Analysis of the Decapole Systematic Error in the Dipoles and of the Correctors

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The equations of motion in the presence of a decapole term are

\[ x'' + K_4 x = \frac{b_4}{p} (x^4 - 6x^2y^2 + y^4) \quad (1a) \]

\[ y'' + K_4 y = -4 \frac{b_4}{p} (x^3y - xy^3) \quad (1b) \]

Where \( p \) is the bending radius in the dipoles and \( b_4 \) is the strength of the decapole term.

We can calculate the first-order contribution to the turn-shift \( x \) due to axial angle-dipole and off-momentum value, by separating the free detonation oscillations from the closed-orbit dynamics.

\[ x = \eta \delta + \tilde{x}, \quad y = \tilde{y} \]

\( \tilde{x}, \tilde{y} \) free detonation oscillations

\( \eta, \delta \), dispersion (only in the horizontal plane)

This term in \( y^4 \) (in \( C_2 \)) cannot be ignored since it does not give the first-order contribution to the turn-shift. Taking the orbit from (1a & 1b) the equations for the closed orbit...
\[ \dddot{x} + \kappa \ddot{x} = -\frac{2q}{p} \left( \ddot{x}^4 + 4\eta \dot{x}^3 + 6\eta^2 \dot{x}^2 \dddot{x} + \right. \\
\left. + 4\eta^3 \dot{x}^2 - 6\dot{x}^2 y^2 - 18\eta \dot{x} \dot{y} \ddot{y}^2 - 6\eta^2 \ddot{y}^2 \right) \quad (2\eta) \]

\[ \dddot{y} + \kappa \ddot{y} = -\frac{2q}{p} \left( \ddot{x}^2 y + 3\eta \dot{x} \dddot{y} + 3\eta^2 \dot{y}^2 + \right. \\
\left. + \eta^3 \dot{y} - \dddot{x} y - \eta \ddot{y} \right) \quad (2\eta) \]

These two shifts are

\[ \Delta Q_m = -\frac{1}{4\pi p} \int_{\mathcal{H}} \left( \dot{x}^3 + 4\eta \dot{x} \ddot{x}^2 + 6\eta^2 \dot{x} \dddot{x} + 4\eta^3 \dddot{x} + \right. \\
\left. - 6\dot{x}^2 y^2 - (2\eta \dot{y} \ddot{y}) \right) \right] \quad (2\eta) \]

where we have ignored the last two terms at the r.h. side of \( \Delta Q_m \) since they do not give contribution to first order.

\[ \Delta Q_i = \frac{1}{4\pi p} \int_{\mathcal{H}} \left( \dot{x}^3 + 3\eta \dot{x} \ddot{x}^2 + 3\eta^2 \dot{x} \dddot{x} + \right. \\
\left. + \eta^3 \dot{y} - \dddot{x} y - \eta \ddot{y} \right) \right] \quad (2\eta) \]

where the integrals are taken over the shell circumference of the ring.
To first-order, the tune shifts are averaged over several betatron oscillations, thus the terms in $\tilde{x}^2$ and $\tilde{\psi}^2$ in eq. (3a) give in average zero contribution. Similarly.

Again in first-order, we can set

$$\tilde{x}^2 = \frac{\varepsilon_H}{\beta} \cos^2 \psi_H$$

$$\tilde{y}^2 = \frac{\varepsilon_V}{\eta} \beta_V \cos^2 \psi_V$$

Finally

$$\Delta Q_H = - \frac{\delta^3}{\pi \rho} \int \frac{1}{4} \beta_H \eta^3 ds +$$

$$- \frac{\delta \varepsilon_H}{\pi \rho} \int \frac{1}{4} \beta_H \eta^2 \cos^2 \psi_H ds +$$

$$+ \frac{3 \delta \varepsilon_V}{\pi \rho} \int \frac{1}{4} \beta_H \beta_V \eta \cos^2 \psi_V ds$$

and
\[ \Delta Q_v = \frac{\delta^3}{\pi^3} \int b_4 \beta \gamma \eta \, ds + \]

\[ = \frac{\delta \bar{c}n}{\pi} \int b_4 \beta v \gamma \cos^2 \psi_v \, ds + \]

\[ + \frac{3 \delta \bar{c}n}{\pi} \int b_4 \beta \eta \gamma \cos^2 \psi_n \, ds \] (4.1)

The term in $\delta^3$ has an equivalent in the second-order contribution from the sextupoles. By properly arranging the sextupoles in femtorigies, this term should be compensated for by the sextupole strength alone (caution: to be done, though). We will neglect these for the term in $\delta^3$ in both eqs (4a and b). For the remaining terms, it is sufficient to take

\[ \langle \cos^2 \psi_v \rangle = \langle \cos^2 \psi_n \rangle = \frac{1}{2} \]

and

\[ \Delta Q_{3n} = -\frac{\delta \bar{c}n}{2\pi} \int b_4 \beta n \gamma \, ds + \frac{3 \delta \bar{c}n}{2\pi} \int b_4 \beta n \beta n \gamma \, ds \] (5a)
\[ \Delta Q_V = - \frac{\delta \varepsilon_V}{2\pi p} \int b_4 \beta_V \eta \, ds + \frac{3 \delta \varepsilon_H}{2\pi p} \int b_4 \beta_H \eta \, ds \quad (5) \]

A. Contribution from dipole error (systematic)

\[ b_4 = -4.7 \times 10^{-4} \text{ m}^4 \]

\[ \begin{cases} 
\Delta Q_H &= - b_4 \langle \eta \beta_H^2 \rangle \delta \varepsilon_H + 3 b_4 \langle \eta \beta_H \beta_V \rangle \delta \varepsilon_V \\
\Delta Q_V &= 3 b_4 \langle \eta \beta_H \beta_V \rangle \delta \varepsilon_H - b_4 \langle \eta \beta_V^2 \rangle \delta \varepsilon_V 
\end{cases} \quad (6) \]

where \( \langle \cdots \rangle \) is the average value over a single dipole

\[ \langle \eta \beta_H^2 \rangle = 862.1 \text{ m}^2 \]

\[ \langle \eta \beta_V^2 \rangle = 682.6 \text{ m}^2 \]

\[ \langle \eta \beta_H \beta_V \rangle = 505.3 \text{ m}^3 \]
<table>
<thead>
<tr>
<th></th>
<th>$\beta_H$</th>
<th>$\beta_N$</th>
<th>$\mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>first end dipole</td>
<td>38.3 m</td>
<td>11.8 m</td>
<td>1.31 m</td>
</tr>
<tr>
<td>middle dipole</td>
<td>23.0</td>
<td>23.0</td>
<td>1.07</td>
</tr>
<tr>
<td>second end</td>
<td>11.1</td>
<td>40.3</td>
<td>0.80</td>
</tr>
<tr>
<td>diametrical DF</td>
<td>45.0</td>
<td>10.0</td>
<td>1.45</td>
</tr>
<tr>
<td>diametrical DD</td>
<td>10.0</td>
<td>45.0</td>
<td>0.80</td>
</tr>
</tbody>
</table>

Beam parameters for RHIC CD

Gold - 100 GeV

After 10 hours with $1.1 \times 10^9$

$\phi_N = 30 \, \mu\text{m.mrad}$ (95% of beam, $H$ and $V$)

For a $60_{H,V}$ good-field criterion

$\phi_H = \phi_V = 1.8 \, \mu\text{m.mrad}$

Also the rf-bucket size with design rf system

$\delta = \pm 0.27\%$
\[ \delta \varepsilon_H \]
\[ \delta \varepsilon_V \]

\[ \Delta Q_H = -0.0158 \quad 0.0278 \]
\[ \Delta Q_V = 0.0278 \quad -0.0125 \]

There is clearly a need for correction.

Two families. (1) before \( Q_F \) (2) before \( Q_D \).

Two-shift contribution from correctors.

\[
\Delta Q_H = -\frac{ML}{2\pi \rho} \left( b_F \langle \beta_H^2 \eta \rangle_F + b_D \langle \beta_H^2 \eta \rangle_D \right) \delta \varepsilon_H +
\]
\[
+ \frac{3ML}{2\pi \rho} \left( b_F \langle \beta_H \beta_{\nu \eta} \rangle_F + b_D \langle \beta_H \beta_{\nu \eta} \rangle_D \right) \delta \varepsilon_V \tag{7a}
\]

\[
\Delta Q_D = \frac{3ML}{2\pi \rho} \left( b_F \langle \beta_H \beta_{\nu \eta} \rangle_F + b_D \langle \beta_H \beta_{\nu \eta} \rangle_D \right) \delta \varepsilon_H +
\]
\[
- \frac{ML}{2\pi \rho} \left( b_F \langle \beta_{\nu \eta}^2 \rangle_F + b_D \langle \beta_{\nu \eta}^2 \rangle_D \right) \delta \varepsilon_V \tag{7b}
\]
There are four terms to be cancelled with only two parameters ($b_4^F$ and $b_4^D$). For $\Delta Q_H$

$$-\frac{3ML}{2\pi \rho} \left( b_4^F <\beta_H^2 >_F + b_4^D <\beta_H^2 >_D \right) = b_4 <\eta \beta_H^2 > \quad (8)$$

$$\frac{3ML}{2\pi \rho} \left( b_4^F <\beta_H \beta_{\nu} >_F + b_4^D <\beta_H \beta_{\nu} >_D \right) = -3 b_4 <\eta \beta_H \beta_{\nu} > \quad (9)$$

And for $\Delta Q_{\nu}$

$$\frac{3ML}{2\pi \rho} \left( b_4^F <\beta_H \beta_{\nu} >_F + b_4^D <\beta_H \beta_{\nu} >_D \right) = -3 b_4 <\eta \beta_H \beta_{\nu} > \quad (10)$$

$$-\frac{ML}{2\pi \rho} \left( b_4^F <\beta_{\nu}^2 >_F + b_4^D <\beta_{\nu}^2 >_D \right) = b_4 <\eta \beta_{\nu}^2 > \quad (11)$$

Number of corrections per family

Length of each correction

Observe that condition (9) and (10) are identical. Therefore there are really only three conditions to be satisfied with only two parameters. I chose only (8) and (9).
\[ \langle \beta^2 \rangle_F = 2936 \, \text{m}^3 \]
\[ \langle \beta_H^2 \eta \rangle_D = 80 \]
\[ \langle \beta_H \beta_H \eta \rangle_F = 652.5 \]
\[ \langle \beta_H \beta_H \eta \rangle_D = 360 \]
\[ \langle \beta_V^2 \rangle_F = 145 \]
\[ \langle \beta_V^2 \rangle_D = 1620 \]

\[ \left\{ \begin{array}{l}
2936 \left( \frac{ML}{2 \pi \rho} \frac{b_{4E}}{b_4} \right) + 80 \left( \frac{ML}{2 \pi \rho} \frac{b_{4D}}{b_4} \right) = -852.1 \\
652.5 \left( \frac{ML}{2 \pi \rho} \frac{b_{4E}}{b_4} \right) + 360 \left( \frac{ML}{2 \pi \rho} \frac{b_{4D}}{b_4} \right) = -505.3
\end{array} \right. \]

\[ M = 144 \]
\[ L = 0.5 \, \text{m} \]
\[ R = 244 \, \text{m} \]
The required corrector strengths are

\[
\begin{align*}
\beta_4^F &= -5.7 \times 10^{-4} / \text{in}^4 \\
\beta_4^b &= -19.5 \times 10^{-4} / \text{in}^4 \\
\end{align*}
\]

There is though a residual tune shift given by

\[
\Delta Q_0 = -\frac{ML}{2\pi p} \left( \beta_4^F \langle \beta^2 \rangle_p + \beta_4^b \langle \beta^2 \rangle_d \right) \delta \varepsilon_v
\]

+ dipole contribution (-0.0125)

\[
= 0.0075 \quad \text{(too large ?!)}
\]

Instead of (8) and (9) make use of (10) and (11)

\[
\begin{align*}
&6.525 \left( \frac{ML}{2\pi p} \frac{\beta_4^F}{b} \right) + 360 \left( \frac{ML}{2\pi p} \frac{\beta_4^b}{b_4} \right) = -505.2 \\
&145. \left( \frac{ML}{2\pi p} \frac{b_4^b}{b} \right) + 1520 \left( \frac{ML}{2\pi p} \frac{b_4^b}{b_4} \right) = -682.6
\end{align*}
\]

From these the required corrector strengths are
\[
\begin{align*}
\beta_4p &= -12.1 \quad \beta_4 q &= -56.9 \times 10^{-4} / \text{in}^4 \\
\beta_4 q &= -7.9 \quad \beta_4 q &= -37.1 \times 10^{-4} / \text{in}^4
\end{align*}
\]

The residual now is

\[
\Delta Q_H = -\frac{ML}{2\pi p} \left( \beta_4 p \langle \beta_H^2 \rangle_F + \beta_4 q \langle \beta_H^2 \rangle_D \right) \eta \varepsilon_H + \text{dipole contribution} (-0.058)
\]

\[
= 0.0155 \text{ (too large)}
\]

It seems the only solution is to shim the dipoles to eliminate the error. Unless one wants to make use of correctors also in the invariants as a KSD family.