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TRANSPORT SOLUTIONS TO THE
MONOENERGETIC CRITICAL PROBLEMS

by

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PREFACE

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TRANSPORT SOLUTIONS TO THE MONOENERGETIC CRITICAL PROBLEMS

by

George J. Mitsis

ABSTRACT

Transport solutions to the monoenergetic plane, spherical, and cylindrical critical problems with isotropic scattering are developed by the method of singular expansion modes. The results are given in the form of exact expressions for the neutron distributions and criticality conditions. These expressions depend on expansion coefficients which are shown to satisfy Fredholm integral equations of the second kind. Successive approximations on the coefficients lead to the asymptotic results of diffusion theory as well as to easily accessible transport corrections.

The expansions for the neutron distributions are derived by two different, but equivalent, methods. In the first method a three-dimensional expansion for the angular density is developed from the elementary solutions of the transport equation and then specialized to the desired geometry. For problems with plane symmetry, the resulting expression are used directly to determine the expansion coefficients, whereas in spherical and cylindrical geometries the specialization of the three-dimensional solutions yields a variety of representations for the respective angular densities.

The second method consists of replacing the angular distributions by suitable density transforms and then determining the transforms completely with the aid of the integral equations for the neutron densities. In plane geometry this method leads to the conversion of the transport equation to a singular integral equation for the angular density. In spherical and cylindrical geometries the appropriate transforms are shown

to satisfy separable equations whose solutions can be represented by unique expansions in terms of complete sets of singular expansion modes. The coefficients in the transform expansions are determined by resulting side conditions and the actual neutron distributions established by simple inversions.

Although the emphasis throughout this work is on obtaining exact expressions for the neutron distributions and criticality conditions, the results are illustrated by approximate expressions and computations.

CHAPTER I

THE PROBLEM

1.0 INTRODUCTION

The importance of the neutron transport equation is reflected by the vast number of papers and many well-known and comprehensive treatises on the subject. In spite of these extensive researches, however, exact solutions are available only for highly idealized cases. Moreover, the various numerical and analytical techniques in common use are based on approximations and, accordingly, are limited in their range of applicability and fail to provide information on the precise structure of the transport spectrum. In particular, the problem of finding a complete set of elementary solutions of the transport equation which can be used to represent an arbitrary neutron distribution and form a uniform basis for rigorous or approximate solutions in special cases still remains an open question.

In this work we propose to show that a number of transport problems can be treated exactly without the need of ab initio approximations on the equation or its boundary conditions. Specifically, we shall develop the stationary transport solutions in uniform, multiplying media possessing plane, spherical, and cylindrical symmetries, and containing neutrons which travel with a single speed and scatter isotropically. The results will be in the form of expressions for the angular densities, neutron densities, the neutron currents, and the critical conditions for these systems. All these expressions will be exact in the sense that they will depend on expansion coefficients which obey Fredholm integral equations of the second kind. Although the explicit form of the coefficients cannot be displayed, successive approximations of the integral equations are readily accessible and lead to recognizable results. In particular, the expressions corresponding to the zeroth-order approximation of the coefficients will be identified as the asymptotic results of diffusion theory.

The approach used is a direct outgrowth of the work of K. M. Case (1). This new approach is based on the facts that the neutron transport equation is linear and, at least under certain restrictions, is already or can be made separable. Thus the familiar procedure of expanding the neutron distribution in a set of normal modes generated by the separation of variables and finding the expansion coefficients from the boundary conditions can be used.

The unique feature of this approach is found in the unusual nature of the expansion modes. As shown by Case, the complete spectrum of the one-speed transport equation in plane geometry is composed of discrete terms formed by well-behaved functions and a highly singular but integrable continuum. Following the interpretation of Van Kampen (2), the singular modes can be written as Cauchy principal values with additional arbitrary multiples of the Dirac δ -function to account for the contribution of the singular points. An entirely equivalent, but in many instances less ambiguous, way of representing these singular solutions was given independently by Wigner (3). In this case, the character of the singularities and the prescription for interpreting them are clearly brought out by expressing the singular modes as limits.

Once these singular solutions are recognized and correctly interpreted, they are no more difficult to work with than nonsingular eigenfunctions. Their singular form is of no concern in the final results since the observables characterizing the neutron distribution are always given as integrals over the singular continuum. Furthermore, Case (1) proved that the appropriate discrete and continuous modes of the transport equation in plane geometry form a complete set and, in the process, demonstrated a procedure for finding the expansion coefficients. Consequently, an arbitrary neutron distribution with plane symmetry can be completely represented by an expansion in terms of these modes and the expansion coefficients can be chosen to reproduce given boundary conditions.

Since its introduction, the work of Case has been the focal point for many successful investigations both in treating some hitherto unsolved problems and in simplifying the derivation of previously known results (4-11). In particular, the one-dimensional critical problem, which occupies a large part of Chapter III, was considered independently and about the same time by Zelazny (5), the extension to anisotropic scattering by Mika (6), problems pertaining to multiregion and multigroup calculations by Kuszell (7) and by Zelazny and Kuszell (8), respectively, and similar problems in related transport phenomena by Aamodt (9-10) and Shure (11).

In attempting to extend this approach to non-planar geometries we run immediately into the obstacle of non-separability. Specifically, the monoenergetic transport equation, when written in the conventional space and direction-of-motion variables in spherical and cylindrical geometries, cannot be separated. Hence, suitable sets of expansion modes cannot be generated directly as in the case of plane geometry. This difficulty will be circumvented in this work by two different but equivalent methods.

The first method consists in noting that the elementary solutions of the three-dimensional integrodifferential transport equation can be determined by separation of variables and are completely analogous in structure with the singular modes in plane geometry. Such similarity is not surprising for, in essence, the method employs the procedure - common in many linear problems of mathematical physics - of constructing solutions for non-planar geometries by superposing plane solutions. The outcome of the superposition is an expression for the angular density which is independent of geometrical configuration. Specialization of this expression by imposing the desired symmetries yields the neutron distributions in spherical and cylindrical geometries.

The usefulness of this method arises from the fact that it provides a common basis for treating transport problems irrespective of

geometry and that it separates at the outset the asymptotic behavior of the solutions. Also, the method clarifies the reason for the similar features in problems of different geometries which appear in the method of spherical harmonics (12, p. 148).

These merits are, however, overshadowed by three limitations. First, the existence of the general expansion coefficients has not been demonstrated to date. Thus the completeness of the three-dimensional eigenfunctions remains in doubt. Secondly, a general procedure for determining the expansion coefficients from the boundary conditions has yet to be found. Until this is accomplished, the three-dimensional expansion for the angular density can best be described as a formal transformation which shifts the problem of solving the original transport equation to the hopefully less formidable one of finding the expansion coefficients. In the spherical and cylindrical critical problems both of these limitations are removed here by showing that the expansion coefficients are also defined by equations similar to the one obtained in the plane critical problem. The coefficients are therefore both unique and amenable to systematic approximations. The last limitation pertains to the failure of the superposition procedure to produce directly solutions with singular behavior at zero spherical and cylindrical radii. Although such solutions do not enter in the description of the neutron distribution for uniform multiplying media, the possibility of constructing them by modifying the range of the superposition variables is considered in Sec. 2A of Appendix A.

Spherically symmetric transport solutions have been constructed from plane solutions by Wilson (13) and, more recently, by Kofink (14). Since in the first of these papers the continuous part of the spectrum is overlooked, the resulting expansions for the neutron distribution are only asymptotically correct. The presence of the continuum as given by Wigner (3) is taken into account in the second paper. Some of Kofink's results are re-derived in Sec. 4.2 in order to

establish their peaceful coexistence with the other representations for the spherical distribution presented in that section.

The second, and by far more fruitful, method of circumventing the non-separability of the spherical and cylindrical transport equations exploits some interesting transform properties of the transport integral equation. The procedure used differs from the standard transform techniques only in the selection of suitable transforms. Accordingly, the first step is to replace the actual neutron distribution by a pseudo distribution which is a well-defined transform of the neutron density. If such a transform is properly chosen, it can be completely determined from the properties it inherits from the integral equation. The corresponding neutron distribution can then be established by inversion.

The success of this method is a consequence of two observations. First, transforms can be chosen so that their inverses, namely, the neutron densities, are easily accessible with the aid of the transport integral equation. Secondly, the pseudo distributions obey separable equations with side conditions which are, in essence, the transforms of the physical boundary conditions already contained in the integral equation. Thus, the transforms can be represented as expansions in terms of appropriate complete sets of eigenfunctions determined by separation of variables, and the expansion coefficients can be chosen from the given side conditions. Inversion of these expansions provides the corresponding expansions for the neutron densities, which, in turn, can be inserted in the transport integral equations to produce the desired expressions for the angular densities.

The idea of replacing the actual neutron distribution by a suitable density transform was suggested by the recent work of Leonard and Mullikin (15). These authors show that complex transforms of the plane and spherical neutron densities, when used in conjunction with the respective integral equations, satisfy singular integral equations of the type discussed by Muskhelishvili (16). From these equations it is then possible to extract regular integral equations for the transforms as

well as the criticality relations for these two geometries. Moreover, the plane critical equation is shown to coincide with the result deduced previously from Case's formulation (4).

It should be observed that this alternative treatment of the critical problems reported by Leonard and Mullikin avoids the necessity of establishing a unique expansion for the neutron distribution. In fact, the singular nature of the solutions appears only indirectly in the equation for the transforms. Leonard and Mullikin's investigation, however, deals only with the question of criticality. Thus, although the transform in plane geometry is identified as the emerging angular density at the boundary, it is not apparent how this quantity or the transform for the sphere can be used to construct expressions for the angular densities for the entire range of the space variables. It is shown in Secs. 3.3 and 4.5 that this limitation can be easily remedied by modifying the definition of the transform so as to include the space variable. An interesting consequence of such modification is the conversion of the transport equation in plane geometry to a singular integral equation for the angular density. Hence, transport problems in this geometry can also be treated from this entirely different point of view.

The plan of this work is as follows: Chapter II is devoted to the development of the three-dimensional solutions and the discussion of their properties. The contents comprise the determination of the expansion modes, the superposition of these modes to get expressions for the angular density, the neutron density, and the neutron current, and a check on the solutions by showing that they are in accord with the transport integral equation.

The subsequent three chapters deal with the solutions in plane, spherical, and cylindrical geometries in that order and thus contain the main results of this investigation. These chapters are organized in a similar pattern, each being composed of three interrelated parts. In the first part, the three-dimensional solutions of Chapter II are

subjected to the desired symmetries and the resulting representations for the neutron distribution are examined. In particular, the expansion for the angular density in Chapter III reduces to the form given by Case (1), whereas in Chapters IV and V we show that the spherical and cylindrical symmetries require that the expansion coefficients be independent of the superposition variables. Moreover the distributions in the two non-planar geometries considered admit to a variety of equivalent representations in the form of infinite series or integrals.

The solutions from the transport integral equation are discussed in the second part of each chapter. The form of the integral equation in each geometry is first established and the correspondence of this formulation of the transport equation with the superposition solutions derived in the first part is proved. Then follows the definition of a suitable transform of the neutron density and the procedure for determining it from its properties. Inasmuch as the angular density in plane geometry already possesses the required transform properties, the introduction of a pseudo distribution in Chapter III becomes unnecessary. Instead, we show that by considering the integral equation for the angular density and the definition of the neutron density as a transform pair, we can arrive at a singular integral equation for the angular density. In Chapter IV, the form of the integral equation for the spherical neutron density permits us to choose a transform which converts the spherical problem into the already solved plane problem. The transform in cylindrical geometry is shown to satisfy a second-order, separable integrodifferential equation. An expansion for the transform is then obtained in terms of the eigenfunctions of the equation. Furthermore, the completeness of the eigenfunctions is readily deduced from their striking resemblance to the singular modes of plane geometry.

In the last part of each of these three chapters we consider the solutions of the critical problem. The application of the physical or pseudo boundary condition yields, in each case, a singular integral equation for the expansion coefficients. Each is then reduced to a

regular Fredholm equation coupled with a criticality condition. The critical plane and sphere problems are also treated by a generalization of the method used by Leonard and Mullikin (15), and the results of the two methods are compared. Finally, the major conclusions of the entire investigation are recapitulated in Chapter VI.

To summarize, it is the purpose of this work to specify the behavior and properties of the stationary, monoenergetic neutron distributions for the plane, spherical, and cylindrical critical problems, and to establish systematic approximation procedures for the computation of the desired quantities to any accuracy. The overall approach consists of determining unique expansions for the angular densities either directly from the integrodifferential transport equation or through the use of suitable transforms of the transport integral equation.

1.1 NOTATION

The fundamental quantity in neutron transport theory is the neutron distribution ψ . In general, ψ depends upon the three space variables denoted by the position vector $\underline{r}$, the neutron energy E , the unit vector $\underline{\Omega} = \underline{v}/v$ along the direction of the velocity $\underline{v}$, and the time t , and is defined so that $\psi(\underline{r}, E, \underline{\Omega}, t) d^3r dE d\Omega$ represents the expected number of neutrons in the volume element d^3r about $\underline{r}$, with energy E in dE and direction $\underline{\Omega}$ in $d\Omega$ at time t .

For stationary, monoenergetic problems in uniform media, ψ is no longer a function of E and t and satisfies the simplified Boltzmann equation

$$v \underline{\Omega} \cdot \nabla \psi(\underline{r}, \underline{\Omega}) + \Sigma_t v \psi(\underline{r}, \underline{\Omega}) = \Sigma_s \int_{\underline{\Omega}'} v \psi(\underline{r}, \underline{\Omega}') f(\underline{\Omega}' \rightarrow \underline{\Omega}) d\Omega' + S(\underline{r}, \underline{\Omega}). \quad (1)$$

In Eq. (1), Σ_t and Σ_s are the macroscopic total and scattering cross-sections, which, in this case, are constants, $f(\underline{\Omega}' \rightarrow \underline{\Omega})$ is the neutron scattering frequency, and $S(\underline{r}, \underline{\Omega})$ denotes the source term. We shall further restrict our attention to problems with isotropic scattering, so that

$f(\underline{\Omega}' \rightarrow \underline{\Omega}) = 1/4 \pi$, and to multiplying media where the only source is fission:

$$S(\underline{r}, \underline{\Omega}) = \frac{\nu \Sigma_f}{4 \pi} \int_{\underline{\Omega}'} \nu \psi(\underline{r}, \underline{\Omega}') d\Omega'.$$

Here ν is the average number of neutrons produced per fission and Σ_f stands for the macroscopic fission cross section. The notation becomes more compact if we measure distance in units of total meanfree paths by replacing $\underline{r}$ by $\Sigma_t \underline{r}$ and introduce the mean number of secondaries per collision:

$$c = (\Sigma_s + \nu \Sigma_f) / \Sigma_t.$$

Equation (1) then assumes the form

$$\underline{\Omega} \cdot \underline{\nabla} \psi(\underline{r}, \underline{\Omega}) + \psi(\underline{r}, \underline{\Omega}) = \frac{c}{4 \pi} \int_{\underline{\Omega}'} \psi(\underline{r}, \underline{\Omega}') d\Omega'. \quad (2)$$

The neutron density $\rho(\underline{r})$ and the neutron current $\underline{j}(\underline{r})$ are defined as the zeroth and first angular moments of the angular density $\psi(\underline{r}, \underline{\Omega})$, respectively:

$$\rho(\underline{r}) = \int_{\underline{\Omega}} \psi(\underline{r}, \underline{\Omega}) d\Omega, \quad (3)$$

$$\underline{j}(\underline{r}) = \int_{\underline{\Omega}} \underline{\Omega} \psi(\underline{r}, \underline{\Omega}) d\Omega, \quad (4)$$

and are related by the continuity equation

$$\underline{\nabla} \cdot \underline{j}(\underline{r}) + (1 - c) \rho(\underline{r}) = 0, \quad (5)$$

which follows from the integration of Eq. (2) over $\underline{\Omega}$.

Since the first term of Eq. (2) is the directional derivative in the direction of $\underline{\Omega}$, we can express this equation as a first-order linear differential equation in terms of the coordinate s , where $\underline{r} = \underline{r}' - s \underline{\Omega}$, and then solve it formally to get the transport integral equation (12, p. 25):

$$\psi(\underline{r}, \underline{\Omega}) = \psi(\underline{r} - s_0 \underline{\Omega}, \underline{\Omega}) e^{-s_0} + \frac{c}{4\pi} \int_0^{s_0} \rho(\underline{r} - s \underline{\Omega}) e^{-s} ds. \quad (6)$$

In Eq. (6), $s_0 = |\underline{r}_0 - \underline{r}|$ is the distance from $\underline{r}$ (or from $\underline{r}'$ in Fig. 1) to the surface point $\underline{r}_0$, and $\psi(\underline{r} - s_0 \underline{\Omega}, \underline{\Omega})$ represents the neutrons entering the system. The boundary condition stating that no neutrons enter the system requires

$$\psi(\underline{r} - s_0 \underline{\Omega}, \underline{\Omega}) = 0, \quad (7)$$

and Eq. (6) reduces to

$$\psi(\underline{r}, \underline{\Omega}) = \frac{c}{4\pi} \int_0^{s_0} \rho(\underline{r} - s \underline{\Omega}) e^{-s} ds. \quad (8)$$

Fig. 1
Integral Equation Coordinates

Integration of Eq. (8) over $\underline{\Omega}$ and replacement of $\underline{r} - s \underline{\Omega}$ by $\underline{r}'$ gives the integral equation for the neutron density

$$\rho(\underline{r}) = \frac{c}{4\pi} \int_{\underline{r}'} \frac{\rho(\underline{r}') e^{-|\underline{r} - \underline{r}'|}}{|\underline{r} - \underline{r}'|^2} d^3 r', \quad (9)$$

where the integration is over the entire volume of the system.

CHAPTER II

SOLUTIONS IN THREE DIMENSIONS

2.0 INTRODUCTION

Before turning our attention to the development of transport solutions with specific symmetries, we shall show in this chapter that it is possible to establish the fundamental solutions of the transport equation (1.1-2)¹ and to examine their properties without regard to geometrical configuration. Then, using the superposition technique, we shall construct formal expansions for $\psi(\underline{r}, \underline{\Omega})$, $\rho(\underline{r})$, and $\underline{j}(\underline{r})$, which will be used in subsequent chapters as the bases for the determination of special expressions for these quantities.

Since the definition of suitable density transforms, used in the second method for treating the critical problems, depends on the form of the integral equation for the density in each geometry, this approach will not be illustrated here. The discussion of the integral equation will be limited to a proof that the three-dimensional expansions produced from the superposition are consistent with Eq. (1.1-8). Besides the question of interest in confirming the equality of the two formulations of the transport equation, the procedure for this proof will be used in Chaps. IV and V to show that the coefficients in the transform expansions and the corresponding coefficients from the three-dimensional expansions are identical.

2.1 EIGENFUNCTIONS OF THE TRANSPORT EQUATION

Separation of the variables $\underline{r}$ and $\underline{\Omega}$ shows that the fundamental solutions of the transport equation

$$\underline{\Omega} \cdot \underline{\nabla} \psi(\underline{r}, \underline{\Omega}) + \psi(\underline{r}, \underline{\Omega}) = \frac{c}{4\pi} \int_{\underline{\Omega}'} \psi(\underline{r}, \underline{\Omega}') d\Omega' \quad (1.1-2)$$

¹This label will be used throughout to designate equations exhibited in previous sections, i.e., Eq. (x-y) means Eq. (y) of Sec. x.

have the form

$$\psi_{\underline{k}}(\underline{r}, \underline{\Omega}) = e^{-\underline{k} \cdot \underline{r}} \phi_{\underline{k}}(\underline{\Omega}). \quad (1)$$

It is convenient to express the arbitrary vector $\underline{k}$ as

$$\underline{k} = \underline{u}/\nu, \quad (2)$$

where ν is restricted by the condition that (1) is a solution of Eq. (1.1-2), and $\underline{u}$ is a unit vector with components

$$u_x = \sin \alpha \cos \beta; \quad u_y = \sin \alpha \sin \beta; \quad u_z = \cos \alpha. \quad (2a)$$

For fixed $\underline{u}$, the solutions (1) lie on the family of planes defined by the equation

$$\underline{r} \cdot \underline{u} = \text{constant}.$$

Since the transport equation is linear, a more general solution of Eq. (1.1-2) can be constructed by superposing these plane solutions over all possible orientations of the vector $\underline{u}$:

$$\psi(\underline{r}, \underline{\Omega}) = \frac{1}{4\pi} \int_{\underline{u}} A(\underline{u}) \psi_{\underline{k}}(\underline{r}, \underline{\Omega}) d\underline{u}, \quad (3)$$

where $A(\underline{u})$ is an arbitrary weighting function and the integration over $\underline{u}$ stands for the double integral over the polar angle α and azimuthal angle β (see Fig. 2).

Fig. 2

Superposition Coordinates

The explicit form of $\phi_{\underline{k}}(\underline{\Omega})$ and the permissible values of ν can be determined by the procedure used by Case (1) in plane geometry. If we

substitute Eq. (1) into Eq. (1.1-2), we find

$$\left(1 - \frac{\underline{u} \cdot \underline{\Omega}}{\nu}\right) \phi_{\underline{k}}(\underline{\Omega}) = \frac{c}{4\pi} \int_{\underline{\Omega}'} \phi_{\underline{k}}(\underline{\Omega}') d\Omega'. \quad (4)$$

The linearity of the transport equation permits us to normalize these eigenfunctions so that

$$\frac{1}{2\pi} \int_{\underline{\Omega}} \phi_{\underline{k}}(\underline{\Omega}) d\Omega = 1. \quad (5)$$

Then Eq. (4) reduces to

$$(\nu - \underline{u} \cdot \underline{\Omega}) \phi_{\underline{k}}(\underline{\Omega}) = c \nu / 2. \quad (6)$$

The quantity $\underline{u} \cdot \underline{\Omega}$ is simply the cosine of the angle between two unit vectors and thus lies in the interval $[-1, 1]$. Since ν is as yet unrestricted, it can also take values in this interval. This introduces singularities at the points $\nu = \underline{u} \cdot \underline{\Omega}$ which are taken into account by writing the general solution of Eq. (6) as

$$\phi_{\underline{k}}(\underline{\Omega}) \equiv \phi_{\nu}(\underline{u} \cdot \underline{\Omega}) = \frac{c}{2} P \frac{\nu}{\nu - \underline{u} \cdot \underline{\Omega}} + \lambda(\nu) \delta(\underline{u} \cdot \underline{\Omega} - \nu), \quad (7)$$

where P denotes Cauchy principal values for integrals involving $\phi_{\nu}(\underline{u} \cdot \underline{\Omega})$ and the δ -function term contributes only at the singular points $\nu = \underline{u} \cdot \underline{\Omega}$. Furthermore, this solution satisfies Eq. (6) for any $\lambda(\nu)$.

If the singular terms in Eq. (7) are expressed as limits (17; 18, p. 783),

$$P \frac{1}{\nu - \underline{u} \cdot \underline{\Omega}} = \lim_{\epsilon \rightarrow 0} \frac{1}{2} \left[\frac{1}{\nu - \underline{u} \cdot \underline{\Omega} + i\epsilon} + \frac{1}{\nu - \underline{u} \cdot \underline{\Omega} - i\epsilon} \right] \quad (8a)$$

and

$$\delta(\underline{u} \cdot \underline{\Omega} - \nu) = \frac{1}{\pi} \lim_{\epsilon \rightarrow 0} \frac{\epsilon}{(\underline{u} \cdot \underline{\Omega} - \nu)^2 + \epsilon^2} \quad (8b)$$

the expression for $\phi_{\nu}(\underline{u} \cdot \underline{\Omega})$ corresponds to the singular solutions of Wigner (3).

For values of ν not in the interval $[-1, 1]$, Eq. (7) reduces to

$$\phi_{\nu}(\underline{u} \cdot \underline{\Omega}) = \frac{c}{2} \frac{\nu}{\nu - \underline{u} \cdot \underline{\Omega}}; \quad \nu \notin [-1, 1].$$

Substitution in the normalization condition (5) gives

$$\frac{c\nu}{4\pi} \int_{\underline{\Omega}} \frac{d\Omega}{\nu - \underline{u} \cdot \underline{\Omega}} = 1. \quad (5a)$$

This integration can easily be carried out if we fix $\underline{u}$ along the z-axis and specify $\underline{\Omega}$ by a polar and an azimuthal angle. Then

$$c\nu \tanh^{-1} \frac{1}{\nu} = 1. \quad (9)$$

The two roots of Eq. (9) will be denoted by $\pm\nu_0$ and are purely imaginary for $c > 1$. The corresponding eigenfunctions are

$$\phi_{0\pm}(\underline{u} \cdot \underline{\Omega}) = \frac{c}{2} \frac{\nu_0}{\nu_0 \mp \underline{u} \cdot \underline{\Omega}}. \quad (10)$$

For ν in the interval $[-1, 1]$ the entire solution (7) must be used. Equation (5) in this case serves to specify $\lambda(\nu)$:

$$\lambda(\nu) = 1 - c\nu \tanh^{-1} \nu, \quad (11)$$

but does not restrict ν further. The solutions of Eq. (6) therefore comprise the two discrete terms $\phi_{0\pm}(\underline{u} \cdot \underline{\Omega})$ and the continuum $\phi_{\nu}(\underline{u} \cdot \underline{\Omega})$ for $\nu \in [-1, 1]$. According to Case (1), these eigenfunctions, when specialized to plane geometry, form a complete set. This completeness theorem could not be extended to the three-dimensional modes presented here primarily because the required superposition in the form of Eq. (3) destroys their singular properties, which were essential in the proof.

The elementary solutions (1) of the transport equation can now be separated in two classes:

a) For $\nu \notin [-1, 1]$ there are two discrete solutions

$$\psi_{0\pm}(\underline{r}, \underline{\Omega}; \underline{u}) = e^{\mp \underline{r} \cdot \underline{u} / \nu_0} \phi_{0\pm}(\underline{u} \cdot \underline{\Omega}) \quad (12a)$$

with ν_0 and $\phi_{0\pm}(\underline{u} \cdot \underline{\Omega})$ defined by Eqs. (9) and (10), respectively.

b) For $\nu \in [-1, 1]$ there is a continuum of solutions:

$$\psi_\nu(\underline{r}, \underline{\Omega}; \underline{u}) = e^{-\underline{r} \cdot \underline{u} / \nu} \phi_\nu(\underline{u} \cdot \underline{\Omega}), \quad (12b)$$

where $\phi_\nu(\underline{u} \cdot \underline{\Omega})$ is given by Eq. (7). A linear combination of them represents the transport solutions in a family of planes in space with fixed axis in the direction of $\underline{u}$:

$$\begin{aligned} \psi_{\underline{u}}(\underline{r}, \underline{\Omega}) &= a_{0+}(\underline{u}) \psi_{0+}(\underline{r}, \underline{\Omega}; \underline{u}) + a_{0-}(\underline{u}) \psi_{0-}(\underline{r}, \underline{\Omega}; \underline{u}) \\ &+ \int_{-1}^1 A(\nu, \underline{u}) \psi_\nu(\underline{r}, \underline{\Omega}; \underline{u}) d\nu, \end{aligned} \quad (13)$$

where $a_{0\pm}(\underline{u})$ and $A(\nu, \underline{u})$ are arbitrary expansion coefficients. The function $A(\nu, \underline{u})$ will be assumed to vanish sufficiently fast as $\nu \rightarrow 0$ so that the integral in Eq. (13) exists. Finally, superposing the plane solutions in accordance with Eq. (3), we obtain

$$\psi(\underline{r}, \underline{\Omega}) = \psi_{d+}(\underline{r}, \underline{\Omega}) + \psi_{d-}(\underline{r}, \underline{\Omega}) + \psi_c(\underline{r}, \underline{\Omega}), \quad (14)$$

where

$$\psi_{d\pm}(\underline{r}, \underline{\Omega}) = \frac{1}{4\pi} \int_{\underline{u}} a_{0\pm}(\underline{u}) \psi_{0\pm}(\underline{r}, \underline{\Omega}; \underline{u}) d\underline{u} \quad (14a)$$

are the three-dimensional solutions from the discrete part of the spectrum and

$$\psi_c(\underline{r}, \underline{\Omega}) = \frac{1}{4\pi} \int_{\underline{u}} d\underline{u} \int_{-1}^1 A(\nu, \underline{u}) \psi_\nu(\underline{r}, \underline{\Omega}; \underline{u}) d\nu \quad (14b)$$

is the solution due to the continuum.

The corresponding expression for the neutron density follows from integration of Eq. (14) over $\underline{\Omega}$ and use of Eq. (5):

$$\begin{aligned} \rho(\underline{r}) &= \frac{1}{2} \int_{\underline{u}} d\underline{u} \left[a_{0+}(\underline{u}) e^{-\underline{r} \cdot \underline{u} / \nu_0} + a_{0-}(\underline{u}) e^{\underline{r} \cdot \underline{u} / \nu_0} \right. \\ &\quad \left. + \int_{-1}^1 A(\nu, \underline{u}) e^{-\underline{r} \cdot \underline{u} / \nu} d\nu \right]. \end{aligned} \quad (15)$$

Higher angular moments of $\psi(\underline{r}, \underline{\Omega})$ can also be obtained from Eq. (14) (19, p. 53). In particular, the first discrete term for the neutron current is

$$\begin{aligned} \underline{j}_{d+}(\underline{r}) &= \int_{\underline{\Omega}} \underline{\Omega} \psi_{d+}(\underline{r}, \underline{\Omega}) d\Omega \\ &= \frac{1}{4\pi} \int_{\underline{u}} a_{0+}(\underline{u}) e^{-\underline{r} \cdot \underline{u} / \nu_0} d\mathbf{u} \int_{\underline{\Omega}} \frac{\left(\frac{c}{2} \nu_0\right) \underline{\Omega} d\Omega}{\nu_0 - \underline{u} \cdot \underline{\Omega}} \end{aligned} \quad (16)$$

The integral over $\underline{\Omega}$ in Eq. (16) is a vector $\underline{R}$ along the direction of $\underline{u}$ (see Sec. 1A of Appendix A), with magnitude

$$\begin{aligned} R &= \frac{c \nu_0}{2} \int_{\underline{\Omega}} \frac{\underline{u} \cdot \underline{\Omega} d\Omega}{\nu_0 - \underline{u} \cdot \underline{\Omega}} \\ &= 2\pi \nu_0 (1 - c). \end{aligned} \quad (16a)$$

Hence,

$$\underline{j}_{d+}(\underline{r}) = \frac{1 - c}{2} \nu_0 \int_{\underline{u}} a_{0+}(\underline{u}) e^{-\underline{r} \cdot \underline{u} / \nu_0} \underline{u} d\mathbf{u}. \quad (16b)$$

Evaluating the remaining terms in a similar manner, we arrive at the following expression for the neutron current:

$$\begin{aligned} \underline{j}(\underline{r}) &= \frac{1 - c}{2} \int_{\underline{u}} \underline{u} d\mathbf{u} \left[a_{0+}(\underline{u}) \nu_0 e^{-\underline{r} \cdot \underline{u} / \nu_0} - a_{0-}(\underline{u}) \nu_0 e^{\underline{r} \cdot \underline{u} / \nu_0} \right. \\ &\quad \left. + \int_{-1}^1 A(\nu, \underline{u}) \nu e^{-\underline{r} \cdot \underline{u} / \nu} d\nu \right]. \end{aligned} \quad (17)$$

A comparison of the derivative with respect to r_i of the components of $\underline{j}_{d+}(\underline{r})$ in Eq. (16b) with the first term of Eq. (15) reveals that

$$\frac{\partial j_{d+}^i(\underline{r})}{\partial r_i} = - \frac{1 - c}{2} \int_{\underline{u}} a_{0+}(\underline{u}) u_i u_i e^{-\underline{r} \cdot \underline{u} / \nu_0} d\mathbf{u}$$

or

$$\underline{\nabla} \cdot \underline{j}_{d+}(\underline{r}) = - (1 - c) \rho_{d+}(\underline{r}).$$

From this and similar results for the other terms, we conclude that the neutron current of Eq. (17) is related to $\rho(\underline{r})$ of Eq. (15) by the continuity equation

$$\nabla \cdot \underline{j}(\underline{r}) + (1-c) \rho(\underline{r}) = 0. \quad (1.1-5)$$

Now, consider the discrete terms of Eqs. (15) and (17), which we denote by $\rho_d(\underline{r})$ and $\underline{j}_d(\underline{r})$, respectively. Since

$$\begin{aligned} \frac{\partial \rho_{d+}(\underline{r})}{\partial r_i} &= -\frac{1}{2} \frac{1}{\nu_0} \int_{\underline{u}} a_{0+}(\underline{u}) u_i e^{-\underline{r} \cdot \underline{u} / \nu_0} du \\ &= -\frac{1}{\nu_0^2 (1-c)} j_{d+}^i(\underline{r}), \end{aligned}$$

it follows that

$$\underline{j}_d(\underline{r}) = -D \nabla \rho_d(\underline{r}), \quad (18)$$

where

$$D = \nu_0^2 (1-c). \quad (18a)$$

This definition of the diffusion coefficient degenerates to the familiar result $D = \frac{1}{3}$ in units of mean free paths for c close to unity (19).

Moreover, substitution of Eq. (18) in the continuity equation (1.1-5) yields the standard diffusion equation for the neutron density:

$$\nabla^2 \rho_d(\underline{r}) - \frac{1}{\nu_0^2} \rho_d(\underline{r}) = 0. \quad (19)$$

Thus, the discrete solutions of the transport equation coincide with the asymptotic results of diffusion theory. However, the generality of the expansion (14) for ψ has not been established. In addition, although the results of this section provide considerable insight as to the nature of the monoenergetic transport spectrum and can be used to derive expressions for ψ in various geometries, the lack of a method for finding the expansion coefficients $a_{0\pm}(\underline{u})$ and $A(\nu, \underline{u})$ from the boundary conditions prevents us from treating specific problems directly.

When Eq. (14) is specialized to spherical and cylindrical geometries (see Secs. 4.2 and 5.2), only the solutions which are regular at the center of these systems are generated. Thus, for example, neutron distributions due to point or line sources, or those appropriate for spherical and cylindrical shells where singular solutions are admissible, cannot be determined directly from Eq. (14). This difficulty can be traced to the assumption that $\underline{u}$ in Eq. (2) can be specified by a polar angle α where $0 \leq \alpha \leq \pi$ and an azimuthal angle β where $0 \leq \beta \leq 2\pi$. In fact, such restrictions on the range of superposition variables are not dictated by the analysis. From the analogous problem in the construction of solutions to the wave equation (20, p. 362), it is expected that singular transport solution can be deduced from the above results if the ranges of α and β are extended to the complex domain. The admission of complex components of the vector $\underline{k}$, however, introduces considerable difficulty in the resolution of the spectrum from Eq. (6). These difficulties and the possibility of obtaining non-regular solutions are examined in Sec. 2A of Appendix A.

2.2 TRANSPORT INTEGRAL EQUATION

We now show that if we substitute Eq. (2.1-15) for the neutron density into the integral equation

$$\psi(\underline{r}, \underline{\Omega}) = \frac{c}{4\pi} \int_0^{s_0} \rho(\underline{r} - s\underline{\Omega}) e^{-s} ds, \quad (1.1-8)$$

we obtain again Eq. (2.1-14) for ψ and an additional expression representing the incoming distribution at the boundary.

Consider first the discrete terms in Eq. (2.1-15):

$$\rho_{d\pm}(\underline{r} - s\underline{\Omega}) = \frac{1}{2} \int_{\underline{u}} a_{0\pm}(\underline{u}) e^{\mp \underline{u} \cdot (\underline{r} - s\underline{\Omega}) / \nu_0} du. \quad (1)$$

Inserting Eq. (1) in Eq. (1.1-8) and carrying out the s integration, we get

$$\begin{aligned}
\frac{c}{4\pi} \int_0^{s_0} \rho_{d\pm}(\underline{r} - s\underline{\Omega}) e^{-s} ds &= \frac{1}{4\pi} \int_{\underline{u}} a_{0\pm}(\underline{u}) \psi_{0\pm}(\underline{r}, \underline{\Omega}; \underline{u}) du \\
&\quad - \frac{e^{-s_0}}{4\pi} \int_{\underline{u}} a_{0\pm}(\underline{u}) \psi_{0\pm}(\underline{r} - s_0 \underline{\Omega}, \underline{\Omega}; \underline{u}) du \\
&= \psi_{d\pm}(\underline{r}, \underline{\Omega}) - e^{-s_0} \psi_{d\pm}(\underline{r} - s_0 \underline{\Omega}, \underline{\Omega}), \quad (2)
\end{aligned}$$

where we have made use of Eqs. (2.1-12a) and (2.1-14a).

If $\rho_c(\underline{r})$ denotes the continuous term in Eq. (2.1-15), then

$$\frac{c}{4\pi} \int_0^{s_0} \rho_c(\underline{r} - s\underline{\Omega}) e^{-s} ds = \frac{1}{4\pi} \int_{\underline{u}} \int_{-1}^1 A(\nu, \underline{u}) \frac{c\nu}{2} \frac{1 - e^{-s_0(\nu - \underline{u} \cdot \underline{\Omega})/\nu}}{\nu - \underline{u} \cdot \underline{\Omega}} e^{-\underline{r} \cdot \underline{u}/\nu} d\nu du. \quad (3a)$$

For well-behaved functions $A(\nu, \underline{u})$ the right side of Eq. (3a) is not singular as it stands, but can be expressed as a difference of two Cauchy principal value terms (16, p. 26):

$$\begin{aligned}
&\frac{1}{4\pi} \int_{\underline{u}} \int_{-1}^1 A(\nu, \underline{u}) \frac{c}{2} P \frac{\nu}{\nu - \underline{u} \cdot \underline{\Omega}} e^{-\underline{r} \cdot \underline{u}/\nu} d\nu du \\
&\quad - \frac{e^{-s_0}}{4\pi} \int_{\underline{u}} \int_{-1}^1 A(\nu, \underline{u}) \frac{c}{2} P \frac{\nu}{\nu - \underline{u} \cdot \underline{\Omega}} e^{-\underline{u} \cdot (\underline{r} - s_0 \underline{\Omega})/\nu} d\nu du. \quad (3b)
\end{aligned}$$

To this we now add and subtract the term

$$\begin{aligned}
&\frac{1}{4\pi} \int_{\underline{u}} \int_{-1}^1 A(\nu, \underline{u}) \lambda(\nu) \delta(\underline{u} \cdot \underline{\Omega} - \nu) e^{-\underline{r} \cdot \underline{u}/\nu} d\nu du \equiv \\
&\quad \frac{e^{-s_0}}{4\pi} \int_{\underline{u}} \int_{-1}^1 A(\nu, \underline{u}) \lambda(\nu) \delta(\underline{u} \cdot \underline{\Omega} - \nu) e^{-\underline{u} \cdot (\underline{r} - s_0 \underline{\Omega})/\nu} d\nu du. \quad (3c)
\end{aligned}$$

This part of the continuum is not generated by this procedure because, as seen from Eq. (3c), it appears unchanged both in the incoming and internal distributions. It will enter the proof automatically if we start from Eq. (1.1-6) instead of Eq. (1.1-8) for $\psi(\underline{r}, \underline{\Omega})$.

The continuous contribution then becomes, on remembering Eq. (2.1-14b),

$$\frac{c}{4\pi} \int_0^{s_0} \rho_c(\underline{r} - s\underline{\Omega}) e^{-s} ds = \psi_c(\underline{r}, \underline{\Omega}) - e^{-s_0} \psi_c(\underline{r} - s_0\underline{\Omega}, \underline{\Omega}). \quad (3)$$

Combining Eqs. (2) and (3), we arrive at the expression for $\psi(\underline{r}, \underline{\Omega})$:

$$\begin{aligned} \psi(\underline{r}, \underline{\Omega}) = & \psi_{d+}(\underline{r}, \underline{\Omega}) + \psi_{d-}(\underline{r}, \underline{\Omega}) + \psi_c(\underline{r}, \underline{\Omega}) \\ & - e^{-s_0} [\psi_{d+}(\underline{r} - s_0\underline{\Omega}, \underline{\Omega}) + \psi_{d-}(\underline{r} - s_0\underline{\Omega}, \underline{\Omega}) + \psi_c(\underline{r} - s_0\underline{\Omega}, \underline{\Omega})]. \end{aligned} \quad (4)$$

This is identical with Eq. (2.1-14), since the bracket represents the incoming distribution $\psi(\underline{r} - s_0\underline{\Omega}, \underline{\Omega})$ and therefore vanishes because of the boundary condition (1.1-7).

The requirement that the bracket in Eq. (4) must be zero provides an equation defining the expansion coefficients:

$$\int_{\underline{u}} du \left[a_{0+}(\underline{u}) \psi_{0+}(\underline{r}_0, \underline{\Omega}; \underline{u}) + a_{0-}(\underline{u}) \psi_{0-}(\underline{r}_0, \underline{\Omega}; \underline{u}) + \int_{-1}^1 A(\nu, \underline{u}) \psi_{\nu}(\underline{r}_0, \underline{\Omega}; \underline{u}) d\nu \right] = 0, \quad (5)$$

where $\underline{r}_0 = \underline{r} - s_0\underline{\Omega}$ is the boundary coordinate for incoming neutrons (see Fig. 1). The coefficients and a critical relation will be established directly from this equation only in the case of plane symmetry. In the other two geometries the specialized forms of Eq. (5) are too cumbersome for direct treatment. Simpler but equivalent equations for the spherical and cylindrical coefficients will be derived with the aid of density transforms.

CHAPTER III

SOLUTIONS WITH PLANE SYMMETRY

3.0 INTRODUCTION

The simple form of the transport equation in plane geometry allows us to investigate the methods of solution and illustrate the solution properties in much greater detail than in other geometries. Especially the expansion for the angular density in terms of the singular normal modes becomes particularly easy to deal with in this case. Also, as shown in Sec. 3.2, the transport equation can be transformed into a singular integral equation for ψ which provides a simpler starting point for the treatment of problems with plane symmetry than the original transport equation. In Sec. 3.3, both the expansion and the singular integral equation are applied to the solution of the critical problem. Approximations of the exact expressions for the neutron distribution and the criticality condition furnished by the two methods are shown to be consistent and are compared with the diffusion theory results.

3.1 TRANSPORT EQUATION IN PLANE GEOMETRY

We consider a one-dimensional, uniform, multiplying system in which the thickness t and half-thickness b (see Fig. 3) are measured in units of mean free paths and the origin of coordinates is taken at the

Fig. 3. Plane Geometry Coordinates

center. The angular density ψ depends on the position variable x and the single direction variable $\mu = \Omega_x$, and satisfies the equation

$$\mu \frac{\partial \psi(x, \mu)}{\partial x} + \psi(x, \mu) = \frac{c}{2} \int_{-1}^1 \psi(x, \mu') d\mu'. \quad (1)$$

The eigenfunctions of Eq. (1) and a general expansion for $\psi(x, \mu)$ have been discussed in detail by Case (1). The same

results also follow from the three-dimensional expansions of Sec. 2.1. Thus, if we fix the direction of $\underline{u}$ along the x-axis (see Fig. 2), then

$$\underline{u} \cdot \underline{r} = x; \quad \underline{u} \cdot \underline{\Omega} = \mu.$$

The expansion modes given by Eqs. (2.1-12a) and (2.1-12b) now assume the forms:

$$a) \psi_{0\pm}(x, \mu) = \phi_{0\pm}(\mu) e^{\mp x/\nu_0}, \quad \nu_0 \notin [-1, 1] \quad (2a)$$

for the two discrete terms, where

$$\phi_{0\pm}(\mu) = \frac{c}{2} \frac{\nu_0}{\nu_0 \mp \mu} \quad (2b)$$

and ν_0 is given by Eq. (2.1-9);

$$b) \psi_\nu(x, \mu) = \phi_\nu(\mu) e^{-x/\nu}, \quad \nu \in [-1, 1] \quad (3a)$$

for the continuum, where

$$\phi_\nu(\mu) = \frac{c}{2} P \frac{\nu}{\nu - \mu} + \lambda(\nu) \delta(\mu - \nu), \quad (3b)$$

with $\lambda(\nu)$ defined by Eq. (2.1-11). In accordance with Eq. (2.1-13), the solution of Eq. (1) can be written as

$$\psi(x, \mu) = a_0 + \psi_{0+}(x, \mu) + a_0 - \psi_{0-}(x, \mu) + \int_{-1}^1 A(\nu) \psi_\nu(x, \mu) d\nu. \quad (4)$$

The completeness theorem (1) for the functions $\phi_{0\pm}(\mu)$ and $\phi_\nu(\mu)$ guarantees the uniqueness of the expansion (4) and, in addition, provides a way for finding the coefficients $A(\nu)$ from the symmetry and boundary conditions

$$\psi(x, \mu) = \psi(-x, -\mu) \quad (5)$$

and

$$\psi(-b, \mu > 0) = \psi(b, \mu < 0) = 0. \quad (6)$$

The neutron density

$$\rho(x) = \int_{-1}^1 \psi(x, \mu) d\mu \quad (7)$$

is obtained by integration of Eq. (4) over μ and use of the normalization (2.1-5):

$$\rho(x) = a_{0+} e^{-x/\nu_0} + a_{0-} e^{x/\nu_0} + \int_{-1}^1 A(\nu) e^{-x/\nu} d\nu. \quad (8)$$

Multiplying Eq. (4) by μ and integrating over μ , we arrive at an expression for the neutron current:

$$\begin{aligned} j(x) &= \int_{-1}^1 \mu \psi(x, \mu) d\mu \\ &= a_{0+} \nu_0 (1-c) e^{-x/\nu_0} - a_{0-} \nu_0 (1-c) e^{x/\nu_0} + (1-c) \int_{-1}^1 A(\nu) \nu e^{-x/\nu} d\nu. \end{aligned} \quad (9)$$

As can be readily verified, these expansions for $\rho(x)$ and $j(x)$ are consistent with the one-dimensional continuity equation

$$\frac{dj(x)}{dx} + (1-c) \rho(x) = 0. \quad (10)$$

An integral equation for $\psi(x, \mu)$ can be obtained by considering Eq. (1) as a first-order differential equation with the right side known. It also follows from the transport integral equation

$$\psi(\underline{r}, \underline{\Omega}) = e^{-s_0} \psi(\underline{r} - s_0 \underline{\Omega}, \underline{\Omega}) + \frac{c}{4\pi} \int_0^{s_0} \rho(\underline{r} - s \underline{\Omega}) e^{-s} ds \quad (1.1-6)$$

by noting that, in this case,

$$\psi(\underline{r}, \underline{\Omega}) = \frac{1}{2\pi} \psi(x, \mu); \quad \rho(\underline{r} - s \underline{\Omega}) = \rho(x - s \mu).$$

Referring to Fig. 3, we see that the distance s_0 from the boundary to a point x is

$$s_0 = \frac{b+x}{\mu}, \quad \mu > 0.$$

With the change $y = x - s\mu$, we get

$$\psi(x, \mu) = e^{-\frac{b+x}{\mu}} \psi(-b, \mu) + \frac{c}{2} \frac{1}{\mu} \int_{-b}^x \rho(y) e^{-\frac{x-y}{\mu}} dy, \quad \mu > 0. \quad (11a)$$

The expression for ψ for negative μ can be determined similarly or by replacing x by $-x$ in Eq. (11a) and making use of the symmetries

$$\psi(-x, \mu) = \psi(x, -\mu); \quad \rho(-y) = \rho(y); \quad (5a)$$

$$\psi(x, -\mu) = e^{-\frac{b-x}{\mu}} \psi(b, -\mu) + \frac{c}{2} \frac{1}{\mu} \int_x^b \rho(y) e^{-\frac{y-x}{\mu}} dy, \quad \mu > 0. \quad (11b)$$

The boundary condition (6) requires that the first terms on the right of Eqs. (11a) and (11b) vanish. These equations then reduce to

$$\psi(x, \mu) = \frac{c}{2} \frac{1}{\mu} \int_{-b}^x \rho(y) e^{-\frac{x-y}{\mu}} dy \quad (12a)$$

and

$$\psi(x, -\mu) = \frac{c}{2} \frac{1}{\mu} \int_x^b \rho(y) e^{-\frac{y-x}{\mu}} dy, \quad (12b)$$

where $\mu > 0$. Inserting these expressions for $\psi(x, \mu)$ in Eq. (7), which can also be written as

$$\rho(x) = \int_0^1 [\psi(x, \mu) + \psi(x, -\mu)] d\mu, \quad (13)$$

we arrive at the integral equation for $\rho(x)$:

$$\rho(x) = \frac{c}{2} \int_{-b}^b \rho(y) E_1(|x-y|) dy, \quad (14)$$

where $E_1(z)$ is the first member of the exponential integrals (19, p. 153)

$$E_n(z) = \int_0^1 \mu^{n-2} e^{-z/\mu} d\mu. \quad (14a)$$

If we substitute Eq. (8) and the expansion for $\psi(-b, \mu)$ in Eq. (11a) and carry out the integrations over y , we regenerate the expansion (4) for $\psi(x, \mu)$. Inasmuch as this procedure was discussed in Sec. 2.2, it will not be repeated here.

3.2 A SINGULAR INTEGRAL EQUATION FOR $\psi(x, \mu)$

The integral equations (11a) and (11b) can also be viewed as transforms of $\rho(x)$ with the inverse transform given by Eq. (13). In this section we show that this point of view leads to a singular integral equation for $\psi(x, \mu)$.

Substitution of Eq. (13) into (11a) and change of the order of y and ν integration gives

$$\psi(x, \mu) = e^{-\frac{b+x}{\mu}} \psi(-b, \mu) + \frac{c}{2} \frac{1}{\mu} \int_0^1 d\nu \int_{-b}^x [\psi(y, \nu) + \psi(y, -\nu)] e^{-\frac{x-y}{\mu}} dy, \quad \mu > 0. \quad (1)$$

The integral over y can be performed by inserting the explicit forms for $\psi(x, \pm \nu)$ from Eqs. (11a) and (11b). This is done in Sec. 1B of Appendix B, where we find

$$\begin{aligned} \int_{-b}^x \psi(y, \nu) e^{-\frac{x-y}{\mu}} dy &= \frac{\mu}{\nu - \mu} [\nu \psi(x, \nu) - \mu \psi(x, \mu)] \\ &+ \frac{\mu}{\nu - \mu} e^{-\frac{b+x}{\mu}} [\mu \psi(-b, \mu) - \nu \psi(-b, \nu)] \quad (2a) \end{aligned}$$

and

$$\int_{-b}^x \psi(y, -\nu) e^{-\frac{x-y}{\mu}} dy = \frac{\mu}{\nu + \mu} [\nu \psi(x, -\nu) + \mu \psi(x, \mu)] \\ - \frac{\mu}{\nu + \mu} e^{-\frac{b+x}{\mu}} [\nu \psi(b, \nu) + \mu \psi(-b, \mu)]. \quad (2b)$$

Here, ν and μ lie in the interval $[0, 1]$. Observe that the right side of Eq. (2a) is not singular, since both numerator and denominator vanish at $\mu = \nu$. By the argument which follows Eq. (2.2-3a), the integration of each term separately must be interpreted as a Cauchy principal value. Putting these integrals in Eq. (1) and collecting terms, we obtain

$$\lambda(\mu) \psi(x, \mu) - P \int_0^1 \left(\frac{c}{2} \right) \frac{\nu \psi(x, \nu)}{\nu - \mu} d\nu - \int_0^1 \left(\frac{c}{2} \right) \frac{\nu \psi(x, -\nu)}{\nu + \mu} d\nu = \\ e^{-\frac{b+x}{\mu}} \left[\lambda(\mu) \psi(-b, \mu) - P \int_0^1 \left(\frac{c}{2} \right) \frac{\nu \psi(-b, \nu)}{\nu - \mu} d\nu - \int_0^1 \left(\frac{c}{2} \right) \frac{\nu \psi(b, \nu)}{\nu + \mu} d\nu \right]; \quad \mu > 0, \quad (3)$$

where we have used the fact that

$$\lambda(\mu) = 1 - \frac{c}{2} \mu P \int_{-1}^1 \frac{d\nu}{\mu - \nu}. \quad (2.1-11)$$

With the aid of the symmetry (3.1-5a), we can rewrite Eq. (3) as

$$\lambda(\mu) \psi(x, \mu) - P \int_{-1}^1 \left(\frac{c}{2} \right) \frac{\nu \psi(x, \nu)}{\nu - \mu} d\nu = \\ e^{-\frac{b+x}{\mu}} \left[\lambda(\mu) \psi(-b, \mu) - P \int_{-1}^1 \left(\frac{c}{2} \right) \frac{\nu \psi(-b, \nu)}{\nu - \mu} d\nu \right]; \quad \mu > 0. \quad (4)$$

This is a singular integral equation for $\psi(x, \mu)$ with the space variable x acting as a parameter. Since a general method for treating such equations is available (16), Eq. (4) is a simpler starting point than the original transport equation.

The application of the boundary condition (3.1-6) reduces Eq. (4) to

$$\lambda(\mu)\psi(x, \mu) - P \int_{-1}^1 \frac{\left(\frac{c}{2}\right)^\nu \psi(x, \nu)}{\nu - \mu} d\nu = -e^{-\frac{b+x}{\mu}} \int_0^1 \frac{\left(\frac{c}{2}\right)^\nu \psi(b, \nu)}{\nu + \mu} d\nu; \quad \mu > 0. \quad (5)$$

Using the symmetry condition in the form of Eq. (3.1-5a) on (5), we obtain the equation for ψ for $\mu < 0$:

$$\lambda(\mu)\psi(x, \mu) - P \int_{-1}^1 \frac{\left(\frac{c}{2}\right)^\nu \psi(x, \nu)}{\nu - \mu} d\nu = -e^{\frac{b-x}{\mu}} \int_0^1 \frac{\left(\frac{c}{2}\right)^\nu \psi(b, \nu)}{\nu + \mu} d\nu; \quad \mu < 0. \quad (6)$$

If we let

$$f(b, \mu) = e^{-b/\mu} \int_0^1 \frac{\left(\frac{c}{2}\right)^\nu \psi(b, \nu)}{\nu + \mu} d\nu, \quad (7)$$

then Eqs. (5) and (6) can be written in the combined form

$$\lambda(\mu)\psi(x, \mu) - P \int_{-1}^1 \frac{\left(\frac{c}{2}\right)^\nu \psi(x, \nu)}{\nu - \mu} d\nu = -e^{-x/\mu} f(b, \pm \mu), \quad (8)$$

where the upper and lower signs hold for μ positive and negative, respectively. At $x = b$, Eq. (8) becomes

$$\lambda(\mu)\psi(b, \mu) - P \int_0^1 \frac{\left(\frac{c}{2}\right)^\nu \psi(b, \nu)}{\nu - \mu} d\nu = -e^{-b/\mu} f(b, \mu). \quad (9)$$

This equation for the emerging distribution was derived by Leonard and Mullikin (15) by studying the complex density transform

$$\psi(b, z) = (1/z) \int_{-b}^b \rho(y) e^{-y/z} dy, \quad (10)$$

along with the integral equation (3.1-14) for $\rho(x)$. A constraint on Eq. (9) corresponding to the criticality condition was also deduced from the analytic properties of this transform. By modifying the procedure used in (15) to include the space variable x , we can determine appropriate side conditions for Eq. (8). Thus, consider, instead of Eq. (10),

$$\psi(x, z) = \frac{c}{2} \frac{1}{z} \int_{-b}^x \rho(y) e^{-\frac{x-y}{z}} dy; \quad \text{Re}(z) > 0. \quad (11)$$

As a function of z , $\psi(x, z)$ is analytic in the right half-plane and, in fact, reduces to Eq. (3.1-12a) for z in the real axis segment $[0, 1]$.

Proceeding as in the derivation of Eq. (4), we can show that

$$\lambda(z) \psi(x, z) = \int_{-1}^1 \frac{\left(\frac{c}{2}\right) \nu \psi(x, \nu)}{\nu - z} d\nu - e^{-x/z} f(b, z), \quad (12)$$

which is recognized as the extension of Eq. (5) to the complex μ plane. But the left side of Eq. (12) vanishes at the two roots $\pm \nu_0$ of $\lambda(z) = 0$, since $\psi(x, z)$ remains finite at these points. The right side must also vanish. Hence,

$$\int_{-1}^1 \frac{\left(\frac{c}{2}\right) \nu \psi(x, \nu)}{\nu \pm \nu_0} d\nu = e^{\mp x/\nu_0} f(b, \pm \nu_0). \quad (13)$$

The same two conditions also follow from the extension of Eq. (3.1-12b) to the left half-plane.

Setting $x = b$ in Eq. (13) and remembering the definition of $f(b, \mu)$, we get the constraint for the emerging distribution given in (15):

$$f(b, \nu_0) = f(b, -\nu_0). \quad (14)$$

The solution of Eq. (8) subject to the conditions (13) and (14) is discussed in the next section as one of the two methods for treating the critical problem. Since $\psi(x, \mu)$ and a critical relation can be established directly from the foregoing results, the question of finding a unique expansion for $\psi(x, \mu)$ in terms of a complete set of eigenfunctions does not enter in this approach. What is more important, the success of this method here serves as the prime motivation for investigating the transform properties of the integral equation in other geometries where the eigenfunction expansions become intractable.

3.3 THE ONE-DIMENSIONAL CRITICAL PROBLEM

We now apply the results of the previous sections to the determination of $\psi(x, \mu)$ and the criticality condition for a one-dimensional, multiplying system (see Fig. 3). In the first part of this section we show that the application of the conditions (3.1-5) and (3.1-6) to the general expansion (3.1-4) yields a singular integral equation for the coefficients $A(\nu)$. This equation is then reduced to a Fredholm integral equation and an auxiliary condition which can be systematically approximated. Although the reduction procedure was given in a preliminary report of this investigation (4), it is presented again here in detail because many of its features will be applicable in subsequent parts. Specifically, the solution of Eq. (3.2-8) given in Sec. 3.3d requires only minor changes of the reduction procedure.

3.3a Determination of the Coefficients $A(\nu)$

The general solution of the transport equation in plane geometry, (3.1-1), is given by the expansion (3.1-4). It remains to determine the expansion coefficients from the conditions

$$\psi(x, \mu) = \psi(-x, -\mu) \quad (3.1-5)$$

and

$$\psi(-b, \mu > 0) = 0. \quad (3.1-6)$$

Inserting Eq. (3.1-4) in (3.1-5) and using the symmetry properties of the expansion modes:

$$\psi_{0\pm}(-x, -\mu) = \psi_{0\mp}(x, \mu);$$

$$\psi_{\nu}(-x, -\mu) = \psi_{-\nu}(x, \mu),$$

we find after some rearrangement that

$$\begin{aligned} & (a_{0+} - a_{0-}) \psi_{0+}(x, \mu) + (a_{0-} - a_{0+}) \psi_{0-}(x, \mu) \\ & + \int_{-1}^1 [A(\nu) - A(-\nu)] \psi_{\nu}(x, \mu) d\nu = 0. \end{aligned} \quad (1)$$

Since this is to hold for all $|x| \leq b$ and all $|\mu| \leq 1$, we conclude that

$$a_{0+} = a_{0-}; \quad A(\nu) = A(-\nu). \quad (2)$$

Using the condition at $x = -b$, we obtain the following equation for the coefficients:

$$a_{0+}[\psi_{0+}(-b, \mu) + \psi_{0-}(-b, \mu)] + \int_{-1}^1 A(\nu) \psi_{\nu}(-b, \mu) d\nu = 0; \quad \mu > 0. \quad (3)$$

It should be noted that Eq. (3) is equally valid at $x = b$.

Equation (3) can be put in a standard form for a non-homogeneous singular integral equation by decomposing the integral term and using the fact that both $A(\nu)$ and $\lambda(\nu)$ are even functions:

$$\begin{aligned} A'(\mu) \lambda(\mu) + P \int_0^1 \frac{\left(\frac{c}{2}\right) \nu A'(\nu)}{\nu - \mu} d\nu &= -a [\phi_{0+}(\mu) + \phi_{0-}(\mu) e^{-t/\nu_0}] \\ &- \int_0^1 \frac{\left(\frac{c}{2}\right) \nu A'(\nu) e^{-t/\nu}}{\nu + \mu} d\nu; \quad \mu > 0, \end{aligned} \quad (4)$$

where we have defined

$$A'(\nu) = A(\nu) e^{b/\nu} \quad \text{and} \quad a = a_{0+} e^{b/\nu_0}. \quad (5)$$

The theory of reducing equations of the form of Eq. (4) to Fredholm equations is given by Muskhelishvili (16). In essence, the procedure consists of rewriting Eq. (4) as

$$A'(\mu) \lambda(\mu) + P \int_0^1 \frac{\left(\frac{c}{2}\right) \nu A'(\nu)}{\nu - \mu} d\nu = \psi'(\mu) \quad (6)$$

with

$$\psi'(\mu) = -a [\phi_{0+}(\mu) + \phi_{0-}(\mu) e^{-t/\nu_0}] - \int_0^1 \frac{\left(\frac{c}{2}\right) \nu A'(\nu) e^{-t/\nu}}{\nu + \mu} d\nu \quad (6a)$$

and assuming temporarily that the $\psi'(\mu)$ is a known function. The unknown coefficient $A'(\nu)$ is then related to the boundary values of a sectionally holomorphic (analytic) function as it approaches the cut $[0,1]$ from above and below. Finally, such a function is constructed from its properties, and $A'(\nu)$ is determined from the boundary values of the constructed function. The discrete coefficient a (or a_{0+}) remains arbitrary, as indeed it should.

It is pointed out in Chapter 14 of Ref. 16 that the procedure of reducing singular to regular integral equations holds for sufficiently well-behaved functions $A'(\nu)$ and $\psi'(\mu)$. Specifically, we assume that both A' and ψ' satisfy the H^* condition, i.e., they satisfy the Holder condition (H condition):

$$|A'(\mu_1) - A'(\mu_2)| \leq M |\mu_1 - \mu_2|^\alpha \text{ with } 0 < \alpha \leq 1 \quad (7)$$

on the open interval $0 < \mu < 1$, with possible singularities at the ends $\beta = 0$ or 1 , which are weaker, however, than $1/\mu$:

$$A'(\mu) = \frac{A^*(\mu)}{|\mu - \beta|^\gamma}; \quad \gamma < 1. \quad (7a)$$

In the above conditions, M is a positive constant and $A^*(\mu)$ is an arbitrary function which obeys the H condition (7) in the closed interval $0 \leq \mu \leq 1$.

Following Ref. 1 closely, we introduce the sectionally holomorphic function

$$N(z) = \left(\frac{1}{2\pi i}\right) \int_0^1 \frac{\left(\frac{c}{2}\right) \nu A'(\nu)}{\nu - z} d\nu \quad (8)$$

having the following properties:

- a) $N(z)$ is holomorphic in the plane cut from 0 to 1.
- b) $N(z) \sim \frac{1}{z}$ as $z \rightarrow \infty$.
- c) $N(z) < \frac{\text{const}}{|z - \beta|^\gamma}$, $\gamma < 1$, as $z \rightarrow \beta$.

The boundary values of $N(z)$ as it approaches the line of discontinuity $(0,1)$ are given by the Plemelj formula (16):

$$N^\pm(\mu) = \pm \frac{1}{2} \frac{c}{2} \mu A'(\mu) + \left(\frac{1}{2\pi i}\right) P \int_0^1 \frac{\left(\frac{c}{2}\right) \nu A'(\nu)}{\nu - \mu} d\nu,$$

from which it follows that

$$N^+ + N^- = \frac{1}{\pi i} P \int_0^1 \frac{\left(\frac{c}{2}\right) \nu A'(\nu)}{\nu - \mu} d\nu \quad (8a)$$

and

$$N^+ - N^- = \frac{c}{2} \mu A'(\mu). \quad (8b)$$

Substitution of Eq. (8a) and (8b) into (6) gives

$$G(\mu) N^+(\mu) - N^-(\mu) = \frac{\left(\frac{c}{2}\right) \mu \psi'(\mu)}{\lambda(\mu) - \left(\frac{c \pi i \mu}{2}\right)}, \quad (9)$$

where we have defined

$$G(\mu) = \frac{\lambda(\mu) + \frac{c \pi i \mu}{2}}{\lambda(\mu) - \frac{c \pi i \mu}{2}}. \quad (9a)$$

The reduction of Eq. (6) is now equivalent to the following boundary value problem: to find a sectionally holomorphic function $N(z)$ having the aforementioned properties on the complex plane and subject to Eq. (9) on the boundary. This problem is referred to as the non-homogeneous Hilbert problem and is discussed extensively in Ref. 16. Its solution involves the construction of a particular solution to the homogeneous Hilbert problem which results by equating the right side of (9) to zero. Let $[X(z)]^{-1}$ be that particular solution. By definition, it satisfies the properties of $N(z)$ in the complex plane, and its boundary values are subject to the homogeneous part of Eq. (9):

$$X^+(\mu) = G(\mu) X^-(\mu). \quad (10)$$

The explicit form of $X(z)$ can now be found by taking logarithms of Eq. (10), using the Plemelj formula, and requiring that the resulting expression has the correct behavior at the ends. From Table I of Ref. 1, the appropriate $X(z)$ is

$$X(z) = \frac{1}{1-z} e^{\Gamma(z)}, \quad (11)$$

where

$$\Gamma(z) = \left(\frac{1}{2\pi i}\right) \int_0^1 \frac{\ln G(\mu)}{\mu - z} d\mu. \quad (11a)$$

Inserting $G(\mu)$ from Eq. (10) in Eq. (9), we obtain

$$X^+N^+ - X^-N^- = \gamma(\mu) \psi'(\mu), \quad (12)$$

where

$$\gamma(\mu) = \frac{\left(\frac{c}{2}\right)\mu X^-(\mu)}{\lambda(\mu) - \left(\frac{c\pi i\mu}{2}\right)}. \quad (12a)$$

The solution of the nonhomogeneous Hilbert problem now follows immediately from (12) and the use of Plemelj's formula:

$$N(z) = \frac{1}{2\pi i X(z)} \int_0^1 \frac{\gamma(\mu) \psi'(\mu)}{\mu - z} d\mu + \frac{P_k(z)}{X(z)}, \quad (13)$$

where $P_k(z)$ is an arbitrary polynomial of degree k . Since $X(z) \sim 1/z$ at infinity, it follows from Eq. (13) that $N(z)$ will have this behavior only if $P_k(z) \equiv 0$ and

$$\int_0^1 \gamma(\mu) \psi'(\mu) d\mu = 0. \quad (14)$$

To complete the reduction of Eq. (6), it is now only necessary to find $N^\pm(\mu)$ from Eq. (13) and use Eq. (8b). The result is

$$A'(\mu) = \lambda(\mu)g(c, \mu)\psi'(\mu) - \frac{1}{X^-(\mu) \left[\lambda(\mu) + \frac{c\pi i\mu}{2} \right]} \int_0^1 \frac{\gamma(\nu) \psi'(\nu)}{\nu - \mu} d\nu, \quad (15)$$

where

$$g(c, \mu) = \frac{1}{\lambda^2(\mu) + \left(\frac{c^2\pi^2\mu^2}{4}\right)}. \quad (15a)$$

The proposed integral equation for the coefficients now follows by inserting Eq. (6a) for $\psi'(\mu)$. In spite of the singular appearance of the kernel, this equation falls into the category of regular Fredholm equations (16, p. 338). The final form of Eq. (15), as well as many subsequent results, is considerably simplified by four identities

for the function $X(z)$. These identities were first proved by Case (10). The proofs are sketched in Sec. 2B of Appendix B.

$$1. X(z) = \int_0^1 \frac{\gamma(\mu) d\mu}{\mu - z}.$$

$$2. X(z) X(-z) = \frac{1 - c z \tanh^{-1} z}{(\nu_0^2 - z^2)(1 - c)}.$$

$$3. X(-\nu) = \exp \left[-\frac{c}{2} \int_0^1 g(c, \mu) \left(1 + \frac{c \mu^2}{1 - \mu^2} \right) \ln(\mu + \nu) d\mu \right].$$

$$4. X(-\nu) = \frac{c}{2} \frac{1}{1 - c} \int_{-1}^0 \frac{\mu d\mu}{(\nu_0^2 - \mu^2) X(\mu)(\mu - \nu)}, \quad \nu > 0.$$

The discrete terms of Eq. (6a) give rise to integrals of the form

$$\int_0^1 \frac{\phi_{0\pm}(\nu) \gamma(\nu) d\nu}{\nu - \mu} = \frac{c}{2} \nu_0 \int_0^1 \frac{\gamma(\nu) d\nu}{(\nu_0 \mp \nu)(\nu - \mu)}.$$

Decomposing the denominator by partial fractions and using the first identity, we find

$$\int_0^1 \frac{\phi_{0\pm}(\nu) \gamma(\nu) d\nu}{\nu - \mu} = \frac{c}{2} \nu_0 \frac{1}{\nu_0 \mp \mu} [X(\mu) - X(\pm \nu_0)],$$

where

$$X(\mu) = \frac{1}{2} [X^+(\mu) + X^-(\mu)].$$

Similarly, the integral term of Eq. (6a) gives

$$\int_0^1 \frac{\gamma(\nu) d\nu}{\nu - \mu} \int_0^1 \frac{\left(\frac{c}{2}\right) \alpha A'(\alpha) e^{-t/\alpha}}{\alpha + \nu} d\alpha = \int_0^1 \frac{\left(\frac{c}{2}\right) \alpha A'(\alpha) e^{-t/\alpha}}{\alpha + \mu} [X(\mu) - X(-\alpha)] d\alpha.$$

Finally, from the second identity, we see that

$$\frac{1}{X^-(\mu) \left[\lambda(\mu) + \frac{c\pi i \mu}{2} \right]} = g(c, \mu) (\nu_0^2 - \mu^2) (1 - c) X(-\mu).$$

Substitution in Eq. (15) and cancellation of common terms gives

$$\begin{aligned} A'(\mu) = & -a \left[\phi_{0+}(\mu) X(\nu_0) + \phi_{0-}(\mu) X(-\nu_0) e^{-t/\nu_0} \right] \\ & (x) (\nu_0^2 - \mu^2) (1 - c) X(-\mu) g(c, \mu) \\ & - (\nu_0^2 - \mu^2) (1 - c) g(c, \mu) X(-\mu) \int_0^1 \frac{\left(\frac{c}{2}\right) \nu X(-\nu) A'(\nu) e^{-t/\nu}}{\nu + \mu} d\nu. \end{aligned} \quad (16)$$

By a similar procedure, the auxiliary condition (14) can also be put in a more explicit form:

$$a \frac{c\nu_0}{2} [X(\nu_0) - e^{-t/\nu_0} X(-\nu_0)] = \int_0^1 \frac{c}{2} \nu X(-\nu) A'(\nu) e^{-t/\nu} d\nu. \quad (17)$$

Equation (17) states that a solution of the transport equation with the given boundary conditions exists only if there is a definite relation between material concentrations (given by c or ν_0) and the size given by t . This, therefore, corresponds to an exact statement of the criticality condition.

To summarize, we have shown that the angular neutron density of a one-dimensional multiplying medium can be exactly represented by

$$\psi(x, \mu) = a_{0+} [\psi_{0+}(x, \mu) + \psi_{0-}(x, \mu)] + \int_{-1}^1 A(\nu) \psi_\nu(x, \mu) d\nu, \quad (18)$$

where a_{0+} remains arbitrary and $A(\nu)$ obeys the Fredholm integral equation (16). The criticality condition is given by Eq. (17).

The expression for $\rho(x)$ can also be put in a more recognizable form by recalling that ν_0 is purely imaginary and $A(\nu)$ is an even function:

$$\rho(x) = 2 a_{0+} \cos(x/|\nu_0|) + 2 \int_0^1 A(\nu) \cosh(x/\nu) d\nu, \quad (19)$$

where $\nu_0 = i|\nu_0|$. The first term of (19) is immediately identified as the asymptotic solution from diffusion theory. Also, $\rho(x)$ possesses the expected symmetry about the origin.

A similar procedure for Eq. (3.1-9) gives the following expression for the neutron current:

$$j(x) = 2a_0 + (c-1)|\nu_0| \sin(x/|\nu_0|) + 2(c-1) \int_0^1 A(\nu) \nu \sinh(x/\nu) d\nu. \quad (20)$$

As expected, j vanishes at the origin.

3.3b Approximate Solutions

It is clear from the results of Sec. 3.3a that an explicit expression for $A(\nu)$ cannot be given. The major advantage of the approach, however, is that it affords a systematic approximation procedure by which the desired results can be computed to any accuracy. Moreover, a number of very interesting results emerge from the approximations.

The approximation procedure which will be used in this section involves keeping successive terms of the Neumann series solution (21) of Eq. (16). It is shown in Sec. 3B of Appendix B that such a series converges very rapidly for values of $1 \leq c \leq 2$, which are those of practical interest.

We start by writing Eq. (16) as

$$A'(\nu) = A_1'(\nu) + \lambda \int_0^1 K(\nu, \alpha) A'(\alpha) d\alpha, \quad (21)$$

where $A_1'(\nu)$ denotes the free term of (16),

$$\lambda = \frac{c}{2} (1 - c), \quad (21a)$$

and

$$K(\nu, \alpha) = - \frac{(\nu_0^2 - \nu^2) X(-\nu) g(c, \nu) X(-\alpha) \alpha e^{-t/\alpha}}{\nu + \alpha} \quad (21b)$$

is the kernel. The Neumann series of Eq. (16) has the form

$$A'(\nu) = A'_1(\nu) + \lambda \phi_1(\nu) + \lambda^2 \phi_2(\nu) + \dots, \quad (22)$$

where

$$\phi_n(\nu) = \int_0^1 K_n(\nu, \alpha) A'_1(\alpha) d\alpha \quad (22a)$$

and

$$K_n(\nu, \alpha) = \int_0^1 K(\nu, \beta) K_{n-1}(\beta, \alpha) d\beta. \quad (22b)$$

For the zeroth or diffusion approximation, we take $A'(\nu) = 0$. Since this is strictly true for $c = 1$ (see Sec. 3B of Appendix B), the results would be expected to hold for large systems.

The criticality condition in this case becomes

$$\frac{X(\nu_0)}{X(-\nu_0)} - e^{-t_0/\nu_0} = 0. \quad (23)$$

The ratio $X(-\nu_0)/X(\nu_0)$ is simply related to the Milne-problem extrapolation distance $z_0(c)$ [see Appendix B]:

$$\frac{X(-\nu_0)}{X(\nu_0)} = -e^{-2z_0/\nu_0}. \quad (24)$$

Combining Eqs. (23) and (24), we get

$$e^{2z_0/\nu_0} + e^{-t_0/\nu_0} = 0$$

or

$$\cos(t_0/2|\nu_0| + z_0/|\nu_0|) = 0, \quad (25)$$

from which emerges the familiar result

$$t_0 = \pi|\nu_0| - 2z_0, \quad (26)$$

where t_0 is the thickness in mean free paths for $A'(\mu) = 0$.

The neutron distribution in this approximation is described by the following expressions:

a) The angular density

$$\begin{aligned}\psi_0(x, \mu) &= a_{0+} [\phi_{0+}(\mu) e^{-x/\nu_0} + \phi_{0-}(\mu) e^{x/\nu_0}] \\ &= \frac{a_{0+} c |\nu_0|}{|\nu_0|^2 + \mu^2} \left[|\nu_0| \cos(x/|\nu_0|) + \mu \sin(x/|\nu_0|) \right].\end{aligned}\quad (27)$$

This expression clearly possesses the required symmetry in x and μ .

b) The neutron density

$$\rho_0(x) = 2 a_{0+} \cos(x/|\nu_0|). \quad (28)$$

c) The neutron current

$$j_0(x) = 2 a_{0+} |\nu_0| (c - 1) \sin(x/|\nu_0|). \quad (29)$$

A comparison of Eqs. (28) and (29) gives

$$j_0(x) = -D \frac{d\rho_0(x)}{dx}, \quad (29a)$$

where, as in Eq. (2.1-18a),

$$D = |\nu_0|^2 (c - 1). \quad (29b)$$

Combining Eqs. (29a) and (29b) with the continuity equation (3.1-10), we obtain the diffusion equation

$$\frac{d^2 \rho_0(x)}{dx^2} + \frac{1}{|\nu_0|^2} \rho_0(x) = 0, \quad (30)$$

where $1/|\nu_0|$ plays the role of the "buckling."

For the first approximation we let $A'(\nu)$ equal the first term of Eq. (22):

$$\begin{aligned}A'(\nu) &= A'_1(\nu) = \frac{ac \nu_0}{2} (c - 1) X(-\nu) g(c, \nu) \\ &\quad (x) \left[\nu_0 \left\{ X(\nu_0) + X(-\nu_0) e^{-t_1/\nu_0} \right\} + \nu \left\{ X(\nu_0) - X(-\nu_0) e^{-t_1/\nu_0} \right\} \right].\end{aligned}\quad (31)$$

Inserting this in the criticality statement, we find

$$X(\nu_0) - X(-\nu_0) e^{-t_1/\nu_0} = \frac{c}{2} (c-1) \left\{ \nu_0 \left[X(\nu_0) + X(-\nu_0) e^{-t_1/\nu_0} \right] I_1 + \left[X(\nu_0) - X(-\nu_0) e^{-t_1/\nu_0} \right] I_2 \right\}, \quad (32)$$

where

$$I_1 = \int_0^1 \nu X^2(-\nu) g(c, \nu) e^{-t_1/\nu} d\nu; \quad I_2 = \int_0^1 \nu^2 X^2(-\nu) g(c, \nu) e^{-t_1/\nu} d\nu. \quad (32a)$$

A rearrangement of Eq. (32) gives

$$\frac{X(\nu_0) - X(-\nu_0) e^{-t_1/\nu_0}}{X(\nu_0) + X(-\nu_0) e^{-t_1/\nu_0}} = \nu_0 \alpha(c, t_1), \quad (32b)$$

where the function $\alpha(c, t_1)$ is defined by

$$\alpha = \frac{\frac{c}{2} (c-1) I_1}{1 - \frac{c}{2} (c-1) I_2}. \quad (32c)$$

With the aid of Eq. (24), we finally arrive at the criticality condition corrected to first order:

$$\tan \frac{1}{2|\nu_0|} [t_1 + 2z_0 - \pi|\nu_0|] = -|\nu_0|\alpha. \quad (33)$$

Further insight is gained by examining this result for small values of $\alpha|\nu_0|$. Then

$$\begin{aligned} t_1 &\simeq \pi|\nu_0| - 2z_0 - 2\alpha(c, t_1)|\nu_0|^2 \\ &\simeq t_0 - 2|\nu_0|^2 \alpha(c, t_1), \end{aligned} \quad (34)$$

where t_0 is the zeroth-order thickness. Since $\alpha(c, t_1)$ is positive, it follows from Eq. (34) that the transport correction introduced by the first-order approximation has the effect of decreasing the critical thickness. Moreover, since $\alpha(c, t)$ is small, even for $c = 2.00$, it is possible to estimate the correction term in Eq. (34) without introducing an appreciable error in t_1 . Noting that $I_2 \ll I_1$, we can rewrite (32c) as

$$\alpha(c, t) \approx \frac{c}{2} (c - 1) I_1. \quad (34a)$$

An examination of the integrand of I_1 shows that the greatest contribution to this integral comes from the neighborhood of the upper limit.

Also, from Fig. 4 and (19, p. 84), we have, near $\nu = 1$,

$$X(-\nu) \approx \frac{X(0)}{2}; \quad g(c, \nu) \approx \left(\frac{2}{c\pi} \right)^2.$$

Consequently,

$$I_1 \approx \frac{X^2(0)}{4} \left(\frac{2}{c\pi} \right)^2 E_3(t), \quad (34b)$$

where $E_n(x)$ is given by Eq. (3.1-12a). Substitution of (34a) into (34) and use of (34b) gives

$$t_1' = t_0 - \frac{E_3(t_1')}{c\pi^2}. \quad (34c)$$

Before presenting the expressions for ψ , $\rho(x)$, and $j(x)$ in this approximation, it is convenient to simplify the form of $A_1'(\mu)$ by judicial grouping of constants. First, we note that the quantity $X(\nu_0) e^{t_1/\nu_0}/X(-\nu_0)$ can be expressed in terms of $\alpha(c, t_1)$ with the aid of (32b):

$$\frac{X(\nu_0)}{X(-\nu_0)} e^{t_1/\nu_0} = \frac{1 + \alpha\nu_0}{1 - \alpha\nu_0},$$

from which it follows that

$$X(-\nu_0) e^{-b_1/\nu_0} = \left[X(\nu_0) X(-\nu_0) \frac{1 - \alpha\nu_0}{1 + \alpha\nu_0} \right]^{1/2}.$$

Recalling the definitions of a and $A'(\mu)$ from Eq. (5), we obtain

$$A_1(\mu) = a_0 + B(1 + \alpha\mu) X(-\mu) g(c, \mu) e^{-b_1/\mu}, \quad (35)$$

where

$$B = -c |\nu_0|^2 (c - 1) \left[\frac{X(\nu_0) X(-\nu_0)}{1 - \alpha^2 \nu_0^2} \right]^{1/2} \quad (35a)$$

is a negative constant.

The neutron distribution is now given by the following expressions:

a) Angular density

$$\psi_1(x, \mu) = \psi_0(x, \mu) - q(x, \mu), \quad (36)$$

where $\psi_0(x, \mu)$ is given by Eq. (27) and

$$q(x, \mu) = -A_1(\mu) \lambda(\mu) e^{\mp x/\mu} - P \int_0^1 \frac{c}{2} A_1(\nu) \nu \left[\frac{e^{-x/\nu}}{\nu - \mu} + \frac{e^{x/\nu}}{\nu + \mu} \right] d\nu \quad (36a)$$

is a quantity representing a first-order transport correction. The upper and lower signs in (36a) refer to positive and negative values of μ , respectively.

b) Neutron density

$$\rho_1(x) = \rho_0(x) - h(x), \quad (37)$$

where

$$h(x) = -2a_0 + B \int_0^1 X(-\nu) g(c, \nu) (1 + \alpha\nu) e^{-b/\nu} \cosh(x/\nu) d\nu$$

is a positive correction to the asymptotic density. For c close to unity, a procedure similar to that leading to Eq. (34c) gives

$$h(x) \approx -\frac{BX(0)}{4} [E_2(b-x) + E_2(b+x)]. \quad (37b)$$

c) Neutron current

$$j_1(x) = j_0(x) - 2a_0 + (1 - c)B \int_0^1 \nu X(-\nu) g(c, \nu) (1 + \alpha\nu) e^{-b/\nu} \sinh(x/\nu) d\nu, \quad (38)$$

where the correction term is again positive.

Approximations of higher order become too unwieldy for hand computations and do not seem to add further analytical insight to the neutron distribution or criticality condition. In addition, it is shown in Sec. 3B of Appendix B that the rapid convergence of the Neumann series makes the contribution of such terms negligible except for systems with dimensions less than one mean free path.

3.3c Numerical Results

The first-order correction terms were computed, first, in order to examine their contribution and, secondly, to demonstrate the applicability of the approximation procedure. The numerical results for the neutron distribution are normalized so that

$$\rho_0(0) = 2 a_{0+} = 1. \quad (39)$$

The various integrals were evaluated with a four-point Gaussian quadrature formula (22) which corresponded to an approximation of the integrands by a seventh-order polynomial. The numerical values for $X(-\nu)$, the critical thickness, $\rho(x)$, and $\psi(x, \mu)$ for various values of c will be given in order.

Computations of the function $X(-\nu)$ were made by use of the third identity given in Sec. 3.3a. The results for five values of c are shown in Fig. 4. A comparison of $X(0)$ from the numerical integration and the exact value

$$X(0) = \sqrt{\frac{1}{\nu_0^2 (1 - c)}},$$

which follows readily from the second identity, shows a deviation of one per cent for $c = 1.01$ and of three per cent for $c = 2.00$.

The critical thickness t was determined by first plotting $\alpha(c, t_1)$ and then solving Eq. (33) numerically. The results for five values of c are given in Table I. The values of t_1' from Eq. (34c) are also given in Table I for comparison. In all cases, t is in units of mean free paths. The conclusion of major interest derived from these

results is that even for the extreme case of $c = 2.00$ the first-order transport correction for t is of the order of two per cent.

Fig. 4

The Function $X(-\nu)$

Table I

CRITICAL THICKNESS AS COMPUTED FROM
EQS. (33), (34c), AND (26)

c	t_1	t_1'	t_0	% error = $\frac{t_0 - t_1}{t_1} \times 100$
1.01	16.69	16.69	16.69	$\sim 6 \times 10^{-11}$
1.10	4.24	4.24	4.24	$\sim 4 \times 10^{-4}$
1.30	1.780	1.780	1.783	0.170
1.60	1.022	1.025	1.030	0.775
2.00	0.621	0.625	0.634	2.05

The numerical results for the neutron density are summarized in Table II. A plot of $\rho(x)$ vs. x for three values of c is given in Fig. 5. In Table II we have also tabulated the ratio

$$\frac{\rho_0(x)}{\rho_1(x)} = 1 + \frac{h(x)}{\rho_1(x)}, \quad (40)$$

which serves as a measure of the contribution of the first-order transport correction to the asymptotic density.

Table II
NEUTRON DENSITY AS A FUNCTION OF POSITION

x/b	$c = 1.01$			$c = 1.10$			$c = 1.30$			$c = 1.60$			$c = 2.00$		
	ρ_0	ρ_1	ρ_0/ρ_1	ρ_0	ρ_1	ρ_0/ρ_1	ρ_0	ρ_1	ρ_0/ρ_1	ρ_0	ρ_1	ρ_0/ρ_1	ρ_0	ρ_1	ρ_0/ρ_1
0	1.000	1.000	1.000	1.000	0.9978	1.002	1.000	0.9804	1.020	1.000	0.9525	1.050	1.000	0.9247	1.082
0.25	0.9345	0.9345	1.000	0.9547	0.9517	1.003	0.9742	0.9490	1.023	0.9783	0.9208	1.054	0.9837	0.9054	1.086
0.50	0.7485	0.7485	1.000	0.8231	0.8170	1.007	0.8842	0.8556	1.032	0.9143	0.8518	1.073	0.9352	0.8468	1.104
0.75	0.4722	0.4718	1.001	0.6177	0.6018	1.026	0.7453	0.7008	1.064	0.8104	0.7309	1.109	0.8562	0.7417	1.154
0.85	0.3342	0.3332	1.003	0.5217	0.4972	1.049	0.6774	0.6202	1.094	0.7596	0.6628	1.144	0.8166	0.6836	1.195
0.95	0.1985	0.1916	1.036	0.4130	0.3689	1.120	0.6022	0.5183	1.1619	0.7015	0.5700	1.230	0.7729	0.6113	1.264
1.00	0.1215	0.0969	1.254	0.3577	0.2844	1.1257	0.5640	0.4430	1.271	0.6712	0.5202	1.289	0.7495	0.5706	1.313

Fig. 5
Neutron Density as a
Function of Position

For $c = 1.10$, the maximum difference in the values of $\rho_1(x)$ from Table II and from Eq. (37b) is about 2.4 per cent. Since the constant B is negative, $h(x)$ is positive and increases with c and x . The maximum correction to $\rho_0(x)$, therefore, occurs at the boundary, which, of course, was anticipated.

Finally, the angular density was computed as a function of μ at the origin and at the boundary for three values of c . The results are shown in Figs. 6 and 7. Qualitatively, the shape of $\psi(x, \mu)$ has the following features. At the origin, $\psi(0, \mu)$ always has a maximum which occurs at $\mu = 0$. At the boundary, a maximum appears only when c is greater than a certain value, and shifts from higher to lower values of μ as c increases. This behavior is explained by the nonuniformity of the source distribution $\nu \Sigma_f \rho(x)$. If the source was uniform, the maximum would always occur at $\mu = 0$, because neutrons from other directions have to travel greater distances to reach the boundary. As c increases, $\rho(0) - \rho(b)$ decreases, so that the source is more flat and the peak value moves towards $\mu = 0$.

Fig. 6. Angular Distribution at the Origin

Fig. 7. Angular Distribution at the Boundary

Quantitative verification of these observations can be easily obtained in the zeroth-order approximation. Thus, by letting $d\psi_0/d\mu = 0$ in Eq. (27), we find

$$\mu_{\max} = |\nu_0| \tan (x/2|\nu_0|) \quad (41)$$

and

$$\psi_0(x, \mu_{\max}) = \frac{c}{2} \frac{1}{1 + \tan^2 (x/2|\nu_0|)}. \quad (42)$$

For $x = 0$,

$$\mu_{\max} = 0 \text{ and } \psi(0,0) = c/2.$$

For $x = b$, a maximum exists only if

$$|\nu_0| \tan (b/2|\nu_0|) \leq 1. \quad (42a)$$

By means of Eq. (2.1-9) this condition reduces to

$$cb \leq 2$$

and corresponds to $c \approx 1.13$. Thus, for $c < 1.13$, $\psi_0(b, \mu)$ increases monotonically, whereas for $c \gtrsim 1.13$ it has a maximum. For $c \gg 1$, Eq. (41) becomes

$$\mu_{\max} \sim \frac{2}{\pi c} \tan \frac{b\pi c}{4},$$

and the maximum value approaches $c/2$.

We have shown that the one-dimensional critical problem can be treated exactly by means of Case's normal mode expansion method. The method provides exact expressions for the neutron distribution and criticality condition which can be used to compute these quantities to any desirable accuracy. Moreover, the results are adaptable to systematic approximations. In particular, the zeroth-order approximation for the expansion coefficients leads to the results of diffusion theory and that first-order transport corrections are possible even by hand calculations.

3.3d Solutions from the Integral Equation

We now determine the neutron distribution and the criticality condition from the singular integral equation for $\psi(x, \mu)$,

$$\lambda(\mu) \psi(x, \mu) - P \int_{-1}^1 \frac{\left(\frac{c}{2}\right) \nu \psi(x, \nu)}{\nu - \mu} d\nu = -e^{-x/\mu} f(b, \pm \mu); \quad -1 \leq \mu \leq 1, \quad (3.2-8)$$

and the conditions

$$\int_{-1}^1 \frac{\left(\frac{c}{2}\right) \nu \psi(x, \nu)}{\nu \mp \nu_0} d\nu = e^{\mp x/\nu_0} f(b, \pm \nu_0) \quad (3.2-13)$$

and

$$f(b, \nu_0) = f(b, -\nu_0), \quad (3.2-14)$$

where $f(b, \mu)$ is defined by Eq. (3.2-7).

A close comparison of this equation for $\psi(x, \mu)$ and Eq. (6) for $A'(\nu)$ uncovers three differences. First, the presence of the parameter x here must be kept in mind. Secondly, μ in Eq. (3.2-8) ranges over the entire interval $[-1, 1]$. Lastly, there is a difference in the sign of the principal value integrals whose net effect is to make these two singular integral equations adjoint (16, p. 331). Although these differences do not alter the essential steps in the reduction procedure discussed in Sec. 3.3a, they introduce enough novel features to warrant a re-examination of some of the parts.

We start again by considering the complex transformation

$$N(x, z) = \left(\frac{1}{2\pi i}\right) \int_{-1}^1 \frac{\left(\frac{c}{2}\right) \nu \psi(x, \nu)}{\nu - z} d\nu. \quad (43)$$

From the symmetry conditions, it follows that

$$N(x, z) = N(-x, -z). \quad (43a)$$

As a function of z , $N(x, z)$ possesses the properties listed after Eq. (8) with the exception that the plane must be cut from -1 to 1 . Its boundary values as it approaches this cut from above and below are related by the equations

$$N^+(x, \mu) + N^-(x, \mu) = \frac{1}{\pi i} P \int_{-1}^1 \frac{\left(\frac{c}{2}\right) \nu \psi(x, \nu)}{\nu - \mu} d\nu \quad (43b)$$

and

$$N^+(x, \mu) - N^-(x, \mu) = \frac{c}{2} \mu \psi(x, \mu) \quad (43c)$$

which serve to convert Eq. (3.2-8) to the nonhomogeneous Hilbert problem:

$$N^+(x, \mu) - G(\mu) N^-(x, \mu) = - \frac{\left(\frac{c}{2}\right) \mu e^{-x/\mu} f(b, \pm \mu)}{\lambda^-(\mu)}, \quad (44)$$

where

$$G(\mu) = \frac{\lambda^+(\mu)}{\lambda^-(\mu)}; \quad \lambda^\pm(\mu) = \lambda(\mu) \pm \frac{c\pi i \mu}{2}. \quad (44a)$$

Proceeding as in Sec. 3.3a, we find that the solution of Eq. (44) can be written as

$$N(x, z) = \frac{X_1(z)}{2\pi i} \left[P_k(z) - \int_{-1}^1 \frac{\gamma_1(\nu) e^{-x/\nu} f(b, \pm \nu)}{\nu - z} d\nu \right], \quad (45)$$

where

$$\gamma_1(\mu) = \frac{\left(\frac{c}{2}\right) \mu}{X_1^+(\mu) \lambda^-(\mu)}, \quad (45a)$$

$P_k(z)$ denotes an arbitrary polynomial of degree k :

$$P_k(z) = \sum_0^k a_i(x) z^i, \quad (45b)$$

whose coefficients $a_i(x)$, in general, depend on the parameter x , and $X_1(z)$ is the solution of the homogeneous part of Eq. (44):

$$X_1^+(\mu) = G(\mu) X_1^-(\mu). \quad (45c)$$

This function is given explicitly (1) in terms of $\lambda(z)$ by the second identity of Sec. 3.3a:

$$X_1(z) = X(z) X(-z) = \frac{\lambda(z)}{(\nu_0^2 - z^2)(1 - c)}. \quad (46)$$

Equation (46) shows that $X_1(z)$ goes like $1/z^2$ for large z whereas Eq. (43) requires $N(x, z)$ to behave as $1/z$. This behavior will be incorporated in Eq. (45) if we require all but the first two coefficients $a_i(x)$ to be zero. Then

$$N(x, z) = \frac{X_1(z)}{2\pi i} \left[a_0(x) + a_1(x)z - \int_{-1}^1 \frac{\gamma_1(\nu) e^{-x/\nu} f(b, \pm \nu)}{\nu - z} d\nu \right]. \quad (47)$$

The two remaining coefficients can be expressed in terms of $f(b, \mu)$ by means of the two conditions (3.2-14). In view of Eq. (43) these conditions take the form

$$2\pi i N(x, \pm \nu_0) = e^{\mp x/\nu_0} f(b, \pm \nu_0). \quad (48)$$

Inserting $N(x, \pm \nu_0)$ from Eq. (47) in Eq. (48) and observing that $X_1(z)$ is even, we arrive at the two simultaneous equations for $a_0(x)$ and $a_1(x)$:

$$X_1(\nu_0) \left[a_0(x) + a_1(x)\nu_0 - \int_{-1}^1 \frac{\gamma_1(\nu) e^{-x/\nu} f(b, \pm \nu)}{\nu - \nu_0} d\nu \right] = e^{-x/\nu_0} f(b, \nu_0) \quad (48a)$$

and

$$X_1(\nu_0) \left[a_0(x) - a_1(x)\nu_0 - \int_{-1}^1 \frac{\gamma_1(\nu) e^{-x/\nu} f(b, \pm \nu)}{\nu + \nu_0} d\nu \right] = e^{x/\nu_0} f(b, -\nu_0). \quad (48b)$$

The value of $X_1(\nu_0) = X(\nu_0) X(-\nu_0)$ is given in Sec. 2B of Appendix B.

From the definition of $f(b, \pm \nu)$ and the fact that $\gamma_1(-\nu) = -\gamma_1(\nu)$, we can write the integrals over ν as

$$\int_{-1}^1 \frac{\gamma_1(\nu) e^{-x/\nu} f(b, \pm \nu)}{\nu \mp \nu_0} d\nu = \int_0^1 \frac{\gamma_1(\nu) e^{-x/\nu} f(b, \nu)}{\nu \mp \nu_0} d\nu + \int_0^1 \frac{\gamma_1(\nu) e^{x/\nu} f(b, \nu)}{\nu \pm \nu_0} d\nu.$$

Furthermore, it follows from Eqs. (45a) and (46) that

$$\gamma_1(\nu) \left[\frac{1}{\nu - \nu_0} + \frac{1}{\nu + \nu_0} \right] = c \nu^2 (c - 1) g(c, \nu)$$

and

$$\gamma_1(\nu) \left[\frac{1}{\nu - \nu_0} - \frac{1}{\nu + \nu_0} \right] = c \nu \nu_0 (c - 1) g(c, \nu),$$

where $g(c, \nu)$ is given by Eq. (15a). When these results and Eq. (3.2-14) are used in the solutions of Eqs. (48a) and (48b), we find

$$a_0(x) = \frac{f(b, \nu_0)}{X_1(\nu_0)} \cosh\left(\frac{x}{\nu_0}\right) + c(c - 1) \int_0^1 \nu^2 g(c, \nu) f(b, \nu) \cosh\left(\frac{x}{\nu}\right) d\nu \quad (49a)$$

and

$$a_1(x) = -\frac{f(b, \nu_0)}{\nu_0 X_1(\nu_0)} \sinh\left(\frac{x}{\nu_0}\right) - c(c - 1) \int_0^1 \nu g(c, \nu) f(b, \nu) \sinh\left(\frac{x}{\nu}\right) d\nu. \quad (49b)$$

As a check, note that

$$a_0(x) = a_0(-x); \quad a_1(x) = -a_1(-x),$$

which also follow from Eq. (43a).

Returning to Eq. (47), we can now find $N^\pm(x, \mu)$ by the Plemelj formula and insert the expressions in Eq. (43c) to establish $\psi(x, \mu)$. After simplifications involving terms of the form $\gamma_1(\mu)$ $[X_1^+(\mu) \pm X_1^-(\mu)]$, using the properties of $X_1(z)$, we get

$$\psi(x, \mu) = \frac{a_0(x) + a_1(x)\mu}{(\nu_0^2 - \mu^2)(1 - c)} - \lambda(\mu)g(c, \mu) e^{-x/\mu} f(b, \pm \mu) - \frac{1}{(\nu_0^2 - \mu^2)(1 - c)} \int_{-1}^1 \frac{\gamma_1(\nu) e^{-x/\nu} f(b, \pm \nu)}{\nu - \mu} d\nu. \quad (50)$$

Since $a_0(x)$ and $a_1(x)$ depend on the function f , which, in turn, depends on $\psi(b, \mu)$, Eq. (50) states that the angular density everywhere in the system can be expressed in terms of the emerging distribution at the boundary.

The neutron density can be determined by integration of Eq. (50) or by noting that, from Eq. (43), we have

$$N(x, 0) = \frac{1}{2\pi i} \int_{-1}^1 \left(\frac{c}{2} \right) \psi(x, \nu) d\nu = \frac{1}{2\pi i} \frac{c}{2} \rho(x),$$

while from Eq. (47)

$$N(x, 0) = \frac{X_1(0)}{2\pi i} \left[a_0(x) - \int_{-1}^1 \left(\frac{1}{\nu} \right) \gamma_1(\nu) e^{-x/\nu} f(b, \nu) d\nu \right].$$

Equating these two expressions and using (49a) for $a_0(x)$, we get

$$\rho(x) = \frac{2}{c} \frac{X_1(0)}{X_1(\nu_0)} f(b, \nu_0) \cosh \left(\frac{x}{\nu_0} \right) - 2 \int_0^1 g(c, \nu) f(b, \nu) \cosh \left(\frac{x}{\nu} \right) d\nu. \quad (51)$$

If we multiply $N(x, z)$ by z and let z approach infinity, we obtain, from Eq. (43),

$$\lim_{z \rightarrow \infty} zN(x, z) = - \frac{1}{2\pi i} \int_{-1}^1 \left(\frac{c}{2} \right) \nu \psi(x, \nu) d\nu = - \frac{1}{2\pi i} \frac{c}{2} j(x),$$

and from Eq. (47)

$$\lim_{z \rightarrow \infty} zN(x, z) = - \frac{a_1(x)}{2\pi i}.$$

Hence,

$$j(x) = - \frac{2}{c} \frac{f(b, \nu_0)}{\nu_0 X_1(\nu_0)} \sinh \left(\frac{x}{\nu_0} \right) - 2(c - 1) \int_0^1 \nu g(c, \nu) f(b, \nu) \sinh \left(\frac{x}{\nu} \right) d\nu. \quad (52)$$

Differentiating Eq. (52) by x and recalling that $X_1(0) = X^2(0) = [\nu_0^2(1-c)]^{-1}$, we arrive again at the continuity equation (3.1-10).

The critical problem is now solved, for if we put $x = b$ in (50) we get $\psi(b, \mu)$ in terms of $f(b, \mu)$. When this is inserted in Eq. (3.2-7), the result is a Fredholm equation for $f(b, \mu)$. Finally, the criticality condition is established by finding $f(b, \nu_0)$ and $f(b, -\nu_0)$ from $\psi(b, \mu)$, and imposing the requirement that these two quantities are equal. A simpler equation for $f(b, \mu)$, which also facilitates the comparison with the results of the preceding sections, is obtained by proceeding along the lines of the work of Leonard and Mullikin (15).

Let $x = b$ in Eq. (43), so that

$$N(b, z) = \frac{1}{2\pi i} \int_0^1 \frac{\left(\frac{c}{2}\right) \nu \psi(b, \nu)}{\nu - z} d\nu, \quad (53)$$

since $\psi(b, \nu < 0) = 0$. The boundary values $N^\pm(b, \mu)$ of this function, when used with the singular integral equation (3.2-9) for $\psi(b, \mu)$, produces a Hilbert problem whose solution is

$$N(b, z) = \frac{X(z)}{2\pi i} \left[\alpha(b) - \int_0^1 \frac{\left(\frac{c}{2}\right) \nu e^{-b/\nu} f(b, \nu)}{X^+(\nu) \lambda^-(\nu)(\nu - z)} d\nu \right]. \quad (54)$$

In Eq. (54), $\alpha(b)$ is a constant and the appropriate $X(z)$ is the same as the function used in Sec. 3.3a, because $0 \leq \mu \leq 1$ here also. Furthermore, a comparison of Eq. (53) with the extension of Eq. (3.2-7) to the complex plane,

$$f(b, z) = e^{-b/z} \int_0^1 \frac{\left(\frac{c}{2}\right) \nu \psi(b, \nu)}{\nu + z} d\nu,$$

reveals that

$$2\pi i N(b, -z) = e^{b/z} f(b, z). \quad (55)$$

Consequently,

$$f(b, z) = e^{-b/z} X(-z) \left[\alpha(b) - \int_0^1 \frac{\left(\frac{c}{2}\right) \nu e^{-b/\nu} f(b, \nu)}{X^+(\nu) \lambda^-(\nu)(\nu + z)} d\nu \right]. \quad (56)$$

For $z = \mu$, Eq. (56) reduces to

$$f(b, \mu) = e^{-b/\mu} X(-\mu) \left[\alpha(b) - \int_0^1 \frac{\left(\frac{c}{2}\right) \nu e^{-b/\nu} f(b, \nu)}{X^+(\nu) \lambda^-(\nu)(\nu + \mu)} d\nu \right], \quad (56a)$$

and for $z = \pm \nu_0$ to

$$f(b, \pm \nu_0) = e^{\mp b/\nu_0} X(\mp \nu_0) \left[\alpha(b) - \int_0^1 \frac{\left(\frac{c}{2}\right) \nu e^{-b/\nu} f(b, \nu)}{X^+(\nu) \lambda^-(\nu)(\nu \pm \nu_0)} d\nu \right]. \quad (56b)$$

Apart from differences in notation, these equations for f are the same as those given in (15). Substitution of Eq. (56b) in

$$f(b, \nu_0) = f(b, -\nu_0) \quad (3.2-14)$$

yields the criticality condition

$$\begin{aligned} \alpha(b) \left[X(-\nu_0) e^{-b/\nu_0} - X(\nu_0) e^{b/\nu_0} \right] &= X(-\nu_0) e^{-b/\nu_0} \int_0^1 \frac{\left(\frac{c}{2}\right) \nu e^{-b/\nu} f(b, \nu)}{X^+(\nu) \lambda^-(\nu)(\nu + \nu_0)} d\nu \\ &\quad - X(-\nu_0) e^{b/\nu_0} \int_0^1 \frac{\left(\frac{c}{2}\right) \nu e^{-b/\nu} f(b, \nu)}{X^+(\nu) \lambda^-(\nu)(\nu - \nu_0)} d\nu. \end{aligned} \quad (57)$$

All the quantities characterizing the one-dimensional critical problem are therefore expressed in terms of the function f which, in analogy to $A'(\nu)$, satisfies the regular integral equation (56a) and an arbitrary constant $\alpha(b)$ corresponding to a_{0+} of Sec. 3.3a. Approximate expressions for these quantities can be obtained by keeping

successive terms of the Neumann series of Eq. (56a). For example, if for the zeroth approximation we take $f_0(b, \mu) = 0$, we get, from Eq. (57),

$$\frac{X(\nu_0)}{X(-\nu_0)} - e^{-t/\nu_0} = 0 \quad (58)$$

and from Eq. (50)

$$\psi_0(x, \mu) = \frac{\alpha(b) e^{-b/\nu_0} X(-\nu_0)}{\nu_0 X_1(\nu_0)(1-c)(\nu_0^2 - \nu^2)} \left[\nu_0 \cosh(x/\nu_0) - \mu \sinh(x/\nu_0) \right] \quad (59)$$

since, from Eq. (56a),

$$f_0(b, \nu_0) = \alpha(b) X(-\nu_0) e^{-b/\nu_0}.$$

These results correspond to Eqs. (23) and (27), respectively. Furthermore, the criticality condition from the next approximation, where $f(b, \mu)$ is taken equal to the free term in Eq. (56a), is identical with Eq. (32) (15).

Thus, the two methods for treating the one-dimensional critical problem give comparable expressions for the neutron distribution and criticality statement, and involve the same type of analysis. Specifically, the final results are acquired in both methods by reducing a singular integral equation either for $A'(\nu)$ or for $\psi(x, \mu)$. As the computations of the desired quantities also require about the same effort, the choice of one method over the other for this particular problem is a matter of individual preference.

3.3e Asymptotic Critical Thickness

In view of the agreement of the zeroth-order approximation results with the asymptotic results of diffusion theory, it is of interest to examine the consequences when this approximation is applied at the outset. We therefore put $A(\nu) = 0$ in Eq. (3.1-4) so that

$$\psi_0(x, \mu) = a_{0+} \left[\phi_{0+}(\mu) e^{-x/\nu_0} + \phi_{0-}(\mu) e^{x/\nu_0} \right]. \quad (60)$$

Since the exact condition at the boundary can no longer be satisfied, we replace it by the commonly used condition of no re-entrant current:

$$j_+(-b) = \int_0^1 \mu \psi_0(-b, \mu) d\mu = 0. \quad (60a)$$

Substitution of Eq. (60) in (60a) gives, instead of Eq. (26),

$$t_0^1 = \pi |\nu_0| - 2z_0^1, \quad (61)$$

where

$$z_0^1 = |\nu_0| \tan^{-1} \frac{c-1}{c|\nu_0| \ln \sqrt{1+(1/|\nu_0|^2)}}. \quad (61a)$$

Comparing Eq. (61) and (26), we conclude that the replacement of the exact boundary condition by Eq. (60a) consists of replacing the Milne-problem extrapolation distance z_0 by z_0^1 of Eq. (61a). A numerical comparison of these two quantities is given in Table III. The values of cz_0 in this table were taken from Table 23 of Ref. 19.

Table III

VALUES FOR cz_0 AND cz_0^1

c	cz_0	cz_0^1
1.01	0.7104	0.5930
1.10	0.7106	0.6240
1.30	0.7113	0.7150
1.50	0.7123	0.7300
2.00	0.7151	0.7720

For $(c-1) \ll 1$, Eq. (61a) reduces to

$$z_0^1 \approx \frac{1}{\sqrt{3(c-1)}} \tan^{-1} 2 \left(\frac{c-1}{3} \right)^{1/2} \quad (61b)$$

$$\approx \frac{2}{3} \left[1 + \frac{4}{9} (1 - c) + \dots \right], \quad (61b)$$

where we have used $|\nu_0|^2(c-1) \sim 1/3$. Equation (61b) is identical with the extrapolation distance in the P_1 approximation (12, p. 141). A more detailed comparison of the exact expansion (3.1-4) and the expansion for $\psi(x, \mu)$ determined by the method of spherical harmonics is given in Sec. 4B of Appendix B.

CHAPTER IV

SOLUTIONS WITH SPHERICAL SYMMETRY

4.0 INTRODUCTION

For problems possessing spherical symmetry the neutron distribution remains invariant under rotation of the $(\underline{r}, \underline{\Omega})$ system about the center of the sphere and is azimuthally symmetric about the radius vector $\underline{r}$. Consequently, the angular density ψ depends only on the distance $r = |\underline{r}|$ and the cosine of the angle θ between $\underline{r}$ and $\underline{\Omega}$:

$$\mu = \cos \theta = \underline{\Omega} \cdot \underline{\Omega}_r; \quad \underline{\Omega}_r = \underline{r}/r.$$

In terms of these variables the transport equation (1.1-2) takes the form (12, p. 146)

$$\mu \frac{\partial \psi(r, \mu)}{\partial r} + \frac{1 - \mu^2}{r} \frac{\partial \psi(r, \mu)}{\partial \mu} + \psi(r, \mu) = \frac{c}{2} \int_{-1}^1 \psi(r, \mu') d\mu'. \quad (1)$$

The present chapter deals with the twofold problem of finding a unique expansion for $\psi(r, \mu)$ which satisfies Eq. (1) and then applying such an expansion to a uniform multiplying sphere of radius R . Since Eq. (1) is not amenable to separation of the r and μ variables, we start instead from the three-dimensional solutions of Chap. II and explore the possibility of constructing spherically symmetric solutions from them. The product of this venture is a number of equivalent representations for $\psi(r, \mu)$ for which the expansion coefficients $a_{0\pm}(\underline{u})$ and $A(\nu, \underline{u})$ do not depend on $\underline{u}$. Although these representations are new and possess some interesting properties, their usefulness is impaired by their failure to produce a tractable equation for the coefficients when the boundary condition is imposed.

A simplification in the treatment of the critical sphere problem is achieved in Sec. 4.5 through the introduction of a transform for $\rho(r)$ and subsequent use of the transport integral equation. As shown in that section, a transform can be chosen so that it obeys the transport

equation in plane geometry. The transform therefore can be represented by an expansion in terms of the complete set of plane eigenfunctions and the coefficients of this expansion can be determined by the method of the preceding chapter. Furthermore, an inversion of the density transform shows that these coefficients are the same as those in the representations for $\psi(r, \mu)$ acquired from the superposition method. Thus, besides simplifying the treatment of the critical sphere problem, the examination of the properties of the transport integral equation also provides the proof that the expansions of the spherical distribution derived here are unique.

4.1 SPHERICAL EXPANSION COEFFICIENTS

The application of spherical symmetry on the three-dimensional expansion of Sec. 2.1,

$$\psi(\underline{r}, \underline{\Omega}) = \psi_{d+}(\underline{r}, \underline{\Omega}) + \psi_{d-}(\underline{r}, \underline{\Omega}) + \psi_c(\underline{r}, \underline{\Omega}), \quad (2.1-14)$$

causes the coefficients $a_{0\pm}(\underline{u})$ and $A(\nu, \underline{u})$ to be independent of $\underline{u}$. The proof of this assertion is most easily achieved by expanding each term of Eq. (2.1-14) in the orthonormal set of spherical harmonics:

$$Y_{\ell}^m(\underline{\Omega}) = \left[\frac{2\ell+1}{4\pi} \frac{(\ell-m)!}{(\ell+m)!} \right]^{1/2} P_{\ell}^m(\cos \theta) e^{im\phi}, \quad (1)$$

so that

$$\int_{\underline{\Omega}} Y_{\ell}^{\alpha}(\underline{\Omega}) Y_{\ell}^{\beta*}(\underline{\Omega}) d\Omega = \delta_{\ell m} \delta_{\alpha \beta}, \quad (1a)$$

where θ and ϕ are the polar and azimuthal angles specifying the unit vector $\underline{\Omega}$, $P_{\ell}^m(\cos \theta)$ denotes the associated Legendre function (23, p. 110), $Y_{\ell}^{\beta*}(\underline{\Omega})$ is the complex conjugate harmonic, and $\delta_{\ell m}$ is the Kronecker delta. The addition theorem for the Legendre polynomials $P_{\ell}(\cos \theta)$ becomes, in this notation,

$$P_{\ell}(\underline{\Omega} \cdot \underline{\Omega}') = \frac{4\pi}{2\ell+1} \sum_{\alpha=-\ell}^{\ell} Y_{\ell}^{\alpha}(\underline{\Omega}) Y_{\ell}^{\alpha*}(\underline{\Omega}'). \quad (2)$$

In the first term of Eq. (2.1-14),

$$\psi_{d+}(\underline{r}, \underline{\Omega}) = \frac{1}{4\pi} \frac{c\nu_0}{2} \int_{\underline{u}} a_{0+}(\underline{u}) \frac{e^{-\underline{r} \cdot \underline{u}/\nu_0}}{\nu_0 - \underline{u} \cdot \underline{\Omega}} d\underline{u}, \quad (3)$$

let

$$a_{0+}(\underline{u}) = \frac{1}{\sqrt{4\pi}} \sum_{\ell=0}^{\infty} \sum_{\alpha=-\ell}^{\ell} a_{0+}^{\ell\alpha} Y_{\ell}^{\alpha}(\underline{u}), \quad (4)$$

where $a_{0+}^{\ell\alpha}$ are constants. The expansions for the remaining two factors in the integrand of Eq. (3) are well-known. Specifically, the exponential term admits to the representation (20, p. 409)

$$\begin{aligned} e^{-\underline{r} \cdot \underline{u}/\nu_0} &= \sum_{m=0}^{\infty} (-1)^m (2m+1) i_m(r/\nu_0) P_m(\underline{u} \cdot \underline{\Omega}_r) \\ &= 4\pi \sum_{m=0}^{\infty} \sum_{\beta=-m}^m (-1)^m i_m(r/\nu_0) Y_m^{\beta}(\underline{\Omega}_r) Y_m^{\beta*}(\underline{u}) \end{aligned} \quad (5)$$

In Eq. (5), $i_m(z)$ is the spherical Bessel function:

$$i_m(z) = \sqrt{\frac{\pi}{2z}} I_{m+1/2}(z); \quad i_0(z) = \frac{\sinh z}{z} \quad (5a)$$

where $I_m(z)$ denotes the modified Bessel function of the first kind (24, p. 77).

Using Heine's expansion for $(t-z)^{-1}$ (25, p. 322), we can also write

$$\begin{aligned} \frac{1}{\nu_0 - \underline{u} \cdot \underline{\Omega}} &= \sum_{n=0}^{\infty} (2n+1) Q_n(\nu_0) P_n(\underline{u} \cdot \underline{\Omega}) \\ &= 4\pi \sum_{n=0}^{\infty} \sum_{\gamma=-n}^n Q_n(\nu_0) Y_n^{\gamma}(\underline{u}) Y_n^{\gamma*}(\underline{\Omega}). \end{aligned} \quad (6)$$

Then,

$$\psi_{d+}(\underline{r}, \underline{\Omega}) = \sqrt{4\pi} \left(\frac{c\nu_0}{2} \right) \sum_{\ell\alpha} \sum_{m\beta} \sum_{n\gamma} (-1)^m a_{0+}^{\ell\alpha} B_{\ell mn}^{\alpha\beta\gamma} i_m(r/\nu_0) Q_n(\nu_0) Y_m^\beta(\underline{\Omega}_r) Y_n^{\gamma*}(\underline{\Omega}), \quad (7)$$

where

$$B_{\ell mn}^{\alpha\beta\gamma} = \int_{\underline{u}} Y_\ell^\alpha(\underline{u}) Y_m^{\beta*}(\underline{u}) Y_n^\gamma(\underline{u}) du. \quad (7a)$$

The spherically symmetric function $\psi_{d+}(r, \mu) = 2\pi\psi_{d+}(\underline{r}, \underline{\Omega})$ is given by those terms of Eq. (7) for which $\ell = \alpha = 0$ or else when all the coefficients except a_{0+}^{00} are zero. The proof follows by first observing that $\psi_{d+}(r, \mu)$ is completely represented by the expansion

$$\begin{aligned} \psi_{d+}(r, \mu) &= \sum_{m=0}^{\infty} \left(\frac{2m+1}{4\pi} \right) \psi_{d+}^m(r) P_m(\mu) \\ &= \sum_{m\beta} \psi_{d+}^m(r) Y_m^\beta(\underline{\Omega}_r) Y_m^{\beta*}(\underline{\Omega}). \end{aligned} \quad (8)$$

Equation (7) will have this form only if the coefficients of $Y_n^{\gamma*}(\underline{\Omega})$ vanish for $m \neq n$ and $\beta \neq \gamma$:

$$\sum_{\ell\alpha} a_{0+}^{\ell\alpha} B_{\ell mn}^{\alpha\beta\gamma} = 0; \quad m \neq n, \quad \beta \neq \gamma. \quad (9)$$

Take $n = \gamma = 0$ and note that

$$B_{\ell m 0}^{\alpha\beta 0} = \frac{1}{\sqrt{4\pi}} \delta_{\ell m} \delta_{\alpha\beta} \quad (9a)$$

because of Eq. (1a) and the fact that $Y_0^0(\underline{u}) = (4\pi)^{-1/2}$. The condition (9) for spherical symmetry then reduces to

$$a_{0+}^{m\beta} = 0 \quad \text{unless} \quad m = \beta = 0. \quad (10)$$

When this result is used in Eq. (4) we conclude that $a_{0+}(\underline{u})$ must be independent of $\underline{u}$ if ψ is to be a function of r and μ only. The same conclusion can be inferred from the integral of Eq. (7) over $\underline{\Omega}$:

$$\begin{aligned}\rho_{d+}(\underline{r}) &= \int_{\underline{\Omega}} \psi_{d+}(\underline{r}, \underline{\Omega}) d\Omega \\ &= \sqrt{4\pi} \left(\frac{c\nu_0}{2} \right) \sum_{\ell \alpha} (-1)^\ell a_{0+}^{\ell\alpha} i_\ell(r/\nu_0) Q_0(\nu_0) Y_\ell^\alpha(\underline{\Omega}_r).\end{aligned}\quad (11)$$

For $\rho_{d+}(\underline{r})$ to depend on the radial distance r only, Eq. (10) must hold.

If we replace $a_{0+}(\underline{u})$ and ν_0 by $a_{0-}(\underline{u})$ and $-\nu_0$, respectively, in Eqs. (3) and (7), and employ the properties

$$i_m(-z) = (-1)^m i_m(z); \quad Q_n(-\nu_0) = (-1)^{n+1} Q_n(\nu_0), \quad (12)$$

we arrive at the expansion for $\psi_{d-}(\underline{r}, \underline{\Omega})$. The argument leading to Eq. (10) shows again that for spherical symmetry all the coefficients but a_{0-}^{00} must vanish.

The continuous solution

$$\psi_c(\underline{r}, \underline{\Omega}) = \frac{1}{4\pi} \int_{\underline{u}} du \int_{-1}^1 A(\nu, \underline{u}) \left[\frac{c}{2} P \frac{\nu}{\nu - \underline{u} \cdot \underline{\Omega}} + \lambda(\nu) \delta(\underline{u} \cdot \underline{\Omega} - \nu) \right] e^{-\underline{r} \cdot \underline{u}/\nu} d\nu \quad (13)$$

can be treated similarly. In particular, the expansion for the exponential is the same as Eq. (5) with ν in place of ν_0 , and $A(\nu, \underline{u})$ can be expanded as

$$A(\nu, \underline{u}) = \frac{1}{\sqrt{4\pi}} \sum_{\ell \alpha} A^{\ell\alpha}(\nu) Y_\ell^\alpha(\underline{u}). \quad (13a)$$

Furthermore, by Eq. (6) we can write

$$P \frac{1}{\nu - \underline{u} \cdot \underline{\Omega}} = 4\pi \sum_{n\gamma} Q_n(\nu) Y_n^\gamma(\underline{u}) Y_n^{\gamma*}(\underline{\Omega}), \quad (13b)$$

where the principal value serves to define $Q_n(\nu)$ in the interval $[-1, 1]$ (23, p. 116):

$$Q_n(\nu) = \frac{1}{2} P \int_{-1}^1 \frac{P_n(\mu) d\mu}{\nu - \mu} = \frac{1}{2} \left[Q_n^+(\nu) + Q_n^-(\nu) \right]. \quad (14)$$

For the δ -function term in (13) we use the identity (12, p. 233)

$$\begin{aligned}\delta(\underline{u} \cdot \underline{\Omega} - \nu) &= \sum_{n=0}^{\infty} \left(\frac{2n+1}{2} \right) P_n(\nu) P_n(\underline{u} \cdot \underline{\Omega}) \\ &= 2\pi \sum_{n\gamma} P_n(\nu) Y_n^\gamma(\underline{u}) Y_n^{\gamma*}(\underline{\Omega}).\end{aligned}\quad (15)$$

Hence,

$$\begin{aligned}\psi_c(\underline{r}, \underline{\Omega}) &= \sqrt{4\pi} \sum_{\ell\alpha} \sum_{m\beta} \sum_{n\gamma} (-1)^m B_{\ell mn}^{\alpha\beta\gamma} Y_m^\beta(\underline{\Omega}_r) Y_n^{\gamma*}(\underline{\Omega}) \int_{-1}^1 A^{\ell\alpha}(\nu) \\ &\quad (x) \left[\frac{c\nu}{2} Q_n(\nu) + \frac{1}{2} \lambda(\nu) P_n(\nu) \right] i_m(r/\nu) d\nu.\end{aligned}\quad (16)$$

As in the derivation of Eq. (10), the spherically symmetric function $\psi_c(r, \mu)$ is obtained from Eq. (16) when

$$\int_{-1}^1 A^{\ell\alpha}(\nu) i_\ell(r/\nu) \left[\frac{c\nu}{2} Q_0(\nu) + \frac{1}{2} \lambda(\nu) P_0(\nu) \right] d\nu = 0; \quad \ell \neq 0, \quad \alpha \neq 0, \quad (16a)$$

or, since

$$c\nu Q_0(\nu) + \lambda(\nu) P_0(\nu) = 1,$$

when

$$\int_{-1}^1 A^{\ell\alpha}(\nu) i_\ell(r/\nu) d\nu = 0 \quad \text{unless} \quad \ell = \alpha = 0. \quad (17)$$

Thus, only the terms for which $\ell = \alpha = 0$ in Eq. (16) are needed to represent $\psi_c(r, \mu)$. For this case, Eq. (13a) gives $A(\nu, \underline{u}) = (4\pi)^{-1} A^{00}(\nu)$, so that the continuous coefficients are also independent of $\underline{u}$.

To summarize, we have shown that the three-dimensional expansion (2.1-14) can also be written as

$$\psi(\underline{r}, \underline{\Omega}) = \sqrt{4\pi} \sum_{\ell\alpha} \sum_{m\beta} \sum_{n\gamma} D_{\ell mn}^{\alpha\beta\gamma}(r) Y_m^\beta(\underline{\Omega}_r) Y_n^{\gamma*}(\underline{\Omega}), \quad (18)$$

where, explicitly,

$$D_{lmn}^{\alpha\beta\gamma}(r) = B_{lmn}^{\alpha\beta\gamma} \left[\left\{ (-1)^m a_{0+}^{\ell\alpha} + (-1)^n a_{0-}^{\ell\alpha} \right\} \frac{c\nu_0}{2} Q_n(\nu_0) i_m(r/\nu_0) \right. \\ \left. + (-1)^m \int_{-1}^1 A^{\ell\alpha}(\nu) \left\{ \frac{c\nu}{2} Q_n(\nu) + \frac{1}{2} \lambda(\nu) P_n(\nu) \right\} i_m(r/\nu) d\nu \right]. \quad (18a)$$

More important for this investigation is the conclusion that spherically symmetric solutions can be constructed by superposition merely by ascribing weighting functions $a_{0\pm}$ and $A(\nu)$ which are independent of the superposition variables.

4.2 REPRESENTATIONS FOR $\psi(r, \mu)$

Setting $\ell = \alpha = 0$ and $\psi(r, \underline{\Omega}) = (2\pi)^{-1} \psi(r, \mu)$ in Eq. (18) and using the addition theorem, we obtain

$$\psi(r, \mu) = \sum_{m=0}^{\infty} (-1)^m \left(\frac{2m+1}{2} \right) P_m(\mu) \\ (x) \left[a_0 c\nu_0 Q_m(\nu_0) i_m(r/\nu_0) + \frac{1}{2} \int_{-1}^1 A(\nu) i_m(r/\nu) \{ c\nu Q_m(\nu) + \lambda(\nu) P_m(\nu) \} d\nu \right], \quad (1)$$

where we have defined

$$a_0 = \frac{1}{2} \left[a_{0+}^{00} + a_{0-}^{00} \right]; \quad A(\nu) = A^{00}(\nu). \quad (2)$$

Since $i_m(r/\nu) [c\nu Q_m(\nu) + \lambda(\nu) P_m(\nu)]$ is an even function of ν , Eq. (1) can also be written as

$$\psi(r, \mu) = \sum_{m=0}^{\infty} (-1)^m \left(\frac{2m+1}{2} \right) P_m(\mu) \\ (x) \left[a_0 c\nu_0 Q_m(\nu_0) i_m(r/\nu_0) + \int_0^1 A_s(\nu) i_m(r/\nu) \{ c\nu Q_m(\nu) + \lambda(\nu) P_m(\nu) \} d\nu \right] \quad (3)$$

with

$$A_s(\nu) = \frac{1}{2} [A(\nu) + A(-\nu)]. \quad (3a)$$

If the continuous term in Eq. (3) is disregarded [$A_s(\nu) = 0$], the remaining series corresponds to the spherical solutions of Wilson (13).

Other interesting forms of this expansion follow from the decomposition (23, p. 109)

$$Q_m(x) = P_m(x) Q_0(x) - W_{m-1}(x), \quad (4)$$

where $W_m(x)$ is a polynomial of degree m comprising the nonsingular part of $Q_m(x)$. Inserting Eq. (4) in (3) and noting that

$$\begin{aligned} c\nu_0 Q_m(\nu_0) &= P_m(\nu_0) - c\nu_0 W_{m-1}(\nu_0); \\ c\nu Q_m(\nu) + \lambda(\nu) P_m(\nu) &= P_m(\nu) - c\nu W_{m-1}(\nu) \end{aligned} \quad (4a)$$

because of the definition of ν_0 and $\lambda(\nu)$, we find

$$\begin{aligned} \psi(r, \mu) &= \sum_{m=0}^{\infty} (-1)^m \left(\frac{2m+1}{2} \right) P_m(\mu) \left[a_0 P_m(\nu_0) i_m(r/\nu_0) + \int_0^1 A_s(\nu) i_m(r/\nu) P_m(\nu) d\nu \right] \\ &\quad - \sum_{m=1}^{\infty} (-1)^m \left(\frac{2m+1}{2} \right) P_m(\mu) \\ &\quad (x) \left[a_0 c\nu_0 W_{m-1}(\nu_0) i_m(r/\nu_0) + \int_0^1 A_s(\nu) i_m(r/\nu) c\nu W_{m-1}(\nu) d\nu \right] \end{aligned} \quad (5)$$

as $W_{-1}(\nu) = 0$. The first series in (5) can be summed with the aid of the Gegenbauer addition theorem (26, V. II, p. 102):

$$e^{-z \cos \alpha \cos \beta} I_0(z \sin \alpha \sin \beta) = \sum_0^{\infty} (-1)^m (2m+1) i_m(z) P_m(\cos \alpha) P_m(\cos \beta). \quad (6)$$

Hence,

$$\begin{aligned} \psi(r, \mu) &= \frac{1}{2} e^{-r\mu} \left[a_0 I_0 \left(\frac{r}{\nu_0} \sqrt{(1-\mu^2)(1-\nu_0^2)} \right) + \int_0^1 A(\nu) I_0 \left(\frac{r}{\nu} \sqrt{(1-\mu^2)(1-\nu^2)} \right) d\nu \right] \\ &\quad - \sum_{m=1}^{\infty} (-1)^m \left(\frac{2m+1}{2} \right) P_m(\mu) \\ &\quad (x) \left[a_0 c\nu_0 W_{m-1}(\nu_0) i_m(r/\nu_0) + \int_0^1 A_s(\nu) i_m(r/\nu) c\nu W_{m-1}(\nu) d\nu \right]. \end{aligned} \quad (7)$$

Alternatively, Eq. (5) can be expressed in terms of the function

$$G_n(\nu) = P_n(\nu) - c \nu W_{n-1}(\nu) \quad (8)$$

which occurs in the solution of the transport equation (4.0-1) produced by the spherical harmonics method (12, p. 147). In fact, if we replace the integral term in Eq. (5) by a sum (see Sec. 4B of Appendix B), we obtain the regular spherical harmonics solutions.

Returning to the three-dimensional expansion (2.1-14) and imposing the fact that the coefficients are constants with respect to $\underline{u}$, we get

$$\psi(r, \mu) = a_{0+}^{00} \psi_{0+}(r, \mu) + a_{0-}^{00} \psi_{0-}(r, \mu) + \int_{-1}^1 A^{00}(\nu) \psi_\nu(r, \mu) d\nu, \quad (9)$$

where

$$\psi_{0\pm}(r, \mu) = \frac{c \nu_0}{16\pi} \int_0^\pi \int_0^{2\pi} \frac{e^{\mp \underline{r} \cdot \underline{u} / \nu_0}}{\nu_0 \mp \underline{u} \cdot \underline{\Omega}} \sin \alpha \, d\alpha \, d\beta \quad (9a)$$

and

$$\psi_\nu(r, \mu) = \frac{1}{8\pi} \int_0^\pi \int_0^{2\pi} \left[\frac{c}{2} P \frac{\nu}{\nu - \underline{u} \cdot \underline{\Omega}} + \lambda(\nu) \delta(\underline{u} \cdot \underline{\Omega} - \nu) \right] e^{-\underline{r} \cdot \underline{u} / \nu} \sin \alpha \, d\alpha \, d\beta. \quad (9b)$$

The integrations over $\underline{u}$ are simplified when we choose a coordinate system with $\underline{\Omega}$ along the z-axis and $\underline{r}$ on the xz-plane (see

Fig. 8). Then

$$\underline{u} \cdot \underline{\Omega} = \cos \alpha = \eta \quad (10a)$$

and

$$\begin{aligned} \underline{r} \cdot \underline{u} &= r [\cos \alpha \cos \theta + \sin \alpha \sin \theta \cos \beta] \\ &= r \left[\eta \mu + \sqrt{(1 - \mu^2)(1 - \eta^2)} \cos \beta \right]. \end{aligned} \quad (10b)$$

Fig. 8. Spherical Coordinates

Substitution in Eqs. (9a) and (9b) and use of the expansion (24, p. 369)

$$e^{-z \cos \beta} = \sum_{n=-\infty}^{\infty} (-1)^n I_n(z) e^{-in\beta} \quad (11)$$

for the integration over β gives

$$\psi_{0\pm}(r, \mu) = \frac{c\nu_0}{8} \int_{-1}^1 \frac{e^{\mp r\mu\eta/\nu_0}}{\nu_0 \mp \eta} I_0\left(\frac{r}{\nu_0} \sqrt{(1-\mu^2)(1-\eta^2)}\right) d\eta \quad (12a)$$

and

$$\psi_\nu(r, \mu) = \frac{1}{4} \int_{-1}^1 \left[\frac{c}{2} P \frac{\nu}{\nu - \eta} + \lambda(\nu) \delta(\eta - \nu) \right] I_0\left(\frac{r}{\nu} \sqrt{(1-\mu^2)(1-\eta^2)}\right) e^{-r\mu\eta/\nu} d\eta. \quad (12b)$$

A change from η to $-\eta$ in Eq. (12a) for $\psi_{0-}(r, \mu)$ gives $\psi_{0+}(r, \mu)$. The two discrete solutions are therefore identical. Similarly, from Eq. (12b) we see that $\psi_\nu(r, \mu) = \psi_{-\nu}(r, \mu)$. When these results are introduced into Eq. (9), we obtain

$$\begin{aligned} \psi(r, \mu) = & a_0 \frac{c\nu_0}{4} \int_{-1}^1 \frac{e^{-r\mu\eta/\nu_0}}{\nu_0 - \eta} I_0\left(\frac{r}{\nu_0} \sqrt{(1-\mu^2)(1-\eta^2)}\right) d\eta \\ & + \frac{1}{2} \int_0^1 A_S(\nu) d\nu \int_{-1}^1 \left[\frac{c}{2} P \frac{\nu}{\nu - \eta} + \lambda(\nu) \delta(\eta - \nu) \right] e^{-r\mu\eta/\nu} \\ & (x) I_0\left(\frac{r}{\nu} \sqrt{(1-\mu^2)(1-\eta^2)}\right) d\eta, \end{aligned} \quad (13)$$

where a_0 and $A_S(\nu)$ are defined by Eqs. (2) and (3a), respectively.

Substitution of the expansion (6) in Eq. (13) and use of the Neumann integral (4.1-14) for $Q_n(x)$ converts this integral representation for $\psi(r, \mu)$ to the series solution of Eq. (3). On the other hand, by evaluating the integrals (9a) and (9b) in a different way, we obtain a somewhat more complicated but equivalent series for $\psi(r, \mu)$ in terms of Bessel functions of integer order. Inasmuch as such a series agrees with the spherical solutions derived by Kofink (14), we shall illustrate the procedure only for the discrete integral (9a). To this

end, we start from a new coordinate system in which $\underline{r}$ is along the z -axis and $\underline{\Omega}$ lies on the xz -plane. From Fig. 8 we see that this system is obtained by a clockwise rotation by θ about the y -axis. By definition of spherical symmetry, such rotations do not affect the value of $\psi(r, \mu)$.

We now have

$$\underline{r} \cdot \underline{u} = r \cos \alpha' = rs; \quad s = \cos \alpha' \quad (14a)$$

and

$$\begin{aligned} \underline{u} \cdot \underline{\Omega} &= \cos \alpha' \cos \theta + \sin \alpha' \sin \theta \cos \beta' \\ &= s\mu + \sqrt{(1 - \mu^2)(1 - s^2)} \cos \beta', \end{aligned} \quad (14b)$$

where α' and β' denote the new coordinates of $\underline{u}$. Since $\psi_{0+}(r, \mu) = \psi_{0-}(r, \mu)$, the two discrete terms in Eq. (9) can be combined in the form $a_0 \psi_0(r, \mu)$ where

$$\psi_0(r, \mu) = \frac{c\nu_0}{8\pi} \int_{-1}^1 e^{-rs/\nu_0} ds \int_0^{2\pi} \frac{d\beta'}{\nu_0 - [s\mu + \sqrt{(1 - \mu^2)(1 - s^2)} \cos \beta']}. \quad (15)$$

An expansion of the integrand in (15) in spherical harmonics shows that this integral for $\psi_0(r, \mu)$ is also consistent with the series representation of Eq. (3). The integral over β' can, however, be exhibited in closed form, since

$$\int_0^{2\pi} \frac{d\beta'}{a + b \cos \beta'} = \frac{2\pi}{\sqrt{a^2 - b^2}}; \quad \frac{a}{b} \notin [-1, 1]. \quad (15a)$$

Putting

$$a = \nu_0 - s\mu; \quad b = \sqrt{(1 - \mu^2)(1 - s^2)}$$

and simplifying, we get

$$\psi_0(r, \mu) = \frac{c\nu_0}{4} \int_{-1}^1 \frac{e^{-rs/\nu_0}}{[\nu_0^2 - 1 + \mu^2 + s^2 - 2\nu_0\mu s]^{1/2}} ds. \quad (16)$$

By completing the square on s we can write the denominator in (16) as

$$[\nu_0^2 - 1 + \mu^2 + s^2 - 2\nu_0\mu s]^{1/2} = i\sqrt{(1-\nu_0^2)(1-\mu^2)} \left[1 - \left(\frac{s - \nu_0\mu}{\sqrt{(1-\nu_0^2)(1-\mu^2)}} \right)^2 \right]^{1/2}. \quad (17)$$

This form now suggests a change of variables from s to t , where

$$\cos t = \frac{s - \nu_0\mu}{\sqrt{(1-\nu_0^2)(1-\mu^2)}}. \quad (18)$$

Then

$$\frac{ds}{[\nu_0^2 - 1 + \mu^2 + s^2 - 2\nu_0\mu s]^{1/2}} = i dt \quad (18a)$$

and the limits of integration transform from $s = \pm 1$ to

$$t_2 = -i \ln \left(\frac{1-\mu}{1+\mu} \frac{1+\nu_0}{1-\nu_0} \right)^{1/2}; \quad t_1 = -i \ln e^{i\pi} \left(\frac{1-\mu}{1+\mu} \frac{1-\nu_0}{1+\nu_0} \right)^{1/2}, \quad (18b)$$

respectively, with the aid of the formula

$$\cos^{-1} z = -i \ln \left(z + \sqrt{z^2 - 1} \right).$$

Consequently,

$$\psi_0(r, \mu) = \left(\frac{ic\nu_0}{4} \right) e^{-r\mu} \int_{t_1}^{t_2} \exp \left[-\frac{r}{\nu_0} \sqrt{(1-\mu^2)(1-\nu_0^2)} \cos t \right] dt. \quad (19)$$

Inserting in Eq. (19) the expansion (11) in the form

$$e^{-z \cos t} = I_0(z) + 2 \sum_{\ell=1}^{\infty} (-1)^{\ell} I_{\ell}(z) \cos \ell t \quad (19a)$$

and carrying out the integration over t , we find

$$\begin{aligned} \psi_0(r, \mu) = & \left(\frac{ic\nu_0}{4} \right) e^{-r\mu} I_0 \left(\frac{r}{\nu_0} \sqrt{(1-\mu^2)(1-\nu_0^2)} \right) [t_2 - t_1] \\ & + \left(\frac{ic\nu_0}{2} \right) e^{-r\mu} \sum_{\ell=1}^{\infty} (-1)^{\ell} I_{\ell} \left(\frac{r}{\nu_0} \sqrt{(1-\mu^2)(1-\nu_0^2)} \right) \left[\frac{\sin \ell t_2 - \sin \ell t_1}{\ell} \right]. \end{aligned} \quad (20)$$

But

$$\sin \ell t_2 - \sin \ell t_1 = 2 \sin \frac{\ell}{2} (t_2 - t_1) \cos \frac{\ell}{2} (t_2 + t_1). \quad (20a)$$

Furthermore, it follows from Eq. (18b) and the definition of ν_0 that

$$t_2 - t_1 = -2i \tanh^{-1} \left(\frac{1}{\nu_0} \right) = -\frac{2i}{c\nu_0}; \quad t_2 + t_1 = 2i \tanh^{-1} \left(\frac{1}{\mu} \right). \quad (20b)$$

Introducing these results in Eq. (20), we get finally

$$\begin{aligned} \psi_d(r, \mu) = a_0 \psi_0(r, \mu) &= \frac{a_0}{2} e^{-r\mu} I_0 \left(\frac{r}{\nu_0} \sqrt{(1-\mu^2)(1-\nu_0^2)} \right) \\ &+ a_0 c \nu_0 e^{-r\mu} \sum_1^{\infty} \frac{(-1)^\ell}{\ell} I_\ell \left(\frac{r}{\nu_0} \sqrt{(1-\mu^2)(1-\nu_0^2)} \right) \sinh \left(\frac{\ell}{c\nu_0} \right) \cosh \left[\ell \tanh^{-1} \left(\frac{1}{\mu} \right) \right]. \end{aligned} \quad (21)$$

In spite of its appearance, Eq. (21) converges at the end points $\mu = \pm 1$, for although as $\mu \rightarrow 1$ $\cosh \left[\ell \tanh^{-1} \left(\frac{1}{\mu} \right) \right] \rightarrow (1-\mu)^{-\ell/2}$, the Bessel function $I_\ell (\sqrt{1-\mu^2})$ behaves as $(1-\mu)^\ell$ in this limit.

If in Eq. (20) we express $\sin \ell t$ in terms of exponentials, we find, instead of Eq. (20a),

$$\begin{aligned} (-1)^\ell [\sin \ell t_2 - \sin \ell t_1] &= \frac{1}{2i} \left[\left(\frac{1+\nu_0}{1-\nu_0} \frac{1+\mu}{1-\mu} \right)^{\ell/2} - \left(\frac{1-\nu_0}{1+\nu_0} \frac{1-\mu}{1+\mu} \right)^{\ell/2} \right. \\ &\quad \left. + (-1)^\ell \left(\frac{1+\nu_0}{1-\nu_0} \frac{1-\mu}{1+\mu} \right)^{\ell/2} - (-1)^\ell \left(\frac{1-\nu_0}{1+\nu_0} \frac{1+\mu}{1-\mu} \right)^{\ell/2} \right] \end{aligned} \quad (22)$$

Substitution of Eq. (22) in (20) produces an expression for $\psi_0(r, \mu)$ which corresponds, apart from notation, to the discrete spherical solution given by Kofink (14).

The integral (9b) for $\psi_\nu(r, \mu)$ can be treated in precisely the same manner if we use the limiting forms (2.1-8a) for the principal value term and (2.1-8b) for the δ -function term. This procedure will be demonstrated in Sec. 5.2 where the resulting expansions for the cylindrical distribution are more tractable for computations than the

other cylindrical representations. Such advantage is not evident here since Eq. (21) is not simpler to deal with than the equivalent part of Eq. (7).

Integration of Eq. (3) over μ gives

$$\begin{aligned}\rho(r) &= a_0 c \nu_0 Q_0(\nu_0) i_0(r/\nu_0) + \int_0^1 A_S(\nu) i_0(r/\nu) [c \nu Q_0(\nu) + \lambda(\nu)] d\nu \\ &= a_0 \nu_0 \frac{\sinh(r/\nu_0)}{r} + \int_0^1 A_S(\nu) \nu \frac{\sinh(r/\nu)}{r} d\nu,\end{aligned}\quad (23)$$

where we have used the orthogonality of $P_m(\mu)$ and Eq. (4a). The expansion for the radial neutron current is obtained by multiplying Eq. (3) by μ and integrating:

$$j(r) = -a_0 c \nu_0 Q_1(\nu_0) i_1(r/\nu_0) - \int_0^1 A_S(\nu) i_1(r/\nu) [c \nu Q_1(\nu) + \lambda(\nu) P_1(\nu)] d\nu. \quad (24)$$

But

$$Q_1(x) = x Q_0(x) - 1; \quad P_1(x) = x. \quad (24a)$$

Hence,

$$j(r) = a_0 \nu_0 (c - 1) i_1(r/\nu_0) + (c - 1) \int_0^1 A_S(\nu) \nu i_1(r/\nu) d\nu, \quad (25)$$

where

$$i_1(z) = \frac{di_0(z)}{dz} = \frac{1}{z} \left[\cosh z - \frac{\sinh z}{z} \right]. \quad (25a)$$

These expansions for $\rho(r)$ and $j(r)$ can be verified directly by noting that they are consistent with the continuity equation (1.1-5) in spherical geometry:

$$\frac{dj(r)}{dr} + \frac{2}{r} j(r) + (1 - c) \rho(r) = 0. \quad (26)$$

4.3 PROPERTIES OF THE SPHERICAL SOLUTIONS

Since the various expressions for $\psi(r, \mu)$ developed in the preceding section are equivalent, it is sufficient for the investigation of their properties to consider Eq. (4.2-3). We first verify that this representation satisfies the transport equation (4.0-1). Specifically, if

$$\psi(r, \mu) = a_0 \psi_0(r, \mu) + \int_0^1 A_s(\nu) \psi_\nu(r, \mu) d\nu, \quad (1)$$

where

$$\psi_0(r, \mu) = \sum_{n=0}^{\infty} (-1)^n \left(\frac{2n+1}{2} \right) i_n(r/\nu_0) c\nu_0 Q_n(\nu_0) P_n(\mu) \quad (1a)$$

and

$$\psi_\nu(r, \mu) = \sum_0^{\infty} (-1)^n \left(\frac{2n+1}{2} \right) i_n(r/\nu) [c\nu Q_n(\nu) + \lambda(\nu) P_n(\nu)] P_n(\mu), \quad (1b)$$

then

$$T\psi(r, \mu) = \frac{c}{2} \rho(r), \quad (2)$$

where

$$T = \mu \frac{\partial}{\partial r} + \frac{1 - \mu^2}{r} \frac{\partial}{\partial \mu} + 1$$

and $\rho(r)$ is given by Eq. (4.2-23). The proof follows readily from the recurrence relations of $i_n(z)$ and of the Legendre functions. The details are given in Sec. 1C of Appendix C.

If, instead, we start from Eq. (4.2-7), which we write as

$$\psi(r, \mu) = \psi_1(r, \mu) + \psi_2(r, \mu), \quad (3)$$

where

$$\psi_1(r, \mu) = \frac{1}{2} e^{-r\mu} \left[a_0 I_0 \left(\frac{r}{\nu_0} \sqrt{(1 - \mu^2)(1 - \nu_0^2)} \right) + \int_0^1 A_s(\nu) I_0 \left(\frac{r}{\nu} \sqrt{(1 - \mu^2)(1 - \nu^2)} \right) d\nu \right] \quad (3a)$$

and

$$\psi_2(r, \mu) = - \sum_{n=1}^{\infty} (-1)^n \left(\frac{2n+1}{2} \right) P_n(\mu) \quad (x) \left[a_0 c \nu_0 W_{n-1}(\nu_0) i_n(r/\nu_0) + \int_0^1 A_s(\nu) c \nu W_{n-1}(\nu) i_n(r/\nu) d\nu \right], \quad (3b)$$

and proceed likewise, we find that

$$T\psi_1(r, \mu) = 0; \quad \int_{-1}^1 \psi_1(r, \mu) d\mu = \rho(r), \quad (4a)$$

and

$$T\psi_2(r, \mu) = \frac{c}{2} \rho(r); \quad \int_{-1}^1 \psi_2(r, \mu) d\mu = 0. \quad (4b)$$

In other words, the spherically symmetric distribution can be split into two parts: $\psi_1(r, \mu)$, which satisfies the transport equation with $c = 0$ but contributes entirely to the neutron density, and $\psi_2(r, \mu)$, which obeys Eq. (2) but whose integral over μ is zero. This interesting decomposition was observed by Kofink (14) in the expansions of the form of Eq. (4.2-21).

Replacing $(-1)^n i_n(z)$ in Eqs. (1a) and (2a) by

$$k_n(z) = \sqrt{\frac{\pi}{2z}} K_{n+1/2}(z); \quad k_0(z) = \frac{\pi}{2z} e^{-z}, \quad (5)$$

where $K_n(z)$ is the modified Bessel function of the second kind (24, p. 78), we find that the resulting expansion for $\psi(r, \mu)$ remains a solution of the transport equation even though it is not generated from the superposition analysis. Furthermore, this solution becomes infinite at the origin, whereas all the results of Sec. 4.2 are finite at this point. As pointed out at the end of Sec. 2.1, this apparent failure of the superposition procedure to generate all possible solutions of Eq. (4.0-1)

is a consequence of the restriction on the range of the superposition variables. A plausibility argument on the construction of non-regular solutions is presented in Sec. 2A of Appendix A.

Up to now we have been concerned with the development of suitable expressions for the neutron distribution. For these solutions to be useful in specific problems, it is further necessary to determine the expansion coefficients $A_S(\nu)$ from given boundary conditions. Although an equation defining the coefficients can be readily established by subjecting the above solutions to the boundary conditions, these spherical expansions do not submit to the computational methods used in the preceding chapter. To illustrate, consider a uniform multiplying sphere of radius R . The appropriate condition at the boundary is

$$\psi(R, \mu < 0) = 0. \quad (6)$$

Imposing Eq. (6) on Eq. (4.2-13), we get

$$\int_0^1 A_S(\nu) K(\nu, \mu; R) d\nu + a_0 \left(\frac{c\nu_0}{2} \right) \int_{-1}^1 \frac{e^{R\mu\eta/\nu_0}}{\nu_0 - \eta} I_0 \left(\frac{R}{\nu_0} \sqrt{(1-\mu^2)(1-\eta^2)} \right) d\eta = 0; \quad \mu > 0, \quad (7)$$

where

$$K(\nu, \mu; R) = \int_{-1}^1 \left[\frac{c}{2} P \frac{\nu}{\nu - \eta} + \lambda(\nu) \delta(\eta - \nu) \right] I_0 \left(\frac{R}{\nu} \sqrt{(1-\mu^2)(1-\eta^2)} \right) e^{R\mu\eta/\nu} d\eta. \quad (7a)$$

Equation (7) is also an integral equation for $A_S(\nu)$. Furthermore, its solution is expected to be dependent on an auxiliary (critical) condition on the parameter R . However, since the kernel $K(\nu, \mu; R)$ does not have the useful singular properties which were instrumental in the determination of the critical thickness and expansion coefficients in plane geometry, the method of Sec. 3.3a cannot be used here.

Estimates for $A_S(\nu)$ and the critical radius can be obtained by approximating the integrals in Eq. (7) by finite sums and then proceeding along the lines of the conventional spherical harmonics method

(see Sec. 4B of Appendix B). A more satisfactory approach emerges from the investigation of the properties of the transport integral equation, to which we now turn.

4.4 INTEGRAL EQUATION WITH SPHERICAL SYMMETRY

To specialize the integral equation

$$\psi(\underline{r}, \underline{\Omega}) = \frac{c}{4\pi} \int_0^{s_0} \rho(\underline{r} - s\underline{\Omega}) e^{-s} ds \quad (1.1-8)$$

to problems with spherical symmetry, we replace $\psi(\underline{r}, \underline{\Omega})$ by $(2\pi)^{-1} \psi(r, \mu)$ and observe that ρ is a function only of the distance:

$$|\underline{r} - s\underline{\Omega}| = \sqrt{r^2 + s^2 - 2rs\mu},$$

where $\mu = \cos \theta$. Furthermore, it is seen from Fig. 9 that

$$s_0 = AB + BC.$$

But

$$AB = r\mu; \quad BC = \sqrt{R^2 - r^2(1 - \mu^2)}.$$

Hence,

$$s_0(r, \mu) = r\mu + \sqrt{R^2 - r^2(1 - \mu^2)}.$$

Fig. 9
The Distance s_0
for a Sphere

The integral equation then becomes

$$\psi(r, \mu) = \frac{c}{2} \int_0^{s_0(r, \mu)} \rho \left(\sqrt{r^2 + s^2 - 2rs\mu} \right) e^{-s} ds. \quad (1)$$

For $r = R$ and $\mu < 0$, $s_0(R, \mu < 0) = 0$ and Eq. (1) satisfies the boundary condition (4.3-6).

An integral equation for $\rho(r)$ can be established either by specializing Eq. (1.1-9) to spherical symmetry (12, p. 96) or by integrating Eq. (1) over μ and simplifying the double integral. The latter procedure is carried out in Sec. 2C of Appendix C, where we find

$$\rho(r) = \frac{c}{2} \frac{1}{r} \int_{-R}^R t \rho(t) E_1(|t - r|) dt. \quad (2)$$

Here $E_1(x)$ denotes the exponential integral and the range of r has been extended to negative values by the definition $\rho(r) = \rho(-r)$.

It is of interest to verify that this integral formulation for the neutron distribution agrees with the previous representations for $\psi(r, \mu)$. This is easily proved if we substitute Eq. (4.2-23) for $\rho(r)$ in Eq. (1):

$$\begin{aligned} \psi(r, \mu) = a_0 \left(\frac{c}{2} \right) \int_0^{s_0} i_0 \left(\frac{1}{\nu_0} \sqrt{r^2 + s^2 - 2rs\mu} \right) e^{-s} ds \\ + \left(\frac{c}{2} \right) \int_0^1 A_s(\nu) d\nu \int_0^{s_0} i_0 \left(\frac{1}{\nu} \sqrt{r^2 + s^2 - 2rs\mu} \right) e^{-s} ds. \end{aligned} \quad (3)$$

Use of the expansion (24, p. 363)

$$i_0 \left(\frac{1}{\kappa} \sqrt{r^2 + s^2 - 2rs\mu} \right) = \sum_{n=0}^{\infty} (-1)^n (2n+1) i_n(r/\kappa) i_n(s/\kappa) P_n(\mu) \quad (4)$$

and the fact that

$$i_n(s/\kappa) = \frac{1}{2} \int_{-1}^1 e^{st/\kappa} P_n(t) dt, \quad (5)$$

which follows from the integration of Eq. (4.1-5), gives

$$\int_0^{s_0} i_0 \left(\frac{1}{\kappa} \sqrt{r^2 + s^2 - 2rs\mu} \right) e^{-s} ds = \sum_0^{\infty} (-1)^n \left(\frac{2n+1}{2} \right) i_n(r/\kappa) P_n(\mu) \int_0^{s_0} e^{-s} ds \int_{-1}^1 e^{st/\kappa} P_n(t) dt. \quad (6)$$

Performing the integration over s and recalling Eq. (4.2-6), we arrive at

$$\begin{aligned} \int_0^{s_0} i_0 \left(\frac{1}{\kappa} \sqrt{r^2 + s^2 - 2rs\mu} \right) e^{-s} ds &= \frac{\kappa}{2} \sum_0^{\infty} (-1)^n \left(\frac{2n+1}{2} \right) i_n(r/\kappa) P_n(\mu) \\ &\quad (x) \left[\int_{-1}^1 \frac{P_n(t)}{\kappa-t} dt - e^{-s_0} \int_{-1}^1 \frac{e^{s_0 t/\kappa} P_n(t)}{\kappa-t} dt \right] \\ &= \frac{\kappa}{2} \int_{-1}^1 \frac{e^{-r\mu t/\kappa}}{\kappa-t} I_0 \left(\frac{r}{\kappa} \sqrt{(1-\mu^2)(1-t^2)} \right) dt \\ &\quad - \frac{\kappa}{2} e^{-s_0} \int_{-1}^1 \frac{e^{t(s_0-r\mu)/\kappa}}{\kappa-t} I_0 \left(\frac{r}{\kappa} \sqrt{(1-\mu^2)(1-t^2)} \right) dt. \end{aligned} \quad (7)$$

But from Fig. 9 we have

$$r \sqrt{1-\mu^2} = OB = R \sqrt{1-\mu_0^2}, \quad (8a)$$

where $\mu_0 = \cos \theta_0$, which is equal to μ for $\mu < 0$ and $r = R$, and

$$s_0 - r\mu = \sqrt{R^2 - r^2(1-\mu^2)} = BC = -R\mu_0. \quad (8b)$$

Substitution of these results in Eq. (3) yields

$$\begin{aligned} \psi(r, \mu) &= a_0 \left(\frac{c\nu_0}{4} \right) \int_{-1}^1 \frac{e^{-r\mu t/\nu_0}}{\nu_0-t} I_0 \left(\frac{r}{\nu_0} \sqrt{(1-\mu^2)(1-t^2)} \right) dt \\ &\quad + \frac{1}{2} \int_0^1 A_s(\nu) d\nu \int_{-1}^1 \left(\frac{c}{2} P \frac{\nu}{\nu-t} \right) e^{-r\mu t/\nu} I_0 \left(\frac{r}{\nu} \sqrt{(1-\mu^2)(1-t^2)} \right) dt \\ &\quad - e^{-s_0} \left[a_0 \left(\frac{c\nu_0}{4} \right) \int_{-1}^1 \frac{e^{-r\mu_0 t/\nu_0}}{\nu_0-t} I_0 \left(\frac{R}{\nu_0} \sqrt{(1-\mu_0^2)(1-t^2)} \right) dt \right. \\ &\quad \left. + \frac{1}{2} \int_0^1 A_s(\nu) d\nu \int_{-1}^1 \left(\frac{c}{2} P \frac{\nu}{\nu-t} \right) e^{-R\mu_0 t/\nu} I_0 \left(\frac{R}{\nu} \sqrt{(1-\mu_0^2)(1-t^2)} \right) dt \right]. \end{aligned} \quad (9)$$

To Eq. (9) we now add and subtract the term

$$\begin{aligned} & \frac{1}{2} \int_0^1 A_S(\nu) d\nu \int_{-1}^1 \lambda(\nu) \delta(t-\nu) e^{-r\mu t/\nu} I_0\left(\frac{r}{\nu} \sqrt{(1-\mu^2)(1-t^2)}\right) dt \\ & \equiv \frac{1}{2} e^{-s_0} \int_0^1 A_S(\nu) d\nu \int_{-1}^1 \lambda(\nu) \delta(t-\nu) e^{t(s_0-r\mu)/\nu} I_0\left(\frac{r}{\nu} \sqrt{(1-\mu^2)(1-t^2)}\right) dt \end{aligned} \quad (10)$$

and make use of the invariant quantities in Eqs. (8a) and (8b). Then,

$$\begin{aligned} \psi(r, \mu) &= a_0 \psi_0(r, \mu) + \int_0^1 A_S(\nu) \psi_\nu(r, \mu) d\nu \\ &- e^{-s_0} \left[a_0 \psi_0(R, \mu_0) + \int_0^1 A_S(\nu) \psi_\nu(R, \mu_0) d\nu \right], \end{aligned} \quad (11)$$

where $\psi_0(r, \mu) = 2\psi_{0\pm}(r, \mu)$ and $\psi_\nu(r, \mu)$ are given by Eqs. (4.2-12a) and (4.2-12b), respectively. The bracket in Eq. (11) is identical with Eq. (4.3-7) and therefore vanishes. The remaining terms in this equation are then recognized as the integral representation (4.2-13) for $\psi(r, \mu)$.

In the derivation of Eq. (11) we had to resort to the trick of adding and subtracting the term in (10). As in Sec. 2.2, the reason lies in the fact that in Eq. (1) the boundary condition (4.3-6) has been taken into account. The expression (4.2-23) for $\rho(r)$, on the other hand, has not been subjected to this condition. Thus, a combination of these two equations gives both the distribution within the sphere and an expansion for $e^{-s_0} \psi(R, \mu_0)$, which was neglected in Eq. (1). Since the term in Eq. (10) occurs unaltered in the internal and incoming surface distribution, it cancels out.

4.5 SPHERICAL TRANSFORMATIONS

In this section we discuss the transformations of the neutron density which, when used in conjunction with the integral equation (4.4-2), convert the spherical critical problem to a problem similar to

that treated in the preceding chapter. The simplest transformation is suggested by writing Eq. (4.4-2) in the form

$$r\rho(r) = \frac{c}{2} \int_0^1 \frac{d\mu}{\mu} \left[\int_{-R}^r e^{-(r-t)/\mu} t\rho(t) dt + \int_r^R e^{-(t-r)/\mu} t\rho(t) dt \right]. \quad (1)$$

Introduce the functions

$$\Phi(r, \mu) = \frac{c}{2} \frac{1}{\mu} \int_{-R}^r e^{-(r-t)/\mu} t\rho(t) dt \quad (2a)$$

and

$$\Phi(r, -\mu) = \frac{c}{2} \frac{1}{\mu} \int_r^R e^{-(t-r)/\mu} t\rho(t) dt, \quad (2b)$$

where $\mu > 0$. Replacing r by $-r$ in (2b), we see that

$$\Phi(r, \mu) = -\Phi(-r, -\mu), \quad (3)$$

since $\rho(r) = \rho(-r)$. Also,

$$\Phi(-R, \mu) = \Phi(R, -\mu) = 0; \quad \mu > 0. \quad (4)$$

Furthermore, if $\Phi(r, \mu)$ for $(-1 \leq \mu \leq 1)$ is known, the neutron density $\rho(r)$ can be determined by integrating over μ ; for when we substitute Eqs. (2a) and (2b) in Eq. (1), we find

$$\begin{aligned} r\rho(r) &= \int_0^1 d\mu [\Phi(r, \mu) + \Phi(r, -\mu)] \\ &= \int_{-1}^1 \Phi(r, \mu) d\mu. \end{aligned} \quad (5)$$

A comparison of Eq. (5) with the definition of $\rho(r)$ reveals, further, that

$$\frac{1}{r} \int_{-1}^1 \Phi(r, \mu) d\mu = \int_{-1}^1 \psi(r, \mu) d\mu. \quad (6)$$

Thus, the transform $\Phi(r, \mu)$ acts as a pseudo distribution taking the place of $\psi(r, \mu)$. Moreover, the physical boundary condition $\psi(R, \mu < 0) = 0$, which is incorporated in Eq. (1), is inherited by $\Phi(r, \mu)$ in the form of Eq. (4). For a different condition at the boundary, the integral equation for $\rho(r)$ changes. The definition of the density transform must therefore also be modified if the inverse (5) is to remain valid.

The advantage of working with $\Phi(r, \mu)$ instead of the angular density becomes evident when we compare Eqs. (2a) and (2b) with the integral equations (3.1-12a) and (3.1-12b) for $\psi(x, \mu)$. It is seen that $\Phi(r, \mu)$ is, apart from the nonessential difference in symmetry, the same as the angular density in plane symmetry. Hence, the methods of determining $\psi(x, \mu)$ apply here also. In particular, differentiation of Eqs. (2a) and (2b) with respect to r reveals that $\Phi(r, \mu)$ obeys the one-dimensional transport equation

$$\mu \frac{\partial \Phi(r, \mu)}{\partial r} + \Phi(r, \mu) = \frac{c}{2} \int_{-1}^1 \Phi(r, \mu') d\mu'; \quad -1 \leq \mu \leq 1, \quad (7)$$

subject to the symmetry condition (3) and boundary condition (4). Consequently, this transform is completely represented by the expansion

$$\Phi(r, \mu) = b_{0+} \phi_{0+}(\mu) e^{-r/\nu_0} + b_{0-} \phi_{0-}(\mu) e^{r/\nu_0} + \int_{-1}^1 B(\nu) \phi_{\nu}(\mu) e^{-r/\nu} d\nu, \quad (8)$$

where $b_{0\pm}$ and $B(\nu)$ are to be chosen from the conditions in $\Phi(r, \mu)$ and $\phi_{0\pm}(\mu)$ and $\phi_{\nu}(\mu)$ are the complete set of plane singular modes of Sec. 3.1.

Imposing the symmetry condition (4) in Eq. (8) and proceeding as in the beginning of Sec. 3.3a, we find that

$$b_{0+} = -b_{0-} \quad \text{and} \quad B(\nu) = -B(-\nu). \quad (9)$$

With this and the normalization of $\phi_\nu(\mu)$ in mind, the inverse of Eq. (8) becomes

$$\rho(r) = 2b_{0-} \frac{\sinh(r/\nu_0)}{r} + 2 \int_0^1 B(-\nu) \frac{\sinh(r/\nu)}{r} d\nu. \quad (10)$$

A comparison of this expression for the neutron density with Eq. (4.2-23):

$$\rho(r) = a_0 \nu_0 \frac{\sinh(r/\nu_0)}{r} + \int_0^1 \nu A_S(\nu) \frac{\sinh(r/\nu)}{r} d\nu, \quad (4.2-23)$$

suggests that the two sets of expansion coefficients are related by

$$b_{0-} = \frac{a_0 \nu_0}{2}; \quad B(-\nu) = \frac{1}{2} \nu A_S(\nu). \quad (11)$$

The proof of Eq. (11) follows from the substitution of Eq. (10) into the integral equation (4.4-1) for $\psi(r, \mu)$. A procedure completely analogous to that leading to Eq. (4.4-11) then yields

$$\psi(r, \mu) = f(r, \mu) - e^{-s_0} f(R, \mu_0), \quad (12)$$

where

$$f(r, \mu) = \frac{2b_{0-}}{\nu_0} \psi_0(r, \mu) + 2 \int_0^1 \frac{B(-\nu)}{\nu} \psi_\nu(r, \mu) d\nu. \quad (12a)$$

Furthermore, the function $f(r, \mu)$ in (12a) must be non-negative or non-positive because $\psi(r, \mu)$ and the distance s_0 are, by definition, non-negative. Integrating Eq. (12) over μ , we get

$$\begin{aligned} \rho(r) = & 2b_{0-} \frac{\sinh(r/\nu_0)}{r} + 2 \int_0^1 B(-\nu) \frac{\sinh(r/\nu)}{r} d\nu \\ & - \int_{-1}^1 e^{-s_0} f(R, \mu_0) d\mu. \end{aligned} \quad (12b)$$

For Eq. (12b) to be the same as Eq. (10), from which we started, the integral over μ must vanish. But both factors in the integrand are non-negative. The integral, therefore, cannot be zero unless the integrand vanishes identically. As $\exp[-s_0(r, \mu)]$ is not zero for all r , we must have

$$\begin{aligned} f(R, \mu_0) &= 0 \\ &= 2 \frac{b_{0-}}{\nu_0} \psi_0(R, \mu_0) + 2 \int_0^1 \frac{B(-\nu)}{\nu} \psi_\nu(R, \mu_0) d\nu. \end{aligned} \quad (13)$$

This last result is now recognized as Eq. (4.3-7), which defines the coefficients $A_s(\nu)$. We can, therefore, conclude that Eq. (11) is simply a relabelling of the spherical expansion coefficients. Moreover, since, by Case's theorem (1), $b_{0\pm}$ and $B(\nu)$ in an expansion of the form of Eq. (8) exist and can be uniquely determined, the same holds true for a_0 and $A_s(\nu)$. This equivalence proof for the two sets of the expansion coefficients then also establishes the uniqueness of the representations for $\psi(r, \mu)$ given in Sec. 4.2.

The transform $\Phi(r, \mu)$ can also be determined by converting Eqs. (2a) and (2b) into singular integral equations as in Sec. 3.2. Specifically, if we introduce Eq. (5) into (2a), we obtain

$$\Phi(r, \mu) = \frac{c}{2} \frac{1}{\mu} \int_0^1 d\nu \int_{-R}^r [\Phi(t, \nu) + \Phi(t, -\nu)] e^{-(r-t)/\mu} dt. \quad (14)$$

Instead of the identities (3.2-2a) and (3.2-2b), we have in this case

$$\int_{-R}^r \Phi(t, \nu) e^{-(r-t)/\mu} dt = \frac{\mu}{\nu - \mu} [\nu \Phi(r, \nu) - \mu \Phi(r, \mu)] \quad (15a)$$

and

$$\int_{-R}^r \Phi(t, -\nu) e^{-(r-t)/\mu} dt = \frac{\mu}{\nu + \mu} \left[\nu \Phi(r, -\nu) + \mu \Phi(r, \mu) + e^{-(r+R)/\mu} \Phi(R, \nu) \right]. \quad (15b)$$

Equation (14) then becomes

$$\lambda(\mu) \Phi(r, \mu) - P \int_{-1}^1 \frac{\left(\frac{c}{2}\right) \nu \Phi(r, \nu)}{\nu - \mu} d\nu = e^{-r/\mu} f(R, \mu); \quad \mu > 0, \quad (16)$$

where

$$f(R, \mu) = e^{-R/\mu} \int_0^1 \frac{\left(\frac{c}{2}\right) \nu \Phi(R, \nu)}{\nu + \mu} d\nu. \quad (16a)$$

Changing from r to $-r$ in (16) and using

$$\Phi(-r, \mu) = -\Phi(r, -\mu); \quad \mu > 0 \quad (17)$$

we arrive at the singular integral equation corresponding to Eq. (2b):

$$\lambda(\mu) \Phi(r, \mu) - P \int_{-1}^1 \frac{\left(\frac{c}{2}\right) \nu \Phi(r, \nu)}{\nu - \mu} d\nu = -e^{-r/\mu} f(R, -\mu); \quad \mu < 0. \quad (18)$$

Extension of μ to the complex z -plane in the above equations for Φ and evaluation of z at the points $\pm\nu_0$ gives again two restraining conditions:

$$\int_{-1}^1 \frac{\left(\frac{c}{2}\right) \nu \Phi(r, \nu)}{\nu \mp \nu_0} d\nu = \mp e^{\mp r/\nu_0} f(R, \pm\nu_0). \quad (19)$$

When $r = R$, Eqs. (16) and (19) reduce to

$$\lambda(\mu) \Phi(R, \mu) - P \int_0^1 \frac{\left(\frac{c}{2}\right) \nu \Phi(R, \nu)}{\nu - \mu} d\nu = e^{-R/\mu} f(R, \mu); \quad \mu > 0, \quad (20)$$

and

$$f(R, \nu_0) = -f(R, -\nu_0), \quad (21)$$

respectively. The last two equations were derived in a slightly different way by Leonard and Mullikin (15) and used to determine an expression for the critical radius.

We have thus shown that by introducing a suitable transform of $\rho(r)$ and taking advantage of the properties of the integral equation, we can treat the spherical critical problem by the methods employed in plane geometry. An important consequence of this procedure is the proof that the spherical coefficients a_0 and $A_s(\nu)$ exist and can be determined from the boundary conditions.

It should be stressed that the introduction of $\Phi(r, \mu)$ in the form of Eqs. (2a) and (2b), as well as the subsequent simplifications, was motivated by the resemblance of the integral equation (4.4-2) for $\rho(r)$ and Eq. (3.1-14) for $\rho(x)$. The method of solving integral equations of this type by defining suitable two-dimensional transforms, however, appears to have much wider applicability. In particular, the method seems to work for integral equations whose kernels can be expressed as integrals containing the Green's function of a separable differential operator. To illustrate, let us write Eq. (1) as

$$r\rho(r) = c \int_t^R t\rho(t) K(t, r) dt; \quad 0 \leq t \leq R, \quad (22)$$

where

$$K(t, r) = \left\{ \begin{array}{ll} \int_0^1 e^{-r/\mu} \sinh(t/\mu) \frac{d\mu}{\mu}; & 0 \leq t \leq r; \\ \int_0^1 e^{-t/\mu} \sinh(r/\mu) \frac{d\mu}{\mu}; & r \leq t \leq R. \end{array} \right. \quad (23)$$

But the kernel $K(t, r)$ can also be given in terms of the function

$$G(t, r; \mu) = \begin{cases} \mu e^{-r/\mu} \sinh(t/\mu); & 0 \leq t \leq r \\ \mu e^{-t/\mu} \sinh(r/\mu); & r \leq t \leq R, \end{cases} \quad (24)$$

which is the Green's function of the operator $[(d^2/dr^2) - (1/\mu^2)]$:

$$K(t, r) = \int_0^1 G(t, r; \mu) d\mu/\mu^2. \quad (25)$$

Consequently,

$$r\rho(r) = c \int_0^1 \frac{d\mu}{\mu^2} \int_0^R t\rho(t) G(t, r; \mu) dt. \quad (26)$$

The selection of a transform of $\rho(r)$ which annihilates a separable operator and whose inverse is readily available now becomes obvious. Thus, consider the function

$$\begin{aligned} F(r, \mu) &= c \int_0^R t\rho(t) G(t, r; \mu) dt \\ &= c\mu \int_0^r e^{-r/\mu} \sinh(t/\mu) t\rho(t) dt \\ &\quad + \int_r^R e^{-t/\mu} \sinh(r/\mu) t\rho(t) dt. \end{aligned} \quad (27)$$

Then, by Eq. (26), we have, for the inverse,

$$r\rho(r) = \int_0^1 F(r, \mu) d\mu/\mu^2. \quad (28)$$

Furthermore, Eq. (27) is the solution of an inhomogeneous differential equation with the inhomogeneity being proportional to $t\rho(t)$. As can be verified by differentiation, $F(r, \mu)$ satisfies the equation

$$\frac{\partial^2 F(r, \mu)}{\partial r^2} - \frac{1}{\mu^2} F(r, \mu) = -c \int_0^1 F(r, \mu') d\mu'/\mu'^2; \quad \mu > 0 \quad (29)$$

and is subject to the conditions

$$F(0, \mu) = 0; \quad F'(R, \mu) + \frac{1}{\mu} F(R, \mu) = 0, \quad (30)$$

where the prime denotes differentiation with respect to r .

Assuming that the fundamental solutions of Eq. (29) have the form

$$F_\nu(r, \mu) = R_\nu(r) f_\nu(\mu), \quad (31)$$

we find upon substitution in (29) that $R_\nu(r)$ is a solution of

$$R_\nu''(r) - \frac{1}{\nu^2} R_\nu(r) = 0, \quad (32)$$

where $1/\nu^2$ is the separation constant. Hence, an appropriate $R_\nu(r)$ which satisfies the condition at $r = 0$ is

$$R_\nu(r) = \sinh(r/\nu). \quad (32a)$$

If we normalize $f_\nu(\mu)$ so that

$$\int_0^1 f_\nu(\mu) d\mu / \mu^2 = 1, \quad (33)$$

we find that the μ -eigenfunctions are defined by

$$(\nu^2 - \mu^2) f_\nu(\mu) = c\nu^2 \mu^2. \quad (34)$$

Equation (34) again generates a discrete and a continuous spectrum depending on the values of ν . Inasmuch as this procedure will be used to establish the cylindrical solutions, we defer further discussion to Sec. 5.5. What we wish to point out here is: first, the existence of a more general procedure for selecting appropriate transforms and, secondly, that spherically symmetric solutions could have also been constructed by exploiting the properties of the kernel in the integral equation for $\rho(r)$ instead of the similarity of this equation to the one for $\rho(x)$.

4.6 THE CRITICAL SPHERE PROBLEM

The neutron distribution and criticality condition for a uniform multiplying sphere of radius R , immersed in a vacuum, will now be

determined from the results of the preceding section. First, we shall obtain an integral equation for the coefficients $B(\nu)$ from the condition (4.5-4) on $\Phi(r, \mu)$. In view of the relation (4.5-11), the coefficients $A_s(\nu)$ are also given by this equation. Furthermore, the function $\Phi(r, \mu)$ as well as expressions for $\rho(r)$ and for the critical radius will be determined in Sec. 4.6b by solving Eqs. (4.5-16) and (4.5-18). Finally, we shall compare the critical radius from the zeroth approximation of the exact results with that from the asymptotic critical condition.

4.6a Determination of the Expansion Coefficients

With the aid of Eq. (4.5-9), we can write the expansion (4.5-8) for $\Phi(r, \mu)$ as

$$\begin{aligned} \Phi(r, \mu) = & b_{0+} \left[\phi_{0+}(\mu) e^{-r/\nu_0} - \phi_{0-}(\mu) e^{r/\nu_0} \right] \\ & + \int_0^1 B(\nu) \phi_\nu(\mu) e^{-r/\nu} d\nu - \int_0^1 B(\nu) \phi_\nu(-\mu) e^{r/\nu} d\nu. \end{aligned} \quad (1)$$

Imposing the condition

$$\Phi(-R, \mu > 0) = 0 \quad (4.5-4)$$

and using the explicit form (3.1-3b) of $\phi_\nu(\mu)$, we get

$$\lambda(\mu) B'(\mu) + P \int_0^1 \frac{\left(\frac{c}{2}\right) \nu B'(\nu)}{\nu - \mu} d\nu = \Phi'(\mu), \quad (2)$$

where

$$\Phi'(\mu) = -b \left[\phi_{0+}(\mu) - \phi_{0-}(\mu) e^{-d/\nu_0} \right] + \int_0^1 \frac{\left(\frac{c}{2}\right) \nu B'(\nu) e^{-d/\nu}}{\nu + \mu} d\nu; \quad (3)$$

$$b = b_{0+} e^{R/\nu_0}; \quad B'(\nu) = B(\nu) e^{R/\nu}, \quad (4)$$

and $d = 2R$ denotes the spherical diameter.

A comparison of $\Phi(\mu)$ in Eq. (3) with $\psi'(\mu)$ in Eq. (3.3-6a) reveals that the only differences between these two functions are the signs in front of $\phi_{0-}(\mu) e^{-d/\nu_0}$ and the integral term. Therefore, the reduction of the singular integral equation (2) for $B'(\mu)$ can be written down by simply modifying the solution of Eq. (3.3-6) to account for these differences. We get

$$B'(\mu) = -b \left[\phi_{0+}(\mu) X(\nu_0) - \phi_{0-}(\mu) X(-\nu_0) e^{-d/\nu_0} \right] (\nu_0^2 - \mu^2)(1-c) X(-\mu) g(c, \mu) \\ + (\nu_0^2 - \mu^2)(1-c) X(-\mu) g(c, \mu) \int_0^1 \frac{\left(\frac{c}{2}\right) \nu X(-\nu) B'(\nu) e^{-d/\nu}}{\nu + \mu} d\nu \quad (5)$$

and the auxiliary condition

$$b \left(\frac{c\nu_0}{2} \right) \left[X(\nu_0) + e^{-d/\nu_0} X(-\nu_0) \right] = - \int_0^1 \frac{c}{2} \nu X(-\nu) B'(\nu) e^{-d/\nu} d\nu, \quad (6)$$

which corresponds to the exact criticality condition for the sphere.

Substitution of Eq. (4.5-11) in Eq. (4) gives

$$B'(\nu) = \frac{1}{2} \nu A_S(\nu) e^{R/\nu}; \quad b = \frac{1}{2} \nu_0 a_0 e^{R/\nu_0}. \quad (7)$$

Hence, Eqs. (5) and (6) can also be written in terms of a_0 and $A_S(\nu)$.

In addition, the approximation procedure of Sec. 3.3b is applicable here also. Thus the zeroth approximation, which is obtained by assuming

$B'(\nu) = 0$, reduces Eq. (6) to

$$\frac{X(\nu_0)}{X(-\nu_0)} + e^{-d_0/\nu_0} = 0. \quad (8)$$

Expressing the ratio $X(\nu_0)/X(-\nu_0)$ in terms of the Milne-problem extrapolation distance z_0 , we find, instead of Eq. (3.3-25),

$$\sin [(d_0/2|\nu_0|) + (z_0/|\nu_0|)] = 0, \quad (9)$$

from which we conclude that

$$d_0 = 2\pi|\nu_0| - 2z_0, \quad (10)$$

where d_0 is the critical diameter, in units of mean free paths, in the absence of the continuum.

The first approximation consists of taking $B'(\mu)$ equal to the first term on the right of Eq. (5). Inserting this in Eq. (6), we obtain

$$\frac{X(\nu_0) + X(-\nu_0) e^{-d_1/\nu_0}}{X(\nu_0) - X(-\nu_0) e^{-d_1/\nu_0}} = \nu_0 \alpha(c, d_1), \quad (11)$$

which can be simplified to

$$\tan \frac{1}{2|\nu_0|} [d_1 - d_0] = -|\nu_0| \alpha(c, d_1). \quad (12)$$

For small values of $|\nu_0| \alpha$, d_1 is given approximately by

$$d_1 = d_0 - 2|\nu_0|^2 \alpha(c, d_1). \quad (13)$$

The notation used here is the same as in Sec. 3.3b.

It is evident from the foregoing results that estimates for the spherical critical quantities are no more difficult to obtain than those in plane geometry. The single exception is the expression for the angular density which, even in the zeroth approximation, consists of the infinite series (4.3-1a) or the integral (4.2-12a) for $\psi_0(r, \mu)$.

4.6b Determination of $\Phi(r, \mu)$

The two half-range singular integral equations (4.5-16) and (4.5-18) for $\Phi(r, \mu)$ can be combined in the form

$$\lambda(\mu) \Phi(r, \mu) - P \int_{-1}^{-1} \frac{\left(\frac{c}{2}\right) \nu \Phi(r, \nu)}{\nu - \mu} d\nu = e^{-r/\mu} h(R, \mu); \quad -1 \leq \mu \leq 1, \quad (14)$$

with

$$h(R, \mu) = \pm f(R, \pm \mu), \quad (15)$$

where $f(R, \mu)$ is defined by Eq. (4.5-16a), and the upper and lower signs hold for μ greater or less than zero, respectively.

To solve Eq. (14), we have only to note that it differs from Eq. (3.2-8) in the sign of the right-hand term. When this difference is introduced in Eq. (3.3-50) we find

$$\begin{aligned} \Phi(r, \mu) = & \frac{\beta_0(r) + \mu\beta_1(r)}{(\nu_0^2 - \mu^2)(1 - c)} + \lambda(\mu)g(c, \mu) e^{-r/\mu} h(R, \mu) \\ & + \frac{1}{(\nu_0^2 - \mu^2)(1 - c)} \int_{-1}^1 \frac{\gamma_1(\nu)h(R, \nu) e^{-r/\nu}}{\nu - \mu} d\nu, \end{aligned} \quad (16)$$

where $\beta_0(r)$ and $\beta_1(r)$ take the place of $a_0(x)$ and $a_1(x)$ in Eq. (3.3-50),

$$\gamma_1(\nu) = \frac{\left(\frac{c}{2}\right) \nu}{X_1^+(\nu) \lambda^-(\nu)}, \quad (3.3-45a)$$

and $X_1(z)$ is given by Eq. (3.3-46). As in Sec. 3.3d, the reduction of Eq. (14) involved the introduction and subsequent construction of the function

$$N(r, z) = \frac{1}{2\pi i} \int_{-1}^1 \frac{\left(\frac{c}{2}\right) \nu \Phi(r, \nu)}{\nu - z} d\nu \quad (17a)$$

$$= \frac{X_1(z)}{2\pi i} \left[\beta_0(r) + z\beta_1(r) + \int_{-1}^1 \frac{\gamma_1(\nu)h(R, \nu) e^{-r/\nu}}{\nu - z} d\nu \right]. \quad (17b)$$

Evaluating Eq. (17a) at $z = \pm \nu_0$ and using the definition of $f(R, \mu)$, as well as the conditions (4.5-19) and (4.5-21), we obtain, instead of Eq. (3.3-48),

$$2\pi i N(r, \pm \nu_0) = \mp e^{\mp r/\nu_0} f(R, \nu_0). \quad (18)$$

Substitution of Eq. (17b) for $N(r, \pm \nu_0)$ in (18) produces two simultaneous equations for $\beta_0(r)$ and $\beta_1(r)$, which can be readily solved to give

$$\beta_0(r) = \frac{f(R, \nu_0)}{X_1(\nu_0)} \sinh(r/\nu_0) + c(c-1) \int_0^1 \nu^2 g(c, \nu) f(R, \nu) \sinh(r/\nu) d\nu \quad (19a)$$

and

$$\beta_1(r) = -\frac{f(R, \nu_0)}{\nu_0 X_1(\nu_0)} \cosh(r/\nu_0) - c(c-1) \int_0^1 \nu g(c, \nu) f(R, \nu) \cosh(r/\nu) d\nu. \quad (19b)$$

Furthermore, the analysis leading to Eq. (3.3-56) yields, in this case,

$$f(R, z) = e^{-R/z} X(-z) \left[\beta(R) + \int_0^1 \frac{\left(\frac{c}{2}\right) \nu e^{-R/\nu} f(R, \nu)}{X^+(\nu) \lambda^-(\nu)(\nu+z)} d\nu \right], \quad (20)$$

where $\beta(R)$ is a constant. For $z = \mu$, Eq. (20) becomes an integral equation for $f(R, \mu)$:

$$f(R, \mu) = e^{-R/\mu} X(-\mu) \left[\beta(R) + \int_0^1 \frac{\left(\frac{c}{2}\right) \nu e^{-R/\nu} f(R, \nu)}{X^+(\nu) \lambda^-(\nu)(\nu+\mu)} d\nu \right], \quad (21)$$

which can be systematically approximated by the procedure of Sec. (3.3b). Also, setting $z = \pm \nu_0$ in Eq. (20) and inserting the resulting expressions for $f(R, \pm \nu_0)$ into Eq. (4.5-21), we arrive at the criticality condition:

$$\begin{aligned} \beta(R) \left[X(-\nu_0) e^{-R/\nu_0} + X(\nu_0) e^{R/\nu_0} \right] = & -X(-\nu_0) e^{-R/\nu_0} \int_0^1 \frac{\left(\frac{c}{2}\right) \nu e^{-R/\nu} f(R, \nu)}{X^+(\nu) \lambda^-(\nu)(\nu + \nu_0)} d\nu \\ & - X(\nu_0) e^{R/\nu_0} \int_0^1 \frac{\left(\frac{c}{2}\right) \nu e^{-R/\nu} f(R, \nu)}{X^+(\nu) \lambda^-(\nu)(\nu - \nu_0)} d\nu. \end{aligned} \quad (22)$$

Equations (20) through (22) are consistent with the results of Leonard and Mullikin (15). Their zeroth and first approximations of Eq. (21) lead to expressions for the critical diameter which are identical with Eqs. (10) and (13), respectively.

Finally, putting $z = 0$ in Eqs. (17a) and (17b) and recalling Eq. (4.5-5), we get

$$\begin{aligned} N(r, 0) &= \frac{c}{2} \frac{1}{2\pi i} r \rho(r) \\ &= \frac{X_1(0)}{2\pi i} \left[\beta_0(r) + \int_{-1}^1 \left(\frac{1}{\nu}\right) \gamma_1(\nu) e^{-R/\nu} h(R, \nu) d\nu \right]. \end{aligned} \quad (23)$$

An expression for the neutron density now follows from (23) by inserting Eq. (19a) for $\beta_0(r)$ and using the properties of $X_1(z)$:

$$\rho(r) = \frac{2}{c} \frac{X_1(0)}{X_1(\nu_0)} f(R, \nu_0) \frac{\sinh(r/\nu_0)}{r} + 2 \int_0^1 g(c, \nu) f(R, \nu) \frac{\sinh(r/\nu)}{r} d\nu. \quad (24)$$

4.6c Asymptotic Critical Sphere Diameter

The zeroth approximation [$A_s(\nu) = 0$] of Sec. 4.6a assumes that the neutron distribution can be described by the discrete terms

$$\begin{aligned} \psi_d(r, \mu) &= a_0 \psi_0(r, \mu) \\ &= a_0 c \nu_0 \sum_0^\infty (-1)^n \left(\frac{2n+1}{2}\right) i_n(r/\nu_0) Q_n(\nu_0) P_n(\mu) \end{aligned} \quad (25)$$

and

$$\rho_d(r) = a_0 i_0(r/\nu_0) = a_0 \nu_0 \frac{\sinh(r/\nu_0)}{r}. \quad (26)$$

An asymptotic critical condition can be determined from these expressions if we use the condition of no re-entrant current:

$$j_d(R) = \int_0^1 \mu \psi_d(R, -\mu) d\mu = 0. \quad (27)$$

Before inserting Eq. (25) in (27), it is convenient to re-write the infinite series in terms of even and odd values of the index n :

$$\begin{aligned} \psi_d(r, \mu) = a_0 c \nu_0 \left[\sum_{n=0}^{\infty} \left(\frac{4n+1}{2} \right) i_{2n}(r/\nu_0) Q_{2n}(\nu_0) P_{2n}(\mu) \right. \\ \left. - \sum_{n=0}^{\infty} \left(\frac{4n+3}{2} \right) i_{2n+1}(r/\nu_0) Q_{2n+1}(\nu_0) P_{2n+1}(\mu) \right]. \end{aligned} \quad (25a)$$

The integrals over μ can now be performed with the aid of the formulae (27, V.II, p. 315)

$$\int_0^1 \mu P_{2n+1}(\mu) d\mu = \frac{1}{3} \delta_{n0}, \quad (28)$$

and

$$\int_0^1 \mu P_{2n}(\mu) d\mu = \alpha_n = \frac{(-1)^{n+1}}{2\sqrt{\pi}(2n-1)(n+1)} \frac{\Gamma(n+1/2)}{\Gamma(n+1)}.$$

Equation (27) then gives

$$\frac{1}{2} i_1(R'_0/\nu_0) Q_1(\nu_0) = - \sum_{n=0}^{\infty} \left(\frac{4n+1}{2} \right) \alpha_n i_{2n}(R'_0/\nu_0) Q_{2n}(\nu_0), \quad (29)$$

where R'_0 is the critical radius in this approximation and the fact that $P_n(-\mu) = (-1)^n P_n(\mu)$ has been used. Recalling the value of $Q_1(\nu_0)$

from Eq. (4.2-24a) and utilizing the fact that (24, p.79)

$$i_n(z) = \left(\frac{d}{dz}\right)^n [i_0(z)], \quad (30)$$

we can also write Eq. (29) as

$$\frac{1}{4} \rho_d(R_0') + \frac{D}{2} \left[\frac{d\rho_d(r)}{dr} \right]_{r=R_0'} = -c\nu_0 \sum_1^{\infty} \left(\frac{4n+1}{2} \right) \alpha_n Q_{2n}(\nu_0) \rho_d^{2n}(R_0'), \quad (31)$$

where $D = \nu_0^2(1-c)$ is the diffusion coefficient and $\rho_d^{2n}(R_0') = a_0 i_{2n}(R_0'/\nu_0)$.

For c close to unity, both ν_0 and R_0' go to infinity, but the ratio (R_0'/ν_0) approaches a constant [see Eq. (10)]. But for large ν_0 , $Q_{2n}(\nu_0)$ goes to zero at least as $(1/\nu_0)^{2n+1}$ (25, p. 317). Hence, the right side of Eq. (30) vanishes as $(1/\nu_0^2)$. Neglecting it completely, we arrive at the familiar diffusion theory result:

$$\frac{1}{4} \rho_d(R_0') + \frac{D}{2} \left[\frac{d\rho_d(r)}{dr} \right]_{r=R_0'} \simeq 0. \quad (32)$$

The critical diameter $d_0' = 2R_0'$ from Eq. (32) is compared numerically with d_0 of Eq. (10) in Table IV. From this comparison we conclude that the two asymptotic values of the critical diameter are in close agreement even for spheres with radii of the order of two mean free paths. For $c \gtrsim 1.30$, however, $|\nu_0|$ becomes less than unity and the argument leading to Eq. (32) is no longer justified.

Table IV
ASYMPTOTIC CRITICAL SPHERE
DIAMETER

c	d_0	d_0'
1.01	34.79	34.86
1.05	14.56	14.54
1.10	9.77	9.65
1.20	6.34	6.27
1.30	4.66	4.52

CHAPTER V

SOLUTIONS WITH CYLINDRICAL SYMMETRY

5.0 INTRODUCTION

The final geometrical configuration treated in this work consists of a circular cylinder of radius R and infinite height. In general, the space vector $\underline{r}$ in cylindrical geometry will be specified by the coordinates (r, ϕ_1, z) and the direction of motion $\underline{\Omega}$ by the two angles (θ, ϕ_2) . If z is taken as the axis of the cylinder, then symmetry requires that ψ remains invariant under rotation about this axis and reflection from the rz -plane. Thus ψ can depend only on the difference $\phi = \phi_2 - \phi_1$ of the azimuthal angles and, further, it must be an even function of this difference (12, p. 169). Under these conditions the transport equation (1.1-2) becomes

$$\begin{aligned} & \sin \theta \left[\cos \phi \frac{\partial \psi(r, \theta, \phi, z)}{\partial r} - \frac{\sin \phi}{r} \frac{\partial \psi(r, \theta, \phi, z)}{\partial \phi} \right] + \cos \theta \frac{\partial \psi(r, \theta, \phi, z)}{\partial z} \\ & + \psi(r, \theta, \phi, z) = \frac{c}{4\pi} \int_0^\pi \sin \theta' d\theta' \int_0^{2\pi} \psi(r, \theta', \phi', z) d\phi'. \end{aligned} \quad (1)$$

For infinite cylinders the angular density is also independent of translation along the z -axis and reflection from a plane perpendicular to this axis. These additional symmetries eliminate the z variation and reduce Eq. (1) to

$$\begin{aligned} & \sin \theta \left[\cos \phi \frac{\partial \psi(r, \theta, \phi)}{\partial r} - \frac{\sin \phi}{r} \frac{\partial \psi(r, \theta, \phi)}{\partial \phi} \right] \\ & + \psi(r, \theta, \phi) = \frac{c}{4\pi} \int_{\underline{\Omega}'} \psi(r, \theta', \phi') \sin \theta' d\theta' d\phi'. \end{aligned} \quad (2)$$

In this chapter we shall first show that the three-dimensional solution (2.1-14), under the restrictions of total cylindrical symmetry, assumes the form

$$\psi(r, \theta, \phi) = a_0 \psi_0(r, \theta, \phi) + \int_0^1 A_c(\nu) \psi_\nu(r, \theta, \phi) d\nu, \quad (3)$$

where the arbitrary coefficients a_0 and $A_c(\nu)$ are independent of the superposition variables, and then derive various representations for the cylindrical solution $\psi_0(r, \theta, \phi)$ and $\psi_\nu(r, \theta, \phi)$. Next, we shall use the transport integral equation to define a density transform. Although the transform in this case will obey a second-order integrodifferential equation subject to side conditions which are of the mixed type, the singular modes of this equation are very similar to those in plane geometry. Furthermore, the application of the transform expansion to the solution of the cylindrical critical problem follows the procedure presented in Sec. 3.3a and yields an integral equation for $A_c(\nu)$, as well as an exact criticality condition. Finally, the zeroth approximation of the criticality condition will be compared with the asymptotic critical results.

5.1 CYLINDRICAL EXPANSION COEFFICIENTS

If α and β denote the coordinates of the unit vector $\underline{u}$ (see Fig. 2), then, in cylindrical geometry, we have

$$\underline{u} \cdot \underline{r} = r \sin \alpha \cos(\phi_1 - \beta) + z \cos \alpha, \quad (1)$$

and the fundamental transport solutions (2.1-1) become

$$\psi_{\underline{k}}(\underline{r}, \underline{\Omega}) = \exp \left[-\frac{1}{\nu} (r \sin \alpha \cos(\phi_1 - \beta) + z \cos \alpha) \right] \phi_\nu(\underline{u}, \underline{\Omega}). \quad (2)$$

For the construction of cylindrical solutions to Eq. (5.0-2), it is sufficient to superpose the part of Eq. (2) which is defined on planes parallel to the cylindrical (z) axis (20, p. 365). Accordingly, we take $\alpha = \pi/2$ so that $\underline{u}$ lies on the xy -plane. Hence,

$$\psi_{\underline{k}}(\underline{r}, \underline{\Omega}) = e^{-\frac{r}{\nu} \cos(\phi_1 - \beta)} \phi_\nu(\underline{u}, \underline{\Omega}), \quad (3)$$

where

$$\underline{u} \cdot \underline{\Omega} = \sin \theta \cos (\phi_2 - \beta). \quad (3a)$$

Superposing Eq. (3) over $\underline{k}$, we get

$$\psi(\underline{r}, \underline{\Omega}) = \psi_{d+}(\underline{r}, \underline{\Omega}) + \psi_{d-}(\underline{r}, \underline{\Omega}) + \psi_c(\underline{r}, \underline{\Omega}), \quad (2.1-14)$$

where the three-dimensional solutions $\psi_{d+}(\underline{r}, \underline{\Omega})$ and $\psi_c(\underline{r}, \underline{\Omega})$ applicable for cylinders of infinite height are given by

$$\psi_{d+}(\underline{r}, \underline{\Omega}) = \frac{1}{2\pi} \int_0^{2\pi} a_{0+}(\beta) \phi_{0+}(\sin \theta \cos(\phi_2 - \beta)) e^{\mp \frac{r}{\nu_0} \cos(\phi_1 - \beta)} d\beta \quad (4a)$$

and

$$\psi_c(\underline{r}, \underline{\Omega}) = \frac{1}{2\pi} \int_0^{2\pi} d\beta \int_{-1}^1 A(\nu, \beta) \phi_\nu(\sin \theta \cos(\phi_2 - \beta)) e^{-\frac{r}{\nu} \cos(\phi_1 - \beta)} d\nu. \quad (4b)$$

In Eq. (4a) for $\psi_{d+}(\underline{r}, \underline{\Omega})$ let

$$a_{0+}(\beta) = \frac{1}{2\pi} \sum_{-\infty}^{\infty} a_{0+}^{\ell} e^{i\ell\beta}. \quad (5)$$

From the expansion (4.1-6) it also follows that

$$\phi_{0+}(\underline{u} \cdot \underline{\Omega}) = 2\pi c\nu_0 \sum_{n=0}^{\infty} \sum_{-n}^n Q_n(\nu_0) Y_n^m(\underline{u}) Y_n^{m*}(\underline{\Omega}), \quad (6)$$

where, for $\alpha = \pi/2$,

$$Y_n^m(\underline{u}) = H_n^m P_n^m(0) e^{im\beta} \quad (6a)$$

with

$$H_n^m = \left[\frac{2n+1}{4n} \frac{(n-m)!}{(n+m)!} \right]^{1/2}.$$

Consequently,

$$\psi_{d+}(\underline{r}, \underline{\Omega}) = \frac{c\nu_0}{2\pi} \sum_{\ell=-\infty}^{\infty} \sum_{nm} a_{0+}^{\ell} Q_n(\nu_0) Y_n^{m*}(\underline{\Omega}) \int_0^{2\pi} Y_n^m(\underline{u})$$

$$(\underline{x}) e^{-\frac{r}{\nu_0} \cos(\phi_1 - \beta) + i\ell\beta} d\beta. \quad (7)$$

But from the expansion (4.2-11) we can readily show that

$$\frac{1}{2\pi} \int_0^{2\pi} e^{-\frac{r}{\nu_0} \cos(\phi - \beta) + is\beta} d\beta = (-1)^s I_s(r/\nu_0) e^{is\phi}. \quad (8)$$

Hence,

$$\psi_{d+}(\underline{r}, \underline{\Omega}) = c\nu_0 \sum_{\ell} \sum_{nm} (-1)^{\ell+m} a_{0+} (H_n^m)^2$$

$$(\underline{x}) Q_n(\nu_0) I_{\ell+m}(r/\nu_0) P_n^m(0) P_n^m(\cos \theta) e^{i\ell\phi_1 - im(\phi_2 - \phi_1)}. \quad (9)$$

Proceeding similarly with the integrals for $\psi_{d-}(\underline{r}, \underline{\Omega})$ and $\psi_c(\underline{r}, \underline{\Omega})$, we arrive at the expansion

$$\psi(\underline{r}, \underline{\Omega}) = \sum_{\ell=-\infty}^{\infty} \sum_{n=0}^{\infty} \sum_{m=-n}^n D_{\ell mn}(r) (H_n^m)^2 P_n^m(0) P_n^m(\cos \theta)$$

$$(\underline{x}) e^{i\ell\phi_1} e^{-im(\phi_2 - \phi_1)}, \quad (10)$$

where

$$D_{\ell mn}(r) = \left[(-1)^{\ell+m} a_{0+}^{\ell} + (-1)^n a_{0-}^{\ell} \right] c\nu_0 Q_n(\nu_0) I_{\ell+m}(r/\nu_0)$$

$$+ (-1)^{\ell+m} \int_{-1}^1 A^{\ell}(\nu) \left[c\nu Q_n(\nu) + \lambda(\nu) P_n(\nu) \right]$$

$$(\underline{x}) I_{\ell+m}(r/\nu) d\nu. \quad (10a)$$

Integration of Eq. (10) over $\underline{\Omega}$ gives the expression for the neutron density:

$$\rho(\underline{r}) = \sum_{\ell=-\infty}^{\infty} D_{\ell 00}(r) e^{i\ell\phi_1}. \quad (11)$$

The angular density $\psi(r, \theta, \phi)$ with total cylindrical symmetry comprises those terms of Eq. (10) for which $\ell = 0$. This conclusion can be deduced by the comparison method employed in Sec. 4.1. It also follows from Eq. (11) by using the fact that the cylindrically symmetric density depends only on the radial distance r . Replacing $\rho(\underline{r})$ by $\rho(r)$, multiplying by $e^{-im\phi_1}$, and integrating over ϕ_1 , we get

$$D_{\ell 00}(r) = 0 \quad \text{unless } \ell = 0. \quad (12)$$

When we put $\ell = 0$ in the expansions having the form of Eq. (5) for $a_{0\pm}(\beta)$ and $A(\nu, \beta)$, we conclude further that

$$a_{0\pm}(\beta) = \frac{1}{2\pi} a_{0\pm}^0; \quad A(\nu, \beta) = \frac{1}{2\pi} A^0(\nu). \quad (13)$$

Thus for cylindrically symmetric distributions the expansion coefficients in Eqs. (4a) and (4b) are independent of β .

5.2 REPRESENTATIONS FOR $\psi(r, \theta, \phi)$

Setting $\ell = 0$ in Eq. (10), we obtain a double series representation for $\psi(r, \theta, \phi)$. The double series can be arranged in a more convenient form as follows. First note that, with $\ell = 0$, the expansion (9) for ψ_{d+} can be written as

$$\begin{aligned} \psi_{d+}(r, \theta, \phi) &= \frac{a_{0+}^0 + c\nu_0}{4\pi} \sum_{n=0}^{\infty} \sum_{m=0}^n (-1)^m \epsilon_m (2n+1) \\ &\quad (x) \frac{(n-m)!}{(n+m)!} Q_n(\nu_0) I_m(r/\nu_0) P_n^m(0) \\ &\quad (x) P_n^m(\cos \theta) \cos m\phi, \end{aligned} \quad (1)$$

where $\phi = \phi_2 - \phi_1$ and $\epsilon_m = 2 - \delta_{m0}$. Next, change the order of summation in (1) with the aid of the theorem (24, p. 363)

$$\sum_{n=0}^{\infty} \sum_{m=0}^n u_n^m = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} u_{n+m}^m \quad (2)$$

and use the fact that¹

$$P_n^m(0) = (-1)^m \cos \left(\pi \frac{n+m}{2} \right) \frac{2^m}{\sqrt{\pi}} \frac{\Gamma \left(\frac{1}{2} + \frac{n+m}{2} \right)}{\Gamma \left(1 + \frac{n-m}{2} \right)} \delta_{(n+m), 2k};$$

$$k = 0, 1, 2, \dots \quad (3)$$

The summation change produces $P_{n+m}^m(0)$. But, by Eq. (3) $P_{n+m}^m(0)$ vanishes unless n is even; otherwise the sum $n+2m$ of the upper and lower indices cannot be even. Hence, with $n = 2\ell$, Eq. (1) reduces to

$$\psi_{d+}(r, \theta, \phi) = a_{0+}^0 \frac{c\nu_0}{4\pi} \sum_{m=0}^{\infty} \sum_{\ell=0}^{\infty} (-1)^m \epsilon_m d_{m\ell} Q_{2\ell+m}(\nu_0) I_m(r/\nu_0)$$

$$(x) P_{2\ell+m}^m(\cos \theta) \cos m\phi, \quad (4)$$

where, if we recall the duplication formula (26, V. I, p. 5),

$$\Gamma \left(z + \frac{1}{2} \right) = \frac{\sqrt{\pi} \Gamma(2z)}{2^{2z-1} \Gamma(z)}, \quad (4a)$$

we can write explicitly

$$d_{m\ell} \equiv \frac{(2\ell)!}{(2\ell + 2m)!} (4\ell + 2m + 1) P_{2\ell+m}^m(0)$$

$$= (-1)^\ell \frac{(4\ell + 2m + 1)}{2^m \sqrt{\pi}} \frac{\Gamma \left(\ell + \frac{1}{2} \right)}{\Gamma \left(\ell + m + \frac{1}{2} \right)}. \quad (4b)$$

¹See (26, V.I., p. 145). A factor $(-1)^m$ was inserted in the formula quoted in this reference in order to conform with Ferrer's definition of $P_n^m(\mu)$ given in (23, p. 111).

A similar procedure gives

$$\psi_{d-}(r, \theta, \phi) = a_0^0 - \frac{c\nu_0}{4\pi} \sum_{m=0}^{\infty} \sum_{\ell=0}^{\infty} (-1)^m \epsilon_m d_{m\ell} Q_{2\ell+m}(\nu_0) I_m(r/\nu_0) \\ (x) P_{2\ell+m}^m(\cos \theta) \cos m\phi \quad (5)$$

for the other discrete solution and

$$\psi_c(r, \theta, \phi) = \frac{1}{4\pi} \sum_{m=0}^{\infty} \sum_{\ell=0}^{\infty} (-1)^m \epsilon_m d_{m\ell} P_{2\ell+m}^m(\cos \theta) \cos m\phi \\ (x) \int_{-1}^1 A^0(\nu) [c\nu Q_{2\ell+m}(\nu) + \lambda(\nu) P_{2\ell+m}(\nu)] I_m(r/\nu) d\nu \quad (6)$$

for the continuum. The symmetry properties of the factors in the integrand of Eq. (6) enable us to write the integral over ν as

$$\int_0^1 A_C(\nu) [c\nu Q_{2\ell+m}(\nu) + \lambda(\nu) P_{2\ell+m}(\nu)] I_m(r/\nu) d\nu, \quad (6a)$$

where

$$A_C(\nu) = A^0(\nu) + A^0(-\nu). \quad (6b)$$

Finally, collecting the foregoing expansions, we can write the double series representation for $\psi(r, \theta, \phi)$ as

$$\psi(r, \theta, \phi) = a_0 \psi_0(r, \theta, \phi) + \int_0^1 A_C(\nu) \psi_\nu(r, \theta, \phi) d\nu, \quad (5.0-3)$$

where

$$a_0 = a_0^0 + a_0^0-, \quad (7a)$$

and the cylindrical eigenfunctions are given by

$$\psi_0(r, \theta, \phi) = \frac{c\nu_0}{4\pi} \sum_{m\ell} (-1)^m \epsilon_m d_{m\ell} Q_{2\ell+m}(\nu_0) I_m(r/\nu_0) P_{2\ell+m}^m(\cos \theta) \\ (x) \cos m\phi \quad (7b)$$

and

$$\psi_\nu(r, \theta, \phi) = \frac{1}{4\pi} \sum_{m\ell} (-1)^m \epsilon_m d_{m\ell} [c\nu Q_{2\ell+m}(\nu) + \lambda(\nu) P_{2\ell+m}(\nu)]$$

$$(x) I_m(r/\nu) P_{2\ell+m}^m(\cos \theta) \cos m\phi. \quad (7c)$$

Equivalent integral representations for $\psi_0(r, \theta, \phi)$ and $\psi_\nu(r, \theta, \phi)$ can be deduced from Eqs. (5.1-4a) and (5.1-4b). When we use the fact that the coefficients $a_{0\pm}(\beta)$ and $A(\nu, \beta)$ are no longer functions of β , these equations reduce to

$$\psi_{d\pm}(r, \theta, \phi) = \frac{a_{0\pm}^0}{(2\pi)^2} \int_0^{2\pi} \left(\frac{c\nu_0}{2} \right) \frac{e^{\mp \frac{r}{\nu_0} \cos(\phi_1 - \beta)}}{\nu_0 \mp \sin \theta \cos(\phi_2 - \beta)} d\beta \quad (8a)$$

and

$$\psi_c(r, \theta, \phi) = \frac{1}{(2\pi)^2} \int_{-1}^1 A^0(\nu) d\nu \int_0^{2\pi} (x) \left[\frac{c}{2} P \frac{\nu}{\nu - \sin \theta \cos(\phi_2 - \beta)} + \lambda(\nu) \delta(\sin \theta \cos(\phi_2 - \beta) - \nu) \right]$$

$$(x) e^{-\frac{r}{\nu} \cos(\phi_1 - \beta)} d\beta. \quad (8b)$$

In Eq. (8a) let $\gamma = \beta - \phi_2$. Since the integrand is an analytic function of β with period 2π on the unit circle, this rotation does not affect the integration limits. Then

$$\psi_{d\pm}(r, \theta, \phi) = \frac{a_{0\pm}^0}{(2\pi)^2} \int_0^{2\pi} \left(\frac{c\nu_0}{2} \right) \frac{e^{\mp \frac{r}{\nu_0} \cos(\phi + \gamma)}}{\nu_0 \mp \sin \theta \cos \gamma} d\gamma. \quad (9)$$

Replacing γ by $\gamma + \pi$ in the integral in Eq. (9) corresponding to $\psi_{d-}(r, \theta, \phi)$, we find that it becomes the same as the integral corresponding to $\psi_{d+}(r, \theta, \phi)$. Hence, the two discrete solutions can be combined as

$$\begin{aligned}\psi_d(r, \theta, \phi) &= \psi_{d+}(r, \theta, \phi) + \psi_{d-}(r, \theta, \phi) \\ &= a_0 \psi_0(r, \theta, \phi),\end{aligned}\quad (10)$$

where

$$\psi_0(r, \theta, \phi) = \frac{1}{(2\pi)^2} \int_0^{2\pi} \left(\frac{c\nu_0}{2} \right) \frac{e^{-\frac{r}{\nu_0} \cos(\phi + \gamma)}}{\nu_0 - \sin\theta \cos\gamma} d\gamma \quad (10a)$$

and

$$a_0 = a_{0+}^0 + a_{0-}^0.$$

The same treatment applied to Eq. (8b) gives

$$\psi_c(r, \theta, \phi) = \int_0^1 A_c(\nu) \psi_\nu(r, \theta, \phi) d\nu, \quad (11)$$

where

$$\begin{aligned}\psi_\nu(r, \theta, \phi) &= \frac{1}{(2\pi)^2} \int_0^{2\pi} \left[\frac{c}{2} P \frac{\nu}{\nu - \sin\theta \cos\gamma} + \lambda(\nu) \delta(\sin\theta \cos\gamma - \nu) \right] \\ &\quad e^{-\frac{r}{\nu} \cos(\phi + \gamma)} d\gamma \\ &= \psi_{-\nu}(r, \theta, \phi)\end{aligned}\quad (11a)$$

and $A_c(\nu)$ is given by Eq. (6b).

If we replace the denominators in Eqs. (10a) and (11a) and the δ -function term by their spherical harmonics expansions and make use of Eq. (5.1-8), we can easily show that these integrals are equivalent to the double series representations of Eqs. (7b) and (7c). A more

interesting and useful single series representation for $\psi_0(r, \theta, \phi)$ and $\psi_\nu(r, \theta, \phi)$ follows from the evaluation of these integrals by the method of contour integration. We consider, first, Eq. (10a), in which we introduce the expansion (4.2-11) for the exponential:

$$\psi_0(r, \theta, \phi) = \frac{c\nu_0}{2(2\pi)^2} \sum_{-\infty}^{\infty} (-1)^n I_n(r/\nu_0) e^{in\phi} \quad (x) \int_0^{2\pi} \frac{e^{in\gamma}}{\nu_0 - \sin\theta \cos\gamma} d\gamma. \quad (12)$$

But if $B_n(\nu_0, \theta)$ denotes the remaining integral in (12), then by replacing γ by $2\pi - \gamma$ we see that

$$\begin{aligned} B_n(\nu_0, \theta) &= \int_0^{2\pi} \frac{e^{in\gamma}}{\nu_0 - \sin\theta \cos\gamma} d\gamma \\ &= B_{-n}(\nu_0, \theta). \end{aligned} \quad (13)$$

Equation (12) can now be written as

$$\psi_0(r, \theta, \phi) = \frac{c\nu_0}{2(2\pi)^2} \sum_0^{\infty} (-1)^n \epsilon_n I_n(r/\nu_0) \cos n\phi B_n(\nu, \theta), \quad (14)$$

where $\epsilon_n = 2 - \delta_{n0}$.

Let $z = e^{i\gamma}$. By the residue theorem the integral in Eq. (13) becomes

$$\begin{aligned} B_n(\nu_0, \theta) &= -\frac{2}{i \sin\theta} \int_C \frac{z^n dz}{z - 2z \cos \alpha_0 + 1} \\ &= -\left(\frac{2}{i \sin\theta}\right) 2\pi i \sum_j R_n(z_j), \end{aligned} \quad (15)$$

where the contour C is the unit circle, $R_n(z_j)$ denotes the residue at the pole z_j within C , and α_0 is defined by

$$\cos \alpha_0 = \frac{\nu_0}{\sin \theta}. \quad (15a)$$

For $n \geq 0$ the integrand in (15) has two poles occurring at the two roots of the denominator:

$$z_{1,2} = \frac{\nu_0}{\sin \theta} \pm \sqrt{\frac{\nu_0^2}{\sin^2 \theta} - 1} = e^{\pm i \alpha_0}. \quad (16)$$

One of these roots, $z_2 = e^{-i \alpha_0}$, lies within the unit circle. The residue there is

$$\begin{aligned} R_n(z_2) &= - (1/2i) \frac{e^{-i n \alpha_0}}{\sin \alpha_0} \\ &= - \frac{1}{2i} \left[\frac{T_n(\cos \alpha_0)}{\sin \alpha_0} - i U_{n-1}(\cos \alpha_0) \right], \end{aligned} \quad (17)$$

where $\sin \alpha_0 = \sqrt{1 - \cos^2 \alpha_0}$ and $T_n(x)$ and $U_n(x)$ denote the Tchebichef polynomials of the first and second kind, respectively (26, V. 2, p. 184):

$$T_n(\cos \theta) = \cos n \theta; \quad U_n(\cos \theta) = \frac{\sin(n+1)\theta}{\sin \theta}. \quad (18)$$

Finally,

$$\begin{aligned} B_n(\nu_0, \theta) &= \frac{2\pi}{i \sin \theta \sin \alpha_0} e^{-i n \alpha_0} \\ &= - \frac{2\pi i}{\sin \theta} \left[\frac{T_n(\cos \alpha_0)}{\sin \alpha_0} - i U_{n-1}(\cos \alpha_0) \right]. \end{aligned} \quad (19)$$

Substitution in Eq. (14) yields

$$\psi_0(r, \theta, \phi) = \frac{c \nu_0}{4\pi \sqrt{\nu_0^2 - \sin^2 \theta}} \sum_{n=0}^{\infty} (-1)^n \epsilon_n I_n(r/\nu_0) e^{-i n \alpha_0} \cos n \phi, \quad (20)$$

where explicitly

$$e^{-i n \alpha_0} = \left[\frac{\sin \theta}{\nu_0 + \sqrt{\nu_0^2 - \sin^2 \theta}} \right]^n. \quad (20a)$$

Alternatively, Eq. (20) can be written as

$$\psi_0(r, \theta, \phi) = \frac{c\nu_0}{4\pi} \sum_0^{\infty} (-1)^n \epsilon_n I_n(r/\nu_0) \cos n\phi$$

$$(x) \left[\frac{T_n(\cos \alpha_0)}{\sqrt{\nu_0^2 - \sin^2 \theta}} - \frac{U_{n-1}(\cos \alpha_0)}{\sin \theta} \right]. \quad (21)$$

Let S denote the part of the series in (21) which contains $T_n(\cos \alpha_0)$. Then with the aid of Eqs. (18) and (4.2-19a) we can give S in closed form:

$$S = \frac{c\nu_0}{4\pi\sqrt{\nu_0^2 - \sin^2 \theta}} \sum_{n=0}^{\infty} (-1)^n \epsilon_n \cos n\phi \cos n\alpha_0$$

$$= \frac{c\nu_0}{4\pi\sqrt{\nu_0^2 - \sin^2 \theta}} \left(\frac{1}{2} \right) \left[e^{-\frac{r}{\nu_0} \cos(\phi + \alpha_0)} + e^{-\frac{r}{\nu_0} \cos(\phi - \alpha_0)} \right]$$

$$= \frac{c\nu_0}{4\pi\sqrt{\nu_0^2 - \sin^2 \theta}} e^{-r \frac{\cos \phi}{\sin \theta}} \cos \left[\frac{r}{\nu_0} \frac{\sin \phi}{\sin \theta} \sqrt{\nu_0^2 - \sin^2 \theta} \right]. \quad (22)$$

Therefore,

$$\psi_0(r, \theta, \phi) = \frac{c\nu_0}{4\pi\sqrt{\nu_0^2 - \sin^2 \theta}} e^{-r \frac{\cos \phi}{\sin \theta}} \cos \left[\frac{r}{\nu_0} \frac{\sin \phi}{\sin \theta} \sqrt{\nu_0^2 - \sin^2 \theta} \right]$$

$$- \frac{c\nu_0}{2\pi} \sum_1^{\infty} (-1)^n I_n(r/\nu_0) \cos n\phi \frac{U_{n-1}(\nu_0/\sin \theta)}{\sin \theta} \quad (23)$$

as $U_{-1}(x) = 0$. Furthermore, we see from Eqs. (20) and (20a) that for θ equal to zero or π ,

$$\psi_0 = \frac{c}{4\pi} I_0(r/\nu_0); \sin \theta = 0. \quad (23a)$$

We now consider the integral

$$\psi'_\nu(r, \theta, \phi) = \frac{c\nu}{2(2\pi)^2} \mathcal{P} \int_0^{2\pi} \frac{e^{-\frac{r}{\nu} \cos(\phi + \gamma)}}{\nu - \sin\theta \cos\gamma} d\gamma; \quad \nu \in [-1, 1], \quad (24)$$

occurring in the continuous solution (11a). By Eq. (2.1-8a) we can write the principal value as

$$\mathcal{P} \frac{1}{\nu - \sin\theta \cos\gamma} = \lim_{\epsilon \rightarrow 0} \frac{1}{2} \left[\frac{1}{\nu_+ - \sin\theta \cos\gamma} + \frac{1}{\nu_- - \sin\theta \cos\gamma} \right], \quad (24a)$$

where $\nu_\pm = \nu \pm i\epsilon$. The foregoing contour integration procedure now gives

$$\psi'_\nu(r, \theta, \phi) = \frac{c\nu}{4\pi i \sin\theta} \sum_{n=0}^{\infty} (-1)^n \epsilon_n I_n(r/\nu) \cos n\phi \left\{ \lim_{\epsilon \rightarrow 0} \frac{1}{2} \right. \\ \left. (x) \left[\left(\frac{T_n(\cos\alpha_+)}{\sin\alpha_+} + \frac{T_n(\cos\alpha_-)}{\sin\alpha_-} \right) - i \left(U_{n-1}(\cos\alpha_+) + U_{n-1}(\cos\alpha_-) \right) \right] \right\},$$

where

$$\cos\alpha_\pm = \nu_\pm / \sin\theta. \quad (25a)$$

To evaluate the limit in Eq. (25) we first recall that the functions T_n and U_{n-1} are polynomials of $\nu/\sin\theta$. Therefore they are continuous on the real positive axis and are not affected by the limiting process. But, since both ν and $\sin\theta$ are non-negative, the function $\sin\alpha = \sqrt{1 - (\nu^2/\sin^2\theta)}$ changes sign upon crossing the cut $[0, 1]$ whenever $\nu/\sin\theta$ lies in that interval. Hence,

$$\lim_{\epsilon \rightarrow 0} \left[\frac{1}{\sin\alpha_+} + \frac{1}{\sin\alpha_-} \right] = \begin{cases} 0 & \text{if } \nu < \sin\theta \\ \frac{2}{\sqrt{1 - \frac{\nu^2}{\sin^2\theta}}} & \text{if } \nu > \sin\theta. \end{cases} \quad (25b)$$

Displaying this discontinuity by the Heaviside function

$$h(\nu - \sin \theta) = \begin{cases} 0 & \nu < \sin \theta \\ 1 & \nu > \sin \theta, \end{cases} \quad (25c)$$

we can rewrite Eq. (25) as

$$\psi'_\nu(r, \theta, \phi) = \frac{c\nu}{4\pi} \sum_0^\infty (-1)^n \epsilon_n I_n(r/\nu) \cos n\phi$$

$$(x) \left[\frac{T_n(\nu/\sin \theta) h(\nu - \sin \theta)}{\sqrt{\nu^2 - \sin^2 \theta}} - \frac{U_{n-1}(\nu/\sin \theta)}{\sin \theta} \right] \quad (26a)$$

$$= \frac{c\nu}{4\pi\sqrt{\nu^2 - \sin^2 \theta}} e^{-r \frac{\cos \phi}{\sin \theta}} \cos \left[\frac{r \sin \phi}{\nu \sin \theta} \sqrt{\nu^2 - \sin^2 \theta} \right]$$

$$(x) h(\nu - \sin \theta) - \frac{c\nu}{2\pi} \sum_1^\infty (-1)^n I_n(r/\nu) \cos n\phi \frac{U_{n-1}(\nu/\sin \theta)}{\sin \theta}. \quad (26b)$$

The δ -function integral in Eq. (11a)

$$\psi''_\nu(r, \theta, \phi) = \frac{\lambda(\nu)}{(2\pi)^2} \int_0^{2\pi} \delta(\sin \theta \cos \gamma - \nu) e^{-\frac{r}{\nu} \cos(\phi + \gamma)} d\gamma, \quad (27)$$

can be treated in precisely the same manner because from Eq. (2.1-8b) we can write

$$\delta(\sin \theta \cos \gamma - \nu) = \frac{1}{2\pi i} \lim_{\epsilon \rightarrow 0} \left[\frac{1}{\nu_- - \sin \theta \cos \gamma} - \frac{1}{\nu_+ - \sin \theta \cos \gamma} \right]. \quad (28)$$

However, this integral can be evaluated directly. Thus, Eq. (27) can also be written as

$$\psi''_\nu(r, \theta, \phi) = \frac{\lambda(\nu)}{(2\pi)^2} \int_0^\pi \left[e^{\frac{r}{\nu} \cos(\phi + \gamma)} + e^{-\frac{r}{\nu} \cos(\phi - \gamma)} \right] d\gamma$$

$$(x) \delta(\sin \theta \cos \gamma - \nu) d\gamma. \quad (29)$$

With $t = \cos \gamma$, Eq. (29) becomes

$$\psi_{\nu}''(r, \theta, \phi) = \frac{\lambda(\nu)}{(2\pi)^2} \int_{-1}^1 \left[e^{-\frac{r}{\nu} \cos(\phi + \gamma)} + e^{-\frac{r}{\nu} \cos(\phi - \gamma)} \right] (x) \delta(t \sin \theta - \gamma) \frac{dt}{\sqrt{1 - t^2}}, \quad (29a)$$

which vanishes unless $\nu/\sin \theta \in [-1, 1]$, or, since both ν and $\sin \theta$ are non-negative, unless $\nu \leq \sin \theta$. In the previous notation $\psi_{\nu}''(r, \theta, \phi)$ is given by

$$\begin{aligned} \psi_{\nu}''(r, \theta, \phi) &= \frac{2\lambda(\nu)}{(2\pi)^2 \sqrt{\sin^2 \theta - \nu^2}} e^{-r \frac{\cos \phi}{\sin \theta}} \\ (x) \cos \left[\frac{r}{\nu} \frac{\sin \phi}{\sin \theta} \sqrt{\nu^2 - \sin^2 \theta} \right] h(\sin \theta - \nu) \\ &= \frac{2\lambda(\nu)}{(2\pi)^2} \sum_{n=0}^{\infty} (-1)^n \epsilon_n I_n(r/\nu) \cos n\phi \frac{T_n(\nu/\sin \theta)}{\sqrt{\sin^2 \theta - \nu^2}} h(\sin \theta - \nu). \end{aligned} \quad (30)$$

Finally, the continuous solution is given by a combination of Eqs. (26) and (30):

$$\psi_{\nu}(r, \theta, \phi) = \psi'_{\nu}(r, \theta, \phi) + \psi''_{\nu}(r, \theta, \phi). \quad (31)$$

The equivalence of the expansions for $\psi_0(r, \theta, \phi)$ and $\psi_{\nu}(r, \theta, \phi)$ discussed above with the double series expansions of Eqs. (7b) and (7c) for these solutions follows from the fact that both representations are generated from the corresponding integrals given by Eqs. (10a) and (11a). When we equate coefficients of $I_m(r/\nu_0) \cos m\phi$ in Eqs. (7a) and (20) for $\psi_0(r, \theta, \phi)$, we see that

$$\frac{1}{\sqrt{\nu_0^2 - \sin^2 \theta}} \left[\frac{\sin \theta}{\nu_0 + \sqrt{\nu_0^2 - \sin^2 \theta}} \right]^m = \sum_{\ell=0}^{\infty} d_{m\ell} Q_{2\ell+m}(\nu_0) P_{2\ell+m}^m(\cos \theta). \quad (32)$$

It is of interest to note that this identity can be established independently, simply by considering the Fourier series for γ of the even function $(\nu_0 - \sin\theta \cos\gamma)^{-1}$:

$$\frac{1}{\nu_0 - \sin\theta \cos\gamma} = \sum_{m=0}^{\infty} E_m(\nu_0, \sin\theta) \cos m\gamma. \quad (33)$$

Then

$$E_m(\nu_0, \sin\theta) = \frac{1}{\pi} \int_0^{2\pi} \frac{\cos m\gamma}{\nu_0 - \sin\theta \cos\gamma} d\gamma. \quad (34)$$

Evaluation of the integral in Eq. (34) by the contour integration procedure leading to Eq. (20) yields $2(\nu_0^2 - \sin^2\theta)^{-1/2} \exp[-im\alpha_0]$, which is identified as the left side of Eq. (32). The coefficients E_m can also be expressed as a series by expanding the denominator in the integrand with the aid of Heine's expansion (4.1-6) and making use of the orthogonality of the trigonometric functions to perform the γ integration:

$$E_m(r, \theta, \phi) = 2 \sum_{s=0}^{\infty} (2s+1) Q_s(\nu_0) \frac{(s-m)!}{(s+m)!} P_s^m(0) P_s^m(\cos\theta). \quad (34a)$$

The first m terms in (34a) are zero because the Legendre functions vanish unless $s \geq |m|$. This series can therefore be written in terms of the index $p = s - m$, where p ranges from zero to infinity. Furthermore, by Eq. (3), p must be even. With $p = 2\ell$ the series (34a) converts to a form which agrees with the right side of Eq. (32) and the proof of this identity is established.

To summarize, we have shown so far that the specialization of the three-dimensional solutions of Chapter II to a circular cylinder of infinite height produces the expansion

$$\psi(r, \theta, \phi) = a_0 \psi_0(r, \theta, \phi) + \int_0^1 A_C(\nu) \psi_\nu(r, \theta, \phi) d\nu \quad (5.0-3)$$

and have developed various expressions for the cylindrical expansion modes $\psi_0(r, \theta, \phi)$ and $\psi_\nu(r, \theta, \phi)$.

The cylindrical neutron density can be determined simply by setting $\ell = 0$ in Eq. (5.1-11) and using the subsequent definitions of the coefficients:

$$\rho(r) = a_0 I_0(r/\nu_0) + \int_0^1 A_c(\nu) I_0(r/\nu) d\nu. \quad (35)$$

Multiplying the double series representation for $\psi(r, \theta, \phi)$ by

$$\begin{aligned} \Omega_r &= \sin\theta \cos\phi \\ &= \sqrt{\frac{2\pi}{3}} \left[Y_1^1(\underline{\Omega}) + Y_1^{-1}(\underline{\Omega}) \right] \end{aligned}$$

and using the orthogonality of the spherical harmonics to integrate over θ and ϕ , we obtain

$$j(r) = (c-1) \left[a_0 \nu_0 I_1(r/\nu_0) + \int_0^1 A_c(\nu) \nu I_1(r/\nu) d\nu \right] \quad (36)$$

for the radial neutron current. It can be readily verified that these expressions for $\rho(r)$ and $j(r)$ are consistent with the continuity equation (1.1-5) which, in this case, has the form

$$\frac{dj(r)}{dr} + \frac{1}{r} j(r) + (1-c)\rho(r) = 0. \quad (37)$$

5.3 PROPERTIES OF THE CYLINDRICAL SOLUTIONS

A set of properties similar to those discussed in Sec. 4.3 for the spherical solutions can also be given for the cylindrical solutions. To start with, we show that the expansion (5.0-3) for $\psi(r, \theta, \phi)$ obeys the transport equation (5.0-2):

$$T\psi(r, \theta, \phi) = \frac{c}{4\pi} \rho(r), \quad (1)$$

where

$$T = \sin\theta \left[\cos\phi \frac{\partial}{\partial r} - \frac{\sin\phi}{r} \frac{\partial}{\partial \phi} \right] + 1 \quad (1a)$$

and $\rho(r)$ is given by Eq. (5.2-35).

Consider first the discrete solution

$$\psi_d(r, \theta, \phi) = a_0 \psi_0(r, \theta, \phi) \quad (5.2-10)$$

and use Eq. (5.2-20) for $\psi_0(r, \theta, \phi)$:

$$\psi_0(r, \theta, \phi) = \frac{c\nu_0}{4\pi} \sum_0^{\infty} F_n(\nu_0, \theta) I_n(r/\nu_0) \cos n\phi, \quad (2a)$$

where

$$F_n(\nu_0, \theta) = (-1)^n \frac{\epsilon_n e^{-in\alpha_0}}{\sqrt{\nu_0^2 - \sin^2\theta}}. \quad (2b)$$

From the recurrence relations of $I_n(z)$ (24, p. 79), it then follows that

$$\begin{aligned} T\psi_d(r, \theta, \phi) &= \frac{a_0 c}{4\pi} \sum_0^{\infty} F_n(\nu_0, \theta) \\ (x) \quad &\left[\frac{1}{2} I_{n+1}(r/\nu_0) \sin\theta \cos(n+1)\phi + \frac{1}{2} I_{n-1}(r/\nu_0) \sin\theta \cos(n-1)\phi + \nu_0 I_n(r/\nu_0) \cos n\phi \right] \\ &= \frac{a_0 c}{4\pi} \sum_{n=0}^{\infty} I_n(r/\nu_0) \cos n\phi \\ (x) \quad &\left[\frac{1}{2} F_{n-1} \sin\theta (1 - \delta_{n0} + \delta_{n1}) + \frac{1}{2} F_{n+1} \sin\theta + \nu_0 F_n \right]. \end{aligned} \quad (3)$$

But from the definition (2b) of F_n we can easily show that

$$\frac{1}{2} F_{n-1} \sin\theta (1 - \delta_{n0} + \delta_{n1}) + \frac{1}{2} F_{n+1} \sin\theta + \nu_0 F_n = \delta_{n0}. \quad (3a)$$

Hence,

$$T\psi_d(r, \theta, \phi) = \frac{a_0 c}{4\pi} I_0(r/\nu_0) = \frac{c}{4\pi} \rho_d(r). \quad (4)$$

Since the r and ϕ dependence of $\psi_\nu(r, \theta, \phi)$ in Eq. (5.2-31) is the same as $\psi_0(r, \theta, \phi)$, the above procedure can also be applied to the continuous solution

$$\psi_c(r, \theta, \phi) = \int_0^1 A_c(\nu) \psi_\nu(r, \theta, \phi) d\nu \quad (5)$$

in order to complete the proof of Eq. (1).

A combination of Eqs. (5.2-23), (5.2-26b), and (5.2-30) for $\psi_0(r, \theta, \phi)$ and $\psi_\nu(r, \theta, \phi)$ reveals that the cylindrical distribution can also be written as

$$\psi(r, \theta, \phi) = \psi_1(r, \theta, \phi) + \psi_2(r, \theta, \phi), \quad (6)$$

where

$$\begin{aligned} \psi_1(r, \theta, \phi) = & \frac{1}{4\pi} e^{-r \frac{\cos \phi}{\sin \theta}} \left[\frac{a_0 c \nu_0}{\sqrt{\nu_0^2 - \sin^2 \theta}} \cos \left[\frac{r}{\nu_0} \frac{\sin \phi}{\sin \theta} \sqrt{\nu_0^2 - \sin^2 \theta} \right] \right. \\ & \left. + \int_0^1 A_c(\nu) \left(\frac{c \nu h(\nu - \sin \theta)}{\sqrt{\nu^2 - \sin^2 \theta}} + \frac{2}{\pi} \lambda(\nu) \frac{h(\sin \theta - \nu)}{\sqrt{\sin^2 \theta - \nu^2}} \right) \cos \left[\frac{r}{\nu} \frac{\sin \phi}{\sin \theta} \sqrt{\nu^2 - \sin^2 \theta} \right] d\nu \right] \end{aligned} \quad (6a)$$

and

$$\begin{aligned} \psi_2(r, \theta, \phi) = & -\frac{1}{2\pi} \sum_1^\infty (-1)^n \frac{\cos n\phi}{\sin \theta} \left[a_0 c \nu_0 I_n(r/\nu_0) U_{n-1}(\nu_0/\sin \theta) \right. \\ & \left. + \int_0^1 A_c(\nu) c \nu I_n(r/\nu) U_{n-1}(\nu/\sin \theta) d\nu \right]. \end{aligned} \quad (6b)$$

If, as in Eq. (5.2-22), we express $\psi_1(r, \theta, \phi)$ as an expansion in α_0 , we can show that

$$T\psi_1(r, \theta, \phi) = 0; \int_0^\pi \sin \theta d\theta \int_0^{2\pi} \psi_1(r, \theta, \phi) d\phi = \rho(r). \quad (7a)$$

Furthermore, from Eq. (6b) it follows that

$$T\psi_2(r, \theta, \phi) = \frac{c}{4\pi} \rho(r); \int_0^\pi \sin \theta d\theta \int_0^{2\pi} \psi_2(r, \theta, \phi) d\phi = 0. \quad (7b)$$

Thus, the decomposition discussed in Sec. 4.3 for the spherical solutions is also true in this case.

The two limitations associated with the solutions constructed by the superposition method are also present here. First, observe that the verification of the expansions for $\psi(r, \theta, \phi)$ as solutions to the transport equation depended only on the recurrence relations of $I_n(r/\nu)$. Since the modified Bessel function of the second kind, $K_n(z)$, obeys similar recurrence relations (24, p. 79), the replacement of $(-1)^n I_n(z)$ by $K_n(z)$ in the foregoing series expansions results in expressions for $\psi(r, \theta, \phi)$ which also satisfy the transport equation but are singular at $r = 0$. Such singular transport solutions were not generated by the analysis presented in the previous sections. It is expected, however, that these singular solutions can be obtained by appropriate modification of the superposition variable β in Eqs. (5.1-4a) and (5.1-4b) (see Sec. 2A of Appendix A).

More important is the limitation concerning the determination of the expansion coefficients a_0 and $A_C(\nu)$ from the boundary conditions. For an infinite circular cylinder of radius R , the incoming distribution at the surface is described by $\psi(R, \theta, \phi_0)$ where $\frac{\pi}{2} \leq \phi_0 \leq \frac{3\pi}{2}$. The condition for this quantity to vanish is

$$a_0 \psi_0(R, \theta, \phi_0) + \int_0^1 A_C(\nu) \psi_\nu(R, \theta, \phi_0) d\nu = 0; \quad \frac{\pi}{2} \leq \phi_0 \leq \frac{3\pi}{2}. \quad (8)$$

An exact criticality statement as well as an equation for $A_C(\nu)$ and information on the uniqueness of these coefficients could not be extracted from Eq. (8). Even a spherical harmonics type of approximation obtained by using the double series representations of Eqs. (5.2-7b) and (5.2-7c) for ψ_0 and ψ_ν and replacing the integral over ν by a sum (see Sec. 4B of Appendix B) becomes very tedious in this geometry because of the presence of the double sum (28, 29, 30). It is this difficulty that forces us to look at the integral equation for a simpler formulation of the cylindrical critical problem.

Fig. 10
Cylindrical Coordinates for the Integral
Equation.

since $OA = R$ and $AC = s_0 \sin \theta$. Then

$$s_0(r, \theta, \phi) = \frac{r \cos \phi + [R^2 - r^2 \sin^2 \phi]^{1/2}}{\sin \theta}, \quad (2)$$

where the sign of the square root was so chosen that for $r = R$ and $\phi = \phi_0$ in $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$, $s_0(R, \theta, \phi_0) = 0$ and the boundary condition $\psi(R, \theta, \phi_0) = 0$ is satisfied.

The integral equation for ψ now takes the form

$$\psi(r, \theta, \phi) = \frac{c}{4\pi} \int_0^{s_0(r, \theta, \phi)} \rho \left(\sqrt{r^2 + s^2 \sin^2 \theta - 2sr \sin \theta \cos \phi} \right) e^{-s} ds. \quad (3)$$

Integration of Eq. (3) over θ and ϕ yields the integral equation for the cylindrical density:

$$\rho(r) = \frac{c}{2\pi} \int_0^R t \rho(t) dt \int_0^{2\pi} d\alpha \int_1^\infty K_0 \left(y \sqrt{r^2 + t^2 - 2rt \cos \alpha} \right) dy, \quad (4)$$

where $K_0(x)$ is the modified Bessel function of the second kind. The details for the integration to arrive at Eq. (4) are given in Appendix D.

The integral over α in Eq. (4) can be eliminated by first splitting the integral over t into two integrals ranging from 0 to r and r to R , and then using the addition theorem for K_0 (26, V. 2, p. 102):

$$K_0 \left(\sqrt{r^2 + t^2 - 2rt \cos \alpha} \right) = \sum_{-\infty}^{\infty} e^{in\alpha} \left\{ \begin{array}{ll} K_n(r) I_n(t); & r \geq t \\ K_n(t) I_n(r); & r \leq t. \end{array} \right. \quad (5)$$

Then

$$\rho(r) = c \int_1^\infty dy \left[\int_0^r K_0(ry) I_0(ty) t \rho(t) dt + \int_r^R K_0(ty) I_0(ry) t \rho(t) dt \right]. \quad (6)$$

Another form of (6) is obtained by replacing the product of Bessel functions by the integral (26, V. 2, p. 96)

$$I_0(ry) K_0(ty) = \int_0^\infty \frac{J_0(rx) J_0(tx) x dx}{x^2 + y^2}; \quad t \geq r; \quad y > 0, \quad (7)$$

where $J_0(x)$ denotes the regular Bessel function of the first kind. Substitution of (7) into (6) yields

$$\rho(r) = c \int_0^R t \rho(t) dt \int_0^\infty J_0(rx) J_0(tx) \tan^{-1}x dx \quad (8)$$

since

$$\int_1^\infty \frac{dy}{x^2 + y^2} = \frac{1}{x} \tan^{-1}x.$$

We shall now verify that the representations $\psi(r, \theta, \phi)$ presented in Sec. 5.2 are consistent with the integral formulation of Eq. (3). Following the procedure of Sec. 4.4, we put the expansion

$$\rho(r) = a_0 I_0(r/\nu_0) + \int_0^1 A_C(\nu) I_0(r/\nu) d\nu \quad (5.2-35)$$

in Eq. (3). Then

$$\begin{aligned} \psi(r, \theta, \phi) = & \frac{a_0 c}{4\pi} \int_0^{s_0} I_0 \left(\frac{1}{\nu_0} \sqrt{r^2 + s^2 \sin^2 \theta - 2sr \sin \theta \cos \phi} \right) e^{-s} ds \\ & + \frac{c}{4\pi} \int_0^1 A_C(\nu) d\nu \int_0^{s_0} I_0 \left(\frac{1}{\nu} \sqrt{r^2 + s^2 \sin^2 \theta - 2sr \sin \theta \cos \phi} \right) \\ & (x) e^{-s} ds. \end{aligned} \quad (9)$$

The addition theorem (26, V. 2, p. 102)

$$I_n \left(\frac{1}{\kappa} \sqrt{r^2 + t^2 - 2rt \cos \phi} \right) = \sum_{-\infty}^{\infty} (-1)^n I_n(r/\kappa) I_n(t/\kappa) e^{in\phi} \quad (10)$$

and Eq. (5.1-8) transform the integrals over s to

$$\int_0^{s_0} I_0 \left(\frac{1}{\kappa} \sqrt{r^2 + s^2 \sin^2 \theta - 2sr \sin \theta \cos \phi} \right) e^{-s} ds = \frac{\kappa}{2\pi} \sum_{-\infty}^{\infty} (-1)^n I_n(r/\kappa)$$

$$(x) \left[\int_0^{2\pi} \frac{e^{-in\gamma}}{\kappa - \sin \theta \cos(\phi + \gamma)} \left(1 - \exp \left[s_0 \left(\frac{\sin \theta \cos(\phi + \gamma)}{\kappa} - 1 \right) \right] \right) d\gamma \right] \quad (11)$$

Replacing γ by $\gamma - \phi$ and using Eq. (4.2-11), we can rewrite the right side of Eq. (11) as

$$\frac{\kappa}{2\pi} \int_0^{2\pi} \frac{d\gamma}{\kappa - \sin \theta \cos \gamma}$$

$$(x) \left[\exp \left\{ -r \cos(\phi + \gamma)/\kappa \right\} - e^{-s_0} \exp \left[-\frac{1}{\kappa} \left\{ r \cos(\phi + \gamma) - s_0 \sin \theta \cos \gamma \right\} \right] \right] \quad (12)$$

But

$$r \cos(\phi + \gamma) - s_0 \sin \theta \cos \gamma = -\cos \gamma (s_0 \sin \theta - r \cos \phi) - r \sin \phi \sin \gamma. \quad (12a)$$

Also, from Fig. 10 it follows that

$$r \sin \phi = OF = R \sin \phi_0; \quad \frac{\pi}{2} \leq \phi_0 \leq \frac{3\pi}{2} \quad (12b)$$

and

$$s_0 \sin \theta - r \cos \phi = FG = -R \cos \phi_0. \quad (12c)$$

Hence,

$$r \cos(\phi + \gamma) - s_0 \sin \theta \cos \gamma = R \cos(\phi_0 + \gamma). \quad (12d)$$

Substitution of these results in Eq. (9) gives

$$\psi(r, \theta, \phi) = a_0 \psi_0(r, \theta, \phi) + \int_0^1 A_C(\nu) d\nu P \int_0^{2\pi} \frac{e^{-r \cos(\phi + \gamma)/\nu}}{\nu - \sin \theta \cos \gamma} d\gamma$$

$$- e^{-s_0} \left[a_0 \psi_0(R, \theta, \phi_0) + \int_0^1 A_C(\nu) d\nu P \int_0^{2\pi} \frac{e^{-R \cos(\phi_0 + \gamma)}}{\nu - \sin \theta \cos \gamma} d\gamma \right], \quad (13)$$

where $\psi_0(r, \theta, \phi)$ is given by Eq. (5.2-10a). Adding and subtracting the invariant term

$$\begin{aligned} & \frac{1}{(2\pi)^2} \lambda(\nu) \int_0^{2\pi} \delta(\sin\theta \cos\gamma - \nu) e^{-r \cos(\phi + \gamma)/\nu} d\gamma \\ & \equiv \frac{1}{(2\pi)^2} \lambda(\nu) e^{-s_0} \int_0^{2\pi} \delta(\sin\theta \cos\gamma - \nu) e^{-R \cos(\phi_0 + \gamma)/\nu} d\gamma, \end{aligned} \quad (14)$$

to the last integrand, we get finally

$$\begin{aligned} \psi(r, \theta, \phi) = & a_0 \psi_0(r, \theta, \phi) + \int_0^1 A_C(\nu) \psi_\nu(r, \theta, \phi) d\nu \\ & + e^{-s_0} \left[a_0 \psi_0(R, \theta, \phi_0) + \int_0^1 A_C(\nu) \psi_\nu(R, \theta, \phi_0) d\nu \right]. \end{aligned} \quad (15)$$

Since the term in the bracket vanishes by Eq. (5.3-8), this expression for $\psi(r, \theta, \phi)$ is identical with Eq. (5.0-3).

5.5 CYLINDRICAL TRANSFORMATION

With the change of variables $y = 1/\mu$, the integral equation (5.4-6) for $\rho(r)$ takes the form

$$\begin{aligned} \rho(r) = & c \int_0^1 \frac{d\mu}{\mu^2} \\ (x) \left[\int_0^r t \rho(t) K_0(r/\mu) I_0(t/\mu) dt + \int_r^R t \rho(t) K_0(t/\mu) I_0(r/\mu) dt \right]. \end{aligned} \quad (1)$$

This equation differs from its counterparts in plane and spherical geometries in the fact that the kernel here involves products of Bessel functions instead of exponentials. It is still possible, however, to introduce a suitable transform for $\rho(r)$, which converts Eq. (1) into a separable integrodifferential equation.

Following the idea outlined at the end of Sec. 4.5, we start by writing Eq. (1) as

$$\rho(r) = c \int_t^R t \rho(t) K(r, t) dt; \quad 0 \leq t \leq R, \quad (2)$$

where

$$K(r, t) = \int_0^1 \frac{1}{\mu^2} G(r, t; \mu) d\mu \quad (2a)$$

and

$$G(r, t; \mu) = \begin{cases} K_0(r/\mu) I_0(t/\mu); & r \geq t \\ K_0(t/\mu) I_0(r/\mu); & r \leq t. \end{cases} \quad (2b)$$

The function $G(r, t; \mu)$ is now identified as the Green's function of the operator $[d^2/dr^2 + 1/r(d/dr) - 1/\mu^2]$ (31, p. 826). In terms of G , Eq. (1) is

$$\rho(r) = c \int_0^1 \frac{d\mu}{\mu^2} \int_0^R t \rho(t) G(r, t; \mu) dt. \quad (3)$$

Introduce the function $\Phi(r, \mu)$ by

$$\Phi(r, \mu) = c \int_0^R t \rho(t) G(r, t; \mu) dt \quad (4a)$$

$$\begin{aligned} &= c \int_0^r t \rho(t) K_0(r/\mu) I_0(t/\mu) dt \\ &+ c \int_r^R t \rho(t) K_0(t/\mu) I_0(r/\mu) dt. \end{aligned} \quad (4b)$$

Multiplying (4a) by $(1/\mu^2)$ and integrating over μ , we get

$$\rho(r) = \int_0^1 \frac{1}{\mu^2} \Phi(r, \mu) d\mu. \quad (5)$$

The functions $\rho(r)$ and $\Phi(r, \mu)$ therefore form a transform pair. In addition to the fact that if Φ is known the neutron density can be determined by a simple integration, Eq. (5) suggests that $\Phi(r, \mu)$ can be interpreted as a pseudo neutron distribution taking the place of $\psi(r, \theta, \phi)$.

Use of the integral (5.4-7) reduces Eq. (4b) to

$$\Phi(r, \mu) = c \mu^2 \int_0^R t \rho(t) dt \int_0^\infty \frac{J_0(tx) J_0(rx)}{x^2 \mu^2 + 1} dx, \quad (6)$$

from which it follows that

$$\Phi(-r, \mu) = \Phi(r, \mu); \quad \Phi(r, -\mu) = \Phi(r, \mu). \quad (7)$$

Furthermore, from Eq. (4a) we conclude that $\Phi(r, \mu)$ obeys an inhomogeneous differential equation where the operator is associated with $G(r, t; \mu)$ and the inhomogeneity is proportional to $\rho(r)$. Differentiation of Eq. (4b) with respect to r and use of the relation (24, p. 80)

$$K_{n+1}(z)I_n(z) + I_{n+1}(z)K_n(z) = 1/z \quad (8)$$

yields

$$\begin{aligned} \frac{\partial^2 \Phi(r, \mu)}{\partial r^2} + \frac{1}{r} \frac{\partial \Phi(r, \mu)}{\partial r} - \frac{1}{\mu^2} \Phi(r, \mu) &= -c \rho(r) \\ &= -c \int_0^1 \frac{1}{(\mu')^2} \Phi(r, \mu') d\mu'. \end{aligned} \quad (9)$$

The boundary conditions for this equation also follow from the definition of $\Phi(r, \mu)$. Thus, setting $r = 0$ in Eq. (4b) and observing that the first term vanishes since $K_0(r)$ diverges as $\ln r$, whereas the integral goes to zero as r , we find that $\Phi(0, \mu)$ is finite at the origin. Also, at $r = R$,

$$K_0(R/\mu) \Phi'(R, \mu) + \frac{1}{\mu} K_1(R/\mu) \Phi(R, \mu) = 0, \quad (10)$$

where the prime indicates differentiation with respect to r .

These two conditions and Eq. (9) suffice to define $\Phi(r, \mu)$ completely. Moreover, it is now possible to determine a complete set of eigenfunctions for this density transform. For let us assume that the solutions of Eq. (9) have the form

$$\Phi_\nu(r, \mu) = R_\nu(r) \eta_\nu(\mu). \quad (11)$$

Then, substituting Eq. (11) in (9) and separating the r and μ dependence, we obtain

$$R_\nu''(r) + \frac{1}{r} R_\nu'(r) - \frac{1}{\nu^2} R_\nu(r) = 0 \quad (12)$$

and

$$\left(\frac{1}{\mu^2} - \frac{1}{\nu^2} \right) \eta_\nu(\mu) = c \int_0^1 \frac{1}{(\mu')^2} \eta_\nu(\mu') d\mu'; \quad \mu > 0, \quad (13)$$

where $(1/\nu^2)$ is the separation constant.

The solution of the zeroth-order Bessel equation (12) which remains finite at the origin is

$$R_\nu(r) = I_0(r/\nu). \quad (14)$$

If we normalize $\eta_\nu(\mu)$ by taking

$$\int_0^1 \frac{1}{\mu^2} \eta_\nu(\mu) d\mu = 1, \quad (15)$$

we can write Eq. (13) as

$$(\nu^2 - \mu^2) \eta_\nu(\mu) = c \nu^2 \mu^2; \quad 0 \leq \mu \leq 1. \quad (16)$$

For ν in the interval $[-1, 1]$, there are two singularities at the points $\nu = \pm \mu$. The general solution of (16) is therefore

$$\eta_\nu(\mu) = c P \frac{\nu^2 \mu^2}{\nu^2 - \mu^2} + \xi(\nu) \delta(\mu - \nu) + \xi(\nu) \delta(\mu + \nu), \quad (16a)$$

where P denotes the Cauchy principal value of integrals involving this term and $\zeta(\nu)$ and $\xi(\nu)$ are arbitrary. But from Eq. (16) we see that $\eta_\nu(\mu) = \eta_\nu(-\mu)$. Applying this symmetry in Eq. (16a) we get $\zeta(\nu) = \xi(\nu)$. Replacement of these functions by $\nu^2 \lambda(\mu)$ then yields

$$\eta_\nu(\mu) = c P \frac{\nu^2 \mu^2}{\nu^2 - \mu^2} + \nu^2 \lambda(\nu) [\delta(\mu - \nu) + \delta(\mu + \nu)]; \mu > 0. \quad (17)$$

For ν not in $[-1, 1]$, the normalization (15) gives $\nu = \pm \nu_0$, where ν_0 is defined by Eq. (2.1-9). The corresponding eigenfunction is

$$\eta_0(\mu) = \frac{c \nu_0^2 \mu^2}{\nu_0^2 - \mu^2}. \quad (18)$$

For $\nu \in [-1, 1]$ substitution of Eq. (17) in (15) shows that $\lambda(\nu)$ is the same as the quantity defined by Eq. (2.1-11).

Thus we have again a discrete spectrum comprising the single term $\eta_0(\mu)$ and a continuum $\eta_\nu(\mu)$. Furthermore, this set of solutions has the same completeness property as that proved for the plane eigenfunctions $\phi_\nu(\mu)$ by Case. In fact, this property can be proved simply by comparing an expansion in $\eta_\nu(\mu)$ with one in $\phi_\nu(\mu)$. To this end, consider the expansion for an even function $\Phi(\mu)$ defined in the interval $-1 \leq \mu \leq 1$:

$$\Phi(\mu) = a_0 \eta_0(\mu) + \int_{-1}^1 \bar{A}(\nu) \eta_\nu(\mu) d\nu, \quad (19a)$$

or, because $\eta_{-\nu}(\mu) = \eta_\nu(\mu)$,

$$\Phi(\mu) = a_0 \eta_0(\mu) + \int_0^1 A(\nu) \eta_\nu(\mu) d\nu, \quad (19b)$$

with $A(\nu) = \bar{A}(\nu) + \bar{A}(-\nu)$. We wish to show that for given $\Phi(\mu)$, the coefficients $A(\nu)$ in (19b) exist and can be uniquely determined.

From Eqs. (17) and (18) it follows that

$$\eta_0(\mu) = \mu^2 [\phi_{0+}(\mu) + \phi_{0-}(\mu)] \quad (20a)$$

and

$$\eta_\nu(\mu) = \mu^2 [\phi_\nu(\mu) + \phi_{-\nu}(\mu)], \quad (20b)$$

where $\phi_{0\pm}(\mu)$ and $\phi_\nu(\mu)$ are the plane expansion modes given by Eqs. (3.1-2b) and (3.1-3b), respectively. Then,

$$\Phi(\mu) = \mu^2 \left\{ a_0 [\phi_{0+}(\mu) + \phi_{0-}(\mu)] + \int_0^1 A(\nu) [\phi_\nu(\mu) + \phi_{-\nu}(\mu)] d\nu \right\}. \quad (21)$$

But Eq. (21) is simply the expansion of the even function $\psi(\mu) = \Phi(\mu) \mu^{-2}$ in terms of the complete set $\phi_\nu(\mu)$. By the theorem of Case (1), a unique set of coefficients a_0 and $A(\nu)$ exists. The expansion modes $\eta_0(\mu)$ and $\eta_\nu(\mu)$ are therefore also complete.

Returning to the cylindrical density transform, we can now express the general solution of Eq. (9) by the expansion

$$\Phi(r, \mu) = b_0 \eta_0(\mu) I_0(r/\nu_0) + \int_{-1}^1 \bar{B}(\nu) \eta_\nu(\mu) I_0(r/\nu) d\nu \quad (22a)$$

or else

$$\Phi(r, \mu) = b_0 \eta_0(\mu) I_0(r/\nu_0) + \int_0^1 B(\nu) \eta_\nu(\mu) I_0(r/\nu) d\nu, \quad (22b)$$

with $B(\nu) = \bar{B}(\nu) + \bar{B}(-\nu)$. Multiplying this by $(1/\mu^2)$ and integrating over μ , we obtain

$$\rho(r) = b_0 I_0(r/\nu_0) + \int_0^1 B(\nu) I_0(r/\nu) d\nu \quad (23)$$

because of Eqs. (5) and (15). Moreover, a comparison of Eq. (23) with

$$\rho(r) = a_0 I_0(r/\nu_0) + \int_0^1 A_C(\nu) I_0(r/\nu) d\nu \quad (5.2-35)$$

suggests that

$$a_0 = b_0; \quad A_C(\nu) = B(\nu). \quad (24)$$

The procedure for constructing a rigorous proof of this identity between the two sets of cylindrical expansion coefficients is completely analogous to that used in Sec. 4.5 for the spherical coefficients. Thus, if we insert Eq. (23) in the integral equation (5.4-3) and proceed as in Sec. 5.4, we shall find that $B(\nu)$ must satisfy Eq. (5.3-8) which defines $A_c(\nu)$.

From the results of this section we can therefore draw two conclusions. First, by introducing a suitable density transform we can treat problems with cylindrical symmetry using Case's method of singular expansion modes. Secondly, from the completeness property of $\eta_\nu(\mu)$ and Eq. (24) it follows that the representations for the cylindrical distribution $\psi(r, \theta, \phi)$ developed in Sec. 5.2 are unique if the boundary condition is $\psi(R, \theta, \phi_0) = 0$.

5.6 THE CYLINDRICAL CRITICAL PROBLEM

The complete determination of the neutron distribution in a multiplying circular cylinder of radius R and infinite height involves the determination of the expansion coefficients $A_c(\nu)$ and a criticality condition. In view of the results of the preceding section, this problem is equivalent to finding $B(\nu) = A_c(\nu)$ in the expansion for $\Phi(r, \mu)$ by applying the condition (5.5-10). Although the mixed character of this condition introduces some complications in the simplification of the final results, the procedure of finding a Fredholm integral equation of the second kind as well as an auxiliary condition for $A_c(\nu)$, which are adaptable to systematic approximations, is the same as that discussed in Sec. 3.3a. In particular, the zeroth-order approximations for the critical radius again submits to a comparison with the asymptotic critical result.

5.6a Determination of the Expansion Coefficients

In terms of the coefficients $A_c(\nu)$, the general expansion for $\Phi(r, \mu)$ is

$$\Phi(r, \mu) = a_0 \eta_0(\mu) I_0(r/\nu_0) + \int_0^1 A_C(\nu) \eta_\nu(\mu) I_0(r/\nu) d\nu. \quad (1)$$

Evaluating Φ and its first derivative at $r = R$ and introducing these values in the condition

$$K_0(R/\mu) \Phi(R, \mu) + \frac{1}{\mu} K_1(R/\mu) \Phi(R, \mu) = 0; \quad \mu > 0, \quad (5.5-10)$$

we obtain

$$\begin{aligned} a_0 \eta_0(\mu) & \left[\frac{1}{\nu_0} K_0(R/\mu) I_1(R/\nu_0) + \frac{1}{\mu} K_1(R/\mu) I_0(R/\nu_0) \right] \\ & + \int_0^1 A_C(\nu) \eta_\nu(\mu) \left[\frac{1}{\nu} K_0(R/\mu) I_1(R/\nu) + \frac{1}{\mu} K_1(R/\mu) I_0(R/\nu) \right] d\nu = 0. \end{aligned} \quad (2)$$

Let

$$q(\nu, \mu) = R \left[\frac{1}{\nu} K_0(R/\mu) I_1(R/\nu) + \frac{1}{\mu} K_1(R/\mu) I_0(R/\nu) \right]. \quad (3)$$

From Eq. (5.5-8) it follows that $q(\nu, \nu) = 1$. With this notation Eq. (2) assumes the form

$$a_0 \eta_0(\mu) q(\nu_0, \mu) + \int_0^1 A_C(\nu) \eta_\nu(\mu) q(\nu, \mu) d\nu = 0; \quad \mu > 0. \quad (4)$$

If in Eq. (4) we insert Eqs. (5.5-20a) and (5.5-20b) for $\eta_0(\mu)$ and $\eta_\nu(\mu)$, respectively, and recall that both ν and μ are non-negative, we arrive at the singular integral equation

$$\begin{aligned} \lambda(\mu) A_C(\mu) + P \int_0^1 \frac{\left(\frac{c}{2}\right) \nu A_C(\nu) q(\nu, \mu)}{\nu - \mu} d\nu = \\ -a_0 [\phi_{0+}(\mu) + \phi_{0-}(\mu)] q(\nu_0, \mu) - \int_0^1 \frac{\left(\frac{c}{2}\right) \nu A_C(\nu) q(\nu, \mu)}{\nu + \mu} d\nu. \end{aligned} \quad (5)$$

This equation can be cast in the form considered in Sec. 3.3a (16) by introducing the function

$$H(\nu, \mu) = \frac{q(\nu, \mu) - q(\nu, \nu)}{\nu - \mu}, \quad (6)$$

so that

$$q(\nu, \mu) = 1 + (\nu - \mu) H(\nu, \mu), \quad (6a)$$

since $q(\nu, \nu) = 1$. Then Eq. (5) can be written as

$$\lambda(\mu) A_c(\mu) + P \int_0^1 \frac{\left(\frac{c}{2}\right) \nu A_c(\nu)}{\nu - \mu} d\nu = \Phi'(\mu); \quad \mu > 0, \quad (7)$$

where $\Phi'(\mu)$ is the nonsingular part defined by

$$\begin{aligned} \Phi'(\mu) = & -a_0 [\phi_{0+}(\mu) + \phi_{0-}(\mu)] q(\nu_0, \mu) \\ & - \int_0^1 \frac{\left(\frac{c}{2}\right) \nu A_c(\nu) [2\nu H(\nu, \mu) + 1]}{\nu + \mu} d\nu. \end{aligned} \quad (7a)$$

By the method of Sec. 3.3a, Eq. (7) can be reduced to the Fredholm type of integral equation:

$$A_c(\mu) = \lambda(\mu) g(c, \mu) \Phi'(\mu) - \frac{1}{X^-(\mu) \lambda^+(\mu)} \int_0^1 \frac{\gamma(\nu) \Phi'(\nu)}{\nu - \mu} d\nu \quad (8)$$

and the auxiliary condition

$$\int_0^1 \gamma(\mu) \Phi'(\mu) d\mu = 0. \quad (9)$$

Thus, the cylindrical expansion coefficients $A_c(\nu)$ are given as the solutions of the integral equation (8), whereas Eq. (9) represents an exact statement of the criticality condition for this geometry.

Further simplifications of these two expressions, which were made in the corresponding expressions in the previous two chapters, are not possible because of the presence of the function $q(\nu, \mu)$. However, successive approximations for $A_c(\nu)$ and the critical dimensions can be determined from these results by the procedure of Sec. 3.3.b. For illustrations we shall examine the zeroth-order approximation which is equivalent to the neglect of the contribution from the continuous solutions.

Setting $A_c(\nu) = 0$ in Eq. (7a), we get

$$\Phi'_0(\mu) = -a_0 [\phi_{0+}(\mu) + \phi_{0-}(\mu)] q(\nu_0, \mu). \quad (10)$$

Substitution of Eq. (10) in (9) gives

$$I_0(R_0/\nu_0) + (1/\nu_0) S(R_0, \nu_0) I_1(R_0/\nu_0) = 0, \quad (11)$$

where

$$S = Q_1/Q_2;$$

$$Q_1(R_0, \nu_0) = \int_0^1 \frac{\gamma(\mu) K_0(R_0/\mu)}{\nu_0^2 - \mu^2} d\mu; \quad Q_2(R_0, \nu_0) = \int_0^1 \frac{\gamma(\mu) K_1(R_0/\mu)}{\mu(\nu_0^2 - \mu^2)} d\mu, \quad (11a)$$

and R_0 denotes the critical cylinder radius for $A_c(\nu) = 0$. From the asymptotic properties of $K_n(z)$ it also follows that the function $S(R_0, \nu_0)$ can be approximated by unity. Hence, for large systems, the critical relation (11) reduces to

$$I_0(R_0/\nu_0) + \frac{1}{\nu_0} I_1(R_0/\nu_0) \approx 0. \quad (12)$$

Higher estimates can be readily constructed by the following iteration scheme:

$$A_c^0(\mu) = 0;$$

$$A_c^{n+1}(\mu) = \lambda(\mu) g(c, \mu) \Phi'_n(\mu) - \frac{1}{X^-(\mu) \lambda^+(\mu)} \int_0^1 \frac{\gamma(\nu) \Phi'_n(\nu)}{\nu - \mu} d\nu, \quad (13)$$

where, by Eqs. (7a) and (10),

$$\Phi_n'(\mu) = \Phi_0'(\mu) - \int_0^1 \frac{\left(\frac{c}{2}\right) \nu A_c^n(\nu) [2H(\nu, \mu) + 1]}{\nu + \mu} d\nu. \quad (14)$$

5.6b Asymptotic Critical Radius

In the zeroth approximation of the preceding section, the neutron distribution is described by the discrete solutions only:

$$\psi_d(r, \theta, \phi) = a_0 \psi_0(r, \theta, \phi) \quad (15)$$

and

$$\rho_d(r) = a_0 I_0(r/\nu_0). \quad (16)$$

Imposing the condition of no re-entrant current:

$$\begin{aligned} j_d^-(R) &\equiv \int_0^\pi d\theta \int_{\pi/2}^{3\pi/2} \psi_d(R, \theta, \phi_0) \sin^2 \theta \cos \phi_0 d\phi_0 \\ &= 0 \end{aligned} \quad (17)$$

on Eq. (15) and using the form of Eq. (5.2-1) for $\psi_d(r, \theta, \phi)$, we obtain

$$\begin{aligned} \frac{a_0 c \nu_0}{4\pi} \sum_{n=0}^{\infty} \sum_{m=0}^n (-1)^m \epsilon_m (2n+1) \frac{(n-m)!}{(n+m)!} I_m(R/\nu_0) Q_n(\nu_0) P_n^m(0) \\ (x) \int_{-1}^1 (1-\mu^2)^{1/2} P_n^m(\mu) d\mu \int_{\pi/2}^{3\pi/2} \cos m \phi_0 \cos \phi_0 d\phi_0 = 0. \end{aligned} \quad (17a)$$

But

$$\int_{\pi/2}^{3\pi/2} \cos m \phi_0 \cos \phi_0 d\phi_0 = \frac{\pi}{2} \delta_{m1} + (-1)^k \frac{2}{(2k+1)(2k-1)} \delta_{2k,m}. \quad (18)$$

Inserting Eq. (18) in (17a), we get

$$(1-c)\pi I_1(R/\nu_0) = c \sum_{\ell=0}^{\infty} \sum_{k=0}^{\ell} (-1)^k \epsilon_{2k} \frac{(4\ell+1)}{(2k+1)(2k-1)} \frac{(2\ell-2k)!}{(2\ell+2k)!} C_{\ell}^k \\ (x) P_{2\ell}^{2k}(0) Q_{2\ell}(\nu_0) I_{2k}(R/\nu_0), \quad (19)$$

where

$$C_{\ell}^k = \int_{-1}^1 (1-\mu^2)^{1/2} P_{2\ell}^{2k}(\mu) d\mu \\ = \frac{1}{2^{2\ell}} \sum_{t=0}^{\ell-k} (-1)^{t+\ell+k} \frac{\Gamma(2t+2\ell+2k+1) \Gamma(k+\frac{3}{2}) \Gamma(t+\frac{1}{2})}{\Gamma(t+\ell+k+1) \Gamma(\ell-t-k+1) \Gamma(2t+1) \Gamma(t+k+2)}. \quad (19a)$$

This integral was evaluated by putting the definition (23, p. 111)

$$P_{2\ell}^{2k}(\mu) = \frac{(1-\mu^2)^k}{2^{2\ell}(2\ell)!} \frac{d^{2\ell+2k}(\mu^2-1)^{2\ell}}{d\mu^{2\ell+2k}} \\ = \frac{(1-\mu^2)^k}{2^{2\ell}} \sum_{s=\ell+k}^{2\ell} (-1)^s \frac{\Gamma(2s+1)(\mu)^{2s-2\ell-2k}}{\Gamma(s+1) \Gamma(2s-2\ell-2k+1) \Gamma(2\ell-s+1)} \quad (19b)$$

in (19a), performing the resulting integral of the trigonometric functions and rearranging the final series.¹

For c close to unity, ν_0 becomes large and the double series (19) converges rapidly because of the factor $Q_{2\ell}(\nu_0)$, which goes like $(1/\nu_0)^{2\ell+1}$. Keeping only the $\ell = 0$ term, we find

$$I_0(R'_0/\nu_0) + 2\nu_0(1-c) I_1(R'_0/\nu_0) \simeq 0, \quad (20)$$

where R'_0 is the critical radius in these approximations. Equation (20) can also be written in the familiar form

$$\frac{1}{4} \rho_d(R'_0) + \frac{D}{2} \left[\frac{d\rho_d(r)}{dr} \right]_{r=R'_0} \simeq 0, \quad (21)$$

¹General integrals of this form have been considered by W. W. Clendenin (30).

where $D = \nu_0^2(1 - c)$. Since for $(c - 1) \ll 1$, $D \rightarrow 1/3$, it is seen by comparing Eqs. (20) and (12) that these two approximate results differ by a factor of $2/3$ in the coefficient of $I_1(R/\nu_0)$.

CHAPTER VI

SUMMARY AND CONCLUSIONS

We have shown that the stationary solutions of the monoenergetic transport equation with isotropic scattering applicable to uniform multiplying media in plane, spherical, and cylindrical geometries can be represented uniquely by an expansion of the form

$$\psi = a_0\psi_0 + \int_0^1 A(\nu)\psi_\nu d\nu.$$

The expansion modes ψ_0 and ψ_ν can be deduced by first establishing the elementary solutions of the transport equation in three dimensions and then specializing them to obtain solutions with the desired symmetry. We have further demonstrated that the powerful method of singular integral equations, employed by Case, can be extended to the spherical and cylindrical critical problems by judicious choice of density transforms.

In Chapter III, the specialization of the three-dimensional solutions to plane geometry resulted in the general expansion for $\psi(x, \mu)$ given by Case (1). The application of the condition $\psi(-b, \mu > 0)$ to this expansion then led to a Fredholm integral equation of the second kind for the coefficients $A(\nu)$ as well as an exact criticality condition which depended on $A(\nu)$. Successive iterations of the integral equation by means of a Neumann series were illustrated both analytically and numerically. The results of diffusion theory with the Milne-problem extrapolation distance emerged as the zeroth-order approximation of the Neumann series. The contribution of the first-order transport correction remained small for the critical thickness, but became significant for the angular density and neutron density in systems with dimensions of the order of one mean free path or less.

The investigation of the plane transport integral equation, by generalizing the transform method of Leonard and Mullikin (15), produced a singular integral equation for $\psi(x, \mu)$ where the space variable x played

the role of a parameter. Although the complete identity between the solutions generated by this singular equation and the results from the normal mode expansion was not established here, the approximations for the neutron distribution and the critical thickness were shown to be consistent.

The spherical and cylindrical expansions resulting from the three-dimensional transport solutions were represented, in Chapters IV and V, respectively, in various equivalent forms. For the determination of the expansion coefficients in these geometries, however, we were forced to reformulate the critical problems by introducing suitable density transforms and utilizing the properties of the respective transport integral equations. In the process, we demonstrated a method of solving integral equations whose kernels can be expressed as integrals involving known Green's functions.

In Chapter IV we found that it was possible, although not necessary, to introduce a pseudo distribution which satisfied the transport equation in plane geometry. The spherical critical problem could then be treated by the methods of Chapter III. In the case of the infinite circular cylinder, the appropriate density transform obeyed a second-order integrodifferential equation with mixed boundary conditions. The completeness of the singular expansion modes generated by this equation was then proved by comparison with the known complete set in plane geometry. Furthermore, the equation for the cylindrical expansion coefficients was also reduced to an integral equation which submitted to systematic approximations. Thus the transform approach enabled us to extend the method of plane geometry to the two non-planar geometries considered.

The correspondence of the treatment of the plane and spherical critical problems was made possible by the similarity in the integral equations for the neutron densities in these two geometries. Although this similarity has been noted and used previously (12,15), its

full impact on the development of spherical solutions was realized here by the introduction of the density transform $\Phi(r, \mu)$. Of particular interest in this treatment is the reappearance of the Milne-problem extrapolation distance $z_0(c)$ as the appropriate extrapolation distance in the zeroth-order expression for the critical sphere diameter. It is tempting to conclude from this that z_0 is the correct extrapolation distance for asymptotic criticality calculations independent of geometry. We feel, however, that the reappearance of this quantity in spherical geometry is more a consequence of the particular method used than any fundamental significance of $z_0(c)$. Thus, if instead of $\Phi(r, \mu)$ we had used the alternative spherical transform $F(r, \mu)$, discussed at the end of Sec. 4.5, z_0 would not enter in the critical sphere dimensions. Furthermore, although the use of $F(r, \mu)$ leads to more cumbersome analysis because of the mixed boundary conditions associated with its equation, it is possible that the accuracy of the spherical quantities could be improved by using $F(r, \mu)$ rather than $\Phi(r, \mu)$.

It is, perhaps, a truism to state that an investigation of this type creates or suggests more problems than it solves. Among the areas which require more attention is the possibility of exhibiting complete three-dimensional transport solutions either by the method of superposition or through transformations of the integral equation. In particular, the generality of the solutions given in Chapter II and the removal of the limitations associated with their failure to generate the non-regular transport solutions and with the absence of a method for choosing the expansion coefficients to satisfy given side conditions deserves further study.

The successful use of particular density transforms here also suggests the extension of this approach to multigroup and multiregion problems or to the determination of anisotropic distributions in non-planar geometries. Furthermore, recent reports on the adaptation of Case's singular modes method to energy dependent problems (32,33)

and to time-dependent neutron thermalization problems (34,35) indicate that the restrictions of time- and energy-independent neutron distributions can be relaxed or eliminated. Finally, the generalization of related transport problems as suggested by Shure (11, p. 109) can now be visualized.

Thus, while the main effort of this work was devoted to the determination of the neutron distributions and criticality conditions for monoenergetic critical problems, it is expected that the techniques explored and developed here will be found useful in widening further the rather exclusive class of transport problems which can be treated exactly.

A. APPENDIX TO CHAPTER II

1A. PROOF OF EQUATION (2.1-16a)

In obtaining Eq. (2.1-17) for the neutron current, we used the fact that

$$\underline{R} = \frac{c\nu_0}{2} \int_{\underline{\Omega}} \frac{\underline{\Omega} d\Omega}{\nu_0 - \underline{u} \cdot \underline{\Omega}} = 2\pi\nu_0(1-c)\underline{u}. \quad (1)$$

Proof. By Eq. (4.1-6), the denominator of Eq. (1) can be written as

$$\frac{1}{\nu_0 - \underline{u} \cdot \underline{\Omega}} = 4\pi \sum_{n\gamma} Q_n(\nu_0) Y_n^\gamma(\underline{u}) Y_n^{\gamma*}(\underline{\Omega}). \quad (4.1-6)$$

Also, if $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ denotes a set of orthonormal base vectors, then

$$\underline{\Omega} = \Omega_1 \underline{e}_1 + \Omega_2 \underline{e}_2 + \Omega_3 \underline{e}_3, \quad (2a)$$

where

$$\begin{aligned} \Omega_1 &= \sin \theta \cos \phi = \sqrt{\frac{2\pi}{3}} \left[Y_1^1(\underline{\Omega}) + Y_1^{-1}(\underline{\Omega}) \right]; \\ \Omega_2 &= \sin \theta \sin \phi = \frac{1}{i} \sqrt{\frac{2\pi}{3}} \left[Y_1^1(\underline{\Omega}) - Y_1^{-1}(\underline{\Omega}) \right]; \\ \Omega_3 &= \cos \theta = \sqrt{\frac{4\pi}{3}} Y_1^0(\underline{\Omega}). \end{aligned} \quad (2b)$$

Substitution in Eq. (1) and use of the orthogonality of the spherical harmonics gives

$$\begin{aligned} \underline{R} &= 2\pi c\nu_0 Q_1(\nu_0) \left[\sqrt{\frac{2\pi}{3}} \left(Y_1^1(\underline{u}) + Y_1^{-1}(\underline{u}) \right) \underline{e}_1 + \frac{1}{i} \sqrt{\frac{2\pi}{3}} \left(Y_1^1(\underline{u}) - Y_1^{-1}(\underline{u}) \right) \underline{e}_2 + \sqrt{\frac{4\pi}{3}} Y_1^0(\underline{u}) \underline{e}_3 \right] \\ &= 2\pi c\nu_0 Q_1(\nu_0) \underline{u}, \end{aligned} \quad (3)$$

which completes the proof.

2A. REMARKS ON THE SUPERPOSITION SOLUTIONS

Regular spherical and cylindrical solutions have been constructed in Chapter II by superposing fundamental solutions of the form

$$\psi_{\underline{k}}(\underline{r}, \underline{\Omega}) = e^{-\underline{r} \cdot \underline{u}/\nu} \phi_{\nu}(\underline{u} \cdot \underline{\Omega}), \quad (1)$$

where $\underline{u}$ was assumed to range over the surface of a unit sphere. It was further pointed out at the end of Sec. 2.1 that such restriction on the range of $\underline{u}$ is artificial and could well be the cause for the failure of this method to generate non-regular transport solutions.

A formal generalization of the analysis in Chapter II can be obtained by considering $\underline{u}$ as representing the three components

$$u_1 = \sin \alpha \cos \beta; u_2 = \sin \alpha \sin \beta; u_3 = \cos \alpha, \quad (2)$$

where α and β can take complex values:

$$\alpha = \alpha_1 + i\alpha_2; \beta = \beta_1 + i\beta_2. \quad (2a)$$

Substituting of Eq. (1) in the transport equation and retaining the normalization (2.1-5) for $\phi_{\nu}(\underline{u} \cdot \underline{\Omega})$, we find again that

$$(\nu - \underline{u} \cdot \underline{\Omega}) \phi_{\nu}(\underline{u} \cdot \underline{\Omega}) = \frac{c}{2} \nu. \quad (3)$$

Until the range of α and β are further restricted, the quantity

$$\underline{u} \cdot \underline{\Omega} = \cos \alpha \cos \theta + \sin \alpha \sin \theta \cos(\beta - \phi) \quad (4)$$

remains unspecified, and the domains of a discrete and a continuous spectrum are not evident from Eq. (3).

Assuming that values of ν exist such that Eq. (3) gives a continuous and a discrete solution according to whether ν can equal $\underline{u} \cdot \underline{\Omega}$ or not, we find that $\phi_{\nu}(\underline{u} \cdot \underline{\Omega})$ is again given by the expressions of Sec. 2.1. The normalization integral

$$\frac{1}{2\pi} \int_{\underline{\Omega}} \phi_{\nu}(\underline{u} \cdot \underline{\Omega}) d\Omega = 1 \quad (2.1-5)$$

can now be evaluated with the aid of the expansion (4.1-6) - which holds for complex values of the angles (25, p.326) - and gives again the roots $\pm\nu_0$ of Eq. (2.1-9) for the two discrete points and Eq. (2.1-11) for $\lambda(\nu)$. Thus, apart from the fact that the range of ν is not as yet specified, $\psi(\underline{r}, \underline{\Omega})$ is given by an expansion of the form of Eq. (2.1-14) and the neutron density by

$$\rho(\underline{r}) = \frac{1}{2} \int_{\alpha} \int_{\beta} \left[a_{0+}(\underline{u}) e^{-\underline{r} \cdot \underline{u}/\nu_0} + a_{0-}(\underline{u}) e^{\underline{r} \cdot \underline{u}/\nu_0} + \int_{\nu} A(\nu, \underline{u}) e^{-\underline{r} \cdot \underline{u}/\nu} d\nu \right] \sin \alpha d\alpha d\beta \quad (5)$$

From the construction of the different solutions to the wave equation by a similar method (20, p.367), it is expected that the range of α and β in the above results can be chosen so as to produce transport solutions having the desired behavior. For illustration, we consider the following plausibility argument for arriving at spherical and cylindrical densities which are singular at $r = 0$ and vanish at infinity.

For the sphere we keep $0 \leq \beta \leq 2\pi$ in a coordinate system where $\underline{r}$ is along the z -axis. Then, since the spherically symmetric coefficients $a_{0\pm}$ and A are independent of α and β , Eq. (5) reduces to

$$\rho(r) = \pi \left[a_{0+} \int_a^b e^{-rs/\nu_0} ds + a_{0-} \int_a^b e^{rs/\nu_0} ds + \int_{\nu} A(\nu) d\nu \int_a^b e^{-rs/\nu} ds \right], \quad (6)$$

where $s = \cos \alpha$. The limits a and b (and, hence, the range of α) are now chosen by requiring each term in Eq. (6) and the corresponding neutron current term, say

$$j_{d+}(r) = \pi(1 - c) a_{0+} \int_a^b s e^{-rs/\nu_0} ds, \quad (6a)$$

to be consistent with the continuity equation

$$\frac{dj(r)}{dr} + \frac{2}{r} j(r) + (1 - c) \rho(r) = 0. \quad (4.2-26)$$

This requirement gives, for the first discrete terms,

$$\frac{2\nu_0}{r} \int_a^b s e^{-rs/\nu_0} ds + \int_a^b (1 - s^2) e^{-rs/\nu_0} ds = 0. \quad (7)$$

Integration of the second integral by parts reduces Eq. (7) to

$$\left[(1 - s^2) e^{-rs/\nu_0} \right]_a^b = 0. \quad (8)$$

This condition is clearly satisfied if, as in Sec. 2.1, $0 \leq \alpha \leq \pi$, so that $a = -1$ and $b = 1$. But for ν_0 real and greater than unity ($c < 1$), Eq. (8) will also be satisfied if $a = 1$ and $b = \infty$. The second discrete term in Eq. (6) diverges for such choice, so we must set $a_{0-} = 0$. The discrete part of the neutron density then assumes the form

$$\rho_d(r) = \frac{\pi a_{0+} \nu_0}{r} e^{-r/\nu_0}. \quad (9)$$

Using the result of Eq. (4.2-14b), we can write Eq. (4) as

$$\underline{u} \cdot \underline{\Omega} = s\mu + i\sqrt{(1 - \mu^2)(s^2 - 1)} \cos \beta \quad (10)$$

which, for $1 \leq s \leq \infty$ and $0 \leq \beta \leq 2\pi$, can take any value in the complex plane. Consequently, the range of ν in the continuous spectrum is still unspecified. But the integral

$$\int_{\nu} A(\nu) d\nu \int_1^{\infty} e^{-rs/\nu} ds$$

in Eq. (6) will converge for large r only if $\text{Re}(1/\nu) \geq 0$. To assure a convergence of this integral which is faster than the asymptotic part given by Eq. (9), we take $A(\nu) = 0$ except when $0 \leq \nu \leq 1$. Thus a possible expression for the spherical density possessing the desired behavior at the origin and infinity is

$$\rho(r) = \frac{\pi a_{0+} \nu_0}{r} e^{-r/\nu_0} + \pi \int_0^1 A_s(\nu) \frac{e^{-r/\nu}}{r} \nu d\nu. \quad (11)$$

It should be noted that Eq. (11) coincides with the expression for $\rho(r)$ obtained by replacing $(-1)^n i_n(r/\nu)$ by $k_n(r/\nu)$ in the results of Sec. 4.2 (see Eq. 4.3-5).

For the infinite cylinder we take again $\alpha = \pi/2$, but now let β be pure imaginary. To simplify the integration over $\underline{u}$ in Eq. (6), we choose a coordinate system where $\underline{r}$ is along the x-axis and $\underline{\Omega}$ lies on the xz-plane. Then Eq. (6) gives

$$\rho(r) = a_0 \int_t \frac{e^{-rt/\nu_0}}{\sqrt{t^2 - 1}} dt + a_1 \int_t \frac{e^{rt/\nu_0}}{\sqrt{t^2 - 1}} + \int_\nu A(\nu) d\nu \int_t \frac{e^{-rt/\nu}}{\sqrt{t^2 - 1}} dt, \quad (12)$$

where $t = \cosh \beta_2$. Also, Eq. (4) becomes in this case

$$\underline{u} \cdot \underline{\Omega} = t \sin \theta. \quad (13)$$

If β , ($\beta = i\beta_2$), can take any value on the imaginary axis, then $1 \leq t \leq \infty$. For these limits of t the second integral diverges (since $\nu_0 \geq 1$ in this case), and therefore we must have $a_1 = 0$. Moreover, from Eq. (13) it follows that $\underline{u} \cdot \underline{\Omega}$ lies on the entire real axis. Hence a continuum exists for $-\infty \leq \nu \leq \infty$. To guarantee a convergence of the integral over ν for large r and also avoid the overlapping of the continuum with the discrete term, we restrict ν to the interval $[0, 1]$ by taking $A(\nu) = 0$ for ν outside this interval. Then Eq. (12) becomes

$$\rho(r) = a_0 K_0(r/\nu_0) + \int_0^1 A_C(\nu) K_0(r/\nu) d\nu, \quad (14)$$

which agrees with the form of the singular solutions expected from the discussion following Eq. (5.3-7b).

The preceding two examples were considered in order to illustrate the possibility of constructing non-regular transport solutions by the superposition method as well as the difficulties in cases other than when α and β are restricted to the surface of a unit sphere.

B. APPENDIX TO CHAPTER III

1B. PROOF OF EQS. (3.2-2a) AND (3.2-2b)

In the derivation of the singular integral equations for $\psi(x, \mu)$ we made use of the identities

$$\begin{aligned} \int_{-b}^x \psi(y, \nu) \exp \left[-\left(\frac{x-y}{\mu}\right) \right] dy &= \frac{\mu}{\nu-\mu} [\nu \psi(x, \nu) - \mu \psi(x, \mu)] \\ &+ \frac{\mu}{\nu-\mu} \exp \left[-\left(\frac{b+x}{\mu}\right) \right] [\mu \psi(-b, \mu) - \nu \psi(-b, \nu)] \end{aligned} \quad (3.2-2a)$$

and

$$\begin{aligned} \int_{-b}^x \psi(y, -\nu) \exp \left[-\left(\frac{x-y}{\mu}\right) \right] dy &= \frac{\mu}{\nu+\mu} [\nu \psi(x, -\nu) + \mu \psi(x, \mu)] \\ &- \frac{\mu}{\nu+\mu} \exp \left[-\left(\frac{b+x}{\mu}\right) \right] [\nu \psi(b, \nu) + \mu \psi(-b, \mu)], \end{aligned} \quad (3.2-2b)$$

where $\psi(y, \pm\nu)$ are given by Eqs. (3.1-11a) and (3.1-11b), respectively, and ν and $\mu \in [0, 1]$.

Proof. Inserting

$$\begin{aligned} \psi(y, \nu) &= \exp \left[-\left(\frac{b+y}{\nu}\right) \right] \psi(-b, \nu) + \frac{c}{2} \frac{1}{\nu} \int_{-b}^y \rho(z) \\ &\quad (x) \exp \left[-\left(\frac{y-z}{\nu}\right) \right] dz \end{aligned} \quad (3.1-11a)$$

in Eq. (3.2-2a), we get

$$\begin{aligned} \int_{-b}^x \psi(y, \nu) \exp \left[-\left(\frac{x-y}{\mu}\right) \right] dy &= \psi(-b, \nu) \int_{-b}^x \exp \left[-\left(\frac{b+y}{\nu}\right) - \left(\frac{x-y}{\mu}\right) \right] dy \\ &+ \frac{c}{2} \frac{1}{\nu} \int_{-b}^x \exp \left[-\left(\frac{x-y}{\mu}\right) \right] dy \int_{-b}^y \rho(z) \exp \left[-\left(\frac{y-z}{\nu}\right) \right] dz. \end{aligned} \quad (1)$$

The first integral on the right of Eq. (1) yields

$$\frac{\nu\mu}{\nu-\mu} \psi(-b, \nu) \left[\exp \left[-\left(\frac{b+x}{\nu}\right) \right] - \exp \left[-\left(\frac{b+x}{\mu}\right) \right] \right]. \quad (1a)$$

In the second integral we interchange the order of the y and z integration:

$$\begin{aligned} & \frac{c}{2} \frac{1}{\nu} \int_{-b}^x \exp \left[-\left(\frac{x-y}{\mu}\right) \right] dy \int_{-b}^y \rho(z) \exp \left[-\left(\frac{y-z}{\nu}\right) \right] dz \\ &= \frac{c}{2} \frac{1}{\nu} \int_{-b}^x \rho(z) dz \int_z^x \exp \left[-\left(\frac{x-y}{\mu}\right) - \left(\frac{y-z}{\nu}\right) \right] dy \\ &= \frac{\mu}{\nu-\mu} \left[\frac{c}{2} \int_{-b}^x \rho(z) \exp \left[-\left(\frac{x-z}{\nu}\right) \right] dz - \frac{c}{2} \int_{-b}^x \rho(z) \exp \left[-\left(\frac{x-z}{\mu}\right) \right] dz \right]. \end{aligned}$$

But from Eq. (3.1-11a) we see that

$$\frac{c}{2} \frac{1}{\nu} \int_{-b}^x \rho(z) \exp \left[-\left(\frac{x-z}{s}\right) \right] dz = \left[s\psi(x, s) - s\psi(-b, s) \exp \left[-\left(\frac{b+x}{s}\right) \right] \right].$$

Consequently,

$$\begin{aligned} & \frac{c}{2} \frac{1}{\nu} \int_{-b}^x \exp \left[-\left(\frac{x-y}{\mu}\right) \right] dy \int_{-b}^y \rho(z) \exp \left[-\left(\frac{y-z}{\nu}\right) \right] dz \\ &= \frac{\mu}{\nu-\mu} \left[\nu\psi(x, \nu) - \mu\psi(x, \mu) \right] - \frac{\mu}{\nu-\mu} \left[\nu\psi(-b, \nu) \exp \left[-\left(\frac{b+x}{\nu}\right) \right] \right. \\ & \quad \left. - \mu\psi(-b, \mu) \exp \left[-\left(\frac{b+x}{\mu}\right) \right] \right]. \end{aligned} \quad (1b)$$

Putting (1a) and (1b) in Eq. (1), we get Eq. (3.2-2a). The same procedure can then be repeated to prove Eq. (3.2-2b).

2B. PROOFS FOR THE IDENTITIES OF $X(z)$

1. First Identity:

$$X(z) = \int_0^1 \frac{\gamma(\mu) d\mu}{\mu - z}$$

where

$$\gamma(\mu) = \frac{c}{2} \frac{\mu X^-(\mu)}{\lambda(\mu) - \frac{c\pi i \mu}{2}}.$$

Proof. Since $X(z)$ is analytic over the whole plane with a branch cut on the real axis segment $[0, 1]$ and behaves as $1/z$ at infinity, we can use Cauchy's theorem to write

$$X(z) = \frac{1}{2\pi i} \int_C \frac{X(z') dz'}{z' - z}, \quad (1)$$

where the contour C consists of two parts, as shown in the accompanying figure. If C_2 is taken at infinity and C_1 around the cut, Eq. (1) reduces to

$$\begin{aligned} X(z) &= \frac{1}{2\pi i} \int_0^1 \frac{d\mu}{\mu - z} [X^+(\mu) - X^-(\mu)] \\ &= \frac{1}{2\pi i} \int_0^1 \frac{d\mu X^-(\mu)}{\mu - z} [(X^+/X^-) - 1]. \end{aligned} \quad (2)$$

But from Eq. (3.3-10) we have

$$\frac{X^+}{X^-} - 1 = G(\mu) - 1 = \frac{c\pi i \mu}{\lambda(\mu) - \frac{c\pi i \mu}{2}}. \quad (3)$$

Inserting (3) into (2) gives the first identity.

2. Second Identity

$$X(z) X(-z) = \frac{1 - cz \tanh^{-1} z}{(\nu_0^2 - z^2)(1 - c)}; \quad \tanh^{-1} z = \begin{cases} \frac{1}{2} \ln \frac{1+z}{1-z}; & z \in [-1, 1] \\ \frac{1}{2} \ln \frac{z+1}{z-1}; & z \notin [-1, 1]. \end{cases}$$

Proof: This identity was proved by Case by using the properties of $X(z)$. Here we show that it also follows from the definition of $X(z)$:

$$X(z) = \frac{1}{1-z} e^{\Gamma(z)}. \quad (3.3-11)$$

Consequently,

$$X(z) X(-z) = \frac{1}{1-z^2} e^{\Gamma(z) + \Gamma(-z)}. \quad (4)$$

But from Eq. (3.3-11a), we can write

$$\Gamma(z) + \Gamma(-z) = \frac{1}{2\pi i} \int_{-1}^1 \frac{\ln G(\mu)}{\mu - z} d\mu. \quad (5)$$

The integral in (5) was evaluated in Appendix A of Ref. 1:

$$\Gamma(z) + \Gamma(-z) = \ln \left[\frac{1-z^2}{\nu_0^2 - z^2} \frac{1 - cz \tanh^{-1} z}{1-c} \right]. \quad (6)$$

Substitution of (6) in (4) completes the proof. Moreover, at $z = 0$ the identity reduces to

$$X^2(0) = \frac{1}{\nu_0^2(1-c)} = \frac{1}{D}. \quad (7)$$

Also, by taking the limit as $z \rightarrow \nu_0$, we find

$$X(\nu_0) X(-\nu_0) = \frac{1}{2} \frac{1}{\nu_0^2(1-c)} \left[\frac{1 - \nu_0^2(1-c)}{1 - \nu_0^2} \right] = \frac{1}{2} \frac{X^2(0) - 1}{1 - \nu_0^2}. \quad (8)$$

3. Third Identity

$$X(z) = \exp \left[-\frac{c}{2} \int_0^1 g(c, \mu) \left(1 + \frac{c\mu^2}{1-\mu^2} \right) \ln(\mu - z) d\mu \right],$$

where

$$g(c, \mu) = \frac{1}{\lambda^2(\mu) + \frac{c^2 \pi^2 \mu^2}{4}}.$$

Proof: In Eq. (3.3-11a) let

$$G(\mu) = e^{2i\theta(\mu)},$$

where

$$\theta(\mu) = \arg \left[\lambda(\mu) + \frac{c\pi i \mu}{2} \right] \quad (9)$$

and $\theta(0) = 0$; $\theta(1) = \pi$. Then

$$\Gamma(z) = \frac{1}{\pi} \int_0^1 \frac{\theta(\mu) d\mu}{\mu - z}.$$

Integration by parts gives

$$\Gamma(z) = \ln(1-z) - \frac{1}{\pi} \int_0^1 \frac{d\theta}{d\mu} \ln(\mu - z) d\mu. \quad (10)$$

But

$$\frac{d\theta}{d\mu} = \frac{c\pi}{2} g(c, \mu) \left(1 + \frac{c\mu^2}{1 - \mu^2} \right). \quad (11)$$

Combining (10), (11), and Eq. (3.3-11), we obtain the third identity.

A similar procedure can be used to relate the ratio $X(-\nu_0)/X(\nu_0)$ to the Milne-problem extrapolation distance:

$$z_0 = \nu_0 \frac{c}{2} \int_0^1 g(c, \mu) \left(1 + \frac{c\mu^2}{1 - \mu^2} \right) \tanh^{-1} \left(\frac{\mu}{\nu_0} \right) d\mu. \quad (12)$$

From Equations (3.3-11) and (9), we have

$$\begin{aligned} \frac{X(-\nu_0)}{X(\nu_0)} &= \frac{1 - \nu_0}{1 + \nu_0} \exp \left[-\frac{2\nu_0}{\pi} \int_0^1 \frac{\theta(\mu)}{\mu^2 - \nu_0^2} d\mu \right] \\ &= \frac{1 - \nu_0}{1 + \nu_0} \exp \left[\frac{2}{\pi} \int_0^1 \theta(\mu) \left(\frac{d}{d\mu} \tanh^{-1} \frac{\mu}{\nu_0} \right) d\mu \right]. \end{aligned} \quad (13)$$

Integration by parts and use of (11) for $d\theta(\mu)/d\mu$ gives

$$\begin{aligned}\frac{X(-\nu_0)}{X(\nu_0)} &= \frac{1 - \nu_0}{1 + \nu_0} \exp [2 \tanh^{-1} (1/\nu_0) - (2z_0/\nu_0)] \\ &= -e^{-2z_0/\nu_0}.\end{aligned}\tag{14}$$

4. Fourth Identity:

$$X(-\mu) = \frac{c}{2} \frac{1}{1-c} \int_{-1}^0 \frac{\mu' d\mu'}{(\nu_0^2 - \mu'^2) X(\mu') (\mu' - \mu)}; \quad \mu > 0.$$

Proof: This integral equation follows directly from the first two identities. From the second identity we find

$$\frac{X^-(\mu)}{\lambda(\mu) - \frac{c\pi i\mu}{2}} = \frac{1}{1-c} \frac{1}{\nu_0^2 - \mu^2} \frac{1}{X(-\mu)}.\tag{15}$$

Substitution of (15) into the first identity and change of μ to $-\mu'$ gives the integral equations.

3B. ON THE APPROXIMATIONS FOR $A'(\nu)$

We first investigate the convergence of the series (3.3-22). The discussion follows the method given in Tricomi (21) since the kernel $K(\nu, \alpha)$ clearly satisfies the required property of square integrability. The parameter λ is chosen as

$$\lambda = \frac{c}{2} (1 - c).\tag{1}$$

The general solution of Eq. (3.3-21) is

$$\begin{aligned}A'(\nu) &= A'_1(\nu) + \sum_{n=1}^{\infty} \lambda^n \phi_n(\nu) \\ &= A'_1(\nu) + \lambda \sum_{n=1}^{\infty} \lambda^{n-1} \int_0^1 K_n(\nu, \alpha) A'_1(\alpha) d\alpha.\end{aligned}\tag{2}$$

The convergence properties of (2) are examined by investigating the series

$$H(\nu, \alpha; \lambda) = - \sum_{n=1}^{\infty} \lambda^{n-1} K_n(\nu, \alpha). \quad (3)$$

Define the norm of the kernel by

$$\begin{aligned} ||K^2|| &= \int_0^1 \int_0^1 K^2(\nu, \alpha) d\nu d\alpha \\ &= \int_0^1 F^2(\nu) d\nu = \int_0^1 G^2(\alpha) d\alpha \leq N^2, \end{aligned} \quad (4)$$

where N^2 is an upper bound and

$$F^2(\nu) = \int_0^1 K^2(\nu, \alpha) d\alpha; \quad G^2(\alpha) = \int_0^1 K^2(\nu, \alpha) d\nu. \quad (4a)$$

By Schwartz' inequality, it follows that

$$|K_{n+2}(\nu, \alpha)| \leq |F(\nu)| |G(\alpha)| N^n. \quad (5)$$

Inserting (5) into (3), we find

$$|H(\nu, \alpha; \lambda) - K(\nu, \alpha)| \leq |\lambda| |F(\nu)| |G(\alpha)| \sum_{m=0}^{\infty} |\lambda N|^m. \quad (6)$$

The geometric series in (6) converges if

$$|\lambda N| < 1. \quad (7)$$

It remains now to estimate the upper bound N . Using the behavior of $X(-\nu)$ near $\nu = 0$ and $\nu = 1$ and of $g(c, \nu)$ from Ref. 19, it readily follows from (4) that

$$||K^2|| \leq \frac{e^{-2t}}{4(1-c)^2} = N^2. \quad (8)$$

Substitution of (1) and (8) in (7) yields

$$1 < \frac{4e^t}{c}, \quad (9)$$

where t and c are related by the criticality condition and Eq. (2.1-9). Since in practical applications $c < 2$, the convergence condition of Eq. (9) is clearly satisfied.

An upper bound of the error introduced by keeping the first n terms of the series in (2) can also be estimated. First, we note that $A_1'(\nu) \leq 0$ and the kernel $K(\nu, \alpha)$ is positive for $c > 1$. From this it follows that $\phi_n(\nu) \leq 0$. In view of the fact that λ is also negative, the series in (2) is alternating and can be written as

$$\sum_{n=1}^{\infty} \lambda^n \phi_n(\nu) = \sum_{n=1}^{\infty} (-1)^{n+1} a_n(\nu), \quad (10)$$

where $a_n(\nu)$ is positive and $a_{n+1} < a_n$. By the well-known theorem of convergent alternating series of this type, the error made in stopping at n terms is less in absolute value than the first term neglected. Thus, if we denote the absolute value of the error by $\epsilon_n(\nu)$, then

$$\begin{aligned} \epsilon_n(\nu) &< a_{n+1}(\nu) \\ &< |\lambda|^{n+1} \left| \phi_{n+1}(\nu) \right|. \end{aligned} \quad (11)$$

Using the definition of $\phi_n(\nu)$ and Eq. (5), we find

$$|\phi_{n+1}(\nu)| \leq |F(\nu)| |N|^{n-1} \int_0^1 |G(\alpha)| |A_1'(\alpha)| d\alpha. \quad (12)$$

A rough estimate of the upper bound of the factors on the right of (12) gives

$$|\phi_{n+1}(\nu)| \leq \frac{|\nu_0|^2}{2} e^{-t} E_2(t) |X^3(0)| |A_1'(0)| |N|^{n-1} p(\nu), \quad (13)$$

where we have defined

$$p(\nu) = \frac{(|\nu_0|^2 + \nu^2)X(-\nu)g(c, \nu)}{1 + \nu} \quad (13a)$$

and $E_2(t)$ is the exponential integral tabulated in Ref. 19. Finally, substitution of (13) in (11) and use of the explicit expressions for $X(0)$, $|\lambda N|$, and $A_1(0)$ gives

$$\epsilon_n(\nu) < 2a_0 + \left(\frac{c}{4}\right)^{n+1} |B| \frac{e^{-nt} E_2(t)}{|\nu_0|^2} p(\nu), \quad (14)$$

where B is the constant given in Eq. (3.3-35a). In particular, the upper bound of the error in the first approximation is

$$\epsilon_0(\nu) < 2a_0 + \frac{c}{4} \frac{|B|}{|\nu_0|^2} E_2(\nu) p(\nu), \quad (15)$$

and the expansion coefficients can be written as

$$A'(\nu) = A_1'(\nu) + \epsilon_0(\nu). \quad (16)$$

Furthermore, the actual value of $A'(\nu)$ lies between $A_1'(\nu)$ and $A_1'(\nu) + \epsilon_0(\nu)$. In Table BI we give values for the lower and upper bounds of $A'(\nu)$, viz.,

$$\underline{A}'(\nu) = A_1'(\nu);$$

$$\overline{A}'(\nu) = A_1'(\nu) + \frac{c}{4} \frac{|B|}{|\nu_0|^2} E_2(\nu) p(\nu),$$

where we have used the normalization of Eq. (3.3-39).

The exponential nature of Eq. (14) indicates that these bounds will be reduced considerably in the next higher approximations. Also, since the contribution of terms involving $A'(\nu)$ on the observables $\rho(x)$ and $\psi(x, \mu)$ was shown to be small for $1 \leq c < 2$, small errors in the coefficients will be further reduced in the computations of these quantities.

Table BI

BOUNDS FOR $A'(\nu)$

ν	$c = 2.00$		$c = 1.10$	
	$\underline{A}'(\nu)$	$\bar{A}'(\nu)$	$\underline{A}'(\nu)$	$\bar{A}'(\nu)$
0	-0.586	0.00	-0.160	-0.0720
0.4	-0.143	-0.0688	-0.0814	-0.0674
0.8	-0.0303	-0.0152	-0.0368	-0.0337
1.00	0.00	0.00	0.00	0.00

Another result of interest follows from the above discussion. By taking the limit of Eq. (16) as $c \rightarrow 1$ and recalling that in this limit both t and $|\nu_0|$ become infinite, we find

$$\lim_{c \rightarrow 1} A'(\nu) = 0, \quad (17)$$

from which we conclude that $A(\nu) \equiv 0$ at $c = 1$.

4B. COMPARISON WITH THE METHOD OF SPHERICAL HARMONICS

In view of the fact that an expansion in spherical harmonics is one of the most widely used methods of solving the transport equation, it is of interest to compare the exact normal mode expansion of Sec. 3.1 with the spherical harmonics analysis. The comparison presented here consists of rewriting the expansion (3.1-4) for $\psi(x, \mu)$ in a series of Legendre polynomials and matching this form with the corresponding spherical harmonics expansion.

If we approximate the angular density by a series of the form

$$\psi(x, \mu) = \sum_0^N \left(\frac{2n+1}{2} \right) \psi_n(x) P_n(\mu) \quad (1)$$

and substitute (1) in the transport equation (3.1-1), we find (12, p. 116)

$$\psi(x, \mu) = \sum_0^N \sum_j A_j G_n(\nu_j) e^{-x/\nu_j} P_n(\mu), \quad (2)$$

where, as in Eq. (4.3-8),

$$G_n(\nu_j) = P_n(\nu_j) - c\nu_j W_{n-1}(\nu_j), \quad (3)$$

A_j are arbitrary constants, and ν_j are the roots of

$$G_{N+1}(\nu_j) = 0. \quad (4)$$

For $c > 1$, two of the roots of Eq. (4) are imaginary conjugates, which approach the roots $\pm\nu_0$ of Eq. (2.1-9) in the limit as $N \rightarrow \infty$ (36, p. 250) and the other roots lie in the interval $[-1, 1]$.

The coefficients A_j are determined by approximating the exact boundary condition $\psi(-b, \mu > 0) = 0$ either by Mark's conditions (12, p. 129):

$$\psi(-b, \mu_j) = 0; \quad j = 1, 2, \dots, \frac{1}{2}(N+1), \quad (4a)$$

or by the more convenient Marshak conditions (12, p. 133):

$$\int_0^1 \psi(-b, \mu) P_{2j-1}(\mu) d\mu = 0. \quad (4b)$$

Imposing either of these conditions in Eq. (2), we obtain a homogeneous set of equations for A_j . The requirement that the determinant of the coefficients in these equations vanish then corresponds to the criticality condition.

To bring the exact representation

$$\psi(x, \mu) = a_{0+} \psi_{0+}(x, \mu) + a_{0-} \psi_{0-}(x, \mu) + \int_{-1}^1 A(\nu) \psi_\nu(x, \mu) d\nu \quad (3.1-4)$$

to the form of Eq. (2), we first write

$$\begin{aligned}
\psi_{0\pm}(\mathbf{x}, \mu) &= \frac{c}{2} \frac{\nu_0}{\nu_0 \mp \mu} e^{\mp \mathbf{x}/\nu_0} \\
&= \sum_0^{\infty} \left(\frac{2n+1}{2} \right) c \nu_0 Q_n(\nu_0) P_n(\pm \mu) e^{\mp \mathbf{x}/\nu_0}
\end{aligned} \tag{5a}$$

and

$$\begin{aligned}
\psi_{\nu}(\mathbf{x}, \mu) &= \left[\frac{c}{2} P \frac{\nu}{\nu - \mu} + \lambda(\nu) \delta(\mu - \nu) \right] e^{-\mathbf{x}/\nu} \\
&= \sum_0^{\infty} \left(\frac{2n+1}{2} \right) [c \nu Q_n(\nu) + \lambda(\nu) P_n(\nu)] P_n(\mu) e^{-\mathbf{x}/\nu}.
\end{aligned} \tag{5b}$$

Then

$$\psi(\mathbf{x}, \mu) = \sum_0^{\infty} \left(\frac{2n+1}{2} \right) C_n(\mathbf{x}) P_n(\mu), \tag{6}$$

where

$$\begin{aligned}
C_n(\mathbf{x}) &= a_0 + G_n(\nu_0) e^{-\mathbf{x}/\nu_0} + (-1)^n a_0 - G_n(\nu_0) e^{\mathbf{x}/\nu_0} + \\
&\quad \int_{-1}^1 A(\nu) G_n(\nu) e^{-\mathbf{x}/\nu} d\nu,
\end{aligned} \tag{7}$$

since $P_n(-\mu) = (-1)^n P_n(\mu)$ and, by Eq. (4.2-4a),

$$\begin{aligned}
G_n(\nu) &= P_n(\nu) - c \nu W_{n-1}(\nu) \\
&= c \nu Q_n(\nu) + \lambda(\nu) P_n(\nu),
\end{aligned} \tag{7a}$$

so that $G_n(\nu_0) = c \nu_0 Q_n(\nu_0)$.

Next, we approximate the integral over ν in Eq. (7) by a Gauss quadrature (22, p. 61):

$$\int_{-1}^1 A(\nu) G_n(\nu) e^{-\mathbf{x}/\nu} d\nu \approx \sum_j^M a_j A(\nu_j) G_n(\nu_j) e^{-\mathbf{x}/\nu_j}, \tag{8}$$

where a_j are tabulated weights and ν_j are the roots of $P_{2M}(\nu_j) = 0$. Hence,

$$C_n(x) \simeq \sum_j A_j G_n(\nu_j) e^{-x/\nu_j}. \quad (9)$$

In Eq. (9), A_j and ν_j are redefined to incorporate $a_{0\pm}$ and $\pm\nu_0$. Inserting Eq. (9) into (6) and terminating the expansion at $n = N$, we get

$$\psi(x, \mu) = \sum_0^N \sum_j A_j G_n(\nu_j) e^{-x/\nu_j} P_n(\mu), \quad (10)$$

which, apart from the values of ν_j , agrees with the spherical harmonics expansion (2). In this case, ν_j comprise the exact discrete pair $\pm\nu_0$ and the zeroes of $P_{2M}(\nu)$, which also lie in the interval $[-1, 1]$.

The essential step in the transformation of Eq. (3.1-4) to the spherical harmonics expansion was the approximation of the continuum by a sum. This procedure suggests an alternative numerical treatment of the critical problems. By approximating the continuous solutions at the outset, we can write Eq. (3.1-4) as

$$\psi(x, \mu) \simeq a_{0+} \psi_{0+}(x, \mu) + a_{0-} \psi_{0-}(x, \mu) + \sum_j^M A_j(\nu_j) \psi_{\nu_j}(x, \mu). \quad (11)$$

The coefficients can then be computed by approximating the exact boundary condition by enough conditions having the form of Eq. (4b). It is evident that the same approximation procedure is applicable to Eq. (4.3-7) for the spherical distribution and Eq. (5.3-8) for the cylindrical distributions. The asymptotic approximations corresponding to $A_j = 0$ for all three geometries have been discussed in Secs. (3.3e), (4.6c), and (5.6b).

C. APPENDIX TO CHAPTER IV

1C. VERIFICATION OF THE SPHERICAL SOLUTIONS

Application of the operator

$$T = \mu \frac{\partial}{\partial r} + \frac{1 - \mu^2}{r} \frac{\partial}{\partial \mu} + 1 \quad (1)$$

to the discrete spherical solution $a_0 \psi_0(r, \mu)$ [see Eq. (4.3-1)] gives

$$T a_0 \psi_0(r, \mu) = a_0 \frac{c}{2} \sum_0^{\infty} (-1)^n Q_n(\nu_0) \quad (x) \left[(2n+1) \frac{di_n(r/\nu_0)}{d(r/\nu_0)} \mu P_n(\mu) + (2n+1) \frac{\nu_0}{r} i_n(r/\nu_0) (1 - \mu^2) \frac{dP_n(\mu)}{d\mu} + (2n+1) \nu_0 i_n(r/\nu_0) P_n(\mu) \right]. \quad (2)$$

But from the recurrence relations of $I_n(z)$ (24, p. 79) and the definition (4.1-5a) of $i_n(z)$, we can show that

$$\frac{2n+1}{z} i_n(z) = i_{n-1}(z) - i_{n+1}(z) \quad (3a)$$

and

$$(2n+1) \frac{di_n(z)}{dz} = n i_{n-1}(z) + (n+1) i_{n+1}(z). \quad (3b)$$

Also, we need (23, p. 114)

$$(1 - \mu^2) \frac{dP_n(\mu)}{d\mu} = n P_{n-1}(\mu) - n \mu P_n(\mu) \quad (4a)$$

and

$$(2n+1) \mu P_n(\mu) = (n+1) P_{n+1}(\mu) + n P_{n-1}(\mu). \quad (4b)$$

With the aid of these relations, Eq. (2) simplifies to

$$T a_0 \psi_0(r, \mu) = a_0 \frac{c}{2} \sum_0^{\infty} (-1)^n Q_n(\nu_0) \quad (x) \left[(n+1) P_{n+1}(\mu) i_{n+1}(r/\nu_0) + n P_{n-1}(\mu) i_{n-1}(r/\nu_0) + (2n+1) \nu_0 P_n(\mu) i_n(r/\nu_0) \right]. \quad (5)$$

A change of indices from n to $(n - 1)$ in the first term on the right and to $(n + 1)$ in the second reduces Eq. (5) to

$$T a_0 \psi_0(r, \mu) = a_0 \left(\frac{c}{2} \right) \left[\nu_0 Q_0(\nu_0) - Q_1(\nu_0) \right] + a_0 \frac{c}{2} \sum_1^{\infty} (-1)^n i_n(r/\nu_0) P_n(\mu) \\ (x) \left[(2n + 1) \nu_0 Q_n(\nu_0) - n Q_{n-1}(\nu_0) - (n + 1) Q_{n+1}(\nu_0) \right]. \quad (6)$$

Use of the recurrence relation for $Q_n(\nu_0)$ now shows that the first bracket in Eq. (6) is unity and the second is zero. Hence,

$$T a_0 \psi_0(r, \mu) = a_0 \frac{c}{2} i_0(r/\nu_0). \quad (7)$$

Since the r and μ dependence in Eq. (4.3-1b) is the same as in Eq. (4.3-1a), the result of the operation by T on the continuous solution can be written down by inspection:

$$T \int_0^1 A_S(\nu) \psi_\nu(r, \mu) d\nu = \frac{c}{2} \int_0^1 A_S(\nu) i_0(r/\nu) d\nu. \quad (8)$$

A combination of Eqs. (7) and (8) then proves Eq. (4.3-2).

2C. PROOF OF EQ. (4.4-2)

The integral equation (4.4-2) for $\rho(r)$ can be derived by integrating Eq. (4.4-1) for $\psi(r, \mu)$ over μ :

$$\rho(r) = \frac{c}{2} \int_{-1}^1 d\mu \int_0^{s_0(r, \mu)} \rho \left(\sqrt{r^2 + s^2 - 2rs\mu} \right) e^{-s} ds, \quad (1)$$

where

$$s_0(r, \mu) = r\mu + \left[R^2 - r^2(1 - \mu^2) \right]^{1/2}.$$

The coordinate transformation

$$t = \sqrt{r^2 + s^2 - 2rs\mu}; s=s, \quad (2)$$

maps the integration area on the $s\mu$ plane to a trapezoid on the st -plane as shown in the accompanying diagrams.

The Jacobian of transformation is

$$J\left(\frac{\mu, s}{t, s}\right) = \begin{vmatrix} \frac{\partial \mu}{\partial t} & \frac{\partial \mu}{\partial s} \\ \frac{\partial s}{\partial t} & \frac{\partial s}{\partial s} \end{vmatrix} = -\frac{t}{rs}.$$

The integration on the st-plane then becomes

$$\begin{aligned} \rho(r) &= \frac{c}{2} \frac{1}{r} \int_t t \rho(t) dt \int_s e^{-s} \frac{ds}{s} \\ &= \frac{c}{2} \frac{1}{r} \left[\int_0^r t \rho(t) dt \int_{r-t}^{r+t} e^{-s} \frac{ds}{s} + \int_r^R t \rho(t) dt \int_{t-r}^{t+r} e^{-s} \frac{ds}{s} \right]. \end{aligned}$$

But

$$\begin{aligned} \int_{r-t}^{r+t} e^{-s} \frac{ds}{s} &= \int_{r-t}^{\infty} e^{-s} \frac{ds}{s} - \int_{r+t}^{\infty} e^{-s} \frac{ds}{s} \\ &= E_1(r-t) - E_1(r+t). \end{aligned} \quad (4)$$

Then

$$\rho(r) = \frac{c}{2} \frac{1}{r} \int_0^R t \rho(t) [E_1(|r-t|) - E_1(r+t)] dt. \quad (5)$$

We can now write Eq. (5) in the form of Eq. (4.4-2) by introducing negative values of r so that $\rho(-r) = \rho(r)$.

D. APPENDIX TO CHAPTER V

1D. PROOF OF EQ. (5.4-4)

Integration of Eq. (5.4-3) for $\psi(r, \theta, \phi)$ over θ and ϕ gives

$$\rho(r) = \frac{c}{4\pi} \int_0^\pi \sin \theta \, d\theta \int_0^{2\pi} d\phi \int_0^{s_0} \rho(\sqrt{r^2 + s^2 \sin^2 \theta - 2sr \sin \theta \cos \phi}) e^{-s} \, ds, \quad (1)$$

where

$$s_0(r, \theta, \phi) = \frac{r \cos \phi + \sqrt{R^2 - r^2 \sin^2 \phi}}{\sin \theta}. \quad (5.4-2)$$

Equation (1) can be converted to Eq. (5.4-4) by the following series of transformations. First note that the integral over ϕ and s in Eq. (1) can be written as

$$\begin{aligned} I &= \int_0^{2\pi} d\phi \int_0^{s_0} \rho(r, s, \theta, \phi) e^{-s} \, ds \\ &= 2 \int_{-1}^1 \frac{dv}{\sqrt{1-v^2}} \int_0^{s_0} \rho(r, s, \theta, v) e^{-s} \, ds, \end{aligned} \quad (2)$$

with $v = \cos \phi$. Applying the transformation

$$t = \sqrt{r^2 + s^2 \sin^2 \theta - 2sr v \sin \theta}; \quad s = s, \quad (3)$$

and proceeding as in Sec. 2C of Appendix C, we get

$$I = 4 \int_0^r t \rho(t) \, dt \int_{s_1}^{s_2} e^{-s} \frac{ds}{p(s, t)} + 4 \int_r^R t \rho(t) \, dt \int_{-s_1}^{s_2} e^{-s} \frac{ds}{p(s, t)}, \quad (4)$$

where

$$s_1 = \frac{r-t}{\sin \theta}; \quad s_2 = \frac{r+t}{\sin \theta}; \quad (4a)$$

and

$$p(s, t) = \left\{ \left[(r + s \sin \theta)^2 - t^2 \right] \left[t^2 - (r - s \sin \theta)^2 \right] \right\}^{1/2}. \quad (4b)$$

A change from s to x by $x = s\sqrt{1 - \mu^2}$, where $\mu = \cos \theta$, and substitution in Eq. (1) yields

$$\rho(r) = \frac{c}{\pi} \int_0^R t \rho(t) dt \int_{|r-t|}^{r+t} \frac{dx}{p(x, t)} \int_{-1}^1 \frac{\exp\left[-x/\sqrt{1 - \mu^2}\right]}{\sqrt{1 - \mu^2}} d\mu. \quad (5)$$

Now set

$$z = \frac{x\mu}{\sqrt{1 - \mu^2}}, \quad (6)$$

so that

$$\frac{d\mu}{\sqrt{1 - \mu^2}} = \frac{xdz}{x^2 + z^2}; \quad \frac{x}{\sqrt{1 - \mu^2}} = \sqrt{x^2 + z^2}. \quad (6a)$$

Then, by further obvious transformations,

$$\begin{aligned} \int_{-1}^1 \frac{\exp\left[-x/\sqrt{1 - \mu^2}\right]}{\sqrt{1 - \mu^2}} d\mu &= \frac{2}{x} \int_0^\infty \frac{\exp\left[-x\sqrt{1 + (z/x)^2}\right]}{1 + (z/x)^2} dz \\ &= \frac{2}{x} \int_0^\infty \frac{dz}{\sqrt{1 + (z/x)^2}} \int_x^\infty \exp\left[-u\sqrt{1 + (z/x)^2}\right] du \\ &= 2 \int_x^\infty K_0(u) du \\ &= 2x \int_1^\infty K_0(xy) dy, \end{aligned} \quad (7)$$

since

$$\int_0^\infty \frac{e^{-u\sqrt{1 + v^2}}}{\sqrt{1 + v^2}} dv = K_0(u).$$

Inserting Eq. (7) in (5), we get

$$\rho(r) = \frac{2c}{\pi} \int_0^R t \rho(t) dt \int_{|r-t|}^{r+t} \frac{x dx}{p(x, t)} \int_1^\infty K_0(xy) dy. \quad (8)$$

Finally, changing from x to α , where

$$x^2 = t^2 + r^2 - 2rt \cos \alpha, \quad (9)$$

we find

$$p(x, t) = p(\alpha, t) = 2rt \sin \alpha,$$

and Eq. (8) becomes

$$\rho(r) = \frac{c}{2\pi} \int_0^R t \rho(t) dt \int_0^{2\pi} d\alpha \int_1^\infty K_0\left(y\sqrt{r^2 + t^2 - 2rt \cos \alpha}\right) dy,$$

which is the result quoted in Eq. (5.4-4).

The derivation of Eq. (5.4-4) is considerably simpler if we start from

$$\rho(r) = \frac{c}{4\pi} \int_{\underline{r}'} \frac{\rho(\underline{r}') e^{-|\underline{r} - \underline{r}'|}}{|\underline{r} - \underline{r}'|^2} d^3r'. \quad (1.1-9)$$

Choosing coordinates so that $\underline{r}$ is along the x -axis and denoting the cylindrical coordinates of $\underline{r}'$ by (t, α, z) , we get

$$|\underline{r} - \underline{r}'|^2 = r^2 + t^2 - 2rt \cos \alpha + z^2. \quad (10)$$

Consequently,

$$\rho(r) = \frac{c}{4\pi} \int_0^R t \rho(t) dt \int_0^{2\pi} d\alpha \int_{-\infty}^\infty \frac{\exp\left[-\sqrt{r^2 + t^2 - 2rt \cos \alpha + z^2}\right]}{r^2 + t^2 - 2rt \cos \alpha + z^2} dz. \quad (11)$$

The integral over z can easily be reduced to

$$\begin{aligned} \int_{-\infty}^\infty \frac{\exp\left[-\sqrt{r^2 + t^2 - 2rt \cos \alpha + z^2}\right]}{r^2 + t^2 - 2rt \cos \alpha + z^2} dz = \\ 2 \int_1^\infty K_0\left(y\sqrt{r^2 + t^2 - 2rt \cos \alpha}\right) dy. \end{aligned} \quad (12)$$

Substitution of Eq. (12) in (11) gives Eq. (5.4-4).

BIBLIOGRAPHY

1. K. M. Case, *Ann. Phys.* 9, 1 (1960).
2. N. G. Van Kampen, *Physica*, 21, 949 (1955).
3. E. P. Wigner, "Mathematical Problems of Nuclear Reactor Theory," *Proceedings of Symposia in Applied Mathematics* Vol. XI, American Mathematical Society, Providence, R. I. (1961).
4. G. J. Mitsis, "Transport Solutions to the One-Dimensional Critical Problem" Argonne National Laboratory Report, ANL-6459 (December, 1961).
5. R. Zelazny, *J. Math. Phys.* 2, 538 (1961).
6. J. R. Mika, *Nuclear Sci. and Eng.*, 11, 415 (1961).
7. A. Kuszell, *Acta Physica Polonica*, Vol. XX, Fasc. 7, 567 (1961).
8. R. Zelazny and A. Kuszell, *Ann. Phys.* 16, 81 (1961).
9. R. E. Aamodt, "Three Soluble Problems in Linear Transport Theory," Thesis, The Univ. of Mich., Ann Arbor, Mich. (1962). Also available as NP-12458 (microcard).
10. R. E. Aamodt and K. M. Case, *Ann. Phys.* 21, 284 (1963).
11. F. C. Shure, "Boundary Value Problems in Plasma Oscillations," Thesis, The Univ. of Mich., Ann Arbor, Mich. (1962).
12. B. Davison, Neutron Transport Theory, Oxford Univ. Press (1957).
13. A. H. Wilson, "The General Properties of the Transport Equation and Its Uses in Determining the Multiplication in Bodies Having Spherical Symmetry," A.E.R.E. Harwell Report, M. S. 105A, (December 1950).
14. W. Kofink, "New Solutions of the Boltzman Equation for Mono-energetic Neutron Transport in Spherical Geometry," Oak Ridge National Laboratory Report, ORNL-3216 (November 1961).

15. A. Leonard and T. W. Mullikin, "Solutions to the Criticality Problems for Spheres and Slabs," The Rand Corporation Memorandum, RM-3256-PR, (July 1962).
16. N. I. Muskhelishvili, Singular Integral Equations, Noordhoff, Groningen, Holland (1953).
17. H. J. Bremermann and L. Durand III, J. of Math. Phys. 2, 240 (1961).
18. R. Courant and D. Hilbert, Methods of Mathematical Physics, Vol. II, Interscience, New York (1962).
19. K. M. Case, F. DeHoffmann and G. Placzek, Introduction to the Theory of Neutron Diffusion, Vol. I, U. S. Gov. Printing Office (1953).
20. J. A. Stratton, Electromagnetic Theory, McGraw Hill, New York (1941).
21. F. G. Tricomi, Integral Equations, Interscience, New York (1957).
22. S. Chandrasekhar, Radiative Transfer, Dover, New York (1960).
23. E. Jahnke and F. Emde, Tables of Functions, Dover, New York (1945).
24. G. N. Watson, A Treatise on the Theory of Bessel Functions, Cambridge, Univ. Press (1945).
25. E. T. Whittaker and G. N. Watson, A Course of Modern Analysis, Cambridge Univ. Press (1940).
26. A. Erdélyi, Higher Transcendental Functions, Vols. I-III, McGraw Hill, New York (1953).
27. A. Erdélyi, Tables of Functions, Vols. I-III, McGraw Hill, New York (1954).
28. B. Davison, Can. J. of Phys. 35, 576 (1957).
29. W. Kofink, Nuclear Sci. and Eng. 6, 475 (1959).
30. W. W. Clendenin, Nuclear Sci. and Eng. 13, 46 (1962).

31. P. M. Morse and H. Feshbach, Methods of Theoretical Physics, McGraw Hill, New York (1953).
32. J. H. Fertziger and A. Leonard, Ann. Phys. 22, 192 (1963).
33. G. C. Summerfield and P. F. Zweifel, Nuclear Sci. and Eng., 15, 476 (1963).
34. N. Corngold, P. Michael and W. Wollman, Nuclear Sci. and Eng., 15, 13 (1963).
35. J. U. Koppel, Nuclear Sci. and Eng., 16, 101 (1963).
36. A. M. Weinberg and E. P. Wigner, The Physical Theory of Neutron Chain Reactors, The Univ. of Chicago Press (1958).