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Theory and Structural Design Applications of Weibull Statistics

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Theory and Structural Design Applications of Weibull Statistics

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UNCLASSIFIED NERVA RESEARCH
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ABSTRACT

This report describes the basic theory of Weibull statistics and its application to structural design. Included is a description of the Weibull parameters and their effects on design considerations. Several methods of reducing fracture data to obtain the Weibull parameters are described and compared.

The size and stress distribution effects predicted by Weibull's theory are presented in detail along with working curves which may be used in design. Relationships between reliability and safety factor and methods of determining allowable material strengths are presented.

Several example problems illustrating the use of Weibull statistics in design considerations are also included.

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1. INTRODUCTION

In the testing of brittle materials for ultimate tensile strength, two experimental findings predominate which are not directly accounted for by the classical theory of strength: 1) Repetitive tests under presumably similar experimental conditions do not yield identical results, and 2) The mean strength as determined from a multiplicity of similar tests depends on the volume of material stressed, the shape of the test specimen, and the manner of loading.

If, for example, we consider the experimental strengths obtained from a series of specimens, we will find that there is a definite relationship between the probability that a specimen will fracture and the stress to which it is subjected. A plot of this relationship is often called the cumulative probability of fracture distribution and is shown in Figure 1-1 for a unit volume of material.

Figure 1-1 Typical Cumulative Probability of Fracture Distribution

2. BASIC THEORY

The basic hypothesis for all statistical theories is that materials contain flaws, and that these flaws prevent the material from experimentally achieving its potential strength. Dispersion of test results are due to the dispersion of flaws. Some statistical theories hypothesize the nature of the flaws and their distribution. Weibull's analysis employs the flaw concept only in the arbitrary formulation of an analytical expression known as the "risk of rupture".

If the proposition is allowed that a distribution function exists for the fracture strength of a material, a theory of strength can be developed. Let P be the probability that fracture will occur before the stress σ is reached for a specimen of unit volume. Then, the probability, S , that it will survive the stress σ is $(1-P)$. Similarly, the probability that the second unit element will survive the stress σ is also $(1-P)$. For the combination of the two elements to survive the stress, both must survive it. Since the probability that two events will both occur is the product of the probabilities that each will occur, it follows that

$$(1-P_2) = (1-P) (1-P) = (1-P)^2 \quad (2-1)$$

where P_2 is the probability of fracture of the double element. If the two elements are in parallel, and the applied load is doubled, the stress will still be σ and the probability of fracture of the double element will still be defined by equation (2-1). Thus, whether the elements are in series or in parallel, the same probability of fracture occurs for a double volume subjected to the stress σ . If a number of such elements, having a volume V , are subjected to the stress σ , then whether they are in series or parallel, the probability of fracture P_V is defined by

$$(1-P_V) = (1-P)^V \quad (2-2)$$

or taking logarithms

$$\ln (1-P_V) = V \ln (1-P) \quad (2-3)$$

Weibull defines the risk of rupture as $R \equiv -\ln (1-P_V)$, or from equation (2-3)

$$R = -V \ln (1-P) \quad (2-4)$$

Combining equations (2-3) and (2-4) gives the probability of fracture as

$$P_V = 1 - e^{-R} \quad (2-5)$$

Consider now a body of arbitrary shape and stress distribution. Each small element of volume has a definite risk of rupture which can be expressed as

$$dR = -\ln (1-P) dV \quad (2-6)$$

Note that dR is proportional to the volume and the quantity $(1-P)$, which for randomly distributed flaws, is a function of the tensile stress level alone. If we let $f(\sigma) = -\ln (1-P)$ and integrate equation (2-6) over the entire volume of the body, the risk of rupture for the body is given by

$$R = \int_V f(\sigma) dV \quad (2-7)$$

Including this definition of R in the expression for the probability of fracture, equation (2-5) gives

$$P_V = 1 - e^{-\int_V f(\sigma) dV} \quad (2-8)$$

The mean fracture strength, σ_m , is defined as the first moment of the distribution about $\sigma = 0$. Similarly, the variance, σ_m^2 , defined as the square of the standard deviation, is equivalent to the second moment of the distribution about the mean. After some statistical manipulations [1]* we find that the mean fracture strength may be calculated from

$$\sigma_m = \int_0^{\infty} \sigma e^{-R} d\sigma \quad (2-9)$$

*The numbers in brackets refer to the references found in Section 12.

and the standard deviation from

$$\sigma^2 = \int_0^{\infty} e^{-R} d(\sigma^2) - \sigma_m^2 \quad (2-10)$$

where, again,

$$R = \int_V f(\sigma) dV$$

These equations express the fundamental law of an isotropic brittle material, and the material function, $f(\sigma)$, expresses the state of stress. In anisotropic materials, the stress-coordinate system and direction must be taken into account. In the foregoing derivations, the assumption has been made that the risk of rupture is independent of stress direction, that is, σ is a one-dimensional tensile stress. If there are arbitrarily distributed forces, then σ must be replaced by another scalar which is determined by the principal stresses.

3. WEIBULL DISTRIBUTION FUNCTION

At this point, the theory is perfectly general. It is in the definition of $f(\sigma)$ that the major problems arise. From the previous section the risk of rupture of a given structure subjected to a particular state of stress may be expressed as

$$R = \int_V f(\sigma) dV$$

where $f(\sigma)$ is a function of the state of stress alone. Theoretically correct closed form expressions for R are not available at this time. Weibull^[2] proposed using a simple empirical relation for $f(\sigma)$, the use of which he justified by comparison with experimental data.

The only necessary general condition that the function has to satisfy is that it be positive and non-decreasing. The original function used by Weibull was of two-degrees-of-freedom

$$f(\sigma) = \left(\frac{\sigma}{\sigma_0}\right)^m \quad (3-1)$$

so that the risk of rupture (reduced form) becomes

$$R = \int_V \left(\frac{\sigma}{\sigma_0}\right)^m dV \quad (3-2)$$

The probability of fracture, from equation (2-8) then becomes

$$P = 1 - e^{-R} = 1 - \exp \left[- \int_V \left(\frac{\sigma}{\sigma_0}\right)^m dV \right] \quad (3-3)$$

This distribution function is simple enough so that it may be handled mathematically and has been shown empirically to fit many situations well.

After considerable experience with the above distribution function, Weibull came to the conclusion that agreement with many distributions would be better if a lower bound were added. This new form was

$$f(\sigma) = \left(\frac{\sigma - \sigma_u}{\sigma_0}\right)^m \quad (3-4)$$

where σ_u is a stress level below which no failures are to be expected. The three Weibull distribution parameters, m , σ_u , and σ_o are discussed more fully in Section 4.

Combining the Weibull distribution given in equation (3-4) with equation (2-7) gives the general form of the risk of rupture as

$$R = \int_V \left(\frac{\sigma - \sigma_u}{\sigma_o} \right)^m dV \quad (3-5)$$

The probability of fracture, from equation (2-8) can then be expressed as

$$P = 1 - e^{-R} = 1 - \exp \left[- \int_V \left(\frac{\sigma - \sigma_u}{\sigma_o} \right)^m dV \right], \quad \sigma \geq \sigma_u \quad (3-6)$$

$$P = 0, \quad \sigma < \sigma_u$$

In addition, the mean failure stress, from equation (2-9), expressed in terms of the Weibull parameters, becomes

$$\sigma_m = \sigma_u + \int_{\sigma_u}^{\infty} e^{-R} d\sigma \quad (3-7)$$

and the variance becomes

$$\sigma^2 = \int_{\sigma_u}^{\infty} e^{-R} d(\sigma^2) + \sigma_u^2 - \sigma_m^2 \quad (3-8)$$

4. WEIBULL DISTRIBUTION PARAMETERS

The three distribution parameters, m , σ_u , and σ_o are associated with the material and are independent of size and manner of loading. Figure 4-1 graphically shows the probability density curves for various values of the Weibull parameters.

4.1 THE MATERIAL CONSTANT

The value of the material constant "m" characterizes the nature, severity, and dispersion of flaws contained in the material. It is an index of homogeneity or relative number of flaws in the material and, consequently, is related to the amount of scatter observed in the test data. A low value of "m" indicates either a material in which the severity of the flaws is highly variable or a material in which the flaws are dispersed very nonuniformly. Low values of "m" are characteristic of brittle materials. A high value of "m" indicates a material with a uniform distribution of highly homogeneous flaws. A material with a high value of "m" exhibits a small scatter in fracture stresses and the distribution curve is relatively narrow as seen in Figure 4-1(a). Typical values of the material constant "m", for several materials, are given in Table 4-1.

TABLE 4-1
TYPICAL WEIBULL MATERIAL CONSTANTS

<u>Material</u>	<u>Material Constant, m</u>			
Glass Fibers	1.3			
Dense Silicon Carbide	4.3			
ATJ Graphite (Bare) [4], [5]	6.3			
Nickel-Bonded Titanium Carbide K151A	7			
Hydro-Stone Plaster [6]	15			
Steel at Temperature of Liquid Air	24			
Champion's Porcelain [7]	35			
Steel at Room Temperature	58			
Classically Perfect Material	∞			
		↓ Increasing Uniformity of Flaws	↓ Increasing Homogeneity	↓ Narrowing Distribution Curve

Figure 4-1 - Effect of Variation of Weibull Parameters [3]

The classically perfect material is defined as an ideally homogeneous and isotropic body, that is, every unit volume of the material is identical to every other unit volume. Weibull shows that as m approaches infinity, the distribution curve narrows. At m equal infinity, the probability of fracture in equation (3-3) will be zero for $\sigma < \sigma_0$ and 100% for $\sigma > \sigma_0$. This means that every body of such a material would fracture at one, and only one, stress, σ_0 . As predicted by the so-called classical (phenomenological) theories of strength, there would be no scatter of fracture stresses.

4.2 THE ZERO STRENGTH

The zero strength " σ_u " represents the limit of the stress below which the material is not expected to fail regardless of the size and stress distribution or type of loading. In a sense, it is a measure of the inherent strength of the material and indicates the lower bound of the distribution function. The effect of the variation of σ_u on the probability density function is shown in Figure 4-1(b).

The significance of " σ_u " lies in the fact that if some single stress value has to be chosen for the strength of a brittle material, the zero strength may best serve this purpose. Actually, this strength cannot be measured in a test since extremely large amounts of fracture data are required. The only way to obtain the zero strength of a brittle material would be to find the distribution function that properly describes the data and then to determine mathematically the stress corresponding to the zero probability of fracture.

4.3 THE NORMALIZING STRESS

The constant σ_0 is something of a reference level stress, that is, it accounts for the relative magnitude of the stress involved. In the limit, as m approaches infinity, the variance σ^2 approaches zero and σ_0 represents the ultimate strength as determined by classical theory, " P/A ".

In general, materials with high values of " m ", " σ_u ", and " σ_0 " are usually preferred in the design of structural components.

5. LOADING FACTORS

In use, the practical value of the distribution stems from the fact that the reduced risk of rupture expression

$$R = \int_V \left(\frac{\sigma}{\sigma_o} \right)^m dV \quad (5-1)$$

may for many cases of interest be integrated to the form

$$R = KV \left(\frac{\sigma}{\sigma_o} \right)^m \quad (5-2)$$

where V is the volume of material under load and K is a constant related to the type of loading. When the lower bound, σ_u , is not taken to be zero the integration of equation (3-6) becomes more complex. Generally, it is assumed that equation (3-6) may be approximated by the form

$$P = 1-S = 1-\exp \left[-KV \left(\frac{\sigma-\sigma_u}{\sigma_o} \right)^m \right] \quad (5-3)$$

or

$$P = 1-S = 1-e^{-R} \quad (5-4)$$

where

$$R = KV \left(\frac{\sigma-\sigma_u}{\sigma_o} \right)^m \quad (5-5)$$

In many cases, although it is theoretically incorrect, it is possible to use the K-factors obtained with $\sigma_u = 0$ in the more general equation (5-3) without introducing serious error [3].

The risks of rupture for several basic cases have been calculated for both σ_u equal to and greater than zero. These integrations have been performed for a fundamental set of structures and loading conditions. It is expected that any given structure may be broken down into elements, each of which may be approximated well by one of these simpler conditions. The probabilities of success of each element may then be multiplied together to give the probability that failure will not occur in the complete structure.

5.1 BASIC REDUCED LOADING FACTORS

Several loading (or K-) factors are evaluated and presented in the following cases. In addition, these K-factors, developed for $\sigma_u = 0$, are only a function of the material constant "m" and are also presented graphically.

Case 1 - Uniform Tension

In the case of uniform tension, the stress " σ " is constant throughout the volume. Therefore, the risk of rupture for uniform tension is

$$R = \int_V \left(\frac{\sigma}{\sigma_o} \right)^m dV = V \left(\frac{\sigma}{\sigma_o} \right)^m$$

and since $R = KV \left(\frac{\sigma}{\sigma_o} \right)^m$, $K = 1$ for uniform tension. V is the volume under tension.

Case 2 - Pure Bending

Consider the rectangular beam subjected to pure bending as illustrated in Figure 5-1. The bending moment is constant along the length " L " and the stress varies linearly through the height of the beam only. The tensile stress at a distance " y " from the neutral axis and the volume element can be expressed as

$$\sigma = \sigma_B \left(\frac{y}{h/2} \right)$$

$$dV = b L dy$$

Substituting these expressions into equation (5-1) and integrating only in the region of tensile stress gives

$$R = \int_0^{h/2} \left(\frac{\sigma_B 2y}{\sigma_o h} \right)^m b L dy$$

Figure 5-1 - Uniform Bending Model

or

$$R = \frac{V}{2(m+1)} \left(\frac{\sigma_B}{\sigma_o} \right)^m$$

From equation (5-2),

$$K = \frac{1}{2(m+1)}$$

and V is the beam volume.

The following cases of bending of rectangular beams are of primary interest in the reduction of test data. The maximum bending stress, σ_B , is given by $(6M)/(bh^2)$.

Case 3 - Center-Point Loading

$$K = \frac{1}{2(m+1)^2}$$

Case 4 - Third-Point Loading

$$K = \frac{m+3}{6(m+1)^2}$$

Case 5 - Fourth-Point Loading

$$K = \frac{m+2}{4(m+1)^2}$$

Case 6 - Sixth-Point Loading

$$K = \frac{2m+3}{6(m+1)^2}$$

Case 7 - General Linear Stress Distribution Across Rectangular Beam of Constant Cross-Section

$$K = \frac{1}{m+1}$$

Case 7 may be used for any linear stress distribution that has a point of zero stress. As the compressive stress does not contribute to failure, this expression must be used with the actual volume of material under tensile stress. It is possible to use this for the case where the tensile load never reaches zero by computing the probability of success for the fictitious beam with zero stress (see h_{eff} in sketch) and dividing that probability by the probability of success of the fictitious portion. If $h_{fict}/h_{eff} > 0.9$, it is recommended that an average uniform stress be assumed and Case 1 be used.

Plots of these functions (all for rectangular beams of constant cross-section) are given in Figure 5-2. Similar K-factors for use with bars of circular cross-section^[3] are graphically presented in Figure 5-3.

5.2 GENERALIZED LOADING FACTORS

For several basic loading cases, it is possible to integrate the risk of rupture, equation (3-5), for $\sigma_u \neq 0$. These integrations have been performed^[8] for several cases of a rectangular beam under three- and four-point loading. The risk of rupture expressions, to be used with equation (3-6), are given below:

a) Pure bending

$$R = \frac{V}{2(m+1)} \left(1 - \frac{\sigma_u}{\sigma_B}\right) \left(\frac{\sigma_B - \sigma_u}{\sigma_o}\right)^m$$

b) Fourth-point loading

$$R = \frac{V}{4(m+1)} \left(1 - \frac{\sigma_u}{\sigma_B}\right) \left(\frac{\sigma_B - \sigma_u}{\sigma_o}\right)^m (1 + \sum) \\ + \frac{V}{4(m+1) \sigma_B \sigma_o} (-\sigma_u)^{m+1} \ln\left(\frac{h}{2y_u}\right)$$

Figure 5-2 - Loading (K-) Factors for Basic Stresses in Rectangular Beams [3]

Figure 5-3 - Loading (K-) Factors for Basic Stresses in Circular Bars ^[3]

c) Third-point loading

$$\begin{aligned}
 R = & \frac{V}{6(m+1)} \left(1 - \frac{\sigma_u}{\sigma_B}\right) \left(\frac{\sigma_B - \sigma_u}{\sigma_o}\right)^m (1+2\Sigma) \\
 & + \frac{V}{3(m+1)} \frac{1}{\sigma_B \sigma_o} {}^m (-\sigma_u)^{m+1} \ln \left(\frac{h}{2y_u}\right)
 \end{aligned}$$

d) Three-point loading (center-point loading)

$$\begin{aligned}
 R = & \frac{V}{2(m+1)} \left(1 - \frac{\sigma_u}{\sigma_B}\right) \left(\frac{\sigma_B - \sigma_u}{\sigma_o}\right)^m \Sigma + \\
 & \frac{V}{2(m+1)} \frac{1}{\sigma_B \sigma_o} {}^m (-\sigma_u)^{m+1} \ln \left(\frac{h}{2y_u}\right)
 \end{aligned}$$

In the above expressions

V = entire volume of beam

$$y_u = \frac{h}{2} \frac{\sigma_u}{\sigma_B}$$

$$\Sigma = \sum_{r=0}^m \frac{1}{m+1-r} \left(1 - \frac{\sigma_B}{\sigma_u}\right)^{-r}$$

The complexity of these expressions, even for the most basic loadings, suggest why the simpler K-factors are used as an approximation.

5.3 ADDITIONAL REDUCED LOADING FACTORS

In many practical situations the state of stress in a component is such that the risk of rupture cannot be approximated well by a simple model. In this case the risk of rupture expression, Equation (3-5), must be evaluated for the component in question.

Robinson^[9] evaluated the risk of rupture for round tubes under internal pressure and under three-point symmetric bending. Manson and Smith^[10] investigated the resistance of brittle materials to thermal shock. Since the stress distribution in thermal shock specimens is usually not expressible in simple analytical form, a graphical approach is used to derive an expression for the risk of rupture. Barnett, et. al.^[11] investigated the integrity of brittle elements subjected to time-dependent thermal and mechanical loads. They determined the temperature distribution, thermal and mechanical stresses, and the statistical description of the resistance of beams and discs to transient thermal and mechanical loads.

5.4 LOADING FACTORS BASED ON SURFACE FLAW DISTRIBUTIONS

In its standard form, Weibull's theory states that the risk of fracture is related to the volume of the body under stress. However, it is not unusual that the rupture of a brittle body starts on the surface, where properties may differ from those of the material in the interior of the body, for instance, owing to the method of manufacture. For cases where fracture always is initiated at the surface, the volume integral of Equation (2-7) may be replaced by a surface integral^[1],

$$R = \int_A f(\sigma) dA \quad (5-6)$$

In case separate material functions apply to the surface layer and the interior, Equation (2-8) becomes

$$P = 1 - e^{-\left(\int_V f_1(\sigma) dV + \int_A f_2(\sigma) dA\right)} \quad (5-7)$$

As with the volume integral, Equation (5-6) may be integrated for many cases of interest to the form

$$R = KA \left(\frac{\sigma}{\sigma_0}\right)^m \quad (5-8)$$

where A is the surface area of the material under load and K is a constant related to the type of loading. Loading (or K -) factors for several basic cases have been evaluated for use in Equation (5-8). For the rectangular beam described in Section 5.1, the loading factors are as follows:

Case 1 - Uniform Tension

$$K = 1$$

Case 2 - Pure Bending

$$K = \frac{1}{2(\lambda+1)} \left(\frac{1}{m+1} + \lambda \right)$$

Case 3 - Center-Point Loading

$$K = \frac{1}{2(\lambda+1)(m+1)} \left(\frac{1}{m+1} + \lambda \right)$$

Case 4 - Third-Point Loading

$$K = \frac{m+3}{6(\lambda+1)(m+1)} \left(\frac{1}{m+1} + \lambda \right)$$

Case 5 - Fourth-Point Loading

$$K = \frac{m+2}{4(\lambda+1)(m+1)} \left(\frac{1}{m+1} + \lambda \right)$$

Case 6 - Sixth-Point Loading

$$K = \frac{2m+3}{6(\lambda+1)(m+1)} \left(\frac{1}{m+1} + \lambda \right)$$

In the above expressions, λ is defined as the beam width-to-depth ratio, b/h .

Generalized risk of rupture expressions (i.e., for $\sigma_u \neq 0$) based on surface flaw distributions have also been developed for several simple cases [8]. As with the generalized volume risk of rupture expressions of Section 5.2, the complexity of these expression limits their usefulness.

6. MEAN STRENGTH AND STANDARD DEVIATION

The strength of a body, in the concept of Weibull's theory, is the composite result of the stress state at every point in the body, as well as the gross size of the body. The expression for the risk of rupture, as given in equation (5-2),

$$R = KV \left(\frac{\sigma}{\sigma_o} \right)^m$$

can be combined with equation (3-7), with $\sigma_u = 0$, to give the mean failure stress as

$$\sigma_m = \frac{\sigma_o}{(KV)^{1/m}} \int_0^{\infty} e^{-Z^m} dZ \quad (6-1)$$

where $Z = \frac{\sigma}{\sigma_o} (KV)^{1/m}$. The integral in equation (6-1) is also expressible as the gamma function, $\Gamma(1 + 1/m)$. The mean failure stress (also called the ultimate strength) is then

$$\sigma_m = \frac{\sigma_o \Gamma(1 + 1/m)}{(KV)^{1/m}} \quad (6-2)$$

where σ_m is also known as the first statistical moment. The gamma function, $\Gamma(1 + 1/m)$, defined as A_1 , is plotted as a function of m in Figure 6-1. If σ_u is not taken to be zero, the quantity σ_m in equation (6-2) may be replaced by $(\sigma_m - \sigma_u)$. Similarly, the variance, σ_a^2 , can be evaluated from equation (3-8), with $\sigma_u = 0$, to be

$$\sigma_a^2 = \frac{\sigma_o^2 \Gamma(1 + 2/m)}{(KV)^{2/m}} - \sigma_m^2 \quad (6-3)$$

or using the results obtained in equation (6-2)

$$\sigma_a^2 = \frac{\sigma_o^2}{(KV)^{2/m}} [\Gamma(1 + 2/m) - \Gamma^2(1 + 1/m)] \quad (6-4)$$

The term in the square brackets in equation (6-4), defined as A_2 , is plotted in Figure 6-2 as a function of m . Equation (6-4) can also be expressed in terms of the mean strength, σ_m , by combining equations (6-2) and (6-3) to give the following equation:

Figure 6-1 - Plot of the Gamma Function A_1 as a Function of m

Figure 6-2 - Plot of the Coefficient A_2 as a Function of m

$$\frac{\sigma^2}{\sigma_m^2} = \frac{\Gamma(1 + 2/m)}{\Gamma^2(1 + 1/m)} - 1 = A_4 \quad (6-5)$$

A graph of the right hand side of this equation, defined as A_4 , is plotted in Figure 6-3 as a function of m . In addition, a useful approximation of equation (6-5) is given by

$$A_4 \approx \left(\frac{1.2}{m}\right)^2 \quad (6-6)$$

Note that A_4 is also the square of the coefficient of variation, C_{vm}^2 .

The third statistical moment about the mean, μ_3 , derived in Reference [5], is given by

$$\mu_3 = \frac{\sigma_o^3}{3/m} \left[\Gamma(1 + 3/m) - 3 \Gamma(1 + 2/m) \Gamma(1 + 1/m) + 2 \Gamma^3(1 + 1/m) \right] \quad (6-7)$$

The statistical moments presented here will be used in determining the Weibull parameters in Section 9.

Figure 6-3 (a) - Plot of the Coefficient A_4 as a Function of m

Figure 6-3(b) - Plot of the Coefficient A_4 as a Function of m

Figure 6-3(c) - Plot of the Coefficient A_4 as a Function of m

7. THE SIZE AND STRESS DISTRIBUTION EFFECTS

The decrease of strength with increase in size and the dependence of strength on the stress distribution are well known characteristics of a brittle material. These two effects can be understood and formulated by analyzing the equations presented in the previous section.

7.1 THE VOLUME EFFECT

Suppose that the mean failure strengths of two different volumes of the same material, subjected to different stress states, are to be compared. Equation (6-2) can be written for the two volumes, defined as V_1 and V_2 , and the corresponding stress states, K_1 and K_2 . The ratio of the two mean failure strengths is then given by

$$\frac{(\sigma_m)_1}{(\sigma_m)_2} = \left(\frac{K_2 V_2}{K_1 V_1} \right)^{1/m} \quad (7-1)$$

since m and σ_0 remain constant. If, in addition, the stress states are the same,

$$\frac{(\sigma_m)_1}{(\sigma_m)_2} = \left(\frac{V_2}{V_1} \right)^{1/m} \quad (7-2)$$

Hence, the stresses at fracture are inversely proportional to the m^{th} root of the reciprocal of the volume ratio. The larger the volume, the lower the fracture stress. As previously mentioned, if σ_u is not taken to be zero the quantity σ_m may be replaced by $(\sigma_m - \sigma_u)$.

As a numerical example, m will be taken to be 10 (a reasonable value for beryllium). If the volume ratio between the two specimens is 10, equation (7-2) gives, for the same stress states,

$$\frac{(\sigma_m)_1}{(\sigma_m)_2} = 1.26$$

Figure 7-1 - The Volume Effect on the Mean Fracture Strength [4]

Hence, the larger specimen is expected to be only about 80 percent as strong as the smaller specimen. Figure 7-1 is a carpet plot of equation (7-2) showing the size effect for equal stress states with parameters of stress ratio, volume ratio, and material constant m .

The ratio of the standard deviations, from equation (6-5), is also proportional to the m^{th} root of the reciprocal of the volume ratio, or

$$\frac{a_1}{a_2} = \frac{(\sigma_m)_1}{(\sigma_m)_2} = \left(\frac{K_2 V_2}{K_1 V_1} \right)^{1/m} \quad (7-3)$$

7.2 THE STRESS DISTRIBUTION EFFECT

The relationships between stress states can also be determined from the above equations.

From Section 5.1 of this report, the loading factor for uniform tension (for any cross section) is $K = 1$ and for pure bending (rectangular cross section), $K = \frac{1}{2(m+1)}$.

The mean failure stresses in tension, σ_A , and in bending, σ_B , are then given by equation (6-2) as

$$\sigma_A = \frac{\sigma_o \Gamma(1 + 1/m)}{V_A^{1/m}} \quad (7-4)$$

and

$$\sigma_B = \frac{\sigma_o \Gamma(1 + 1/m)}{\left(\frac{V_B}{2m+2} \right)^{1/m}} \quad (7-5)$$

A third simple stress state of pure torsion is also often of interest. For this case the risk of rupture is not readily reducible to the form given by equation (5-2). The mean failure stress of a circular torsion specimen, however, as derived in References [6] and [7], is expressible in the following form,

$$\sigma_T = \frac{\sigma_o \Gamma(1 + 1/m)}{\left[\frac{(2m+1) \beta_1 V_T}{2(m+2)} \right]^{1/m}} \quad (7-6)$$

where β'_1 is defined as

$$\beta'_1 = \frac{\Gamma(m+1) \Gamma(\frac{m+1}{2})}{\Gamma(\frac{2m+1}{2} + 1) \Gamma(\frac{m}{2} + 1)}$$

Values of β'_1 for various values of m are given in Table 7-1.

TABLE 7-1

VALUES OF β'_1 FOR VARIOUS VALUES OF m	
Material Constant, m	β'_1
1	0.8488
2	0.5333
3	0.3880
4	0.3048
5	0.2509
6	0.2131
7	0.1852
8	0.1638
9	0.1468
10	0.1330
11	0.1216
12	0.1120
13	0.1038
14	0.0967
15	0.0948
16	0.08504
17	0.08022
18	0.07603
19	0.07205
23	0.05985
35	0.03985

Weibull's theory states that the material constant m should not vary with stress state, so that m determined in one stress state may be used in any other stress state. The following relations for predicting the strength (or standard deviation) in one stress state from the strength (or standard deviation) in another can then be established from equations (7-4), (7-5), and (7-6).

$$\frac{\sigma_B}{\sigma_A} = \frac{a_B}{a_A} = \left[(2m+2) \sqrt[3]{V_A/V_B} \right]^{1/m} \quad (7-7)$$

$$\frac{\sigma_A}{\sigma_T} = \frac{a_A}{a_T} = \left[\frac{(2m+1) \beta'_1}{2(m+2)} \frac{V_T}{V_A} \right]^{1/m} \quad (7-8)$$

$$\frac{\sigma_B}{\sigma_T} = \frac{a_B}{a_T} = \left[\frac{(2m+1) (2m+2) \beta'_1}{2 (m+2)} \frac{V_T}{V_B} \right]^{1/m} \quad (7-9)$$

Plots of the stress ratios given in equations (7-7), (7-8), and (7-9) are shown in Figures 7-2, 7-3, and 7-4, respectively, for volume ratios equal to one. Other volume ratios may be accounted for by using Figure 7-1 in conjunction with above figures.

As an example of the stress distribution effect, consider the mean fracture strength ratio for rectangular beams, of the same material, tested both in uniaxial tension and in pure bending. For equal test volumes, equation (7-7) can be used in the form

$$\frac{\sigma_B}{\sigma_A} = (2m+2)^{1/m}$$

Using again the value of $m = 10$, the expected strength difference is

$$\frac{\sigma_B}{\sigma_A} = 1.36$$

This indicates that the tensile strength should be about 73.5 percent of the pure bending strength. This stress ratio could also have been obtained from Figure 7-2.

It may be noted that the ratio of center-point bending (for a rectangular specimen) to axial tension is also shown on Figure 7-2. For the same test volume and Weibull material constant as in the previous example, the center-point bending to tension ratio is seen to be 1.73. This indicates that a material having an "m" value of 10 should have a center-point

Figure 7-2 - The Variation of Bending to Tension Stress Ratio with Weibull Material Constant, m [4]

flexure strength 73 percent greater than the uniaxial strength or 27 percent $(1.73/1.36 = 1.27)$ greater than the pure bending flexure strength.

Figure 7-3 - The Variation of Tension to Torsion Stress Ratio with Weibull Material Constant, m [4]

Figure 7-4 - The Variation of Bending to Torsion Stress Ratio with Weibull Material Constant, m [4]

8. RELATIONSHIP BETWEEN RELIABILITY AND SAFETY FACTOR

The probability of fracture, P , is given in Equation (3-6) as

$$P = 1 - e^{-R}$$

and the risk of rupture, R , from Equation (5-2) as

$$R = KV \left(\frac{\sigma}{\sigma_0} \right)^m$$

The material reliability, defined as the probability of survival, S , is then given by

$$S = 1 - P = e^{-KV \left(\frac{\sigma}{\sigma_0} \right)^m} \quad (8-1)$$

The mean failure stress, σ_m , has been defined in Equation (6-2) as

$$\sigma_m = \sigma_0 \frac{\Gamma \left(1 + \frac{1}{m} \right)}{(KV)^{1/m}} \quad (8-2)$$

Combining Equations (8-1) and (8-2) by eliminating the quantity KV gives

$$S = e^{-\left[\sigma_s / \sigma_m \Gamma \left(1 + \frac{1}{m} \right) \right]^m} \quad (8-3)$$

or

$$S = e^{-R} \quad \text{where}$$

$$R = \left[\beta \Gamma \left(1 + \frac{1}{m} \right) \right]^m \quad (8-4)$$

and

$$\beta = \frac{\sigma_s}{\sigma_m}$$

The risk of rupture, R , expressed in the form of Equation (8-4), is plotted in Figure 8-1 as a function of β for various value of the parameter, m . The stress ratio β is defined as the normalized stress (for $\sigma_u = 0$). The reciprocal of β , or σ_m / σ_s , is defined as the material safety factor, k_s . It represents the ratio of the mean fracture stress, σ_m , to the fracture stress corresponding to a given probability of survival, σ_s , or

$$k_s = \frac{\sigma_m}{\sigma_s} = \Gamma(1 + \frac{1}{m}) / [\ln(1/S)]^{1/m} \quad (8-5)$$

In a similar manner, Equation (8-3) can be generalized to include the effect of the zero strength, σ_u . Theoretically, it must be assumed that σ_u varies throughout the volume of the material in the same manner as the applied stress, so the two bear a constant relation to one another. This allows the quantity $(\sigma - \sigma_u)^m$ to be factored out of the volume integral as was done with $(\sigma)^m$ so that the term β is the risk of rupture expression, Equation (8-4), is given by

$$\beta = \frac{\sigma_s - \sigma_u}{\sigma_m - \sigma_u}$$

Equation (8-3) represents the desired relationship between the material reliability, S , and the material safety factor, σ_m / σ_s , as a function of the material constant m . Figure 8-2 is a plot of Equation (8-3), where $\beta = \sigma_s / \sigma_m$. It can be seen that high safety factors (low values of β) are necessary to insure a high reliability for materials with low Weibull constants.

Figure 8-1 - Risk of Rupture Vs. Normalized Stress Ratio [9]

Figure 8-2 (a) - The Normalized Weibull Distribution [9]

Figure 8-2 (b) - The Normalized Weibull Distribution [9]

Figure 8-3 - The Variation of Safety Factor with Weibull Material Constant, m , for Various Material Reliabilities [4]

Figure 8-3 is another plot of Equation (8-3) for five high reliability values. It can readily be seen from Figure 8-3 that the material safety factor, σ_m / σ_s , decreases sharply with increasing values of m . For example, a material with a Weibull constant of 1.3 (such as glass fibers) should have a maximum allowable strength of $\sigma_{.99} = \sigma_m / 25$ to insure a 99% reliability. A material such as mild steel, $m = 58$, should have a $\sigma_{.99}$ of $\sigma_m / 1.1$. Hence, Weibull theory predicts that glass fibers are a poor risk as a structural material when compared to steel.

8.1 DETERMINING ALLOWABLE MATERIAL STRENGTHS

Once the Weibull material constant, m , is determined from a particular test, it is possible to calculate the allowable strength associated with a particular material reliability. In many cases it is desirable to limit the allowable material strength to a 99% reliability value, that is $\sigma_{.99}$. Note that if the strength distribution is not representative of population statistics, the confidence level upon which the allowable material strength is based should also be given. For population statistics, $\sigma_{.99}$ represents the strength above which 99% of the remaining strength distribution lies, as shown in Figure 8-4.

Figure 8-4 - Allowable Material Strength Based on a 99% Weibull Reliability

As previously described, the ratio of the mean material strength to the 99% allowable strength is only a function of the Weibull material constant, m , and can be obtained from curve ① of Figure 8-3. Equation (8-5) can be used for reliabilities higher than those presented in Figure 8-3. In fact, for high reliabilities, the following approximation can be made

$$\ln (1/S) = -\ln(S) = -[(S-1) - 1/2(S-1)^2 + \dots]$$

and since $P = 1-S$

$$\ln (1/S) = P + P^2/2 + \dots \approx P, \text{ for } P < 10^{-3}$$

Equation (8-5) can then be approximated as

$$k_s = \frac{\Gamma(1+1/m)}{\ln(1/S)^{1/m}} \approx (1/P)^{1/m} \quad (8-6)$$

since $\Gamma(1+1/m) \approx 1$ for most values of m (see Figure 6-1) and then

$$P \approx (\sigma_s / \sigma_m)^m \quad (8-7)$$

If, in addition, a structure is composed of n subvolumes, each with a volume equal to that of the test specimen and the Weibull parameters obtained from these test specimens are m and σ_m , we get from Equations (7-2), (8-3) and (8-10)

$$k_s = \frac{\sigma_m}{\sigma_s} = \frac{(n)^{1/m} \Gamma(1+1/m)}{\ln(1/S)^{1/m}} \approx (n/P)^{1/m} \quad (8-8)$$

or

$$P = 1-S \approx n \left(\frac{\sigma_s}{\sigma_m} \right)^m \quad (8-9)$$

where σ_s is the stress corresponding to the probability of success, S , and σ_m is the mean strength of the test specimens from which the Weibull parameters were derived. If $\sigma_u \neq 0$, k_s in Equations (8-6) and (8-8) may be replaced by

$$k_s = \frac{\sigma_m - \sigma_u}{\sigma_s - \sigma_u}$$

It has been assumed in the previous derivations that the state of stress in the structure is the same as that in the test specimens. If this is not so, the K-factor modification given in Equation (7-1) must also be included. It is also assumed that the stress σ_s is not, itself, statistically distributed.

Consider, for example, a structure consisting of BeO ($m \approx 10$) to operate with a reliability of 0.95. If the structure volume is 100 times the test specimen volume and is subjected to the same state of stress as the test specimens (whose mean strength is given by σ_m), find the allowable stress. Since $P = 1 - .99999 = 10^{-5}$, from Equation (8-8),

$$k_s = \frac{\sigma_m}{\sigma_s} \approx \left(\frac{100}{10^{-5}} \right)^{1/10} = 5$$

or

$$\sigma_s = \sigma_m / 5$$

8.2 EXTREME VALUE STATISTICS

In design applications of brittle materials the extreme (minimum) value of strength is an important quantity. It is evident, looking at Figure 8-3, that although the mean values may be comfortably high, when m values are low there may be an uncomfortably high probability of observing relatively weak specimens. The occurrence of weak specimens increases as the sample size increases and as m decreases. It is possible to predict the most likely least value in a set of n specimens and, in fact, to describe the distribution of the least values. Following the method of Epstein [12] as outlined by Robinson [9],

we see that the probability of n specimens simultaneously surviving the stress σ is given by

$$S_n = [S(\sigma)]^n \quad (8-10)$$

where S gives the survival probability of any one specimen. The probability density is found by differentiating once:

$$\frac{dS_n}{d\beta} = nS^{n-1}S'$$

where S' is $\frac{dS}{d\beta}$ and $\beta = \sigma/\sigma_m$.

The most probable least value corresponds to the maximum of the probability density which is obtained by another differentiation and equating to zero, giving the formula

$$(n-1)(S')^2 = -S''S \quad (8-11)$$

The value of β satisfying this equation will be the most likely least value. Substitution of Equation (8-3) and its derivatives into Equation (8-11) gives for the most likely least value, β^*

$$\beta^* = \frac{\sigma^*}{\sigma_m} = \left(\frac{m-1}{mn}\right)^{1/m} \frac{1}{\Gamma(1+1/m)} \quad (8-12)$$

Equation (8-12) is plotted in Figure 8-5.

This equation and graph are extremely useful for design purposes since the most likely least value in a set of n specimens of Weibull material constant m , may be determined. In addition, Figure 8-5 may also be used to make a first estimate of the m -value from the lowest observed specimen strength.

Figure 8-5 - Normalized Extreme Values (Most Likely
Least Values) of the Weibull Distribution [9]

The extreme safety factor, k_x , may also be useful in design. It is defined as the ratio of the least likely strength of the material, σ^* , to the probable design stress, σ_s , or

$$k_x = \frac{\sigma^*}{\sigma_s} \quad (8-13)$$

Equation (8-13) can be rearranged as follows

$$k_x = \frac{\sigma^*}{\sigma_s} = \frac{\sigma^*}{\sigma_m} \cdot \frac{\sigma_m}{\sigma_s} = \beta^* k_s$$

and from Equations (8-12) and (8-8)

$$k_x = \left(\frac{m-1}{m} \right)^{1/m} \ln(1/S)^{-1/m} \quad (8-14)$$

Using the approximation for high reliabilities, Equation (8-14) can be expressed as

$$k_x \approx \left(\frac{m-1}{mP} \right)^{1/m} \quad (8-15)$$

Considering again the structure of BeO ($m \approx 10$) designed to operate at $P = 10^{-5}$, the extreme safety factor is given by

$$k_x = (9 \times 10^4)^{1/10} \approx 3$$

or the operating stress should be approximately one-third of the least likely observed test specimen strength.

The very core of the difficulty of designing with brittle materials can be seen from Figure 8-5. As the number of specimens is increased, the least expected value drops fairly rapidly, especially for the lower values of m . Note, for example, that for $m = 6$ a sample of 100 specimens will probably contain one specimen whose strength is only half of the mean. Furthermore, this least value is itself a statistical parameter and is distributed

according to Equation (8-10) as

$$S^* = e^{-n} [\beta \Gamma(1+1/m)]^m \quad (8-16)$$

One method of possibly avoiding the "least-value problem" in a set of specimens is to remove the weakest ones by some suitable screening or proof test. The resulting group would exhibit some threshold stress below which no failures would occur. It must be noted, however, that a selection process, such as a proof test, may degrade the statistics of the survivors. The effect of truncation on a distribution of the form of Equation (8-3) has also been investigated by Robinson [9]. The normalized truncation level, that is, the proof stress divided by the mean of the untruncated (or primitive) distribution, is defined as ψ . The most probable least value of the truncated distribution is given by a formula identical with Equation (8-12) provided the truncation ψ is less than the predicted value β^* . The interpretation is that the least value of the truncated distribution is identical with the least value of the corresponding initial distribution so long as this least value exceeds ψ ; otherwise, the least value is ψ itself. A critical number of specimens can be determined from Equation (8-12), or from Figure 8-5, by inserting ψ for β^* . For sample sizes above this number the most likely least value is ψ itself.

Although the least value remains essentially unaffected, the mean value of the truncated distribution will exceed the primitive mean value. The improvement of the mean value by truncation is shown in Figure 8-6. The ordinate $\hat{\beta}$ is defined as the ratio of the mean value of the truncated distribution to the primitive mean value. It appears, judging from Figure 8-6, that for most values of m a relatively high truncation level is required to cause a significant change in the mean value. It is not expected, however, that even this improvement can be realized. It has been observed that brittle materials subjected to a stress below the ultimate will exhibit microcracking and slip-band formation and upon subsequent testing will exhibit a reduced strength [13].

Figure 8 - 6 Improvement of the Mean Strength by Truncation [9]

It should be noted that the equations and figures previously developed in this section for $\sigma_u = 0$ can be applied to cases where $\sigma_u \neq 0$ by modifying the β term to read

$$\beta = \frac{\sigma - \sigma_u}{\sigma_m - \sigma_u}$$

8.3 RELATIONSHIP BETWEEN WEIBULL MATERIAL CONSTANT AND THE NORMAL COEFFICIENT OF VARIATION

The Weibull material constant, m , is analogous to the coefficient of variation, C_{vm} , used with materials exhibiting a normal strength distribution. The maximum allowable strength, σ_s , at a given reliability, S , for a normal distribution, is given by

$$\sigma_s = \sigma_m (1 - X_s C_{vm}) \quad (8-17)$$

where X_s is a function of the desired reliability (such as $X_s = 2.326$ for $S = 99\%$) and C_{vm} is the ratio of the material standard deviation to the mean strength.

By rearranging Equation (8-17) as follows

$$X_s = (1 - \sigma_s / \sigma_m) / C_{vm} \quad (8-18)$$

the terms in Equation (8-18) can be compared with the terms in Equation (8-3). Both m and C_{vm} represent the scatter in the observed strengths. For a 99% material reliability, the strength ratio can be determined as a function of m from Equation (8-3) or Figure 8-3. Using this ratio in Equation (8-18), with $X_s = 2.326$, gives the corresponding coefficient of variation. Figure 8-7 gives the coefficient of variation for a normally distributed material corresponding to the Weibull material constant for equal allowable strength ratios and a material reliability of 99%.

It is interesting to note that a material with a Weibull constant of 8 has a corresponding normal coefficient of variation of 17.7%. The low value of m and the high scatter in strength are both representative of a brittle material. For an m value of 50, the corresponding C_{vm} value is 4.3%. Both of these values are typical of a ductile material.

Figure 8-7 - Relationship Between Normal Coefficient
of Variation and Weibull Material Constant

9. EVALUATION OF WEIBULL PARAMETERS

The success of the statistical strength theory depends on how well the probability function fits the actual material data. An accurate evaluation of the parameters appearing in equation (5-3) is, therefore, very important, especially the evaluation of the material constant "m".

There are several methods which may be used to evaluate these parameters from experimental fracture data. For practical purposes, only three will be discussed here.

9.1 McCLINTOCK'S METHOD

The principle of this method is to choose the distribution constants, i.e, the Weibull parameters m , σ_o , and σ_u , so that the first three moments of the distribution function coincide with those of the data. The first step of this method is to calculate the first three moments, σ_m , σ^2 , and μ_3 , of the experimental data. These are defined as follows:

First moment (mean strength corresponding to 50% probability)

$$\sigma_m = \frac{1}{N} \sum_{n=1}^N \sigma_n \quad (9-1)$$

Second moment about the mean (square of the standard deviation)

$$\sigma^2 = \frac{1}{N} \sum_{n=1}^N (\sigma_n - \sigma_m)^2 = \frac{1}{N} \sum \sigma_n^2 - (\sigma_m)^2 \quad (9-2)$$

Third moment about the mean

$$\mu_3 = \frac{1}{N} \sum_{n=1}^N (\sigma_n - \sigma_m)^3 = \frac{1}{N} \sum \sigma_n^3 - \frac{3\sigma_m}{N} \sum \sigma_n^2 + 2(\sigma_m)^3 \quad (9-3)$$

where N is the total number of the statistical test specimens and σ_n is the fracture strength of the "n" specimen.

By taking moments of all points on the probability curve about the smallest point, σ_u , and using the relationship between moments taken at an arbitrary point to those taken at the mean, it can be shown that the first, second and third moments can also be expressed as follows:

$$\sigma_m = \sigma_u + \frac{\sigma_o}{(KV)^{1/m}} \Gamma\left(1 + \frac{1}{m}\right) \quad (9-4)$$

$$\sigma^2 = \frac{\sigma_o^2}{(KV)^{2/m}} \left[\Gamma\left(1 + \frac{2}{m}\right) - \Gamma^2\left(1 + \frac{1}{m}\right) \right] \quad (9-5)$$

$$\mu_3 = \frac{\sigma_o^3}{(KV)^{3/m}} \left[\Gamma\left(1 + \frac{3}{m}\right) - 3 \Gamma\left(1 + \frac{2}{m}\right) \Gamma\left(1 + \frac{1}{m}\right) + 2 \Gamma^3\left(1 + \frac{1}{m}\right) \right] \quad (9-6)$$

These equations have also been presented in Section 6.

By dividing equation (9-6) by equation (9-5) raised to the 3/2 power, we obtain a variable " A_3 " defined as

$$A_3 = \frac{\mu_3}{(\sigma^2)^{3/2}} = \frac{\Gamma\left(1 + \frac{3}{m}\right) - 3 \Gamma\left(1 + \frac{2}{m}\right) \Gamma\left(1 + \frac{1}{m}\right) + 2 \Gamma^3\left(1 + \frac{1}{m}\right)}{\left[\Gamma\left(1 + \frac{2}{m}\right) - \Gamma^2\left(1 + \frac{1}{m}\right) \right]^{3/2}} \quad (9-7)$$

Both σ^2 and μ_3 can be determined from the experimental data and equations (9-2) and (9-3), respectively. Thus, the value of A_3 can be determined from the test data. A curve giving the value of A_3 for different values of "m" is presented in Figure 9-1. From the known value of A_3 , m can readily be obtained.

By rearranging equation (9-5), we have

$$A_2 = \frac{\sigma^2 (KV)^{2/m}}{\sigma_o^2} = \left[\Gamma\left(1 + \frac{2}{m}\right) - \Gamma^2\left(1 + \frac{1}{m}\right) \right] \quad (9-8)$$

Figure 9-1 - Plot of the Coefficient A_3 As a Function of m

The curve of Figure 6-2 on Page 6-3 gives the value of the right hand side of equation (9-8), A_2 , as a function of m . With the value of m previously determined, σ_o can be obtained from equation (9-8) and Figure 9-2. It is also assumed that both the test specimen volume and the type of test (to determine K) are known.

Defining $A_1 = \Gamma (1 + \frac{1}{m})$, equation (9-4) may be written as

$$\sigma_u = \sigma_m - \frac{\sigma_o A_1}{(KV)^{1/m}} \quad (9-9)$$

The curve of Figure 6-1 on Page 6-2 gives the value of A_1 as a function of m . Hence, with the value of A_1 taken from the curve and σ_m determined from the experimental data by equation (9-1), the zero strength, σ_u , can be calculated.

It should be noted that if we initially assume $\sigma_u = 0$, the value of m may be obtained from the ratio of equation (9-5) to equation (9-4), or

$$\frac{\sigma_m^2}{\sigma_o^2} = \frac{\Gamma (1 + \frac{2}{m})}{\Gamma^2 (1 + \frac{1}{m})} - 1 = A_4 = C_{vm}^2 \quad (9-10)$$

The first and second moments can be calculated from the experimental data and equation (9-1) and (9-2), respectively. The corresponding value of m may be found from equation (9-10) and Figure 6-3 where A_4 is plotted as a function of m . As noted in Section 6, a useful approximation to equation (9-10) is that $A_4 \approx (1.2/m)^2$. The constant σ_o can then be determined from equation (9-8) as before.

9.2 GRAPHICAL METHODS

9.2.1 ln-Plots

Equation (5-3) can be rearranged as follows:

$$\frac{1}{1-P} = e^{KV (\frac{\sigma - \sigma_u}{\sigma_o})^m} \quad (9-11)$$

and in terms of logarithms

$$\ln \ln \left(\frac{1}{1-P} \right) = m \ln (\sigma - \sigma_u) + \ln \left(\frac{KV}{\sigma_o^m} \right) \quad (9-12)$$

It can be seen from equation (9-12) that a plot of the distribution function will be linear in a system of rectangular coordinates in which $\ln \ln \left(\frac{1}{1-p} \right)$ is the ordinate and $\ln (\sigma - \sigma_u)$ is the abscissa. In addition, m will be the slope of the distribution function in these coordinates. Also, from equation (9-12), a change in the volume of the test specimen merely shifts the distribution curve vertically.

The general principle of this method consists of plotting the fracture data in this system of coordinates, drawing the best straight line through the data, and determining the constants m and σ_o from the slope and intercepts of the line.

The application of this method presupposes the knowledge of the constant σ_u , which, as a rule, is not known. In this case, it is advisable to plot a test curve at first in the system of coordinates previously described. If the curve happens to follow a straight line, this implies that $\sigma_u = 0$. If, on the other hand, the test curve is not straight (say concave downwards), it follows that $\sigma_u \neq 0$, and a tentative value is taken for σ_u . If this value is too high, the curve will be bent in the opposite direction (concave upwards), and further tentative value must be taken until the curve approximates a straight line as closely as possible. It has been found that the curvature of the line is very sensitive to variations of σ_u , provided the test series is reasonably large, so that the value of σ_u can be determined with reasonable accuracy. Once m and σ_u are determined, σ_o can be evaluated from equation (9-12).

A problem also arises when plotting fracture data in the selection of the value of the probability corresponding to a given test value stress. Weibull determines the probability, P_n , corresponding to a fracture stress, σ_n , by the formula

$$P_n = \frac{n}{N+1} \quad (9-13)$$

Here, N represents the total number of tests in the series, where n observed fracture stresses are arranged in increasing order from 1 to N . Then, P_n corresponds to the n th observed fracture stress, σ_n . If equation (9-13) is substituted into the left hand side of equation (9-12) we get

$$\ln \ln \left(\frac{1}{1-P} \right) = \ln \ln \left(\frac{N+1}{N+1-n} \right)$$

so that equation (9-12) can be rewritten as

$$\ln \ln \left(\frac{N+1}{N+1-n} \right) = m \ln (\sigma - \sigma_u) + \ln \left(\frac{KV}{\sigma_o^m} \right) \quad (9-14)$$

If the fracture of brittle materials is complex, the logarithmic distribution curve may consist of two straight portions. This may be attributed to one of the following:

- 1) The fabrication of some of the test specimens are different from the rest.
- 2) Some fundamental change takes place in the nature of the fracture at a certain stress level.

It should be noted that the In-plot method, as previously outlined, assumes that the risk of rupture is expressible as

$$R = KV \left(\frac{\sigma - \sigma_u}{\sigma_o} \right)^m$$

The risk of rupture, however, does not actually integrate to this form, except for the case of uniform tension. As an example, consider the case of uniform bending, i.e., symmetric four-point loading. The risk of rupture, from Section 5.2(a), is

$$R = \frac{V}{2(m+1)} \left(\frac{\sigma_B - \sigma_u}{\sigma_o} \right)^m \left(1 - \frac{\sigma_u}{\sigma_B} \right)$$

Writing this risk of rupture into a logarithmic form similar to equation (9-12) gives

$$\ln \ln \left(\frac{1}{1-P} \right) = (m+1) \ln (\sigma_B - \sigma_u) - \ln \sigma_B + \ln \frac{V}{2(m+1)\sigma_o^m}$$

The extra term involving the variable σ_B shows that there is no reason to expect a straight line on the coordinates of equation (9-12) from bending test data. The plot will be linear, however, in a system of rectangular coordinates in which $\ln \ln (1/(1-P)) + \ln \sigma_B$ is the ordinate and $\ln (\sigma_B - \sigma_u)$ is the abscissa. The slope of the distribution function will be $(m+1)$ in these coordinates.

9.2.2 Normalized Stress Method

In the following method, the equations of Weibull's theory are re-expressed with the mean strength, σ_m , substituted for σ_o in the distribution function and σ_u assumed to be zero. It can then be shown that the failure probability distribution is explicitly dependent only on σ_m and m , regardless of the stress distribution or the specimen volume. This relation allows a straightforward and simple determination of m in cases where the integration of the applied stress distribution is not possible. This general form of the reduced Weibull distribution, given previously as equation (8-3), can be written

$$S = e^{-\left[\beta \Gamma\left(1 + \frac{1}{m}\right)\right]^m} \quad (9-15)$$

where $\beta = \sigma_s / \sigma_m$ and σ_s is the failure stress corresponding to a particular probability of survival level, S . Equation (9-15) is plotted on Figure 8-2 as a function of m for several values of the parameter β . For comparison, the logarithmic form is

$$\ln \ln \left(\frac{1}{S}\right) = m \ln \beta + m \ln \Gamma\left(1 + \frac{1}{m}\right) \quad (9-16)$$

and plots as the straight lines shown in Figure 9-2. A more sensitive graphical approach, however, is afforded by the use of Figure 8-2.

The procedure for evaluating a set of data by this method is to first normalize the experimentally obtained strength values, i.e., divide by the mean strength, and then work these values in a monotonic order of size. The cumulative survival probabilities may then be computed from equation (9-13). The normalized stress, β , and the corresponding probability of survival values, S , are tabulated and plotted on a graph as shown in Figure 8-2. Care must be taken in plotting values of S in the region approaching $\beta = 1$ since these β lines become nearly horizontal and a small uncertainty in probability has a very wide band of m values associated with it. If the resulting plot can be approximated by a vertical line, then the Weibull distribution with a single m value describes the data.

Figure 9-2 - Theoretical Weibull Lines

Curving of the line to either side may be due to a complex distribution involving at least two values of m . Also possible is the fact that the sample is biased by a positive zero strength, σ_u . In this case β is defined as $(\sigma - \sigma_u) / (\sigma_m - \sigma_u)$ and values of σ_u must be arbitrarily selected before plotting to obtain the "best" straight line fit.

9.3 EFFECT OF SAMPLE SIZE ON WEIBULL PARAMETERS

It is obvious that the statistical values obtained from equations (9-1), (9-2), and (9-3) may vary with sample size. The larger the number of samples, the more consistent and accurate are these values. An investigation was conducted into the adequate sample size by C. B. shields^[14] of Vought Astronautics Division. The results of this investigation are shown in Figure 9-3. Examination of the curves show that at low sample sizes there is a large bias and high scatter present in the evaluation of the Weibull parameters by McClintock's method. It also shows that more consistent values for Weibull parameters can be obtained from a statistical sample size of at least 80-100 specimens for this set of data.

Robinson^[15] also investigated the effect of sample size on the Weibull parameters. A study was also made of the different methods of estimating m , using computer-generated samples from known, controlled parent distributions. No one method appeared consistently superior for the range of m -values (3-30) and sample sizes (10-100) studied. Broadly speaking, the estimates of m were normally distributed about the true value with a coefficient of variation $\sqrt{1/n}$. That is, a variation of m of 31.6% may be expected from a sample size of 10. Increasing the sample size to 100 reduces the expected m -variation to 10%.

Figure 9-3 - Weibull Parameters Derived from Varying Sample Sizes [14]

10. EXAMPLES OF PARAMETER EVALUATIONS

In order to further illustrate the three methods of evaluating the Weibull parameters described in Section 9, an example of each is presented below. The methods, both for $\sigma_u = 0$ and $\sigma_u \neq 0$, are applied to the same experimental data and the resulting parameters compared.

10.1 L_n-PLOT METHOD

A sample evaluation of the Weibull parameters by the ln-plot method is presented in Table 10-1, Figure 10-1 and Figure 10-2. The fracture data shown in Table 10-1 are obtained from tensile tests on ring specimens. ^[16] The material tested is a cobalt alloyed tungsten carbide steel designated as K94. The test specimen volume is 0.15315 in³ and for pure tension tests, K is taken to be one in the risk of rupture expression.

TABLE 10-1
EVALUATION OF WEIBULL PARAMETERS BY LN-PLOT METHOD

Fracture Stresses σ_n , ksi	Rank n	Failure Probability $P = n/(N+1)$	$\frac{N+1}{N+1-n}$	$\ln \ln \left(\frac{N+1}{N+1-n} \right)$	For $\sigma_u = 0$ $\ln \sigma$	For $\sigma_u = 74.7$ ksi $\sigma - \sigma_u$ $\ln (\sigma - \sigma_u)$	
109.20	1	0.0769	1.0833	-2.526	11.6009	34.70	10.4545
122.46	2	0.1538	1.1818	-1.790	11.7155	47.76	10.7739
127.34	3	0.2307	1.3000	-1.338	11.7546	52.64	10.8712
136.11	4	0.3077	1.4444	-1.001	11.8219	61.41	11.0253
140.01	5	0.3846	1.6250	-0.723	11.8494	65.31	11.0869
145.86	6	0.4615	1.8571	-0.480	11.8904	71.26	11.1741
148.01	7	0.5384	2.1666	-0.257	11.9050	73.31	11.2024
149.37	8	0.6154	2.6000	-0.069	11.9142	74.67	11.2208
153.27	9	0.6923	3.2500	+0.164	11.9399	78.57	11.2717
167.51	10	0.7692	4.3333	+0.372	12.0287	92.81	11.4383
168.48	11	0.8461	6.5000	+0.627	12.0345	93.78	11.4487
177.84	12	0.9230	13.0000	+0.942	12.0888	103.14	11.5438

Two values of σ_u are assumed in Table 10-1, $\sigma_u = 0$ and 74.7×10^3 psi. The latter value was arbitrarily selected to agree with the value of σ_u obtained by McClintock's method as described in Section 10.2. The data of Table 10-1 are plotted on Figure 10-1 for $\sigma_u = 0$ and on Figure 10-2 for $\sigma_u = 74.7 \times 10^3$ psi.

Figure 10-1 - Logarithmic Distribution Curve for $\sigma_u = 0$ [16]

Figure 10-2 - Logarithmic Distribution Curve for $\sigma_u \neq 0$ [16]

The data of Figure 10-1 can be approximated by a straight line of slope, m , determined as follows:

$$m = \frac{-0.4 - (-2.4)}{11.9 - 11.625} = 7.3$$

The value of σ_o is readily calculated from equation (9-14), noting that, from Figure 10-1, $\ln \ln \left(\frac{N+1}{N+1-n} \right) = 0$ when $\ln \sigma = 11.955$. Then for $K = 1$, $V = 0.15315 \text{ in.}^3$ and $m = 7.3$,

$$\ln \sigma_o = \ln (\sigma - \sigma_u) + 1/m \ln (KV)$$

$$\ln \sigma_o = 11.955 + 1/7.3 \ln (0.15315) = 11.738$$

$$\text{or } \sigma_o = 125 \times 10^3 \text{ psi}$$

The probability of failure can then be expressed, from equation (5-3), as

$$P = 1 - \exp \left[-0.15315 \left(\frac{\sigma}{125,000} \right)^{7.3} \right]$$

The data of Figure 10-2, for $\sigma_u = 74.7 \times 10^3 \text{ psi}$, are approximated by a straight line of slope,

$$m = \frac{0.0 - (-1.8)}{11.275 - 10.775} = 3.6$$

Again, for $\ln \ln \left(\frac{N+1}{N+1-n} \right) = 0$ at $\ln (\sigma - \sigma_u) = 11.275$ and $KV = 0.15315$, equation (9-14) gives

$$\ln \sigma_o = 11.275 + 1/3.6 \ln (0.15315) = 10.753$$

$$\text{or } \sigma_o = 46.7 \times 10^3 \text{ psi}$$

The probability of failure can be expressed, from equation (5-3), as

$$P = 1 - \exp \left[-0.15315 \left(\frac{\sigma - 74,700}{46,700} \right)^{3.6} \right]$$

10.2 MC CLINTOCK'S METHOD

A sample evaluation of the Weibull parameters by McClintock's method for the same K94 data is given here for comparison. Table 10-2 gives the 12 observed fracture stresses and the multiples required.

TABLE 10-2
EVALUATION OF WEIBULL PARAMETERS BY MC CLINTOCK'S METHOD

Fracture Stresses σ_n, ksi	σ_n^2, ksi^2	σ_n^3, ksi^3
109.20	11,924.64	1,302,170.69
122.46	14,996.45	1,836,465.26
127.34	16,215.47	2,064,877.95
136.11	18,525.93	2,521,564.33
140.01	19,602.80	2,744,588.03
145.86	21,275.14	3,103,191.92
148.01	21,906.96	3,242,449.15
149.37	22,311.40	3,332,653.82
153.27	23,491.69	3,600,571.32
167.51	28,059.60	4,700,263.60
168.48	28,385.51	4,782,390.12
177.84	31,627.06	5,624,556.35

The mean observed stress can be obtained from equation (9-1) as

$$\sigma_m = 1/N \sum \sigma_n = \frac{1745.5}{12} = 145.5 \text{ ksi}$$

The second moment about the mean (square of the standard deviation) is, from equation (9-2),

$$\begin{aligned} \sigma^2 &= 1/N \sum \sigma_n^2 - (\sigma_m)^2 \\ &= 1/12 (258.3 \times 10^3) - (145.5)^2 = 369.7 \text{ ksi}^2 \end{aligned}$$

The third moment about the mean is, from equation (9-3),

$$\begin{aligned}\mu_3 &= 1/N \sum \sigma_n^3 - 3 \sigma_m / N \sum \sigma_n^2 + 2 (\sigma_m)^3 \\ &= 1/12 (38.86 \times 10^6) - \frac{3(145.5)}{12} (258.3 \times 10^3) + 2(145.5)^3 \\ &= -772.3 \text{ ksi}^3\end{aligned}$$

From equation (9-7),

$$A_3 = \frac{\mu_3}{\frac{3/2}{(\sigma^2)}} = \frac{-772.3}{\frac{3/2}{(369.7)}} = -0.1086$$

and from the curve of Figure 9-1, $m = 4.3$.

The value of A_2 obtained from Figure 9-2, for $m = 4.3$, is 0.06. Since $K=1$ and $V=0.15315 \text{ in.}^3$, we get from equation (9-8),

$$\begin{aligned}\sigma_o &= \left(\frac{\sigma^2}{A_2} \right)^{1/2} (KV)^{1/m} \\ \sigma_o &= \left(\frac{369.7}{0.06} \right)^{1/2} (0.15315)^{1/4.3} \\ \sigma_o &= 50,700 \text{ psi}\end{aligned}$$

From the curve of Figure 6-1, for $m = 4.3$, $A_1 = 0.90$. The minimum strength, σ_u , can be determined from equation 9-9, as

$$\begin{aligned}\sigma_u &= \sigma_m - \frac{\sigma_o A_1}{(KV)^{1/m}} \\ \sigma_u &= 145.5 - \frac{(50.74) (0.90)}{(0.15315)^{1/4.3}} \\ \sigma_u &= 74,700 \text{ psi}\end{aligned}$$

The probability of failure can then be expressed, from equation (5-3), as

$$P = 1 - \exp \left[-0.15315 \left(\frac{\sigma - 74,700}{50,700} \right)^{4.3} \right]$$

If we assume $\sigma_u = 0$, the value of m may be found from the ratio of the first and second moments only. From equation (9-10)

$$A_4 = a^2 / \sigma_m^2$$

$$A_4 = \frac{369.7}{(145.5)^2} = 0.01749$$

From Figure 6-3, for $A_4 = 0.01749$, we find that $m = 8.9$. The corresponding value of A_2 is found from Figure 6-2 to be 0.015. Since $K = 1$ and $V = 0.15315 \text{ in.}^3$, we get from equation (9-8),

$$\sigma_o = \left(\frac{a^2}{A_2} \right)^{1/2} (KV)^{1/m}$$

$$\sigma_o = \left(\frac{369.7}{0.015} \right)^{1/2} (0.15315)^{1/8.9}$$

$$\sigma_o = 127,000 \text{ psi}$$

The probability of failure can then be expressed, from equation (5-3), as

$$P = 1 - \exp \left[-0.15315 \left(\frac{\sigma}{127,000} \right)^{8.9} \right]$$

10.3 NORMALIZED STRESS METHOD

Using the normalized stress distribution method, the data is organized as shown in Table 10-3. The normalized stresses and their corresponding survival probabilities are plotted as shown in Figure 10-3. The data can be approximated by a straight line of value, $m = 7.9$. From Figure 6-1 the value of A_1 , for $m = 7.9$, is 0.94. Since $K = 1$ and $V = 0.15315 \text{ in.}^3$, we get from equation (9-9), with $\sigma_u = 0$,

$$\sigma_o = \frac{\sigma_m (KV)^{1/m}}{A_1}$$

$$\sigma_o = \frac{145.5 (0.15315)^{1/7.9}}{0.94}$$

$$\sigma_o = 122,000 \text{ psi}$$

TABLE 10-3

EVALUATION OF WEIBULL PARAMETERS BY NORMALIZED STRESS METHOD

<u>Ordered Fracture Stresses, $\sigma_n \times 10^{-3}$ psi</u>	<u>Rank, n</u>	<u>Normalized* Stresses, 10^3 psi</u>	<u>Survival** Probability</u>
109.20	1	0.750	0.9231
122.46	2	0.840	0.8462
127.34	3	0.875	0.7693
136.11	4	0.935	0.6923
140.01	5	0.962	0.6154
145.86	6	1.001	0.5385
148.01	7	1.015	0.4616
149.37	8	1.025	0.3846
153.27	9	1.052	0.3077
157.51	10	1.108	0.2308
168.48	11	1.16	0.1539
177.84	12	1.22	0.0770

Mean Stress, $\sigma_m = 145.45$ ksi

*Normalized Stress, $\beta = \sigma_n / \sigma_m$

** Survival Probability, $S = 1 - P = \frac{N+1-n}{N+1}$

Figure 10-3 - The Normalized Weibull Distribution

The probability of failure can then be expressed, from equation (5-3), as

$$P = 1 - \exp \left[-0.15315 \left(\frac{\sigma}{122,000} \right)^{7.9} \right]$$

Similarly, it is possible to obtain a corresponding set of Weibull parameters for $\sigma_u \neq 0$. As with the graphical method previously described, it is first necessary to select a value for σ_u that will produce the best straight line fit through the plotted data. The value of σ_u is arbitrarily chosen as 74,700 psi to be consistent with the previous values. The normalized stress, β , and the corresponding survival probability can then be plotted on Figure 10-3, this time with β defined as

$$\beta = \frac{\sigma_n - \sigma_u}{\sigma_m - \sigma_u}$$

The values of m and σ_o are obtained as previously described and are given by 3.9 and 48,300 psi, respectively. The corresponding probability of failure expression is

$$P = 1 - \exp \left[-0.15315 \left(\frac{\sigma - 74,700}{48,300} \right)^{3.9} \right]$$

10.4 COMPARISON OF PARAMETER EVALUATION METHODS

The three Weibull parameters obtained by the various methods are shown for comparison in Table 10-4. Examination of Table 10-4 shows a significant variation in the parameters m and σ_o as calculated by the three different methods. This variation may be attributed to the following causes:

- 1) The inherent error in graphical methods,
- 2) Method of ranking, i.e., could have used median ranks, etc.,
- 3) Small experimental sample size.

TABLE 10-4

COMPARISON OF WEIBULL PARAMETERS OBTAINED BY THE THREE METHODS

Method of Parameter Evaluation	$\sigma_u = 0$			$\sigma_u = 74,700 \text{ psi}$		
	m	σ_o , psi	σ^* , psi	m	σ_o , psi	σ^* , psi
Ln - Plot	7.3	125,000	107,500	3.6	46,700	106,200
McClintock's	8.9	127,000	113,500	4.3	50,700	111,100
Normalized Stress	7.9	122,000	110,500	3.9	48,300	108,600

Mean Strength = 145,455 psi

Lowest Observed Fracture Stress = 109,200 psi

Number of Samples = 12

Variations in the individual parameters m and σ_o , when calculated by the same method, for $\sigma_u = 0$ and $\sigma_u \neq 0$, are large. However, variations in calculated results, based on whole parameter groups (i.e., m , σ_o , and σ_u), are not extremely significant. This may be seen from the curves and Weibull parameters of Figure 10-4. Probability of failure curves, for two of the six parameter groups, are plotted on Figure 10-4. The two curves are based on parameters obtained from the ln-plot method with $\sigma_u = 0$ and McClintock's method with $\sigma_u = 74,700 \text{ psi}$. The experimental failure data are also presented. The probability curves based on the other four Weibull parameter groups, for the most part, fall within the band formed by the two curves shown. These curves approximate the actual data reasonably well.

The variation in the mean strength due to the volume effect can be calculated from the different Weibull parameter groups. Assume that a mean strength of 145,455 psi is obtained from a series of test specimens of volume V_1 with Weibull parameters $m = 8.9$, $\sigma_o = 127,000 \text{ psi}$, and $\sigma_u = 0$. The expected mean strength from a series of specimens, tested in a like manner, of volume $V_2 = 10 V_1$ can be calculated from equation (7-2) as

$$\frac{(\sigma_m)_2}{(\sigma_m)_1} = \left(\frac{V_1}{V_2} \right)^{1/m}$$

$$\text{or } (\sigma_m)_2 = \frac{145,455}{(10)^{1/8.9}} \quad 112,000 \text{ psi}$$

If the calculation is based on the Weibull parameters obtained from McClintock's method with $\sigma_u = 74,700$ psi, $m = 4.3$, and $\sigma_o = 50,700$ psi, the modified form of equation (7-2) gives

$$\frac{(\sigma_m - \sigma_u)_2}{(\sigma_m - \sigma_u)_1} = \left(\frac{V_1}{V_2}\right)^{1/m}$$

$$\text{or } (\sigma_m)_2 = \frac{145,455 - 74,700}{(10)^{1/4.3}} + 74,700$$

$$(\sigma_m)_2 = 116,000 \text{ psi}$$

and lies within the expected variation.

Also shown in Table 10-4 is a comparison of the predicted least likely strength values, σ^* . These values were calculated from Figure 8-5. All of the predicted least likely strengths compare reasonably well with the lowest observed fracture stress.

Figure 10-4 - Failure Probability Plot of Data and Theoretical Curves

11. APPLICATION TO DESIGN

The steps which could be followed in conducting the reliability analysis of a component are as follows:

- 1) Establish the distribution function for the tensile strength of the material to be used. If the Weibull parameters are not available, determine them as explained in Section 9.
- 2) Conduct a complete stress analysis showing the stress conditions throughout the volume of the component.
- 3) If the material is not known to be a pure series material, subdivide the component into a convenient number of small elements so that the volume of each element should not be smaller than the gage volume of the test specimen.
- 4) Calculate the maximum tensile stress in each volume. Assume that the stresses throughout this volume are uniform and equal to the maximum stress. If the component is subjected to biaxial stresses, determine the principal stresses. Disregard the compressive stresses which do not contribute to failure.
- 5) Determine the reliability of each volume by equation (5-3). The reliability, also defined as the probability of survival, S , is given by $1-P$, where P is the probability of failure defined by equation (5-3).

If there are two principal stresses present, the reliability of the element is obtained by multiplying the reliabilities calculated for the two principal stresses.

- 6) Find the entire reliability of the component by multiplying the reliabilities of all the individual elements.

Numerical Examples

Example (1)

In the acceptance tests of a bar component, the bending strength is determined by loading a specimen of 1.1 inch diameter. The load, W , is applied at the midpoint of the span which is 2.5 inch as shown in Figure 11-1. The Weibull parameters

Figure 11-1 Center-Point Loading of a Bar Specimen

obtained from the standard transverse rupture tests of the bar material are as follows:

$$m = 8.57, \sigma_0 = 131,100 \text{ psi}, \sigma_U = 0$$

Determine the probability of failure for a prescribed bending stress of 250,000 psi.

Solution:

The volume of the bar component is

$$V = \frac{\pi}{4} (1.1)^2 (2.5) = 2.3757 \text{ in.}^3$$

The K-factor for center-point loading of a circular bar corresponding to $m = 8.57$ is obtained from Figure 5-3 as $K = 0.0024$. Substituting the appropriate values into

equation (5-3) gives,

$$\begin{aligned}
 P &= 1 - e^{-KV \left(\frac{\sigma - \sigma_u}{\sigma_c} \right)^m} \\
 P &= 1 - e^{-(0.0024)(2.3757) \left(\frac{250,000}{131,000} \right)^{8.75}} \\
 P &= 1 - 0.27 = 0.73
 \end{aligned}$$

The reliability of the component is given by $S = 1 - P = 0.27$ and is relatively low.

Example (2)

The material constant and the average bending strength obtained from center-point loading, transverse rupture tests of grade K9 tungsten carbide steel are found to be

$$m = 7.5, \quad \sigma_B = 375,760 \text{ psi}, \quad \sigma_u = 0,$$

based on a gage volume of 0.0225 in.^3

Find the average tensile strength expected from a uniform tension test on a gage volume of 0.1531 in.^3 of the same grade of steel.

Solution:

The relationship between the mean tensile strength, σ_A , and the mean center-point bending strength, σ_B , is given by equation (7-1) as

$$\frac{\sigma_A}{\sigma_B} = \left(\frac{K_B V_B}{K_A V_A} \right)^{1/m} = \left(\frac{\left(\frac{1}{2(7.5+1)^2} \right) (0.0225)}{(1) (0.1531)} \right)^{1/7.5} = 0.339$$

where $K_B = \frac{1}{2(m+1)^2}$ and $K_A = 1$ as determined from Cases 3 and 1 of Section 5.1, respectively. The average tensile strength expected for this size test specimen is then

$$\sigma_A = (375,760) (0.339) = 127,446 \text{ psi}$$

Example (3)

A hollow disc of uniform thickness, Figure 11-2, rotates at 20,000 rpm. It is made of a material having Weibull parameters of

$$m = 7.3, \quad \sigma_o = 120,300 \text{ psi}, \quad \sigma_u = 0.$$

Figure 11-2 Rotating Hollow Disc of Uniform Thickness

Determine the reliability of the disc due to stresses produced by centrifugal forces.

Solution:

Step (1)

From equation (5-3),

$$S = e^{-KV((\sigma - \sigma_u)/\sigma_o)^m} = e^{-V(\sigma/120,300)^{7.3}}$$

Step (2)

Radial and tangential stresses are induced in the disc during rotation. Radial stresses are compressive and tangential stresses are tensile.

The tangential stress at a distance "r" from the center is given by

$$\sigma_t = \frac{\gamma v^2}{g} \cdot \frac{3+\mu}{8} \left(1 + \alpha^2 - \frac{1+\mu}{3+\mu} \cdot x^2 + \frac{\alpha^2}{x^2} \right)$$

where

- v = Circumferential velocity of disc (in./sec)
- γ = Specific weight of material (0.509 lb/in.³)
- g = Gravity constant (386.4 in./sec²)
- μ = Poisson's ratio (0.282)
- α = a/b = 1/4
- x = r/b = r/4

The circumferential velocity of the disc at 20,000 rpm is

$$v = r\omega = 4 \frac{(2\pi)}{60} (20,000) = 8377.6 \text{ in./sec}$$

Substituting these values into the stress equation gives

$$\sigma_t = 37,929 (1.0625 - 0.0976 r + 1/r^2)$$

The radial stresses are disregarded since they do not contribute to failure.

Step (3)

The disc is divided into six small rings with the dimensions and volumes shown in Table 11-1.

TABLE 11-1
ROTATING DISC RELIABILITY CALCULATIONS

Element No. n	r (in.)	V (cu. in.)	σ_t (psi)	$S = \text{EXP} \left[-V \left(\frac{\sigma_t}{120,300} \right)^{7.3} \right]$
1	1.0	3.9270	74,326	0.88771
2	1.5	5.4978	51,602	0.98868
3	2.0	7.0684	42,378	0.99651
4	2.5	8.6393	37,113	0.99840
5	3.0	10.2102	33,408	0.99910
6	3.5	11.7810	30,438	0.99950

Steps (4) and (5)

The maximum tangential stress in each element is at the inside radius. The values of the maximum stress and reliability of each element are given in columns 4 and 5, respectively, of Table 11-1.

Step (6)

The overall reliability of the disc is obtained by multiplying the reliabilities of the six elements. Hence,

$$S = 0.88771 \times 0.98868 \times 0.99651 \times 0.99846 \times 0.99910 = 0.87197 \approx 87.2\%$$

The probability of failure is

$$P = 1 - S = 1 - 0.87197 = 0.12803 \approx 12.8\%$$

Example (4)

The end force, W , acting on the cantilever bar of Figure 11-3 is 2800 lb. The bar is made of a tungsten carbide steel with the following Weibull parameters obtained from transverse rupture tests: $m = 7.5$, $\sigma_o = 116,000$ psi, and $\sigma_u = 0$.

Determine the reliability of the bar.

Solution:

Step (1)

Substituting the given data into equation (5-3) gives

$$S = e^{-KV \left(\frac{\sigma}{116,000} \right)^{7.5}}$$

Step (2)

Bending stresses are produced by the end force. The fiber stress at any point of a given section is calculated by the known formula,

$$\sigma = \frac{Mc}{I}$$

Step (3)

The bar is divided into 11 elements as shown in Figure 11-3(a).

Step (4) and (5)

The maximum tensile stress in the extreme fiber at Section A-A of the first element is as follows.

(a)

(b)

(c)

Figure 11-3 Cantilever Bar with End Load

$$\sigma_1' = \frac{Mc}{I} = \frac{32 P L D}{\pi (D^4 - d_1^4)}$$

$$\sigma_1' = \frac{(32)(2800)(11)(1.5)}{\pi ((1.5)^4 - (0.5)^4)} = 94,117 \text{ psi}$$

The stress at the extreme fiber of the hole is

$$\sigma_1'' = \frac{32 P L d_1}{\pi (D^4 - D_1^4)} = 94,117 \frac{(0.5)}{1.5} = 31,372 \text{ psi}$$

The stress distribution at Section A-A is illustrated in Figure 11-3(b).

We first assume that the element is solid and the bending stress at Section A-A is uniform throughout the element with a maximum value equal to $\sigma_1' = 94,117$ psi. The reliability of such a solid, circular element, under pure bending, can be calculated readily.

The volume of the element is $V_1' = \frac{\pi (1.5)^2 (1)}{4} = 1.767 \text{ in.}^3$. The loading factor, K , for pure bending of a circular beam, obtained from Figure 5-3 for $m = 7.5$, is $K = 0.03$. Substituting the values of σ_1' , K , and V_1' into the reliability equation of step (1) give

$$S_1' = e^{-\left[0.03 (1.767) \left(\frac{94,117}{116,000}\right)^{7.5}\right]} = 0.98898$$

Now, we assume that a fictitious element represents the hole; an element having a volume of $V_1'' = \pi (0.5)^2 (1)/4 = 0.196 \text{ in.}^3$ and a maximum bending stress equal to $\sigma_1'' = 31,372$ psi (see Figure 11-3(c)). The reliability of the first fictitious element is

$$S_1'' = e^{-\left[0.03 (0.196) \left(\frac{31,372}{116,000}\right)^{7.5}\right]} \approx 1.00$$

The actual reliability of the first element is obtained by

$$S_1 = \frac{S_1'}{S_1''} = \frac{0.98898}{1.00} = 0.98898$$

The reliabilities of the other elements which were calculated in the same way are presented in Table 11-2.

TABLE 11-2

CANTILEVER BAR RELIABILITY CALCULATIONS

Element No. n	σ_n' psi	σ_n'' psi	V_n' (Solid) cu. in.	V_n'' (Fict.) cu. in.	K	$S_n' = \exp -KV \left(\frac{\sigma}{116,000} \right)^{7.5}$	$S_n'' = \exp -KV \left(\frac{\sigma}{116,000} \right)^{7.5}$	$S_n = S_n' / S_n''$
1	94,117	31,372	1.75715	0.19635	0.03	0.98898	1.000	0.98898
2	85,561	28,520	1.75715	0.19635	0.03	0.99460	1.000	0.99460
3	77,005	25,668	1.75715	0.19635	0.03	0.99760	1.000	0.99760
4	68,449	22,816	1.75715	0.19635	0.03	0.99898	1.000	0.99898
5	59,893	19,964	1.75715	0.19635	0.03	0.99962	1.000	0.99962
6	51,337	29,775	1.75715	0.59447	0.03	0.99990	1.000	0.99990
7	42,781	24,830	1.75715	0.59447	0.03	1.000	1.000	1.000
8	34,224	19,853	1.75715	0.59447	0.03	1.000	1.000	1.000
9	25,668	14,887	1.75715	0.59447	0.03	1.000	1.000	1.000
10	17,112	9,925	1.75715	0.59447	0.03	1.000	1.000	1.000
11	8,556	4,965	1.75715	0.59447	0.03	1.000	1.000	1.000

11-9

Step (6)

The overall reliability of the bar is determined by multiplying together the reliabilities of all 11 elements. Hence,

$$\begin{aligned} S &= (0.98898) (0.99460) (0.99760) (0.99892) (0.99962) \\ &\quad (0.99990) (1) (1) (1) (1) (1) (1) \\ &= 0.9798 \approx 98\% \end{aligned}$$

The probability of failure is then

$$P = 1 - S = 1 - 0.9798 = 0.0202 \approx 2\%$$

Example (5)

A rectangular beam with a shoulder fillet is loaded as shown in Figure 11-4. The Weibull parameters for this particular grade of steel are:

$$m = 7.5, \sigma_o = 132,000 \text{ psi}, \sigma_u = 0.$$

Determine the reliability of the bar.

Solution:

Step (1)

From the Weibull data and equation (5-3),

$$S = e^{-KV \left(\frac{\sigma}{132,000} \right)^{7.5}}$$

Step (2)

Tensile and bending stresses are produced by forces P and F, respectively. The combined stresses at any section are given by

$$\sigma = \sigma_A \pm \sigma_B = \frac{P}{A} \pm \frac{Mc}{I}$$

Step (3)

Due to symmetry, only one side of the beam is divided into seven elements with 1 inch lengths and numbered from 1 to 7 as shown in Figure 11-4(a).

(a)

Figure 11-4 Rectangular Beam with Shoulder Fillet

Steps (4) and (5)

The reliability of the first element

The maximum tensile stress on the lower extreme fiber is

$$\begin{aligned}\sigma_1' &= \frac{P}{A} + \frac{Mc}{I} = \frac{P}{Hb} + \frac{(F/2)(c)}{bH^2/6} \\ &= \frac{18,000}{(1)(0.5)} + \frac{400/2(6)(6)}{(0.5)(1)^2} \\ &= 50,400 \text{ psi}\end{aligned}$$

The minimum tensile stress on the upper extreme fiber is

$$\sigma_1'' = \frac{P}{A} - \frac{Mc}{I} = 36,000 - 14,400 = 21,600 \text{ psi}$$

The stress distribution at Section A-A is illustrated in Figure 11-4(b).

The reliability of this element may be calculated by assuming a fictitious volume where the stress distribution varies linearly from zero to σ_1' .

The volume of the entire fictitious element is

$$V_1' = (1.75)(1)(1) = 0.875 \text{ in.}^3$$

The loading factor, K, for this case, taken from Case 7 of Section 5.1, is

$$K = \frac{1}{m+1} = 1/8.5 = 0.1176$$

The reliability of the entire fictitious element is

$$\begin{aligned}S_1' &= e^{-0.1176(0.875) \left(\frac{50,400}{132,000}\right)^{7.5}} \\ &= 0.99986\end{aligned}$$

The volume of the fictitious portion only is

$$V_1'' = (1.750 - 1)(0.5)(1.0) = 0.375 \text{ in.}^3$$

The maximum stress in this portion is equal to the actual minimum stress in the first element. Using the same loading factor, the reliability of the fictitious portion becomes

$$S_1'' = e^{-0.1176 (0.375) \left(\frac{21,600}{132,000}\right)^{7.5}} \approx 1.00$$

The reliability of the first element becomes

$$S_1 = \frac{S_1'}{S_1''} = \frac{0.99986}{1.00} = 0.99986$$

The reliability of the second and third elements can be calculated in the same way. The results of these calculations are given in Table 11-3.

The reliability of the fourth element

The maximum tensile stresses in the actual and fictitious portions of the fourth element are, respectively,

$$\sigma_4' = \frac{18,000}{(0.5)(0.6)} + \frac{400/2 (3)(6)}{(0.5)(0.6)^2}$$

$$\text{or } \sigma_4' = 80,000 \text{ psi}$$

$$\text{and } \sigma_4'' = 60,000 - 20,000 = 40,000 \text{ psi}$$

Due to the fillet, the stress concentration must be taken into account. For $r/h = 0.15/0.6 = 0.25$ and $H/h = 1/0.6 = 1.6$, the stress concentration factor ^[17] is $K_t = 1.65$. The maximum and minimum stresses are therefore

$$(\sigma_4')_{\max} = (80,000)(1.65) = 132,000 \text{ psi}$$

$$(\sigma_4'')_{\max} = (40,000)(1.65) = 66,000 \text{ psi}$$

TABLE 11-3

RECTANGULAR BEAM RELIABILITY CALCULATIONS

Element No. <u>n</u>	σ_A <u>psi</u>	σ_B <u>psi</u>	$\sigma_{\max},$ $(\sigma_A + \sigma_B)$ <u>psi</u>	$\sigma_{\min},$ $(\sigma_A + \sigma_B)$ <u>psi</u>	V_n' Entire Fict. <u>cu. in.</u>	V_n'' Fict. Portion <u>cu. in.</u>	<u>K</u>
1	36,000	14,400	50,400	21,600	0.875	0.375	0.1176
2	36,000	12,000	48,000	24,000	1.000	0.500	0.1176
3	36,000	9,600	45,600	26,400	1.1875	0.6875	0.1176
4	60,000	20,000	*132,000	*66,000	0.600	0.300	0.1176
5	60,000	13,333	73,333	46,667	0.825	0.525	0.1176
6	60,000	6,666	66,666	53,334	1.500	1.200	0.1176
7	60,000	0	60,000	60,000	0.300	0	1.000

Element No. <u>n</u>	$S_n' = \exp \left[-KV \left(\frac{\sigma_{\max}}{132,000} \right)^{7.5} \right]$	$S_n'' = \exp \left[-KV \left(\frac{\sigma_{\max}}{132,000} \right)^{7.5} \right]$	$S = \frac{S_n'}{S_n''}$
1	0.99986	1.000	0.99986
2	0.99962	1.000	0.99962
3	0.99964	1.000	0.99964
4	0.93192	0.99982	0.93209
5	0.99888	0.99996	0.99886
6	0.99894	0.99984	0.99910
7	0.99920	1.000	0.99920

(*) These stresses are obtained by including the stress concentration factor, $K_t = 1.65$.

The volume of the entire fictitious element, calculated from the geometry and the stress distribution, is

$$V_4' = 0.60 \text{ in.}^3$$

The reliability is therefore

$$\begin{aligned}
 S_4' &= e^{-(0.1176)(0.600) \left(\frac{132,000}{132,000} \right)^{7.5}} \\
 &= 0.93192
 \end{aligned}$$

The volume of the fictitious portion only is $V_4'' = 0.300 \text{ in.}^3$ and the corresponding reliability is

$$\begin{aligned}
 S_4'' &= e^{-0.1176 (0.300) \left(\frac{66,000}{132,000} \right)^{7.5}} \\
 &= 0.99982
 \end{aligned}$$

The reliability of the actual fourth element is

$$\begin{aligned}
 S_4 &= \frac{S_4'}{S_4''} = \frac{0.93192}{0.99982} \\
 &= 0.93209
 \end{aligned}$$

The reliability of the fifth and sixth elements calculated in the same way are given in Table 11-3.

The reliability of the seventh element

This element is essentially only under tension since $\sigma_7''/\sigma_7' > 0.9$ and the loading factor, K , is 1.00. The reliability is thus

$$\begin{aligned}
 S_7 &= e^{-(0.3) \left(\frac{60,000}{132,000} \right)^{7.5}} \\
 &= 0.9992
 \end{aligned}$$

Step (6)

The reliability of one side of the bar is obtained by multiplying the reliabilities of the seven elements together. Hence,

$$\begin{aligned}
 S \text{ (one side)} &= (0.99986) (0.99962) \\
 &\quad (0.99964) (0.93209) (0.99886) \\
 &\quad (0.99910) (0.99920) = 0.92862
 \end{aligned}$$

The total reliability of the bar is

$$\begin{aligned}
 S &= [S \text{ (one side)}]^2 = (0.92862)^2 \\
 S &= 0.86236
 \end{aligned}$$

The probability of failure is then $1 - 0.86236 = 0.13764$ or ≈ 13.76 percent.

Example (6)

A solid shaft is to carry a load W as shown in Figure 11-5. The shaft material is tungsten carbide having the following Weibull parameters:

$$m = 8.8, \sigma_o = 156,200 \text{ psi}, \sigma_u = 0$$

Determine the maximum load W for a reliability of 0.9999.

Figure 11-5 Solid Shaft With Mid-Point Load

Solution:

The volume of the shaft subjected to the bending stress is

$$V = \frac{\pi D^2 L}{4} = \frac{\pi (2)^2 (20)}{4} = 62.832 \text{ in.}^3$$

The loading factor, K , for center-point loading of a circular shaft and $m = 8.8$ is, from Figure 5-3, $K = 0.0023$.

Substituting the values of m , σ_o , σ_u , V , and k into equation (5-3) gives

$$\begin{aligned} S &= e^{-0.0023 (62.832) (\sigma/156,200)^{8.8}} \\ &= e^{-0.1445 (\sigma/156,200)^{8.8}} \end{aligned}$$

For a reliability of 0.9999, the equation becomes, after taking logarithms,

$$\log (0.9999) = -0.1445 (\sigma/156,200)^{8.8} \log e$$

Since $\log (0.9999) = -0.00004$ and $\log e = 0.4343$,

$$(\sigma/156,200)^{8.8} = 0.0006374$$

Taking the logarithm again and solving for σ , gives

$$8.8 (\log \sigma - \log 156,200) = \log 0.0006374$$

$$\text{or } \log \sigma = -\frac{3.19559}{8.8} + 5.19363 = 4.83054$$

$$\text{or } \sigma = 67,693 \text{ psi}$$

For a reliability of 0.9999, the maximum bending stress in the shaft should not, therefore, exceed 67,693 psi. Since

$$\sigma_{\max} = \frac{M_c}{I} = \frac{\left(\frac{W}{2}\right) \left(\frac{L}{2}\right)}{\pi D^3/32} = \frac{W (5)}{0.7854} = 67,693 \text{ psi}$$

the maximum value of load is found to be

$$W = \frac{0.7854 (67,693)}{5} = 10,633 \text{ lb.}$$

Example (7)

Consider the problem of calculating the reliability of the circular plate shown in Figure 11-6. For the purpose of this example, the tensile strength distribution is assumed to be of the Weibull type with parameters:

$$m = 3.0, \sigma_o = 5,610 \text{ psi, and } \sigma_u = 4,130 \text{ psi.}$$

The test specimen gage volume from which these parameters were obtained was 0.125 in.^3 .

Figure 11-6 Circular Plate with Internal and External Radial Stresses

Solution:

Assuming a linearly elastic, homogeneous and isotropic material, the radial and circumferential stresses are found from elasticity theory to be

$$\sigma_r = S_i \left[\frac{r_o^2 - r^2}{r_o^2 - r_i^2} \right] \left(\frac{r_i}{r} \right)^2 + S_o \left[\frac{r^2 - r_i^2}{r_o^2 - r_i^2} \right] \left(\frac{r_o}{r} \right)^2$$

$$\sigma_\theta = -S_i \left[\frac{r_o^2 + r^2}{r_o^2 - r_i^2} \right] \left(\frac{r_i}{r} \right)^2 + S_o \left[\frac{r^2 + r_i^2}{r_o^2 - r_i^2} \right] \left(\frac{r_o}{r} \right)^2$$

For	S_o	=	4,956 psi
	S_i	=	4,460 psi
	r_o	=	4.0 in.
	r_i	=	1.0 in.
	h	=	1.0 in.

these expressions reduce to

$$\sigma_r = 4.99 - \frac{0.495}{r^2}$$

$$\sigma_\theta = 4.99 + \frac{0.495}{r^2}$$

The plate is divided into five concentric rings with the dimensions indicated in columns 2 and 3 of Table 11-4. Note that the volume of each ring is greater than the gage volume of 0.125 in.³. The "worst" risk of rupture is computed by using the maximum stresses in the subvolume. These maximum stresses, calculated from the equations above, are listed in columns 5 and 6 of Table 11-4.

TABLE 11-4

CIRCULAR PLATE RELIABILITY CALCULATIONS

Ring No.	Inside Radius	Outside Radius	Volume	Maximum Radial Stress (σ_r)	Maximum Circumferential Stress (σ_θ)	Radial "Risk of Rupture"	Circumferential "Risk of Rupture"	Ring Reliability
n	a_n	b_n	V_n	(σ_{r_n})	(σ_{θ_n})	(R_r) n	(R_θ) n	S_n
	(in)	(in)	(in ³)	(ksi)	(ksi)			
1	1.0	1.5	3.927	4.76	5.48	0.00556	0.05472	0.94148
2	1.5	2.0	5.498	4.86	5.21	0.01211	0.03923	0.94996
3	2.0	2.5	7.069	4.91	5.11	0.02045	0.04056	0.94090
4	2.5	3.0	8.639	4.94	5.06	0.02600	0.03936	0.93674
5	3.0	4.0	21.991	4.95	5.04	0.06868	0.09386	0.85015

The risk of rupture is calculated for each subvolume, n , for both the radial and tangential stresses according to equation (5-5),

$$(R_r)_n = KV_n \left(\frac{(\sigma_r)_n - \sigma_u}{\sigma_o} \right)^m$$

$$(R_\theta)_n = KV_n \left(\frac{(\sigma_\theta)_n - \sigma_u}{\sigma_o} \right)^m$$

Since uniform tension is assumed throughout $K = 1$. The calculated risk of rupture values are listed in columns 7 and 8 of Table 11-4. The reliability of each ring is obtained from equation (5-4), generalized to include the biaxial stress state, as

$$S_n = e^{-(R_r + R_\theta)_n}$$

and listed in column 9.

The overall reliability of the plate is given by the product of all the subvolume reliabilities as

$$S = (0.94148) (0.94996) (0.94090)$$

$$(0.93674) (0.85015)$$

$$S = 0.670$$

If the material in the plate is known to be a pure series material, the restriction that the subvolumes must be greater than the test specimen volume may be dropped. This permits the use of infinitesimal rings which lead to a reliability prediction ^[11] of 0.696. Thus, in this example, the partitioning of the plate into only five volumes results in a fair estimate of the total reliability.

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