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ABSTRACT

The general structure of the two photon processes in colliding beam experiments, $e+e \to e+e+\Gamma$, is studied for an arbitrary hadron final state Γ. The dependence of the scattering amplitudes on the lepton variables are explicitly factored out from the helicity amplitudes for the basic (hadronic) process $\gamma+\gamma \to \Gamma$. General formulas are given for the differential cross-section as well as for important special cases. The most important inclusive channel ($\gamma+\gamma \to$ anything) and exclusive channel ($\gamma+\gamma \to \pi+\pi$) are studied in some detail. The first case can yield information on the fundamental process $\gamma+\gamma \to \gamma+\gamma$. The second case provides a clean method for extracting the s-wave π-π phase shifts.

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INTRODUCTION

Colliding electron (positron) beams are a potentially fruitful source of interesting information about electromagnetic and strong interactions. The initial results from several laboratories have furnished the first clean study of the well known vector mesons and created much incentive for theoretical investigations of the one photon annihilation process, $e^+ + e^- \to \gamma \to$ hadrons.[1] It has been recently realized, however, that the cross-section for "two photon processes" (i.e. $e + e \to e + e +$ hadrons, see fig. 1) is expected to be large[2,3] and will dominate the annihilation process at energies above 1 - 1.5 Gev. This opens up a whole area of new possibilities and information to be gained from such processes. In particular, it provides an important opportunity to study photon-photon interactions[4] and hadronic systems of even charge conjugation.[5]

The purpose of this paper is to explore the general features of colliding beam processes in which an arbitrary final hadron system is produced by two virtual photons and to examine the possibilities of extracting specific information on the fundamental process $\gamma + \gamma \to \gamma + \gamma$ as well as on the S-wave π-π phase shifts. In section II we treat the kinematics. Because of the many particles in the final state, this can be rather complicated. By using the helicity formalism for these current-current scattering amplitudes, we factor out in an explicit manner the dependence on the lepton variables from that of the hadronic variables in the process $\gamma+\gamma \to$ hadrons. This enables us to write down the general formulas for the scattering amplitudes and differential cross-section for an arbitrary final state and give the small angle approximation to such formulas (Weizsacker-Williams approximation[6]). In section III we consider the experimental situation when none of the outgoing hadrons is observed. The cross-section for that case (when either one or both of the outgoing electrons are observed) can be related to the absorptive parts of the forward $\gamma+\gamma \to \gamma+\gamma$ scattering amplitudes. Valuable information on the latter process can be gained from such measurements. We also study, in this section, the full consequences of gauge invariance for these γ-γ amplitudes and some of their scaling properties. In section IV we study the most important final hadron state for the two photon process - the two pion state. The emphasis is on ways to extract the S-wave π-π phase shifts from this interesting channel. In Appendix A we give some detailed formulas on the kinematics and the differential cross-section for a general two photon process. Appendix B contains the definition of invariant amplitudes for photon-photon forward scattering.

II GENERAL KINEMATICS

We consider the process

$$e + e \rightarrow e + e + \Gamma \tag{1}$$

in the two photon exchange approximation[2,3] (Fig. 1). In (1) e represents either e^- or e^+ and Γ stands for some arbitrary hadron state. The process can be either exclusive (all particles in Γ measured) or inclusive (final state Γ partially measured or unmeasured). As shown in Fig. 1, the incoming and outgoing momenta of one electron are labelled by k_1 and k_2; those of the other by q_1 and q_2. The momenta of the virtual photons are therefore,

$$\begin{aligned} k &= k_1 - k_2 \\ q &= q_1 - q_2. \end{aligned} \tag{2}$$

We also define

$$\begin{aligned} K &= k_1 + k_2 \\ Q &= q_1 + q_2. \end{aligned} \tag{3}$$

The total four-momenum of the hadronic system Γ is given by $p\,(=k+q)$ while the individual particle momenta will be denoted by p_1, $p_2 \ldots . p_n$. The squared effective mass of the state Γ is designated by s with,

$$s = -\,p^2 = -\,(k+q)^2 \tag{4}$$

The kinematics of the overall process (1) is considerably simplified by the assumed two photon exchange structure of the amplitude (Fig. 1). The amplitudes for (1) can be expressed in terms of those describing the simpler process,

$$\gamma + \gamma \to \Gamma \ . \tag{5}$$

To take advantage of this simplifying feature, it is essential to choose a set of variables which completely separates the known dependence on the lepton vertices from the unknown dependence on the hadronic amplitude, (5), in an explicit manner. The obvious choice of hadronic variables are those appropriate for the process (5) with virtual (space-like) photons of "masses" k^2 and q^2 respectively. It is not hard to see that the natural variables for the leptons are those specifying the lepton configurations in the "rest frame"[7] of the corresponding virtual photons. We shall define those variables explicitly in subsection B and derive the connection between processes (1) and (5) for a general final state Γ in subsection C. First, however, let us familiarize ourselves with the process (1) by introducing the laboratory frame variables which are the experimentally measured variables.

A. The Laboratory Frame Variables

In colliding beam experiments, the Lab. frame is just the C.M. frame for the incoming particles. We choose the z-axis to be that of the incoming beams. We further specify the x-z plane to be defined by a vector $\underset{\sim}{p}_1$ belonging to the hadronic system Γ. Then the Lab. frame kinematics is as illustrated in Fig. 2. The independent variables in this frame can be chosen as,

E: energy of the incoming particles,

ε, θ, ϕ: energy and polar angles of the outgoing particle with 4-momentum k_2. (6)

ε', θ', ϕ': energy and polar angles (with respect to a set of axes related to the above by 180° rotation around the y-axis) for the outgoing particle q_2.

plus other "intrinsic" hadronic variables, if any.

Although experimentally the Lab. variables are directly measured, they are not very useful in exhibiting the structure of the two-photon exchange amplitude, Fig. 1. The expression for the cross-section is exceedingly complicated and without explicit physical interpretation. We shall therefore go over to the "natural variables" alluded to previously.

B. Natural Variables

The use of lepton Brick Wall frame variables for current-current amplitudes and its close association with helicity amplitudes (or form factors) for current-hadron scattering has been discussed in the context of electron-hadron and neutrino-hadron scattering.[8,9]

The same analysis can be easily carried over to our process, Eq. (1). There are two separate B.W. frames ("rest frames" for the virtual photons[7]) associated with the two pairs of leptons. In either frame we define the photon three-momenta to be along the z-axis. The two frames are further specified by requiring the time component of one or the other photon 4-momentum to be zero. Thus in the B.W. frame of the "k-electron" we have

$$k = \sqrt{k^2}(0,\ 0,\ 0,\ 1) \tag{7}$$

and $$q = \sqrt{q^2}(\sinh u,\ 0,\ 0,\ -\cosh u)$$

where $\cosh u = -k\cdot q/(k^2q^2)^{\frac{1}{2}}$. An arbitrary configuration of the k-electrons in this frame can be specified by two variables (ψ,χ) which parameterize the transformation $O(\psi,\chi)$ which brings the vectors $\underset{\sim}{k}_1$ and $\underset{\sim}{k}_2$ along the z-axis and which leaves k, Eq. (7), invariant. Explicitly, $O(\psi,\chi)$ consists of a rotation around the z-axis by the azimuthal angle χ and a boost along the x-axis by the hyperbolic angle ψ. The 4-momenta k_1, k_2 are therefore specified by Eq. (7) together with

$$K = \sqrt{k^2}(\cosh\psi,\ \sinh\psi\cos\chi,\ \sinh\psi\sin\chi,\ 0). \tag{8}$$

Similar definitions of the variables (ψ',χ') associated with the "q-electron" in its own B.W. frame when transformed[10] to the "k-electron" B.W. frame yield the following form for the vector Q.

$$Q = \sqrt{q^2}(\cosh\psi'\cosh u, -\sinh\psi'\cos\chi', \sinh\psi'\sin\chi', -\cosh\psi'\sinh u). \tag{9}$$

Eqs. (7)-(9) specify all the relevant vectors in one frame in terms of the independent variables,

$$q^2,\ k^2,\ s(\text{or } k\cdot q) \qquad\qquad (10)$$

and $\quad \psi,\ \chi, \psi', \chi'$.

The kinematic configuration in this frame is depicted in Fig. 3. The configuration of these vectors as well as their explicit expressions in terms of the variables (10) in the "q-electron" B.W. frame are exactly the same as shown here with the roles of the k-electron and q-electron reversed. The connection between these variables and the Lab. variables is given in Appendix A along with other detailed kinematic facts.

The utility of the set of variables (10) lies in the fact that each electron-photon vertex function can be written as a known function of its B.W. frame variables while the rest of the overall amplitude is independent of these variables. For instance, it is easy to see that

$$\begin{aligned}\langle k_2\lambda_2|j^\mu(0)|k_1,\lambda_1\rangle &= \varepsilon^\mu_{(\alpha)}(k)\langle k_2\lambda_2|\varepsilon^{(\alpha)*}\cdot j(0)|k_1\lambda_1\rangle \\ &= \varepsilon^\mu_{(\alpha)}(k)\langle \bar{k}_2\lambda_2|O^{-1}(\psi,\chi)\varepsilon^{(\alpha)*}\cdot j(0)O(\psi,\chi)|\bar{k}_1\lambda_1\rangle \\ &= \varepsilon^\mu_{(\alpha)}(k)D(\psi,\chi)^\alpha{}_\beta\langle \bar{k}_2\lambda_2|\varepsilon^{(\beta)*}\cdot j(0)|\bar{k}_1\lambda_1\rangle \\ &= \varepsilon^\mu_{(\alpha)}(k)D(\psi,\chi)^\alpha{}_\beta\ j^{(\beta)}{}_{\lambda_2\lambda_1}(k^2)\end{aligned} \qquad (11)$$

Here $\bar{k}_{1,2} = (1/2)\sqrt{k^2}(1,\ 0,\ 0,\ \pm 1)$; and $\varepsilon_{(\alpha)}{}^\mu(k)$ $(\alpha=0,1,2,3)$ are a set of helicity polarization vectors for the virtual photon. $O(\psi,\chi)$

is the SO(2,1) transformation introduced earlier and $D(\psi,\chi)^{\alpha}{}_{\beta}$ is the transformation matrix for j in the spherical basis.[12] The lepton "form factors" $j^{(\beta)}_{\lambda_2\lambda_1}(k)$ can be easily calculated. The only non-zero ones are,

$$j^{(+)}_{\frac{1}{2},\frac{1}{2}}(k^2) = j^{(-)}_{-\frac{1}{2},\frac{1}{2}}(k^2) = -\sqrt{2k^2} \tag{12}$$

C. Scattering Amplitude and Cross-Section

With these preliminaries, we can now write down the transition amplitude for process (1).

$$T = e^4 \langle k_2\lambda_2|j_\mu(0)|k_1\lambda_1\rangle\langle q_2\lambda_2'|j_\nu(0)|q_1\lambda_1'\rangle \frac{1}{k^2}\frac{1}{q^2}$$

$$\int d^4x\, e^{ikx}\langle\Gamma|T(J^\mu(x)J^\nu(0)|0\rangle \tag{13}$$

$$=(e^4/k^2q^2)T_{\alpha\beta}(k^2,q^2,s\ldots)D(\psi,\chi)^{\alpha}{}_{\alpha'}D(\psi,\chi)^{\beta}{}_{\beta'}\,j^{(\alpha')}_{\lambda_2\lambda_1}(k^2).j^{(\beta')}_{\lambda_2'\lambda_1'}(q^2).$$

The amplitude is thus in a manifestly factorized form with the dependence on the leptonic variables explicitly displayed. The hadronic part is isolated in the factor $T_{\alpha\beta}(k^2,q^2,s\ldots)$ which is nothing but the helicity amplitude for the process $\gamma+\gamma\to\Gamma$ with virtual photons of masses k^2, q^2 and helicities α,β respectively,

$$T_{\alpha\beta}(k^2,q^2,s\ldots) = \int d^4x\langle\Gamma|T(J_\mu(x)J_\nu(0))|0\rangle e^{ikx}\varepsilon^{\mu}_{(\alpha)}(k)\varepsilon^{\nu}_{(\beta)}(q) \tag{14}$$

We note that for e^-e^- collisions, because of identical particles, an additional term in Eq. (13) is required. It can be obtained from the expression given by interchanging k_2 and q_2.

For a general experiment of the type (1), let us split the final phase space for the final hadron system into two factors.

$$d\Gamma = d\Gamma' \, d\Gamma'' \tag{15}$$

where $d\Gamma'$ is associated with the observed variables while $d\Gamma''$ with the unobserved variables. The differential cross-section, after averaging (summing) over the initial (final) lepton polarizations can then be written,

$$\begin{aligned} d\sigma = \frac{\alpha^4}{2^9\pi^2} \; \frac{1}{E^2} \; \frac{dk^2}{k^2} \; \frac{dq^2}{q^2} \; d(\cosh\psi)d(\cosh\psi')d\chi d\chi' d\Gamma' \\ \times W_{\ell j, mn}(k^2,q^2,s\ldots)\; e^{-i(m-\ell)\chi} e^{-i(n-j)\chi'} \\ \times[d(\psi)^{\ell}{}_{1} d(\psi)^{m}{}_{1} + d(\psi)^{\ell}{}_{-1} d(\psi)^{m}{}_{-1}][d(\psi')^{j}{}_{1} d(\psi')^{n}{}_{1} + d(\psi')^{j}{}_{-1} d(\psi')^{n}{}_{-1}] \end{aligned} \tag{16}$$

where we have used (12) and the explicit form for $D(\psi,\chi)$.[12]

We also introduced,

$$W_{\ell j,mn}(k^2,q^2,s\ldots) = \int d\Gamma''(2\pi)^4\delta^4(k+q-p)T^*_{\ell j}(k^2,q^2,s\ldots)T_{mn}(k^2,q^2,s\ldots) \tag{17}$$

which is clearly the contribution to the absorptive part of γ-γ forward elastic scattering amplitude due to the state Γ summed over $d\Gamma''$. These functions satisfy the hermiticity relation

$$W_{\ell j,mn} = W^*_{mn,\ell j} \tag{18}$$

and the parity relation

$$W_{\ell,j,m,n} = (-1)^{\ell-j+m-n}W_{-\ell,-j,-m,-n} \tag{19}$$

Eq. (16) can obviously be further expanded out using the explicit expressions for $\bar{d}(\psi)^m{}_n$ as given in footnote 12. We give the full expression in Appendix A. For purposes of discussion, Eq. (16) is, in fact, more concise and clear.

D. Small Angle Approximation

Because of the presence of the factor $(k^2q^2)^{-1}$, the bulk of the cross-section is confined to the small momentum transfer region, as in the famous Mott cross-section formula. In other words, the photons are mostly near mass shell and the laboratory scattering angles θ and θ'

for the leptons are small. In this limit our B.W. frame variables are simply related to the laboratory variables,

$$
\begin{aligned}
k^2 &= q^2 = 0 \\
\chi = \phi \quad & \quad \chi' = \phi' \\
s = -2(k\bullet q) &= 4(E-\varepsilon)(E-\varepsilon') \\
\cosh\psi &= (E+\varepsilon)/(E-\varepsilon) \\
\cosh\psi' &= (E+\varepsilon')/(E-\varepsilon').
\end{aligned}
\tag{20}
$$

Thus, if the outgoing leptons are not observed one can easily integrate over the small angle region, setting $k^2=q^2=0$ in the photon-photon amplitudes $W_{\ell j, mn}$, and obtain the well known expression for the cross-section of process (1) in the "Weizsacker-Williams approximation".

$$\frac{d\sigma}{d\Gamma'} = 2\left(\frac{\alpha}{\pi}\right)^2 \left(\ln\frac{E}{m_e}\right)^2 \int_{s_0}^{4E^2} \frac{ds}{s}\, f\left[\left(\frac{s}{4E^2}\right)^{\frac{1}{2}}\right] \frac{d\sigma_{\gamma\gamma}}{d\Gamma'} \tag{21}$$

where $$\frac{d\sigma_{\gamma\gamma}}{d\Gamma'} = \frac{4\pi^2\alpha^2}{s}\left(W_{11,11} + W_{1-1,1-1}\right) \tag{22}$$

and $$f(x) = (2+x^2)^2 \ln(1/x) - (1-x^2)(3+x^2).$$

The total cross-sections for various hadron processes based on these approximate formulas were calculated by various authors before.[2,3,13]

III. CONNECTION WITH FORWARD ELASTIC PHOTON-PHOTON SCATTERING AMPLITUDE

If none of the final state hadrons are observed, we sum over all possible states Γ in Eqs. (14) and (17). The functions $W_{\ell j, mn}(k^2, q^2, s\ldots)$ then become the absorptive part of the forward elastic helicity amplitudes for the fundamental process,[4] Fig. 4,

$$\gamma(k,m)+\gamma(q,n) \to \gamma(k,\ell)+\gamma(q,j). \tag{23}$$

Explicitly, we have

$$W_{\ell j, mn}(k^2 q^2 s) = \epsilon^{*\mu}_{(\ell)}(k)\,\epsilon^{*\nu}_{(j)}(q)\,W_{\mu\nu, \lambda\sigma}(k,q)\,\epsilon^{\lambda}_{(m)}(k)\,\epsilon^{\sigma}_{(n)}(q) \tag{24}$$

where

$$W_{\mu\nu, \lambda\sigma}(k,q) = \int d^4x\, d^4y\, d^4z\; e^{-ik(x-z)-iqy} \langle 0|T^*(J_\mu(x)J_\nu(y))T(J_\lambda(z)J_\sigma(0))|0\rangle. \tag{25}$$

In addition to the symmetry relations (18) and (19), these forward elastic amplitudes satisfy,

$$W_{\ell j, mn} = \delta_{(\ell-j),(m-n)} W_{\ell j, mn} \qquad \text{(angular momentum conservation)} \tag{26}$$

and

$$W_{\ell j, mn} = W^*_{\ell j, mn} \qquad \text{(time reversal invariance)} \tag{27}$$

By making use of these symmetry relations, one can write down the explicit formula for the differential cross-section for the overall process (1) summed over Γ,

$$d\sigma = \frac{\alpha^4}{2^8\pi E^2}\,\frac{dk^2}{k^2}\,\frac{dq^2}{q^2}\,d(\cosh\psi)d(\cosh\psi')d\,\chi \tag{28}$$

$$\times\,\{(\cosh^2\psi+1)(\cosh^2\psi'+1)\tfrac{1}{2}(W_{11,11}+W_{1-1,1-1})$$

$$+(\cosh^2\psi+1)(\cosh^2\psi'-1)\,W_{10,10}+(\cosh^2\psi-1)(\cosh^2\psi'+1)W_{01,01}$$

$$+(\cosh^2\psi-1)(\cosh^2\psi'-1)[W_{00,00}+\tfrac{1}{2}\cos 2\chi\quad W_{11,-1-1}]$$

$$+\sinh 2\psi\,\sinh 2\psi'\,\cos\chi\quad\tfrac{1}{2}(W_{11,00}-W_{10,0-1})\}\,.$$

where now the x-z plane is defined by the outgoing q-electron. It is obvious from the above expression that by measuring the outgoing leptons, one can determine, in principle, six combinations of the eight independent forward elastic (virtual) photon-photon scattering amplitudes.

In view of the importance of the fundamental process (23), it is worthwhile to look into the structure of the helicity amplitudes, Eq.(24), arising from the gauge invariance properties of the photon interaction. These can be written

$$\begin{aligned} k^\mu W_{\mu\nu,\lambda\sigma} &= q^\nu W_{\mu\nu,\lambda\sigma} \\ &= W_{\mu\nu,\lambda\sigma}k^\lambda = W_{\mu\nu,\lambda\sigma}q^\sigma = 0. \end{aligned} \tag{29}$$

It is well known that these conditions give rise to low energy theorems for the helicity amplitudes. The full content of (29) can be explicitly displayed by expanding $W_{\mu\nu,\lambda\sigma}$ into a set of minimal polynomial tensor basis $\{L_i\}$ which satisfy all the requirements of Lorentz transformation, gauge invariance and symmetry conditions (18), (19), (26) and (27),

$$W^{\mu\nu,\lambda\sigma} = \sum_i W_i(k^2,q^2,s)\, L_i^{\mu\nu,\lambda\sigma}(k,q). \tag{30}$$

The coefficients $W_i(k^2,q^2,s)$ are then (absorptive-parts of) invariant amplitudes which are free from all kinematic singularities and zeros (constraints). A general precedure for constructing such a tensor basis exists.[14] Some of the explicit formulas for the resulting gauge invariant fourth rank tensors are rather lengthy. We give the detailed results in Appendix B. Here we only exhibit the kinematic structure of the helicity amplitudes by expressing them in terms of the analytic invariant amplitudes W_i,

$$\begin{aligned}
W_{00,00} &= k^2q^2\{W_2+W_3+2W_6+q^2W_4+k^2W_5\} \\
W_{10,10} &= -q^2\{k^2[W_3+k^2W_5+W_6+(k\cdot q)W_8] + (k\cdot q)^2W_4\} \\
W_{01,01} &= -k^2\{q^2[W_3+q^2W_4+W_6+(k\cdot q)W_8] + (k\cdot q)^2W_5\} \\
W_{11,-1-1} &= (k\cdot q)^2W_2+k^2q^2[W_6-(k\cdot q)W_8] \\
\tfrac{1}{2}(W_{11,11}+W_{1-1,1-1}) &= (k\cdot q)^2[\tfrac{1}{2}W_2+q^2W_4+k^2W_5+W_6+(k\cdot q)W_8] \\
&\quad + \tfrac{1}{2}k^2q^2[2W_3+W_6-(k\cdot q)W_8] \\
\tfrac{1}{2}(W_{11,00}-W_{10,0-1}) &= -\tfrac{1}{2}(k^2q^2)^{\frac{1}{2}}[(k\cdot q)(W_2+W_6)-k^2q^2W_8] \\
\tfrac{1}{2}(W_{11,11}-W_{1-1,1-1}) &= \tfrac{1}{2}(k\cdot q)[2W_1+(k\cdot q)W_2] + \tfrac{1}{2}k^2q^2[W_6-2W_7-(k\cdot q)W_8] \\
\tfrac{1}{2}(W_{11,00}+W_{10,0-1}) &= -(k^2q^2)^{\frac{1}{2}}[W_1+\tfrac{1}{2}(k\cdot q)(W_2-W_6+2W_7) + k^2q^2W_8]
\end{aligned} \tag{31}$$

One well known consequence of these relations is the low energy theorems for $W_{\ell j,mn}$ that they display explicitly. Thus the factors of $(k^2)^{\frac{1}{2}}$ and $(q^2)^{\frac{1}{2}}$ associated with each longitudinal virtual photon emerges as expected. In addition, Eq. (31) shows that the first six amplitudes, which are the ones appearing in the cross-section formula (28), each vanishes as the square of the momentum components when either $k_\mu \to 0$ or $q_\mu \to 0$.

With gauge invariance and other kinematical requirements explicitly taken care of, one may examine dynamical problems associated with the photon-photon elastic scattering amplitudes. We shall not go into much of these more speculative topics and confine ourselves to a simple discussion of possible scaling behavior for this process.

The motivation for looking into the scaling behavior comes, of course, from the rather dramatic results found in deep inelastic electron-nucleon scattering.[15,16] In trying to apply similar considerations to the two photon process, one immediately recognizes some differences. For instance, the amplitudes for our process, e+e → e+e+anything, depend on three invariant variables (k^2, q^2, s) (cf. Eqs. (24) and (30)). One must, therefore, decide on which of the many possible scaling limits to take. Here, our freedom of choice is limited by some serious practical considerations. Since the cross-section drops off very fast when either k^2 or q^2 become large, it is impractical to look in the region where both variables are large. This suggests that we should look at the situation where one electron (say, the k-electron) is detected at large 4-momentum transfers and the other (the q-electron) remains inside the small momentum transfer forward peak. This situation is then very much similar to the inelastic electron nucleon scattering case with the low energy photon emitted by the q-electon playing the role of the target nucleon in the previous case.

Since the most striking aspect of scaling, so far, has been its experimental verification rather than its theoretical elucidation, it appears appropriate to apply the simplest possible arguments in order to find the scaling behavior of the various amplitudes. In the present case where there is no mass variable to set the scale, dimensional analysis is almost sufficient. One can invoke analogues of Bjorken's heuristic relations to argue that the relevant quantities tend to non-vanishing limits in the scaling region. Thus, for q^2 almost on mass shell, one can obtain,

$$\nu^{N_i} W_i(k^2,0,s) \to F_i(\omega) \tag{32}$$

where

$$\nu = - k\cdot q$$

$$\omega = - k\cdot q/k^2$$

$$N_i = \begin{cases} 1 & \text{for } i=1 \\ 2 & \text{for } i=2,3,6,7 \\ 3 & \text{for } i=4,5,8 \end{cases}$$

and the limit is taken with $k^2 \to \infty$, $(k\cdot q) \to \infty$ and ω fixed.

The expression for the cross-section then takes the form

$$\frac{d\sigma}{dk^2 d\varepsilon d\varepsilon'} = \frac{\alpha^4}{16E^4}\left(\ln\frac{E}{m_\ell}\right)\frac{1}{k^2}\left(\cosh^2\psi'+1\right)$$
$$\times\left\{(\cosh^2\psi+1)F(\omega)-\frac{2}{\omega}F_5(\omega)\right\} \tag{33}$$

where $F(\omega) = \frac{1}{2}F_2+F_6-F_8$. We note that when $q^2 \sim 0$, the connection between the natural variables and the Lab. variables are rather simple. From Appendix A, we have

$$\begin{aligned}
\cosh\psi &\simeq (E+\varepsilon\cos^2\tfrac{1}{2}\theta)/(E-\varepsilon\cos^2\tfrac{1}{2}\theta)\\
\cosh\psi' &\simeq (E+\varepsilon')/(E-\varepsilon)\\
k^2 &\simeq 2E\varepsilon(1-\cos\theta)\\
-k\cdot q &\simeq 2(E-\varepsilon')(E-\varepsilon\cos^2\tfrac{1}{2}\theta).
\end{aligned} \tag{34}$$

If the q-electron is not detected experimentally, the above expression for the differential cross-section must, of course, be integrated over ε'. This integration can not be done without knowing the ω-dependence of the structure functions $F(\omega)$ and $F_5(\omega)$.

IV. THE TWO PI FINAL STATE AND S-WAVE PI-PI PHASE SHIFTS

As mentioned previously, when two high energy electrons collide in process (1), the bulk of the cross-section is confined to the region with small scattering angles for the electrons. In this region, the electrons also carry away most of the incoming energy, leaving relatively small energy transfer to the hadron system. This is reflected, for instance, in Equation (21) by the damping factor (1/s) in the hadron mass spectrum. Consequently, as a probe for hadronic structure, this process is most useful in studying low energy hadronic systems which have the quantum numbers of two photons. The most important of these, by far, is the two pion channel which must be in a I=0 or 2, J=even and P=+ state. We shall investigate the possibilities of extracting the s-wave π-π phase shifts from this type of experiment.[5]

For practical reasons, we assume the outgoing electrons are not observed. Then, we can make the small angle approximation, and relate the observed cross-section to that of the process,

$$\gamma+\gamma \rightarrow \pi+\pi \tag{35}$$

for almost on shell photons. The kinematics for this process in the C.M. frame is illustrated in Fig. 5. We denote by θ_π the scattering angle in this frame and define

$$\begin{aligned} p&=p_1+p_2=k+q \\ r&=\tfrac{1}{2}(p_1-p_2). \end{aligned} \tag{36}$$

Let $T_{m,n}$ (m,n=±1) be the helicity amplitudes for the processes (35) (cf. Eq. (14)). Then the differential cross-section for the overall process e+e → e+e+π+π becomes,

$$\frac{d\sigma}{ds\,d(\cos\theta_\pi)} = \frac{\alpha^4}{2\pi}\left(\ln\frac{E}{m_e}\right)^2 \frac{1}{s^2}\left(1-\frac{4\mu^2}{s}\right)^{\frac{1}{2}} f\left[\left(\frac{s}{4E^2}\right)^{\frac{1}{2}}\right]\left(|T_{1,1}|^2+|T_{1,-1}|^2\right) \tag{37}$$

In the absence of strong final state interactions, the amplitudes T_{mn} are given by the gauge invariant Born term. Thus for the charged $\pi^+\pi^-$ final state (superscript c) we get,

$$\begin{aligned} T^{(B)c}_{1,1} &= 2 - 2p^2\sin^2\theta_\pi\left(\frac{1}{\mu^2-t}+\frac{1}{\mu^2-u}\right) = \frac{2\mu^2 s}{(\mu^2-t)(\mu^2-u)} \\ T^{(B)c}_{1,-1} &= 2p^2\sin^2\theta_\pi\left(\frac{1}{\mu^2-t}+\frac{1}{\mu^2-u}\right) \end{aligned} \tag{38}$$

where
$$p^2 = \tfrac{1}{4}(s-4\mu^2)$$
$$\mu^2-t = p\sqrt{s}[(\sqrt{s}/2p)-\cos\theta_\pi]$$
$$\mu^2-u = p\sqrt{s}[(\sqrt{s}/2p)+\cos\theta_\pi].$$

The corresponding amplitudes for the neutral $\pi^\circ\pi^\circ$ final state (superscript n) are obviously,

$$T^{(B)n}_{mn} = 0. \tag{39}$$

Strong interaction effects modify these simple expressions. In the low energy region where elastic unitarity approximately holds (from threshold ($s=4\mu^2$) to slightly above the inelastic threshold ($s=16\mu^2$)). Most of these strong interaction corrections can be incorporated in the form of final state interaction through Watson's theorem.[17] We expect such corrections to be only important in the lowest partial wave (i.e. the s-wave) and for the isospin channel I=0.[18] Therefore, we can write the scattering amplitudes as,

$$\begin{aligned}
T_{1,1}{}^{c} &= |a^{(0)}(s)|e^{i\delta_0^{(0)}} - \tfrac{2}{3}a^{(B)c}(s) + T_{1,1}{}^{(B)c}(s,\cos\theta_\pi) \\
T_{1,-1}{}^{c} &= T_{1,-1}{}^{(B)c}(s,\cos\theta_\pi) \\
T_{1,1}{}^{n} &= -|a^{(0)}(s)|e^{i\delta_0^{(0)}} + \tfrac{2}{3}a^{(B)c}(s) \\
T_{1,-1}{}^{n} &= 0,
\end{aligned} \tag{40}$$

where $a^{(0)}(s)$ is the I=0 s-wave partial amplitude, $\delta_0^{(0)}$ is its phase and $a^{(B)c}(s)$ is the s-wave projection of the Born term for the charged final state ($\pi^+\pi^-$), Eq. (38). In deriving Eqs, (40) we have used the relations,

$$\begin{aligned}
T^c &= 2^{-\frac{1}{2}}\, T^{(2)} + T^{(0)} \\
T^n &= 2^{\frac{1}{2}}\, T^{(2)} - T^{(0)}
\end{aligned} \tag{41}$$

where $T^{(I)}$ are amplitudes with given isospin I. We can now write down the differential cross-section for these two final states:

$$\frac{d\sigma^{c}}{ds\ d(\cos\theta\pi)} - \frac{d\sigma^{(B)c}}{ds\ d(\cos\theta\pi)} = N(s)\{|\quad a^{(0)}(s) - \frac{2}{3}\, a^{(B)c}(s)|^2 \tag{42}$$

$$+ 2\ [\quad |a^{(0)}(s)|\cos\delta_0{}^{(0)} - \frac{2}{3}\, a^{(B)c}(s)]\ T_{1,1}{}^{(B)c}(s\,,\cos\theta\pi)\}$$

$$\frac{d\sigma^{n}}{ds\ d(\cos\theta\pi)} = N(s)|-a^{(0)}(s) + \frac{2}{3}a^{(B)c}(s)|^2$$

where

$$N(s) = \frac{\alpha^4}{2\pi}\ (\ln\frac{E}{m_e})^2\ \frac{1}{s^2}\ (1 - \frac{4\mu^2}{s})^{\frac{1}{2}} f[(\frac{s}{2E^2})^{\frac{1}{2}}]$$

$$\frac{d\sigma^{(B)c}}{ds\ d(\cos\theta\pi)} = N(s)(|T_{1,1}{}^{(B)c}|^2 + |T_{1,-1}{}^{(B)c}|^2)$$

$$a^{(B)c}(s) = \frac{2\mu^2}{p\sqrt{s}}\ \ln\frac{\sqrt{s}+2p}{\sqrt{s}-2p}$$

and $T_{m,n}{}^{(B)c}$ are given by Eq. (38).

There are two unknowns in these formulas, $|a^{(0)}(s)|$ and $\delta_0{}^{(0)}(s)$. By making full use of the interference effect between the s-wave correction term and the Born term in the first equation of (42), one can determine these two unknowns <u>independently</u>. Thus one can fit the data with an expression like

$$A(s) + B(s)\ T_{11}{}^{(B)c}(s,\cos\theta\pi) \tag{43}$$

for the right hand side and solve for the unknowns from A(s) and B(s).

If one is willing to use more dynamical imput, he can do better than outlined above by attempting to relate $|a^{(0)}(s)|$ to $\delta_0^{(0)}(s)$ through dispersion relations. To do this, we first need amplitudes which are free from kinematic singlarities and zeros. For on-shell photons, two such amplitudes (A_1, A_2) can be defined by:

$$T^{\mu\nu} = [(k\cdot q)g^{\mu\nu} - q^\mu k^\nu]\, A_1 \tag{44}$$

$$+[(k\cdot q)r^\mu r^\nu - (k\cdot r)q^\mu r^\nu - (q\cdot r)r^\mu k^\nu + (k\cdot r)(q\cdot r)g^{\mu\nu}]A_2.$$

It is easy to verify,

$$T_{11} = \tfrac{1}{2}s[A_1 + \tfrac{1}{2}p^2 A_2]$$

$$T_{1-1} = -\frac{1}{2}\, s\; p^2 \sin^2\theta\pi A_2. \tag{45}$$

This implies (T_{11}/s) and $T_{1-1}/s(s-4\mu^2)\sin^2\theta\pi$ are regular amplitudes. We shall be only interested in the first one. In writing down the fixed t dispersion relation for T_{11}, we assume: (i) the right hand cut is dominated by the elastic unitarity and (ii) the only important contribution to the left hand cut comes from the gauge invariant Born term. The kinematic zero at s=0 enables us to make a free subtraction at s=0. We have, therefore,

$$T_{11}^{I}(s,t) = T_{11}^{(B)I} + \frac{s}{\pi} \int_{4\mu^2}^{\infty} ds' \frac{\mathrm{Im}T_{11}^{I}(s',t)}{s'(s'-s)} \tag{46}$$

Projecting out the s-wave part of this equation, we get:

$$a^{I}(s) = a^{(B)I} + \frac{s}{\pi} \int_{4\mu^2}^{\infty} ds' \frac{a^{I}(s)\sin\delta_0^{(I)} e^{-i\delta_0^{(I)}}}{s'(s'-s)} . \tag{47}$$

This is a standard Omnes equation which has the solution

$$a^{I}(s) = a^{(B)I}(s) + \frac{se^{\Delta^{I}(s)}}{\pi} \int_{4\mu^2}^{\infty} ds' \frac{a^{(B)I}(s')\sin\delta_0^{(I)}(s')e^{-\mathrm{Re}\Delta(s')}}{s'(s'-s)} \tag{48}$$

where

$$\Delta^{I}(s) = \frac{s}{\pi} \int_{4\mu^2}^{\infty} ds' \frac{\delta_0^{(I)}(s')}{s'(s'-s)} \tag{49}$$

The dependence of $a^{I}(s)$ on $\delta_0^{(I)}(s)$ as expressed in Eq. (48) is too indirect to be of any practical value.

To simplify this solution into a manageable form, we notice that, because of the rapid convergence factor, only the low energy portion of the integral on the right hand side is important. (This is, of course, necessary in order for the elastic unitarity approximation to hold in the first place.) Over this range, the Born term can be well approximated by a pole on the negative real axis.[19]

By using the explicit formula for $a^{(B)}$, Eq. (42), we obtain an approximate expression.

$$a^{(B)c}(s) \simeq \frac{b}{s+c}, \qquad b=14.7\mu^2,\ c=3.35\mu^2 \tag{50}$$

which is accurate to within 4% throughout the elastic unitarity range, $s=4\mu^2$ to $16\mu^2$. Now, rewrite our solution to the Omnes equation, Eq. (48), in the form

$$a^I(s) = a^{(B)I}(s) - \frac{se^{\Delta^I(s)}}{2\pi i} \int_{c_R} ds' \frac{a^{(B)I}(s')e^{-\Delta(s')}}{s'(s'-s)} \tag{51}$$

with the contour c_R circling the cut on the positive real axis as depicted in Fig. 6. Inserting the approximate form for the Born term, Eq. (50), into the right hand side of this equation one can deform the contour of integration until it is reduced to three circles (c_1, c_2 and c_3, Fig. 6) around the three poles of the integrand. One obtains then:

$$\begin{aligned} a^I(s) &= b^I \left\{ \frac{1}{s+c} - se^{\Delta^I(s)}\left[\frac{e^{-\Delta^I(s)}}{(s+c)s} - \frac{e^{-\Delta^I(0)}}{cs} + \frac{e^{-\Delta^I(-c)}}{c(s+c)} \right] \right\} \\ &= \frac{b^I}{c}\, e^{\Delta^I(s)}\left[1 - \frac{s}{s+c}\, e^{-\Delta^I(-c)}\right] \end{aligned} \tag{52}$$

where

$$b^{(0)} = (2/3)b = 9.8\mu^2$$

$$b^{(2)} = (\sqrt{2}/3)b = 6.94\mu^2$$

Eq. (52) furnishes a simple relation for the magnitude of $a^I(s)$ in terms of its phase:

$$|a^I(s)| = \left|\frac{b^I}{c}\right| \left| 1 - \frac{s}{s+c}\, e^{-\Delta^I(-c)}\right| e^{\mathrm{Re}\Delta(s)} \tag{53}$$

where $$\mathrm{Re}\Delta^I(s) = \frac{s}{\pi} P \int_{4\mu^2}^{\infty} ds' \frac{\delta_0^{(I)}(s')}{s'(s'-s)}$$

and P denotes the principal part of the integral. The constant $[-\Delta^I(-c)]$ is also, in principle, given once the phase shift is known. But in practice one can regard it as a parameter (same for all energies) which is positive and small.[20]

Eq. (53) can be used as constraints to the program for extracting the phase shifts from data as discussed in the first part of this section (where $|a^{(0)}(s)|$ and $\delta_0^{(0)}$ were regarded as independent of each other). With these constraints available one may even try to put in correction effects in the I=2 channel, which has been neglected so far, and experimentally verify whether it is indeed small. Thus Eq. (42) will become,

$$\frac{d\sigma^c}{ds\, d(\cos\theta_\pi)} - \frac{d\sigma^{(B)c}}{ds\, d(\cos\theta_\pi)} = N(s) \{ | a^{(0)}(s) + \frac{1}{\sqrt{2}} a^{(2)}(s) - a^{(B)c}(s)|^2$$

$$+ T_{11}^{(B)c}(s,\cos\theta_\pi)[2\,|a^{(0)}(s)|\cos\delta_0^{(0)} + \sqrt{2}\,|a^{(2)}(s)|\cos\delta_0^{(2)} - 2a^{(B)c}(s)]\}$$

$$\frac{d\sigma^n}{ds\ d(\cos\theta_\pi)} = N(s)\ |-a^{(0)}(s) + \sqrt{2}\ a^{(2)}(s)|^2 \tag{54}$$

where the various quantities have the same meaning as defined there. These equations must be used in conjunction with the constraint equations, (53), in order to be useful for the extraction of information on the phase shifts $\delta_0^{(0)}$ and $\delta_0^{(2)}$. If the correction effects due to strong interaction are small, then we can neglect the square of the correction terms in Eqs. (42) and (54) for $d\sigma^c/ds\ d(\cos\theta_\pi)$. With only the interference term on the right hand sides of these equations, and with $|a^I(s)|$ related to $\delta_0^{(I)}(s)$ through Eq. (53), one may even attempt to invert these equations and express $\delta_0^{(I)}(s)$ directly in terms of the measured quantities. Such a program, however, does not seem to be especially efficient and practical. We shall not pursue it here.[21]

We close this section by remarking that although the extracting of s-wave π-π phase shifts from the two photon process is not without its difficulties, the present method does have considerable advantage over the other available methods. We only point to the absence of other strongly interacting particles in the final state which could make the extraction of relevent information theoretically ambiguous (as in the $\pi N \rightarrow \pi\pi N$ case[22]) as well as the absence of the p-wave final channel which strongly dominates over the s-wave channel and makes the extraction of information practically difficult (as in K_{l4} decay[22]).

CONCLUDING REMARKS

In this paper, we have only exhbited the most general kinematic structure of two photon processes in colliding beam experiments and examined possibilities for extracting interesting information from the most important inclusive ($\gamma+\gamma \rightarrow$ anything) and exclusive ($\gamma+\gamma \rightarrow \pi+\pi$) channels. There are obviously many more interesting possibilities that can be explored.

The kinematics for these processes are a bit more complicated than the cases we are used to, because of the presence of two outgoing electrons accompanying the final hadron state. Here, the helicity formalism proves to be instrumental in explicitly factoring out the dependence on the lepton variables from the essential hadron amplitudes to keep the physics transparent and the kinematics always manageable. The possibilities of learning something about $\gamma\gamma$ elastic scattering and π-π s-wave phase shifts, while in neither case is this the only method available, do seem to be extremely interesting and theoretically clean. It is hoped that in the near future, these possibilities will also prove to be experimentally practical.

APPENDIX A

We give the details of the kinematics for two photon processes in colliding beam experiments.

Relation Between Various Sets of Variables

The momentum labels are assigned in the text and in Fig. 1. We introduced two sets of independent variables; the lab. variables $\{E, \varepsilon, \theta, \phi, \varepsilon', \theta', \phi'\}$ and the B.W. frame variables $\{s, k^2, \psi, \chi, q^2, \psi', \chi'\}$. There is a third set, the invariant variables $\{s, k^2, q^2, q\cdot K, k\cdot Q, K\cdot Q, k\cdot p_1\}$, which is useful in providing the connection between the two previous sets. It is easy to verify (we neglect the letpon mass throughout),

$$
\begin{aligned}
k^2 &= 4E\varepsilon \sin^2 \tfrac{1}{2}\theta \\
q^2 &= 4E\varepsilon' \sin^2 \tfrac{1}{2}\theta' \\
s &= 4(E-\varepsilon)(E-\varepsilon') - 4\varepsilon\varepsilon' \sin^2 \tfrac{1}{2}\Theta \\
-q\cdot K &= 2(E+\varepsilon)(E-\varepsilon') + 2\varepsilon\varepsilon' \sin^2 \tfrac{1}{2}\Theta - \tfrac{1}{2}(k^2-q^2) \\
&= [(k\cdot q)^2 - k^2q^2]^{\frac{1}{2}} \cosh\psi \\
-k\cdot Q &= 2(E-\varepsilon)(E+\varepsilon') + 2\varepsilon\varepsilon' \sin^2 \tfrac{1}{2}\Theta + \tfrac{1}{2}(k^2-q^2) \\
&= [(k\cdot q)^2 - k^2q^2]^{\frac{1}{2}} \cos\psi' \\
-K\cdot Q &= 2(E+\varepsilon)(E+\varepsilon') - 2\varepsilon\varepsilon' \sin^2 \tfrac{1}{2}\Theta - \tfrac{1}{2}(k^2+q^2) \\
&= -(k\cdot q)\cosh\psi\cosh\psi' - (k^2q^2)^{\frac{1}{2}} \sin\psi\sin\psi' \cos(\chi+\chi')
\end{aligned}
\tag{A-1}
$$

where Θ is the angle between $\underset{\sim}{k}_2$ and $\underset{\sim}{q}_2$ in the laboratory frame, i.e.

$$\cos\Theta = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi+\phi').$$

These relations enables one to solve one set of variables in terms of the other. We shall not do this explicitly. The formulas become considerably simplified when one of the electrons (say, the q-electron) is scattered in the forward cone where one can use the small angle approximation, then

$$
\begin{aligned}
&\theta' \simeq 0, \quad q^2 \simeq 0, \quad \cos\Theta \simeq \cos\theta \\
&s \simeq 4(E-\varepsilon)(E-\varepsilon') - 4\varepsilon\varepsilon' \sin^2 \tfrac{1}{2}\theta \\
&\cosh\psi \simeq (E+\varepsilon\cos^2\tfrac{1}{2}\theta)/(E-\varepsilon\cos^2\tfrac{1}{2}\theta) \\
&\cosh\psi' \simeq (E+\varepsilon')/(E-\varepsilon') \\
&E^2 \simeq \tfrac{1}{8}(-k\cdot q)(\cosh\psi+1)(\cosh\psi'+1).
\end{aligned}
\tag{A-2}
$$

Phase Space

The invariant phase space element

$$d\rho = (d^3k_2/k_2^\circ)(d^3q_2/q_2^\circ) \tag{A-3}$$

takes the form

$$d\rho = \varepsilon\varepsilon'\, d\varepsilon\, d\varepsilon'\, d(\cos\theta)d(\cos\theta')d\phi d\phi' \tag{A-4}$$

in terms of the lab. variables and the form

$$d\rho = (1/64)dk^2 dq^2 d(\cosh\psi)d(\cosh\psi')d\chi d\chi' \tag{A-5}$$

in terms of the B.W. frame variables.

The region of integration in the lab. variables is defined by: (i) E fixed

(ii) $\theta, \theta', \phi, \phi'$ their usual range (A-6)

(iii) $\varepsilon, \varepsilon'$ bounded by

$$\begin{cases} \varepsilon=0, \ \varepsilon'=0 \\ (E-\varepsilon\cos^2 \tfrac{1}{2}\Theta)(E-\varepsilon' \cos^2 \tfrac{1}{2}\Theta)=\mu^2\cos^2 \tfrac{1}{2}\Theta + E^2\sin^2 \tfrac{1}{2}\Theta \end{cases}$$

and the corresponding region in terms of the B.W. frame variables is specified by:

(i) $\chi, \ \chi'$: their usual range

(ii) $k^2 > 0, \ q^2 > 0, \ s > 4\mu^2$ (A-7)

(iii) $-(k\cdot q)(1+\cosh\psi \cosh\psi')+[(k\cdot q)^2-k^2q^2]^{\frac{1}{2}} (\cosh\psi + \cosh \psi') - (k^2q^2)^{\frac{1}{2}} \sinh \psi \sinh \psi' \cosh(\chi+\chi')=8E^2$ fixed.

The last condition simplifies considerably if $k^2 \simeq 0$ or $q^2 \simeq 0$. One gets

$$4\mu^2 < s < 4E^2 - \tfrac{1}{2}(k^2+q^2)$$

$$(\cosh\psi +1)(\cosh \psi'+1) = \frac{8E^2}{(-k\cdot q)} < \frac{16E^2}{k^2+q^2+4\mu^2}. \qquad \text{(A-8)}$$

Differential Cross-Section

The general expression for the differential cross-section, Eq. (16), can be expanded out using the explicit expressions for $\partial (\psi)^m{}_n$. The result is:

$$d\sigma = \frac{\alpha^4}{8\pi^2 E^2} \quad \frac{d\rho}{k^2 q^2} \quad d\Gamma' \tag{A-9}$$

$$\times \quad \{(\cosh^2\psi+1)(\cosh^2\psi'+1)\ \tfrac{1}{2}\ [W_{11,11} + W_{1-1,1-1}]$$

$$+(\cosh^2\psi+1)(\cosh^2\psi'-1)[W_{10,10} - \cos 2\chi'\ \tfrac{1}{2}\ (W_{11,1-1} + W_{-11,-1-1})]$$

$$+(\cosh^2\psi-1)(\cosh^2\psi'+1)[W_{01,01} - \cos 2\chi\ \tfrac{1}{2}\ (W_{11,-11} + W_{1-1,-1-1})]$$

$$+(\cosh^2\psi-1)(\cosh^2\psi'-1)[W_{00,00} - \cos 2\chi\ W_{10,-10} - \cos 2\chi'\ W_{01,0-1}$$

$$+ \tfrac{1}{2}\cos 2(\chi+\chi')W_{11,-1-1} + \tfrac{1}{2}\cos 2(\chi-\chi')W_{1-1,-11}]$$

$$+ 2\sinh 2\psi\ \sinh 2\psi'[\tfrac{1}{2}\cos(\chi+\chi')\ \tfrac{1}{2}\ \mathrm{Re}(W_{11,00} - W_{10,0-1})$$

$$+ \tfrac{1}{2}\cos(\chi-\chi')\ \tfrac{1}{2}\ \mathrm{Re}(W_{10,01} - W_{1-1,00})]$$

$$+ \sqrt{2}\sinh 2\psi\ (\cosh^2\psi'+1)\cos\chi\ \tfrac{1}{2}\ \mathrm{Re}(W_{01,-11} + W_{0-1,-1-1})$$

$$+ \sqrt{2}(\cosh^2\psi+1)\sinh 2\psi'\ \cos\chi'\ \tfrac{1}{2}\ \mathrm{Re}(W_{10,1-1} + W_{-10,-1-1})$$

$$+ \sqrt{2}\sinh 2\psi(\cosh^2\psi'-1)[\tfrac{1}{2}\cos(\chi+2\chi')\mathrm{Re}\ W_{11,0-1} + \cos\chi\ \ \mathrm{Re}\ W_{00,-10}$$

$$+ \tfrac{1}{2}\cos(\chi-2\chi')\mathrm{Re}\ W_{1-1,01}]$$

$$+ \sqrt{2}(\cosh^2\psi-1)\sinh 2\psi'[\tfrac{1}{2}\cos(\chi'+2\chi)\mathrm{Re}\ W_{11,-10} + \cos\chi'\mathrm{Re}W_{00,0-1}$$

$$+ \tfrac{1}{2}\cos(\chi'-2\chi)\mathrm{Re}\ W_{-11,10}]\ \}\ .$$

The quantities $W_{\ell j,mn}$ contain kinematic factors $(k^2)^{\frac{1}{2}}$ or $(q^2)^{\frac{1}{2}}$ whenever the helicity indices are zero (longitudinal virtual photon). Therefore, Eq. (A-9) simplifies when one or both of the photon is near their mass shell. Also, if none of the final hadrons are measured, we can choose the x-z plane to be defined by one of the outgoing leptons. All terms in the above expression which depend on $\chi-\chi'$ then drop out.

APPENDIX B

Following the procedure of reference 14, we can construct the minimal gauge invariant tensor basis $\{L_i^{\mu\nu\lambda\sigma}\}$ for the forward photon-photon scattering amplitude defined by (25). Through a straight-forward, albeit tedious, calculation, we obtain the following set:

$$L_1 = g^{\mu\sigma}k^\nu q^\lambda + q^\mu k^\sigma g^{\nu\lambda} - g^{\mu\nu}q^\lambda k^\sigma - q^\mu k^\nu g^{\lambda\sigma} + (k\cdot q)(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\sigma}g^{\nu\lambda})$$

$$L_2 = (q^\mu k^\nu - k\cdot q\ g^{\mu\nu})(q^\lambda k^\sigma - k\cdot q\ g^{\lambda\sigma})$$

$$L_3 = (k^2 g^{\mu\lambda} - k^\mu k^\lambda)(q^2 g^{\nu\sigma} - q^\nu q^\sigma)$$

$$L_4 = [k^2 q^\mu q^\lambda - (k\cdot q)(k^\mu q^\lambda + q^\mu k^\lambda) + (k\cdot q)^2 g^{\mu\lambda}](q^2 g^{\nu\sigma} - q^\nu q^\sigma)$$

$$L_5 = (k^2 g^{\mu\lambda} - k^\mu k^\lambda)[q^2 k^\nu k^\sigma - (k\cdot q)(k^\nu q^\sigma + q^\nu k^\sigma) + (k.q)^2 g^{\nu\sigma}]$$

$$\begin{aligned} L_6 = {} & q^2k^2g^{\mu\nu}g^{\lambda\sigma} + (q\cdot k)^2 g^{\mu\lambda}g^{\nu\sigma} + k^2q^\mu q^\lambda g^{\nu\sigma} + q^2 g^{\mu\lambda}k^\nu k^\sigma \\ & + g^{\mu\nu}[(q\cdot k)k^\lambda q^\sigma - k^2q^\lambda q^\sigma - q^2k^\lambda k^\sigma] + [(q\cdot k)k^\mu q^\nu - k^2q^\mu q^\nu - q^2k^\mu k^\nu]g^{\lambda\sigma} \\ & -(q\cdot k)[g^{\mu\lambda}(q^\nu k^\sigma + k^\nu q^\sigma) + (q^\mu k^\lambda + k^\mu q^\lambda)g^{\nu\sigma}] \\ & + k^\mu k^\nu q^\lambda q^\sigma + q^\mu q^\nu k^\lambda k^\sigma \end{aligned}$$

$$\begin{aligned} L_7 = {} & k^2q^2(g^{\mu\sigma}g^{\nu\lambda} - g^{\mu\nu}g^{\lambda\sigma}) - (q\cdot k)(g^{\mu\nu}k^\lambda q^\sigma + k^\mu q^\nu g^{\lambda\sigma} - g^{\mu\sigma}q^\nu k^\lambda - k^\mu q^\sigma g^{\nu\lambda}) \\ & + g^{\mu\nu}(k^2q^\lambda q^\sigma + q^2k^\lambda k^\sigma) + (k^2q^\mu q^\nu + q^2k^\mu k^\nu)g^{\lambda\sigma} \end{aligned}$$

$$- g^{\mu\sigma}(k^2q^\nu q^\lambda+q^2k^\nu k^\lambda)-(k^2q^\mu q^\sigma+q^2k^\mu k^\sigma)g^{\nu\lambda}$$

$$+ k^\mu q^\nu q^\lambda k^\sigma+q^\mu k^\nu k^\lambda q^\sigma-k^\mu k^\nu q^\lambda q^\sigma-q^\mu q^\nu k^\lambda k^\sigma$$

$$L_8 = k^2q^2(g^{\mu\nu}q^\lambda k^\sigma+q^\mu k^\nu g^{\lambda\sigma})+(k\cdot q)^3g^{\mu\lambda}g^{\nu\sigma}$$

$$+(k\cdot q)[q^2g^{\mu\lambda}k^\nu k^\sigma+k^2q^\mu q^\lambda g^{\nu\sigma}-k^2q^2g^{\mu\nu}g^{\lambda\sigma}]$$

$$-(k\cdot q)^2[g^{\mu\lambda}(k^\nu q^\sigma+q^\nu k^\sigma)+(k^\mu q^\lambda+q^\mu k^\lambda)g^{\nu\sigma}]$$

$$+(k\cdot q)(q^\mu k^\lambda+k^\mu q^\lambda)(k^\nu q^\sigma+q^\nu k^\sigma)-k^2q^\mu q^\lambda(k^\nu q^\sigma+q^\nu k^\sigma)$$

$$-q^2(q^\mu k^\lambda+k^\mu q^\lambda)k^\nu k^\sigma$$

We see that, although the first five tensors can be easily anticipated, the last three involve entirely non-trivial combinations of the elementary tensors. Not all of the terms written down in these equations contirbute to the physical helicity amplitudes (24). When contracting with the helicity polarization vectors, terms with k_μ, q_ν, k_λ or q_σ vanish.

FOOTNOTES AND REFERENCES

1. See, for example, M. Gourdin, Proceedings of 11th Scottish Universities' Summer School in Physics, 1970. (To be published).

2. V.E. Balakin, V.M. Budnev, and I.F. Ginzberg, ZhETF Pid. Red. 11, 559 (1970). [Translation: JETP Letters 11, 388 (1970)].

3. S. J. Brodsky, T. Kinoshita, and H. Terazawa, Phys. Rev. Lett. 25, 972 (1970).

4. See, for instance, L. Stodolsky, Phys. Rev. Letters 26, 404 (1971), and references cited therein.

5. P.C. De Celles and J.F. Goehl, Jr., Phys. Rev. 184, 1617 (1969).

6. W. Heitler, The Quantum Theory of Radiation, Third Edition, Oxford University Press (1954), page 414.

7. "Rest Frame" is put in quotation marks for obvious reasons. Both k^2 and q^2 are spacelike, so the natural simple frame analogous to a rest frame (for timelike momenta) is the frame in which k (or q) has only the z-component of its four-momentum non-zero.

8. I. Muzinich, J.M. Wang, and L.L. Wang, Phys. Rev. D2, 1985 (1970).

9. T.P. Cheng and Wu-Ki Tung, Phys. Rev. D3, 733 (1971).

Footnotes and References (continued)

10. The two frames are related by the Lorentz transformation,

$$L^{\mu}{}_{\nu} = \begin{pmatrix} \cosh u & 0 & 0 & \sinh u \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh u & 0 & 0 & -\cosh u \end{pmatrix}.$$

11. Explicitly, in the B.W. frame of k, $\varepsilon_{(\pm)}(k)=(0, \mp 1, -i, 0)/\sqrt{2}$, $\varepsilon_{(0)}(k)=(1, 0, 0, 0)$, and $\varepsilon_{(3)}(k)=(0, 0, 0, 1)$. Still in the B.W. frame of k, the polarization vectors for the q-photon are $\varepsilon_{(\pm)}(q)= =(0, \pm 1, -i, 0)/\sqrt{2}$, $\varepsilon_{(0)}(q)=(\cosh u, 0, 0, -\sinh u)$, and $\varepsilon_{(3)}(q)=(\sinh u, 0, 0, -\cosh u)$.

12. Let m, n = 1, 0, -1. Then $D(\psi,\chi)^{3}{}_{m}=D(\psi,\chi)^{m}{}_{3}=0$, $D(\psi,\chi)^{3}{}_{3}=1$, $D(\psi,\chi)^{m}{}_{n}=e^{-im\chi}\,\bar{d}(\psi)^{m}{}_{n}$, and

$$\bar{d}(\psi)^{m}{}_{n} = \begin{pmatrix} \frac{1}{2}(\cosh\psi+1) & -\sinh\psi/\sqrt{2} & -\frac{1}{2}(\cosh\psi-1) \\ -\sinh\psi/\sqrt{2} & \cosh\psi & \sinh\psi/\sqrt{2} \\ -\frac{1}{2}(\cosh\psi-1) & \sinh\psi/\sqrt{2} & \frac{1}{2}(\cosh\psi+1) \end{pmatrix}$$

13. F.E. Low, Phys. Rev. 120, 582 (1960).

14. W.A. Bardeen and Wu-Ki Tung, Phys. Rev. 173, 1423 (1968).

15. J.D. Bjorken, Phys. Rev. 179, 1547 (1969).

16. For a recent summary of experimental data, see E. Bloom et. al., Contribution to the XV International Conference on High Energy Physics, Kiev, U.S.S.R. (to be published).

Footnotes and Refs. (cont.)

17. K.M. Watson, Phys. Rev. 88, 1163 (1952).

18. We note two relevent facts on this point: (i) the next higher partial wave is d-wave, p-wave being forbidden by charge conjugation, (ii) all available estimates of s and d π-π-phase shifts support this expectation. On the second point, see for instance D. Morgan and G. Shaw, Phys. Rev. D, 2, 521 (1970) and the many contributions to the Proceedings of the Conference on $\pi\pi$ and πK Interactions, Argonne National Laboratories 1969 (unpublished).

19. The precise formula for $a^{(B)}$, Eq. (42), has a cut from $-\infty$ to 0 with discontinuity proportional to $(s-4\mu^2)^{\frac{1}{2}}/s^{\frac{1}{2}}$.

20. Assuming $\delta^I(s) = \pi \frac{s-4\mu^2}{s+16\mu^2}$ (corresponding to a broad resonance at $s^{\frac{1}{2}} \simeq 5\mu$) we found $-\Delta^I(-c)=0.2$. In channels where there is no low energy resonances, it should be much smaller than this.

21. After the completion of this work, we noticed a preprint by D.H. Lyth on the process $\gamma\gamma \to \pi\pi$. His general approach is very similar to ours. In particular, he presented some numerical estimates which supports the approximations which led to the dispersion relation (46) and also wrote down our equation (48). However, he approximates this equation by neglecting the dependence of $e^{\Delta^I(s)}$ on s and obtains an expression which holds for small values of s where $\delta^I(s)$ remain very small. This may render his subsequent inversion formulas unreliable because they involve

Footnotes and Refs. (cont).

integrating over the full range of s. This is true especially if a broad resonance is present near the elastic unitarity region as is believed to be the case. Our results relating $|a^I(s)|$ to $\delta_0{}^I(s)$, Eqs. (52), (53), do not suffer from this difficulty.

22. See the references quoted at the end of footnote 18.

VI. CONCLUSIONS

We have seen that leading Regge trajectories $\alpha(t)$ with square-root branch points at t=0 are generally expected to be present in amplitudes which require faster than logarithmic shrinkage of the diffraction peak.[29] The more detailed conditions have been summarized in Section III. Note that we are talking here about shrinkage which is required by unitarity because of the growth of the amplitude. It is not an extra feature as in the case of ordinary Regge poles. In our arguments, we use s-and t-channel properties of the amplitudes, and we assume that the relevant singularities in the complex angular momentum plane are non-essential. But isolated essential singularities and natural boundaries are also considerably restricted by the analytic properties of $F(t,\lambda)$ as a function of two complex variables. In certain cases they can be excluded, but in this paper we have not discussed these problems.[30]

We note that trajectories of the form $\alpha(t)=\alpha(0)+\text{const.}\sqrt{t}+O(t)$ can, of course, also be present in cases where they are not required by general principles.[31]

Continued partial wave amplitudes with complex singular surfaces of the type (1.1) are most naturally related to representations of scattering amplitudes in terms of superpositions of Bessel functions of the argument $\xi\sqrt{-at}\,\lg s$, $0 \leq \xi \leq 1$. In this sense, the picture of high energy scattering involving these complex trajectories is actually rather intuitive from the point of view of the s-channel. It is possible to relate it to quasi-classical pictures.

Using a rather general Ansatz for the partial wave amplitude near $(t,\lambda)=(0,1)$, we have evaluated the high energy limits of the amplitude in terms of Bessel function representations. These representations are very useful for the construction of specific models.

The character of the singular surfaces (1.1) of $F_{\pm}(t,\lambda)$ is dependent upon the specific asymptotic properties of the amplitude $F(s,t)$, in particular for $t=0$. Although this high energy limit is physical for $t \leq 0$, the general notions of dispersion theory require also that we take into account the t-channel properties of the amplitude, in particular the unitarity constraints. To do this requires some analytic continuation, and it is most simply done by using the continued partial wave amplitudes. We have shown that, unless we want to introduce very special shielding cuts, or singular surfaces with t-dependent character, the actual character of the trajectories (1.1) should be either such that they are by themselves compatible with t-channel unitarity, or that they represent the degenerate limit of a pole-cut relationship. In a pole-cut relationship of this type, the Regge poles and branch points are of the form (1.1) near $t=0$, but they are different near the threshold $t=t_0$, where the poles develop a branch point and the branch point trajectories are weak and do not disturb unitarity.

Incorporating the constraints mentioned above, we have constructed a one parameter family of explicit examples for amplitudes with complex trajectories (1.2). We have chosen the particular case of constant asymptotic total cross-sections σ and $\bar{\sigma}$, with $\sigma \neq \bar{\sigma}$. In the complex

FIGURE CAPTIONS

Fig. 1: Two photon processes in colliding beam experiments. The dashed lines are the incoming and outgoing leptons, the wavy lines are the virtual photons and the solid lines are final state hadrons.

Fig.2: Kinematics in the laboratory frame which is also the C.M. frame of the colliding beams. Note the x-z plane is determined by one of the hadronic momenta p_1.

Fig. 3: Kinematics in the brick wall frame of the k-electron. The hadronic vector p_1 which defines the x-z plane is not shown. The azimuthal angle between the two planes for the k-and q-electrons is $\chi+\chi'$.

Fig. 4: Forward photon-photon scattering; m, n, l, j are helicity indices.

Fig. 5: Kinematics in the pion-pion C.M. frame. The lepton momenta are not shown. Their configuration is the same as in Fig. 3 since the two frames are related simply by a boost along the z-axis.

Fig. 6: The singularities of the integrand in Eq. (48) in the s-plane and contours of integration for Eq. (51) and Eq. (52).

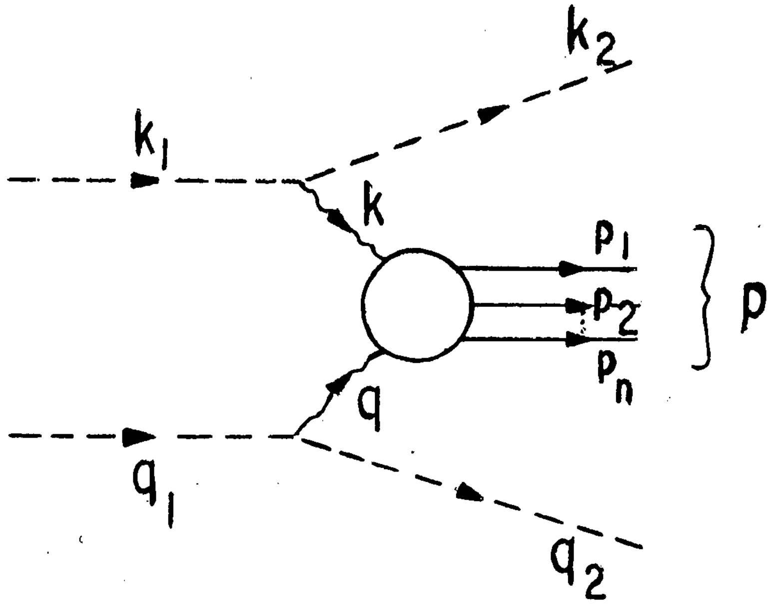

Fig. 1

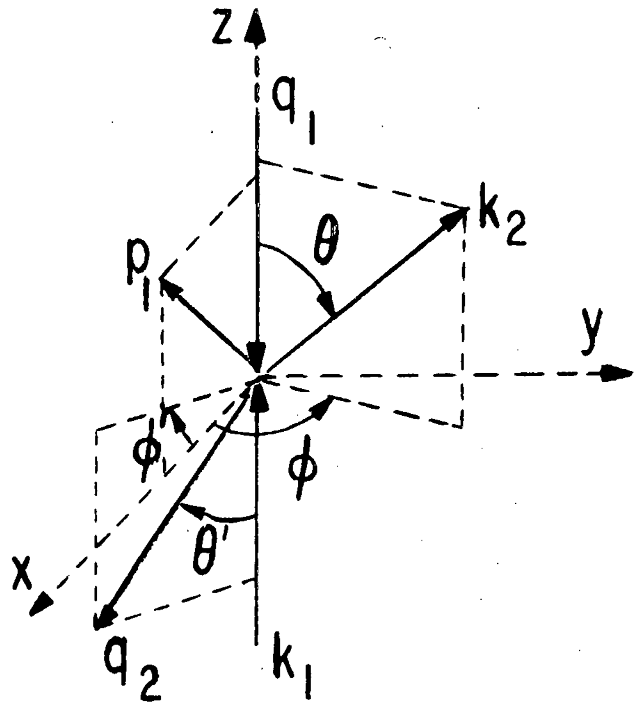

Fig. 2

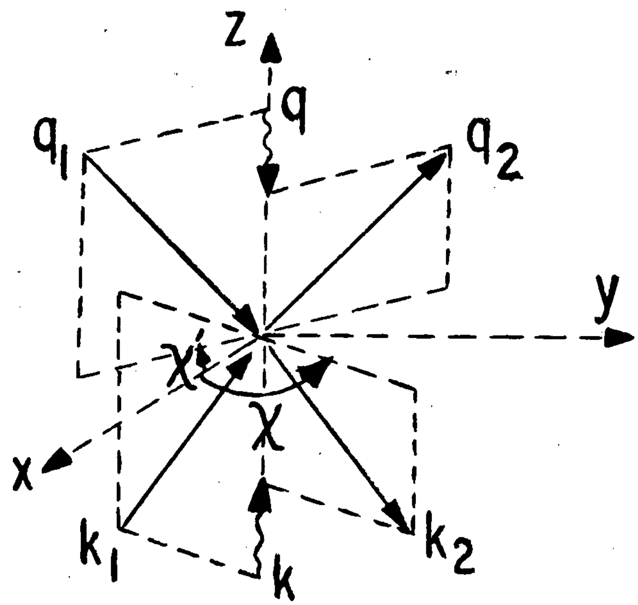

Fig.3

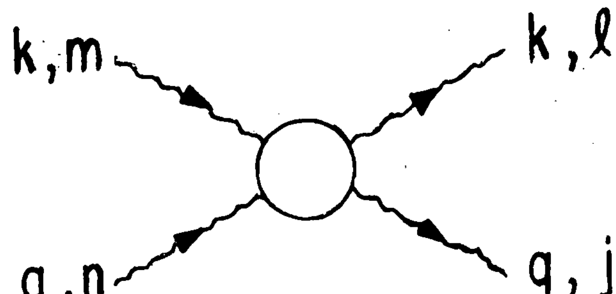

Fig.4

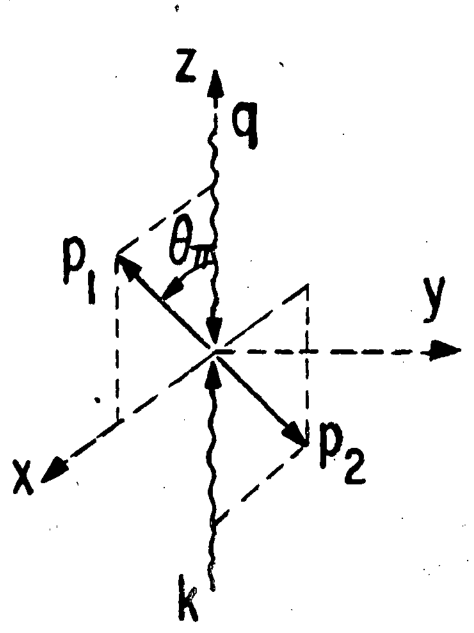

Fig.5

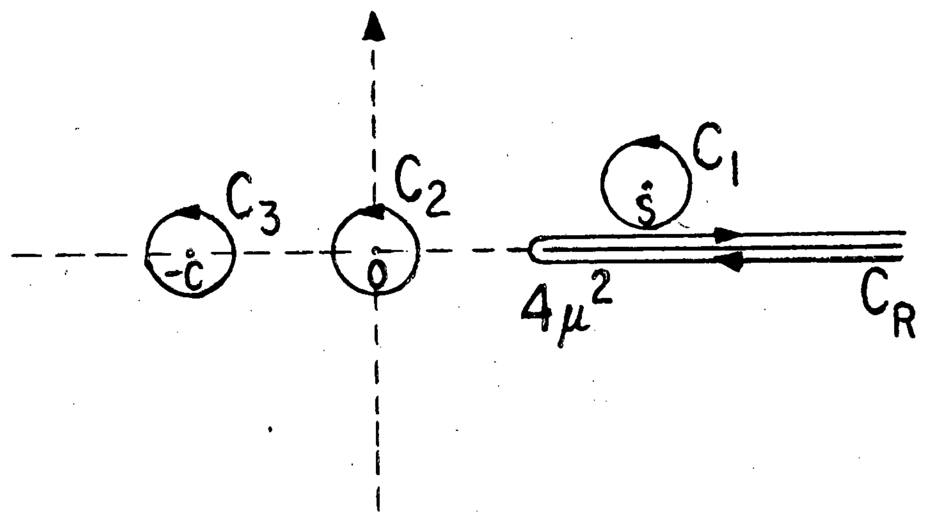

Fig.6