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THE CROSSCORRELATION METHOD

OF MEASURING
SYSTEM DYNAMIC RESPONSE

MASTER

By

J. Douglas Balcomb
University of California
Los Alamos Scientific Laboratory
Los Alamos, New Mexico

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I. BASIC RELATIONSHIPS

The crosscorrelation method is one of the several techniques for measuring the dynamic response characteristics of a system. The method can be used to obtain accurate information about system response in the presence of extraneous noise signals in a minimum time. The method is particularly useful in cases where test perturbations to the system output are largely corrupted by system noise and it is desirable to minimize the duration of the experiment. The method consists of performing the following operations:

- A. The system, for which a dynamic response is desired, is excited by introducing a noise-like input signal. This signal contains, in equal magnitude, each frequency for which the system has a significant response.
- B. The crosscorrelation function of the system input and system output signals is computed. This correlation function is equal to the average value of the product of the system output and the system input delayed by a given time. This average is then a function of the time delay used.

The basic mathematical result utilized in the crosscorrelation method is that the function computed in (B), above, is proportional to the impulse response of the system. The basic virtue of the method is the tendency of the effects of extraneous noise in the signals to cancel out in the final results. The impulse response is a complete

description of the dynamics of a linear system and from it other useful descriptions of system dynamics, such as the transfer function, can be developed.

The basic relationships of the crosscorrelation method have been derived many times in the literature.^{1,2,3} If a linear system with an impulse response or weighting function, $h(t)$, has a stationary input, $a(t)$, and the corresponding response signal is $b(t)$, then the crosscorrelation function between $a(t)$ and $b(t)$ is given by:

$$\phi_{ab}(\tau) = \int_0^{\infty} d\lambda h(\lambda) \phi_{aa}(\tau - \lambda) \quad (1)$$

where

$$\phi_{ab}(\tau) \equiv \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T dt a(t) b(t + \tau) \quad (2)$$

= crosscorrelation function of $a(t)$ and $b(t)$

and

$$\phi_{aa}(\tau) \equiv \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T dt a(t) a(t + \tau) \quad (3)$$

= autocorrelation function of $a(t)$.

If the time function of $a(t)$ is white noise then its autocorrelation function is a delta function, $\phi_{aa}(\tau) = K\delta(\tau)$, and the crosscorrelation function, $\phi_{ab}(\tau)$, is proportional to the system impulse response.

$$\phi_{ab}(\tau) = K h(\tau) \quad (4)$$

Thus the impulse response of a system can be measured by providing a white noise input to a system and computing the crosscorrelation function between the input and the output.

The analysis can also be carried out in terms of the spectral representations or Laplace transforms of the signals. This approach results in the relationship

$$\Phi_{ab}(i\omega) = H(i\omega) \Phi_{aa}(i\omega) \quad (5)$$

where

$$\Phi_{ab}(i\omega) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} d\tau e^{-i\omega\tau} \phi_{ab}(\tau) \quad (6)$$

= cross power spectral density between $a(t)$ and $b(t)$

and

$$\Phi_{aa}(i\omega) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} d\tau e^{-i\omega\tau} \phi_{aa}(\tau) \quad (7)$$

= power spectral density of $a(t)$

and

$$H(i\omega) = \int_0^{\infty} dt e^{-i\omega t} h(t) \quad (8)$$

= system transfer function.

Thus the system transfer function can be measured by computing the cross power spectrum of the input and the output signals and the power spectrum of the input signal and then computing the ratio

$$H(i\omega) = \frac{\Phi_{ab}(i\omega)}{\Phi_{aa}(i\omega)} \quad (9)$$

In the special case where the input signal, $a(t)$, is white noise, the power spectrum is a constant, $K/2\pi$, and the transfer function is given by:

$$H(i\omega) = 2\pi \Phi_{ab}(i\omega) / K \quad (10)$$

The purpose of this paper is to illustrate the application of these techniques in practical cases where there are three complications which cause the above idealized relations to be inaccurate. These complications are as follows:

1. The experiment is of finite length.
2. The input signal has a finite bandwidth.
3. There is uncorrelated noise present in the system response signal.

II. ILLUSTRATIVE EXAMPLE

For purposes of illustrating the method and also for predicting results in advance of actual experiments, a digital computer code has been written to simulate experiments. A hypothetical wide band gaussian input signal is generated using a random number code. The response of hypothetical systems to this random input signal is computed by representing the systems by linear differential equations and using

numerical integration techniques to solve the equations. Thus the output of the computation is a series of values of system input and system output such as might be obtained in an actual experiment utilizing a sampled data recording system. The computation also allows for the addition of hypothetical uncorrelated gaussian noise signals to the system response.

The signals so computed are analysed using a standard digital code designed for computing correlation functions and power spectra of recorded signals.

The hypothetical systems simulated in the calculation are as follows:

Case 1.

First order lag with a break frequency of 100 cps.

$$db/dt = 628 (a - b)$$

$$h(t) = 628 e^{-628 t}$$

$$H(i\omega) = \frac{628}{(i\omega) + 628}$$

Case 2.

Second order lag with a break frequency of 100 cps and a damping factor of 1/10.

$$d^2b/dt^2 = 628^2 (a - b) - 125.6 db/dt$$

$$h(t) = 631 e^{-62.8t} \sin(625t)$$

$$H(i\omega) = \frac{628^2}{(i\omega)^2 + 125.6(i\omega) + 628^2}$$

The rms value of the input signal is taken as unity. Five thousand data points were recorded at a spacing of 1/2000 second. Thus the total record length is 2.5 seconds. Figure 1 shows the autocorrelation function and power spectral density of the input signal and the crosscorrelation function between the input and output for each of the two cases. These results indicate that a straightforward application of the method utilizing equation (4) and a reasonable length of signal will produce poor results. The reason the results are poor is because of the finite length of signal; this essentially results in a non-uniformity in the effective spectrum of the input signal. However, good results can be obtained from this same data by utilizing equation (9); this is because a correction is made for the non-uniformity of the input spectrum when the cross power spectrum is divided by the input power spectrum. Figure 2 shows the cross power spectra for each case and the ratio $\bar{\phi}_{ab}(i\omega)/\bar{\phi}_{aa}(i\omega)$ for each case. This ratio is essentially equal to transfer function of the system in both cases as predicted by equation (9).

III. THE USE OF PSEUDO-RANDOM INPUT SIGNALS

It has been demonstrated in the previous section that for the case of random input signals it is usually necessary to carry the analysis through to the computation of power spectra in order to obtain good results. Because of the large number of computations involved, this leads to the use of automatic computing equipment, either analog or digital, and the answers are obtained off-line after an appreciable time delay. For many applications, however, it is desirable to perform the crosscorrelation on-line using analog equipment and to utilize equation (4) directly. In order to do this it is necessary either to have extremely long experiments or to synthesize input signals which have a uniform power spectral density. A method of synthesizing such signals has been developed and is discussed in several of the references.^{5,6} These signals are generated from either of two types of pseudo-random binary sequences. The waveform of such a synthesized signal is binary and appears very much like a square wave which has been modified so that the sign in each time interval is chosen at random. The signal is periodic, however, the period can be made very long. The signal has the very desirable property that the autocorrelation function is very nearly a delta function.

IV. ILLUSTRATIVE EXAMPLE

A hypothetical experiment was performed utilizing the same technique as outlined in section II except with a pseudo-random binary input signal having a value of either +1 or -1 generated from pseudo-random binary chain of length 251 and having a periodicity of

0.1255 seconds. Figure 3 shows the autocorrelation function and power spectrum of the input signal and the crosscorrelation function between system input and system output for each of the two cases. The periodicity in the crosscorrelation function is a result of the periodicity in the input signal. These results indicate that good data can be obtained utilizing equation (4) if the special synthesized input signals are used. By the use of special techniques, such as selecting the crosscorrelation integral to cover an integer number of periods of the input signal, even more accurate results can be obtained if required. These techniques were not utilized in the case presented in Fig. 3.

V. THE EFFECTS OF NOISE

Presumably, the reason for using the crosscorrelation method in a given experiment is because system noise tends to corrupt answers and, ideally, answers obtained by the crosscorrelation method are not affected by noise signals which are uncorrelated with the system input signal. The finite duration of the experiment, however, will result in some errors in the results due to system noise. The purpose of this section is to discuss a relationship which can be used to estimate these errors.

In an experiment, the signal $a(t)$ is applied to the system. The observed output is $c(t)$ which is made up of two components -- the response of the system to $a(t)$ which we have called $b(t)$, and a noise signal, $n(t)$. Presumably, the signal $n(t)$ is a stochastic noise signal whose characteristics could be determined from a sample obtained by recording the system output with no applied input signal. Thus:

$$c(t) = b(t) + n(t) \quad (11)$$

For an experiment of finite duration, the crosscorrelation function is redefined in terms of the correlation time, P , as follows:

$$\phi_{ac}(\tau) = \frac{1}{P} \int_0^P dt c(t) a(t-\tau)$$

The system response to $a(t)$ is $b(t)$ which is:

$$b(t) = \int_{t_0}^t d\lambda a(\lambda) h(t-\lambda) = \int_0^{t+t_0} d\lambda h(\lambda) a(t-\lambda) \quad (12)$$

where t_0 is the time when the signal $a(t)$ started.

Thus:

$$\phi_{ac}(\tau) = \phi_{ab}(\tau) + X(\tau) \quad (13)$$

where

$$\phi_{ab}(\tau) = \int_0^\infty d\lambda h(\lambda) \phi_{aa}(\tau-\lambda) \quad (14)$$

and

$$X(\tau) \equiv \frac{1}{P} \int_0^P n(t) a(t-\tau) dt \quad (15)$$

The assumptions which have been made are as follows:

1. The system is linear.
2. The signal $a(t)$ is periodic of period P .
3. $h(t) \approx 0$ for $t > -t_0$.

We desire to obtain an expression for the magnitude of $X(\tau)$, the component of $\phi_{ac}(\tau)$ due to noise, as compared to $\phi_{ab}(\tau)$, the desired answer. If we knew $n(t)$ explicitly we could find $X(\tau)$ explicitly, or for that matter, subtract $n(t)$ from $c(t)$ prior to the crosscorrelation, and obtain an exact answer for $\phi_{ab}(\tau)$ and ultimately $h(t)$. However, $n(t)$ is not known explicitly; the general properties of $n(t)$ are known, such as the autocorrelation or the power spectrum. Thus only average properties of $X(\tau)$ can be determined. Two properties of $X(\tau)$ which are useful and which can be determined are the mean-square-average, $\overline{X^2}$, and the power spectrum $\Phi_{xx}(i\omega)$. Expressions for these quantities are derived in reference (4) as follows:

$$\Phi_{xx}(i\omega) = A(i\omega) \Phi_{nn}(i\omega) \quad (16)$$

$$\overline{X^2} = \int_{-\infty}^{\infty} d\omega A(\omega) \Phi_{nn}(\omega) \quad (17)$$

where $A(i\omega)$ is a property of the input

$$A(i\omega) = \frac{1}{P} \int_0^P d\beta \frac{1}{P} \int_0^P dt e^{i\omega(t-\beta)} \phi_{aa}(t-\beta) \quad (18)$$

In order to obtain an accurate value for $\Phi_{xx}(i\omega)$ it is necessary to evaluate $A(i\omega)$. A particular example of $A(i\omega)$ for a pseudo-random binary input signal is given in reference (4). For signals which have $\phi_{aa}(\tau)$ nearly equal to a delta function, then $A(i\omega)$ is equal to a constant over a wide range of frequencies. For estimating purposes it is adequate to use the following expression:

$$A(i\omega) \cong \frac{2\pi}{P} \Phi_{aa}(i\omega) \quad (19)$$

Thus, the power spectral density and the mean-square of the error, $X(\tau)$, can be estimated from the relationships:

$$\Phi_{xx}(i\omega) \cong \frac{2\pi}{P} \Phi_{aa}(i\omega) \Phi_{nn}(i\omega) \quad (20)$$

$$\overline{X^2} \cong \int_{-\infty}^{\infty} d\omega \Phi_{xx}(\omega) \quad (21)$$

An estimate of the magnitude of the errors in the transfer function can be obtained from the relation:

$$\Phi_{xx}(i\omega) = \frac{X(i\omega) X(-i\omega)}{2\pi} \quad (22)$$

Thus

$$|X(i\omega)| = \sqrt{\Phi_{xx}(i\omega) (2\pi)} \quad (23)$$

$$\frac{\Phi_{ac}(i\omega)}{\Phi_{aa}(i\omega)} = \frac{\Phi_{ab}(i\omega) + X(i\omega)}{\Phi_{aa}(i\omega)} \quad (24)$$

$$= H(i\omega) + H_{\text{error}}(i\omega) \quad (25)$$

where

$$|H_{\text{error}}(i\omega)| \approx 2\pi \sqrt{\frac{\Phi_{nn}(i\omega)}{P \Phi_{aa}(i\omega)}} \quad (26)$$

These results indicate that errors in the crosscorrelation function (X_{rms} , equation 21), and the transfer function, ($H_{\text{error}}(i\omega)$, equation 26), both vary inversely as the square root of the crosscorrelation integration interval. Therefore, even if the spectrum of the noise has not been determined, it is possible to estimate from one experiment the integration time required to produce a given error in a future experiment.

VI. ILLUSTRATIVE EXAMPLE

As an illustration of the results of the previous section, the digitally simulated experiment described in Section II is repeated except with a gaussian noise signal added to the system response signal. For both systems, the signal-to-noise ratio is equal to unity, that is, the rms value of the added noise is numerically equal to the rms value of the system response to the input signal. The rms values of the noise is as follows: Case 1. - 0.391, Case 2. - 0.890. The results of the example are given in Fig. 4.

VII. COMMENTS

Two variations of the crosscorrelation method have been discussed. A selection of one of these methods will depend upon various factors as follows:

1. Whether the results are desired off-line or on-line.
2. Equipment available, for example:
 - a. Special pseudo-random binary signal generator.
 - b. Analog computer equipment.
 - c. High speed digital computer.
 - d. Data recording and digitizing equipment compatible with c.
3. Precision required vs length of experiment and noise level.

The two general categories for which the crosscorrelation method has been applied differ widely and indicate different approaches. In one application, the method is used as an on-line technique for determination of some characteristic parameter of the system. In this case it is desirable to utilize the results of equation (4) directly.

Therefore it is almost necessary to utilize an input signal with the special properties of the pseudo-random binary signals. This requires construction of special equipment to generate these signals and also to perform the crosscorrelation. The use of a binary signal for an input signal is also desirable because it facilitates both the time delay and the multiplication operations required in the computation of the crosscorrelation function.

For the other application the input and output signals are recorded and the processing is performed in a digital computer-off line. In this way the computation can be carried through to spectral analysis of the signals and thus errors due to vagaries of the input signal can be corrected by the use of equation (9). Therefore, it is not necessary to utilize the special pseudo-random binary input signal. In some cases, where the internal system noise is wide band and large, it is possible to avoid the use of a separate input signal by utilizing a natural observable signal at the system input as the system input signal.

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