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A STUDY OF TURBULENT BOUNDARY LAYER MODELS
PART II, MEAN TURBULENT FIELD CLOSURE

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Prepared by

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This report supersedes SC-CR-70-6125, dated May 1970. Revisions have been made in the discussion of the "Viscous terms" beginning on page 14. As a result, there are slight changes in the equations in this and subsequent sections and in one of the relations given in Figure 1.

ABSTRACT

In order to close the mean velocity equations appeal is made to the mean turbulent equations of motion obtained by taking the first moment (in velocity) of the Navier Stokes equations. To close the resulting equation set, assumptions are required, the consequence of which has been labeled Mean Turbulent Field closure. Whenever possible, an attempt at supplying a rational basis for each assumption is made. In any event, the central hypothesis is that already proposed by Rotta. Various simplified sub-models may be identified including Mean Velocity Field closure.

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1. INTRODUCTION

Although in Part I of this series considerable success was claimed for Mean Velocity Field (MVF) closure, several major difficulties were encountered. The most important was that the very concept of an eddy viscosity, although heuristically plausible in the spirit of Prandtl's mixing-length argument did not seem to evolve directly from the equations of motion. Part II is, therefore, devoted to a discussion of the turbulent transport equations in an attempt to disclose an underlying basis for some type of constitutive equation, to determine guide lines for a conceptional view of turbulent flow and to increase predictive capability.

The most plausible overall framework for this analysis is that proposed by Prandtl and Rotta and quite a bit of the material included here will be found in the papers by Rotta (1951). The basis of Mean Turbulent Field (MTF) closure is the complete equations for both the mean velocity field and the mean turbulent field. Flux-gradient constitutive equations are proposed and their coefficients are related directly to mean turbulent field variables rather than the mean velocity field. Although more empirical constants are required, the additional capability is provided of calculating the turbulent field quantities for comparison with experiment. This is important because it would seem that as models become complex the additional complexity should be supported by additional data.

As mentioned above, the overall framework described in this last part was originally conceived by Prandtl and developed much further by

Rotta. Variants of the MTF closure have been applied by Glushko (1965), Beckwith and Bushnell (1968), Mellor and Herring (1968-II), which is described further in this paper, and most recently Donaldson and Rosenbaum (1968). It is clear that, given the general framework, the theory has nevertheless not evolved to a point of consensus as has the MVF closure. Furthermore, within the MTF closure there does appear to be more than one valid level of empiricism which differ in degree of complexity. One purpose of this paper is to emphasize what are believed to be the important elements of MTF closure so as to provide some organization for future research.

As in Part I, the Method of Matched Asymptotic Expansions suggests a framework to systematically determine the necessary empirical constitutive coefficients.

2. THE BASIC EQUATIONS

The field equations of motion for incompressible flow when decomposed into ensemble mean velocities, U_i , plus fluctuation velocities, u_i , and their concomitant pressures P and p are

$$\frac{\partial U_i}{\partial x_i} = 0, \quad (1)$$

$$\frac{\partial U_j}{\partial t} + \frac{\partial}{\partial x_k} (U_k U_j + \overline{u_k u_j}) = \frac{\partial}{\partial x_k} (-\delta_{kj} P + T_{kj}), \quad (2)$$

$$\frac{\partial u_i}{\partial x_i} = 0, \quad (3)$$

$$\begin{aligned} \frac{\partial u_j}{\partial t} + \frac{\partial}{\partial x_k} (U_k u_j + U_j u_k + u_k u_j - \overline{u_k u_j}) \\ = \frac{\partial}{\partial x_k} (-\delta_{kj} p + \tau_{kj}), \end{aligned} \quad (4)$$

where

$$T_{ij} \equiv \nu \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right); \quad \tau_{ij} \equiv \nu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad (5a,b)$$

are the mean and fluctuating viscous stresses respectively. In the MVF closure of Part I only equation (1) and (2) were used and closure was achieved by empirically relating $\overline{u_i u_j}$ directly to the mean velocity strain where only one component of $\overline{u_i u_j}$ was required by virtue of the boundary layer approximation. Here, however, an equation for $\overline{u_i u_j}$ is obtained by taking a moment of (4) with u_i and adding this to another moment of (4)

after the indices have been interchanged. The result is,

$$\begin{aligned}
 \frac{\partial \overline{u_i u_j}}{\partial t} + \frac{\partial}{\partial x_k} [u_k \overline{u_i u_j} + \overline{u_k u_i u_j}] &= - \overline{u_k u_i} \frac{\partial u_j}{\partial x_k} - \overline{u_k u_j} \frac{\partial u_i}{\partial x_k} \\
 &\quad - \overline{u_i} \frac{\partial p}{\partial x_j} - \overline{u_j} \frac{\partial p}{\partial x_i} \\
 &\quad + \overline{u_i} \frac{\partial \tau_{kj}}{\partial x_k} + \overline{u_j} \frac{\partial \tau_{ki}}{\partial x_k} .
 \end{aligned} \tag{6}$$

Another form of (6) is,

$$\begin{aligned}
 \frac{\partial \overline{u_i u_j}}{\partial t} + \frac{\partial}{\partial x_k} [u_k \overline{u_i u_j} + \overline{u_k u_i u_j} + \overline{u_i (\delta_{kj} p - \tau_{kj})} + \overline{u_j (\delta_{ki} p - \tau_{ki})}] \\
 = - \overline{u_k u_i} \frac{\partial u_j}{\partial x_k} - \overline{u_k u_j} \frac{\partial u_i}{\partial x_k} \\
 + \overline{p} \frac{\partial u_i}{\partial x_j} + \overline{p} \frac{\partial u_j}{\partial x_i} \\
 - \overline{\tau_{kj}} \frac{\partial u_i}{\partial x_k} - \overline{\tau_{ki}} \frac{\partial u_j}{\partial x_k} .
 \end{aligned} \tag{7}$$

Whether (6) or (7) is the preferable form is more easily decided with regard to the trace of (6) or (7). If we contract (7), we obtain

$$\frac{\partial}{\partial t} \left(\frac{q^2}{2} \right) + \frac{\partial}{\partial x_k} \left[U_k \frac{q^2}{2} + \overline{u_k \frac{u_i^2}{2}} + \overline{u_k p} - \overline{u_i \tau_{ki}} \right] \quad (8)$$

local rate of change of turbulent energy	turbulent energy advection	turbulent kinetic diffusion	turbulent pressure diffusion	turbulent viscous diffusion
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$$= \overline{(-u_k u_i)} \frac{\partial U_i}{\partial x_k} - \overline{\tau_{ki} \frac{\partial u_i}{\partial x_k}},$$

turbulent energy production	turbulent energy dissipation
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where $q^2 \equiv \overline{u_i^2}$. The above is a relatively familiar* identification of terms. The reason that (8), as opposed to the trace of (6), is attractive is that the dissipation term

$$\epsilon = \overline{\tau_{ki} \frac{\partial u_i}{\partial x_k}} = \frac{\nu}{2} \overline{\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right)^2} \quad (9)$$

is positive definite; in (8) it is always a "sink" term. In fact, consideration of the second law of thermodynamics shows that this must be so and the assumed constitutive equation (5a,b) of Navier and Stokes, where ν is positive, satisfies this requirement. Similarly, a "second law of turbulence" might be hypothesized to require that the production, $-\overline{u_k u_i} \partial U_i / \partial x_k$ be a positive, source term.† A linear constitutive equation for $-\overline{u_k u_i}$ involving a positive, scalar eddy viscosity would satisfy this requirement. Of course, a "second law of turbulence", if valid at all, can only be approximate and, for example, small regions of

* However, as noted by Corrsin (1953), the viscous terms are often misinterpreted. This continues to be true.

† And which appears as a sink term in the mean velocity kinetic energy equation; see Table I.

negative production in the case of wall jets have apparently been measured. Nevertheless, the predominant role of the production term as a positive source is hardly disputable.

Table I displays the complete hierarchy of energy equations where the energy flows from the mean velocity kinetic energy through the turbulent kinetic energy and finally to internal energy. By adding all equations the conservative nature of the system may be seen.

It should be pointed out that the diagonal equations in (7), either as summed or individually, all display a conservative nature, thus coloring our semantics. However, this property disappears when off diagonal equations are considered.

TABLE I

<u>Advection</u>	<u>Turbulent Diffusion and Heat Transfer</u>	<u>Turbulent Energy Transfer</u>	<u>Mean Viscous Transfer</u>	<u>Turbulent Viscous Transfer</u>
$\frac{D(U_i^2/2)}{Dt}$	$+ \frac{\partial}{\partial x_k} [U_k p + U_j (-\overline{u_k u_j} + T_{kj})]$	$= - (-\overline{u_i u_j}) \frac{\partial U_j}{\partial x_i}$	$- T_{ij} \frac{\partial U_j}{\partial x_i}$	
$\frac{D(q^2/2)}{Dt}$	$+ \frac{\partial}{\partial x_k} [\overline{u_k p} + \overline{u_k u_j^2/2} - \overline{u_j \tau_{ki}}]$	$= (\overline{u_i u_j}) \frac{\partial U_j}{\partial x_i}$		$- \overline{\tau_{ij} \frac{\partial u_j}{\partial x_i}}$
$+ \frac{DE}{Dt}$	$+ \frac{\partial}{\partial x_k} (\overline{u_k e} + Q_k)$	$=$	$+ T_{ij} \frac{\partial U_j}{\partial x_i}$	$+ \overline{\tau_{ij} \frac{\partial u_i}{\partial x_i}}$
<hr/>				
$\frac{D}{Dt} \left[E + \frac{U_i^2}{2} + \frac{q^2}{2} \right] + [\text{Turbulent Diffusion and Heat Transfer}]^* = 0$				

[†] E = mean internal energy; e = fluctuating internal energy; Q_k = mean heat flux.

* If the equation is integrated through a volume bounded by stationary and adiabatic surfaces, this term integrates to zero.

3. KINETIC THEORY AND NEARLY ISOTROPIC TURBULENT FLOW

The kinetic theory of dilute gases represents a complete (so long as intermolecular forces are known) and purely mechanistic theory of molecular scale fluctuations which yields an energy equation similar to (8) but lacking terms involving p and τ_{ki} , and where u_i may be interpreted as the random molecular or "peculiar" velocity. However, kinetic theory also yields simple constitutive equations for $\overline{u_i u_j}$ and $\overline{u_k u_\ell^2}/2$ in the case of nearly isotropic molecular velocity fields; that is where $\left| \overline{u_i u_j} - \delta_{ij} q^2/3 \right| \ll q^2$ and $\left| \overline{u_i u_j u_k} - \frac{\delta_{ij}}{3} \overline{u_k u_\ell^2} \right| \ll q^3$. The fact that similar constitutive equations do not seem to be available for turbulent flow represents one of its major difficulties. Therefore, it is instructive to compare the kinetic theory equations (see, for example, Chapman and Cowling, 1958 or Patterson, 1956) with the turbulent equations, insofar as is possible. For this purpose, equation (6) seems to be more convenient than (7). To specifically identify departures from isotropy, a non-dimensional tensor, a_{ij} , maybe defined, such that

$$\overline{u_i u_j} = (\delta_{ij} + a_{ij}) \frac{q^2}{3}, \quad (10)$$

where $a_{ii} = 0$. Next, one can contract (6), (which, of course, is (8)), multiply each term by $\delta_{ij}/3$ and subtract from (6) again. Using (10) and rearranging, the result is

$$\begin{aligned}
& \frac{q}{3} \frac{\partial U_m}{\partial x_n} (\delta_{jm} \delta_{in} + \delta_{im} \delta_{jn} + a_{nj} a_{im} + a_{ni} \delta_{jm} - \frac{2}{3} \delta_{ij} a_{nm}) \\
& + \frac{D}{Dt} (a_{ij} \frac{q^2}{3}) + \frac{\partial}{\partial x_k} (\overline{u_k u_i u_j} - \frac{\delta_{ij}}{3} \overline{u_k u_l^2}) \\
& = - \overline{u_i \frac{\partial p}{\partial x_j}} - \overline{u_j \frac{\partial p}{\partial x_i}} + \frac{2}{3} \delta_{ij} \overline{u_k \frac{\partial p}{\partial x_k}} \\
& + \overline{u_i \frac{\partial \tau_{kj}}{\partial x_k}} + \overline{u_j \frac{\partial \tau_{ki}}{\partial x_k}} - \frac{2}{3} \delta_{ij} \overline{u_l \frac{\partial \tau_{kl}}{\partial x_k}}, \tag{11}
\end{aligned}$$

where

$$\frac{Df}{Dt} \equiv \frac{\partial f}{\partial t} + \frac{\partial}{\partial x_k} (U_k f).$$

Note that if equation (11) is contracted the result is null so that (11) is clearly independent of (8).

Now the point is that the terms appearing on the left side of (11) are identical to terms appearing in the kinetic theory, transport equations. Attention may then be directed to the right-hand side of (11). In kinetic theory, the viscous terms do not appear. In turbulent flow it can be argued that the viscous terms are negligible since they relate to the smallest scales of turbulent motion which tend more to isotropy than the larger scales. Formally, this means that $\overline{u_i \partial \tau_{kj} / \partial x_k} \propto \delta_{ij}$ and the viscous terms in (11) cancel. Therefore, it may be seen that it is the velocity pressure gradient correlations that distinguish the turbulent flow transport equations from their kinetic theory counterpart where the Boltzman collision integrals appear

instead. Before returning to this point it is enlightening to examine the consequences of isotropy on the left side of (11); that is, to determine its asymptotic form as $a_{ij} \rightarrow 0$.

The first term in (11) is $(q^2/3) (\partial U_m / \partial x_n) [\delta_{jm} \delta_{in} + \delta_{im} \delta_{jn} + O_{ijmn}(a)]$ where, say $a^2 = a_{ij}^2$; therefore, its asymptotic behavior is clear. To estimate the behavior of the second term, equation (8) may be used to demonstrate that $D(q^2/2)/Dt$ is at most of the order of the turbulent energy production or dissipation term. Thus, $D(a_{ij} q^2/3)/Dt = (q^2/3) (\partial U_k / \partial x_m) O_{ijkm}(a^2)$. For the third diffusion-like term, one should appeal to a triple correlation equation; however, in kinetic theory the diffusion term in the equation corresponding to (11) is $O(a^2)$ and this will be assumed to be the case here. Therefore, as $a \rightarrow 0$, equation (11) reduces to

$$\frac{q^2}{3} \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) = - \overline{u_i \frac{\partial p}{\partial x_j}} - \overline{u_j \frac{\partial p}{\partial x_i}} + \frac{2}{3} \delta_{ij} \overline{u_k \frac{\partial p}{\partial x_k}} \quad (12)$$

asymptotically as $a \rightarrow 0$ in both near isotropic turbulent flow and kinetic theory; however, in kinetic theory the right side is replaced by the Boltzman integral.*

* The Boltzman integral can be evaluated and shown to be proportional to $(q/\lambda) (-\overline{u_i u_j} + \delta_{ij} q^2/3)$; here λ is interpreted as the mean free path and $\overline{u_i u_j}$ and q^2 the molecular velocity correlations. If $\partial U_i / \partial x_j \equiv O_{ij}(U/L)$ then it may be seen that $a = O[(\lambda/L)(U/q)] = O[(\text{Knudsen number})(\text{Mach number})]$.

In principle, one can conceive of a nearly isotropic flow field on which is superposed a small mean strain (see Rose, 1966). In such a case, equations (1), (2), (8), and (12) should apply and it appears that a closure scheme must involve assumptions for the turbulent diffusion transport terms and the dissipation term in (8). Since (12) is the Reynolds stress transport equation,[†] a closure assumption must relate $\overline{u_1 \partial p / \partial x_j}$ (or, what is nearly equivalent, an assumption for $\overline{p \partial u_1 / \partial x_j}$ as discussed later) to $\overline{u_1 u_j}$. If this is done by analogy to kinetic theory, a conventional Newtonian or Boussinesq constitutive relation will result.

[†] The logic behind the constructions of Bradshaw, Ferriss, and Atwell (1967), is not at all clear, in light of this, since an equation for $-uv$ is obtained from the energy equation. It would seem that assumptions are being made about the wrong terms in the wrong equation.

4. TYPICAL TURBULENT FLOWS

The purpose of examining nearly isotropic turbulent flows was to preceive, under such simplifying circumstances, a framework for a model theory. Now an attempt will be made to set up model equations which certainly would include but not be restricted to nearly isotropic flow, and where, typically, the characteristic length scale is not particularly small. However, in seeking generality assumptions must be used as economically as possible. That is, wherever justified terms will be neglected in order to reduce the empiricism required while maintaining a maximum correlation with experiment. This will also serve to identify the most important empirical ingredients which can then be the focus of further research and improvement.

Although most flows of interest are of the parallel flow or boundary layer type where simplifications of the basic equations are possible, in the hope of achieving greater generality the initial treatment will be in terms of the full tensorial forms of the equations similar to the approach of Rotta (1951) and Donaldson and Rosenbaum (1968). The object will be to relate all terms in equation (7) to $\overline{u_i u_j}$.

Viscous terms: The viscous diffusion terms should be simple to deal with since

$$\overline{u_i \tau_{kj}} + \overline{u_j \tau_{ki}} = \nu \left[\overline{u_i \left(\frac{\partial u_k}{\partial x_j} + \frac{\partial u_j}{\partial x_k} \right)} + \overline{u_j \left(\frac{\partial u_k}{\partial x_i} + \frac{\partial u_i}{\partial x_k} \right)} \right]$$

and it appears that the right side could be expressed in terms of gradients of $\overline{u_k u_i}$, etc., with no assumptions. However, using the notation^{*}

$$R_{ij}(\bar{x}, \underline{r}) = u_i(\underline{x}) u_j(\underline{x}'); \quad \bar{x} = \frac{1}{2} (\underline{x} + \underline{x}'), \quad \underline{r} = \underline{x}' - \underline{x}, \quad (13)$$

it may be shown that

$$\begin{aligned} \overline{u_i \tau_{kj}} + \overline{u_j \tau_{ki}} = v \left[\left(\frac{\partial R_{ij}}{\partial \bar{x}_k} \right)_0 + \frac{1}{2} \left(\frac{\partial R_{ik}}{\partial \bar{x}_j} \right)_0 + \frac{1}{2} \left(\frac{\partial R_{jk}}{\partial \bar{x}_i} \right)_0 \right. \\ \left. + \left(\frac{\partial R_{ik}}{\partial r_j} \right)_0 + \left(\frac{\partial R_{jk}}{\partial r_i} \right)_0 \right], \quad (14) \end{aligned}$$

where the subscript $_0$ refers to the limit $r \rightarrow 0$. Equation (14) contains terms which are not simple to model. Similarly, the last pair of terms in equation (7), which corresponds to the dissipation in the turbulent energy equation,

$$\begin{aligned} \overline{\tau_{kj} \frac{\partial u_i}{\partial x_k}} + \overline{\tau_{ki} \frac{\partial u_j}{\partial x_k}} = -2v \left(\frac{\partial^2 R_{ij}}{\partial r_x \partial r_k} \right)_0 + v \frac{\partial}{\partial x_k} \left[\left(\frac{\partial R_{ik}}{\partial r_j} \right)_0 + \left(\frac{\partial R_{jk}}{\partial r_i} \right)_0 \right] \\ + \frac{1}{2} v \frac{\partial}{\partial x_k} \left[\left(\frac{\partial R_{ij}}{\partial \bar{x}_k} \right)_0 + \left(\frac{\partial R_{ik}}{\partial \bar{x}_j} \right)_0 + \left(\frac{\partial R_{jk}}{\partial \bar{x}_i} \right)_0 \right], \quad (15) \end{aligned}$$

* The properties of R_{ij} seem to be well-known only for homogeneous turbulence. For example, in nonhomogeneous turbulence $R_{ij}(\bar{x}, \underline{r}) \neq R_{ji}(\bar{x}, -\underline{r})$ in contrast to a homogeneous flow field although, of course, $R_{ij}(\bar{x}, 0) = R_{ji}(\bar{x}, 0)$. Furthermore,

$$\frac{\partial}{\partial x_k} \overline{u_i(\underline{x}) u_j(\underline{x}')} = \frac{1}{2} \frac{\partial R_{ij}}{\partial \bar{x}_k} - \frac{\partial R_{ij}}{\partial r_k}, \quad \frac{\partial}{\partial x_k} \overline{u_j(\underline{x}) u_i(\underline{x}')} = \frac{1}{2} \frac{\partial R_{ji}}{\partial \bar{x}_k} + \frac{\partial R_{ji}}{\partial r_k}.$$

The two expressions must be equal as $r \rightarrow 0$ and therefore it is evident that $\partial R_{ij}(\bar{x}, 0) / \partial x_k = \partial R_{ji}(\bar{x}, 0) / \partial \bar{x}_k$, and also $(\partial R_{ij} / \partial r_k)_{r \rightarrow 0} = -(\partial R_{ji} / \partial r_k)_{r \rightarrow 0}$. From continuity $\partial R_{ij} / \partial r_i = \frac{1}{2} \partial R_{ij} / \partial \bar{x}_i$ so that in inhomogeneous flow R_{ij} is not solenoidal with respect to $\underline{r}$ as it is in homogeneous flow. Finally, by equating $\partial^2 u_j(\underline{x}) u_i(\underline{x}') / \partial x_k \partial x_\ell$ and $\partial^2 u_i(\underline{x}) u_j(\underline{x}') / \partial x_k' \partial x_\ell'$ as $r \rightarrow 0$ it may be established that $(\partial^2 R_{ij} / \partial r_k \partial r_\ell)_0 = (\partial^2 R_{ji} / \partial r_k \partial r_\ell)_0$. These properties are useful in obtaining the relations following equation (13).

is quite complicated. On the other hand, the expression obtained by combining the terms from equations (14) and (15),

$$\begin{aligned} \overline{u_i \frac{\partial \tau_{kj}}{\partial x_k}} + \overline{u_j \frac{\partial \tau_{ki}}{\partial x_k}} &= \frac{\partial}{\partial x_k} (\overline{u_i \tau_{kj}} + \overline{u_j \tau_{ki}}) - \tau_{kj} \frac{\partial u_i}{\partial x_k} - \tau_{ki} \frac{\partial u_j}{\partial x_k} \\ &= \nu \left[\frac{1}{2} \frac{\partial^2 \overline{u_i u_j}}{\partial x_k \partial x_k} + 2 \left(\frac{\partial^2 R_{ij}}{\partial r_k \partial r_k} \right)_0 \right], \end{aligned} \quad (16)$$

is remarkably simple.

Because the first term has the characteristics of a diffusion term and the second the characteristics of a dissipation term, a cursory inspection suggests that such an identification might be possible. However, it is clear from equations (14) and (15) that each of these two terms represents a complex combination of both "diffusion" and "dissipation." Nevertheless, in the succeeding development one is prone to refer to the first term on the right as the viscous "diffusion" and the second term as the viscous "dissipation."

There now exists a requirement to model the term $(2\nu \partial^2 R_{ij} / \partial r_k^2)_0$. It is widely believed that the small scale turbulence is isotropic; or in spectral wave number space, the spectral density, $\Phi_{ij}(k) \sim$ isotropic form as $k \rightarrow \infty$. The term in question may be written $-2\nu \int k^2 \Phi_{ij} dk$ and, therefore, tends to be isotropic; higher derivatives would presumably be more so. Therefore, it is reasonable to set

$$-2\nu \left(\frac{\partial^2 R_{ij}}{\partial r_k^2} \right)_0 = \frac{2}{3} \epsilon \delta_{ij}, \quad \epsilon = \nu \frac{5q^2}{3\lambda^2}, \quad (17a,b)$$

where λ is the Taylor microscale.

The expression for the dissipation is now assumed to be of the form,

$$\epsilon = \frac{q^3}{\Lambda}, \quad (18)$$

where later on the stipulation will be made that the dissipation length, Λ , is independent of viscosity except near a wall. The jump from (17b) to (18) is, of course, an important step which most researchers who have designed turbulent flow models have adopted. It is based on the experimental evidence in wakes, jets, and in homogeneous turbulence that, so long as the Reynolds number is large, the dissipation of energy is essentially independent of viscosity. The accepted explanation is that ϵ is determined by the large scale eddies, that is to say it is fixed by (18). In spectral wave number space the energy cascades toward larger wave numbers (and smaller eddies) until it is dissipated according to (17b) at some large wave number proportional to λ^{-1} , so that λ adjusts itself in order that (17b) is satisfied. Thus, in a given flow with fixed large scales and velocities (Λ and q are fixed) it develops that $\lambda^2 \propto \nu$. Although this is a widely accepted explanation, it lacks mathematical precision. (It is likely that this situation can be corrected by application of the Method of Matched Asymptotic Expansion to the two point correlation equations; this point of view will be developed in a future paper.)

Very near a wall, viscosity should reappear in equation (18) so that $\Lambda \sim (q/\nu) (\text{length scale})^2$ as $y \rightarrow 0$.

Summarizing, (17a) and (18) can be inserted in equation (16) to yield

$$\overline{u_i \frac{\partial \tau_{kj}}{\partial x_k}} + \overline{u_j \frac{\partial \tau_{ki}}{\partial x_k}} = \frac{\nu}{2} \frac{\partial^2 \overline{u_i u_j}}{\partial x_k \partial x_k} - \frac{2}{3} \delta_{ij} \frac{q^3}{\Lambda} \quad (19)$$

which represents a model for the viscous terms.

Pressure-velocity terms: The important hypothesis which will be adopted for the pressure-velocity terms in equation (7) is that due to Rotta (1951);* that is

$$-\overline{p \frac{\partial u_i}{\partial x_j}} = \frac{1}{6} \frac{q}{\ell_1} \left(\overline{u_i u_j} - \frac{\delta_{ij}}{3} q^2 \right), \quad (20)$$

to which we add the traditional assumption for the flux of a scalar quantity

$$-\overline{p u_i} = \frac{q \ell_2}{2} \frac{\partial q^2}{\partial x_i}. \quad (21)$$

The length scale quantities ℓ_1 and ℓ_2 will be discussed later.

Rotta (1951) discussed the physical basis of (20) so that here it is noted that formally the hypothesis may be broken down into three separate statements: (1) that $\overline{p \frac{\partial u_i}{\partial x_j}} = C_{ijk} \overline{u_k u_m}$; (2) that the constitutive coefficient is an isotropic tensor, and (3) that $\overline{p \frac{\partial u_i}{\partial x_i}}$ vanishes. The first two statements are assumptions; the third, of course, is not.

Note that in equation (7) the component equations for $\overline{u_1^2}$, $\overline{u_2^2}$, $\overline{u_3^2}$, involve the terms $\overline{p \frac{\partial u_1}{\partial x_1}}$, $\overline{p \frac{\partial u_2}{\partial x_2}}$, and $\overline{p \frac{\partial u_3}{\partial x_3}}$, respectively. As pointed out by Rotta, these terms represent energy flows between the

* The material in Section 3, was presented by the first author at the Symposium on Turbulent Structure, Southhampton, 1967, and was the basis for proposing (20) in ignorance of Rotta's work 16 years earlier.

component energies depending on the level of anisotropy. Their sum is zero so that they do not represent a net flow of total kinetic energy. Equation (20) conforms to these notions.

Equation (21) is only one of two candidates[†] which coalesce, however, in the limit as isotropy is approached.

Triple-velocity terms: The terms remaining are $\overline{u_k u_i u_j}$ in equation (7). In principle, one could appeal to higher moments of equation (4) - and so on and so on. However, this model will be closed by assuming that

$$-\overline{u_i u_j u_k} = q \ell_3 \left(\frac{\partial \overline{u_j u_k}}{\partial x_i} + \frac{\partial \overline{u_i u_k}}{\partial x_j} + \frac{\partial \overline{u_i u_j}}{\partial x_k} \right) . \quad (22)$$

Equation (22) is only one of three candidates;* it is chosen simply by analogy to the diffusion term in (16); however $\overline{u_k u_i u_j}$ requires symmetry in i, j and k .

[†] If one assumes that $\overline{p u_i} = C_{ijk m} \partial \overline{u_j u_k} / \partial x_m$, and that $C_{ijk m}$ is an isotropic tensor, we obtain a term proportional to $\partial \overline{u_i u} / \partial x_m$ in addition to (21).

* If one assumes that $\overline{u_i u_j u_k} = C_{ijk \ell m n} \partial u_\ell / \partial x_n$, requires that $C_{ijk \ell m n}$ be isotropic and requires symmetry in i, j and k , $C_{ijk \ell m n}$ would reduce to a form such that we obtain terms proportional to $\delta_{ij} \partial q^2 / \partial x_k + \delta_{jk} \partial q^2 / \partial x_i + \delta_{ki} \partial q^2 / \partial x_j$ and $\delta_{ij} \partial \overline{u_k u} / \partial x_m + \delta_{jk} \partial \overline{u_i u} / \partial x_m + \delta_{ik} \partial \overline{u_j u} / \partial x_m$ in addition to (22). In the isotropic limit all of the expressions would collapse to the first expression.

5. MTF EQUATIONS - VISCOELASTIC FORM

If equations (19), (20), (21), and (22) are substituted into (7) the result is

$$\begin{aligned} \frac{D \overline{u_i u_j}}{Dt} = \frac{\partial}{\partial x_k} & \left[\frac{\nu}{2} \frac{\partial \overline{u_i u_j}}{\partial x_k} + q \ell_3 \left(\frac{\partial \overline{u_i u_k}}{\partial x_i} + \frac{\partial \overline{u_i u_k}}{\partial x_j} + \frac{\partial \overline{u_i u_j}}{\partial x_k} \right) \right. \\ & \left. + \frac{q \ell_2}{2} \left(\delta_{kj} \frac{\partial q^2}{\partial x_i} + \delta_{ki} \frac{\partial q^2}{\partial x_j} \right) \right] - \overline{u_k u_i} \frac{\partial U_j}{\partial x_k} \\ & - \overline{u_k u_j} \frac{\partial U_i}{\partial x_k} - \frac{1}{3} \frac{q}{\ell_1} (\overline{u_i u_j} - \frac{\delta_{ij}}{3} q^2) - \frac{2}{3} \frac{q^3}{\Lambda} \delta_{ij}. \end{aligned} \quad (23)$$

Note that the system of equations represented by (1), (2), and (23) is closed providing empirical statements concerning ℓ_1 , ℓ_2 , ℓ_3 , and Λ can be meaningfully postulated. The energy equation, corresponding to equation (8) may be obtained by contracting (23) and dividing by 2. Thus,

$$\begin{aligned} \frac{D q^2/2}{Dt} = \frac{\partial}{\partial x_k} & \left[\frac{\nu}{2} \frac{\partial q^2/2}{\partial x_k} + q \ell_3 \left(\frac{\partial q^2/2}{\partial x_k} + \frac{\partial \overline{u_i u_k}}{\partial x_i} \right) + q \ell_2 \frac{\partial q^2/2}{\partial x_k} \right] \\ & - \overline{u_k u_i} \frac{\partial U_i}{\partial x_k} - \frac{q^3}{\Lambda}. \end{aligned} \quad (24)$$

The Two-Dimensional Boundary Layer Approximation: The boundary layer approximation to equations (1), (2), and (23), whereby derivatives with respect to x are neglected relative to derivatives with respect to y are as follows:

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0 \quad , \quad (25)$$

$$\frac{DU}{Dt} = - \frac{\partial P}{\partial x} + \frac{\partial}{\partial y} \left(-\overline{uv} + v \frac{\partial U}{\partial y} \right) \quad , \quad (26)$$

$$\begin{aligned} \frac{D\bar{v}^2}{Dt} = \frac{\partial}{\partial y} \left[\left(\frac{v}{2} + 3q\ell_3 \right) \frac{\partial \bar{v}^2}{\partial y} + q\ell_2 \frac{\partial q^2}{\partial y} \right] \\ - \frac{1}{3} \frac{q}{\ell_1} (\bar{v}^2 - \frac{1}{3} q^2) - \frac{2}{3} \frac{q^3}{\Lambda} \quad , \end{aligned} \quad (27)$$

$$\frac{D\bar{u}^2}{Dt} = \frac{\partial}{\partial y} \left[\left(\frac{v}{2} + q\ell_3 \right) \frac{\partial \bar{u}^2}{\partial y} \right] - \frac{1}{3} \frac{q}{\ell_1} (\bar{u}^2 - \frac{1}{3} q^2) - \frac{2}{3} \frac{q^3}{\Lambda} - 2\overline{uv} \frac{\partial U}{\partial y} \quad , \quad (28)$$

$$\frac{D\bar{w}^2}{Dt} = \frac{\partial}{\partial y} \left[\left(\frac{v}{2} + q\ell_3 \right) \frac{\partial \bar{w}^2}{\partial y} \right] - \frac{1}{3} \frac{q}{\ell_1} (\bar{w}^2 - \frac{1}{3} q^2) - \frac{2}{3} \frac{q^3}{\Lambda} \quad , \quad (29)$$

$$\frac{D\overline{uv}}{Dt} = \frac{\partial}{\partial y} \left[\left(\frac{v}{2} + 2q\ell_3 \right) \frac{\partial \overline{uv}}{\partial y} \right] - \frac{1}{3} \frac{q}{\ell_1} \overline{uv} - \bar{v}^2 \frac{\partial U}{\partial y} \quad , \quad (30)$$

The equations have here been specialized to two-dimensional mean flow, but, of course, the full three-dimensional equations are easily obtained.

6. MTF EQUATIONS - NEWTONIAN FORM

The Newtonian version of equation (23) is attractive since the number of equations to be solved is considerably reduced. The equations are

$$\frac{Dq^2/2}{Dt} = \frac{\partial}{\partial x_k} \left[\left(\frac{v}{2} + \frac{5}{3} q\ell_3 + q\ell_2 \right) \frac{\partial q^2/2}{\partial x_k} \right] - \overline{u_k u_i} \frac{\partial U_i}{\partial x_k} - \frac{q^3}{\Lambda}, \quad (31)$$

and

$$- \overline{u_i u_j} + \frac{\delta_{ij}}{3} q^2 = q\ell_1 \left(\frac{\partial U_j}{\partial x_i} + \frac{\partial U_i}{\partial x_j} \right), \quad (32)$$

Together with (1) and (2), (31) and (32) comprise a closed set of equations after ℓ_1 , ℓ_2 , ℓ_3 , and Λ have been specified.

As discussed in Section 3, equation (31) may be obtained formally from (23) by contraction. Then one substitutes $\overline{u_i u_j} = (\delta_{ij} + a_{ij}) q^2/3$ in (23) and subtracts the product of (31) and $\delta_{ij}/3$ - term by term. One then lets $a = [a_{ij}]^{1/2} = (U/q) O(\ell/L)$ as $\ell/L \rightarrow 0$ and obtains equation (32).

Boundary Layer Approximation: The boundary layer approximation to (31) and (32) is

$$\frac{Dq^2/2}{Dt} = \frac{\partial}{\partial y} \left[\left(q\ell_2 + \frac{v}{2} + \frac{5}{3} q\ell_3 \right) \frac{\partial q^2/2}{\partial y} \right] - \overline{uv} \frac{\partial U}{\partial y} - \frac{q^3}{\Lambda} \quad (33)$$

$$- \overline{uv} = q\ell_1 \frac{\partial U}{\partial y}, \quad (34)$$

which together with (25) and (26) form a closed set of boundary layer equations. Furthermore, (34) may be inserted into (26) so that (26) and

(33) are similar in form and may be solved simultaneously for $U(x,y)$ and $q(x,y)$; such a relatively simple system is attractive from a computational point of view.

Now (33) and (34) may be compared with (27) - (30). First of all, (33) can be obtained directly from (27), (28), and (29) by addition and by approximating $\overline{3v^2} + \overline{u^2} + \overline{w^2} = (1.67) q^2$ which is, of course, true only for isotropic turbulence. Based on the channel flow data of Laufer (1954) and the boundary layer data of Klebanoff (1955) one finds that

$$a_{ij} \approx \begin{bmatrix} .70, & -.40, & 0 \\ -.40, & -.55, & 0 \\ 0, & 0, & -.15 \end{bmatrix} \quad (35)$$

over a large portion of the layers. Therefore, one obtains $\overline{3v^2} + \overline{u^2} + \overline{w^2} = (1.30) q^2$. In jets and wakes a result closer to the isotropic expression is obtained.

However, the most significant difference between the "viscoelastic" and "Newtonian" forms is, in fact, that associated with their labels; the former contains the term $\overline{Duv}/Dt$ (which, if equated to $\overline{v^2} \partial U / \partial y$ would represent elastic behavior) in (30), whereas it has been neglected in (34). To assess the importance of this Reynolds stress advective term the best test might be to compare the ratio of $\overline{D(q^2/2)}/Dt$ to $\overline{uv} \partial U / \partial y$ with the ratio of $\overline{Duv}/Dt$ to $\overline{v^2} \partial U / \partial y$. Thus, from the values of a_{ij} given above and neglecting $D a_{ij} / Dt$ we obtain

$$\frac{\overline{Duv}/Dt}{\overline{Dq^2/2}/Dt} \frac{\overline{v^2} \partial U / \partial y}{\overline{uv} \partial U / \partial y} \approx \frac{2a_{12}^2}{3(1 + a_{22})} \approx 0.25. \quad (36)$$

Similarly, in assessing the relative role of the diffusion terms in (30) and (34) one obtains a value of about 0.15.

Finally, it should be noted that after neglecting the advective and diffusion terms in (30) one would obtain $-uv = 3(\overline{v^2}/q) \ell_1 \partial U/\partial y$. Again referring to the a_{ij} numbers one finds that $3\overline{v^2}/q = .45 q$. Thus, due to the isotropy assumption, the constitutive coefficient in (34) is "in error" by factor of about 2. However, in practice this factor is simply absorbed into the definition of ℓ_1 .

7. MVF CLOSURE REVISITED

The relationship between the Newtonian form of the MTF equation and the viscoelastic form has been shown. Structurally, the difference amounts to the neglect of the diffusion-like term and the advective term in the Reynolds stress transport equation. By the same token one can now see how the still simpler MVF closure is related to MTF closure. Thus, if one also neglects the diffusion and advective terms in the energy equation (33) and combines with (34), the result is

$$-\overline{uv} = (\Lambda^{1/3} \ell)^{3/2} \left| \frac{\partial U}{\partial y} \right| \frac{\partial U}{\partial y},$$

including the correct placement of the modulus symbol. The same result may be obtained by neglecting the diffusion and advection terms in equations (27) - (30). Thus, MVF closure, as discussed in Part I, may now be regarded as an approximation to MTF closure.*

The question is now: which of the three possible closure schemes (or even some intermediate scheme) is to be preferred? There is no clear answer to this question. Obviously, since MVF closure yields such good results in the case of boundary layer flows and since it is relatively simple, it will continue to be useful. Also, it stands as a warning with respect to the evaluation of new or more complicated models; that is, models that are more complex or purport to contain more physics may seem to work well even if the physics are wrong so long as the model is dominated by MVF-like elements.

* It may also be shown that the functional forms cited in Section 8 may be transformed to the functional forms for v_t in Part I. To a fair approximation the specific functions are the same.

8. EMPIRICAL POSTULATES FOR l_1 , l_2 , l_3 , AND Λ AS USED IN EQUATIONS (33) AND (34)

There are, however, several incentives to move on to the consideration of more complicated models - and the authors believe that this automatically implies MTF closure as the next logical step. The first is the purely scientific motivation to try and understand the nature of turbulent flow fields as well as their concomitant velocity fields; furthermore, it is satisfying to create the ability to predict turbulent fields for comparison with data. And, as has been shown, the consideration of MTF closure allows one to look back and view MVF closure in proper perspective; later an attempt will be made to understand why MVF closure seems to work well. A second incentive is simply that one would hope to continue embracing a larger range of flow cases with less empirical content or at least empirical content which need not be altered to fit specific classes of flows.

Having decided to pursue MTF closure, it was natural to exploit the simpler Newtonian form first and leave the viscoelastic form for future work. It was felt that the Newtonian form of MTF closure deserved clear identification in a systematic investigation of closure schemes. The Newtonian form is closest to classic concepts of transport physics and it will be seen that, at this level, there are things to be learned. Specifically, it is a relatively simple basis on which to seek out empirical determinations of l_1 , l_2 , l_3 , and Λ which, according to this paper, can be corrected and applied to the viscoelastic form. Furthermore, when work is extended to the viscoelastic form there will be some background against which to measure its efficacy.

Trial and error calculations indicate that $(5\ell_3/3 + \ell_2)/\ell_1 = 1 \pm .10$. Therefore, a first simplifying assumption that $\ell = \ell_1 = (5\ell_3/3 + \ell_2)$ may be made. Thus one need only search for empirical statements for ℓ and Λ . From experience gained in Part I and in Mellor (1970) there is very little doubt as to the next step. Therefore,

- i. near a wall one assumes that $\ell = \text{fcn}(y, q, \nu)$. Therefore^{*}

$$\frac{\ell q}{\nu} = \varphi_\ell(\chi) ; \chi = \frac{qy}{\nu} ,$$

- ii. far from the wall one assumes that $\ell = \text{fcn}(y, q, Q)$ where Q is as yet undefined except that it will have dimensions of kinematic viscosity. Thus,

$$\frac{\ell q}{Q} = \bar{\varphi}_\ell(X) ; X = \frac{qy}{Q} ,$$

- iii. it is assumed that an overlap layer exists such that $\nu\varphi = Q\bar{\varphi}$.

Since φ_ℓ and $\bar{\varphi}_\ell$ are universal and the right side of this expression is independent of viscosity, then

$$\varphi_\ell = k_\ell \chi \text{ and } \bar{\varphi}_\ell = k_\ell X$$

in the overlap layer. k_ℓ is an arbitrary constant. Now as before the above procedure is regarded as a way of obtaining inner and outer functions as in the MMAE (the small parameter can be identified as ν/Q). Thus φ_ℓ is the

* Note the question of "eddy viscosity" models versus "mixing length" models is now eliminated since $\nu_t \equiv q\ell$.

inner function whose outer asymptote is $k_\ell \chi$ whereas $\bar{\Phi}_\ell$ is the outer function whose inner asymptote is $k_\ell X$. Therefore, a composite function can be written

$$\ell q = v\varphi_\ell(\chi) + Q\bar{\Phi}_\ell(X) = k_\ell \left\{ \begin{array}{l} v\chi \\ \text{or} \\ QX \end{array} \right\}, \quad (37)$$

where $\chi = X(v/Q)$.

Similarly, two functions φ_Λ and $\bar{\Phi}_\Lambda$ are needed to specify Λ according to

$$\Lambda q = v\varphi_\Lambda(\chi) + Q\bar{\Phi}_\Lambda(X) = k_\Lambda \left\{ \begin{array}{l} v\chi \\ \text{or} \\ QX \end{array} \right\}. \quad (38)$$

As with the quantity N in Part I the quantity Q can probably be defined in a number of ways; viz, $Q \equiv \delta q$ where δ is some edge of the boundary layer or $Q \equiv \int_0^\infty q \, dy$ where the assumption is made that $q \rightarrow 0$ as $y \rightarrow \infty$ (otherwise use the quantity $q - q_\infty$). Here, the latter was chosen for the reason that it is well defined mathematically, and it is consistent with the authors' earlier MVF work which enjoyed considerable success. However, the former definition would probably work well.

9. COMPARISON WITH DATA

With due regard for the asymptotic behavior of ϕ_ℓ and ϕ_λ as $\chi \rightarrow 0^*$ (see Mellor and Herring 1968), the functions shown in Figure 1 are proposed where the constants have been adjusted to best match the velocity and kinetic energy profiles of Klebanoff's (1955) constant pressure case and, in the inner layer, the pipe flow data of Laufer (1954). The calculations (see Herring and Mellor 1970 for numerical program details) are based on equations (1), (2), (33), (34), (37), and (38) of the present paper.

The boundary conditions are

$$U(x,0) = 0 , \quad (39a)$$

$$V(x,0) = 0 , \quad (39b)$$

$$U_\infty(x) - U(x,y) \sim y^{-n} \text{ as } y \rightarrow \infty , \quad (39c)$$

$$U(y, x_0) = U_0(y) , \quad (39d)$$

$$q(x,0) = 0 , \quad (40a)$$

$$q(x,y) \sim y^{-n} \text{ as } y \rightarrow \infty , \quad (40b)$$

$$q(x_0, y) = q_0(y) , \quad (40c)$$

where $n > 1$.

* Near the wall (33) reduces to $(\nu/2) \partial^2(q^2/2)/\partial y^2 = q^3/\Lambda$. In order that $q^2 = O(y^2)$ one must have $\Lambda \sim 2qy^2/\nu$ or $\phi_\Lambda \sim 2\chi^2$ as $y \rightarrow 0$ which is consistent with the previous discussion of equation (18). The factor 1/2 was not carried along in Mellor and Herring (1968), but the final results are equivalent.

Figure 1. Inner and outer functions which together with equations (37) and (38) completely specify l and Λ .

Thus, the five free constants in φ_ℓ , φ_Λ , Φ_ℓ , Φ_Λ have been adjusted so that the close agreement of theory and data shown in Figure 2 is achieved. As in previous MVF experience the hope is that, without further empirical adjustment, the basic theory will correctly predict velocity and turbulent energy profiles for arbitrary pressure gradients and other boundary layer variables.

A good deal of comparison with data has fortunately already been achieved, since in Mellor and Herring (1968 II), calculations showed that all of the integral parameters of the Stanford Symposium could be predicted quite well. The surprising result was that the predictions were no better nor worse than predictions using the simple MVF model. This will be discussed below. Lacking in the material presented at the Stanford meeting - aside from the supporting analytical work presented here - was a comparison of turbulent energy profile predictions with data. Therefore, in Figure 3 and 4 calculated profiles are compared with some data obtained by Bradshaw (1965, 1969). The mean velocity profile comparison of Figure 3 is not especially favorable but it must be noted that the calculations of this particular flow by almost all participants of the Stanford Conference compared badly with the data. Despite the peculiarly poor prediction of mean velocities in Figure 3, the overall evidence (including a great many other comparisons of velocity profiles where turbulence measurements were not available), indicates that both velocity and turbulent energy profiles can be predicted quite well.

A somewhat more detailed way of looking at the output of the model is to examine profiles of advection, production, dissipation, and diffusion. Thus we define,

Figure 2a. Mean velocity and turbulent energy in the wall region.
Data by Klebanoff (1955).

Figure 2b. Mean velocity and turbulent energy in the outer region.
Data by Klebanoff (1955).

Figure 3. Comparison of mean velocity, Reynolds stress, mean turbulent energy calculations, and data for a boundary layer responding to a sudden removal of pressure gradient. Data by Bradshaw and Ferriss (1965).

Figure 4. Comparison of mean velocity, Reynolds stress, mean turbulent energy calculations, and data for a boundary layer responding to a sudden application of an adverse pressure gradient. Data by Bradshaw (1969).

$$A^* \equiv \frac{v}{u_\tau^4} \left[U \frac{\partial q^2/2}{\partial x} + v \frac{\partial q^2/2}{\partial y} \right], \quad (41a)$$

$$P^* \equiv \frac{v}{u_\tau^4} \left[- \overline{uv} \frac{\partial U}{\partial y} \right], \quad (41b)$$

$$D^* \equiv \frac{v}{u_\tau^4} \left[\frac{q^3}{\Lambda} \right], \quad (41c)$$

$$C^* \equiv \frac{v}{u_\tau^4} \left[\frac{\partial}{\partial y} \left(\frac{v}{2} + q\ell \right) \frac{\partial q^2/2}{\partial y} \right], \quad (41d)$$

where $A^* + P^* + D^* + C^* = 0$. Typical results are given in Figures 5 and 6.

The results shown in Figure 5 can be compared to the data obtained by Klebanoff (1955) which is also cited and discussed by Townsend (1956) and Rotta (1962). First of all, A^* and P^* appear to be correct. This is not surprising since these are terms about which no empiricism is required. Correspondingly, the data are relatively easy to obtain. However, there are differences in D^* and C^* which might indicate that future adjustments may be in order (for example, the value of D^* appears too large in the vicinity of $yu_\tau/v \approx 10$). However, it is just these terms about which there exists some experimental uncertainty. Therefore, judgement on such a detailed level as that required by Figure 5 will, for the present, be withheld.

Figure 5a. Advection, production, dissipation, and diffusion terms in the wall region. From calculations also plotted in Figure 2a. A* is essentially zero in this region.

Figure 5b. Advection, production, dissipation, and diffusion terms in the outer region. From calculations also plotted in Figure 2b.

Figure 6a. Advection, production, dissipation, and diffusion terms in the wall region. From calculations also plotted in Figure 3. A^* is essentially zero in this region.

Figure 6b. Advection, production, dissipation, and diffusion terms in the couter region. From calculations also plotted in Figure 4.

10. HISTORY

At the Stanford Symposium there was a great deal of discussion concerning the importance of "history" in turbulent boundary layer development although the definition of the word itself remained somewhat obscure. Here, "history" is interpreted to mean the terms DU/Dt , Dq^2/Dt (or its components) and $\overline{Duv}/Dt$ in the mean momentum, energy and Reynolds stress equations. From the discussion in Section 6, $\overline{Duv}/Dt$ was concluded to be relatively less important than Dq^2/Dt . Here the assertion is made that in the prediction of mean velocity profiles it is possible to neglect Dq^2/Dt ; while this can result in errors in q^2 , the error in predicting all of boundary layer velocity profile data collected at the Stanford Symposium is negligible. Thus, MVF closure does represent a useful and simple procedure in many cases.

11. CONCLUDING REMARKS

While MVF models perform rather well, MTF models hold considerable promise in advancing our understanding and the capability to predict a wider range of turbulent flows. Furthermore, the turbulent energy profiles themselves are also predicted.

The basic framework for modeling transiting flows exists in MTF models. Although Glushko (1965) and Donaldson (1969) did not succeed quantitatively in predicting transition in their MTF models, nevertheless the possibility for future improvements does exist. Perhaps the greatest potential exists in treating flow with body force effects, such as centrifugal or buoyancy terms, in which case new terms appear that feed energy directly into $\overline{v^2}$, equation (27), and therefore directly effect $\overline{uv}$ through equation (30). The question is: can equations (25) - (30) correctly account for these effects without empirical alterations of the functions, ℓ and Λ ? (With MVF closure empirical alterations are clearly necessary. For stratified flow the necessity of introducing a Monin-Obukhov length, for example, exists.) This question remains to be answered.

This paper has been concerned with the proposal of a reasonable framework for MTF closure. Central to MTF models (and MVF models as well since they are a simplification of MTF models) is the Rotta hypothesis, equation (20). Because of its importance one would expect it to be the focus of future research. An integral expression can be obtained for either $\overline{p \partial u_i / \partial x_j}$ or $\overline{u_i \partial p / \partial x_j}$. The latter yields the simplest expression and it is advantageous to proceed on that basis. Note that if $\overline{\partial p u_i / \partial x_j} = 0$ as it would be for homogenous flow the two terms are equal. On the

other hand, the terms, $\overline{u_i \partial p / \partial x_j}$ appear directly in (6) and perhaps, in the first place, need not have been reconstituted as in (7).

By taking the divergence of (4) Poisson's equation for p is obtained. Poisson's integral may then be written

$$p(\underline{x}) = \frac{1}{4\pi} \iiint \left[2 \frac{\partial U'_\ell}{\partial x'_m} \frac{\partial u'_m}{\partial x'_\ell} + \frac{\partial^2}{\partial x'_j \partial x'_m} (u'_\ell u'_m - \overline{u'_\ell u'_m}) \right] \frac{dx'_j}{r},$$

where $\underline{x}' = \underline{x} + \underline{r}$ and $U'_\ell = U_\ell(\underline{x}')$, etc.

Finally,

$$\overline{u_i \frac{\partial p}{\partial x_j}} = \frac{1}{4\pi} \iiint \left[2 \frac{\partial U'_\ell}{\partial r_m} \frac{\partial R_{im}}{\partial r_\ell} + \frac{\partial^2}{\partial r_\ell \partial r_m} S_{ilm} \right] \frac{r_j}{r^3} d\underline{r}, \quad (42)$$

where $R_{im}(\underline{x}, \underline{r}) = u_i(\underline{x}) u_m(\underline{x} + \underline{r})$ and $S_{ilm} = \overline{u_i(\underline{x}) u_\ell(\underline{x} + \underline{r}) u_m(\underline{x} + \underline{r})}$.

No further discussion is offered except to note that it must be considered a challenge of future research to either rationally justify the jump from (41) to the expression, $\overline{u_i \partial p / \partial x_j} \propto u_i u_j - \delta_{ij} q^2/3$ or suggest an alternative.

A further challenge to future research is to provide equations or a more rational means of proscribing the length scales ℓ and Λ . In this regard Rotta (1951) has already made some proposals.

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