, \tag{5.1a}$$

$$t = -2k^2\,(1 - \cos\theta), \tag{5.1b}$$

and

$$\bar{s} = (-k + \sqrt{k^2+1})^2 - 2k^2\,(1 + \cos\theta). \tag{5.1c}$$

Here s is the square of the barycentric energy, and t the square of the corresponding momentum transfer. The differential cross section is

$$\frac{d\sigma}{d\Omega} = \left| \frac{1}{8\pi} \frac{T}{\sqrt{s}} \right|^2, \tag{5.2}$$

where T is the T matrix defined in Eq. (2.3).

A. Helicity Amplitudes

Here we have $\lambda_a = 0 = \lambda_d$ and therefore $\lambda_a = \lambda$, $\lambda_c = \mu$, with the λ and μ values being ± 1. If we denote the helicity amplitude by $f_{\mu\lambda}(\theta)$, we have

$$f_{\mu\lambda}(\theta) = \frac{1}{k} \sum_{j=1} (j + \tfrac{1}{2})\, T_{\mu\lambda}{}^{j}(s)\, d_{\lambda\mu}{}^{j}(\theta) \tag{5.3}$$

and

$$\frac{d\sigma}{d\Omega} = |\, f_{\mu\lambda}(\theta)\, |^2. \tag{5.4}$$

If we denote λ and μ indices by $\pm$, we have

$$T_{++}^{\;j}(s) = T_{--}^{\;j}(s),$$

and

$$T_{+-}^{\;j}(s) = T_{-+}^{\;j}(s).$$

Using Eq. (5.2) with appropriate values for the polarization vectors e_1, and e_2 and comparing it with Eq. (5.4), we obtain

$$a(s,\bar{s},t) = \frac{B(s,\bar{s},t)}{s-\bar{s}} = \frac{8\pi k}{s\bar{s}-1}\;\frac{s}{s-1}\; f_{++}(\theta) \tag{5.5a}$$

and

$$b(s,\bar{s},t) = \frac{1}{4}\left[A(s,\bar{s},t) + \frac{4-t}{s-\bar{s}} B(s,\bar{s},t)\right] = \frac{8\pi k}{t}\;\frac{s}{s-1}\; f_{+-}(\theta) \tag{5.5b}$$

where

$$f_{++}(\theta) = \frac{1}{k}\sum_{j=1}^{\infty}\left(j+\frac{1}{2}\right) T_{++}^{\;j}(s)\, d_{1,1}^{\;j}(\theta) \tag{5.6a}$$

and

$$f_{+-}(\theta) = \frac{1}{k}\sum_{j=1}^{\infty}\left(j+\frac{1}{2}\right) T_{+-}^{\;j}(s)\, d_{1,-1}^{\;j}(\theta). \tag{5.6b}$$

Thus we have

$$a(s,\bar{s},t) = \frac{B(s,\bar{s},t)}{s-\bar{s}} = \frac{4\pi s}{s-1}\sum_{j}(2j+1)\, T_{++}^{\;j}(s)\,\frac{d_{1,1}^{\;j}(\theta)}{s\bar{s}-1} \tag{5.7a}$$

and

$$b(s,\bar{s},t) = \frac{1}{4}\left[A(s,\bar{s},t) + \frac{4-t}{s-\bar{s}} B(s,\bar{s},t)\right]$$

$$= \frac{4\pi s}{s-1}\sum_{j}(2j+1)\, T_{+-}^{\;j}(s)\,\frac{d_{1,-1}^{\;j}(\theta)}{t}. \tag{5.7b}$$

The $d^j(\theta)$ functions are given by Jacob and Wick as

$$d_{1,1}{}^{j}(\theta) = \frac{P'_j(\cos\theta) - P'_{j-1}(\cos\theta) + j^2 P_j(\cos\theta)}{j(j+1)} \qquad (5.8a)$$

and

$$d_{1,-1}{}^{j}(\theta) = \frac{P'_j(\cos\theta) + P'_{j-1}(\cos\theta) - j^2 P_j(\cos\theta)}{j(j+1)}, \qquad (5.8b)$$

where the primes indicate derivatives with respect to $\cos\theta$. In Eqs. (5.7a and b) we have $s\bar{s} - 1$ and t in the denominators, and therefore we can use

$$\frac{d_{1,1}{}^{j}(\theta)}{1+\cos\theta} = \frac{P''_{j-1}(\cos\theta) - P''_j(\cos\theta) + j\,P'_j(\cos\theta)}{j(j+1)} \qquad (5.9a)$$

and

$$\frac{d_{1,-1}{}^{j}(\theta)}{1-\cos\theta} = \frac{P''_{j-1}(\cos\theta) + P''_j(\cos\theta) + j\,P'_j(\cos\theta)}{j(j+1)}. \qquad (5.9b)$$

B. Asymptotic Behavior

Unitarity demands that

$$|T_{++}{}^{j}(s)| \le 1 \qquad (5.10a)$$

and

$$|T_{+-}{}^{j}(s)| \le 1 . \qquad (5.10b)$$

Further, the Legendre functions and their derivatives satisfy the following relations:

$$P_j(1) = 1, \quad P'_j(1) = \frac{j(j+1)}{2} \tag{5.11a}$$

$$P''_j(1) = \frac{(j-1)\, j(j+1)(j+2)}{8} \, . \tag{5.11b}$$

For $\cos\theta = -1$, we use the relation

$$P_j(-\cos\theta) = (-1)^j \, P_j\,(\cos\theta).$$

For $\cos\theta \neq \pm 1$, we have for large values of j

$$P_j\,(\cos\theta) = \frac{h_0(\theta)}{\sqrt{j}} \, , \tag{5.12a}$$

$$P'_j(\cos\theta) = \sqrt{j}\, h_1\,(\theta), \tag{5.12b}$$

and

$$P''_j(\cos\theta) = j\sqrt{j}\, h_2\,(\theta). \tag{5.12c}$$

where $h_0(\theta)$, $h_1(\theta)$, and $h_2(\theta)$ are functions of θ only.

For the a and b amplitudes given in Eqs. (5.7a and b), if we keep t fixed and let s approach infinity, then, since $\cos\theta$ approaches 1, we have from Eqs. (5.9), (5.10), and (5.11)

$$|a| \lesssim \frac{1}{s^2} \; \Sigma\,(j + \tfrac{1}{2}) \rightarrow \frac{1}{s^2}\; {j_{max}}^2 = \frac{1}{s^2}\;(kR)^2 \rightarrow \frac{1}{s} \tag{5.13a}$$

and

$$|b| \lesssim \frac{1}{s} \quad \Sigma(j + \tfrac{1}{2})\, j^2 \rightarrow \frac{1}{s} \; {j_{max}}^4 = \frac{1}{s}\,(kR)^4 \rightarrow s, \tag{5.13b}$$

where R is the interaction radius in the sense of Froissart's analysis[10] and is essentially a constant. Similarly, if we keep $\bar{s}$ fixed and let s approach infinity, then, since $\cos\theta$ approaches -1, we have

$$|a| \lesssim \text{constant} \tag{5.14a}$$

and

$$|b| \lesssim \text{constant.} \tag{5.14b}$$

For $\cos\theta \neq \pm 1$ and $s \to \infty$, we have from Eqs. (5.12a, b, and c)

$$|a| \lesssim \frac{1}{s^2} \Sigma(j + \tfrac{1}{2}) \frac{1}{\sqrt{j}} \to s^{-5/4} \tag{5.15a}$$

and

$$|b| \lesssim \frac{1}{s} \Sigma(j + \tfrac{1}{2}) \frac{1}{\sqrt{j}} \to s^{-1/4} . \tag{5.15b}$$

From these asymptotic conditions for the a and b amplitudes, we have for the A and B amplitudes as s approaches infinity

$$|A| \lesssim s, \quad |B| \lesssim \text{constant} \tag{5.16a}$$

for t fixed, i.e., $\cos\theta = 1$,

$$|A| \lesssim s, \quad |B| \lesssim s \tag{5.16b}$$

for $\bar{s}$ fixed, i.e., $\cos\theta = -1$, and

$$|A| \lesssim s^{+1/4}, \quad |B| \lesssim s^{-1/4} \tag{5.16c}$$

for $\cos\theta \neq \pm 1$.

C. Fixed Momentum-Transfer Dispersion Relations

In Eqs. (5.7a and b) we notice that since B is an odd function of $s - \bar{s}$, no new singularities are introduced in the a and b amplitudes. Moreover, we have $d_{1,1}^{j}(\pi) = 0$ and $d^{j}_{1,-1}(0) = 0$, corresponding to the vanishing of the forward helicity-flip and backward nonhelicity-flip amplitudes. However, these zeroes are absent in the a and b amplitudes because of the presence of the factors $s\bar{s} - 1$ and t in the denominators in (5.7a and b). The a and b amplitudes have the further property that each is expressed in terms of a given type of helicity amplitude.

We shall now proceed to write dispersion relations for the a and b amplitudes rather than the A and B amplitudes because of their simple properties given above. We shall not, however, use the Mandelstam representation in its full generality, but only the part of it obtained by keeping t (the square of the momentum transfer) fixed. In order to derive maximum benefit from the Mandelstam representation, i.e., in order to use information about the singularities of the scattering amplitude in all variables, we write down partial-wave dispersion relations. If we do so in the Compton scattering channel, the total amplitude for $\gamma + \gamma \rightarrow \pi + \pi$ is explicitly involved, corresponding to the cut $t \geq 4$. For the fixed momentum-transfer dispersion relations, however, because of crossing symmetry, only the absorptive part of the $\gamma + \pi \rightarrow \gamma + \pi$ amplitude is involved except for the $J = 0$ amplitude for the $\gamma + \gamma \rightarrow \pi + \pi$ channel. By making proper subtractions (see Section Vb and VIb), we then have, for fixed t,

$$a(s,t) = \frac{4\pi e^2}{(1-s)(1-\bar{s})} + \frac{1}{\pi} \int_4^\infty ds' \, a_1(s',t) \left(\frac{1}{s'-s} + \frac{1}{s'-\bar{s}} \right) \tag{5.17a}$$

and

$$b(s,t) = \frac{4\pi e^2}{(1-s)(1-\bar{s})} + 4\pi \, C_+^{\,0}(t)$$

$$+ \frac{1}{\pi} \int_4^\infty ds' \, b_1(s',t) \left[\frac{1}{s'-s} + \frac{1}{s'-\bar{s}} - \frac{1}{2p_-q_-} \times \ln \left(\frac{s' + p_-^2 + q_-^2 + 2p_-q_-}{s' + p_-^2 + q_-^2 - 2p_-q_-} \right) \right], \tag{5.17b}$$

where $C_+^{\,0}(t)$ is the correction term coming from the $j = 0$, $\gamma + \gamma \rightarrow \pi + \pi$ amplitude continued to negative t values (see Section VIc) and is allowed in b but not in a by the asymptotic conditions (5.13a and b). The correction terms $C_+^{\,0,c}(t)$ and $C_+^{\,0,n}(t)$ for the charged and neutral case respectively are connected through the relation (6.10) to the correction terms $C_+^{\,0,I}(t)$ given in Eq. (6.16). In Eqs. (5.17a and b) we define $a_1(s,t)$, $b_1(s,t)$, p_-, and q_- by

$$a_1(s,t) = \mathrm{Im}\, a(s,t)$$

$$= \frac{4\pi s}{s-1} \sum_{j=1}^{\infty} (2j+1) \, \mathrm{Im}\, T_{++}^{\,j}(s) \, \frac{d_{1,1}^{\,j}(\theta)}{s\bar{s}-1}, \tag{5.18a}$$

$$b_1(s,t) = \mathrm{Im}\, b(s,t)$$

$$= \frac{4\pi s}{s-1} \sum_{j=1}^{\infty} (2j+1) \, \mathrm{Im}\, T_{+-}^{\,j}(s) \, \frac{d_{1,-1}^{\,j}(\theta)}{t}. \tag{5.18b}$$

$$p_- = i\,\frac{\sqrt{4-t}}{2}$$

and

$$q_- = i\frac{\sqrt{-t}}{2} .$$

Using the unitarity of the S matrix we can express $\mathrm{Im}\, a$ and $\mathrm{Im}\, b$ in terms of a sum of the absolute squares of the amplitudes for $\gamma + \pi \rightarrow n$, where n stands for the possible intermediate states. In this preliminary calculation, motivated by the success of the analogous approach for γp scattering,[5] we neglect the contribution of all but the 2π intermediate states. If a 3π resonance or bound state exists, its contribution may be nonneligible. However, because of insufficient information about such a state, we do not consider it in the present discussion. In the above approximation, then, a knowledge of the $\gamma + \pi \rightarrow 2\pi$ amplitude is sufficient to give $\mathrm{Im}\, a$ and $\mathrm{Im}\, b$. This amplitude has recently been studied by H. S. Wong on the basis of the Mandelstam representation.[7] Only a single invariant amplitude is involved, and only odd angular momenta need be considered. We denote the helicity amplitudes $\langle \gamma\pi | T^j(E) | \pi\pi \rangle$ for a given angular momentum j and energy E in the $\gamma + \pi \rightarrow 2\pi$ reaction by $R_{\pm}{}^{j}(s)$, where $\pm$ indicate the photon spin parallel or antiparallel to the photon's direction of motion.[6] From unitarity, we then obtain

$$\mathrm{Im}\; T_{\pm}{}^{j}(s) = \pm\; \frac{1}{2}\; |R^j(s)|^2$$

where

$$R_{+}{}^{j}(s) = -\, R_{-}{}^{j}(s) = R^j(s). \qquad (5.19)$$

The $R^j(s)$ amplitudes are connected as follows to the amplitudes $M_j(s)$ given by Wong:

$$|R^j(s)|^2 = \frac{1}{(64\pi)^2} \left[\frac{(s-1)^2(s-4)}{s} \right]^{3/2} \frac{j(j+1)}{(2j+1)^2} |M_j(s)|^2 . \tag{5.20}$$

Thus from Eqs. (5.18a and b) we obtain

$$a_1(s,t) = - \frac{1}{(32\sqrt{\pi})^2} \left[\frac{(s-4)^3}{s} \right]^{1/2} \sum_{j\ \mathrm{odd}} \frac{j(j+1)}{2j+1} |M_j(s)|^2 \frac{d_{1,1}^{\,j}(\theta)}{1+\cos\theta} \tag{5.21a}$$

and

$$b_1(s,t) = \frac{1}{(32\sqrt{\pi})^2} [s(s-4)^3]^{1/2} \sum_{j\ \mathrm{odd}} \frac{j(j+1)}{2j+1} |M_j(s)|^2 \frac{d_{1,-1}^{\,j}(\theta)}{1-\cos\theta} . \tag{5.21b}$$

In Eqs. (5.21a) and (5.21b) we retain only the $j = 1$ term and substitute the corresponding a_1 and b_1 in the dispersion integrals (5.17a and b). This seems to be a good approximation, since energies under consideration are low. Furthermore, because of the assumed P wave $\pi\pi$ resonance, the amplitude for $j = 1$ is expected to be larger than the higher waves. A similar approximation has been made in proton Compton scattering, $\gamma + p \to \gamma + p$.[5] Here only the πp intermediate state is retained, and by neglecting all but the contribution of the resonance in the $j = 3/2$ and $T = 3/2$ state (T being the isotopic spin), the results obtained are in good agreement with experiments.[5] Reintroducing the charged and neutral superscripts c and n, we have then the following relations:

$$a^c(s,t) = \frac{4\pi e^2}{(1-s)(1-\bar{s})} - \frac{1}{3(32\sqrt{\pi})^2}\,\frac{1}{\pi}$$

$$\times \int_4^\infty ds' \left[\frac{(s'-4)^3}{s'}\right]^{1/2} |M_1(s')|^2 \left(\frac{1}{s'-s} + \frac{1}{s'-\bar{s}}\right) \tag{5.22a}$$

$$b^c(s,t) = \frac{4\pi e^2}{(1-s)(1-\bar{s})} + 4\pi\, C_+^{0,c}(t) + \frac{1}{3(32\sqrt{\pi})^2}\,\frac{1}{\pi}$$

$$\times \int_4^\infty ds' \left[s'(s'-4)^3\right]^{1/2} |M_1(s')|^2$$

$$\times \left[\frac{1}{s'-s} + \frac{1}{s'-\bar{s}} - \frac{1}{2p_- q_\mp}\, \ell n\left(\frac{s' + p_-^2 + q_-^2 + 2p_\mp q_-}{s' + p_-^2 + q_-^2 - 2p_- q_-}\right)\right] \tag{5.22b}$$

$$a^n(s,t) = -\frac{1}{3(32\sqrt{\pi})^2}\,\frac{1}{\pi}\int_4^\infty ds' \left[\frac{(s'-4)^3}{s'}\right]^{1/2}$$

$$\times |M_1(s')|^2 \left(\frac{1}{s'-s} + \frac{1}{s'-\bar{s}}\right) \tag{5.23a}$$

$$b^n(s,t) = 4\pi\, C_+^{0,n}(t) + \frac{1}{3(32\sqrt{\pi})^2}\, \frac{1}{\pi}$$

$$\times \int_4^\infty ds' \,[s'(s'-4)^3]^{1/2} |M_1(s')|^2 \,[\, \frac{1}{s'-s} + \frac{1}{s'-\bar{s}}$$

$$- \frac{1}{2p_-q_-} \,\ell n\, (\frac{s' + p_-^2 + q_-^2 + 2p_-q_-}{s' + p_-^2 + q_-^2 - 2p_-q_-})] \qquad (5.23b)$$

The $M_1(s)$ amplitude has been obtained by Wong using partial-wave dispersion relations.[7] Keeping only the contribution of the 2π $J = 1$ intermediate state, we observe that the phase of $M_1(s)$ is given by the phase of the $\pi\pi$ P wave. By replacing the left cut involved in the partial-wave dispersion relations by a single pole at a, Wong gave the $M_1(s)$ amplitude

$$M_1(s) = \Lambda\, \frac{(1+a)\; D_1(1)}{(s+a)\; D_1(s)}\,, \qquad (5.26)$$

where Λ is a pseudoelementary constant proportional to the residue of the left-cut pole, and $D_1(s)$ is the denominator function of the P-wave $\pi\pi$ system which is necessary to give $M_1(s)$ the required phase. The position a is given by the behavior of the P wave, and is larger for higher values of the P-wave resonance energy. The constant Λ is estimated by Wong on the basis of the π^0 lifetime, where it plays a role.[7] For a π^0 lifetime of $\sim 4 \times 10^{-16}$ sec, he estimates Λ to be $\sim e$. With the Frazer-Fulco value for the P wave $\pi\pi$ resonance position $s_R \simeq 10$ and width $\Gamma = 0.4$ Wong found $a \simeq 5.7$. When these estimates of Λ and a are inserted into the dispersion integrals in Eqs. (5.22a and b) for charged pions, we find by an exact calculation that their contribution is $\lesssim 1\%$. Near

$s \simeq s_R$, we of course expect the imaginary parts of a and b to be important. For any s value, we have from Eq. (5.26)

$$I_a(s) = \text{Im}\, a^c(s,t) = \text{Im}\, a^n(s,t)$$

$$= - \frac{\Lambda^2}{3(32\sqrt{\pi})^2} \left(\frac{1+a}{s+a} \right)^2 \left[\frac{(s-4)^3}{s} \right]^{1/2} \left| \frac{D_1(1)}{D_1(s)} \right|^2 \tag{5.27a}$$

and

$$I_b(s) = \text{Im}\, b^c(s,t) = \text{Im}\, b^n(s,t)$$

$$= \frac{\Lambda^2}{3(32\sqrt{\pi})^2} \left(\frac{1+a}{s+a} \right)^2 [s(s-4)^3]^{1/2} \left| \frac{D_1(1)}{D_1(s)} \right|^2 . \tag{5.27b}$$

The ratios $[I_a(s)/B_f(s)]^2$ and $[I_b(s)/B_f(s)]^2$ are given in Table I, where $B_f(s)$ is the minimum value of the Born term in Eqs. (5.22a and b) attained in the forward direction. We observe that the above ratios are not greater than ~1% near the resonance energy $s_R \simeq 10$. We have, so far discussed the resonance contribution only for $s_R \simeq 10$, but for a higher s_R value $\simeq 20$ (see the Appendix), the situation will not qualitatively change.

The biggest correction to the Born amplitude seems to come from the $C_+^{0,c}(t)$ term and is roughly of the order ~ 10% if we take $\lambda = -0.20$ (see Section VIc). The ratios of the differential cross section $d\sigma/d\Omega$ to $(d\sigma/d\Omega)_B$ is given in Table II for $\theta = 90$ deg and $\theta = 180$ deg, where $(d\sigma/d\Omega)_B$ is the differential cross section obtained by keeping only the Born term. For $\theta = 0$ deg, the b amplitude is

Table I. Values of $(I_a(s)/B_f(s))^2$ and $(I_b(s)/B_f(s))^2$ for $s_R = 10$ and $\Gamma = 0.4$.

s	$(I_a(s)/B_f(s))^2$	$(I_b(s)/B_f(s))^2$
5	2×10^{-9}	6×10^{-8}
10	5×10^{-5}	5×10^{-3}
15	4×10^{-5}	9×10^{-3}
20	10^{-5}	4×10^{-3}

Table II. Values of $\left(\frac{d\sigma^c}{d\Omega}/\frac{d\sigma_B{}^c}{d\Omega}\right)$ at $\theta = 90$ and 180 deg.

s	θ = 90 deg	θ = 180 deg
5	1.06	0.93
10	1.10	0.89
15	1.12	0.88
20	1.13	0.87
25	1.14	0.86

absent, and hence the contribution to $d\sigma/d\Omega$ comes entirely from the Born term. For the neutral case, of course, the contribution of $C_+^{0,n}(t)$ is the only important one. If the correction term for the $I = 2$, $j = 0$, $\gamma + \gamma \rightarrow \pi + \pi$ amplitude is neglected (see Section VIc and the Appendix) we have from Eq. (6.10)

$$C_+^{0,n}(t) \simeq C_+^{0,c}(t).$$

The above results are in great contrast to the results in proton Compton scattering where, as described earlier, the 3-3 resonance in the intermediate pion-nucleon system increases substantially the cross section coming from the Born term.[5] The reason for the negligible contribution of the $\pi\pi$ resonance is, of course, the smallness of the $\gamma + \pi \rightarrow 2\pi$ amplitude as is seen from the factor $1/3(32\sqrt{\pi})^2$ in front of the integrals in 5.22a,b. The normalization of the constant Λ introduced by Wong is evidently misleading, since $\Lambda \simeq e$ suggests a substantial magnitude for the $\gamma + \pi \rightarrow 2\pi$ amplitude. Numerical factors should be absorbed in Λ so as to make it appear small compared to e. The reason Λ should be small is probably associated with the minimal character of the electromagnetic interactions which appear in $\gamma + \pi \rightarrow 2\pi$. This amplitude is essentially the vertex joining a single photon to three pions. Now from minimality we know that a photon line can couple directly only with a charged pair, and then the coupling constant is e, the elementary charge. In the case under consideration, therefore, we need a two-particle intermediate state. A two-pion intermediate state or in fact, any state containing an even number of pions is, however, forbidden because G-conjugation does not allow an even number of pions to go into an odd number. Thus particles heavier than pions (e.g. kaons or nucleon-antinucleon pairs) must be created in intermediate states. The constant Λ should then be of the order e/M (where M is the nucleon or kaon mass) and therefore be small. In Compton scattering, the contribution of the 2π intermediate state is proportional to Λ^2 and thus to $1/M^2$.

VI. PION-PAIR PRODUCTION CHANNEL

In the barycentric system we can write, $k_1 = (q, -\vec{q})$, $k_2 = (q, \vec{q})$, $p_1 = (-q, \vec{p})$, and $p_2 = (-q, -\vec{p})$, where

$$t = 4q^2 = 4(p^2+1), \tag{6.1a}$$

$$s = -q^2 - p^2 + 2qp \cos \phi, \tag{6.1b}$$

and

$$\bar{s} = -q^2 - p^2 - 2qp \cos \phi . \tag{6.1c}$$

Here ϕ is the scattering angle, t is the square of the barycentric energy and s, the square of the momentum transfer. The differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{p}{q} \left| \frac{1}{8\pi} \frac{T}{\sqrt{t}} \right|^2 . \tag{6.2}$$

A. Helicity Amplitudes

If λ_a, λ_b are the helicities of the photons and λ_c, λ_d those of the pions, we have $\lambda_c = 0 = \lambda_d$, and $\lambda = \lambda_a - \lambda_b$. If we denote the helicity amplitudes by $F_{\lambda 0}(\phi)$, we have

$$F_{\lambda 0}(\phi) = \frac{1}{q} \sum_j \left(j + \frac{1}{2}\right) M_{\lambda 0}^{\,j}(\phi)\, d_{\lambda 0}^{\,j}(\phi) \tag{6.3}$$

and

$$\frac{d\sigma}{d\Omega} = \left| F_{\lambda 0}(\phi) \right|^2 . \tag{6.4}$$

If by $M_{+\pm}^{\,j}(t)$ we denote $M_{\lambda 0}^{\,j}(t)$ with $\lambda_a = 1$ and $\lambda_b = 1$

(i.e. $\lambda = 0$) and $\lambda_a = 1$ and $\lambda_b = -1$ (i.e. $\lambda = 2$), respectively, then we have

$$M_{++}{}^{j}(t) = M_{--}{}^{j}(t)$$

and

$$M_{+-}{}^{j}(t) = M_{-+}{}^{j}(t) \ .$$

As in Section Va by comparing Eqs. (6.1) and (6.3) for appropriate polarization vectors, we have

$$a(s,\bar{s},t) = 4\pi\sqrt{\frac{t}{t-4}} \sum_{j=2}^{\infty} (2j+1)\, M_{+-}{}^{j}(t) \frac{d_{2,0}{}^{j}(\phi)}{s\bar{s}-1}$$

$$= 4\pi \sum_{j=2}^{\infty} (2j+1)\, h_{-}{}^{j}(t) \frac{d_{2,0}{}^{j}(\phi)}{1-\cos^2\phi} \qquad (6.5a)$$

and

$$b(s,\bar{s},t) = 4\pi\sqrt{\frac{t}{t-4}} \sum_{j=0}^{\infty} (2j+1)\, M_{++}{}^{j}(t) \frac{d_{0,0}{}^{j}(\phi)}{t}$$

$$= 4\pi \sum_{j=0}^{\infty} (2j+1)\, h_{+}{}^{j}(t)\, d_{0,0}{}^{j}(\phi) \qquad (6.5b)$$

where

$$h_{-}{}^{j}(t) = \frac{4}{t}\sqrt{\frac{t}{(t-4)^3}}\; M_{+-}{}^{j}(t) \qquad (6.6a)$$

and,

$$h_+^j(t) = \frac{1}{t}\sqrt{\frac{t}{t-4}}\; M_{++}^j(t) . \tag{6.6b}$$

In terms of a and b we have

$$h_-^j(t) = \frac{1}{8\pi} \int_{-1}^{1} d\cos\phi \; \bar{d}_{2,0}^{\,j}(\phi)\, a(t, \cos\phi)$$

and

$$h_+^j(t) = \frac{1}{8\pi} \int_{-1}^{1} d\cos\phi \; d_{0,0}^{\,j}(\phi)\, b(t, \cos\phi)$$

where

$$\bar{d}_{2,0}^{\,j}(\phi) = (1 - \cos^2\phi)\, d_{2,0}^{\,j}(\phi) .$$

The $d^j(\phi)$ functions are

$$d_{2,0}^{\,j}(\phi) = \frac{2P'_{j-1}(\cos\phi) - j(j-1)\, P_j(\cos\phi)}{\sqrt{(j-1)\, j(j+1)(j+2)}} . \tag{6.7}$$

and

$$d_{0,0}^{\,j}(\phi) = P_j(\cos\phi), \tag{6.8}$$

where primes indicate derivatives with respect to $\cos\phi$. In Eq. (6.4), we have $(1 - \cos^2\phi)$ in the denominator, and we find

$$\frac{d_{2,0}^{\,j}(\phi)}{1 - \cos^2\phi} = \frac{P''_j(\cos\phi)}{\sqrt{(j-1)\, j(j+1)(j+2)}} .$$

B. Asymptotic Behavior

The general procedure for establishing asymptotic behavior is the same as in Section Vb. Relations (5.11) and (5.12) together with the unitarity limitations

$$|M_{+-}{}^{j}(t)| \leq 1 \qquad (6.9a)$$

and

$$|M_{++}{}^{j}(t)| \leq 1 \qquad (6.9b)$$

then give us for $t \to \infty$

$$|a| \lesssim \text{constant}, \qquad |b| \lesssim \text{constant}$$

for fixed s(or $\bar{s}$), i.e. $\cos \phi = \pm 1$ and

$$|a| \lesssim t^{-5/4}, \qquad |b| \lesssim t^{-1/4}$$

for any other value of $\cos \phi$.

Equivalently, we have

$$|A| \lesssim t, \qquad |B| \lesssim t$$

for $\cos \phi = \pm 1$ and

$$|A| \lesssim t^{-1/4}, \qquad |B| \lesssim t^{-1/4}$$

for $\cos \phi \neq \pm 1$.

C. Partial-Wave Dispersion Relations

Knowing the singularities in the amplitudes a and b, we can write down partial-wave dispersion relations for $h_{-}{}^{j}(t)$ and $h_{+}{}^{j}(t)$. The branch cuts in a and b are, of course, the same as those in A and B. There is a branch cut $t \geq 4$ for the amplitudes $h_{\pm}{}^{j}(t)$.

Table III. Values of $h_{B+}^{0,0}(t)$, $\mathrm{Re}C_{+}^{0,0}(t)$, and $\mathrm{Im}C_{+}^{0,0}(t)$ for $\lambda = -0.20$, in units of e^2.

t	$h_{B+}^{0,0}(t)$	$\mathrm{Re}C_{+}^{0,0}(t)$	$\mathrm{Im}C_{+}^{0,0}(t)$
4.0	0.289	0.395	0
4.5	0.237	0.042	0.208
5.0	0.203	-0.026	0.178
6.0	0.146	-0.059	0.093
7.0	0.109	-0.056	0.056
8.0	0.086	-0.051	0.037
12.0	0.045	-0.032	0.010
16.0	0.027	-0.022	0.002

Table IV. Values of $h_{B+}^{0,2}(t)$, $ReC_{+}^{0,2}(t)$, and $ImC_{+}^{0,2}(t)$ for $\lambda = -0.20$, in units of e^2.

t	$h_{B+}^{0,2}(t)$	$ReC_{+}^{0,2}(t)$	$ImC_{+}^{0,2}(t)$
4.0	0.204	0.027	0
4.5	0.168	0.016	0.017
5.0	0.143	0.009	0.018
6.0	0.103	0.002	0.015
7.0	0.077	-0.001	0.012
8.0	0.061	-0.003	0.011
12.0	0.032	-0.004	0.006
16.0	0.019	-0.003	0.004

Corresponding to the cut $s \geq 4$ as well as $\bar{s} \geq 4$ in A and B, there will also be a cut $t < -9/4$. We shall project out the Born term and write it explicitly as $h_{B\mp}^{j}(t)$. Before we write down partial-wave dispersion relations, however, we introduce amplitudes $h_{\mp}^{j,I}$ corresponding to a definite isotopic spin I of the final two-pion state. From charge conjugation or, equivalently, from crossing symmetry, we notice that only even angular momentum states are allowed and, therefore, only states with $I = 0$ and $I = 2$ need be considered. For the $I = 0$ and $I = 2$ states, we designate the amplitudes by the superscripts 0 and 2, respectively. An elementary calculation then gives their relations with the charged-neutral amplitudes as

$$h_{+}^{j,0}(t) = \frac{1}{\sqrt{3}}\left[2h_{+}^{j,c}(t) + h_{+}^{j,n}(t)\right] \tag{6.10a}$$

and

$$h_{+}^{j,2}(t) = \sqrt{\frac{2}{3}}\left[h_{+}^{j,c}(t) - h_{+}^{j,n}(t)\right], \tag{6.10b}$$

where $h_{+}^{j,c}(t)$ and $h_{+}^{j,n}(t)$ are the charged and neutral amplitudes, respectively. Similar relations hold for $h_{-}^{j}(t)$ amplitudes. We then have

$$h_{\mp}^{j,I}(t) = h_{B\mp}^{j,I}(t) + \frac{1}{\pi}\int_{-\infty}^{-9/4} dt' \frac{\operatorname{Im} h_{\mp}^{j,I}(t')}{t' - t} + \frac{1}{\pi}\int_{4}^{\infty} dt' \frac{\operatorname{Im} h_{\mp}^{j,I}(t')}{t' - t} \tag{6.11}$$

where

$$h_{B-}^{j,I}(t) = \frac{1}{\sqrt{3}} \int_{-1}^{1} d\cos\phi \; \bar{d}_{2,0}^{\;j}(\phi) \; \frac{e^2}{(1-s)(1-\bar{s})} \tag{6.12a}$$

and

$$h_{B+}^{j,I}(t) = \frac{1}{\sqrt{6}} \int_{-1}^{1} d\cos\phi \; d_{0,0}^{\;j}(\phi) \; \frac{e^2}{(1-s)(1-\bar{s})} \, . \tag{6.12b}$$

From Eqs. (6.6) and (6.9), we have $|h_-^{\,j}(t)| \lesssim \frac{1}{t^2}$ and $|h_+^{\,j}(t)| \lesssim \frac{1}{t}$ in the physical region. Hence, in the above relations, no subtraction constants are needed. The integral along the left cut corresponds to the correction to the Born term for the crossed channel $\gamma + \pi \rightarrow \gamma + \pi$. As we have already seen, the correction is probably small and we shall neglect it in this preliminary calculation.

For the integrals in Eq. (6.11) involving positive t values greater than four, we shall use unitarity. In the approximation of including only two-pion intermediate states, we have

$$\mathrm{Im}\; h_{\mp}^{j,I}(t) = h_{\mp}^{j,I}(t)\, A^{*j,I}(t) \sqrt{\frac{t-4}{t}} \tag{6.13}$$

where

$$A^{j,I}(t) = \sqrt{\frac{t}{t-4}} \;\; e^{i\delta_j^{\,I}} \; \sin\delta_j^{\,I} \tag{6.14}$$

is the pion-pion scattering amplitude defined by Chew and Mandelstam for angular momentum j and isotopic spin I; $\delta_j^{\,I}$ being the corresponding phase shift.[3]

At low energies, we shall neglect the integrals in Eq. (6.11) along the right hand cut for $j \geq 2$, since the pion-pion amplitude for D and higher waves is expected to be small. We than have

$$h_-^{j,I}(t) = h_{B-}^{j,I}(t)$$

since always $j \geq 2$;

$$h_+^{j,I}(t) = h_{B+}^{j,I}(t)$$

for $j \geq 2$;

and

$$h_+^{0,I}(t) = h_{B+}^{0,I}(t) + \frac{1}{\pi} \int_4^\infty dt' \sqrt{\frac{t'-4}{t'}} \frac{h_+^{0,I}(t') A^{*0,I}(t')}{t'-t} . \tag{6.15}$$

Thus we consider the $\pi\pi$ interaction correction only to the $j = 0$ state (i.e. S state). Following Chew and Mandelstam, we write[3]

$$A^{0,I}(t) = \frac{N_0^I(t)}{D_0^I(t)},$$

where $N_0^I(t)$ and $D_0^I(t)$ are the numerator and denominator functions in the $\pi\pi$ S amplitudes. From a modification of the form given by Omnes,[8,12] we have

$$h_+^{0,I}(t) = h_{B+}^{0,I}(t) + \frac{1}{\pi} \frac{1}{D_0^I(t)} \int_4^\infty \frac{dt'}{t'-t} \sqrt{\frac{t'-4}{t'}}\, h_{B+}^{0,I}(t') N_0^I(t') \tag{6.16}$$

$$= h_{B+}^{0,I}(t) + C_+^{0,I}(t),$$

where $C_+^{0,I}(t)$ is the rescattering correction term due to the $\pi\pi$ interaction.

In the Appendix using crossing symmetry, we have obtained values for the S amplitudes in terms of the pion-pion coupling constant λ. At present $\lambda \sim -0.20$ seems a reasonable estimate (see the Appendix). For this λ value, we have calculated the correction $C_+^{0,I}(t)$ to the Born term $h_{B+}^{0,I}(t)$. This correction, of course comes from the final-state $\pi\pi$ interaction in the S wave. From Tables III and IV, we find that for the $I = 0$ state the correction is large at low energies corresponding to strong attraction, but for higher energies it quickly changes sign. Such a circumstance can be understood as follows: If we take the Born term $h_{B+}^{0}(t)$ to be approximately a pole at $t = -t_B$, then its slope is $\sim t_B^{-2}$. Since the distance at which the pairs are produced is $\sim t_B^{-1/2}$, the larger the distance, the faster is the decrease of $h_{B+}^{0}(t)$ in the integral in Eq. (6.16). We have here a case in which the pions are produced at a relatively large distance---about a pion Compton wave length---and, therefore, $h_{B+}^{0}(t)$ decreases relatively rapidly giving rise to a sign change, in the principal part of the integral in Eq. (6.16). At higher energies this negative contribution is $\sim 70\%$ of the Born term. The ratios $(\sigma_+^{I}(t)/\sigma_{B+}^{I}(t))$ of the total cross sections for a given I spin with and without the correction terms are given in Table V. Such interactions as discussed above may perhaps be detected by rather accurate experiments on pion-pair production by a photon in the Coulomb field of a nucleus.[13]

Table V. Values of $(\sigma_+{}^0(t)/\sigma_{B+}{}^0(t))$ and $(\sigma_+{}^2(t)/\sigma_{B+}{}^2(t))$ for $\lambda = -0.20$.

t	$\sigma_+{}^0(t)/\sigma_{B+}{}^0(t)$	$\sigma_+{}^2(t)/\sigma_{B+}{}^2(t))$
4.0	5.60	1.29
4.5	2.15	1.21
5.0	1.54	1.16
6.0	0.76	1.05
7.0	0.54	0.99
8.0	0.42	0.94
12.0	0.22	0.83
16.0	0.21	0.77

VII. ACKNOWLEDGMENTS

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APPENDIX

Evidence for a P-wave $\pi\pi$ resonance has recently been found by Anderson et al. in an experiment on peripheral $\pi^- p$ collisions;[14] the resonance position and width are in rough accord with predictions based on nucleon electromagnetic structure.[15] It now becomes possible to make certain assertions about the S-wave $\pi\pi$ phase shifts on the basis of the crossing relations developed by Chew and Mandelstam.[3,16] Recently it has been suggested by Truong[17] that the anomalous peak in the double-pion production in p + d collisions--$p + d \rightarrow He^3 + \pi^+ + \pi^-$--observed by Abashian et al.[18] may perhaps be due to the large enhancement brought about by the interaction of the S-wave pions in the $I = 0$ state, I being the isotopic spin. In this connection therefore, it is, of interest to see whether we can obtain from our solutions large $I = 0$ S-wave amplitudes.

Crossing symmetry gives relations between the derivatives of the S- and P-wave amplitudes at the symmetry point, which are exact if we consider all higher partial waves to be small.[3,16] At this symmetry point, where $\nu = \nu_0 = -2/3$ (ν being the square of the c.m. momentum of a pion), the two S amplitudes are given in terms of the pion-pion coupling constant, λ, and the first derivatives of the S amplitudes are given by the value of the P amplitude. In addition, there is a single relation connecting the second derivatives of the S waves to the first P-wave derivative. A two-parameter form for the P resonance has been given by Frazer and Fulco, the parameters being ν_R and Γ which are related to the position and the width of the resonance.[4] To fit the experiment of reference 14, we need

$\nu_R = 3.5$ and $\Gamma = 0.3$. Such a two-parameter form should be sufficient, we believe, to give a rough first approximation to the P amplitude and its first derivative at ν_0 if the contribution from the left cut is no larger than estimated by Chew and Mandelstam.[16,19] The above crossing relations then largely determine the S-wave amplitudes at low energies in terms of the three parameters λ, ν_R, and Γ.

The crossing relations at ν_0 are[16,20]

$$a_0 = \frac{5}{2} a_2 = -5\lambda \tag{A1}$$

$$a'_0 = -2a'_2 = 6a_1 \tag{A2}$$

and

$$a''_0 - \frac{5}{2} a''_2 = -12 a'_1, \tag{A3}$$

where a_0 and a_2 are the S amplitudes at ν_0 for the isotopic spin 0 and 2, respectively, and a_1 is the P amplitude. The primes indicate derivatives at ν_0. A correction for the D waves has already been made in the second-derivative relation (A3) given above.

If we indicate by $A_0^I(\nu)$ the S amplitude at an energy ν for a given isotopic spin I (= 0 or 2), we can write it in the familiar form[3]

$$A_0^I(\nu) = \frac{N_0^I(\nu)}{D_0^I(\nu)}, \tag{A4}$$

where $N_0^I(\nu)$ and $D_0^I(\nu)$ are the numerator and the denominator functions, respectively. In the approximation in which the left-hand cut is replaced by a pole, Chew and Mandelstam obtained the formulas:[16]

$$N_0^{\ I}(\nu) = a_I + (\nu - \nu_0)\ \frac{\omega_{SI} + \nu_0}{\omega_{SI} + \nu}\ B_I \qquad \text{(A5)}$$

and

$$D_0^{\ I}(\nu) = 1 - (\nu - \nu_0)\ [K(-\nu, -\nu_0)\ a_I + (\omega_{SI} + \nu_0)\ K(\omega_{SI}, -\nu) B_I], \qquad \text{(A6)}$$

where ω_{SI} gives the position of the pole, B_I is proportional to the residue, and K is a known function defined in reference 3.

The corresponding one-pole approximation for $A_1^{\ 1}(\nu)/\nu$ -- the P resonance $(I = 1)$ at an energy ν--was written in the two-parameter resonance form by Frazer and Fulco as[4]

$$\frac{A_1^{\ 1}(\nu)}{\nu} = \frac{\Gamma}{\nu_R - \nu\ [1 - \Gamma\alpha(\nu)] - i\ \Gamma\ \left(\frac{\nu^3}{\nu+1}\right)^{1/2}}, \qquad \text{(A7)}$$

where $\alpha(\nu)$ is a known function. Given ν_R and Γ, we obviously can calculate the values of a_1 and a'_1 needed in Eqs. (A2) and (A3) above. For example, we find $a_1 = 0.074$ and $a'_1 = 0.014$ for $\nu_R = 3.5$ and $\Gamma = 0.3$.[14,21] We have five conditions embodied in the crossing relations (1), (2), and (3) and six parameters to determine in our S-wave effective-range formulas: a_0, a_2, ω_{S0}, ω_{S2}, B_0, and B_2. To achieve a sixth condition, we notice from the above relations that for an a_1 value such as the one given above, the potential for the $I = 0$ state has a long-range repulsion and a short-range attraction. So long as the inner attraction is strong, we find that the interaction is not sensitive to the range of the outer repulsion. We shall, therefore, fix *a priori* the value of ω_{S0} which is proportional to the range of the repulsive potential. A reasonable estimate of $\omega_{S0} \simeq (2a_1/a'_1) - \nu_0$ is given by the approximate crossing conditions of Chew and Mandelstam.[16] Using this estimate, we obtain $\omega_{S0} \simeq 11$ for the a_1 and a'_1 values given above. For the $I = 2$ state, the outer potential is attractive and we may expect the interaction to be sensitive to the range. We shall, therefore, use relation (3) to determine ω_{S2}. It turns out that for the above values of a_1, a'_1 and ω_{S0}, no solutions exist for ω_{S2} when we have $\lambda \gtrsim +0.03$.

The curves for $a_I\,[\nu/\nu+1]^{1/2}\cot\delta_0^{\,I}$, where $\delta_0^{\,I}$ is the S-wave phase shift for a given isotopic spin I are given in Figs. 2 and 3 for various values of λ with $a_1 = 0.074$, $a'_1 = 0.014$, and $\omega_{S0} = 11$. We find that the interaction in the $I = 0$ state is attractive and much stronger than in the $I = 2$ state for both positive and negative values of λ.[22] In general, we observe that the results we obtain here are quite different from the ones in the S-dominant case.[23,24]

Knowing the results of double-pion production in $p + d$ collisions, $p + d \rightarrow He^3 + \pi^+ + \pi^-$, we can obtain additional information about the $I = 0$, S amplitude.[17,18] The enhancement factors $|D_0^{\,0}(0)/D_0^{\,0}(\nu)|^2$--normalized to unity at $\nu = 0$--for the $I = 0$ state for different values of λ are given in Table V together with the corresponding scattering lengths a_{S0}.[25] It is found that for λ in the interval (-0.15, -0.20) corresponding to the scattering length in the interval (2, 3), the enhancement factor together with the usual phase space gives a good fit to the experimental data after the $I = 1$ component of the 2π system is subtracted out.[26] We may further add that in this region of λ values, our assumption of considering the interaction to be insensitive to ω_{SO} is particularly good, since we now have a rather strong inner attraction.

On the basis of the τ-decay spectrum,[27] some authors have observed that the $I = 2$ state should be more attractive than the $I = 0$ state,[28,17] a result which is impossible to obtain within the present framework. The discrepancy may perhaps lie in the assumptions usually made in the τ-decay analysis: (a) considering the three-body problem in terms of simple two-body forces, and (b) considering only the symmetric $I = 1$ final state.

Table VI. Values of $|D_0^{\,0}(0)/D_0^{\,0}(\nu)|^2$ for I = 0 and for different λ, a_{SO} values

ν	$\lambda = -0.20$ $a_{SO} = 2.81$	$\lambda = -0.15$ $a_{SO} = 1.96$	$\lambda = -0.10$ $a_{SO} = 1.32$	$\lambda = +0.01$ $a_{SO} = 0.35$
0	1	1	1	1
0.1	0.491	0.659	0.822	1.088
0.2	0.337	0.503	0.704	1.166
0.3	0.262	0.413	0.619	1.229
0.4	0.217	0.354	0.554	1.273
0.5	0.188	0.311	0.502	1.298
0.6	0.167	0.280	0.461	1.302
0.7	0.149	0.255	0.425	1.288
0.8	0.138	0.235	0.396	1.257
0.9	0.128	0.219	0.370	1.214
1.0	0.120	0.205	0.348	1.164

FOOTNOTES

1. A preliminary account of this work was given at the 1960 Winter meeting of the American Physical Society, December 29-31, 1960 [Bipin R. Desai, Bull. Am. Phys. Soc. 5, 509 (1960)]. We employ units $\hbar = c = \mu = 1$, where μ is the pion mass. For the charge e we use the units $e^2 \sim 1/137$. The metric is defined so that we have $g^{0,0} = 1$ and $g^{ii} = -1$, where $i = 1, 2, 3$.
2. S. Mandelstam, Phys. Rev. 112, 1344 (1959); ibid. 115, 1741 and 1752 (1959).
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4. W. R. Frazer and J. R. Fulco, Phys. Rev. Letters 2, 365 (1959); Phys. Rev. 117, 1603 (1960).
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6. M. Jacob and G. C. Wick, Ann. Phys. 7, 404 (1959).
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9. M. Gourdin and A. Martin, Nuovo cimento 17, 224 (1960). The Cini-Fubini approximate version of the Mandelstam representation has been used by these authors, but no numerical estimates have been attempted.

10. Marcel Froissart, University of California, Berkeley, to be published in Phys. Rev.

11. V. Singh and B. M. Udgaonkar, Theory of π-N Scattering in the Strip Approximation to the Mandelstam Representation, UCRL-9561, February 9, 1961; submitted to Phys. Rev.

12. R. Omnes, Nuovo cimento 8, 316 (1958).

13. Yongduk Kim, Lawrence Radiation Laboratory, Berkeley, private communication.

14. J. A. Anderson, V. X. Bang, P. G. Burke, D. D. Carmony, and N. Schmitz, Phys. Rev. Letters 6, 365 (1961). The resonance energy is observed to be ~ 4.5.

15. Existence of a P-wave $\pi\pi$ resonance on the basis of nucleon electromagnetic structure was first conjectured by W. R. Frazer and J. R. Fulco (see reference 4). In their analysis, these authors attempted to represent the entire anomalous isotopic-vector magnetic moment in terms of the $\pi\pi$ P-wave resonance, and on this basis predicted a resonance energy at about 3.5. However, the actually observed nucleon form factor has a small ($\sim$ 20%) high-momentum component of a sign opposite to the 2π contribution as pointed out by J. Bowcock, W. N. Cottingham, and D. Lurié, Nuovo cimento 16, 918 (1960) and Phys. Rev. Letters 5, 386 (1960). [See also S. C. Frautschi, Phys. Rev. Letters 5, 159 (1960)]. When this effect is included, the resonance energy is correctly predicted to be ~ 4.5.

16. G. F. Chew and S. Mandelstam, Lawrence Radiation Laboratory Report UCRL-9126, March 1960 (unpublished) to be published in

Nuovo cimento. See also G. F. Chew, Proc. 1960 Annual International Conference on High Energy Physics, Rochester, 1960 (Interscience Publishing Co., New York, 1960).

17. T. N. Truong, Phys. Rev. Letters 6, 308 (1961).

18. A. Abashian, N. E. Booth, and K. M. Crowe, Phys. Rev. Letters 5, 258 (1960) and talk presented at the 1960 Conf. on Strong Interactions, University of California, Berkeley, December 1960.

19. Recently, a four-parameter resonance form has been given by J. S. Ball and D. Y. Wong, Phys. Rev. Letters 6, 29 (1961). This resonance form cannot be represented, even roughly, by the Frazer-Fulco two-parameter function. We do not consider here the Ball-Wong form, because it corresponds to a nearby discontinuity on the left cut that is an order of magnitude larger than that estimated by Chew and Mandelstam.[4] Nevertheless, as future developments improve our understanding of the P wave, the calculations described here may correspondingly be improved. This approach is in no way committed to the Frazer-Fulco form.

20. A set of relations for the S and P amplitudes at zero energy have been recently given by J. G. Taylor, Phys. Rev. Letters 6, 237 (1961). These relations are incorrect inasmuch as the contribution of the higher partial waves in the crossed channel is neglected. For example, the relation $a_0^2/a_2^2 = 5/2$ is not consistent with perturbation theory, since we expect that in the limit of weak coupling the above ratio should be $\sim(5/2)^2$, the ratio at the symmetry point, and not 5/2.

21. Notice that the position of the resonance is not ν_R as defined by the relation (7) but it is the value of ν for which the real part of the denominator in Eq. (7) vanishes and is higher than ν_R.

22. A bound state appears for the $I = 0$ state for $\lambda \simeq -0.50$.

23. G. F. Chew, S. Mandelstam, and H. P. Noyes, Phys. Rev. 119, 478 (1960).

24. Such behavior was already indicated by G. F. Chew, Proc. 1960 Annual International Conference on High Energy Physics, Rochester, 1960 (Interscience Publishing Co., New York, 1960).

25. A good approximation to the enhancement factor for small ν values is $\{[1 + a_{SI} \frac{2}{\pi}\sqrt{\frac{\nu}{\nu+1}}\, \ell n (\sqrt{\nu} + \sqrt{\nu+1})]^2 + a_{SI}^{\;2} \frac{\nu}{\nu+1}\}^{-1}$, where a_{SI} is the scattering length for the S wave with isotopic spin I.

26. A. Abashian and N. E. Booth, Lawrence Radiation Laboratory, University of California, Berkeley (private communication). I am told by these authors that the experiment is being analyzed further.

27. R. Dalitz, Phys. Rev. 94, 1046 (1954); E. Fabri, Nuovo cimento 11, 479 (1954); S. McKenna, S. Natali, M. O'Connell, J. Tietge, and N. C. Varshneya, Nuovo cimento 10, 763 (1958).

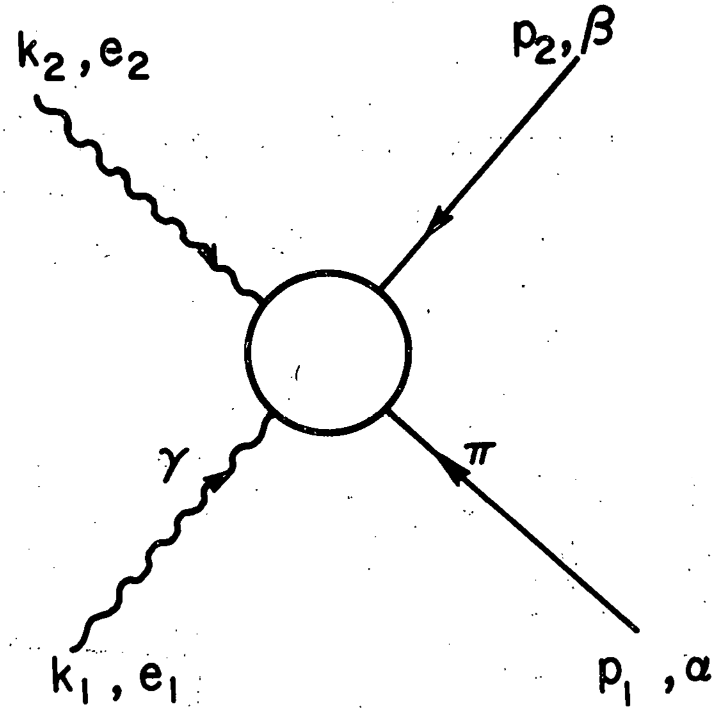
28. B. S. Thomas and W. G. Holladay, Phys. Rev. 115, 1329 (1959); N. N. Khuri and S. B. Treiman, Phys. Rev. 119, 1115 (1960); R. F. Sawyer and K. C. Wali, Phys. Rev. 119, 1429 (1960). See also A. N. Mitra and E. Lomon, Proc. 1960 Annual International Conference on High Energy Physics, Rochester, 1960 (Interscience Publishing Co., New York, 1960).

FIGURE LEGENDS

Fig. 1. The $(2\pi, 2\gamma)$ vertex.

Fig. 2. Product of the cotangent of $\delta_0^{\ 0}$ and $-5\lambda\,[\nu/\nu+1]^{1/2}$ for the λ values -0.20, -0.15, -0.10, and +0.01 with $\omega_{S0} = 11$.

Fig. 3. Product of the cotangent of $\delta_0^{\ 2}$ and $-2\lambda\,[\nu/\nu+1]^{1/2}$ for the (λ, ω_{S2}) values (-0.20, 2.2), (-0.15, 2.3), (-0.10, 2.7), and (+0.01, 6.5).



MU-23461

Fig. 1

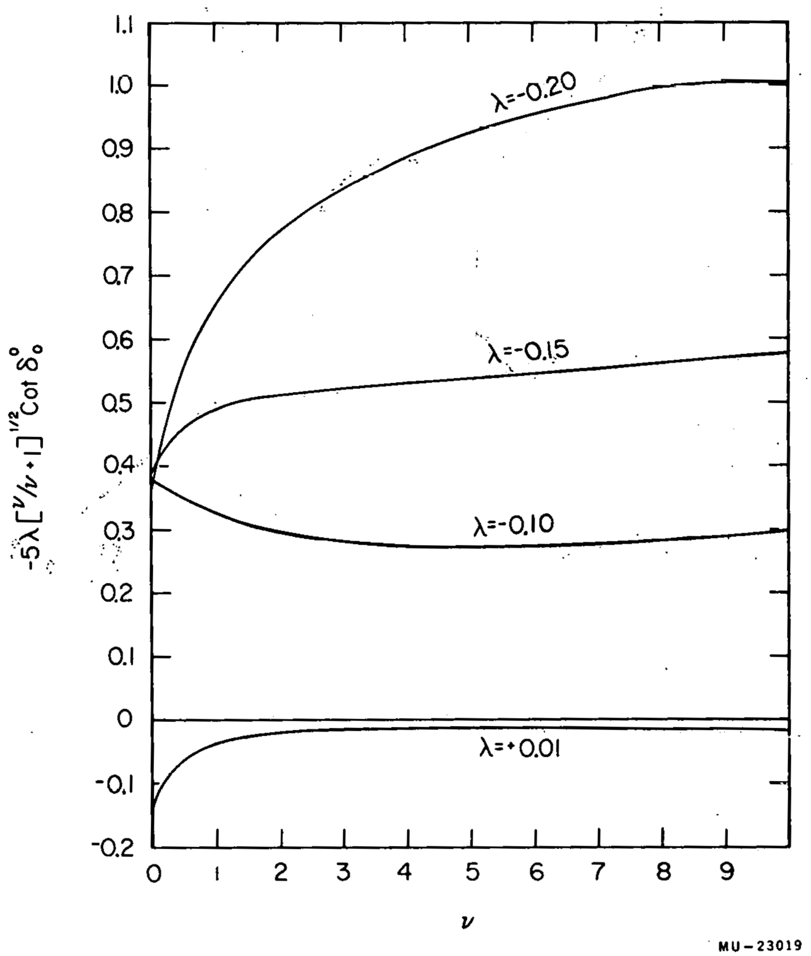

λ=-0.20
λ=-0.15
λ=-0.10
λ=+0.01
$-5\lambda\left[\nu/\nu+1\right]^{1/2}\cot\delta_0^0$
ν
1.1
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
0
1
2
3
4
5
6
7
8
9
MU-23019

Fig. 2

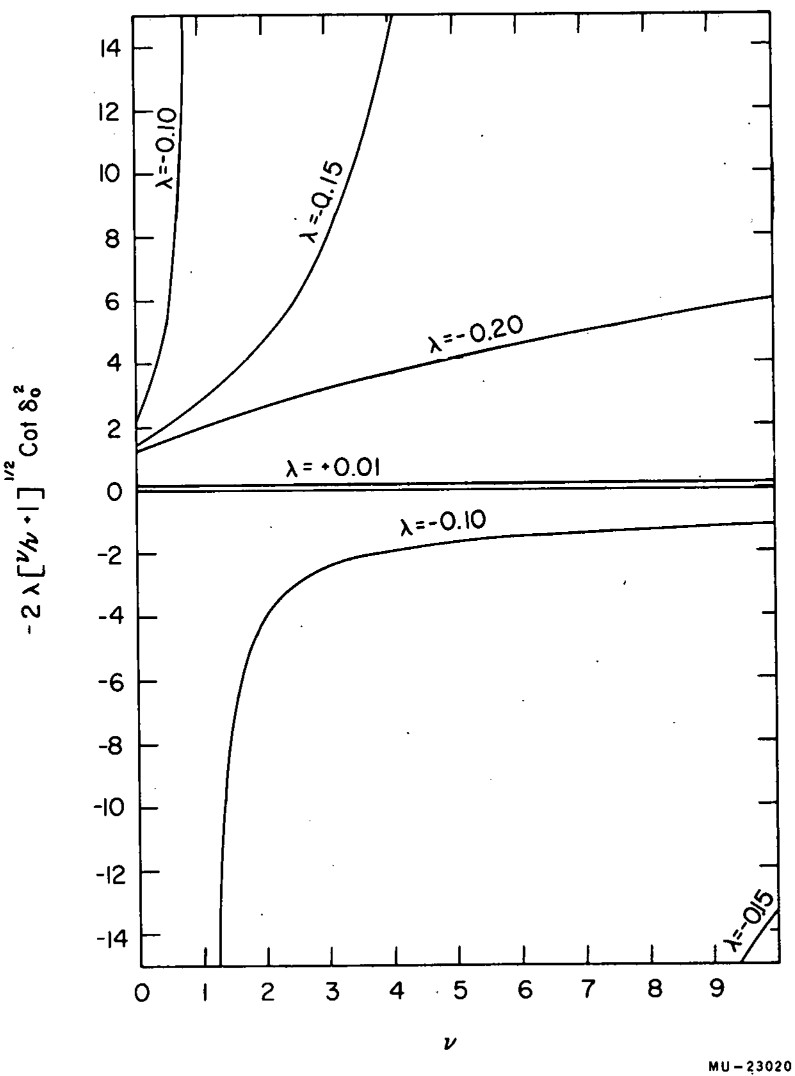


Fig. 3