NUMERICAL STUDIES OF NON-LINEAR
EVOLUTION OF KINK AND TEARING
MODES IN TOKAMAKS

BY

R. WHITE, D. MONTICELLO,
M. N. ROSENBLUTH, H. STRAUSS,
AND B. B. KADOMTSEV

PLASMA PHYSICS
LABORATORY

MASTER

PRINCETON UNIVERSITY
PRINCETON, NEW JERSEY

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Numerical Studies of Non-linear Evolution of Kink and Tearing Modes in Tokamaks

R. White*, D. Monticello, M. N. Rosenbluth[†], H. Strauss
(Institute for Advanced Study, Princeton, New Jersey)

and

B. B. Kadomtsev
(Kurchatov Institute, Moscow)

ABSTRACT

A set of numerical techniques for investigating the full non-linear unstable behavior of low β kink modes of given helical symmetry in Tokamaks is presented. Uniform current density plasmas display complicated deformations including the formation of large vacuum bubbles provided that the safety factor q is sufficiently close to integral. Fairly large $m=1$ deformations, but not bubble formation, persist for a plasma with a parabolic current density profile (and hence shear). Deformations for $m \geq 2$ are however greatly suppressed.

The numerical treatment of MHD instabilities in Tokamaks in a straightforward way is difficult because of the three dimensional character of the modes, because of the various time scales involved, [Alfvén waves and sound waves, relatively slow kinks and even slower resistive modes], and because of the free boundary between plasma and vacuum.

We therefore consider a reduced set of equations which make use of the appropriate Tokamak ordering and have the following features: 1) the cylindrical approximation is employed as is appropriate for these modes, 2) only perturbations of a particular helical symmetry are treated, but with the complete non-linear equations. This is sufficient to reduce the problem to a two-dimensional one and should be accurate at least in the neighborhood of rational q . 3) The MHD fluid equations are used, with the envisaged addition of finite resistivity and 4) because of the large toroidal field, $\nabla_{\perp} \cdot \vec{V} = 0$, and $k B_z \sim \mathcal{O}[(m/r) B_0]$, and perturbed toroidal field $\tilde{B}_z \sim \mathcal{O}(k^2 r^2 B_z)$.

Hence all quantities depend only on r , and $m\theta + kz$ with $\partial/\partial z = (k/m) \partial/\partial \theta$. It is convenient to introduce the stream function $\psi = A_z - (kr/m)A_\theta$ with $B_\theta = -(\partial\psi/\partial r) - (kr/m)B_z$, $B_r = (1/r)(\partial\psi/\partial \theta)$. The value of ψ at the chamber wall (ψ_w) is related to the safety factor q evaluated just outside the plasma surface ($r = a$; the chamber wall is at $r = 1$) through $(q/m) = a^2 \ln a / [(a^2 - 1)/2 - \psi_w]$, and we will use both parameters.

Expanding in the parameter $\epsilon = kr$ and keeping only lowest order terms the equations then become:

$$\frac{\partial \psi}{\partial t} + (\mathbf{v} \cdot \nabla) \psi = \eta \nabla^2 \psi \quad (\text{plasma}) \quad (1)$$

$$\rho \frac{d\mathbf{v}}{dt} = \nabla \left[-\left(p + \frac{B^2}{2}\right) + \frac{(\nabla \psi)^2}{2} - 2 \frac{kB_z}{m} \psi - \frac{1}{2} \left(\frac{krB_z}{m}\right)^2 \right] - \nabla \psi \nabla^2 \psi \quad (2)$$

$$\nabla \cdot \mathbf{v} = 0 \quad (3)$$

$$\nabla^2 \psi = -2 \frac{kB_z}{m} \quad (\text{vacuum}); \quad \psi \equiv 0 \text{ on interface, } \psi = \psi_w \text{ on the outer wall.} \quad (4)$$

$$p + \frac{B^2}{2} \text{ continuous between vacuum and plasma.} \quad (5)$$

We are considering 3 phases of complexity in the problem:

I. $\eta = 0$, and $\psi \equiv 0$ in the plasma. This corresponds to an equilibrium circular state in which the pitch of the perturbation exactly matches the constant pitch of the equilibrium field in the plasma. The plasma has constant current density and no magnetic shear. By adding surface current the equilibrium may be made linearly stable or unstable.

II. $\eta = 0$, but with a general plasma current distribution $J(r)$.

III. $\eta \neq 0$, (tearing modes, and effects of resistivity on kinks).

As of this writing phase I has been completed and the results are presented herein. Phase II is in production and some results will be discussed. Phase III has not been started yet and questions remain concerning the possibility of an adequate grid for the narrow resistive layer.

Phase I

For Phase I $\vec{\nabla} \times \vec{v} = 0$, so that $\vec{v} = \vec{\nabla} \phi$ with

$$\nabla^2 \phi = 0. \quad (6)$$

The problem reduces to following the motion of the plasma-vacuum interface. Thus equations (4) and (6) must be satisfied, and equation (5) becomes in suitable dimensionless units:

$$\frac{\partial \phi}{\partial t} \Big|_{p-v} = -\frac{1}{2} [(\nabla \psi)^2 + (\nabla \phi)^2] \quad (7)$$

Various numerical methods have been tried — there being considerable difficulty because of the rather extreme distortions of the surface which occur and the impossibility of using simple Fourier representations. A satisfactory scheme seems to be the following: The surface is represented by a set of mass points moving with velocity $\nabla\phi$. The necessary potentials ϕ in the plasma and ψ in the vacuum are determined by using Gauss' Theorem to relate the normal derivatives of ϕ and ψ to their values on the surface, and mass points may be slid along the surface with proper interpolation in order to retain an adequate representation. The basic time step then consists of matrix inversions of the integral equation along the surface. Linear growth rates and oscillation periods are determined with accuracies of 10^{-3} with 20 particles on the surface. In Fig. 1a the linear growth rate γ is plotted as a function of ψ_w for a plasma radius $a = 0.8$. Modes $m = 1$ and $m = 2$ are shown, modes with higher m value having a smaller range of instability, but all modes sharing the left hand neutral stable point.

The equations (without resistivity) have a conserved integral (energy), and examination of this quantity has shown that the lowest potential energy state should often be a "bubble" of vacuum field in the center with the plasma moved out towards the wall.¹ The energy of the bubble state is easily calculated as a function of the plasma outer radius r_1 , related to the bubble radius r_b through $r_1^2 - r_b^2 = a^2$.

$$E_b = \frac{-r_1^2 a^2}{2} + \frac{[(1-r_1^2/2)+\psi_w]^2}{\ln r_1} \quad (8)$$

The resulting potential well for bubble formation is shown for a particular example in Fig. 1b. The range of values of ψ_w for which this state is energetically favored is shown in Fig. 1a. Note that it extends beyond the region of linear instability. Formation of a bubble would be an obvious case of disruptive instability. Previous work² has indicated that usually the first non-linear terms in perturbations away from equilibrium are destabilizing. A perturbation expansion of the equations for a $m = 1$ displacement in the vicinity of the right hand neutral stable point (see Fig. 1a) leads to an equation of motion:

$$\delta\ddot{r} = -\omega^2\delta r + C(a)\delta r^3 \quad (9)$$

We find that the term $C(a)$ is stabilizing for $a < 0.65$; otherwise no nearby equilibrium exists. By introducing a damping term $(-\nu\dot{\phi})$ into Eq. (7) the final asymptotic state can be found numerically. The nearby states found in this manner had displacements agreeing within a few percent with the value $\delta r = [\omega^2/C(a)]^{1/2}$ given by Eq. (9).

Equation (9) was also used to investigate the threshold for finite amplitude displacements to the right of the neutral stable point when $C(a)$ is destabilizing, and agreed with our numerical results.

Far from the right hand neutral stable point no simple bifurcation exists and the full non-linear development of the plasma surface must be followed numerically. We have examined the final states thus obtained over a wide range of ψ_w and a .

In Figure 2a,b,c,d the range of final states is shown for $a = 0.8$. Near the right hand neutral stable point, where a small number of modes are

linearly unstable, the plasma exhibits a shallow deformation which gradually deepens as $\psi_w \rightarrow 0$. However the quantity $(\partial\psi/\partial n)$ evaluated on the surface (which determines the skin current since $\psi \equiv 0$ in the plasma interior), always points outward and has a constant magnitude everywhere on the surface. For $\psi_w < 0$, however, $(\partial\psi/\partial n)$ becomes discontinuous, the plasma forming sharp horns at the points of discontinuity which tend to close off the resulting vacuum bubble. For numerical reasons a complete bubble is not seen, but as the number of points is increased the horns approach each other more closely, evidently asymptoting to a completely enclosed bubble, whose size agrees with the energetic minimum predicted analytically by Eq. (8). In Fig. 2e,f are shown final states obtained for $m = 2, 3$, $a = 0.7$.

In the absence of damping the plasma boundary executes large amplitude elastic oscillations. In Fig. 3 is shown the evolution in time of the plasma vacuum boundary following an $m = 1$ perturbation. We have chosen a case for which bubble formation is favored, $\psi_w = -0.08$ ($q/m = 1$), $a = 0.7$. In spite of the rather violent distortions, the plasma behaves quite elastically, cases with less violent bubble formation having been run through several non-linear oscillation periods.

Phase II

For Phase II, $\nabla \times \vec{V} \neq 0$, $\vec{V} = -\hat{z} \times \nabla A$ and the necessary elliptic equation for $(\partial A/\partial t)$ is found by taking the curl of Eq. (2) whose parallel component at the surface also determines $(\partial/\partial t)(\partial A/\partial n)|_s$. Specifically

$$\frac{D(\nabla^2 A)}{Dt} = -\nabla\psi \times \nabla(\nabla^2\psi) \cdot \hat{z} \quad (10)$$

$$\frac{D\vec{V}}{Dt} \cdot \vec{P} = -\frac{1}{2} \vec{P} \cdot \nabla \{ (\nabla\psi)_v^2 - (\nabla\psi)_p^2 \} \quad (11)$$

where $\vec{P}$ is the counterclockwise parallel to the plasma surface. The analysis is more complicated than in Phase I because interior grid points must be retained, necessitating an interior relaxation procedure once the surface values have been determined as in Phase I. Two approaches to the interior finite differencing problem are currently being attempted. One method moves the grid points to maintain an orthogonal coordinate system, the other, which is not presently operational, is a non-orthogonal code that moves grid points to maintain equal spacing. To solve Poisson's equation $(\nabla^2 A = \delta)$ for A with $(\partial A/\partial n)$ known on an arbitrarily shaped boundary the following iteration scheme³ is used,

$$\nabla_{Ave}^2 A = \delta - (\nabla^2 A - \nabla_{Ave}^2 A) \quad (12)$$

where ∇_{Ave}^2 is the Laplacian operator with the scale factors averaged around a $\psi = \text{constant}$ surface (fast Fourier transforms then being used on ∇_{Ave}^2). The convergence of this iteration procedure has been proved by using a variational principle, which also yields the rate of convergence.

A parabolic current (J_z) profile vanishing on the surface and with no skin current ($\partial\psi/\partial n$ continuous) was used for the first runs of this phase. Figure 1c shows a typical linear dispersion relation for $m=2$ and $r_{pl} = 2/3 r_{wall}$. In general shear has a significant effect on the unstable modes. However, it has no effect on the $m=1$ linear dispersion curve because the motion for $m=1$ is just a solid displacement so that the plasma magnetic energy is not affected.

As might be expected, shear greatly effects the non-linear development of the kink modes, even for an $m=1$ perturbation. The results so far indicate that magnetic bubbles are not formed. However, the non-linear distortions are still quite severe for $m=1$. In Figure 4a,b,c,d, we present some typical final states, which are to be compared with the $\psi_p \equiv 0$ final states (Fig. 2).

Probably more relevant to Tokamaks are the $m=2$ final states, examples of which we present in Fig. 4e,f. These show that the non-linear distortions are only moderate elongations of the plasma with little or no concavity. To quantitatively determine the stabilizing effect of shear we are examining other current profiles, intermediate between constant current and parabolic.

Linear growth and oscillation rates produced by the code agree to within a few percent with analytic theory or (in the $m=2$ case) radial numerical quadrature. An analytic nonlinear theory is being developed at this time to check the code in the region near the right hand marginal stable point ($q \neq \text{integer}$). Other checks on the code are energy and area, which are conserved to within a few percent-even during the most severe nonlinear distortions.

In summary we have developed a set of numerical techniques for investigating the full non-linear unstable behavior of low β kink (and possibly tearing) modes of given helical symmetry in Tokamaks. Results to date are with perfect conductivity. For uniform current density unstable plasmas show widely differing types of behavior, simple bifurcation to neighboring helical equilibria; very complicated and distorted, but still elastic motions; or apparent shattering and bubble formation, depending on how close the q value is to integral.

The introduction of a non-uniform plasma current, and consequently shear, stabilizes this behavior to a great extent, but appreciable distortions without bubbles are still observed particularly for $m=1$ modes. Non-linear behavior with parabolic current profiles for $m \geq 2$ is generally mild enough to lend support to the conjecture that ideal MHD kink instabilities with $q > 1$ do not represent a threat to Tokamak confinement.

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- * Present Address, Plasma Physics Laboratory, Princeton University.
- † Also at the Plasma Physics Laboratory, Princeton University.
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Fig. 1a. The linear growth rate γ as a function of ψ_w . The plasma radius $a = 0.8$. Stable and unstable branches are shown for $m = 1, 2$. Also shown is the region in which a single bubble state is energetically favorable. $\psi \equiv 0$ in plasma. The growth rate is in units given by the Alfvén frequency, $\omega_A = (kB_z/\rho^{1/2m})$.

b. Bubble energy E_b (units arbitrary) as a function of plasma outer radius r_1 , $r_1^2 - r_p^2 = a^2$, for various values of ψ_w . Here $a = 0.8$. $\psi \equiv 0$ in plasma.

c. The linear growth rate for $m = 2$ as a function of q for a parabolic current profile, $J(r) \sim [1 - (r^2/a^2)]$, and for the zero shear cases $\psi_p \equiv 0$ and $J(r) = \text{constant}$ with no skin current. Here $a = 0.66$.

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Fig.2abcd. Final plasma-vacuum configurations for $\psi = 0$ in the plasma obtained by adding a damping term to Eq. 7. The direction of $(\partial\psi/\partial n)$ is indicated by arrows.

e,f. Final states for $m = 2, 3$ and ψ in the plasma (ψ_p) equal to zero. Here $a = 0.7$ and $\psi_w = -0.08$ ($q/m = 1$).

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Fig. 3. Undamped temporal evolution of the plasma-vacuum interface in a case where bubble formation is favored. Here $\psi_p = 0$, $a = 0.7$, $\psi_w = -0.08$.

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Fig.4abcd. Final plasma configurations for $m = 1$ for a parabolic current profile. Here $a = 0.8$. These are to be compared to the $\psi \equiv 0$ states shown in Fig. 3. In the center of the plasma fine grid spacing is not necessary since the current profile is approximately constant. This "numerical hole" should not be confused with a vacuum bubble.

e,f. Final plasma configurations for $m = 2$ for a parabolic current profile. Here $a = 0.66$. Undamped oscillations exhibit some degree of concavity, but no bubble formation.

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