MASTER

OAK RIDGE NATIONAL LABORATORY

Operated by

UNION CARBIDE NUCLEAR COMPANY

Division of Union Carbide Corporation

Post Office Box X

Oak Ridge, Tennessee

ORNL
CENTRAL FILES NUMBER

60-7-32

External Distribution Authorized
COPY NO. 84

DATE: July 11, 1960

SUBJECT: Neutron Thermalization and Diffusion in Pulsed Media

TO: Distribution

FROM: S. N. Purohit

LEGAL NOTICE

This report was prepared as an account of Government sponsored work. Neither the United States, nor the Commission nor any person acting on behalf of the Commission

A. Makes any warranty or representation, expressed or implied, with respect to the accuracy, completeness or usefulness of the information contained in this report or that the use of any information, apparatus, method or process disclosed in this report may not infringe privately owned rights, or

B. Assumes any liability with respect to the use of or for damages resulting from the use of any information, apparatus, method or process disclosed in this report.

As used in the above "system" on behalf of the Commission" includes any employee or contractor of the Commission, or employee of such contractor, to the extent that such employee or contractor of the Commission or employee of such contractor discloses information, or provides access to, any information pursuant to the employment or contract with the Commission or his employment with such contractor.

NOTICE

This document contains information of a preliminary nature and was prepared primarily for internal use at the Oak Ridge National Laboratory. It is subject to revision or correction and therefore does not represent a final report. The information is not to be abstracted, reprinted or otherwise given public dissemination without the approval of the ORNL patent branch, Legal and Information Control Department.

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency Thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

DISCLAIMER

Portions of this document may be illegible in electronic image products. Images are produced from the best available original document.

LEGAL NOTICE

This report was prepared as an account of Government sponsored work. Neither the United States, nor the Commission, nor any person acting on behalf of the Commission:

- A. Makes any warranty or representation, expressed or implied, with respect to the accuracy, completeness, or usefulness of the information contained in this report, or that the use of any information, apparatus, method, or process disclosed in this report may not infringe privately owned rights; or
- B. Assumes any liabilities with respect to the use of, or for damages resulting from the use of any information, apparatus, method, or process disclosed in this report.

As used in the above, "person acting on behalf of the Commission" includes any employee or contractor of the Commission, or employee of such contractor, to the extent that such employee or contractor of the Commission, or employee of such contractor prepares, disseminates, or provides access to, any information pursuant to his employment or contract with the Commission, or his employment with such contractor.

**DO NOT
PHOTOSTAT**

Table of Contents

	<u>Page</u>
Abstract	111
1. INTRODUCTION	1
2. MATHEMATICAL FORMULATION	3
3. NONMULTIPLYING MEDIUM	10
DETERMINATION OF EIGENVALUES	10
Eigenvalue λ_0 ($m = 0$ and $n = 0$), One-by-One Determinant ...	11
Eigenvalues λ_0 and λ_1 ($m = 0, 1$ and $n = 0, 1$),	
Two-by-Two Determinant	11
Infinite and 1/v Absorbing Medium ($B_p^2 = 0$)	12
Infinite and Nonabsorbing Medium ($B_p^2 = 0$ and $\sum_a = 0$) ..	12
Comparison Between λ_0 and λ_1	13
Eigenvalues λ_0 , λ_1 , and λ_2 in an Infinite Medium ($B_p^2 = 0$) ..	14
NEUTRON THERMALIZATION IN A HEAVY GAS	17
Determination of Eigenvalues	18
Determination of Eigenfunctions	19
The <u>Zeroth</u> Eigenfunction	19
The First Eigenfunction	21
THERMALIZATION TIME CONSTANT	21
Estimation of t_{th}	23
Relation Between t_{th} and C_0	25
Experimental Determination of t_{th}	26

	<u>Page</u>
4. MULTIPLYING MEDIUM	27
THE MONOENERGETIC CASE	27
Heavy-Element Moderation	27
Hydrogen Moderation	28
THE MULTIENERGY CASE	31
ANALYSIS OF AN EXPERIMENT	32
5. CONCLUSION	33
Acknowledgements	37
Appendix: LAGUERRE POLYNOMIALS $L_n^{(1)}(E)$, MATRIX ELEMENTS W_{mn} , AND INTEGRALS F_{mn}	38
REFERENCES	41
List of Tables	43
List of Figures	44

ABSTRACT

A general formalism for determining the lower time eigenvalues associated with a decaying pulse of neutrons in a finite multiplying as well as nonmultiplying medium has been developed. This formalism is based upon the expansion of each energy eigenfunction by a complete sum of the associated Laguerre polynomials of first order. The eigenvalues are expressed in terms of the energy transfer moments of the scattering kernel of the medium, weighted by the Maxwellian distribution. The importance of the first eigenvalue in the establishment of the final asymptotic energy distribution is discussed. In the case of a nonabsorbing infinite medium, the reciprocal of the first eigenvalue is shown to be equal to the thermalization time constant, with which the Maxwellian velocity distribution of neutrons is attained. The thermalization time constant was estimated for various moderators. For the heavy-gas case, the thermalization time constant was found to be equal to $(1.274 \xi \sum_{s0} v_0)^{-1}$. It is also established in this study that the relations obtained with the help of the Rayleigh-Ritz variational principle, based upon the neutron temperature concept, can be rigorously derived by using only two polynomials. The formalism presented in this paper is general and avoids the concept of neutron temperature in defining the thermalization time constant. The decay of a neutron pulse in a nonmultiplying medium is discussed in detail. For the case of multiplying medium, an analysis of an experiment is presented to indicate the importance of the time-dependent nonleakage probability in the expression of the zeroth eigenvalue.

1. INTRODUCTION

In 1953, von Dardel (1) introduced pulsed-neutron techniques to measure reactor physics parameters, the absorption cross section, and the diffusion coefficient. These parameters are obtained from the measurements of the lowest eigenvalue associated with a decaying pulse of neutrons in a finite medium. Von Dardel and Sjöstrand (2) speculated on the existence of the diffusion cooling phenomenon due to the preferential leakage of the high-energy neutrons from a finite medium. Attempts were made by von Dardel and Sjöstrand (2) and also by Beckurts (3) to establish a connection between the diffusion cooling and thermalization phenomenon. Since then, there has been a great deal of theoretical interest in determining the diffusion cooling coefficient from the lowest eigenvalue associated with a decaying pulse of neutrons. In this paper, the last stage of neutron thermalization and diffusion was studied by determining the lower eigenvalues associated with the energy distribution of a decaying neutron pulse.

A burst of high-energy neutrons, when incident on a medium, undergoes three processes in succession: moderation, thermalization, and diffusion. "Moderation" refers to slowing down in the region above $1/4$ ev; "thermalization" refers to slowing down below this energy. Neutrons lose as well as gain energy by colliding with the atoms of moderator, which have a distribution of velocities and are bound, in the case of solids and liquids. The process of neutron thermalization continues, until the neutrons attain an asymptotic energy distribution. The characteristics of this distribution are governed by the energy exchange scattering kernel and by the absorption and the leakage of neutrons. Neutron thermalization is followed by the

process of diffusion, in which the shape of the asymptotic energy spectrum remains constant but the amplitude decreases with time. This process is terminated by the absorption or the leakage of neutrons from the medium. The lowest eigenvalue, that is, the zeroth, determines the final decay of the asymptotic energy distribution during the diffusion period. The first eigenvalue, however, governs the conditions under which this distribution is attained. It is established in this paper that the first eigenvalue is related to the thermalization time constant. Use of the concept of neutron temperature is avoided in this study. The lowest eigenvalue associated with a decaying neutron pulse has been calculated to determine the diffusion cooling coefficient. A brief review of these studies is presented.

Hurwitz and Nelkin (4) used the perturbation theory to calculate the zeroth eigenvalue in a heavy gas. The theoretical values of the diffusion cooling coefficient for beryllium and graphite under the constant diffusion approximation were found to be smaller than the experimental values. Nelkin (5) and later on Singwi and Kothari (6) employed the Rayleigh-Ritz variational principle to determine the lowest eigenvalue associated with the asymptotic energy distribution, which was assumed to be a modified Maxwellian distribution, based upon the concept of neutron temperature. Nelkin (5) introduced a new reactor physics parameter, which was termed the thermalization parameter M_2 . Singwi (7) recently developed a general theory of diffusion cooling, without using the concept of neutron temperature, which is based upon the expansion of the asymptotic energy distribution by a sum of the associated Laguerre polynomials of order one. Häfele and Dresner (8) have also given a similar theory for the calculation of a diffusion cooling

coefficient in a monoatomic heavy gas, using the same expansion. Kazarnovsky et al. (9) also used the above polynomials in their study of neutron thermalization problems. All these studies were undertaken for the finite medium under the diffusion approximation. Recently, Nelkin (10) studied the decay of a neutron pulse in a semi-infinite medium. Transport correction to the diffusion cooling coefficient was also given by Sjöstrand (11) and Singwi (7).

A general mathematical formalism is developed here to determine the eigenvalues associated with the energy eigenfunctions of a decaying pulse of neutrons in a finite medium. The problem is formulated for a subcritical multiplying medium. The nonmultiplying medium case is obtained by neglecting the neutron regeneration term due to fission. This problem is treated in more detail, as the majority of experiments are carried out with the nonmultiplying media. Neutron thermalization in a monoatomic heavy gas is discussed separately. This paper is part of a more complete study by the author (12).

2. MATHEMATICAL FORMULATION

The distribution of neutrons as a function of energy, space, and time variables in a multiplying and homogeneous medium under the diffusion approximation is given as follows:

$$\begin{aligned} \frac{1}{v} \frac{\partial}{\partial t} \phi(E, \underline{r}, t) = & - \left[\Sigma_a(E) + \Sigma_s(E) - D(E) \nabla^2 \right] \phi(E, \underline{r}, t) \\ & + \int_{E'} \nu \Sigma_f(E') (1 - \beta) \int_{\underline{r}'} \int_{t'} \phi(E', \underline{r}', t') q(\underline{r}' \rightarrow \underline{r}; t' \rightarrow t; E) d\underline{r}' dt' dE' \\ & + \int_{E'} \Sigma_s(E') F(E' \rightarrow E) \phi(E', \underline{r}, t) dE' + S(E, \underline{r}, t) \end{aligned} \quad (1)$$

In the above equation, $P(E' \rightarrow E) dE$ is defined as the probability that a neutron of initial energy E' will have a final energy E in the energy interval between E and $E + dE$ after a collision with an atom of the moderator. $q(\underline{r}' \rightarrow \underline{r}; t' \rightarrow t; E)$ is the slowing-down kernel, which defines the probability of a fission neutron born at $\underline{r}'$ and t' to have slowed down to final energy E at $\underline{r}$ and at time t . The rest of the symbols have their conventional meaning. The external source of neutrons is assumed to be a pulsed source having initial energy E_0 . Expressed mathematically, $S(E, \underline{r}, t)$ is given as follows:

$$S(E, \underline{r}, t) = \delta(E - E_0) \delta(t) S(\underline{r}) \quad (2)$$

For energies E , in the thermal region, the source term disappears from Eq. (1), reducing it to a homogeneous integral equation. $\phi(E, \underline{r}, t)$ is expanded into a complete set of the orthogonal functions $\phi_p(\underline{r})$ of the space variable:

$$\phi(E, \underline{r}, t) = \sum_p \phi_p(\underline{r}) \phi_p(E, t) \quad (3)$$

The p th mode spatial distribution is assumed to vanish at an extrapolated boundary specified by $\tilde{\underline{r}}$. It is further assumed that the extrapolated boundary is constant for neutrons of all energies. $\phi_p(\underline{r})$ is the p th space eigenfunction of the Laplacian operator and has B_p^2 as its eigenvalue:

$$\nabla^2 \phi_p(\underline{r}) = -B_p^2 \phi_p(\underline{r}) \quad (4a)$$

$$\phi_p(\tilde{\underline{r}}) = 0 \quad (4b)$$

where B_p^2 is the geometrical buckling corresponding to p th spatial mode.

The results of the above equations are substituted into Eq. (1) to

$$\begin{aligned} \text{get} \\ \sum_p \phi_p(\underline{r}) \left\{ \frac{1}{v} \frac{\partial}{\partial t} \phi_p(E, t) + \left[\Sigma_a(E) + \Sigma_s(E) + D(E) B_p^2 \right] \phi_p(E, t) \right. \\ \left. - \int_{E'} \Sigma_s(E') \phi_p(E', t) F(E' \rightarrow E) dE' \right\} \\ - \left[\int_{E'} v \Sigma_f(E') (1 - \beta) \int_{\underline{r}'} \int_{t'} \phi_p(\underline{r}') \phi_p(E', t') q(\underline{r}' \rightarrow \underline{r}; t' \rightarrow t; E) d\underline{r}' dt' dE' \right] \\ = 0 \end{aligned} \quad (5)$$

The results of fundamental theorems of the asymptotic reactor theory are employed to simplify the slowing-down integral. These theorems, first given by Weinberg (13), are discussed in detail by Weinberg and Wigner (14). According to these theorems the slowing-down integral can be represented as follows:

$$\sum_p \int_{E'} v \Sigma_f(E') (1 - \beta) \int_{\underline{r}'} \int_{t'} \phi_p(\underline{r}') \phi_p(E', t') q(\underline{r}' \rightarrow \underline{r}; t' \rightarrow t; E) d\underline{r}' dt' dE' \quad (6)$$

$$= \sum_p \phi_p(\underline{r}) \int_{E'} \int_{t'} \bar{q}_{\omega} (B_p^2; t' \rightarrow t; E) \phi_p(E', t') v \Sigma_f(E') (1 - \beta) dt' dE'$$

where $\bar{q}_{\omega} (B_p^2; t' \rightarrow t; E)$ is the Fourier transform of the slowing-down kernel in the infinite medium, with B_p as the Fourier transform variable.

Substituting the above results into Eq. (5), we get the homogeneous integral equation for the p th spatial mode:

$$\begin{aligned} \frac{1}{v} \frac{\partial}{\partial t} \phi_p(E, t) + \left[\Sigma_a(E) + \Sigma_s(E) - D(E) B_p^2 \right] \phi_p(E, t) - \\ - \int_{E'} \Sigma_s(E') \phi_p(E', t) P(E' \rightarrow E) dE' \quad (7) \\ - \int_{E'} \nu \Sigma_f(E') (1 - \beta) \int_{t'} \bar{q}_\infty(B_p^2; t' \rightarrow t; E) \phi_p(E', t') dE' dt' = 0 \end{aligned}$$

The Laplace transform of Eq. (7) with respect to time variable reduces it to an eigenvalue equation, with the transform variable as the eigenvalue. It is equivalent to making the following expansion of $\phi_p(E, t)$ for a subcritical assembly:

$$\phi_p(E, t) = \sum_i \phi_{pi}(E) e^{-\lambda_{pi} t} \quad (8)$$

In Eq. (8) $\phi_{pi}(E)$ is the i th energy eigenfunction associated with the p th spatial mode, and λ_{pi} is the corresponding eigenvalue. On substituting the above expansion into integral Eq. (7),

$$\begin{aligned} \left[-\frac{\lambda_{pi}}{v} + \Sigma_a(E) + \Sigma_s(E) - D(E) B_p^2 \right] \phi_{pi}(E) - \int_{E'} \Sigma_s(E') \phi_{pi}(E') P(E' \rightarrow E) dE' \\ - \int_{E'} \int_{t'} \nu \Sigma_f(E') (1 - \beta) \bar{q}_\infty(B_p^2; t' \rightarrow t; E) \phi_{pi}(E') e^{\lambda_{pi}(t-t')} dt' dE' = 0 \quad (9) \end{aligned}$$

The slowing-down integral can be simplified by the substitution of $t - t'$ equal to t_0 , as the slowing-down kernel is a displacement kernel in time variable:

$$\begin{aligned} \int_{E'} \int_{t'} \nu \Sigma_f(E') (1 - \beta) \bar{q}_\infty(B_p^2; t' \rightarrow t; E) \phi_{p1}(E') e^{\lambda_{p1}(t-t')} dt' dE' \\ = \int_{E'} \int_{t_0=0}^{\infty} \nu \Sigma_f(E') (1 - \beta) \bar{q}_\infty(B_p^2; t_0; E) \phi_{p1}(E') e^{\lambda_{p1}t_0} dt_0 dE' \end{aligned} \quad (10)$$

In the above equation the time-dependent integral is the Laplace-Fourier transform of the slowing-down kernel in the infinite medium, with $-\lambda_{p1}$ as the Laplace transform variable:

$$\begin{aligned} \int_{t_0=0}^{\infty} \bar{q}_\infty(B_p^2; t_0; E) e^{\lambda_{p1}t_0} dt_0 = \text{Laplace-Fourier transform of the slowing-down kernel in the infinite medium.} \\ = \tilde{\bar{q}}_\infty(B_p^2; -\lambda_{p1}; E) \end{aligned} \quad (11)$$

Substitution of Eq. (11) into Eq. (10) results in

$$\begin{aligned} \int_{E'} \int_{t_0=0}^{\infty} \nu \Sigma_f(E') (1 - \beta) \bar{q}_\infty(B_p^2; t_0; E) \phi_{p1}(E') e^{\lambda_{p1}t_0} dt_0 dE' \\ = \int_{E'} \nu \Sigma_f(E') (1 - \beta) \phi_{p1}(E') \tilde{\bar{q}}_\infty(B_p^2; -\lambda_{p1}; E) dE' \end{aligned} \quad (12)$$

The scattering integral is expressed in the alternate form by a Taylor series expansion. Let

$$\phi_{p1}(E) = \Psi_{p1}(E) M(E) \quad (13a)$$

and

$$M(E) dE = E e^{-E} dE \quad (13b)$$

where $M(E)$ is the Maxwellian distribution. The energy E is measured in units of KT , with T as the moderator temperature. According to the detailed balance theorem of the statistical mechanics,

$$\sum_s(E) M(E) F(E \rightarrow E') = \sum_s(E') M(E') F(E' \rightarrow E) \quad (14)$$

According to the definition of total scattering cross section,

$$\sum_s(E) = \int_{E'} \sum_s(E) F(E \rightarrow E') dE' \quad (15)$$

By expanding,

$$\psi_{pi}(E') = \psi_{pi}(E) + \sum_{k=1} \frac{\partial^k \psi_{pi}(E)}{\partial E^k} \frac{(E' - E)^k}{k!} \quad (16)$$

Using the above results, the scattering terms can be expressed as a sum of energy transfer moments:

$$\begin{aligned} \int_{E'} \sum_s(E') \phi_{pi}(E') F(E' \rightarrow E) dE' &= \sum_s(E) \phi_{pi}(E) \\ &= \sum_{k=1} \frac{1}{k!} \frac{\partial^k \psi_{pi}(E)}{\partial E^k} M(E) \sum_s(E) A_k(E) \end{aligned} \quad (17)$$

$$A_k(E) = \int_0^{\infty} (E' - E)^k F(E \rightarrow E') dE' \quad (18)$$

= kth energy transfer moment.

The final form of the integral Eq. (9), after the substitution of results of Eqs. (12) and (17), is as follows:

$$\left[-\frac{\lambda_{pi}}{v} + \sum_a(E) + D(E) B_p^2 \right] \phi_{pi}(E) - \sum_{k=1} \frac{1}{k!} \frac{\partial^k \psi_{pi}(E)}{\partial E^k} M(E) \sum_b(E) A_k(E) \\ - \int_{E'} \nu \sum_f(E') (1 - \beta) \phi_{pi}(E') \frac{\partial}{\partial E} \left(B_p^2 - \lambda_{pi}; E \right) dE' = 0 \quad (19)$$

The exact solution of the above integral equation and the determination of the eigenfunctions $\phi_{pi}(E)$ and the eigenvalues λ_{pi} are not simple. We expand each eigenfunction in a complete sum of the associated Laguerre polynomials of order one, weighted by the Maxwellian distribution:

$$\phi_{pi}(E) = \sum_{n=0}^{\infty} A_{n1p} L_n^{(1)}(E) M(E) \quad (20a)$$

$$\phi_{pi}(0) = 0 \text{ and } \phi_{pi}(\infty) = 0 \quad (20b)$$

By substituting the above expansion into Eq. (19), multiplying by $L_m^{(1)}(E)$, and integrating over all energies,

$$A_{n1p} \left[\left(\sum_a \right)_{mn} + \left(DB_p^2 \right)_{mn} - \frac{\lambda_{pi}}{v_0} W_{mn} - F_{mn} - G_{mn} \right] = 0 \quad (21)$$

with the above quantities defined as

$$W_{mn} = \int_0^{\infty} \frac{L_m^{(1)}(E) L_n^{(1)}(E) M(E) dE}{E^{1/2}} \quad (22)$$

$$F_{mn} = \sum_{k=1}^{\infty} \frac{1}{k!} \left[\int_0^{\infty} \frac{\delta^k L_n^{(1)}(E)}{\delta E^k} L_m^{(1)}(E) \sum_s(E) M(E) A_k(E) dE \right] \quad (23)$$

$$G_{mn} = \int_0^{\infty} \int_0^{\infty} \nu \Sigma_f(E') (1 - \beta) L_m^{(1)}(E) L_n^{(1)}(E') M(E') \tilde{q}_{\infty}^p(B_p^2; -\lambda_{p1}; E) dE' dE \quad (24)$$

$$\left(\sum_a \right)_{mn} = \int_0^{\infty} L_m^{(1)}(E) \sum_a(E) L_n^{(1)}(E) M(E) dE \quad (25)$$

$$E^{1/2} = \frac{v}{v_0} \quad (v_0 = \text{speed corresponding to most probable energy}) \quad (26)$$

$(DB_p^2)_{mn}$ can be defined similarly as $(\sum_a)_{mn}$. Equation (21) is the basic equation for determining the eigenvalues associated with a single p th spatial mode for a decaying pulse of neutrons in a multiplying or a nonmultiplying medium. These two cases are discussed separately.

3. NONMULTIPLYING MEDIUM

The determinant for determining the eigenvalues associated with the p th spatial mode in a nonmultiplying medium is as follows:

$$\left| -\frac{\lambda_{p1} W_{mn}}{v_0} + \left(\sum_a \right)_{mn} + (DB_p^2)_{mn} - F_{mn} \right| = 0 \quad (27)$$

DETERMINATION OF EIGENVALUES

The following energy-dependent laws for the absorption cross section and the diffusion coefficient are used for determining the lower eigenvalues:

$$\Sigma_a(E) = \frac{\Sigma_{a0}}{E^{1/2}} \quad (28a)$$

$$\left(\sum_a\right)_{mn} = \sum_{a0} W_{mn} \quad (28b)$$

$$D(E) = \text{a constant} \quad (29a)$$

$$\left(DB_p^2\right)_{mn} = DB_p^2 \int_0^\infty L_m^{(1)}(E) M(E) L_n^{(1)}(E) dE = DB_p^2 \delta_{mn} \quad (29b)$$

Eigenvalue λ_0 ($m = 0$ and $n = 0$), One-by-One Determinant:

$$\lambda_0 = \sum_{a0} v_0 + v_0 \frac{DB_p^2}{W_{00}} \quad (30)$$

This corresponds to the zeroth eigenfunction as the Maxwellian distribution.

Eigenvalues λ_0 and λ_1 ($m = 0, 1$ and $n = 0, 1$), Two-by-Two Determinant:

$$\lambda_0 = \sum_{a0} v_0 + v_0 \frac{DB_p^2}{W_{00}} \left[1 - 4 \left(\frac{W_{01}}{W_{00}} \right)^2 \frac{DB_p^2}{M_2} \right] \quad (31)$$

$$\begin{aligned} \lambda_1 = \sum_{a0} v_0 + \frac{M_2 W_{00} v_0}{4(W_{11} W_{00} - W_{01}^2)} + \frac{DB_p^2 v_0}{W_{00}} \left(\frac{W_{00}^2 + W_{01}^2}{W_{11} W_{00} - W_{01}^2} \right) \\ + 4D^2 \frac{B_p^4}{W_{00}} \left(\frac{W_{01}}{W_{00}} \right)^2 \frac{v_0}{W_{00} M_2} \end{aligned} \quad (32)$$

where

$$M_2 = -4F_{11} = \int_0^\infty \int_0^\infty (E' - E)^2 \sum_s (E) M(E) F(E \rightarrow E') dE' dE$$

Examination of the above expressions for λ_0 and λ_1 leads to the following observations for various interesting physical cases:

Infinite and $1/v$ Absorbing Medium ($B_p^2 = 0$):

$$\lambda_0 = \sum_a v_0 \quad (33)$$

$$\lambda_1 = \sum_a v_0 + \frac{M_2^W v_0}{4(W_{11}^W - W_{01}^2)} \quad (34)$$

For very small absorptions, λ_1 is greater than λ_0 ; therefore we have an asymptotic energy distribution. The final decay of this spectrum is governed by λ_0 alone. However, for large absorptions, λ_1 is comparable to λ_0 , in which case, the asymptotic energy distribution may not exist at the time of measurement and may be given by λ_0 and λ_1 . It may, therefore, be concluded that the thermalization parameter M_2 plays an important role in determining the conditions under which the asymptotic energy distribution is established in a medium containing $1/v$ absorber.

Infinite and Nonabsorbing Medium ($B_p^2 = 0$ and $\sum_a = 0$):

$$\lambda_0 = 0 \quad (35)$$

$$\lambda_1 = \frac{M_2^W v_0}{4(W_{11}^W - W_{01}^2)} \quad (36)$$

In this case there is an asymptotic energy distribution. The decay of this distribution is governed by λ_0 equal to zero. If the atoms of the moderator have a Maxwellian velocity distribution, then the asymptotic distribution is also the Maxwellian distribution. λ_1 determines the time constant with which the Maxwellian distribution is established. We obtain a simple relation between the coefficient of B_p^4 in the expression of λ_0

as given by Eq. (31) and λ_1 as given by Eq. (32). For the fundamental spatial mode, p is equal to zero. This coefficient is known as the diffusion cooling coefficient C_0 :

$$C_0 = \frac{4v_0 D^2 W_{01}^2}{W_{00}^3 M_2} \quad (37)$$

Substituting the values for W_{00} , W_{01} , and W_{11} as given in the appendix into Eqs. (36) and (37),

$$\lambda_1 = \frac{2}{3} \frac{M_2 v_0}{\sqrt{\pi}} \quad (36a)$$

$$C_0 = \frac{D^2 v_0}{M_2 \sqrt{\pi}} \quad (37a)$$

By eliminating M_2 from the above equations,

$$\lambda_1 = \frac{2}{3\pi} \frac{D^2 v_0^2}{C_0} \quad (38)$$

The significance of the above result is discussed separately.

Comparison Between λ_0 and λ_1 :

When λ_0 and λ_1 are plotted against B_p^2 , the resulting curves are parabolic. Deviation from the straight-line curve, as given by the coefficient of B_p^4 , is equal in magnitude but opposite in sign. We also observe that the deviation of the asymptotic energy distribution from the Maxwellian distribution appears in the coefficient of B_p^4 in the expression of λ_0 . This coefficient determines the deviation of the asymptotic energy distribution in the finite medium from the Maxwellian distribution. The values of λ_0 and λ_1 for the

fundamental as well as next higher spatial modes are required in the analysis of pulsed-neutron experiments. In the experiments of deSaussure and Silver (15), the geometrical buckling corresponding to the lmn th spatial mode was given by $B_{l,m,n}^2$, which is as follows:

$$B_{lmn}^2 = \frac{\pi^2}{\tilde{a}^2} (l^2 + m^2 + n^2) \quad (39)$$

where $\tilde{a} = a + 1.42 \lambda_{tr}$ (extrapolated side of the cube).

The fundamental spatial mode had B_{111}^2 as its geometrical buckling. By the judicious placement of the detector and source, the next higher spatial mode which could contribute was B_{113}^2 . The ratio between B_{113}^2 and B_{111}^2 is 11/3. The values of the ratios between the eigenvalues are listed in Table I, for several cases, for beryllium assemblies used by deSaussure and Silver (15). The ratio of λ_0/λ_1 for the fundamental spatial mode is plotted in Fig. 1.

The values in Table I were obtained by using the following constants for beryllium:

$$\begin{aligned} \Sigma_{a0} v_0 &= 0.288 \times 10^3 \text{ sec}^{-1} \\ M_2 &= 0.36 \text{ cm}^{-1} \quad [\text{Nelkin (5)}] \end{aligned}$$

$$\frac{2Dv_0}{\sqrt{\pi}} = 1.25 \times 10^5 \text{ cm}^2/\text{sec}$$

Eigenvalues λ_0 , λ_1 , and λ_2 in an Infinite Medium ($B_p^2 = 0$)

For $m = 0, 1, 2$ and $n = 0, 1, 2$ we get the following values for λ_0 , λ_1 , and λ_2 :

UNCLASSIFIED
ORNL-LR-DWG 47000

Fig. 1. Variation of λ_0/λ_1 with B^2 for Beryllium.

Table I

Ratios of Eigenvalues in Beryllium

$B_{111}^2 (\text{cm}^{-2})$	$\frac{\lambda_0(B_{111}^2)}{\lambda_1(B_{111}^2)}$	$\frac{\lambda_0(B_{111}^2)}{\lambda_0(B_{113}^2)}$	$\frac{\lambda_0(B_{113}^2)}{\lambda_1(B_{111}^2)}$	$\frac{\lambda_0(B_{113}^2)}{\lambda_1(B_{113}^2)}$
0	9.57×10^{-3}	1.0	9.57×10^{-3}	9.57×10^{-3}
1.05×10^{-2}	5.02×10^{-2}	3.04×10^{-1}	1.65×10^{-1}	1.44×10^{-1}
2.02×10^{-2}	8.40×10^{-2}	2.77×10^{-1}	3.03×10^{-1}	2.41×10^{-1}
3.31×10^{-2}	1.23×10^{-1}	2.59×10^{-1}	4.76×10^{-1}	3.38×10^{-1}
5.36×10^{-2}	1.76×10^{-1}	2.39×10^{-1}	7.37×10^{-1}	4.48×10^{-1}
7.18×10^{-2}	2.13×10^{-1}	2.25×10^{-1}	9.47×10^{-1}	5.19×10^{-1}

$$\lambda_0 = \sum_{a0} v_0 \quad (40)$$

$$\lambda_1 = v_0 \left(\sum_{a0} + 0.903 \left\{ F_{22} \left[\sqrt{\left(1 + \frac{F_{11}}{F_{22}}\right)^2 - 3.334 \left(\frac{F_{11}}{F_{22}} - \frac{F_{12}^2}{F_{22}^2}\right)} - F_{11} - F_{22} \right] \right\} \right) \quad (41)$$

$$\lambda_2 = v_0 \left(\sum_{a0} - 0.903 \left\{ F_{11} + F_{22} + F_{22} \left[\sqrt{\left(1 + \frac{F_{11}}{F_{22}}\right)^2 - 3.334 \left(\frac{F_{11}}{F_{22}} - \frac{F_{12}^2}{F_{22}^2}\right)} \right] \right\} \right) \quad (42)$$

From the above expressions for λ_0 , λ_1 , and λ_2 it can be seen that even the use of a third Laguerre polynomial introduces energy transfer moments greater than 2. Determination of these moments depends upon the approximations involved in the formulation of a physical model. The parameter M_2 has been calculated for a large number of interesting moderators, but the higher energy transfer moments are not readily available. We shall assume that the correction due to the energy transfer moments greater than 2 can be adequately given by the monoatomic heavy-gas model. The problem of neutron thermalization in a monoatomic heavy gas is discussed subsequently.

NEUTRON THERMALIZATION IN A HEAVY GAS

For the special case of a monoatomic heavy gas, the scattering kernel is given by Wigner and Wilkins (16). The first few energy transfer moments based on this kernel are as follows:

$$\sum_s(E) A_1(E) = \sum_{s0} (2 - E) \xi_s \quad (43)$$

$$\sum_s(E) A_2(E) = 2 \xi_s E \sum_{s0} \quad (44)$$

$$\sum_s(E) A_k(E) = 0 \left(\frac{1}{A^2} \right) \text{ for } k \geq 3 \quad (45)$$

where $\bar{\xi}$ = average logarithmic energy decrement per collision and

Σ_{s0} = free atom scattering cross section.

If we use the above energy transfer moments, then the F_{mn} integral, as given by Eq. (23), is transformed in the following manner:

$$F_{mn} = \sum_{k=1}^{\infty} \frac{1}{k!} \int_0^{\infty} \left[\frac{\partial^k L_n^{(1)}(E)}{\partial E^k} L_m^{(1)}(E) \Sigma_s(E) M(E) A_k(E) dE \right] \quad (23)$$

On neglecting terms for $k \geq 3$, we get

$$F_{mn} = \int_0^{\infty} M(E) \bar{\xi} \Sigma_{s0} L_m^{(1)}(E) \left[E \frac{\partial^2}{\partial E^2} + (2 - E) \frac{\partial}{\partial E} \right] L_n^{(1)}(E) dE \quad (46)$$

The associated Laguerre polynomials of order one are the eigenfunctions of the above differential operator, as demonstrated by the following differential equation (25):

$$\left[E \frac{\partial^2}{\partial E^2} + (2 - E) \frac{\partial}{\partial E} \right] L_n^{(1)}(E) = -n L_n^{(1)}(E) \quad (47)$$

Using the above result, the F_{mn} integral reduces to the following simple expression, which can be determined for all m and n :

$$F_{mn} = -n \bar{\xi} \Sigma_{s0} \delta_{mn} \quad (48)$$

Determination of Eigenvalues

The determinant [Eq. (27)] for the heavy-gas case becomes

$$\left| \left(\Sigma_a \right)_{mn} + \left(DB_p^2 \right)_{mn} - \frac{\lambda p_1 W_{mn}}{v_0} + n \bar{\xi} \Sigma_{s0} \delta_{mn} \right| = 0 \quad (49)$$

Häfele and Dresner (8) determined the coefficient of B_0^4 in the expression of λ_0 for the fundamental spatial mode by expanding the determinant in the minors of the first row. The converged value of the diffusion cooling coefficient was found to be about 33% higher than that obtained by using two Laguerre polynomials. The values of λ_1 obtained by using different numbers of Laguerre polynomials are given in Table II. Some of these values were reported previously by the author (17).

The zeroth eigenvalue for the case in Table II is zero in all orders. The second eigenvalue, with three Laguerre polynomials used, is equal to $4.088 \xi \sum_{s0} v_0$. For the case of $1/v$ absorption, we add the extra term $\sum_{s0} v_0$ to all the eigenvalues of all orders. From Table II it is seen that the first eigenvalue obtained by using two Laguerre polynomials is 11.75 and 15.41% higher than the values obtained by using three and four polynomials, respectively.

Determination of Eigenfunctions

The ith energy eigenfunction can be calculated by using the following expansion:

$$\phi_i(E) = \sum_n A_{ni} L_n^{(1)}(E) M(E) \quad (50)$$

The coefficients A_{ni} for the special case of zero absorption and infinite medium are obtained by solving the following set of equations:

$$\sum_{n=0}^{\infty} A_{ni} \left(-\frac{\lambda_1}{v_0} W_{mn} + n \xi \sum_{s0} \delta_{mn} \right) = 0 \quad (51)$$

The Zeroth Eigenfunction:

For $\lambda_0 = 0$, we have

$$\sum_{n=0}^{\infty} A_{n0} \left(n \xi \sum_{s0} \right) = 0 \quad (52)$$

Table II

First Eigenvalue in a Nonabsorbing and Infinite Heavy Gas

Number of Laguerre Polynomials	$\lambda_1(\xi, \sum_{s=0}^{\infty} v_s)$
2	1.506
3	1.329
4	1.274

which leads to all A_{n0} equal to zero, except A_{00} . The zeroth eigenfunction is, therefore, a Maxwellian distribution:

$$\phi_0(E) = A_{00} M(E)$$

The First Eigenfunction:

For λ_1 equal to $1.274 \frac{\xi}{2} \sum_{s0} v_0$, we have the following result, obtained by using the first four associated Laguerre polynomials of order one:

$$\phi_1(E) = A_{01} M(E) (-4.013 + 3.861E - 0.569E^2 + 0.029E^3) \quad (53)$$

$\phi_1(E)$ is plotted in Fig. 2. It has a minimum at 0.4 KT, a zero at 1.27 KT, and a maximum at about 2.5 KT. The other eigenfunctions can be similarly constructed.

THERMALIZATION TIME CONSTANT

The thermalization time constant is defined as the time constant with which the neutrons attain an asymptotic energy distribution by colliding with the atoms of a moderator. As discussed in the previous section, for the special case of a nonabsorbing and infinite medium, the final asymptotic energy distribution is Maxwellian. The reciprocal of λ_1 is equal to the thermalization time constant. In the case of a finite medium, the zeroth eigenvalues associated with the higher spatial modes also play an important role in the establishment of the final asymptotic energy distribution. If the amplitudes of higher spatial modes are very small compared with the fundamental spatial mode, then the first eigenvalue λ_1 would give the thermalization time constant in the finite medium. Our discussion is limited to the time constant with which the Maxwellian distribution is

Fig. 2. First Energy Eigenfunction $\phi_1(E)$ in a Monoatomic Heavy Gas.

established, given by t_{th} .

Estimation of t_{th}

The thermalization time constant, using only two Laguerre polynomials, is given by the reciprocal of λ_1 as given in Eq. (35):

$$t_{th} = \frac{3\sqrt{\pi}}{2M_2v_0} \quad (54)$$

If we use three Laguerre polynomials, then we get the following result from Eq. (41):

$$t_{th} = \frac{3}{2} \frac{\sqrt{\pi}}{M_2v_0} \frac{1.6668}{1 + \frac{F_{22}}{F_{11}} - \frac{F_{22}}{F_{11}} \sqrt{\left(1 + \frac{F_{11}}{F_{22}}\right)^2 - 3.334 \left(\frac{F_{11}}{F_{22}} - \frac{F_{12}^2}{F_{22}^2}\right)}} \quad (55a)$$

$$t_{th} = \frac{3}{2} \frac{\sqrt{\pi}}{M_2v_0} f_t \quad (55b)$$

The correction factor (f_t) due to higher energy transfer moments for a heavy gas using three and four Laguerre polynomials is equal to 1.133 and 1.182, respectively. Table III gives estimates of t_{th} , with f_t equal to 1.182, for various moderators. For the sake of comparison, the values obtained by other authors are also given.

The thermalization time constant for beryllium using the heavy-gas model is equal to 19.51 μsec . Kothari and Singvi (19) reported 21 μsec , and von Dardel and bjostrand (2) gave 22 μsec . Theoretical values by other authors as presented in Table III and also for the heavy-gas model were obtained by using the concept of neutron temperature.

Table III

Thermalization Time Constant, t_{th}

Moderator	Thermalization Parameter M_2 (cm^{-1})	t_{th} (μsec)	Remarks
Beryllium	0.36^a	39.67	Kothari and Singwi (19) obtained theoretically 28 μsec
Beryllium oxide	0.068^a	71.39	Bhandari et al. (20) obtained theoretically 67 μsec
Graphite	0.20^b	209.98	Beckurts (3) obtained $185 \pm 45 \mu\text{sec}$ experimentally $\bar{c}$
Water	0.151^d	9.42	von Dardel (1) obtained 7 μsec experimentally

^aFrom Nelkin (5).^bFrom Singwi and Kothari (6).^cKothari and Khubchandani estimate theoretically 170 μsec [Nuclear Sci. and Eng. 7, 242 (1960)].^dFrom Nelkin (18).

Relation Between t_{th} and C_0

According to Eq. (38), there exists a relation between the first eigenvalue λ_1 and the diffusion cooling coefficient C_0 . Since the reciprocal of λ_1 is the thermalization time constant, we also have the following relation for the constant diffusion coefficient case:

$$t_{th} = \frac{3}{2} \frac{\pi C_0}{D^2 v_0} \quad (56)$$

The above relation was first obtained by Nelkin (5) using the Rayleigh-Ritz variational principle, based upon the neutron temperature concept. Using the same concept, Mani (21) modified it to take into account the variation of the transport mean free path with energy. We have rigorously derived the above relation, without using the neutron temperature concept and using only two associated Laguerre polynomials of order one. We conclude therefore that two polynomials give the same result as the variational principle using the neutron temperature. We generalize the above result for three Laguerre polynomials using the diffusion cooling coefficient as given by Singwi (7) and the thermalization time constant as given by Eq. (55b):

$$t_{th} = \frac{3}{2} \frac{\pi C_0}{D^2 v_0} \frac{f_t}{f_c} \left(1 - 2\sqrt{2} \frac{l_{01}}{l_{00}} \right)^2 \quad (57)$$

where

f_c = correction factor for diffusion cooling coefficient due to higher energy transfer moments,

$$l_{0m} = \frac{1}{\sqrt{2}} \int_0^\infty l_{tr}(E) M(E) L_0^{(1)}(E) L_m^{(1)}(E) dE,$$

$l_{tr}(E)$ = transport mean free path.

On the basis of a monoatomic heavy-gas model, f_t/f_c is equal to 0.954. For the case of constant diffusion coefficient, l_{01} is equal to zero.

Experimental Determination of t_{th}

The thermalization time constant has been determined experimentally (2) by the transmission method, using foils. Nelkin (5) suggested the determination of the thermalization time constant using the experimental value of the diffusion cooling coefficient obtained from pulsed-neutron techniques. This method is being used to estimate the thermalization time constant and also the thermalization parameter M_2 but requires the higher energy transfer moment correction f_t/f_c , the energy-dependent transport mean free path correction l_{01}/l_{00} , and the finite medium correction arising due to the neglect of B^6 and other higher order terms. All these corrections make this method less satisfactory. We propose a direct measurement of the first eigenvalue λ_1 by the $1/v$ absorber method in an infinite medium. λ_0 and λ_1 are as follows:

$$\lambda_0 = \sum a_0 v_0 \quad (58)$$

$$\lambda_1 = \sum a_0 v_0 + \frac{1}{t_{th}} \quad (59)$$

In the presence of a large $1/v$ absorber, for example, boron in water, the decay of the neutron distribution even at long times is governed by λ_0 and λ_1 . By the analysis of the composite decay curve, t_{th} can be determined. It should be mentioned that this method eliminates the finite medium corrections and does not require knowledge of higher energy transfer moments nor of the energy dependence of the transport mean free path.

4. MULTIPLYING MEDIUM

The mathematical formalism for determining the eigenvalues in a multiplying medium has already been given. In the case of a multiplying medium, there is a regeneration of neutrons, as well as a loss of neutrons by absorption and leakage. Krieger and Zweifel (22) have also studied the time behavior of monoenergetic thermal neutrons in a pulsed multiplying medium. We shall discuss the monoenergetic and polyenergetic cases.

THE MONOENERGETIC CASE

The eigenvalue equation (19) for the monoenergetic case is given as follows:

$$\phi_p(E) \left[-\frac{\lambda_p}{v} + \sum_a(E) + D(E) B_p^2 - v \sum_f(E) (1 - \beta) \tilde{q}_{\infty}(B_p^2; -\lambda_p; E) \right] = 0 \quad (60)$$

The eigenvalue corresponding to the p th spatial mode is readily given from Eq. (60):

$$\lambda_p = v \left[\sum_a(E) + D(E) B_p^2 - v \sum_f(E) (1 - \beta) \tilde{q}_{\infty}(B_p^2; -\lambda_p; E) \right]$$

The Laplace-Fourier transform of the slowing-down kernel is discussed for two special cases.

Heavy-Element Moderation

The slowing-down kernel in the infinite medium, according to the Fermi age model, is as follows:

$$q_{\infty}(r, E, t_0) = \frac{e^{-r^2/4\tau}}{(4\pi\tau)^{3/2}} \frac{P_{\text{res}}}{v} \delta(t_0 - t_s) \quad (61)$$

where

τ = age of fission neutrons (in the heavy element) to have slowed down to final energy E ,

t_s = slowing-down time required by the fission neutrons to reach energy E ,

p_{res} = resonance escape probability for the fission neutrons to reach energy E ,

$\delta(t_0 - t_s)$ = Dirac delta function.

We can therefore determine $\tilde{q}_{\infty}(B_p^2; -\lambda_p; E)$:

$$\begin{aligned} \tilde{q}_{\infty}(B_p^2; -\lambda_p; E) &= 4\pi \int_{t_0=0}^{\infty} \int_{r=0}^{\infty} p_{res} \frac{e^{-r^2/4\tau}}{(4\pi\tau)^{3/2}} \delta(t_0 - t_s) \frac{\sin Br}{Br} e^{\lambda_p t_0} r^2 dr dt_0 \\ &= p_{res} e^{-B_p^2 \tau} e^{\lambda_p t_s} \end{aligned} \quad (62)$$

Hydrogen Moderation

The Laplace-Fourier transform of the slowing-down kernel in an infinite hydrogenous medium is given as follows:

$$\tilde{q}_{\infty}(B_p^2; -\lambda_p; E) = \frac{\sum_s(E_0) \exp \left\{ - \int_E^{E_0} \frac{[\sum_a(E') + D(E') B_p^2 - \lambda_p/v'] dE'}{E' [\sum_a(E') + D(E') B_p^2 + \sum_s(E') - \lambda_p/v']} \right\}}{\sum_a(E_0) + \sum_s(E_0) + D(E_0) B_p^2 - \lambda_p/v_0} \quad (63)$$

The above result was discussed in detail by the author elsewhere (12).

It is based upon the Selengut-Goertzel approximation for the determination of neutron current. E_0 is assumed to be the mean energy with which the

fission neutrons are emitted. The expression on the right side can be written as a product of four factors:

$$\frac{\Sigma_s(E_0)}{X(E_0)} = \text{first-flight scattering probability} \quad (64)$$

$$\exp \left[- \int_E^{E_0} \frac{\Sigma_a(E') dE'}{E' X(E')} \right] = \text{resonance capture escape probability} \quad (65)$$

$$\exp \left[- \int_E^{E_0} \frac{D(E') B_p^2 dE'}{X(E') E'} \right] = \text{nonleakage probability} \quad (66)$$

$$\exp \left\{ \lambda_p \int_E^{E_0} \frac{dE'}{v' [X(E') E]} \right\} = \text{decay probability} \quad (67)$$

where

$$X(E) = \Sigma_a(E) + \Sigma_s(E) + D(E) B_p^2 - \frac{\lambda_p}{v}$$

In the case of hydrogenous systems, $\tilde{q}_{\infty}$ is a complicated function of λ_p ; it is expressed as a product of two factors depending upon B_p^2 and $-\lambda_p$ in the following manner:

$$\tilde{q}_{\infty} = \bar{P}(B_p^2) \tilde{P}(-\lambda_p) \quad (68)$$

$$\bar{P}(B_p^2) = \frac{\Sigma_s(E_0) \exp \left\{ - \int_E^{E_0} \left[\frac{\Sigma_a(E') + D(E') B_p^2}{\Sigma_a(E') + \Sigma_s(E') + D(E') B_p^2} \right] \frac{dE'}{E'} \right\}}{\Sigma_a(E_0) + D(E_0) B_p^2 + \Sigma_s(E_0)} \quad (69)$$

In Eq. (69) it is assumed that $\Sigma_a(E) + \Sigma_s(E) + D(E) B_p^2 \gg \lambda_p/v$.

$$\tilde{P}(-\lambda_p) = \frac{\exp \left\{ \lambda_p \int_E^{E_0} \frac{dE'}{v' \left[\sum_a (E') + \sum_s (E') - \lambda_p / v' \right] E'} \right\}}{1 - \frac{\lambda_p}{v_0 \left[\sum_a (E_0) + \sum_s (E_0) \right]}} \quad (70)$$

In Eq. (70) it is assumed that $\sum_a (E) + \sum_s (E) - \frac{\lambda_p}{v} \gg D(E) B_p^2$.

Replacement of $\tilde{P}(-\lambda_p)$ by $e^{\lambda_p t_s(E)}$ is justified only if $t_s(E)$ is given as follows:

$$t_s(E) = \int_E^{E_0} \frac{dE'}{\left[\sum_a (E') + \sum_s (E') - \lambda_p / v' \right] v' E'} \quad (71)$$

For $\sum_a = 0$ and $\lambda_p = 0$ $t_s(E)$ reduces to the familiar expression

$$t_s(E) = \int_E^{E_0} \frac{dE'}{v' E' \sum_s (E')} \quad (72)$$

In the analysis of pulsed-neutron experiments, Eqs. (73) and (74) may be used for the eigenvalues associated with the p th spatial mode for the heavy-element and hydrogen-moderated systems.

For the heavy-element moderator:

$$\lambda_p = v \left[\sum_a (E) + D(E) B_p^2 - \nu \sum_f (E) (1 - \beta) p_{res} e^{-B_p^2 \tau} e^{\lambda_p t_s} \right] \quad (73)$$

For the hydrogen moderator:

$$\lambda_p = v \left[\sum_a (E) + D(E) B_p^2 - \nu \sum_f (E) (1 - \beta) \bar{P}(B_p^2) \tilde{P}(-\lambda_p) \right] \quad (74)$$

THE MULTIENERGY CASE

The determination of eigenvalues involving the multienergy case depends upon the knowledge of G_{mn} integrals as defined by Eq. (24). These integrals appear in the determinant for determining the eigenvalues in the multiplying medium according to Eq. (21):

$$\left| -\frac{\lambda}{v_0} w_{mn} + (\sum_a)_{mn} + \left(DB_p^2 \right)_{mn} - F_{mn} - G_{mn} \right| = 0 \quad (75)$$

where

$$G_{mn} = \int_0^\infty \int_0^\infty v \sum_f(E') (1 - \beta) L_m^{(1)}(E) L_n^{(1)}(E') M(E') \tilde{q}_\infty(B_p^2; -\lambda_{pi}; E) dE' dE$$

The exact determination of the G_{mn} integral requires knowledge of the energy dependence of the fission cross section $\sum_f(E')$ and the dependence of $\tilde{q}_\infty$ upon the final energy E . Two assumptions are made in order to evaluate these integrals:

- (1) Fission takes place at a single energy E :

$$\sum_f(E') = \sum_f(E) \delta(E' - E) \quad (76)$$

- (2) The Laplace-Fourier transform of the slowing-down kernel in the infinite medium depends upon the final energy E , which is given by an average energy E_{th} , during the thermal energy interval:

$$\tilde{q}_\infty(B_p^2; -\lambda_{pi}; E) \cong \tilde{q}_\infty(B_p^2; -\lambda_{pi}; E_{th}) \quad (77)$$

Now the G_{mn} integrals can be simplified in the following manner:

$$G_{mn} = \nu(1 - \beta) \tilde{q}_{\infty}(B_p^2; -\lambda_{pi}; E_{th}) (\Sigma_f)_{mn} \quad (78)$$

where

$$(\Sigma_f)_{mn} = \int_0^{\infty} \Sigma_f(E) L_m^{(1)}(E) M(E) L_n^{(1)}(E) dE$$

Even if we use the above expression for G_{mn} , it is not possible to determine the eigenvalues by solving the determinant, as given by Eq. (75). The difficulty arises from the complicated form of $\tilde{q}_{\infty}$ as a function of B_p^2 and λ_{pi} . We may, however, express $\tilde{q}_{\infty}$ as follows:

$$\tilde{q}_{\infty}(B_p^2; -\lambda_{pi}; E_{th}) \cong p_{res} (1 - B_p^2) (1 + \lambda_{ps}) \quad (79)$$

We do not attempt the determination of higher eigenvalues. The expression for the zeroth eigenvalue for $m = 0$ and $n = 0$ corresponding to the Maxwellian energy distribution is the following:

$$\lambda_p = \frac{v_0}{w_{00}} \left[(\Sigma_a)_{00} + (DB_p^2)_{00} - \nu(1 - \beta) \tilde{q}_{\infty}(B_p^2; -\lambda_p; E_{th}) (\Sigma_f)_{00} \right] \quad (80)$$

It is proposed that the above expression may be used in the analysis of pulsed-neutron experiments in multiplying media for the Maxwellian distribution.

ANALYSIS OF AN EXPERIMENT

To illustrate the importance of using the correct expression for $\tilde{q}_{\infty}$, we analyze the results of an experiment of Campbell and Stelson (23) for a

uranium-plus-water system. The experiment gives λ_0 for different values of B_0^2 . From these values of λ_0 $\tilde{q}_\infty$ can be calculated according to Eq. (80). The nonleakage probabilities calculated on the basis of the experimental results of Campbell and Stelson (24) are given in Table IV. These results are based upon the measured values of the constants reported by Campbell and Stelson (23).

It can be noted from Table IV that $\tilde{P}(-\lambda_0)$ is not equal to 1, but increases rapidly with increase in B_0^2 . Campbell and Stelson (23) assumed $\tilde{q}_\infty$ to be equal to $\bar{P}(B_0^2)$. Assuming $\bar{P}(B_0^2)$ to be given by $\frac{1}{1 + B_0^2 r}$ in the region where B_0^2 is less than 0.14 cm^{-2} , they obtained the age for fission neutrons to thermal energy in water to be equal to 21.2 cm^2 . Using the values of $\bar{P}(B_0^2)$, as given in Table IV, we estimate the age in water to be about 22.82 cm^2 , with the same approximations as mentioned above. It is important to note that the $\tilde{P}(-\lambda_0)$ correction increases the age in the right direction. In small finite systems, the experimental value of $\tilde{q}_\infty$ is more meaningful than the age itself.

5. CONCLUSION

The importance of the first eigenvalue in establishing the asymptotic energy distribution has been discussed for several cases. For the special case of a nonabsorbing and infinite medium, the reciprocal of the first eigenvalue is equal to the thermalization time constant for the establishment of the Maxwellian distribution. The values of thermalization time constants obtained with the help of M_2 and with a correction factor due to higher energy transfer moments given by the heavy-gas model are in good agreement with the results of other authors. Therefore it may be concluded that the thermalization time constant for the establishment of the Maxwellian distribution can be determined

Table IV

Nonleakage Probabilities

$B_0^2(\text{cm}^{-2})$	$\lambda_0(\text{sec}^{-1})$	$\tilde{q}_{\infty}(B_0^2, -\lambda_0; E_{th})$	$\tilde{P}(-\lambda_0)$	$\tilde{P}(B_0^2)$
	$\times 10^{-12}$			
0.0436	0.550	0.524	1.037	0.505
0.0483	0.630	0.490	1.043	0.470
0.0562	0.704	0.469	1.048	0.448
0.072	0.883	0.407	1.065	0.382
0.135	1.386	0.265	1.099	0.241
0.192	1.786	0.174	1.116	0.156
0.296	2.358	0.060	1.168	0.051
0.506	3.194	0.009	1.232	0.007

rigorously without using the concept of neutron temperature. The use of $t_{th} = \frac{3}{2} \left(\pi C_0 / D^2 v_0^2 \right)$, as given by Eq. (56), cannot be justified to determine the thermalization time constant or the thermalization parameter M_2 . This relation, as shown in this study, is obtained by using only two Laguerre polynomials. However, a direct measurement of the first eigenvalue in an infinite medium containing $1/v$ absorber would give the thermalization time constant exactly. Theoretically, the first eigenvalue can be calculated exactly for the monoatomic heavy gas. Using the first four Laguerre polynomials, it is found to be equal to $1.274 \xi \sum_{s=0}^{\infty} v_0$. For other physical models, the first few energy transfer moments would give the first eigenvalue rigorously. The comparison between the calculated and the measured values of the first eigenvalue would prove more valuable in the study of neutron thermalization than the measured and calculated values of the diffusion cooling coefficient. The knowledge of lower eigenvalues and eigenfunctions is adequate to describe the asymptotic behavior of the energy distribution but is not sufficient to describe the complete time behavior of neutrons during the thermalization. Time-dependent energy spectra for times greater than the slowing-down time in a monoatomic heavy gas were generated numerically and reported by the author (12).

It should be remarked here that the pulsed-neutron experiments in a nonmultiplying medium gives the absorption cross section and the diffusion coefficient averaged over the Maxwellian distribution. This conclusion arises from the theoretical and the experimental investigations of the zeroth eigenvalue of a decaying pulse of neutrons in a finite medium. The change in the zeroth eigenvalue due to the deviation of the asymptotic energy dis-

tribution from the Maxwellian is contained in the diffusion cooling term. In the case of a multiplying medium, the time-dependent nonleakage probability prevents the complete analysis of the problem. This quantity can be determined exactly for the monoenergetic neutrons. For polyenergetic neutrons, detailed knowledge is required of the slowing down of neutrons in the thermal energy region, which is not very well understood. Before the reactor physics parameters τ , η , and t_g are determined from the experimental results, a general agreement between the theoretical and experimental values of the time-dependent nonleakage probability factor is desirable. As indicated by the analysis of an experiment, an approximate representation of this quantity can introduce a large uncertainty in the determination of the important physical parameters. It may, however, be added that the experiments involving the multiplying medium can provide direct information about nonleakage probability, which is in itself of great physical interest.

Acknowledgments

The author wishes to express his thanks to Professor P. F. Zweifel, Nuclear Engineering Department, University of Michigan, Ann Arbor, for his valuable advice. He would also like to thank Mr. E. P. Blizard, the director of the Neutron Physics Division, Oak Ridge National Laboratory, for providing him with the research facilities, and Drs. L. Dresner and E. Inönü for their stimulating discussions.

Appendix

LACUERRE POLYNOMIALS $L_n^{(1)}(E)$, MATRIX ELEMENTS W_{mn} , AND INTEGRALS F_{mn}

Associated Laguerre Polynomials of Order One (25)

$$L_n^{(1)}(E) = \sqrt{n+1} \sum_{p=0}^n \frac{(-1)^p n! E^p}{p! (p+1)! (n-p)!}$$

$$L_0^{(1)}(E) = 1$$

$$L_1^{(1)}(E) = \sqrt{2} (1 - E/2)$$

$$L_2^{(1)}(E) = \sqrt{3} \left(1 - E + \frac{E^2}{3!} \right)$$

$$L_3^{(1)}(E) = \sqrt{4} \left(1 - \frac{3}{2} E + \frac{E^2}{2} - \frac{E^3}{4!} \right)$$

$$\left[E \frac{\partial^2}{\partial E^2} + (2 - E) \frac{\partial}{\partial E} \right] L_n^{(1)}(E) = -n L_n^{(1)}(E)$$

Matrix Elements W_{mn}

$$W_{mn} = \frac{1}{\pi \sqrt{m+1} \sqrt{n+1}} \left[\sum_{k=0}^m \frac{\Gamma(m-k+1/2) \Gamma(n-k+1/2) \Gamma(k+3/2)}{(m-k)! (n-k)! k!} \right] \text{ for } n \geq m$$

$$W_{00} = 0.500 \sqrt{\pi}$$

$$W_{01} = W_{10} = 0.1768 \sqrt{\pi}$$

$$W_{11} = 0.4375 \sqrt{\pi}$$

$$W_{02} = W_{20} = 0.1083 \sqrt{\pi}$$

$$W_{12} = W_{21} = 0.1914 \sqrt{\pi}$$

$$W_{22} = 0.3984 \sqrt{\pi}$$

$$W_{03} = W_{30} = 0.0781 \sqrt{\pi}$$

$$W_{31} = W_{13} = 0.1271 \sqrt{\pi}$$

$$W_{32} = W_{23} = 0.1928 \sqrt{\pi}$$

$$W_{33} = 0.3706 \sqrt{\pi}$$

F_{mn} Integrals

$$F_{mn} = \sum_{k=1}^{\infty} \frac{1}{k!} \left\{ \int_0^{\infty} \frac{\partial^k L_n^{(1)}(E)}{\partial E^k} L_m^{(1)}(E) \sum_s(E) M(E) \left[\int_0^{\infty} F(E \rightarrow E') (E' - E)^k dE' \right] dE \right\}$$

Using the detailed balance theorem,

$$F_{00} = F_{10} = F_{01} = F_{20} = F_{02} = 0$$

$$F_{11} = -\frac{M_2}{4}$$

$$F_{21} = F_{12} = -\frac{\sqrt{6}}{4} \left(M_2 - \frac{M_3}{6} \right)$$

$$F_{22} = -\frac{3}{2} \left(M_2 - \frac{M_3}{3} + M_4 \right)$$

where

$$M_2 = \int_0^\infty \int_0^\infty (E' - E)^2 \sum_E(E) M(E) F(E \rightarrow E') dE' dE$$

$$M_3 = \int_0^\infty \int_0^\infty (E' + E)(E' - E)^2 \sum_E(E) M(E) F(E \rightarrow E') dE' dE$$

$$M_4 = \int_0^\infty \int_0^\infty (E' + E)^2 (E' - E)^2 \sum_E(E) M(E) F(E \rightarrow E') dE' dE$$

REFERENCES

1. G. F. von Dardel, Trans. Roy. Inst. Technol. Stockholm, vol 75 (1954).
2. G. F. von Dardel and N. G. Sjöstrand, Phys. Rev. 96, 1245 (1954); p 185
in Progress in Nuclear Energy, Series I, vol 2, Pergamon Press, New York,
1958.
3. K. H. Beckurts, Nuclear Sci. and Eng. 2, 516 (1957).
4. H. Hurwitz, Jr., and M. S. Nelkin, Nuclear Sci. and Eng. 3, 1 (1958).
5. M. S. Nelkin, J. Nuclear Energy 8, 48 (1958).
6. K. S. Singwi and L. S. Kothari, J. Nuclear Energy 8, 59 (1958).
7. K. S. Singwi, Arkiv Fysik 15, 147 (1960); J. Nuclear Energy 11, 19 (1959).
8. W. Häfele and L. Dresner, Neutron Physics Division Annual Progress Report
for Period Ending September 1, 1959, ORNL-2842, p 124.
9. M. V. Kazarnovsky, A. V. Stepanov, and F. L. Shapiro, "Neutron Thermal-
lization and Diffusion in Heavy Media," Paper 2148, Proc. 2nd U. N. Conf.
Peaceful Uses Atomic Energy, Geneva, 1958.
10. M. S. Nelkin, Nuclear Sci. and Eng. 7, 210 (1960).
11. N. G. Sjöstrand, Arkiv Fysik 15, 12, 147 (1959).
12. S. N. Purohit, Time-Dependent Neutron Energy Spectra in Pulsed Media,
Ph.D. Thesis, University of Michigan, Ann Arbor, 1960.
13. A. M. Weinberg, Am. J. Phys. 20, 401 (1952).
14. A. M. Weinberg and E. P. Wigner, The Physical Theory of Neutron Chain
Reactors, p 381, University of Chicago Press, Chicago, 1958.
15. G. deSaussure and E. G. Silver, Determination of the Neutron Diffusion
Parameters in Room Temperature Beryllium, ORNL-2641 (Feb. 27, 1959).

16. E. P. Wigner and J. E. Wilkins, Jr., Effect of the Temperature of the Moderator on the Velocity Distribution of Neutrons with Numerical Calculations for H as Moderator, AECD-2275 (Sept. 14, 1944).
17. S. N. Purchit, Neutron Physics Division Annual Progress Report for Period Ending September 1, 1959, ORNL-2842, p 141.
18. Private communication by E. Parks, General Atomics, San Diego, Calif.
19. L. S. Kothari and K. S. Singwi, J. Nuclear Energy 5, 342 (1957); Solid State Physics, vol 8, p 184, Academic Press, New York, 1959.
20. R. C. Bhandari, L. S. Kothari, and K. S. Singwi, J. Nuclear Energy 7, 45 (1958).
21. G. S. Mani, J. Nuclear Energy 7, 220 (1958).
22. T. J. Krieger and P. F. Zweifel, Nuclear Sci. and Eng. 5, 21 (1959).
23. E. C. Campbell and P. H. Stelson, Physics Division Semiannual Progress Report for Period Ending September 10, 1956, ORNL-2204.
24. Private communication by E. C. Campbell, ORNL.
25. I. N. Sneddon, Special Functions of Mathematical Physics and Chemistry, Interscience, New York, 1956.

List of Tables

- Table I. Ratios of Eigenvalues in Beryllium.
- Table II. First Eigenvalue in a Nonabsorbing and Infinite Heavy Gas.
- Table III. Thermalization Time Constant, t_{th} .
- Table IV. Nonleakage Probabilities.

List of Figures

1. Variation of λ_0/λ_1 with B^2 for Beryllium.
2. First Energy Eigenfunction $\phi_1(E)$ in a Monoatomic Heavy Gas.

Distribution

1. F. S. Alsmiller
- 2-5. E. P. Blizard
6. A. D. Callihan
7. E. C. Campbell
8. G. T. Chapman
9. R. R. Coveyou
10. G. deSaussure
11. L. Dresner
12. R. Gwin
13. K. M. Henry
14. L. B. Holland
15. E. Inonu
16. E. B. Johnson
17. W. H. Jordan
18. P. Kasten
19. T. A. Love
20. R. E. Maerker
21. F. C. Maienschein

22. J. T. Mihalcz
23. F. J. Muckenthaler
24. R. W. Peelle
25. S. K. Penny
26. R. Perez-Belles
- 27-52. S. N. Purohit
53. M. J. Skinner
54. E. G. Silver
55. A. Simon
56. K. S. Singwi
57. J. T. Thomas
58. A. M. Weinberg
59. W. Zobel
- 60-61. Central Research Library
- 62-73. Laboratory Records Department
74. Laboratory Records, ORNL R.C.
75. Document Reference Section,
Y-12

External

76. N. Corngold, Brookhaven National Laboratory, Upton, Long Island, New York
77. J. Chernick, Brookhaven National Laboratory, Upton, Long Island, New York
78. H. J. Gomberg, Nuclear Engineering Dept., University of Michigan, Ann Arbor, Michigan
79. M. Nelkin, General Atomics, P. O. Box 608, San Diego 12, California
80. R. K. Osborn, University of Michigan, Ann Arbor, Michigan
81. R. V. Meghreblan, Jet Propulsion Laboratory, Pasadena, California
82. D. E. Parks, General Atomics, P. O. Box 608, San Diego 12, California
83. P. F. Zweifel, Nuclear Engineering Dept., University of Michigan, Ann Arbor, Michigan
- 84-98. TISE, AEC

**DO NOT
PHOTOSTAT**