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PHYSICAL MECHANISM OF PARAMETRIC
INSTABILITIES NEAR THE LOWER-
HYBRID FREQUENCY IN INHOMOGENEOUS
PLASMA

BY

MIKLOS PORKOLAB

PLASMA PHYSICS
LABORATORY

MASTER

PRINCETON UNIVERSITY
PRINCETON, NEW JERSEY

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Princeton University
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ERRATA

MATT-1069 Physical Mechanism of Parametric Instabilities Near
the Lower-Hybrid Frequency in Inhomogeneous Plasma..

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Page 5:

The paragraph beginning with Eq. 7 should be replaced by:

$$\zeta^2 \approx \frac{1 + (2T_e/T_i) I_1(b_i) \exp(-b_i)}{(T_e/T_i) b_i (\omega^2/\omega_{LH}^2 - 1)} \quad (7)$$

and so for $b_i < 1$ we obtain Eq. 4(b) as before.

However, for $b_i \gg 1$ we obtain

$$\zeta^2 \approx \frac{1 + (T_e/T_i) (2/\pi)^{1/2} b_i^{-1/2}}{(T_e/T_i) b_i (\omega^2/\omega_{LH}^2 - 1)}$$

so that the selection rules for decay instability could be satisfied for sufficiently large values of b_i even for frequencies close to the lower-hybrid frequency.

Physical Mechanism of Parametric Instabilities Near
the Lower-Hybrid Frequency in Inhomogeneous Plasma.*†

Miklos Porkolab
Plasma Physics Laboratory, Princeton University
Princeton, New Jersey 08540 USA

ABSTRACT

The dispersion relation for parametric instabilities near the lower-hybrid frequency is obtained from model fluid equations. The following instabilities are discussed: for rf pump frequencies $\omega_o \gtrsim 3\omega_{LH}$, $\omega_{pe} \approx \Omega_e$, resonant decay into ion sound (ion cyclotron) modes (previously predicted) is found. In the regime of $1 \lesssim \omega_o/\omega_{LH} \lesssim 3$, $\omega_{pe} \approx \Omega_e$, decay into ion quasi-modes is found. In strong magnetic fields decay into quasi-modes is also found for $3 \lesssim \omega_o/\omega_{LH}$. This mechanism is similar to nonlinear Landau damping in weak turbulence theory. In addition, decay into the purely growing mode and fluid-quasi modes may also occur. The results are compared with recent calculations from the Vlasov equation. The effects of plasma-inhomogeneities will also be considered, including effective thresholds due to density gradients and finite pump extent. The implications of these results to rf heating of tokamaks near the lower-hybrid frequency will be discussed.

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I INTRODUCTION

In this lecture we shall discuss the linear theory of parametric instabilities near and above the lower hybrid frequency, i.e., $\omega_{LH} < \omega_o \ll \min(\omega_{pe}, \Omega_e)$. This regime is of considerable importance from the point of view of rf heating of fusion plasmas. In particular, for lower-hybrid waves we have

$$\omega_2^2 \approx \omega_{LH}^2 [1 + (k_{||}^2/k^2) (m_i/m_e)] \quad 1(a)$$

$$\omega_{LH}^2 \approx \frac{\omega_{pi}^2}{1 + \omega_{pe}^2/\Omega_e^2} \quad 1(b)$$

where we assumed $\Omega_i \ll \omega_{pi}$, $v_{te} \ll \omega_2/k_{||}$, (where ω_{LH} is the lower hybrid frequency, $\omega_{pi}(\omega_{pe})$ is the ion (electron) plasma frequency, $\Omega_i(\Omega_e)$ the ion (electron) cyclotron frequency, and $v_{te} = \sqrt{2T_e/m_e}$. Furthermore, $\cos\theta = k_{||}/k$, $k_{||} = (\vec{k} \cdot \vec{B})/B$, $k^2 = k_{||}^2 + k_{\perp}^2 = k_{||}^2 + k_y^2 + k_x^2$. Waves with propagation angles such that $(m_i/m_e)\cos^2\theta \ll 1$ we shall call lower-hybrid waves (i.e., both electrons and ions contribute to the wave motion in a significant way). The regime $1 \ll (m_i/m_e)\cos^2\theta$ is the electron plasma wave regime (or Trivelpiece-Gould modes).² Here Eq. 1(a) reduces to

$$\omega_2^2 \approx \left(\frac{k_{||}^2}{k^2}\right) \frac{\omega_{pe}^2}{(1 + \omega_{pe}^2/\Omega_e^2)} \quad (2)$$

and in this regime the ion contribution to wave motion is not important. We notice that for $(m_i/m_e)\cos^2\theta \ll 1$ thermal effects become important, and in Eq. 1(a) corrections of the order of $k\lambda_{Di}$, $k\lambda_{De}$, $k_{\perp}r_{ce}$ may be necessary (the exact correction will depend on (m_e/m_i) and T_e/T_i). In addition, mode conversion into very short wavelengths (i.e., $k\lambda_{Di} \gg 1$ or $k_{\perp}r_{ce} \gg 1$) may occur,

which would then Landau damp and heat particles.¹ In the present work we shall concentrate on incident pump waves for which $\cos^2 \theta (m_i/m_e) \gg k\lambda_{Di}, k_{ir}r_{ce}$, so that mode conversion can be ignored. We shall show that in the regime $(\omega_{pe}/\Omega_e) \ll 1$, $1 \ll \omega_o/\omega_{LH} \ll 5$ (which in present and future tokamaks corresponds to frequencies of a few gigahertz), the following parametric instabilities may occur;^{3,4} For frequencies, $1 \ll \omega_o/\omega_{LH} \ll 3$, $\omega_{pe} \approx \Omega_e$, nonresonant decay into lower-hybrid waves and kinetic (resistive) ion quasi modes occurs. The driving mechanism of this process is nonlinear wave-wave-particle scattering, and it is characterized by beat phase velocities $\omega/k_{||} = (\omega_o - \omega_2)/k_{||} \approx v_{te}$. In addition, at high pump powers decay into fluid quasi-modes may also occur, i.e., $v_{te} < \omega/k_{||}$. For frequencies $3 \ll \omega_o/\omega_{LH}$, $\omega_{pe} \approx \Omega_e$ resonant decay into lower-hybrid waves and ion acoustic (ion cyclotron) waves occurs due to mode-mode coupling. In addition, decay into quasi modes may occur in strong magnetic fields ($\omega_{pe} < \Omega_e$). Decay into lower-hybrid waves and the purely growing mode is possible in all regimes, as well as decay into drift waves (in inhomogeneous plasmas) and lower-hybrid waves. In this lecture we shall obtain the growth rates and thresholds of the various instabilities from physical arguments. Then some theoretical results of more exact calculations will be presented. In the second half of this lecture we shall examine the effects of density gradients upon these instabilities. In particular, in collisionless plasmas convection due to inhomogeneities or finite size effects determine the threshold for instability.

II. PARAMETRIC DECAY INSTABILITIES IN A HOMOGENEOUS PLASMA.

A. Resonant Decay Instability.

Here we shall obtain from physical arguments the dispersion relation for resonant decay into lower hybrid waves (i.e., $v_{te} \ll \omega_2/k_{||}$),

and ion acoustic waves $\omega_1 = k C_s [1 + 3T_i/T_e]^{1/2}$, $C_s^2 = T_e/m_i$, or ion cyclotron waves $\omega_1 = (\Omega_i^2 + k_\perp^2 C_s^2)^{1/2}$ (where $\omega_1/k_\parallel v_{te} \ll 1$, and we assumed $T_i < T_e$, $k_\perp r_{ci} < 1$, and $\Omega_i \ll \omega_1$). We recall that in such processes the following selection rules have to be satisfied:

$$\omega_0 = \omega_1 + \omega_2; \quad \vec{k}_0 = \vec{k}_1 + \vec{k}_2; \quad (3)$$

for the following we shall assume that $k_0 \approx 0$ (dipole approximation), thus $\vec{k}_1 \approx -\vec{k}_2$. Let us now examine the consequences of the foregoing selection rules and dispersion relations. We may write for the low frequency waves

$$\zeta^2 \equiv \frac{\pi \omega_1^2}{(k_\parallel v_{te})^2} \approx \frac{\pi \Omega_i^2}{k_\parallel^2 v_{te}^2} + \frac{\pi k_\perp^2}{2k_\parallel^2} \frac{m_e}{m_i} \approx \frac{1}{\eta} \left(1 + \frac{1}{\omega_1^2/\Omega_i^2 - 1} \right), \quad 4(a)$$

where $\eta = (k_\parallel^2/k_\perp^2) (m_i/m_e)$, and where for simplicity we assumed $T_i \ll T_e$. Now using the lower hybrid dispersion relation, we may rewrite Eq. (4) as follows:

$$\zeta^2 \approx \frac{1}{(\omega_2^2/\omega_{LH}^2 - 1)} \left(1 + \frac{1}{\omega_1^2/\Omega_i^2 - 1} \right) \quad 4(b)$$

or, since $\omega_2 \approx \omega_0$, we have

$$\zeta^2 \approx \frac{\omega_{LH}^2}{(\omega_0^2 - \omega_{LH}^2)} \left(\frac{\omega_1^2}{\omega_1^2 - \Omega_i^2} \right). \quad 4(c)$$

For resonant decay we require initially well defined weakly damped waves ω_1, ω_2 , and therefore from Eq. 4(a) we must have $\zeta \ll 1$. Thus, for ion acoustic waves $\Omega_i^2 \ll \omega_1^2$, $\zeta \approx \omega_{LH}/(\omega_0^2 - \omega_{LH}^2)^{1/2}$, we see that usually only for $3\omega_{LH} \ll \omega_0$ can we satisfy the selection rules for resonant decay. For ion cyclotron waves,

i.e., $1 \lesssim \omega_1^2 / \Omega_i^2 \lesssim 2$, it is even more stringent to satisfy resonant decay conditions, as is apparent from Eq. 4(b,c). In fact, fixing the minimum value of $k_\perp$ by boundary conditions $k_\perp \sim 1/D$ (where D is the characteristic plasma size, such as the plasma radius) for $(\omega_1^2 - \Omega_i^2) / \Omega_i^2 \ll 1$ we obtain

$$\zeta = \frac{\Omega_i D}{C_s (\omega_0^2 / \omega_{LH}^2 - 1)^{1/2}} \quad (5)$$

where by our assumption $1 \lesssim \Omega_i D / C_s$. Thus, in this case $\zeta^2 \ll 1$ requires even higher values of $\omega_0^2 / \omega_{LH}^2$, i.e., the regime of resonant decay is further restricted to frequencies considerably above the lower hybrid frequency. On the other hand, we see from Eq. 4(c) that for cyclotron waves the condition $\zeta = \omega_1 / k_\perp v_{te} \approx 1$ is allowed for a large range of pump frequencies; in this case decay into quasi-modes dominates.

For finite ion temperature the ion cyclotron dispersion relation becomes

$$\omega_1^2 \approx \Omega_i^2 [1 + 2 \frac{T_e}{T_i} I_1(b_i) \exp(-b_i)] \quad (6)$$

where $b_i = k_\perp^2 r_{ci}^2$, $r_{ci} = (T_i / m_i) / \Omega_i$ is the ion Larmor radius, and I_1 is the modified Bessel function of the first kind. Thus, for $\omega_1 \approx \Omega_i$ Eq. (5) is replaced by

$$\zeta^2 \approx \frac{T_i}{2 T_e I_1(b_i) \exp(-b_i) [\omega_0^2 / \omega_L^2 - 1]} \quad (7)$$

where $\lambda_\perp \lesssim D$ as before. Again, the resonant decay instability regime is restricted to large values of ω_0^2 / ω_L^2 for $T_e \approx T_i$ since $I_1(b_i) \exp(-b_i) \ll 1$ for both $b_i \ll 1$ and $b_i \gg 1$.

Finally, we note that these arguments remain valid qualitatively even for $k_0 \neq 0$, although quantitatively we could expect somewhat larger regions (by factor of $\sqrt{2}$) of (ω_0/ω_{LH}) for resonant decay.

Let us now obtain the threshold and growth rate for resonant decay into ion acoustic waves and electron plasma waves. Consider the geometry shown in Fig. 1.

Fig. 1. Geometry of parametric decay near the lower-hybrid frequency.

Since $\Omega_i \ll \omega_0 \ll \Omega_e$, under the influence of the external electric field E_{0x} , electrons will oscillate due to the $E \times B$ force mainly in the $\hat{y}$ direction. The velocity is $V_{Oy} = -U \cos \omega_0 t$, where $U = cE_0/B$. Thus, the oscillating electron charge at the sideband is due to the beat of the low frequency charge $\delta n_e(\omega_1)$ and the oscillating electrons at ω_0 ,

$$\delta n_e \approx \alpha \left(\frac{\partial n_e}{\partial y} \right)^s \Delta y = \alpha i k_y n_e^s \Delta y \quad (8)$$

where $\Delta y = \int V_{oy} dt = -(U/\omega_o) \sin \omega_o t$, and the superscript s means the slowly varying component. The proportionality constant α depends on subtracting the ion contribution from the charge at the sideband. For $\omega_{LH}^2 \ll \omega_o^2$, ions do not participate in the wave motion, hence we expect $\alpha = 1$. For $\omega_o^2 = \omega_{LH}^2$, in the cold plasma approximation ions and electrons move together¹ so we expect $\alpha = 0$. Thus, it is reasonable that we use

$$\alpha = (1 - \omega_{LH}^2 / \omega_o^2) = \frac{k_{||}^2}{k^2} \frac{m_i}{m_e} \frac{\omega_{LH}^2}{\omega_o^2} \equiv \eta \omega_{LH}^2 / \omega_o^2 \quad (9)$$

In fact it is easy to show from the fluid equations that Eq. (9) is correct. Assuming the harmonic oscillator equation for the oscillating electron charge at the sideband, in the presence of the external electric field this equation becomes

$$\frac{\partial^2 n_e}{\partial t^2} + 2\gamma_2 \frac{\partial n_e}{\partial t} + \omega_2^2 n_e = -i\alpha k_y n_i U \left(\frac{\omega_2}{\omega_o}\right) \sin \omega_o t \quad (10)$$

where from Eq. (8) we obtained for the additional charge

$$\frac{\partial^2}{\partial t^2} (\delta n_e) = i\alpha k_y n_i U \frac{\omega_2^2}{\omega_o} \sin \omega_o t \quad (11)$$

Here we used the fact that for the low frequency ion sound waves $n_e^{(s)} \approx n_i$. Note that ω_2 has been defined by Eq. (1). The force acting on the ions is more complicated. Direct ion motion due to the presence of E_o is smaller than that of electrons by the factor of $\Omega_i / \omega_o \ll 1$, and hence it can be ignored. In fact, we expect that a generalized $\vec{v} \times \vec{B}$ ponderomotive force acts on electrons on the slow time scale, which then couple to the ion wave via pressure. The pressure is mainly transmitted through

parallel electron fluid motion, but since the ions move essentially in straight line orbits perpendicularly to B_0 in the $\hat{y}$ direction (i.e., $k_x \approx k_y$, $\Omega_i \ll \omega_0$) the relevant force is

$$F_y = -\nabla_y (P_e) = -ik_y \frac{k_x^2}{k^2} P \quad (12)$$

The generalized ponderomotive force should be the $\vec{v} \times \vec{B}$ force, so

$$F_y^{(s)} = n_e e \frac{(\vec{V}_0 \times \vec{B}_0)_y}{c} \Big|_y = \frac{n_e e E_0 \omega_0}{\Omega_e} \sin \omega_0 t$$

where we used the equation of motion for $\omega_0 \ll \Omega_e$,

$$V_{0x} \approx - \frac{e}{m} \frac{E_0 \omega_0}{\Omega_e^2} \sin \omega_0 t \quad (13)$$

Thus, equating Eqs. (12) and (13), the additional electron pressure on ion waves is

$$\delta P = \frac{ik_y}{k_x^2} U m_e n_e \omega_0 \sin \omega_0 t. \quad (14)$$

Recognizing that in the equation for ion waves $n_e^{(s)} \approx n_i$, $\omega_1^2 = k^2 (T_e/m_i)$, and that the acoustic term is $\omega_1^2 n_i = k^2 P_e/m_i$, we have for ions the following harmonic oscillator equation:

$$\frac{\partial^2 n_i}{\partial t^2} + 2\gamma_1 \frac{\partial n_i}{\partial t} + \omega_1^2 n_i = - \frac{k^2 \delta P}{m_i}$$

Substituting in for δP from Eq. (14), we obtain

$$\frac{\partial^2 n_i}{\partial t^2} + 2\gamma_1 \frac{\partial n_i}{\partial t} + \omega_1^2 n_i = - \frac{ik_y k^2}{k_x^2} \left(\frac{m_e}{m_i} \right) U n_e \omega_0 \sin \omega_0 t, \quad (15)$$

where γ_1 is the linear ion wave damping rate. We see that Eqs. (10) and (15) describe the coupling of electron plasma waves and ion acoustic waves. We note that the coupling is proportional to U , the $E \times B$ drift velocity of electrons. We also note that in the coupling term in the ion equation the usual (m_e/m_i) factor is multiplied by the term (k^2/k_n^2) . Since $1 \ll k^2/k_n^2$, we see the coupling for waves $\omega_0 \ll (\omega_{pe}, \Omega_e)$ is enhanced over that of Langmuir waves (for which $k^2/k_n^2 \approx 1$).

We now assume perturbations of the form $n(t) \propto N_\omega \exp(-i\omega t)$, use the selection rules Eqs. (3), assume the resonant conditions $\omega = \omega_1$, $\omega_0 - \omega = \omega_2$, and obtain from Eqs. 10 and 15 (for $\gamma = 0$):

$$-2i\omega_1 \gamma_1 N_i(\omega_1) = - \frac{k_y U \omega_0}{2\eta} N_e(\omega_1 - \omega_0) \quad 16(a)$$

$$2i\omega_2 \gamma_2 N_e(\omega_1 - \omega_0) = - \frac{k_y U \eta \omega_{LH}^2}{2\omega_0} N_i(\omega_1) \quad 16(b)$$

where $\eta = (k_n^2/k^2)(m_i/m_e)$. Solving Eqs. 16(a) and 16(b) simultaneously, we obtain for the threshold

$$\frac{U}{C_s} = 4 \frac{\omega_0}{\omega_{pi}} \left[\frac{\gamma_1 \gamma_2}{\omega_1 \omega_2} \left(1 + \frac{\omega_{pe}^2}{\Omega_e^2} \right) \right]^{1/2} \quad (17)$$

where again $U = cE_0/B_0$. The growth rate in the absence of damping is obtained by replacing $\gamma_1 \gamma_2$ with γ^2 , and thus we find $\gamma \propto E_0$.

It can be shown from kinetic theory that had we considered resonant decay into ion cyclotron waves, the threshold would be

$$\frac{U}{C_s} = \frac{4\omega_0}{\omega_{pi}} \left[\frac{\gamma_1 \gamma_2}{\omega_1 \omega_2} \left(1 + \frac{\omega_{pe}^2}{\Omega_e^2} \right) \right]^{1/2} \left(\frac{\omega_1}{(\omega_1^2 - \Omega_i^2)^{1/2}} \right) \quad (18)$$

where for $k_{ir}^2 c_i^2 \ll 1$, $\omega_1^2 - \Omega_i^2 = k_i^2 C_s^2$, but for short wavelength waves $\omega_1^2 - \Omega_i^2 = 2(T_e/T_i) \Omega_i^2 I_1(b_i) \exp(-b_i)$, as given by Eq. (6).

We note, however, that if the dominant damping of low frequency waves is electron Landau damping, (which can be assured by sufficiently long parallel wavelengths) then

$$\left(\frac{\gamma_1}{\omega_1}\right)_{\text{ion sound}} = \frac{\gamma_1}{\omega_1} \left(\frac{\omega_1^2}{\omega_1^2 - \Omega_i^2}\right)_{\text{ion cyclotron}} = \frac{\sqrt{\pi} \omega_1}{2k_{\parallel} v_{te}},$$

so that the thresholds are the same for ion acoustic waves and ion cyclotron waves for a given frequency and parallel wavenumber.

We note that for typical laboratory plasmas Eqs. 17 and 18 predict $E \times B$ velocities U of the order or less than the ion acoustic speed.

B. The Purely Growing Mode.

In the limit $k_o = 0$, the previous equations predict a second instability at $\omega_{Re} \approx 0$, $\omega_2 \approx \omega_o$, the so-called purely growing mode (oscillating two-stream instability, OTS). This instability occurs for $\epsilon(\omega_2) < 0$, i.e., for a given value of $k_{\parallel}$ the instability occurs for perpendicular wavenumbers somewhat smaller than that of the resonant decay instability. In particular, in the limit of $\omega = 0$ the Fourier transform of Eq. 15 becomes (at threshold so that $\gamma = 0$):

$$\omega_{sN_i}^2(\omega) = \frac{k_{\parallel} U \omega_o}{2\eta} [N_e(\omega - \omega_o) - N_e(\omega + \omega_o)] \quad , \quad (19)$$

and Eq. 10 becomes

$$[-\omega_o^2 + 2i\gamma_2 \omega_o + \omega_2^2] N_e(\omega \pm \omega_o) = \mp \frac{\eta k_{\parallel} U \omega_{LH}^2}{2\omega_o} N_i(\omega) \quad (20)$$

where the ($\mp$) signs correspond to the two different sidebands.

In particular, since the upper and lower sidebands almost coincide, (within the damping rate γ_2) they both have to be retained, and thus this is a four-wave process. Due to the beat of the sidebands, the background ion density is modulated in space. Equation (19) simply states a static pressure balance against the generalized $\vec{v} \times \vec{B}$ ponderomotive force. Combining Eqs. (19) and (20) gives

$$\omega_s^2 = - \frac{k_y^2 U^2 \omega_{LH}^2}{4 \omega_o} \left[\frac{\delta}{\delta^2 + \gamma_2^2} \right] \quad (21)$$

where $\delta = \omega_o - \omega_2$. We see that for instability we need $\delta < 0$, i.e., $\omega_o < \omega_2$ (which means $\epsilon(\omega_2) = (1 - \omega_2^2/\omega_o^2) < 0$). The minimum threshold is obtained at $\delta = -\gamma_2$ (i.e., the mismatch is equal to the damping rate of the lower hybrid wave excited at almost the pump frequency), and is given by

$$\frac{U}{C_s} = \frac{2 \omega_o}{\omega_{pi}} \left[\frac{2 \gamma_2}{\omega_o} \left(1 + \frac{\omega_{pe}^2}{\Omega_e^2} \right) \right]^{1/2} \quad (22)$$

Equation (22) predicts a threshold which is a factor of $(\omega_1/2\gamma_1)^{1/2}$ higher than that of the decay instability. Note that the purely growing mode should occur in almost every pump frequency regime since $\omega/k_{\parallel} v_{te} \rightarrow 0$. In fact, since we required initially that $1 < |\omega_1 + i\gamma_1|/k_{\parallel} v_{te}$ (otherwise the fluid model Equation (15) is not valid) we assume $1 < |\gamma_1/k_{\parallel} v_{te}|$, where γ_1 is the ion-neutral or ion-electron collision frequency. We see from Eq. (22) the well known result from the decay of Langmuir waves that the threshold of this instability becomes independent of γ_1 (at least within the limitations of a relaxation-type collisional damping model). Since the wave-number space of the purely growing mode

is more restricted than that of the decay instability, it is not expected to be as important from the point of view of plasma heating as the resonant decay instability.

C. Decay Into Kinetic (Dissipative) Quasi-Modes.

If $\zeta \approx \pi^{1/2} |\omega_0 - \omega_2| / k_{\parallel} v_{te} \approx 1$, then the condition for resonant decay is not satisfied (i.e., $\varepsilon(\omega_0 - \omega_2) \neq 0$). In particular, at the beat wave we obtain strong Landau damping (Nonlinear Landau damping).^{4,6} As we saw previously from Eqs. 4, 5, and 7, this is a rather general condition which is satisfied for a large range of frequencies and wave numbers. Although one needs a treatment based on kinetic equations, we can understand the fundamental aspects of this problem if we introduce resistivity in our previous model equation for the electron pressure. In particular, as is well known from drift wave theory,⁷ in the presence of resistivity the electron density and potential are phase-shifted by the relation

$$n_e/n_0 \approx (e\phi/T_e)(1+iR) \approx \frac{e\phi}{T_e(1-iR)} \quad 23(a)$$

where for $v/k_{\parallel} v_{te} \ll 1$, 23(b)

$$R \approx \sqrt{\pi} (\omega/k_{\parallel} v_{te}) \exp(-\omega^2/k_{\parallel}^2 v_{te}^2)$$

or $R \approx v\omega/k_{\parallel}^2 v_{te}^2$ for $1 < v/k_{\parallel} v_{te}$. Thus, we see that the pressure term (represented by T_e) is modified and for $R \approx 0(1)$ dissipative effects become very important. Thus, in the pressure balance equation we obtain

$$-\nabla_y P \frac{k_{\parallel}^2}{k^2} (1-iR) \approx \frac{en_e}{c} (\vec{V}_{ox} \times \vec{B}) \quad (24)$$

For $\omega_1^2 \ll \omega_s^2$, Eq. 15 yields for the imaginary part

$$\omega_1^2 n_i \approx ik_y \frac{k_R^2}{k_{\parallel}^2} \frac{m_e}{m_i} U n_e \omega_o \sin \omega_o t \quad (25)$$

Combining the Fourier transform of this equation with Eq. 16(b) (resonant lower hybrid wave at the sideband) we obtain for the threshold

$$\frac{U}{C_s} \approx \frac{2\omega_o}{\omega_{pi}} \left[\frac{2\gamma_2}{R\omega_2} \left(1 + \frac{\omega_{pe}^2}{\Omega_e^2} \right) \right]^{1/2} \quad (26)$$

Thus, we see that the threshold is inversely proportional to R , the resistivity (Landau damping) at the beat frequency. We also see that the growth rate is proportional to $\gamma \propto E_o^2$ (which can be obtained by replacing γ_2 with γ above threshold), namely

$$\frac{\gamma}{\omega_2} \approx \frac{RU^2}{8C_s^2} \frac{\omega_{LH}^2}{\omega_o^2} \quad (27)$$

This is in contrast to the resonant decay mode for which $\gamma \propto E_o$ in the absence of losses.

The growth rate for decay into nonresonant ion-quasi modes has been obtained recently from the Vlasov Equation, and is given by⁴

$$\frac{\gamma}{\omega_2} = \frac{U^2}{8C_s^2} \frac{\omega_{LH}^2}{\omega_o^2} \frac{\sqrt{\pi}\omega}{k_{\parallel} v_{te}} \exp(-\omega^2/k_{\parallel}^2 v_{te}^2) \left| \frac{\chi_e(\omega)}{\epsilon(\omega)} - 1 \right|^2 \quad (28)$$

where $\omega = \omega_o - \omega_2$, χ_e is the electron susceptibility, and $\epsilon(\omega) = 1 + \chi_e(\omega) + \chi_i(\omega)$ is the dielectric constant at the beat frequency. We see that Eq. (28) reduces to Eq. (27) for $|\chi_{eR}| \ll |\chi_{iR}|$. Equation 28 is the classic result for nonlinear wave-wave particle

scattering (nonlinear Landau damping). The first term corresponds to the shielding term, and the second term to the four-wave scattering term (Compton scattering). In Fig. 2 these processes are shown by diagrams.

Fig. 2. Diagrammatic representation of nonlinear wave-particle scattering.

Thus, we see that the assumption $|\chi_e(\omega)/\epsilon(\omega)| \ll 1$ corresponds to reducing the shielding potentials by the strong ion response at the beat frequency. This is especially evident near the ion cyclotron frequency $\omega \approx \Omega_i$, where for $k_{\parallel}^2 \ll k_{\perp}^2$, $\chi_i \approx \omega_{pi}^2 / (\Omega_i^2 - \omega^2)$. Thus, we see that the main contribution to the growth rate comes from Compton scattering on electrons. We expect that this process will be dominant for rf pumping near and above the lower-hybrid frequency for wavelengths such that Eqs. (4)-(7) are satisfied for $\zeta \approx 1$.

D. Fluid-Quasi Modes.

Another nonresonant process which is also possible at high pump powers involves nonresonant fluid modes such that $1 < (\omega/k_{\parallel} v_{te})$. The sideband is still a resonant lower-hybrid (electron plasma) mode. In this case (i.e., $3 \lesssim \omega/k_{\parallel} v_{te}$) the whole distribution function participates, and at long wavelengths

and large pump powers it can be a rather strong hydrodynamic instability. The growth rates have been obtained recently, and the result is⁶

$$\gamma \approx \frac{\sqrt{3}}{4} \left(\frac{\omega_c m_i}{2m_e} \right)^{1/3} \left(\frac{kU \cos \theta}{1 + \frac{m_i}{m_e} \frac{k_u^2}{k^2}} \right)^{2/3} \quad (29)$$

so that $\gamma \propto E_0^{2/3}$.

III. PARAMETRIC DECAY INSTABILITIES IN INHOMOGENEOUS PLASMA

A. Resonant Decay Instability.

(i) Uniform Pump, Inhomogeneous Plasma. Without a B-field, density gradients, finite length, and nonuniform (localized) pump have been discussed in a number of recent papers.⁸⁻¹⁶ We shall apply the results of these theories to lower-hybrid decay. Although in the present case we are dealing with plasma in a magnetic field, the foregoing theoretical results can be applied by generalizing the dispersion relation and the wave-vector to three dimensions. In particular, the important effect is that while in a uniform plasma we could take $k_z = k_y \frac{E_{0x}}{E_0}$, in a nonuniform plasma a component of the wavevector k_x develops due to the changing density (and hence dispersion relation) in the x direction. For example, in the present case if we assume a linear variation of density, the lower-hybrid dispersion relation becomes

$$\omega_2^2 = \frac{\omega_{pi}^2(x) [1 + (k_x^2/k^2) (m_i/m_e)]}{1 + \omega_{pe}^2(x)/\Omega_e^2} = \omega_{LH}^2(x) \left((1 + (k_x^2/k^2) (m_i/m_e)) \right) \quad (30)$$

where $\omega_p^2(x) \approx \omega_{po}^2(1+x/L_n)$, and where L_n is the local density gradient scale length, and $x=0$ is the position where the electron plasma wave and ion acoustic (cyclotron) wave match wavenumbers. Thus, we see that for $x \neq 0$ we have a finite value of k_x developing, i.e., $k^2 = k_{\parallel}^2 + k_y^2 + k_x^2$, and for $x/L < 1$ from Eq. (30) we obtain

$$(\omega_2^2 - \omega_{LH}^2) k_x^2 \approx \frac{x \omega_{LH}^2}{(1 + \omega_{pe}^2 / \Omega_e^2) L_n} (k_x^2 + k_y^2 + k_{\parallel}^2 (1 + m_i/m_e)) \quad (31)$$

where we assumed $k_x \ll k_y$ (for $k_x > k_y$ the coupling is too weak), and where we suppressed the subscript 0 on all frequencies which are to be evaluated at $x=0$. Thus, from Eq. (31) we see that¹⁵ the minimum value of k_x is given by setting $2\pi/x \approx k_x$, so that

$$k_{x_{\min}} \approx \left(\frac{2\pi k^2}{L} \right)^{1/3} \frac{\omega_2^{2/3}}{\omega_{pi}^{2/3}} \left(\frac{k^2}{k_{\parallel}^2} \frac{m_e}{m_i} \right)^{1/3} = \left(\frac{2\pi}{L} \right)^{1/3} k^{2/3} \frac{(1 + 1/\eta)^{1/3}}{(1 + \omega_{pe}^2 / \Omega_e^2)^{1/3}} \quad (32)$$

where $\eta = (k_{\parallel}^2 m_i / k^2 m_e)$. We note that the lower-hybrid wave is a backward wave in the x-direction (i.e., the group velocity $\partial \omega_2 / \partial k_x$, and the phase velocity ω_2 / k_x are in the opposite direction). On the other hand, the ion acoustic and long wavelength ($k_{\parallel}^2 r_{ci}^2 < 1$) ion cyclotron waves are forward waves, while short wavelength ion cyclotron waves ($k_{\parallel}^2 r_{ci}^2 > 1.6$) are backward waves. Thus, the selection rules Eq.(3) dictate that in decay into forward type ion waves and lower-hybrid waves the group velocities are in the same direction, i.e., this is a forward-scattering type decay process.⁹ In such a case no feed-back of energy is possible from one region of the system to another, and hence temporal instability is not possible, i.e., only spatial growth

can occur. However, in hot ion plasmas decay into short wavelength ion cyclotron waves, which are backward waves for $k_{\perp} r_{ci} \gg 1.3$, may result in temporal instabilities since $\vec{v}_1 \cdot \vec{v}_2 < 0$.¹⁶

In general, the growth rate can be obtained following Rosenbluth.¹² Assuming a WKB type phase variation, and that

$$\kappa(x) = \Delta k = k_0 - k_1 - k_2, \quad (33)$$

the equations describing the spatial variation of the two coupled modes can be written in the following form:

$$\frac{dE_1}{dx} + \frac{v_1}{v_1} E_1 = \frac{\gamma_0 E_2^*}{v_1} \exp(i \int_0^x \kappa(x) dx) \quad (34(a))$$

$$\frac{dE_2^*}{dx} + \frac{v_2}{v_2} E_2^* = \frac{\gamma_0 E_1}{v_2} \exp(-i \int_0^x \kappa(x) dx) \quad (34(b))$$

where $v_{1,2}$ are the linear damping rates (as before), $v_{1,2} = (\partial \omega / \partial k_x)_{1,2}$ are the components of group velocity in the x direction, and γ_0 is the temporal growth rate in a uniform plasma (obtained in the previous section). Assuming a linear variation of the mismatch $\kappa(x) \approx \kappa' x$, Eqs. 34(a) and 34(b) can be combined into the following equation:

$$\frac{d^2 \psi}{dx^2} - \left[\frac{\zeta^2}{4} + \frac{1}{2} \frac{d\zeta}{dx} + \frac{\gamma_0^2}{v_1 v_2} \right] \psi = 0 \quad (35)$$

where $\zeta = v_2/v_2 - v_1/v_1 - i\kappa(x)$, and where

$$E_1 = \exp \left\{ -\frac{1}{2} \left(\frac{v_1}{v_1} + \frac{v_2}{v_2} \right) x + \frac{i}{2} \int_0^x \kappa(x) dx \right\} \psi \quad (36)$$

Integrating Eq. (35) by WKB techniques, the total spatial amplification of intensity obtained between the turning points

$$x_t = 2\gamma_0 / (\kappa' \sqrt{v_1 v_2})$$

is given by

$$I = I_0 \exp[2\pi\gamma_0^2 / (|\kappa'| v_{1x} v_{2x})] \quad (37)$$

(where we ignored linear damping).

The effective threshold defined as the pump power for which a factor of 2π e-folding of the initial background noise results is given by

$$1 \leq \frac{\gamma_0^2}{|\kappa'| v_{1x} v_{2x}} \quad (38)$$

Thus, in an inhomogeneous plasma the threshold is determined not by collision damping, but by an effective mismatch of wave-vectors due to density gradients. We note that temperature gradients and magnetic field gradients were assumed to be smaller than density gradients, although the present results can be readily modified to include such effects.

(a) Decay Into Ion Acoustic Waves.

Let us now evaluate Eq. (38) for the lower hybrid and ion acoustic wave decay. Noting that $\partial\kappa/\partial x \approx -(\partial k_2/\partial x)$, and hence $v_2(\partial\kappa/\partial x) = (\partial\omega_2/\partial x) = \omega_2/[2L_n(1+\omega_{pe}^2/\Omega_e^2)]$, and $v_{1x} = (k_x/k^2)\omega_1$ for ion acoustic waves, we obtain for the threshold,

$$1 < \frac{\gamma_0^2}{|\kappa'| v_{1x} v_{2x}} = \frac{U^2}{8C_s^2} \frac{\omega_{pi}^2}{\omega_o^2} \frac{k^2 L_n}{k_x} \quad (39)$$

Thus, for weakly damped ion acoustic waves the minimum value of k_x is given by Eq. (32), so the threshold condition becomes

$$\frac{U^2}{C_s^2} \approx \frac{8(2\pi)^{1/3}}{(L_n k)^{4/3}} \frac{(1+\eta)^{4/3}}{(1+\omega_{pe}^2/\Omega_e^2)^{4/3} \eta^{1/3}} \quad 40(a)$$

where $\eta = (k_{\parallel}^2/k^2)(m_i/m_e)$. On the other hand, for $T_i \approx T_e$ the ion waves become strongly damped, k_x becomes imaginary,^{11,15} and $|k_x| \approx k \sqrt{\gamma_1/\omega_1}$ so the threshold becomes

$$\frac{U^2}{C_s^2} \approx \frac{8\omega_o^2}{\omega_{pi}^2} \left(\frac{\gamma_1}{\omega_1}\right)^{1/2} \frac{1}{kL_n} \quad 40(b)$$

In general, the threshold to be used is whichever gives the higher threshold among Eqs. (40) and (41). Note that the minimum value of threshold corresponds to the shortest wavelength (i.e., maximum k), and the limit is determined by Landau damping at the sideband (lower-hybrid wave). Thus, setting $3\zeta(\omega_2/k_{\parallel}v_{te})$ (where $v_{te} = \sqrt{2T_e/m_e}$) as the regime of weak Landau damping, we obtain from the dispersion relation

$$k_{\max}^2 \approx \frac{(1+\omega_{pe}^2/\Omega_e^2)(1+\eta)}{18\lambda_D^2 \eta} \quad (41)$$

so that Eq. (40) gives for the minimum threshold

$$\frac{U}{C_s} \approx \frac{8(1+\eta)^{1/3} \eta^{1/6}}{(1+\omega_{pe}^2/\Omega_e^2)} \left(\frac{\lambda_D}{L_n}\right)^{2/3} \quad (42)$$

For typical small scale laboratory plasmas for $\omega_o \approx 3\omega_{LH}$, $\lambda_D \approx 4 \times 10^{-3}$ cm, $L_n \approx 1$ cm, so that $U/C_s \approx 1/4$, which is comparable or smaller than collisional damping, so collision damping determines the threshold. On the other hand, let us

estimate the threshold for a typical tokamak plasma (where collisional damping is negligible). Taking $n_{e0} \approx 5 \times 10^{13} \text{ cm}^{-3}$, $B = 40 \text{ kG}$, $f_0 \approx 900 \text{ MHz}$, $T_e \approx 1.5 \text{ kV}$, $T_i \approx 0.5 \text{ kV}$, $L_n \approx 10 \text{ cm}$, for $\omega_0/\omega_{LH} \approx 3$ we obtain $(U/C_s)_{th} \approx 0.5$, which for the present parameters corresponds to $E_x \approx 5 \text{ kV/cm}$ electric field. From the theory of propagation of incident electron plasma wave we obtain a net amplification¹⁷

$$E_x \approx E_{z0} \left(\frac{m_i}{m_e} \right)^{1/4} \frac{(1+\eta)^{1/2}}{(1+\omega_{pe}^2/\Omega_e^2)^{1/2} \eta^{3/2}} \quad (43)$$

which for the present case gives $E_x/E_{z0} \approx 4.6$. Thus, if we take $E_{z0} \approx 2.5 \text{ kV/cm}$ (a typical value quoted in literature to prevent microwave breakdown), we obtain $E_x \approx 11.5 \text{ kV/cm}$, well above threshold. Had we considered a temperature of $T_e \approx 5 \text{ kV}$, we would get for the threshold $E_x \approx 12 \text{ kV/cm}$, which is almost the available electric field. Note, however, that the threshold is reduced with increasing density gradient scale length as $L_n^{4/3}$ (or as L_n , according Eq. 40(b)). Combining Eq. (42) and (43) gives for $\eta > 1$

$$U^2/C_s^2 \Big|_{E_{z0}} \propto \left(\frac{\lambda_D}{L_n} \right)^{4/3} \frac{\eta^3}{(1+\omega_{pe}^2/\Omega_e^2)} \quad (44)$$

Thus, as the wave further penetrates the plasma η is decreasing so that for $\omega_{pe}^2/\Omega_e^2 \lesssim 1$, the threshold is decreasing toward the center of the plasma column. This scaling is helpful to reduce unwanted instability near the plasma surface.

(b) Decay Into Ion Cyclotron Waves.

Let us now consider decay into lower-hybrid waves and ion cyclotron waves ($\omega_1/k_{\perp} v_{te} \ll 1$). The real part of the dielectric

function for ion cyclotron waves is given by

$$\epsilon(\omega) = 1 + \frac{1}{k^2 \lambda_{De}^2} - \frac{1}{k^2 \lambda_{De}^2} \left(\frac{T_e}{T_i} \right) \frac{\Omega_i^2}{\omega^2 - \Omega_i^2} \left[2I_1(b_i) \exp(-b_i) \right] \quad 45(a)$$

where I_1 is the modified Bessel function, and where $b_i = k_{\perp}^2 r_{ci}^2$, $r_{ci} = \sqrt{T_i/m_i}/\Omega_i$. Equation (45) predicts ion cyclotron waves ($\epsilon(\omega_1) = 0$) for $k^2 \lambda_{De}^2 \ll 1$:

$$\omega_1^2 = \Omega_i^2 (1 + (T_e/T_i) 2I_1(b_i) \exp(-b_i)) \quad 45(b)$$

The limiting forms of this dispersion relation are:

For $b_i < 1$, $2I_1(b_i) \exp(-b_i) \approx b_i$, so

$$\omega_1^2 \approx \Omega_i^2 + k_{\perp}^2 C_s^2; \quad (C_s^2 = T_e/M_i) \quad 45(c)$$

For $b_i \gg 1$, $2I_1(b_i) \exp(-b_i) \approx \sqrt{2/\pi} / b_i$, so

$$\omega_1^2 = \Omega_i^2 \left(1 + \frac{T_e}{T_i} \sqrt{\frac{2}{\pi}} \frac{1}{k_{\perp} r_{ci}} \right) \quad 45(d)$$

Thus, the long wavelength limit corresponds to forward waves, and the short wavelength limit corresponds to backward waves. The parametric growth rate in a uniform plasma can be obtained from the general theory of parametric instabilities,^{3,4} which for $\gamma_2 < \omega_1$, states

$$\epsilon(\omega_1) \epsilon(\omega_2) = - \frac{\mu^2}{4} \chi_i(\omega_1) (1 + \chi_e(\omega_1)) \quad 46(a)$$

where for $\omega_0 \ll (\omega_{ce}, \omega_{pe})$, $\mu^2 = (U^2/C_s^2) (\omega_{pi}^2/\omega_0^2) (k^2 \lambda_{De}^2)$, $\chi(\omega)$ is the susceptibility, $\epsilon(\omega_1)$ is defined by Eq. 45(a), and $\epsilon(\omega_2) = 1 - \omega_2^2/\omega_0^2$,

where ω_2 is defined by Eq. (1). Thus, the growth rate is

$$\gamma_o^2 = - \frac{\mu^2}{4} \frac{\chi_i(\omega_1)(1+\chi_e(\omega_1))}{\frac{\partial \epsilon(\omega_1)}{\partial \omega_1} \frac{\partial \epsilon(\omega_2)}{\partial \omega_2}} \quad (46(b))$$

which for ion cyclotron waves and lower-hybrid waves reduces to

$$\gamma_o^2 = \frac{U^2}{16C_s^2} \frac{\omega_{LH}^2}{\omega_o^2} \omega_1 \omega_2 \left(\frac{T_e}{T_i}\right) 2I_1(b_i) \exp(-b_i) \quad (47)$$

where we used $\partial \epsilon(\omega_2)/\partial \omega_2 = (2/\omega_2)(1+\omega_{pe}^2/\Omega_e^2)$. If the ion cyclotron waves are damped by electron Landau damping, then it is easy to show that Eq. (47) predicts the same threshold as for ion acoustic waves.

Let us now consider decay in an inhomogeneous plasma, so that the threshold is given by Eq. (38). Combining Eqs. (38) and (46) we obtain for threshold

$$\left| \frac{\gamma_o^2}{\kappa' \frac{\partial \omega_1}{\partial k_{1x}} \frac{\partial \omega_2}{\partial k_{2x}}} \right| = \frac{\mu^2}{4} \frac{\chi_e^2(\omega_1)}{\left| \kappa' \frac{\partial \epsilon(\omega_1)}{\partial k_{1x}} \frac{\partial \epsilon(\omega_2)}{\partial k_{2x}} \right|} > 1 \quad (48)$$

where we used $\chi_i(\omega_1) \approx 1 - \chi_e(\omega_1) \approx \chi_e(\omega_1)$ for resonant decay, and $k^2 \lambda_D^2 \ll 1$. If there are only density gradients present, $|\kappa' \partial \epsilon(\omega_2)/\partial k_{2x}| \approx |\partial \epsilon(\omega_2)/\partial x| = 1/L_n$, $L_n = (\nabla n_o/n_o)^{-1}$, and so Eq. (48) gives for the threshold

$$\frac{U^2}{8C_s^2} \frac{\omega_{pi}^2 L_n (\omega_1^2 - \Omega_i^2)}{\omega_o^2 \omega_1 (\partial \omega_1 / \partial k_x)} > 1. \quad (49)$$

We see that since for long wavelength ion cyclotron waves

$\omega_1^2 - \Omega_i^2 \approx k_{\perp}^2 C_s^2$, and for ion acoustic wave $\omega_1^2 - \Omega_i^2 \approx k^2 C_s^2 \gg \Omega_i^2$, $k^2 \approx k_{\perp}^2$,

and for both waves $\omega_1 (\partial \omega_1 / \partial k_x) = k_x C_s^2$, Eq. (49) becomes

$$\frac{U^2}{C_s^2} > \frac{8\omega_o^2}{\omega_{pi}^2} \frac{k_x}{k_{Ln}^2} \quad (50)$$

which is exactly the result obtained in Eq. (39). The reason for the agreement is that for a given frequency the group velocities of both ion acoustic waves and long wavelength ion cyclotron waves are the same.

Let us now consider decay into short wavelength ion cyclotron waves (backward waves). Since in this case $\vec{v}_1 \cdot \vec{v}_2 < 0$, the threshold $\gamma_o^2 / (k' v_{1x} v_{2x}) \approx 1$ predicts temporal instability (in addition to spatial growth).¹⁶ The group velocity in this case is

$$\frac{\partial \omega_1}{\partial k_{1x}} = \frac{\Omega_i^2}{\omega_1} \frac{T_e}{T_i} \left(\frac{1}{2\pi} \right)^2 \frac{k_x}{k_{1r_{ci}}^3} = (\omega_1^2 - \Omega_i^2) k_{1x} / (2\omega_1 k_{1i}^2) \quad , \quad (51)$$

so from Eq. (49) the threshold becomes

$$\frac{U^2}{C_s^2} > \frac{4\omega_o^2}{\omega_{pi}^2} \frac{k_x}{k_{Ln}^2} \quad (52)$$

which is a factor of two lower than that given by Eq. (50). Thus, we see that because of their smaller group velocity (by a factor of two) decay into short wavelength ion cyclotron waves is more likely than into long wavelength ion waves, and the instability above threshold (52) should become temporal (and convective below). Since the shortest wavelengths are the most unstable, let us examine when the selection rules can be satisfied for decay into backward ion cyclotron modes. Using the relation Eq. (41), we have for the most unstable wave

$$1.6 \lesssim k_{\perp}^2 r_{ci}^2 = \frac{r_{ci}^2 (1 + \omega_{pe}^2 / \Omega_e^2) (1 + \eta)}{18 \lambda_D^2 \eta}$$

or

$$1 \lesssim \frac{1}{30} \left(1 + \frac{\omega_{pe}^2}{\Omega_e^2} \right) \left(\frac{\omega_{pe}}{\Omega_e} \right)^2 \left(\frac{T_i}{T_e} \right) \left(\frac{m_i}{m_e} \right) \left(\frac{1 + \eta}{\eta} \right) \quad (53)$$

Thus, for a typical tokamak parameter of $3T_i \approx T_e$, deuterium gas, and $1 < \eta$, we can obtain decay for densities $0.1 \lesssim (\omega_{pe} / \Omega_e)$. On the other hand, for $T_i / T_e \approx 1$, decay is obtained for even lower densities. Of course, for very short wavelengths even weak ion-ion collisions would provide strong damping due to diffusion, so we should restrict ourselves to $1.3 \lesssim k_{\perp} r_i \lesssim 10$. In any case, these calculations show that instability will occur at electric fields a factor of $1/\sqrt{2}$ lower than those obtained in Section (a).

(ii) Resonant Decay in Pump Field with Finite Extent.

The treatment presented in the foregoing section is valid only if the rf pump fields extend to regions larger than the e-folding scale length of the instability. While this is a good approximation for the whistler wave (which is coupled from an E_0 field at the surface with roughly the same amplification factor as given by Eq. (43)¹⁸) for the slow electrostatic wave the pump fields are localized¹⁷ to finite regions with a width

$\ell_x \approx L_{oz} \cos \theta \approx (n \lambda_{\perp o}) \cos \theta$, where L_{oz} is the length of the generating coil, and $\cos \theta = \sqrt{\eta (m_e / m_i)}$. Some numerical results are shown below in Fig. (3) from the recent work of Bellan and Porkolab.¹⁷

Fig. 3. Plots of electric field generated by a 100 cm long slow wave structure in a "Tokamak-like" plasma. Density $n_0(r) = 10^{14} \times (1 + (r/5)^2)^{-1}$; $2\pi/k_z = 15$ cm; $f = 1$ GHz; $B = 40$ kG; deuterium gas. Hot plasma field (not shown) starts from the line CD and extends to the right.

Thus, for a nonuniform pump the amplification distance is reduced if $\ell_x < 2x_t$ (where x_t is the turning point defined earlier) and the total amplification obtained is essentially that of a uniform plasma.¹³ For a factor of 2π amplification in intensity we have

$$\frac{\gamma_0 \ell_x}{\pi \sqrt{v_1 v_2}} = 1 \quad , \quad 54(a)$$

and

$$\ell_x < \frac{4\gamma_0}{\kappa' \sqrt{v_1 v_2}} \quad . \quad 54(b)$$

Thus, combining Eqs. 54(a) and 54(b) give

$$\ell_x \leq \frac{4}{\sqrt{\pi k_x}} \quad 54(c)$$

as the conditions for using the finite length instability criterion Eq. 54(a) for threshold. For the present case we obtain the following threshold for forward ion sound (cyclotron) waves:

$$\frac{U^2}{C_s^2} = \frac{16\pi^2}{\ell_x^2} \frac{k_x^2}{k^4} \frac{k_{\parallel}^2}{k^2} \frac{m_i}{m_e} \quad (55)$$

Here k_x may be determined either by the mismatch, given earlier by Eq. (32), or by $k_x \approx 2\pi/\ell_x$, whichever is larger for the particular experimental condition (but is limited by $k_x \leq k_y$, otherwise the coupling is lost). The minimum threshold is determined by $k_{\max}$ as given by Eq. (41). We note that for minimum threshold we want sufficiently long generating coil or waveguide array, whose length should exceed condition 54(c) or

$$L_{OZ} \sqrt{\eta} \sqrt{\frac{m_e}{m_i}} \gtrsim \frac{2\omega_2^3 k^4}{\sqrt{\pi} L_n (1 + \frac{\omega_{pe}^2}{\Omega_e^2}) \omega_L^2 k_x k_{\parallel}^2} \quad 56(a)$$

or using Eq. (32) for k_x , we obtain

$$L_{OZ} \sqrt{\eta} \gtrsim 3.7 \frac{k_{\parallel}}{k} \left(\frac{m_i}{m_e}\right) \frac{\omega_{LH}}{\omega_2} \frac{L_n^{1/3}}{k^{2/3}} \left(1 + \frac{1}{\eta}\right)^{1/6} \quad 56(b)$$

which for $\eta > 1$ reduces to

$$L_{OZ} \gtrsim 3.7 \left(\frac{L_n}{\lambda}\right)^{1/3} \lambda_{\parallel} \quad 56(c)$$

Similar results are obtained for the backward cyclotron wave

(within factors of order $\sqrt{2}$). We note that the finite length requirement increases the threshold as $\lambda^{-1/3}$ as the wave is moving forward the plasma center. Using our previous tokamak parameters, we find that for $\eta \approx 8$ (i.e., $\omega_o \approx 3\omega_{LH}$) Eq. 56(c) predicts $L_{oz}/L_n > 2$, so for an index of refraction $N_z \approx 2$, $f_o \approx 1$ GHz, $\lambda_{o||} \approx 15$ cm, $L_n \approx 10$ cm we have $L_{oz}/\lambda_{o||} = n \approx 2$. Thus, for a waveguide array 2 wavelengths or longer along the magnetic field we use density inhomogeneity as a threshold, and for less than two wavelength Eq. (55) gives the threshold. Again, for decay into backward ion cyclotron waves the threshold power for temporal instability is reduced by a factor of two from that given by Eq. (55).

B. Decay Into Ion Quasi-Modes.

(i) Uniform pump, Inhomogeneous Plasma. Let us now consider decay into resistive (kinetic) ion quasi-modes. The inhomogeneous threshold for such modes has been discussed recently by Galeev and Sagdeev.¹⁴ They have pointed out that in this process due to the large width of ω_R (which is not well defined by linear dispersion) we may assume that new values of k are matched at different positions, and the region of instability is defined by $\gamma(\delta)$, where $\delta = (\omega_o - \omega_1 - \omega_2)$ is the mismatch in frequency. Thus, the total amplification obtained is given by the zeroes of γ as determined by δ . Thus, using $dt = dx/v_{2x}$ for the time of traversal of the unstable region, we have

$$I = I_o \exp \left(2 \int_0^x \frac{\gamma(\delta) dx}{v_{2x}} \right) \quad (57)$$

where $|v_{2x}| = |\partial\omega_2/\partial k_x| = (\omega_{LH}^2/\omega_2) \eta (k_x/k^2)$. In our case it can be shown that for the ion quasi-mode

$$\gamma(\delta) \approx \frac{\omega_2^2 U^2}{8C_s^2 \omega_0^2} \frac{\pi}{\omega_0^2} R \left(\frac{\delta}{\omega_R} + 1 \right)$$

where $R = \sqrt{\pi} (\omega_R / k_{\perp} v_{te}) \exp(-\omega_R^2 / k_{\perp}^2 v_{te}^2)$. Now examining the numerical results in Reference (4) we find that for short wavelengths $(\delta / \omega_R) \ll 1$ for $\eta \ll 1$ for a large range of wavenumbers. Hence, in this case the condition Eq. (57) reduces to

$$I = I_0 \exp(2 \gamma_0 \int_0^{x_0} \frac{dx}{v_{2x}})$$

where x_0 is given by the mismatch wavenumber and distance $(x = x_0)$ defined by Eq. (31), and where we take $k_x \approx k_y$ as the cut-off value. Thus, the total amplification is $A \approx 2x_0 \gamma_0 / v_{2x}$ and so, for a total amplification $A = 2\pi$ (defining threshold) we obtain

$$\frac{U^2}{C_s^2} = \frac{4\pi(1+\omega_{pe}^2/\Omega_e^2)}{R L_n k_x} \left(\frac{\omega_0^2}{\omega_2^2} \right) \quad (58)$$

where approximately $\omega_0 \approx \omega_2$, and $k_x \approx k_y$ for minimum threshold. If we take the cut-off value of $k_{\max}$ determined by Eq. (41), we obtain for the minimum threshold

$$\frac{U^2}{C_s^2} \approx \frac{12\pi\sqrt{2}(1+\omega_{pe}^2/\Omega_e^2)^{1/2} \eta^{1/2} \lambda_D}{R(1+\eta)^{1/2} L_n} \quad (59)$$

Using previously given tokamak parameters, $\eta \approx 1$, $R \approx 0.7$, $L_n \approx 10$ cm, $T_e \approx 1.5$ kV, $n_0 \approx 10^{14}$ cm⁻³, $B \approx 40$ kG, we get for the threshold field $U/C_s \approx 0.15$, or $E_{ox} \approx 1.5$ kV/cm which should be compared with resonant decay for which we obtained $U/C_s \approx 0.5$ (or 0.35 for backward waves). In addition, according

to Eq. (43) the incident pump wave should further amplify to $E_x \approx 8E_{z0} \approx 20$ kV/cm if $E_{z0} \approx 2.5$ kV/cm. Thus, we see that this decay process is expected to occur at lower thresholds. For a $T_e \approx 5$ kV plasma we would get a factor of $(T_e/T_{eo})^{3/4}$ higher threshold field, or $E \approx 4$ kV/cm, which is still well below the available electric field. For the small scale laboratory plasma parameters discussed previously Eq. (59) predicts $U/C_s \approx 0.5$, which is comparable to the collisional threshold. Although these estimates are reasonable for the whistler mode of propagation, for the finite extent slow wave propagation we have further restrictions as shown below.

(ii) Quasi Modes in Pump Fields with Finite Extent.

Let us now consider decay into quasi-modes in the localized slow wave field shown in Fig. 3. In this case if $\ell_x < x_0$ (where $x = x_0$ is defined by Eq. (31)), we may replace x_0 with the width of the pump region, ℓ_x . This is immediately apparent if we treat the pump (and hence the growth rate) as $\gamma_0 \exp(-4x^2/\ell_x^2)$, in which case the growth rate integral becomes

$$2 \int_0^\infty \gamma_0(x) \frac{dx}{v_{2x}} = \frac{\gamma_0 \ell_x \sqrt{\pi}}{2v_{2x}} \approx \frac{\gamma_0 \ell_x}{v_{2x}} \quad (60)$$

Thus, if $\ell_x < x_0$, i.e.,

$$\frac{L_{z0} (m_e/m_i)^{1/2}}{(1 + \omega_{pe}^2/\Omega_e^2)} \left(\frac{k^2}{k_x^2} \right) < L_n \quad (61)$$

the threshold for 2π amplification becomes

$$\frac{U^2}{C_s^2} > \frac{16\sqrt{\pi} \eta k_x}{R k^2 \ell_x} \quad (62)$$

where k_x may be determined by density gradient mismatch given by Eq. (32). There is an additional limit on k_x , and is given by $k_x \gtrsim 2\pi/\ell_x$. So, we have to use whichever value of k_x is larger (but $k_x' < k_y$, or the coupling is destroyed). Thus, using Eq. (32), Eq. (61) predicts for the $T_e \approx 1.5$ kV tokamak plasma with $n \approx 1$, $L_{Oz} \approx 30$ cm (2 wavelengths), $L_n \approx 10$ cm, $E_{th} \approx 3$ kV/cm, whereas for $T_e \approx 5$ kV, $E_{th} \approx 11$ kV/cm. While these are higher values than that obtained for whistler-type propagation in the previous section, they are both less than the calculated local pump field of 20 kV/cm (corresponding to $E_{Oz} \approx 2.5$ kV/cm at the surface).

CONCLUSIONS AND SUMMARY

We have considered the parametric decay near and above the lower-hybrid frequency. The growth rates and thresholds in a uniform plasma have been obtained from physically transparent model equations. Then the results of recent theories concerning inhomogeneous plasmas were applied to obtain thresholds associated with density gradients and spatially finite pump wave fields. It was found that in inhomogeneous plasma decay into short wavelength backward ion-cyclotron modes ($k_{\perp} r_{ci} \gtrsim 1.3$) occurs at a threshold power which is a factor of two below that of decay into long wavelength forward ion cyclotron waves ($k_{\perp} r_{ci} < 1$) or ion acoustic waves. Decay into ion-quasi modes occurs at a threshold comparable to, but farther inside the plasma, than the backward wave decay instability. The lower threshold for decay into short wavelength waves, and predominant decay toward the plasma center should be advantageous for plasma heating, and should reduce undesirable

plasma loss. We conclude that fusion plasmas with electron temperatures up to 5 kV, and density gradient scale lengths of the order of 10 cm or more should be unstable to both the resonant decay instability and the ion quasi-modes for applied pump fields of the order of 2.5 kV/cm. Thus, if coupling to drift waves¹⁹ (discussed by Kaw et al) does not produce unacceptable levels of plasma loss, parametric heating appears to be an important mechanism of energy deposition in lower-hybrid heating schemes (in addition to linear mode conversion). Extrapolation to higher temperature plasmas will depend on experimental developments concerning exact density profiles and breakdown electric fields near the plasma surface. We should remark that toroidal effects have not been studied here, and they may modify these conclusions. Also, decay of the pump wave at the linear mode conversion layer has not been studied here.

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REFERENCES

- ¹T. H. Stix, The Theory of Plasma Waves, (McGraw Hill, New York, 1962) Chap. 2. Also, Phys. Rev. Lett 15, 878 (1965).
- ²A. W. Trivelpiece and R. W. Gould, Journal of App. Phys. 30, 1784 (1959).
- ³M. Porkolab, Nuclear Fusion 12, 329 (1972); also Symposium on Plasma Heating and Injection, Varenna, Italy, 1972, (CNR Lab. Di Fisica Del Plasma E Di Elec. Quant., Milano, Italy, p.46 (1973)). J. M. Kindel et al, Phys. Rev. Letters 29, 995, (1972).
- ⁴M. Porkolab, Princeton MATT Report 1028(1974). (To be published in the Physics of Fluids, July, 1974).
- ⁵K. Nishikawa, J. Phys. Soc. Japan, 24, 916 (1968).
- ⁶D. F. Dubois and M. V. Goldman, Phys. Rev. Lett. 14, 544 (1965).
- ⁷B. B. Kadomtsev, Plasma Turbulence, (Academic Press, New York, 1965) Chap. 4. M. Porkolab, Phys. Fluids 11, 834 (1968).
- ⁸N. M. Kroll, J. Appl. Phys. 36, 34 (1965).
- ⁹M. Porkolab and R. P. H. Chang, Phys. Fluids 13, 2054(1970).
- ¹⁰K. J. Harker and F. W. Crawford, J. Geophys. Res. 75, 5459 (1970).
- ¹¹F. W. Perkins and J. Flick, Phys. Fluids 14, 2012 (1971).
- ¹²M. N. Rosenbluth, Phys. Rev. Lett. 29, 565 (1972).
- ¹³D. Pesme et al, Phys. Rev. Lett. 31, 203 (1973).
- ¹⁴A. A. Galeev and R. Z. Sagdeev, Nuclear Fusion 13, 603 (1973).
- ¹⁵C. S. Liu, Private Communication (1974).

¹⁶E. A. Williams, Private Communication. D. F. Dubois, D. W. Forslund, and E. A. Williams, "Parametric Instabilities in Finite Inhomogeneous Media", Los Alamos Scientific Laboratory Preprint LA-UR 74-806 (1973). We note, however, that at the present time there is still some controversy whether without finite boundaries temporal or convective instability results. (R. B. White, private communication).

¹⁷P. Bellan and M. Porkolab, Princeton MATT 1031 (1974), (to be published in Phys. Fluids, August, 1974).

¹⁸F. Troyon, Private Communication (1974).

¹⁹A. K. Sundaram and P. K. Kaw, Nuclear Fusion 13, 901 (1973).

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