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DEFLECTIONS AND LOAD DISTRIBUTIONS IN LINEAR ELASTIC STRUCTURES—CTAC AND MODE CODES

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BETTIS ATOMIC POWER LABORATORY, PITTSBURGH, PA.,
OPERATED FOR THE U. S. ATOMIC ENERGY COMMISSION
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DEFLECTIONS AND LOAD DISTRIBUTIONS IN LINEAR ELASTIC STRUCTURES—CTAC AND MODE CODES

Carl M. Friedrich

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Since the stress analysis of a complicated statically indeterminate structure involves the handling of many simultaneous equations, the S0207 code for the IBM-650 was developed at Bettis to solve this problem by the use of matrix transformations. The code has two parts, each with a title: (a) the CTAC code gives stresses, deflections, and reactions; and (b) the MODE code calculates natural frequencies and modes of vibration. As an aid to users of the code, a review is given of the elasticity and matrix transformation theory utilized in this code.

DEFLECTIONS AND LOAD DISTRIBUTIONS IN LINEAR ELASTIC STRUCTURES—CTAC AND MODE CODES

C. M. Friedrich

I. INTRODUCTION

The determination of forces and deflections for complicated statically indeterminate structures involves the handling of many simultaneous equations. In order to keep track of the variables and equations in successive steps of calculations, a systematic routine is needed.

One such routine was developed by Gabriel Kron (Ref 1) for electrical engineering problems. This routine of matrix transformations was adapted by Borge Langefors (Ref 2) for elastic structure problems. Kron (Ref 3) later extended Langefors' adaptations to take care of nonlinear plastic deformation. A thorough and detailed discussion with applications on energy principles of structure analysis, including the effects of temperature and nonlinear stress-strain relations, was made by J. H. Argyris (Ref 4).

The purposes of this report are:

- 1) To give a simplified explanation of the matrix transformation method for linear elastic structures (preceded by a review of elementary concepts and theorems in elasticity which are used in the method).
- 2) To show the application of this method in determining influence coefficients of structures that may be redundant.
- 3) To show how this method is applied in the CTAC code (first part of the Wolontis-650 code S0207).
- 4) To show how the MODE code (second part of code S0207) can be used to determine the modes of vibration of an elastic structure.

II. FORCE, DEFLECTION, AND ENERGY

A. Force

The basis of any engineering analysis of a complicated structure is the free body diagram of each component. Every force or moment shown on the free body diagram is actually a system of forces and/or moments which are in equilibrium with each other.

Conventional foundation shading lines on a free body diagram are used to indicate the remainder of the force system acting on the body to retain it in equilibrium. For example, the forces shown acting on the structures in Fig. 1A are actually the force systems shown in Fig. 1B.

The fundamental process of analysis involves the subdividing of the free body diagram of a complicated structure into two or more free bodies of simpler geometry, as shown in Fig. 2. Care must be taken to have the force system on each subdivided body in self-equilibrium and to have the force systems on the subdivided free bodies actually equivalent to the force system on the original free body.

Fig. 1 Free Body Equilibrium

Fig. 2 Subdivision of Free Bodies

In this treatment a force or force system may be labeled by one of the forces or moments in the system. Thus, a "force" may actually be a true force or even a bending moment or a torque.

B. Working Deflection

Reciprocal Theorem

Considering three forces F_1 , F_2 , and F_3 acting on a structure, if the structure is assumed linear-elastic, the working deflection of each force is a linear function of all forces. That is,

$$D_{F1} = A_{11} F_1 + A_{12} F_2 + A_{13} F_3, \quad (1a)$$

$$D_{F2} = A_{21} F_1 + A_{22} F_2 + A_{23} F_3, \quad (1b)$$

and

$$D_{F3} = A_{31} F_1 + A_{32} F_2 + A_{33} F_3 \quad (1c)$$

where

D_{Fi} = working deflection of F_i ,

A_{ij} = influence deflection coefficient, equal to the working deflection of F_i produced by the unit load $F_j = 1$, and

i, j = subscripts having values 1, 2, or 3.

A very important relationship concerning these influence coefficients can be determined by changing the order of load applications and comparing final strain energies. The change in strain energy when loads change on the linear elastic structure is, by definition of working deflections,

$$dU' = F'_1 dD'_{F1} + F'_2 dD'_{F2} + F'_3 dD'_{F3} , \quad (2)$$

where U = strain energy of the structure and a prime is used to denote the value of any quantity during a change. Substitution of the deflections of Eqs (1) and (2) gives

$$\begin{aligned} dU' = & F'_1 (A_{11} dF'_1 + A_{12} dF'_2 + A_{13} dF'_3) \\ & + F'_2 (A_{21} dF'_1 + A_{22} dF'_2 + A_{23} dF'_3) \\ & + F'_3 (A_{31} dF'_1 + A_{32} dF'_2 + A_{33} dF'_3) \end{aligned} \quad (3)$$

First, allow a force, say F'_1 , to increase from zero to full value F_1 while the other forces remain zero. The strain energy of the structure becomes

$$U = \int_0^{F_1} [F'_1 (A_{11} dF'_1 + 0 + 0) + 0 + 0]$$

or

$$U = (A_{11}/2) F_1^2 .$$

Next, allow another force, say F'_2 , to increase from zero to full value F_2 while the other forces remain unchanged. The strain energy of the structure is increased to become

$$U = (A_{11}/2) F_1^2 + \int_0^{F_2} [F_1 (0 + A_{12} dF'_2 + 0) + F'_2 (0 + A_{22} dF'_2 + 0) + 0]$$

or

$$U = (A_{11}/2) F_1^2 + (A_{12}) F_1 F_2 + (A_{22}/2) F_2^2 . \quad (4)$$

If the order of load application were reversed, the total strain energy would be

$$U = (A_{22}/2) F_2^2 + (A_{21}) F_2 F_1 + (A_{11}/2) F_1^2 . \quad (5)$$

Since the final strain energy of a linear-elastic structure depends only on the final shape of the structure as determined by the working deflections and the final working deflections depend only on the final forces according to Eqs (1), then the strain energy should be independent of the order of load applications. Equating the energies of Eqs (4) and (5) leads to the result that $A_{21} = A_{12}$.

This very important result states that the working deflection of one force produced by a unit value of a second force equals the working deflection of the second force produced by a unit value of the first force. This relationship is known as the Reciprocal Theorem. In general,

$$A_{ij} = A_{ji} \quad (6)$$

for all values of i and j .

Castigliano's Theorem

A useful generalization of the energy integral can be obtained by application of the Reciprocal Theorem. Corresponding to Eqs (1), the generalized deflection equation is

$$D_{Fi} = \sum_j A_{ij} F_j , \quad (7)$$

and the generalized strain energy equation is

$$U = \int (\sum_i F_i dD_{Fi}) .$$

Substitution of D_{Fi} from Eq (7) gives

$$U = \int (\sum_i F_i \sum_j A_{ij} dF_j) = \int (\sum_{ij} F_i A_{ij} dF_j) .$$

Now, by using Eq (6),

$$\begin{aligned} U &= \frac{1}{2} \int (\sum_{ij} F_i A_{ij} dF_j) + \frac{1}{2} \int (\sum_{ij} F_j A_{ij} dF_i) \\ &= \frac{1}{2} \int \sum_{ij} A_{ij} (F_i dF_j + F_j dF_i) \\ &= \frac{1}{2} \int \sum_{ij} A_{ij} d(F_i F_j) ; \end{aligned}$$

or after integrating and using Eq (7),

$$U = \frac{1}{2} \sum_{ij} A_{ij} F_i F_j = \frac{1}{2} \sum_i F_i D_{Fi} . \quad (8)$$

A very important relationship can be obtained by differentiation of the strain energy U with respect to a force, say F_1 . Expansion of Eq (8) in terms containing F_1 first gives

$$U = \frac{1}{2} \left(A_{11} F_1^2 + A_{12} F_1 F_2 + A_{21} F_2 F_1 + A_{13} F_1 F_3 + A_{31} F_3 F_1 \right) ,$$

so that differentiation with respect to F_1 gives

$$\frac{\partial U}{\partial F_1} = \frac{1}{2} (2 A_{11} F_1 + A_{12} F_2 + A_{21} F_2 + A_{13} F_3 + A_{31} F_3) .$$

Since $A_{ij} = A_{ji}$, according to the Reciprocal Theorem,

$$\frac{\partial U}{\partial F_1} = A_{11} F_1 + A_{12} F_2 + A_{13} F_3 ,$$

which, according to Eqs (1) or Eq (7), is simply D_{F1} ; thus,

$$D_{F1} = \frac{dU}{dF_1} .$$

This result is known as Castigliano's Theorem. The partial derivative of the strain energy of a linear elastic system with respect to a force equals the working deflection of that force. In general,

$$D_{Fi} = \frac{\partial U}{\partial F_i} . \quad (9)$$

Structure Position

The working deflection of a force acting on a structure is actually the total working deflection of all components of the force system acting on the structure. For example, consider the spring shown in Fig. 3. Suppose that while the left end moves through a distance x_1 , the spring force increases from 0 to full value F_1 . If the spring constant is K_1 , the total stretch of the spring s_1 is equal to F_1/K_1 and the total movement of the right end of the spring is $x_1 + s_1$.

The work W_L done by the left force is

$$W_L = \int_0^{F_1} F_1' dx_1' ,$$

A. ORIGINAL POSITION

B. FINAL POSITION

Fig. 3 Changed Position of a Spring

and the work W_R done by the right force is

$$W_R = + \int_0^{F_1} F_1' d(x_1' + s_1') = + \int_0^{F_1} F_1' dx_1' + \int_0^{F_1} F_1' ds_1' .$$

The strain energy U in the spring is equal to the total work done on the structure. Thus,

$$\begin{aligned} U &= W_L + W_R \\ &= \left(- \int_0^{F_1} F_1' dx_1' \right) + \left(+ \int_0^{F_1} F_1' dx_1' + \int_0^{F_1} F_1' ds_1' \right) \\ &= \int_0^{F_1} F_1' ds_1' , \end{aligned}$$

which is independent of the final position of the spring. Since $s_1 = F_1/K_1$, then

$$U = \int_0^{F_1} F_1' d(F_1'/K_1) = \left(\frac{1}{2K_1} \right) F_1^2 .$$

Then, according to Eq (8), the deflection influence coefficient of F_1 is

$$A_{11} = 1/K_1 ,$$

so that the working deflection of F_1 is, according to Eq (7),

$$D_{F1} = A_{11} F_1 = (1/K_1)F_1 = s_1 .$$

Thus, the working deflection of the force system F_1 is simply the stretch of the spring, as is expected.

In general, working deflections of force systems acting on a linear elastic structure are independent of the final position of the structure and depend only on the final deformation of the structure.

C. Strain Energy

Beam Energy

The influence coefficients for one system of forces can be determined from the influence coefficients of an equivalent system of forces by consideration of the strain energy in the structure. Calculations involved in this process will be carried out for a cantilever beam and a simple support beam in order to clarify the development of the necessary formulas.

The strain energy for a uniform beam under the usual engineering approximations (Ref 5) is

$$U = \int_0^L \left[\left(\frac{1}{2EI} \right) M^2 + \left(\frac{1}{2E_s A_s} \right) V^2 + \left(\frac{1}{2EA} \right) F^2 \right] dx, \quad (10)$$

where

- L = length of a beam, in.
- x = distance along beam, in.
- E = tensile modulus of elasticity, lb-in.⁻²
- E_s = shear modulus of elasticity, lb-in.⁻²
- EI = bending stiffness of beam, lb-in.²
- $E_s A_s$ = shear stiffness of beam, lb
- EA = tensile stiffness of beam, lb
- M = bending moment at x , in.-lb
- V = shear force at x , lb
- F = tensile force at x , lb

Fig. 4 Cantilever Beam

Bending in one plane, shear in the same plane, and tension are the only deformations considered in Eq (10).

Cantilever Beam

Consider a cantilever beam loaded with force systems F_1 , V_1 , and M_1 as shown in Fig. 4. The load distribution along this beam is then

$$M = M_1 + V_1 (L - x), \quad (11a)$$

$$V = V_1, \quad (11b)$$

and

$$F = F_1. \quad (11c)$$

Substitution of these relations into the beam energy equation gives

$$U = \int_0^L \left\{ \left(\frac{1}{2EI} \right) [M_1 + V_1 (L - x)]^2 + \left(\frac{1}{2E_s A_s} \right) V_1^2 + \left(\frac{1}{2EA} \right) F_1^2 \right\} dx$$

which, after integration, becomes

$$U = M_1^2 \left(\frac{L}{2EI} \right) + M_1 V_1 \left(\frac{L^2}{2EI} \right) + V_1^2 \left(\frac{L^3}{6EI} + \frac{L}{2E_s A_s} \right) + F_1^2 \left(\frac{L}{2EA} \right) \quad (12)$$

The cantilever orientation consists of having the left end of the beam on and tangent to the reference line from which deflections are measured. With this orientation, the force components at the left end undergo no deflection and do no work.

The working deflection of the force system M_1 in Fig. 4 is simply the slope θ_1 at the right end; the working deflection of the force system V_1 is the deflection y_1 at the right end; and the working deflection of the force system F_1 is the elongation s_1 of the beam. Therefore, application of Castigliano's Theorem, Eq (9), to the cantilever beam energy, Eq (12), gives

$$\theta_1 = D_{M1} = \frac{\partial U}{\partial M_1} = \left(\frac{L}{EI} \right) M_1 + \left(\frac{L^2}{2EI} \right) V_1 , \quad (13a)$$

$$y_1 = D_{V1} = \frac{\partial U}{\partial V_1} = \left(\frac{L^2}{2EI} \right) M_1 + \left(\frac{L^3}{3EI} + \frac{L}{E_s A_s} \right) V_1 , \quad (13b)$$

and

$$s_1 = D_{F1} = \frac{\partial U}{\partial F_1} = \left(\frac{L}{EA} \right) F_1 . \quad (13c)$$

Simple Support Beam

Next, consider a simple support beam loaded with force systems F_a , M_b , M_c as shown in Fig. 5. The load distribution along the beam is

$$M = \left(1 - \frac{x}{L} \right) M_b + \left(\frac{x}{L} \right) M_c , \quad (14a)$$

$$V = (1/L) (M_b - M_c) , \quad (14b)$$

and

$$F = F_a . \quad (14c)$$

The simple support orientation consists of having the ends of the beam on the reference line from which deflections are measured. With this orientation, the shear loads at each end undergo no deflection and do no work. Thus, the working deflection of the force system M_c is the slope θ_c at the right end, the working deflection of the force system M_b is the slope θ_b at the left end, and the working deflection of the force system F_a is the elongation s_a of the beam.

Instead of evaluating the energy integral and then differentiating, the process can be reversed in calculating working deflections by Castigliano's Theorem. Thus, from Eq (9) and Eq (10),

$$s_a = D_{F_a} = \frac{\partial U}{\partial F_a} = \int_0^L \left[\left(\frac{1}{EI} \right) (M) \left(\frac{\partial M}{\partial F_a} \right) + \left(\frac{1}{E_s A_s} \right) (V) \left(\frac{\partial V}{\partial F_a} \right) + \left(\frac{1}{EA} \right) (F) \left(\frac{\partial F}{\partial F_a} \right) \right] dx .$$

Fig. 5 Simple Support Beam

From Eq (14),

$$\frac{\partial M}{\partial F_a} = 0, \quad \frac{\partial V}{\partial F_a} = 0, \quad \text{and} \quad \frac{\partial F}{\partial F_a} = 1,$$

so that

$$s_a = \int_0^L \left[0 + 0 + \left(\frac{1}{EA} \right) (F_a)(1) \right] dx = \left(\frac{L}{EA} \right) F_a.$$

Similarly,

$$\begin{aligned} \theta_b = D_{Mb} = \frac{\partial U}{\partial M_b} &= \int_0^L \left[\left(\frac{1}{EI} \right) (M) \left(\frac{\partial M}{\partial M_b} \right) + \left(\frac{1}{E_s A_s} \right) (V) \left(\frac{\partial V}{\partial M_b} \right) + \left(\frac{1}{EA} \right) (F) \left(\frac{\partial F}{\partial M_b} \right) \right] dx \\ &= \int_0^L \left\{ \left(\frac{1}{EI} \right) \left[M_b \left(1 - \frac{x}{L} \right) + M_c \left(\frac{x}{L} \right) \right] \left(1 - \frac{x}{L} \right) \right. \\ &\quad \left. + \left(\frac{1}{E_s A_s} \right) \left[\left(\frac{1}{L} \right) (M_b - M_c) \right] \left(\frac{1}{L} \right) + \left(\frac{1}{EA} \right) (F_a)(0) \right\} dx \\ &= \left(\frac{L}{3EI} + \frac{1}{E_s A_s L} \right) M_b + \left(\frac{L}{6EI} - \frac{1}{E_s A_s L} \right) M_c, \end{aligned}$$

and θ_c is found the same way. The three working deflections thus evaluated are

$$s_a = \left(\frac{L}{EA} \right) F_a, \quad (15a)$$

$$\theta_b = \left(\frac{L}{3EI} + \frac{1}{E_s A_s L} \right) M_b + \left(\frac{L}{6EI} - \frac{1}{E_s A_s L} \right) M_c, \quad (15b)$$

and

$$\theta_c = \left(\frac{L}{6EI} - \frac{1}{E_s A_s L} \right) M_b + \left(\frac{L}{3EI} + \frac{1}{E_s A_s L} \right) M_c. \quad (15c)$$

Transformation from Cantilever to Simple Support

By comparison of Figs. 4 and 5, the cantilever beam and the single support beam are seen to have the same loading distributions if

$$M_1 = M_c \quad (16a)$$

$$V_1 = (1/L)(M_b - M_c), \quad (16b)$$

and

$$F_1 = F_a. \quad (16c)$$

Then, the cantilever deflections can be found in terms of the simple support loads by putting the loads of Eqs (16) in the cantilever deflections of Eqs (13). After simplification, the equations become

$$\theta_1 = \left(\frac{L}{2EI} \right) M_b + \left(\frac{L}{2EI} \right) M_c, \quad (17a)$$

$$y_1 = \left(\frac{L^2}{3EI} + \frac{1}{E_s A_s} \right) M_b + \left(\frac{L^2}{6EI} - \frac{1}{E_s A_s} \right) M_c, \quad (17b)$$

and

$$s_1 = \left(\frac{L}{EA} \right) F_a. \quad (17c)$$

Then, substitution of deflections from Eqs (17) and forces from Eqs (16) into the energy equation (8) gives

$$U = \frac{1}{2} (M_1 \theta_1 + V_1 y_1 + F_1 s_1) = \frac{1}{2} \left\{ (M_c) \left[\left(\frac{L}{2EI} \right) M_b + \left(\frac{L}{2EI} \right) M_c \right] \right. \\ \left. + (1/L)(M_b - M_c) \left[\left(\frac{L^2}{3EI} + \frac{1}{E_s A_s} \right) M_b + \left(\frac{L^2}{6EI} - \frac{1}{E_s A_s} \right) M_c \right] + (F_a) \left[\left(\frac{L}{EA} \right) F_a \right] \right\}$$

or

$$U = \left(\frac{L}{2EA} \right) F_a^2 + \left(\frac{L}{6EI} + \frac{1}{2E_s A_s L} \right) M_b^2 + \left(\frac{L}{6EI} - \frac{1}{E_s A_s L} \right) M_b M_c \\ + \left(\frac{L}{6EI} + \frac{1}{2E_s A_s L} \right) M_c^2 \quad (18)$$

Differentiation with respect to the force systems gives

$$D_{F_a} = \frac{\partial U}{\partial F_a} = \left(\frac{L}{EA} \right) F_a, \quad (19a)$$

$$D_{M_b} = \frac{\partial U}{\partial M_b} = \left(\frac{L}{3EI} + \frac{1}{E_s A_s L} \right) M_b + \left(\frac{1}{6EI} - \frac{1}{E_s A_s L} \right) M_c, \quad (19b)$$

and

$$D_{M_c} = \frac{\partial U}{\partial M_c} = \left(\frac{L}{6EI} - \frac{1}{E_s A_s L} \right) M_b + \left(\frac{L}{3EI} + \frac{1}{E_s A_s L} \right) M_c, \quad (19c)$$

which are the same as in Eqs (15). Thus, by using the cantilever deflection formulas of Eqs (13) and the force relationships of Eqs (16) in an energy equation, the working deflections of the simple support beam have been obtained.

This transformation from the influence coefficients of the cantilever force system to the influence coefficients of the simple support force system shows that the cantilever and simple support beams are really the same structure with the same loading. The only real difference is the change of the reference line from which deflections are measured, as shown in Fig. 6, to correspond with the change in force system notation.

According to Eqs (17c) and (19a), $s_1 = s_a$. This is true in Fig. 6 if deflections are small compared to structure dimensions so that $L \gg s_1$, and if slopes are small so that $\sin \theta \cong \theta$ and $\cos \theta \cong 1$, where $\theta = \theta_1$, θ_a , or θ_b . As shown in Fig. 6,

Fig. 6 Equivalence of Cantilever and Simple Support Beam

$$\theta_b = (1/L) y_1$$

and

$$\theta_c = \theta_1 - (1/L) y_1$$

Substitution of the values from Eqs (17) into these equations gives

$$\theta_b = \left(\frac{L}{3EI} + \frac{1}{E_s A_s L} \right) M_b + \left(\frac{L}{6EI} - \frac{1}{E_s A_s L} \right) M_c$$

and

$$\begin{aligned} \theta_c &= \left[\left(\frac{L}{2EI} \right) M_b + \left(\frac{L}{2EI} \right) M_c \right] - \left[\left(\frac{L}{3EI} + \frac{1}{E_s A_s L} \right) M_b + \left(\frac{L}{6EI} - \frac{1}{E_s A_s L} \right) M_c \right] \\ &= \left(\frac{L}{6EI} - \frac{1}{E_s A_s L} \right) M_b + \left(\frac{L}{3EI} + \frac{1}{E_s A_s L} \right) M_c \end{aligned}$$

which are the same as obtained in Eqs (15). Thus, the use of strain energy in transforming influence coefficients from one force system to another equivalent force system is checked by direct use of geometry.

III. SYSTEMATIC CALCULATION

A. Cantilever

Influence Coefficients

In order to keep calculations involving many variables in simultaneous equations under control, a systematic procedure involving matrices* will be used. Consider the cantilever deflections of Eqs (13) rewritten with all influence coefficients present:

$$\theta_1 = \left(\frac{L}{EI} \right) M_1 + \left(\frac{L^2}{2EI} \right) V_1 + (0) F_1$$

$$y_1 = \left(\frac{L^2}{2EI} \right) M_1 + \left(\frac{L^3}{3EI} + \frac{L}{E_s A_s} \right) V_1 + (0) F_1$$

$$s_1 = (0) M_1 + (0) V_1 + \left(\frac{L}{EA} \right) F_1$$

In matrix form, these equations become

$$\begin{bmatrix} \theta_1 \\ y_1 \\ s_1 \end{bmatrix} = \begin{bmatrix} \frac{L}{EI} & \frac{L^2}{2EI} & 0 \\ \frac{L^2}{2EI} & \frac{L^3}{3EI} + \frac{L}{E_s A_s} & 0 \\ 0 & 0 & \frac{L}{EA} \end{bmatrix} \times \begin{bmatrix} M_1 \\ V_1 \\ F_1 \end{bmatrix}$$

or simply

$$(D_p) = (A_p) \times (P) \quad , \quad (20)$$

where

$$(P) = \begin{bmatrix} M_1 \\ V_1 \\ F_1 \end{bmatrix} = \text{force matrix of cantilever,}$$

*Elementary properties of matrices are discussed in Ref 6.

$$(D_p) = \begin{bmatrix} \theta_1 \\ y_1 \\ s_1 \end{bmatrix} = \text{deflection matrix of cantilever, and}$$

$$(A_p) = \begin{bmatrix} \frac{L}{EI} & \frac{L^2}{2EI} & 0 \\ \frac{L^2}{2EI} & \frac{L^3}{3EI} + \frac{L}{E_s A_s} & 0 \\ 0 & 0 & \frac{L}{EA} \end{bmatrix} = \text{flexibility matrix of cantilever.}$$

Let $(D_p)_{ij}$ = element in row i and column j of matrix (D_p) . The equations are so arranged that $(D_p)_{i1}$ is the working deflection of force $(P)_{i1}$ for $i = 1, 2$, and 3 . With this arrangement, the flexibility matrix (A_p) is symmetrical, as required by Eq (6).

Energy

According to Eq (8), the cantilever energy is

$$U = \frac{1}{2} (M_1 \theta_1 + V_1 y_1 + F_1 s_1) ;$$

or in matrix form,

$$(U) = \frac{1}{2} \times (P^T) \times (D_p) , \quad (21a)$$

where $(U) = \begin{bmatrix} U \end{bmatrix}$ = the energy matrix of 1 row and 1 column, and (P^T) is the transpose of (P) .

Substitution of the cantilever deflection matrix (D_p) from Eq (20) into Eq (21a) gives

$$U = \frac{1}{2} \times (P^T) \times (A_p) \times (P) . \quad (21b)$$

Equations (21a) and (21b) are matrix equivalents of Eq (8).

B. Simple Support

CTAC Rule

The cantilever matrix deflection equation (20) and the energy equation (21b) can be used to determine the matrix deflection equations for the simple support beam, once the relationship between the force systems is put in matrix form. Equation (16) is equivalent to:

$$\begin{bmatrix} M_1 \\ V_1 \\ F_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \frac{1}{L} & -\frac{1}{L} \\ 1 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} F_a \\ M_b \\ M_c \end{bmatrix}$$

or

$$(P) = (C) \times (Q) , \quad (22)$$

where

$$(Q) = \begin{bmatrix} F_a \\ M_b \\ M_c \end{bmatrix} = \text{force matrix for simple support beam,}$$

and

$$(C) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \frac{1}{L} & -\frac{1}{L} \\ 1 & 0 & 0 \end{bmatrix} = \text{connection matrix relating cantilever forces (P) to simple support forces (Q).}$$

Substituting Eq (22) into Eq (21a) gives

$$(U) = \frac{1}{2} \times [(C) \times (Q)]^T \times (D_p) ;$$

or, since the transposed product of two matrices is the product of the transposes of the two matrices in reverse order,

$$(U) = \frac{1}{2} \times (Q^T) \times (C^T) \times (D_p) . \quad (23)$$

Let

$$(D_q) = \begin{bmatrix} s_a \\ \theta_b \\ \theta_c \end{bmatrix} = \text{simple support deflection matrix corresponding to the simple support force matrix (Q),}$$

so that, as in Eq (21a),

$$(U) = \frac{1}{2} (Q^T) \times (D_q) .$$

Comparison of this matrix equation with Eq (23) shows that

$$(D_q) = (C^T) \times (D_p) . \quad (24a)$$

Next, substitution of (D_p) from Eq (20) gives

$$(D_q) = (C^T) \times (A_p) \times (P) ,$$

and substitution of (P) from Eq (22) gives

$$(D_q) = (C^T) \times (A_p) \times (C) \times (Q) . \quad (24b)$$

Let (A_Q) be the simple support flexibility matrix relating (D_Q) and (Q) in the equation

$$(D_q) = (A_Q) \times (Q) . \quad (24c)$$

Then, from Eqs (24b) and (24c),

$$(A_Q) = (C^T) \times (A_p) \times (C) . \quad (25)$$

This is the most important equation of all—the "CTAC" equation. It gives the flexibility of a system of forces acting on a linear-elastic structure in terms of (1) the flexibility of an equivalent second system of forces and (2) the connection matrix relating the second system to the first.

In more simple notation, let A be the flexibility matrix of a force matrix P , and C be the correction matrix relating P to an equivalent force matrix P' . Then, the flexibility matrix A' of P' is simply

$$A' = C^T A C ,$$

where $P = C P'$.

Flexibility

As a check, the CTAC equation (Eq 25) will be evaluated for the simple support beam. Thus,

$$\begin{aligned}
 (A_q) &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & \frac{1}{L} & 0 \\ 1 & -\frac{1}{L} & 0 \end{bmatrix} \times \begin{bmatrix} \frac{L}{EI} & \frac{L^2}{2EI} & 0 \\ \frac{L^2}{2EI} & \frac{L^3}{3EI} + \frac{L}{E_s A_s} & 0 \\ 0 & 0 & \frac{L}{EA} \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 1 \\ 0 & \frac{1}{L} & -\frac{1}{L} \\ 1 & 0 & 0 \end{bmatrix} \\
 &= \begin{bmatrix} \frac{L}{EA} & 0 & 0 \\ 0 & \frac{L}{3EI} + \frac{1}{E_s A_s L} & \frac{L}{6EI} - \frac{1}{E_s A_s L} \\ 0 & \frac{L}{6EI} - \frac{1}{E_s A_s L} & \frac{L}{3EI} - \frac{1}{E_s A_s L} \end{bmatrix}
 \end{aligned}$$

Substitution into Eq (24c) gives

$$\begin{bmatrix} s_a \\ \theta_b \\ \theta_c \end{bmatrix} = \begin{bmatrix} \frac{L}{EA} & 0 & 0 \\ 0 & \frac{L}{3EI} + \frac{1}{E_s A_s L} & \frac{L}{6EI} - \frac{1}{E_s A_s L} \\ 0 & \frac{L}{6EI} - \frac{1}{E_s A_s L} & \frac{L}{3EI} - \frac{1}{E_s A_s L} \end{bmatrix} \times \begin{bmatrix} F_a \\ M_b \\ M_c \end{bmatrix}$$

This is equivalent to the three simultaneous equations

$$s_a = \left(\frac{L}{EA} \right) F_a + (0) M_b + (0) M_c ,$$

$$\theta_b = (0) F_a + \left(\frac{L}{3EI} + \frac{1}{E_s A_s L} \right) M_b + \left(\frac{L}{6EI} - \frac{1}{E_s A_s L} \right) M_c ,$$

and

$$\theta_c = (0) F_a + \left(\frac{L}{6EI} - \frac{1}{E_s A_s L} \right) M_b + \left(\frac{L}{3EI} - \frac{1}{E_s A_s L} \right) M_c ,$$

which check Eq (15). Thus, the CTAC procedure does lead to the correct result.

IV. APPLICATION OF CTAC

A. Structure in WAPD-115

Given Data

The problem is to determine the influence coefficients and modes of vibration of the structure used as an example in WAPD-115 (Ref 7) and shown in Fig. 7. Geometric and flexibility coefficients are given in the following table.

TABLE OF GEOMETRIC AND FLEXIBILITY COEFFICIENTS

Region No.	1, 2	3, 4	5, 6	7, 8	Units
D_o	12	12	6	7.5	in.
D_i	11	8	0	0	in.
$A = \pi(D_o^2 - D_i^2)/4$	18.06	62.83	28.27	44.18	in. ²
$I = \pi(D_o^4 - D_i^4)/64$	299.2	816.8	63.62	155.3	in. ⁴
$A_s = A/2$ or $3A/4$	9.03	31.42	21.20	33.14	in. ²
L	10	5	7.5	7.5	in.
$W = (0.284 \text{ pci})A\bar{L}$	51.29	89.22	60.22	94.10	lb
$EI = (30 \times 10^6 \text{ psi})I$	8976	24,504	1908.6	4659	10^6 lb-in.^2
$E_s A_s = (12 \times 10^6 \text{ psi})A_s$	108.4	377.0	254.4	397.7	10^6 lb
$L/E_s A_s$	0.09225	0.01326	0.02948	0.01885	10^{-6} in./lb
L/EI	0.001,114	0.000,204	0.003,930	0.001,610	$10^{-6}/\text{lb-in.}$
$L^2/2EI$	0.005,570	0.000,510	0.01474	0.00604	$10^{-6}/\text{lb}$
$L^3/3EI$	0.03714	0.001,700	0.07368	0.03018	10^{-6} in./lb
$(L^3/3EI) + L/E_s A_s$	0.1294	0.01496	0.10316	0.04903	10^{-6} in./lb

Fig. 7 Geometry of Sample Problem

Fig. 8 Separated Free Bodies

The loads, including the statically indeterminate end reaction F_5 between the two cylinders, may be put into the single force matrix:

$$(F) = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \end{bmatrix}$$

Flexibility of a Subdivided Free Body

Consider region No. 1 as a separate free body with cantilever loads V_1 and M_1 as shown in Fig. 8. These loads can be put into the force matrix:

$$(P_1) = \begin{bmatrix} M_1 \\ V_1 \end{bmatrix}$$

The cantilever equations for region No. 1 are

$$D_{M_1} = \left(\frac{L}{EI} \right)_1 M_1 + \left(\frac{L^2}{2EI} \right)_1 V_1 = (1.114 \times 10^{-9} / \text{in.} \cdot \text{lb}) M_1 + (5.57 \times 10^{-9} / \text{lb}) V_1$$

and

$$D_{V_1} = \left(\frac{L^2}{2EI} \right)_1 M_1 + \left(\frac{L^3}{3EI} + \frac{L}{E_s A_s} \right)_1 V_1 = (5.57 \times 10^{-9} / \text{lb}) M_1 + (129.4 \times 10^{-9} \text{ in.} / \text{lb}) V_1$$

so that the flexibility matrix for (P_1) on body No. 1 is

$$(A_1) = \begin{bmatrix} 1.114 \text{ in.}^{-1} & 5.57 \\ 5.57 & 129.4 \text{ in.} \end{bmatrix} \times 10^{-9} \text{ lb}^{-1}$$

By inspection of Figs. 7 and 8, the moment at the right end of body No. 1 is seen to be

$$M_1 = (15 \text{ in.}) F_2 + (20 \text{ in.}) F_5$$

and the shear load on body No. 1 is

$$V_1 = F_1 + F_2 + F_3$$

In expanded form, the equations become

$$M_1 = (0) F_1 + (15 \text{ in.}) F_2 + (0) F_3 + (0) F_4 + (20 \text{ in.}) F_5$$

and

$$V_1 = (1) F_1 + (1) F_2 + (0) F_3 + (0) F_4 + (1) F_5$$

so that

$$\begin{bmatrix} M_1 \\ V_1 \end{bmatrix} = \begin{bmatrix} 0 & 15 \text{ in.} & 0 & 0 & 20 \text{ in.} \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \end{bmatrix}$$

or

$$(P_1) = (C_1) \times (F)$$

where $(C_1) =$

0	15 in.	0	0	20 in.
1	1	0	0	1

The flexibility matrix of (F) produced by deflections of body No. 1 is then, by the CTAC rule,

$$\begin{aligned} (A_{F1}) &= (C_1^T) \times (A_1) \times (C_1) \\ &= \begin{bmatrix} 0 & 1 \\ 15 \text{ in.} & 1 \\ 0 & 0 \\ 0 & 0 \\ 20 \text{ in.} & 1 \end{bmatrix} \times \begin{bmatrix} 1.14 \text{ in.}^{-1} & 5.57 \\ 5.57 & 129.4 \text{ in.} \end{bmatrix} \times 10^{-9} \text{ lb}^{-1} \\ &\quad \times \begin{bmatrix} 0 & 15 \text{ in.} & 0 & 0 & 20 \text{ in.} \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0.1294 & 0.21295 & 0 & 0 & 0.2408 \\ 0.21295 & 0.54715 & 0 & 0 & 0.65855 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0.2408 & 0.65855 & 0 & 0 & 0.7978 \end{bmatrix} \times 10^{-6} \text{ in. / lb} \end{aligned}$$

This is the flexibility matrix giving the deflections of the forces in (F) resulting from the flexibility of body No. 1.

Total Flexibility

In the same way, the deflections due to body No. 2 can be determined. As before, let

$$(P_2) = \begin{bmatrix} M_2 \\ V_2 \end{bmatrix}$$

so that

$$(A_2) = (A_1) = \begin{bmatrix} 1.114 \text{ in.}^{-1} & 5.57 \\ 5.57 & 129.4 \text{ in.} \end{bmatrix} \times 10^{-9} \text{ lb}^{-1}$$

The connection equations for (P_2) in terms of (F) are

and $M_2 = (5 \text{ in.})F_2 + (10 \text{ in.})F_5$

$$V_2 = F_2 + F_5$$

so that

$$(C_2) = \begin{bmatrix} 0 & 5 \text{ in.} & 0 & 0 & 10 \text{ in.} \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Carrying through the multiplications of

$$(A_{F2}) = (C_2^T) \times (A_2) \times (C_2)$$

gives

$$(A_{F2}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0.21295 & 0 & 0 & 0.26865 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0.26865 & 0 & 0 & 0.3522 \end{bmatrix} \times 10^{-6} \text{ in. / lb}$$

The total flexibility of (F) due to both body No. 1 and body No. 2 is

$$(A_{F1,2}) = (A_{F1}) + (A_{F2})$$

$$= \begin{bmatrix} 0.1295 & 0.21295 & 0 & 0 & 0.2408 \\ 0.21295 & 0.7601 & 0 & 0 & 0.9272 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0.2408 & 0.9272 & 0 & 0 & 1.15 \end{bmatrix} \times 10^{-6} \text{ in. / lb}$$

The final flexibility matrix for all eight primitives (basic beam structures) is found to be

$$(A_F) = (A_{F1 \text{ to } 8})$$

$$= \begin{bmatrix} 0.1295 & 0.21295 & 0 & 0 & 0.2408 \\ 0.21295 & 0.77506 & 0 & 0 & 0.94471 \\ 0 & 0 & 2.0239625 & 0.32426 & -2.95355 \\ 0 & 0 & 0.32426 & 0.10316 & -0.43481 \\ -0.2408 & 0.94471 & -2.95355 & -0.43481 & 5.654975 \end{bmatrix} \times 10^{-6} \text{ in. / lb}$$

It is to be noted that all of the flexibility matrices are symmetrical, in agreement with the Reciprocal Theorem.

Redundant Load Calculation

The working deflection D_{F5} is the net working deflection of F_5 on both cantilever beams in the structure. Since there is no net deflection between the beams connected at the end, D_{F5} should be zero. Thus,

$$\begin{aligned} 0 = D_{F5} &\equiv (D_F)_{51} \equiv \sum_i (A_F)_{5i} (F)_{i1} \\ &= (0.2408F_1 + 0.94471F_2 - 2.95355F_3 - 0.43481F_4 + 5.654975F_5) \times 10^{-6} \text{ in./lb} . \end{aligned}$$

This equation is used to obtain the nonworking (redundant) force F_5 in terms of the working (non-redundant) forces F_1 to F_4 . Solving for F_5 gives

$$F_5 = -0.042,582F_1 - 0.167,058F_2 + 0.522,292F_3 + 0.076,890F_4 . \quad (26)$$

By use of this equation, the net working deflections of the other forces F_1 to F_4 can be obtained; for example,

$$\begin{aligned} D_{F1} &= \sum_i (A_F)_{1i} (F)_{i1} \\ &= [0.1295F_1 + 0.21295F_2 + (0)F_3 + (0)F_4 \\ &\quad + 0.2408(-0.042,582F_1 - 0.167,058F_2 + 0.522,292F_3 + 0.076,890F_4)] \times 10^{-6} \text{ in./lb} \\ &= (0.119,246F_1 + 0.172,722F_2 + 0.125,768F_3 + 0.018,515F_4) \times 10^{-6} \text{ in./lb} . \end{aligned}$$

Similarly,

$$D_{F2} = (0.172,722F_1 + 0.617,239F_3 + 0.493,414F_3 + 0.072,639F_4) \times 10^{-6} \text{ in./lb} ,$$

and so forth. The resulting flexibility matrix for the net working deflection is thus determined.

Let

$$(W) = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} \text{ be the working force matrix ,}$$

and

$$(D_W) = \begin{bmatrix} D_{W1} \\ D_{W2} \\ D_{W3} \\ D_{W4} \end{bmatrix} \text{ be the corresponding deflection matrix .}$$

Then, the flexibility matrix in the equation

$$(D_W) = (A_W) \times (W) \quad (27)$$

is

$$(A_W) = \begin{array}{|c|c|c|c|} \hline 0.119,246 & 0.172,722 & 0.125,768 & 0.081,515 \\ \hline 0.172,722 & 0.617,239 & 0.493,414 & 0.072,639 \\ \hline 0.125,768 & 0.493,414 & 0.481,347 & 0.097,162 \\ \hline 0.018,515 & 0.072,639 & 0.087,162 & 0.069,727 \\ \hline \end{array} \times 10^{-6} \text{ in. / lb .}$$

This checks Eq (5) in WAPD-115 to slide rule accuracy.

Systematic Load Calculation

A systematic way to calculate (A_W) is to use the concept of partitioned matrices. The original flexibility matrix equation for the statically determinate free bodies is

$$\begin{array}{|c|} \hline (D_F)_W \\ \hline D_{F1} \\ \hline D_{F2} \\ \hline D_{F3} \\ \hline D_{F4} \\ \hline D_{F5} \\ \hline (D_F)_N \\ \hline \end{array} = \begin{array}{|c|c|c|c|c|} \hline (A_F)_{WW} & (A_F)_{WN} & & & \\ \hline (A_F)_{11} & (A_F)_{12} & (A_F)_{13} & (A_F)_{14} & (A_F)_{15} \\ \hline (A_F)_{21} & (A_F)_{22} & (A_F)_{23} & (A_F)_{24} & (A_F)_{25} \\ \hline (A_F)_{31} & (A_F)_{32} & (A_F)_{33} & (A_F)_{34} & (A_F)_{35} \\ \hline (A_F)_{41} & (A_F)_{42} & (A_F)_{43} & (A_F)_{44} & (A_F)_{45} \\ \hline (A_F)_{51} & (A_F)_{52} & (A_F)_{53} & (A_F)_{54} & (A_F)_{55} \\ \hline (A_F)_{NW} & (A_F)_{NN} & & & \\ \hline \end{array} \times \begin{array}{|c|} \hline (F)_W \\ \hline F_1 \\ \hline F_2 \\ \hline F_3 \\ \hline F_4 \\ \hline F_5 \\ \hline (F)_N \\ \hline \end{array}$$

It is to be noted that double lines partition these matrices into working (W) and nonworking (N) blocks, and that these blocks can be used in separated matrix equations; that is,

$$\begin{array}{|c|} \hline (D_F)_W \\ \hline (D_F)_N \\ \hline \end{array} = \begin{array}{|c|c|} \hline (A_F)_{WW} & (A_F)_{WN} \\ \hline (A_F)_{NW} & (A_F)_{NN} \\ \hline \end{array} \times \begin{array}{|c|} \hline (F)_W \\ \hline (F)_N \\ \hline \end{array}$$

can be expressed as

$$(D_F)_W = (A_F)_{WW} \times (F)_W + (A_F)_{WN} \times (F)_N \quad (28a)$$

and

$$(D_F)_N = (A_F)_{NW} \times (F)_W + (A_F)_{NN} \times (F)_N \quad (28b)$$

Setting $(D_F)_N = (0)$ gives

$$(0) = (A_F)_{NW} \times (F)_W + (A_F)_{NN} \times (F)_N$$

To solve for the redundant loads $(F)_N$, multiply each term by the inverse matrix of $(A_F)_{NN}$; thus

$$(0) = (A_F)_{NN}^{-1} \times (A_F)_{NW} \times (F)_W + (A_F)_{NN}^{-1} \times (A_F)_{NN} \times (F)_N$$

so that

$$(F)_N = - (A_F)_{NN}^{-1} \times (A_F)_{NW} \times (F)_W \quad (29)$$

As a check, this matrix equation will be solved numerically. First,

$$(A_F)_{NN} = (A_F)_{55} = 5.654,975 \times 10^{-6} \text{ in./lb}$$

giving the inverse

$$(A_F^I)_{NN} = 0.176,835,44 \times 10^6 \text{ lb/in.}$$

Then,

$$\begin{aligned} - (A_F^I)_{NN} \times (A_F)_{NW} &= -0.176,835,44 \times 10^6 \text{ lb/in.} \\ &\times \begin{bmatrix} 0.2408 & 0.94471 & -2.95355 & -0.43481 \end{bmatrix} \times 10^{-6} \text{ in./lb} \\ &= \begin{bmatrix} -0.042,582 & -0.167,058 & +0.522,292 & +0.076,890 \end{bmatrix} \end{aligned}$$

which checks Eq (26).

Substitution of Eq (29) into Eq (28a) gives

$$\begin{aligned} (D_F)_W &= (A_F)_{WW} \times (F)_W + (A_F)_{WN} \times \left[- (A_F^I)_{NN} \times (A_F)_{NW} \times (F)_W \right] \\ &= \left[(A_F)_{WW} - (A_F)_{WN} \times (A_F^I)_{NN} \times (A_F)_{NW} \right] \times (F)_W \end{aligned}$$

Since

$$(D_F)_W = (D_W), \quad (F)_W = (W), \quad \text{and} \quad (D_W) = (A_W) \times (W),$$

then,

$$(A_W) = (A_F)_{WW} - (A_F)_{WN} \times (A_F^I)_{NN} \times (A_F)_{NW} \quad (30)$$

Numerically,

$$\begin{aligned} (A_W) &= \begin{bmatrix} 0.1295 & 0.21295 & 0 & 0 \\ 0.21295 & 0.77506 & 0 & 0 \\ 0 & 0 & 2.023,962,5 & 0.32426 \\ 0 & 0 & 0.32426 & 0.10316 \end{bmatrix} \\ &+ \begin{bmatrix} 0.2408 \\ 0.94471 \\ -2.95355 \\ -0.43481 \end{bmatrix} \times \begin{bmatrix} -0.042,582 & -0.167,058 & +0.522,292 & +0.076,890 \end{bmatrix} \times 10^6 \text{ in./lb} \\ &= \begin{bmatrix} 0.11925 & 0.17272 & 0.12577 & 0.01852 \\ 0.17272 & 0.61724 & 0.49341 & 0.07264 \\ 0.12577 & 0.49341 & 0.48135 & 0.09716 \\ 0.01852 & 0.07264 & 0.09716 & 0.06973 \end{bmatrix} \times 10^{-6} \text{ in./lb} \end{aligned}$$

which checks Eq (27).

B. S0207 Code (CTAC and MODE)

Discussion

A WOLONTIS-650 code has been developed to calculate the influence deflection coefficients (flexibility matrix) for an elastic structure with up to 10 applied loads and up to 14 loads when the structure is made statically determinate. An estimate is made of the shock response to a 1-g loading by using weights as the applied loads, and the fundamental frequency is approximated by the formula (Ref 8):

$$f \cong (2\pi)^{-1} \left[g \left(\sum_i W_i Y_i \right) / \left(\sum_i W_i Y_i^2 \right) \right]^{1/2},$$

where

f = natural frequency, cps,

g = 386 in./sec², usually,

W_i = weight at point i ,

and

Y_i = static deflection of W_i .

This formula is good if all deflections are positive. Local forces and moments throughout the structure can also be determined. All this is programmed in S0207 deck 1 and is called the CTAC code.

In S0207, deck 2 is programmed the MODE code, which uses the iteration procedure of WAPD-115 to calculate as many modes of vibration as desired. Local forces and moments for each mode can also be determined.

Arrangement of Decks

CTAC without MODE: Successive CTAC problems can be run with the following arrangement of cards. (Each arrangement must be specified on the back of the requester card.)

<u>Specified Arrangement on Requester Card</u>	<u>Actual Contents of Arrangement (Not Indicated on Requester Card)</u>
S0207 deck 1	On file in 650 room
Input deck	<div style="display: flex; align-items: center;"> <div style="font-size: 3em; margin-right: 10px;">{</div> <div> CTAC transfer card First CTAC data deck -- Last CTAC data deck </div> </div>

CTAC and MODE: The MODE code uses the data from the CTAC code input and output to calculate natural frequencies, mode shapes, and mode amplitudes of an elastic structure. If the MODE problem immediately follows the corresponding CTAC problem, the arrangement of decks is this:

<u>Specified Arrangement on Requester Card</u>	<u>Actual Contents of Arrangement (Not Indicated on Requester Card)</u>
S0207 deck 1	On file in 650 room
Input deck 1	<div style="display: flex; align-items: center;"> <div style="font-size: 3em; margin-right: 10px;">{</div> <div> CTAC transfer card First CTAC data deck -- Last CTAC data deck </div> </div>

(Continued)

<u>Specified Arrangement on Requester Card</u>	<u>Actual Contents of Arrangement (Not Indicated on Requester Card)</u>
S0207 deck 2	On file in 650 room
Input deck 2	MODE data deck cor- responding to last CTAC data deck

MODE Alone: If the corresponding CTAC problem had been done on a separate previous occasion, the arrangement of cards is:

<u>Specified Arrangement on Requester Card</u>	<u>Actual Contents of Arrangement (Not Indicated on Requester Card)</u>
S0207 deck 1	On file
Input deck 1	{ CTAC data deck CTAC output deck MODE transfer card
S0207 deck 2	On file
Input deck 2	MODE data deck

Input Cards

Transfer Cards:

1) CTAC Transfer Card

<u>Columns</u>	<u>Contents</u>
7-10	6850
77-79	xxx (Arbitrary 3-digit problem identification)

2) MODE Transfer Card

<u>Columns</u>	<u>Contents</u>
7-10	5480
77-79	xxx (Arbitrary 3-digit problem identification)

Data Card Format: Each number (called a "word" on a card) has a sign, + or -, and 10 digits, taking up a total of 11 columns on a card. A "fixed" number is put at the right end and is preceded by zeros; for example, +23 (fix) is put on a card as +0000000023. A number that is not fixed is considered to be in the floating form $\pm(a.bcd\text{efgh}) \times 10^{ij-50}$ and is put on a card as $\pm abcdefghij$; for example, -23 is put on a card as -2300000051, and +0.056 is put on a card as +5600000048. Standard WOLONTIS format cards are used as shown on the following page.

Columns	Contents
7-9	Location of first word of card in 650 storage drum.
10	Number of consecutive words on card, 1 to 6. If a row of words is more than 6, the row is broken in groups of 6 or less and the groups put on successive cards. For example, if a row of 9 words is to be put on cards starting at location 101, 5 words could be put on a card with location 101 and 4 words could be put on the next card with location $101 + 5 = 106$.
11-21	First word
22-32	Second word
	--
66-76	Sixth word

CTAC Data Deck (Input)

Location of First Word	Number of Words in Row	First Word	Second Word	Third Word	Explanation
638	1	TRA No. 1			<p>TRA No. 1 = +0203000745, if there is a load calculation for this CTAC problem.</p> <p>TRA No. 1 = +0203000548, if there is no load calculation and if MODE follows this CTAC problem.</p> <p>TRA No. 1 = +0203000685, if there is no load calculation and if MODE does not follow this particular CTAC problem.</p>
540	3	$\bar{P}(\text{fix})$	$W(\text{fix})$	$R(\text{fix})$	<p>$\bar{P}$ = total number of elementary (primitive) structures; $1 \leq \bar{P} < 999$.</p> <p>W = total number of external loads (weights) on structure: $1 \leq W \leq 10$.</p> <p>R = total number of loads (F_R) on statically determinate structure; $W \leq R \leq 14$. The first W loads of R are the original external loads.</p>
					Next come $\bar{P}$ sets of cards for $\bar{P}$ primitives in order from $P = 1$ to $P = \bar{P}$.
543	3	$P(\text{fix})$	TRA No. 2	$N(\text{fix})$	<p>P = number of primitive; $1 \leq P \leq \bar{P}$</p> <p>TRA No. 2 = +0203000602, if partial flexibility matrix A_R due to P primitives is to be punched.</p> <p>TRA No. 2 = +0203000604 otherwise.</p> <p>N = number of loads on primitive.</p>
197	N	A_{11}	A_{12}	etc.	Flexibility matrix A_p of primitive structure P . The first word of each row must start a new card.
$197 + N$	N	A_{21}	A_{22}	etc.	
--					
$197 - N + N^2$	N	A_{N1}	A_{N2}	etc.	
246	R	C_{11}	C_{12}	etc.	Connection matrix C_p giving $(F_P) = (C_p) \times (F_R)$. The first word of each row must start a new card.
$246 + R$	R	C_{21}	C_{22}	etc.	
--					
$246 - R + NR$	R	C_{N1}	C_{N2}	etc.	

CTAC Data Deck (Input), Cont.

Location of First Word	Number of Words in Row	First Word	Second Word	Third Word	Explanation
					These cards come after the P sets of cards.
515 500	W 1	W_1 g	W_2	etc.	Weights (pounds force) Gravity = 386 in./sec ²
					If there are to be load calculations on some primitives, the following sets of cards must be added to the deck. Otherwise, they are not used. There is one set of cards for each primitive to be considered, and these primitives need not be the same as before.
543	3	P(fix)	TRA No. 3	N(fix)	TRA No. 3 = +0203000769, if this is not the last primitive on which load calculations are made. TRA No. 3 = +0203000548, if this is the last primitive considered, and if MODE follows. TRA No. 3 = +0203000685, if this is the last primitive considered and if CTAC follows or if the end of the last problem.
197 -- 197 - N + N ²	N	A_{11}	A_{12}	etc.	Flexibility matrix A_p .
246 -- 246 - R + NR	R R	C_{11} C_{R1}	C_{12} C_{R2}	etc. etc.	Connection matrix C_p .

MODE Data Deck (Input)

Location of First Word	Number of Words in Row	First Word	Second Word	Third Word	Explanation
337	2	E	T		E = relative error for convergence in iterations (10^{-6} corresponds to an error in the 7th digit of a number). T = total number of iterations carried out if results do not converge, say 15.
327	3	V_o	f_o	M(fix)	V_o = maximum starting velocity, ips f_o = cutoff natural frequency, cps M = number of modes to be calculated; $1 \leq M \leq W$. The starting velocity V for each mode of frequency f is determined by the formulas $\begin{cases} V = V_o & \text{if } f \leq f_o \\ V = f_o V_o / f & \text{if } f > f_o \end{cases}$
					If there are to be load calculations on some primitives, the following sets of cards must be used. These cards have precisely the same format as in the CTAC data deck.
543	3	P(fix)	TRA No. 3	N(fix)	
197 -- 197 - N + N ²	N N	A_{11} A_{N1}	A_{12} A_{N2}	etc. etc.	
246 -- 246 - R + NR	R R	C_{11} C_{N1}	C_{12} C_{N2}	etc. etc.	

Output Cards

Output Card Format: Most of the output is in the form of blocks of data (matrices). Before each matrix is printed out, three words in locations 997 to 999 give, respectively, the machine storage location of the first element of the matrix, the number of rows, and the number of columns.

If the structure is redundant, one of the matrices will contain the flexibility matrix A_W of the applied loads and the coefficients of the redundant forces in terms of the applied loads. This matrix is identified by ones along the lower right diagonal. For example, if a structure has $R = 4$ loads when made statically determinate and $W = 2$ applied loads, this matrix will be in the form:

A_{11}	A_{12}	X	X
A_{21}	A_{22}	X	X
K_{31}	K_{32}	-1	X
K_{41}	K_{42}	K_{43}	-1

where the corresponding equations are

$$D_{F1} = A_{11}F_1 + A_{12}F_2 ,$$

$$D_{F2} = A_{21}F_1 + A_{22}F_2 ,$$

$$F_3 = K_{31}F_1 + K_{32}F_2 ,$$

and

$$F_4 = K_{41}F_1 + K_{42}F_2 + K_{43}F_3 .$$

CTAC Output Deck

Location of First Word	Number of Words in Row	First Word	Second Word	Third Word	Fourth Word	Explanation
						If TRA No. 2 for primitive P is +0203000602, the following cards are punched:
540	4	$\bar{P}(\text{fix})$	$W(\text{fix})$	$R(\text{fix})$	$P(\text{fix})$	
997	3	001(fix)	$R(\text{fix})$	$R(\text{fix})$		Flexibility matrix A_R of R loads due to P primitive.
997	3	001(fix)	$R(\text{fix})$	$R(\text{fix})$		If the structure is redundant ($W < R$), this matrix containing A_W and coefficients K_{ij} is punched.
540	3	$\bar{P}(\text{fix})$	$W(\text{fix})$	$R(\text{fix})$		
997	3	001(fix)	$W(\text{fix})$	$W(\text{fix})$		Flexibility matrix A_W of applied loads.
501	W	Y_1	Y_2	etc.		Static deflections of weights, in.
529	3	Y_s	$2\pi f$	f		Y_s = equivalent static deflection $= (\sum_i W_i Y_i^2) / (\sum_i W_i Y_i)$, $2\pi f$ = natural angular frequency $= \sqrt{g/Y_s}$, rad/sec f = frequency, cps
515 + W	R - W	F_{W+1}	F_{W+2}	etc.		If $W < R$, these redundant loads produced by 1 g applied loads are punched.
						If load calculations were made for a primitive, the following oct of cards is punched for this primitive.
543	3	P(fix)	TRA No. 3	N(fix)		
997	3	351(fix)	N(fix)	1(fix)		Matrix of primitive loads under 1 g applied load.
997	3	371(fix)	N(fix)	1(fix)		Matrix of primitive deflections under 1 g applied load.

MODE Output Deck

Location of First Word	Number of Words in Row	First Word	Second Word	Third Word	Fourth Word	Explanation
						The following cards are punched for each mode calculated.
330	3	$2\pi f_J$	f_J	V_J	J(fix)	$2\pi f_J$ = Nat ang freq, rps f_J = Nat frequency, cps V_J = starting velocity, ips J = mode; J = 1 to M.
336	3	T_J	E	T		T_J = number of iterations in calculation of mode J.
297	W	F'_{1J}	F'_{2J}	etc.		F'_{IJ} = force F_I in mode J after T_J iterations, lb.
307	W	F''_{1J}	F''_{2J}	etc.		F''_{IJ} = force F_I in mode J after $T_J - 1$ iterations, lb.
317	W	F'''_{1J}	F'''_{2J}	etc.		$F'''_{IJ} = (W_I/g)(2\pi f_J)^2 Y_{IJ}$; lb.
484	W	Y_{1J}	Y_{2J}	etc.		Y_{IJ} = deflection of F_{IJ} , in.
						If $W < R$, the following cards are punched.
997	3	344 + WM (fix)	1(fix)	M(fix)		Matrix of F_{W+1J} , first redundant load.
--	--	--	--	--	--	--
997	3	[344 + RM - M]	1(fix)	M(fix)		Matrix of F_{RJ} , last redundant load.
						If load calculations were made for a primitive, the following set of cards are punched for each primitive.
543	3	P(fix)	TRA No. 3	N(fix)		
997	3	010(fix)	N(fix)	M(fix)		Matrix of primitive loads; $(F_P)_{IJ}$.
997	3	080(fix)	N(fix)	M(fix)		Matrix of primitive deflections: $(D_{FP})_{IJ}$.

Sample Problem Using Structure in WAPD-115

Printout of Cards: The printout of the cards used in the determination of influence coefficients and modes of vibration starts on page 32. The S0207 input cards were sent to the IBM-650 operator with the following note on the back of the requester card:

Order of Cards

S0207 deck 1

Input deck 1

S0207 deck 2

Input deck 2

Results: The CTAC output gives an estimated natural frequency of 223.50 cps, which compares well with the first MODE frequency of 222.45 cps. For a natural frequency of 223.5 cps, the shock acceleration factor corresponding to a unit starting velocity is

$$N = \frac{2\pi fV}{g} = \frac{2(223.5 \text{ sec}^{-1})(1 \text{ in. sec}^{-1})}{(386 \text{ in. sec}^{-2})} = 3.658$$

Thus, from the CTAC output, the moment at the left end of the outer cylinder is

$$M_O = (7688.1 \text{ in. -lb})(3.638) = 27,969 \text{ in. -lb}$$

and the absolute sum of the four MODE results is

$$M_O = (27,315 + 1,118 + 4686 + 375) \text{ in. -lb} = 33,494 \text{ in. -lb}$$

The MODE code gives a higher moment but, in the usual shock calculations, the higher modes have smaller starting velocities so that the results will be closer.

From CTAC output, the moment at the left end of the inner cylinder is

$$M_i = (2936.5 \text{ in. -lb})(3.638) = 10,683 \text{ in. -lb}$$

and the absolute sum of the four MODE results is

$$M_i = (9131 + 1537 + 4500 + 1978) \text{ in. -lb} = 17,146 \text{ in. -lb}$$

As before, taking only two modes would give a closer check.

A direct comparison of mode calculations from WAPD-115 and from the code S0207 is given in the table on the following page.

The odd numbered modes are not so accurate in slide rule calculations because of small differences between large numbers in the determination of C_J . Otherwise, the agreement is satisfactory.

Sample Problem Cards: The actual cards used in the solution of the sample problem given in WAPD-115 are found on pages 32 through 43 of this report.

COMPARISON TABLE OF MODE CALCULATIONS

Mode J.	1	2	3	4	Source
C_J for $V_J = 1$ in./sec	121.8	-134.8	1776	-298.2	From WAPD-115 (Slide-rule accuracy)
$F_{IJ} = C_J e_{IJ}$ (lb)	122	-135	1776	-298	
	755	-215	- 691	518	
	699	259	30	-644	
	80	164	1067	823	
$D_{FIJ} = C_J a_{IJ}$ (10^6 in.)	234	- 18	116	- 11	
	837	- 16	- 26	11	
	734	18	1	- 14	
	130	18	60	27	
F_{IJ} (lb)	122	-148	1779	-291	From S0207 (IBM-650 machine)
	757	-233	- 676	534	
	698	280	15	-662	
	80	188	1061	808	
D_{FIJ} (10^6 in.)	235	- 19	117	- 11	
	839	- 17	- 26	11	
	733	20	1	- 13	
	131	21	59	25	

		S0207		INPUT			
001	1 685 0		TR CARD			CTAC Input	
	2 638 1	+C203000745					
	3 540 3	+C000000008	+0000000004	+0000000005			
	4 543 3	+C000000001	+0203000602	+0000000002	First Primitive		
	5 197 2	+1114000041	+5570000041				
	6 199 2	+5570000041	+1294000043				
	7 246 3	+0000000000	+1500000051	+0000000000			
	8 249 2	+0000000000	+2000000051				
	9 251 5	+1000000050	+1000000050	+0000000000	+0000000000	+1000000050	
	10 543 3	+0000000002	+0203000602	+0000000002	Second Primitive		
	11 197 2	+1114000041	+5570000041				
	12 199 2	+5570000041	+1294000043				
	13 246 5	+0000000000	+5000000050	+0000000000	+0000000000	+1000000051	
	14 251 5	+0000000000	+1000000050	+0000000000	+0000000000	+1000000050	
	15 543 3	+0000000003	+0203000604	+0000000002	Third Primitive		
	16 197 2	+2040000040	+5100000040				
	17 199 2	+5100000040	+1496000042				
	18 246 5	+0000000000	+0000000000	+0000000000	+0000000000	+5000000050	
	19 251 5	+0000000000	+1000000050	+0000000000	+0000000000	+1000000050	

20	543	3	+0000000004	+0203000604	+0000000002	Fourth Primitive	
21	197	2	+2040000040	+5100000040			
22	199	2	+5100000040	+1496000042			
23	246	5	+0000000000	+0300000000	+0000000000	+0000000000	+0000000000
24	251	5	+0000000000	+0300000000	+0000000000	+0000000000	+1000000050
25	543	3	+0000000005	+0203000604	+0000000002	Fifth Primitive	
26	197	2	+3930000041	+1474000042			
27	199	2	+1474000042	+1331600043			
28	246	5	+0000000000	+0000000000	+1500000051	+0000000000	-2250000051
29	251	5	+0000000000	+0300000000	+1000000050	+1000000050	-1000000050
30	543	3	+0000000006	+0203000604	+0000000002	Sixth Primitive	
31	197	2	+3930000041	+1474000042			
32	199	2	+1474000042	+1331600043			
33	246	5	+0000000000	+0600000000	+7500000050	+0000000000	-1500000051
34	251	5	+0000000000	+0300000000	+1000000050	+0000000000	-1000000050

35	543 3	+0000000007	+0203000604	+0000000002	Seventh Primitive	
36	197 2	+1510000041	-6040000041			
37	199 2	+6040000041	-4903000042			
38	246 5	+0000000000	-0000000000	+0000000000	+0000000000	-7500000050
39	251 5	+0000000000	-0000000000	+1000000050	+0000000000	-1000000050
40	543 3	+0000000008	-0203000602	+0000000002	Last Primitive	
41	197 2	+1510000041	-6040000041			
42	199 2	+6040000041	-4903000042			
43	246 5	+0000000000	+0000000000	+0000000000	+0000000000	+0000000000
44	251 5	+0000000000	-0000000000	+0000000000	+0000000000	-1000000050
45	515 4	+1025800052	-1784400052	+1882000052	+1204400052	Weights
46	500 1	+3360000052	Gravity			
47	543 3	+0000000001	+0203000769	+0000000003	First Primitive in Load Calculation (including the foundation moment)	
48	197 3	+1114000041	+5570000041	+0000000000		
49	200 3	+5570000041	+1294000043	+0000000000		
50	203 3	+0000000000	+0000000000	+0000000000		
51	246 5	+0000000000	+1500000051	+0000000000	+0000000000	+2000000051
52	251 5	+1000000050	+1000000050	+0000000000	+0000000000	+1000000050
53	256 5	+1000000051	+2500000051	+0000000000	+0000000000	+3000000051

54	543	3	+0000000005	+0203000548	+0000000003	Last Primitive in Load Calculation (including the foundation moment)	
55	197	3	+3930000041	+1474000042	+0000000000		
56	200	3	+1474000042	+1031600043	+0000000000		
57	203	3	+0000000000	+0000000000	+0000000000		
58	245	5	+0000000000	+0000000000	+1500000051	+0000000000	-2250000051
59	251	5	+0000000000	+0000000000	+1000000050	+0000000000	-1000000050
60	255	5	+0000000000	+0000000000	+2250000051	+7500000050	-3000000051

61	337	2	+1000000044	+1500000051			MODE Input
62	327	3	+1000000050	+1000000056	+0000000004		
63	543	3	+0000000001	+0203000769	+0000000003		First Primitive in Load Calculation (same as in CTAC)
64	197	3	+1114000041	+5570000041	+0000000000		
65	200	3	+5570000041	+1294000043	+0000000000		
66	203	3	+0000000000	+0000000000	+0000000000		
67	246	5	+0000000000	+1500000051	+0000000000	+0000000000	+2000000051
68	251	5	+1000000050	+1000000050	+0000000000	+0000000000	+1000000050
69	256	5	+1000000051	+2500000051	+0000000000	+0000000000	+3000000051
70	543	3	+0000000005	+0203000548	+0000000003		Last Primitive in Load Calculation (same as in CTAC)
71	197	3	+3930000041	+1474000042	+0000000000		
72	200	3	+1474000042	+1031600043	+0000000000		
73	203	3	+0000000000	+0000000000	+0000000000		
74	246	5	+0000000000	+0000000000	+1500000051	+0000000000	-2250000051
75	251	5	+0000000000	+0000000000	+1000000050	+0000000000	-1000000050
76	256	5	+0000000000	+0000000000	+2250000051	+7500000050	-3000000051

			S0207	DATA		
001	1	540 4	+0000000008	+0000000004	+0000000005	+0000000001 CTAC Output
001	2	997 3	+0000000001	+0000000005	+0000000005	A _R due to 1 primitive
001	3	001 5	+1294000043	+2129500043	+0000000000	+0000000000 +2408000043
001	4	006 5	+2129500043	+5471500043	+0000000000	+0000000000 +6585500043
001	5	011 5	+0000000000	+0000000000	+0000000000	+0000000000 +0000000000
001	6	016 5	+0000000000	+0000000000	+0000000000	+0000000000 +0000000000
001	7	021 5	+2408000043	+6585500043	+0000000000	+0000000000 +7978000043
001	8	540 4	+0000000008	+0000000004	+0000000005	+0000000002
001	9	997 3	+0000000001	+0000000005	+0000000005	A _R due to 2 primitives
001	10	001 5	+1294000043	+2129500043	+0000000000	+0000000000 +2408000043
001	11	006 5	+2129500043	+7501000043	+0000000000	+0000000000 +9272000043
001	12	011 5	+0000000000	+0000000000	+0000000000	+0000000000 +0000000000
001	13	016 5	+0000000000	+0000000000	+0000000000	+0000000000 +0000000000
001	14	021 5	+2408000043	+9272000043	+0000000000	+0000000000 +1150000044

001	15	540	4	+3000000008	+0000000004	+0000000005	+0000000008	
001	16	997	3	+3000000001	+0000000005	+0000000005	A _R due to all primitives	
001	17	001	5	+1294000043	+2129500043	+0000000000	+0000000000	+2408000043
001	18	006	5	+2129500043	+7750600043	+0000000000	+0000000000	+9447100043
001	19	011	5	+3000000000	+0000000000	+2023962544	+3242600043	-2953550044
001	20	016	5	+3000000000	+0000000000	+3242600043	+1031600043	-4348100043
001	21	021	5	+2408000043	+9447100043	-2953550044	-4348100043	+5654975044
001	22	997	3	+3000000001	+0000000005	+0000000005	A _W and K _{-j}	
001	23	001	5	+1191462643	+1727223843	+1257679943	+1851506842	+2408000043
001	24	006	5	+1727223843	+6172384443	+4934147743	+7263858042	+9447100043
001	25	011	5	+1257679943	+4934147843	+4813460043	+9716208042	-2953550044
001	26	016	5	+1851506842	+7263858042	+9716208042	+6972754042	-4348100043
001	27	021	5	-4258197448	-1670582149	+5222923149	+7688981848	-9999999949

001	28	540	3	+0000000008	+0000000004	+0000000005	
001	29	997	3	+0000000001	+0000000004	+0000000004	A _w
001	30	001	4	+1191462643	+1727223843	+1257679943	+1851506842
001	31	005	4	+1727223843	+6172384443	+4934147743	+7263858042
001	32	009	4	+1257679943	+4934147843	+4813460043	+9716208042
001	33	013	4	+1851506842	+7263858042	+9716208042	+6972754042
001	34	501	4	+6894209545	+2294671446	+2032377346	+4154479245
001	35	529	3	+1957346746	+1404299653	+2235012252	Frequency
001	36	519	1	+7337809751	Redundant Load		
001	37	543	3	+0000000001	+0203000769	+0000000003	
001	38	997	3	+00000000351	+0000000003	+0000000001	
001	39	351	1	+4144161953	Load Calculation for Primitive No. 1		
001	40	352	1	+3543981052			
001	41	353	1	+7688142953			
001	42	997	3	+00000000371	+0000000003	+0000000001	
001	43	371	1	+6590593844			
001	44	372	1	+6894209645			
001	45	373	1	+0000000000			

001	46 543 3	+3000000005	+0203000548	+0000000003
001	47 997 3	+3000000351	+0000000003	+0000000001
001	48 351 1	+1171992853	Load Calculation	
001	49 352 1	+1148219052	for Primitive No. 5	
001	50 353 1	+2936457153		
001	51 997 3	+3000000371	+0000000003	+0000000001
001	52 371 1	+5298406544		
001	53 372 1	+2912020145		
001	54 373 1	+3000000000		

001	55	330	4	+1397707353	+2224520352	+1000000050	+0000000001	MODE Output
001	56	336	3	+7000000050	+1000000044	+1500000051		
001	57	297	4	+1217628252	+7572647152	+6977637252	+7961363451	First Mode
001	58	307	4	+1217628252	+7572646952	+6977637352	+7961362351	
001	59	317	4	+1217628252	+7572646852	+6977637452	+7961363451	
001	60	484	4	+2345345446	+8385139946	+7325606746	+1306085246	
001	61	330	4	+5398087653	+8591323352	+1000000050	+0000000002	
001	62	336	3	+1600000051	+1000000044	+1500000051		
001	63	297	4	-1479105352	-2328211152	+2802711752	+1883356652	Second Mode
001	64	307	4	-1479105352	-2328287652	+2802786152	+1883356652	
001	65	317	4	-1479086352	-2328143052	+2802635552	+1883292252	
001	66	484	4	-1910021545	-1728322945	+1972670045	+2071354345	
001	67	330	4	+7572882253	+1205261653	+1000000050	+0000000003	
001	68	336	3	+1600000051	+1000000044	+1500000051		
001	69	297	4	+1778515053	-6760884352	+1524057551	+1061353153	Third Mode
001	70	307	4	+1778515053	-6760173852	+1524057551	+1061353153	
001	71	317	4	+1778493653	-6761204852	+1530871451	+1061210053	
001	72	484	4	+1166956146	-2550332445	+5474998443	+5930557045	

001	73	330	4	+1009491054	+1606654853	+1000000050	+0000000004	
001	74	336	3	+2000000050	+1000000044	+1500000051		
001	75	297	4	-2908838852	+5340940252	-6623016752	+8081641452	Fourth
001	76	307	4	-2908838752	+5340940252	-6623016752	+8081641452	Mode
001	77	317	4	-2908840652	+5340191952	-6621871752	+8079908252	
001	78	484	4	-1074087445	+1133566645	-1332734245	+2541078445	
001	79	997	3	+0000000360	+0000000001	+0000000004	Redundant Load	
001	80	360	4	+2388659252	+2060575852	+1267807252	-3606138652	
001	81	543	3	+0000000001	+0203000769	+0000000003	Load Calculation	
001	82	997	3	+0000000010	+0000000003	+0000000004	for Primitive No. 1	
001	83	010	4	+1613628954	+6288349052	-7605712053	+7991331052	
001	84	014	4	+1117893553	-1746740652	+1229207353	-1174037252	
001	85	018	4	+2731522454	-1117905753	+4686360653	-3749040052	
001	86	997	3	+0000000080	+0000000003	+0000000004		
001	87	080	4	+2420249345	-2724124343	-1626078544	+2362955543	
001	88	084	4	+2345345546	-1910021345	+1166956046	-1074087045	
001	89	088	4	+0000000000	+0000000000	+0000000000	+0000000000	

001	90	543	3	+0000000005	+0203000548	+0000000003	Load Calculation for Primitive No. 5
001	91	997	3	+0000000010	+0000000003	+0000000004	
001	92	010	4	+5091973053	-4322280052	-2623957653	-1820713253
001	93	014	4	+4588978052	+7421359051	-1115401452	-3016878152
001	94	018	4	+9130808053	+1536891453	+4499639653	+1977859053
001	95	937	3	+0000000080	+0000000003	+0000000004	
001	96	030	4	+2677560845	-6047477043	-1195625545	-1160228145
001	97	084	4	+1223955846	+1284833244	-5018361645	-5795942745
001	98	088	4	+0000000000	+0000000000	+0000000000	+0000000000

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