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SEAL-SHELL-2 - A COMPUTER PROGRAM FOR THE STRESS ANALYSIS
OF A THICK SHELL OF REVOLUTION WITH AXISYMMETRIC
PRESSURES, TEMPERATURES, AND DISTRIBUTED LOADS

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SEAL-SHELL-2, an ALTAC-3 computer program in FORTRAN-2 language for a 32,000 word machine, determines stresses and deflections in a thick or thin shell of revolution. The shell is linear-elastic with tensile, bending, and shear strains. Pressures, temperatures, and distributed loads are axisymmetric.

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OF A THICK SHELL OF REVOLUTION WITH AXISYMMETRIC
PRESSURES, TEMPERATURES, AND DISTRIBUTED LOADS

C. M. Friedrich

I. INTRODUCTION

SEAL-SHELL-2 is an improved and extended version of SEAL-SHELL-1 (Ref 1), a digital computer program which determines stresses, loads, and deflections in a thin shell of revolution with axisymmetric end loads and uniform pressure. The shell is divided into 2 to 100 straight segments of uniform thickness, and there may be middle surface shifts between flexible segments.

SEAL-SHELL-2 contains the following improved features:

1. Each segment may have different elastic properties.
2. Each segment may have different pressures.
3. Each segment may have a curved middle surface, or may be straight as in SEAL-SHELL-1, which permits each segment to be part of a flat plate, cone, or cylinder.
4. Each segment may have a temperature distribution varying arbitrarily through the thickness but varying linearly along the segment.
5. Each segment may vary linearly with thickness.
6. Each segment may have a radial load, an axial load, and a bending moment applied externally.
7. Thick shell geometry is used in the stress analysis.

In the SEAL-SHELL-2 program-2 program, an arbitrary set of 11 deformation shapes are applied to each segment and the resulting strain energy is determined under the restrictions: plane cross sections remain plane; and the shear stress varies parabolically through the thickness. By application of the principle of virtual work to the strain energy of each segment, the forces on each segment are determined in terms of the deflections. Then the simultaneous equations are solved to give the influence coefficients for deflections in terms of forces.

The middle surface slope and deflection at all joints of adjacent segments are equated in order to calculate the forces connecting all the segments. Once the forces and deflections are known for each segment, the strain energy relationships already derived give the local stresses and strains.

The output of SEAL-SHELL-2 lists the same variables as does SEAL-SHELL-1, namely:

1. Input
2. Flexibility matrix of end loads
3. Forces at joints
4. Deflections at joints
5. Local stresses in segments:
 - a. Shear stress, τ_{xy}
 - b. Longitudinal stress S_x
 - c. Hoop stress, S_θ
 - d. Thickness stress, S_y
 - e. Generalized Von-Mises stress
 - f. Twice maximum shear stress

Some applications of SEAL-SHELL-2 are:

1. Pressure vessel design
2. Nozzle design
3. Bellows
4. Seal weld
5. Bellmouthing of pipes including nonuniform distribution of thread loads
6. Thermal stresses in any of the preceding

II. NOTATION

- a $\left\{ \begin{array}{l} \text{Position } K-1 \text{ on segment } K \\ \text{Position } K_a \text{ on group of segments } K_a \text{ to } K_b \end{array} \right.$
- a Value of at position a
- ab Change in value of from a to b: $(\text{ }ab) = (\text{ }b) - (\text{ }a)$
- $(A_{gt})_i$ Thermal deflection of $(D_g)_i$ in Eq (41)
- $(A_{gg})_{i,j}$ Flexibility coefficient of $(D_g)_i$ vs $(F_g)_j$ in Eq (41)
- $(A_{kt})_i$ Thermal deflection of $(D_k)_i$ in Eq (49)
- $(A_{kk})_{i,j}$ Flexibility coefficient of $(D_k)_i$ vs $(F_k)_j$ in Eq (49)

- b $\left\{ \begin{array}{l} \text{Position K on segment K} \\ \text{Position } K_b \text{ on group of segments } K_a \text{ to } K_b \end{array} \right.$
- __b Value of __ at position b
- $(B_{gt})_i$ Thermal force of $(F_g)_i$ in Eq (40)
- $(B_{gg})_{i,j}$ Stiffness coefficient of $(F_g)_i$ vs $(D_g)_j$ in Eq (40)
- __c Value of __ at center of ellipse
- $(C_{eg})_{i,j}$ Coefficient of ϵ_x or ϵ_θ vs $(D_g)_j$ in Eq (30)
- $(C'_{eg})_{i,j}$ Coefficient of ϵ_x , ϵ_θ , or ϵ_y vs $(D_g)_j$ in Eq (34)
- $(C_{ek})_i$ Coefficient of γ_{xy} vs $(F_k)_i$ in Eq (56)
- $(C'_{et})_i$ Thermal strain of ϵ_x , ϵ_θ , or ϵ_y in Eq (34)
- $(C_{ev})_{i,j}$ Coefficient of ϵ_x or ϵ_θ vs V, W, or U in Eq (17)
- $(C_{ev})_{3,3}$ Coefficient of ϵ_y vs U in Eq (32)
- $(C_{k-1,k})_{3,j}$ Coefficient of $(F_{k-1})_3$ vs $(F_k)_j$ in Eq (42)
- $(C_{kg})_{i,j} \left\{ \begin{array}{l} \text{Coefficient of } (D_k)_i \text{ vs } (D_g)_j \text{ in Eq (47)} \\ \text{Coefficient of } (F_g)_j \text{ vs } (F_k)_i \text{ in Eq (49)} \end{array} \right.$
- $(C_{kv})_{i,j}$ Coefficient of $(D_k)_i$ vs V, W, or U in Eq (46)
- $(C_{qk})_i$ Coefficient of Q_x vs $(F_k)_i$ in Eq (52)
- $(C'_{st})_i$ Thermal stress of S_x , S_θ , or S_y in Eq (35)
- $(C'_{sg})_{i,j}$ Stiffness coefficient of S_x , S_θ , or S_y vs $(D_g)_j$ in Eq (35)
- $(C_{tk})_i$ Coefficient of τ_{xy} vs $(F_k)_i$ in Eq (54)
- C_{ut} Thermal deflection of U in Eq (33)
- $(C_{ug})_i$ Coefficient of U vs $(D_g)_i$ in Eq (33)
- d__ An infinitesimal change in __ produced by the coordinate changes dX , $d\theta$ and/or dY
- $(D_g)_i$ An arbitrary deformation shape on a segment, $i = 1$ to 11
- $(D_k)_i$ Working deflection of $(F_k)_i$, $i = 1$ to 7 and $k = 1$ to N
- dL_x Length of volume element $dVol$ produced by dX in Fig. 3

dL_y	Length of volume element $dVol$ produced by dY in Fig. 3
dL_θ	Length of volume element $dVol$ produced by $d\theta$ in Fig. 3
D_{mo}	Rotation at 0, + . $D_{mo} = (D_1)_5$ in Fig. 8
D_{mn}	Rotation at n, + . $D_{mn} = (D_n)_7$ in Fig. 8
D_{ro}	Radial deflection at 0, + . $D_{ro} = (D_1)_4$ in Fig. 8
D_{rn}	Radial deflection at n, + . $D_{rn} = (D_n)_6$ in Fig. 8
$dVol$	Volume element produced by dX , $d\theta$, and dY
D_{zn}	Axial deflection of shell, . $D_{zn} = \sum_{k=1}^n k (D_k)_3$ in Fig. 8
E_{ex}	External work
E_{in}	Internal strain energy
E_s	Shear modulus of elasticity
E_t	Tensile modulus of elasticity
E_t'	Modified tensile modulus in Eq (28)
$(F_g)_i$	Generalized force corresponding to the deformation $(D_g)_i$
$(F_k)_i$	Force on segment K, $i = 1$ to 7 and $K = 1$ to N. See Sec. II.F in Theory.
F_{mo}	Bending moment at 0, in.-lb. + . $F_{mo} = (F_1)_5$
F_{mn}	Bending moment at n, in.-lb. + . $F_{mn} = (F_n)_7$
$(F_m)_k$	Bending moment connecting segments K and K+1 at position K $(F_m)_k = (F_k)_7 = (F_{k+1})_5$
F_{ro}	Radial force at 0, lb. + . $F_{ro} = (F_1)_4$
F_{rn}	Radial force at n, lb. + . $F_{rn} = (F_n)_6$
$(F_r)_k$	Radial force connecting segments K and K+1 at position K $(F_r)_k = (F_k)_6 = (F_{k+1})_4$
F_x	Geometric factor for dL_x vs dX in Eq (3)
F_{zn}	Axial force at n, lb, + . $F_{zn} = (F_n)_3$
$(F_z)_k$	Axial force connecting segments K and K+1 at position K $(F_z)_k = (F_k)_3$

F_{θ}	Geometric factor for dL_{θ} vs $Rd\theta$ in Eq (3)
$G_{\underline{\quad}}$	$\begin{cases} \text{Value of } F_{\underline{\quad}} \text{ if } K_{\underline{\quad}} = 1 \\ \text{Value of } D_{\underline{\quad}} \text{ if } K_{\underline{\quad}} = 0 \end{cases}$
H	Thickness, in.
i	Index
$\underline{\quad}_i$	Value of $\underline{\quad}$ at the inside surface $Y = + H/2$
$I_u(Y)$	Integrating factor for differential equation (33)
j	Index
K	Position along X axis, 0 to n
K	Segment along X axis, 1 to n Segment K is between positions K-1 and K Position K is between segments K and K+1
$\underline{\quad}_k$	Value of $\underline{\quad}$ at position K or on segment K
m	Index
N	Number of segments, 2 to 100. Last position
$\underline{\quad}_n$	Value of $\underline{\quad}$ at position N or on segment N
$\underline{\quad}_o$	Value of $\underline{\quad}$ at the outside surface $Y = - H/2$
P_i	Inside pressure, psi
P_o	Outside pressure, psi
Q_x	Shear force per unit length of circumference, lb/in.
R	Radius from axis of revolution for middle surface
R_e	Radial semi-axis of ellipse
R_y	Radius at distance Y from middle surface
S_{ave}	Stress at center of Mohr's circle
S_e	Sign of curvature in a segment
	$S_e = \begin{cases} +1 \\ 0 \\ -1 \end{cases} \quad \text{if } \phi'(x) \begin{cases} > \\ = \\ < \end{cases} 0$

S_i	Stress at inside surface $Y = H/2$
S_{max}	Maximum stress on Von-Mises criteria
S_o	Stress at outside surface $Y = - H/2$
S_{rad}	Radius of Mohr's circle
S_x	Meridional stress
S_y	Thickness stress
S_θ	Hoop stress
T	Temperature, $^{\circ}F$
$T(Y)$	T at distance Y from middle surface
U	Inward displacement at Y relative to middle surface
u	Normalized coordinate along the X axis of a segment $u = (X - X_a)/X_{ab}$
V	Meridional displacement of the middle surface
W	Inward displacement of the middle surface
$(W_r)_k$	Radial load external to segment K at position K , + $\rightarrow$
$(W_m)_k$	Bending load external to segment K at position K , + $\curvearrowright$
$(W_z)_k$	Axial load external to segment K at position K , + $\downarrow$
X	Meridional coordinate of middle surface
X_e	$dX/d\beta$ in ellipse
$__(X)$	Value of $__$ at X
$__'(X)$	$d[__(X)]/dX$
$__''(X)$	$d^2[__(X)]/dX^2$
Y	Inward coordinate from middle surface
Y'	Dummy variable of integration
Z	Axial coordinate of middle surface
Z_e	Axial semi-axis of ellipse

α	Coefficient of linear thermal expansion, (in./in.)/°F
β	Parametric angle of ellipse
γ_{xy}	Shear strain for directions X and Y
δ —	A virtual change in —
ϵ_x	Meridional strain
ϵ_y	Thickness strain
ϵ_θ	Hoop strain
θ	Circumferential angle (hoop direction)
ν	Poisson's ratio. $0 < \nu < 1/2$
τ_{xy}	Shear stress for directions X and Y
τ_{max}	Maximum shear strain
ϕ	Angle of tilt of middle surface of shell. See Fig. 2

III. THEORY

A. Geometry

1. Coordinates

The axisymmetric shell of revolution is geometrically described by its intersection with a meridional plane passing through the axis, as shown in Fig. 1. The intersecting surface is determined by the thickness and by the coordinates of the middle surface. Let:

X = position along the meridional line of the middle surface

$H = H(X)$ = thickness at X

$R = R(X)$ = radius of the middle surface, + $\rightarrow$

$Z = Z(X)$ = elevation of the middle surface, + $\uparrow$

The intersecting line of the middle surface in the RZ plane has two end points, O and N. Instead of the cylindrical coordinates R and Z , curvilinear coordinates X and Y may be used, where:

X = distance along the line from O towards N

Y = normal distance from the middle surface, $\pi/2$ radians.

FIG.1 A MERIDIONAL PLANE IN AN AXISYMMETRIC SHELL

FIG.2 CURVILINEAR COORDINATES OF MIDDLE SURFACE

Arbitrarily, the inside of the shell is at the surface $Y = + H/2$, and the outside surface of the shell is $Y = - H/2$. The determination of inside and outside is therefore set by the choice of end points 0 and N, as shown in Fig. 2.

2. Volume

An important variable in thick-shell theory is the angle ϕ , where

$\phi = \phi(X) =$ angle of the local X axis with respect to the R axis,
 $+ \curvearrowright$; or the angle of the local Y axis with respect
to the - Z axis, $+ \curvearrowright$.

For example, consider the infinitesimal element of volume $dVol$ formed by coordinate changes dX , dY , and circumferential angle $d\theta$, as shown in Fig. 3. In the thin shell theory of SEAL-SHELL-1, sides of the element $dVol$ are approximated by

$$\left. \begin{aligned} dL_{\theta} &\cong R d\theta \\ dL_x &\cong dX \\ dL_y &= dY \end{aligned} \right\} \quad (1)$$

so that

$$dVol = (dL_{\theta})(dL_x)(dL_y) = R d\theta dX dY \quad (2)$$

However, the effect of ϕ is considered in SEAL-SHELL-2; that is,

$$\left. \begin{aligned} dL_{\theta} &= (R - Y \sin \phi) d\theta = F_{\theta} R d\theta \\ dL_x &= dX - Y d\phi = F_x dX \\ dL_y &= dY \end{aligned} \right\} \quad (3)$$

where

$$\begin{aligned} F_{\theta} &= 1 - (Y/R) \sin \phi \\ F_x &= 1 - Y d\phi/dX \end{aligned} \quad (4)$$

FIG. 3 ELEMENT OF VOLUME $dL_x dL_\theta dL_y = d \text{ VOL}$

3. Curved X-Axis

In versions 1 and 2 of SEAL-SHELL, the axisymmetric shell is divided into N segments between positions 0 and N; that is, segment K is bounded by positions K-1 and K, where K = 1 to N. Let:

a = subscript for position K-1 on segment K
b = subscript for position K on segment K
ab = subscript for change in value from a to b

Then for a straight segment K,

$$\left. \begin{aligned} X_{ab} &= \sqrt{R_{ab}^2 + Z_{ab}^2} \\ \cos \phi &= R_{ab}/X_{ab} \\ \sin \phi &= -Z_{ab}/X_{ab} \\ \tan \phi &= -Z_{ab}/R_{ab} \\ \phi'(X) &= d\phi/dX = 0 \\ \phi''(X) &= d^2\phi/dX^2 = 0 \end{aligned} \right\} \quad (5)$$

In SEAL-SHELL-2, each segment may be straight or curved along the X-axis. A curved segment is specified to fit within one quadrant of an ellipse in the RZ plane. If the inside of the curved segment is inside the ellipse, $\phi'(X)$ is positive; otherwise, $\phi'(X)$ is negative. Let:

$$S_e = +1 \text{ if } \phi'(X) > 0$$

$$S_e = 0 \text{ if } \phi'(X) = 0 \text{ (Straight)}$$

$$S_e = -1 \text{ if } \phi'(X) < 0$$

$$R_e = \text{horizontal semi-axis of the ellipse, with the same sign as } S_e$$

$$Z_e = \text{vertical semi-axis of the ellipse, with the same sign as } S_e$$

$$R_c, Z_c = \text{coordinates of the center of the ellipse}$$

$$\beta = \text{parametric angle, as shown in Fig. 4}$$

FIG 4 COORDINATES OF ELLIPSE

Then the parametric equations for the ellipse are

$$\begin{aligned} R &= R_c + R_e \sin \beta \\ Z &= Z_c + Z_e \cos \beta \end{aligned} \quad (6)$$

By differentiation of (6),

$$\begin{aligned} dR/d\beta &= R_e \cos \beta \\ dZ/d\beta &= -Z_e \sin \beta \end{aligned} \quad (7)$$

so that

$$dX/d\beta = X_e$$

where

$$X_e(\beta) = S_e \sqrt{(R_e \cos \beta)^2 + (Z_e \sin \beta)^2}$$

Then

$$X_{ab} = \int_{\beta_a}^{\beta_b} X_e(\beta) d\beta \quad (8)$$

The sign of X_e is determined by S_e because β decreases with X when the inside of the shell is outside of the ellipse.

By the definition of ϕ ,

$$\begin{aligned} dR/dX &= \cos \phi \\ dZ/dX &= -\sin \phi \end{aligned} \quad (9)$$

so that from (7),

$$\left. \begin{aligned} \cos \phi &= (R_e/X_e) \cos \beta \\ \sin \phi &= (Z_e/X_e) \sin \beta \\ \tan \phi &= (Z_e/R_e) \tan \beta \end{aligned} \right\} \quad (10)$$

Differentiation of the last equation gives

$$\left. \begin{aligned} d\phi/d\beta &= Z_e R_e / X_e^2 \\ \phi'(X) &= Z_e R_e / X_e^3 \\ \phi''(X) &= (3Z_e R_e / X_e^6)(R_e^2 - Z_e^2) \sin \beta \cos \beta \end{aligned} \right\} \quad (11)$$

so that

and

B. Strains vs Displacements

The strain energy of each segment is to be expressed in terms of an arbitrary set of 11 deformations. The deformations will be set up in terms of U, V, and W where, as shown in Fig. 6:

V = displacement of middle surface in X direction

W = displacement of middle surface in Y direction

U = movement of the surface at position Y inward with respect to the middle surface. Therefore, U = 0 at Y = 0.

Also, let:

$\Delta __$ = change in $__$ produced by U, V, and W

ϵ_x = strain in X direction

ϵ_θ = strain in circumferential direction

ϵ_y = strain in Y direction

Strains ϵ_x , ϵ_θ , and ϵ_y^* can now be expressed as functions of the displacements U, V, and W. By definition,

$$\begin{aligned}\epsilon_x &= \Delta dL_x / dL_x \\ \epsilon_\theta &= \Delta dL_\theta / dL_\theta \\ \epsilon_y &= \Delta dL_y / dL_y\end{aligned}\tag{12}$$

From Eq (3)

$$\left. \begin{aligned}\Delta dL_x &= \Delta(dX - Yd\phi) = \Delta dX - (Y)\Delta d\phi - (d\phi)\Delta Y \\ \Delta dL_\theta &= \Delta(R - Y \sin \phi)d\theta = [\Delta R - (Y \cos \phi)\Delta \phi - (\sin \phi)\Delta Y] d\theta \\ \Delta dL_y &= \Delta(dY)\end{aligned}\right\}\tag{13}$$

*The shear strain γ_{xy} is discussed in a later section of the theory.

FIG. 5 GEOMETRY OF SEGMENT K

FIG. 6 DISPLACEMENTS

so that

$$\left. \begin{aligned} \epsilon_x &= \frac{\Delta dX - (Y)\Delta d\phi - (d\phi)\Delta Y}{dX - Yd\phi} = \frac{\Delta dX - (Y)\Delta d\phi - (d\phi)\Delta Y}{(F_x) dX} \\ \epsilon_\theta &= \frac{[\Delta R - (Y \cos \phi)\Delta\phi - (\sin \phi)\Delta Y] d\theta}{(R - Y \sin \phi) d\theta} = \frac{\Delta R - (Y \cos \phi)\Delta\phi - (\sin \phi)\Delta Y}{(F_\theta) R} \\ \epsilon_y &= \frac{\Delta(dY)}{dY} \end{aligned} \right\} (14)$$

Next, the Δ quantities will be evaluated in terms of U , V , and W by use of Figs. 3 and 6. In the X , Y , and ϕ directions,

$$\left. \begin{aligned} \Delta dX &= dV - Wd\phi = [V'(X) - \phi'(X)W] dX ; \\ \Delta Y &= U \text{ so that } \Delta dY = (\partial U / \partial Y) dY ; \text{ and} \\ \Delta\phi &= W'(X) + \phi'(X)V \text{ so that} \\ \Delta d\phi &= [W''(X) + \phi''(X)V + \phi'(X)V'(X)] dX \end{aligned} \right\} (15)$$

In the R and Z directions

$$\begin{aligned} \Delta R &= (\cos \phi)V + (-\sin \phi)W \\ \Delta Z &= (-\sin \phi)V + (-\cos \phi)W \end{aligned} \quad (16)$$

Substituting (15) and (16) into (14) gives

$$\begin{aligned} \epsilon_x &= \left[- \left(\frac{Y\phi''(X)}{F_x} \right) V + \frac{dV}{dX} \right] - \left[+ \left(\frac{\phi'(X)}{F_x} \right) W + \left(\frac{Y}{F_x} \right) \frac{dW}{dX} \right] - \left(\frac{\phi'(X)}{F_x} \right) U \\ \epsilon_\theta &= \left[\frac{(\cos \phi)(1 - Y\phi'(X))}{RF_\theta} \right] V - \left[\left(\frac{\sin \phi}{RF_\theta} \right) W + \left(\frac{Y \cos \phi}{RF_\theta} \right) \frac{dW}{dX} \right] - \left(\frac{\sin \phi}{RF_\theta} \right) U \\ \epsilon_y &= \partial U / \partial Y \end{aligned}$$

or in operational form

$$\left. \begin{aligned} \epsilon_x &= (c_{ev})_{1,1} V + (c_{ev})_{1,2} W + (c_{ev})_{1,3} U \\ \epsilon_\theta &= (c_{ev})_{2,1} V + (c_{ev})_{2,2} W + (c_{ev})_{2,3} U \\ \epsilon_y &= \partial U / \partial Y \end{aligned} \right\} \quad (17)$$

where

$$(c_{ev})_{1,1} = - \frac{Y\phi''(X)}{F_x} + \frac{d}{dX}$$

$$(c_{ev})_{1,2} = - \frac{\phi'(X)}{F_x} - \frac{Y}{F_x} \cdot \frac{d}{dX}$$

$$(c_{ev})_{1,3} = - \frac{\phi'(X)}{F_x}$$

$$(c_{ev})_{2,1} = \frac{(\cos \phi) F_x}{R F_\theta}$$

$$(c_{ev})_{2,2} = - \frac{\sin \phi}{R F_\theta} - \frac{Y \cos \phi}{R F_\theta} \cdot \frac{d}{dX}$$

$$(c_{ev})_{2,3} = - \frac{\sin \phi}{R F_\theta}$$

C. Displacements V and W vs Arbitrary Deformations

As mentioned in the Introduction, an arbitrary set of 11 deformations are applied to each segment and the resulting strain energy is determined. The first step in this process is to set up equations giving the deflection distribution of the arbitrary deformations. Let:

$$(D_g)_i = i^{\text{th}} \text{ arbitrary deformation, } i = 1 \text{ to } 11$$

$$(c_{vg})_{1,i} = \text{coefficient for } V \text{ vs } (D_g)_i$$

$$(c_{vg})_{2,i} = \text{coefficient for } W \text{ vs } (D_g)_i$$

$$(c_{vg})_{3,i} = \text{coefficient for } U \text{ vs } (D_g)_i$$

so that

$$\begin{aligned} V &= (C_{vg})_{1,1} (D_g)_1 + (C_{vg})_{2,1} (D_g)_2 + \dots + (C_{vg})_{1,11} (D_g)_{11} \\ W &= (C_{vg})_{w,1} (D_g)_1 + (C_{vg})_{2,2} (D_g)_2 + \dots + (C_{vg})_{2,11} (D_g)_{11} \\ U &= (C_{vg})_{3,1} (D_g)_1 + (C_{vg})_{3,2} (D_g)_2 + \dots + (C_{vg})_{3,11} (D_g)_{11} \end{aligned}$$

or in summation notation,

$$\left. \begin{aligned} V &= \sum_{i=1}^{11} i (C_{vg})_{1,i} (D_g)_i \\ W &= \sum_{i=1}^{11} i (C_{vg})_{2,i} (D_g)_i \\ U &= \sum_{i=1}^{11} i (C_{vg})_{3,i} (D_g)_i \end{aligned} \right\} \quad (18)$$

The set-up of $(C_{vg})_{i,j}$ will be discussed for each deformation $(D_g)_j$.

$(D_g)_1$, Vertical Elongation

Arbitrarily, vertical elongation is measured with respect to position a on each segment. In terms of V and W, let

$$\begin{aligned} V &= (u \sin \phi_b) (D_g)_1 \\ W &= (u \cos \phi_b) (D_g)_1 \end{aligned} \quad (19)$$

where

$$u = (X - X_a)/X_{ab} = \text{normalized coordinate along X axis of segment}$$

Thus $u = 0$ at a and $u = 1$ at b. Substitution of (19) into (15) gives

$$\begin{aligned} \Delta R &= (u)(\cos \phi \sin \phi_b - \sin \phi \cos \phi_b)(D_g)_1 \\ \Delta Z &= (u)(-\sin \phi \sin \phi_b - \cos \phi \cos \phi_b)(D_g)_1 \end{aligned}$$

so that at a,

$$\Delta R = 0$$

$$\Delta Z = 0$$

and at b,

$$\Delta R = 0$$

$$\Delta Z = - (D_g)_1$$

Therefore, the deformations in (19) produce a vertical elongation of the segment. A comparison of (19) and (18) shows that

$$\begin{aligned} (C_{vg})_{1,1} &= u \sin \phi_b \\ (C_{vg})_{2,1} &= u \cos \phi_b \end{aligned} \quad (20)$$

$(D_g)_2$, Radial Deformation at a

For radial deformation at a, let

$$\begin{aligned} V &= \left[(1 - u) \cos \phi_a \right] (D_g)_2 \\ W &= \left[-(1 - u) \sin \phi_a \right] (D_g)_2 \end{aligned}$$

so that

$$\begin{aligned} (C_{vg})_{1,2} &= (1 - u) \cos \phi_a \\ (C_{vg})_{2,2} &= -(1 - u) \sin \phi_a \end{aligned} \quad (21)$$

As a check, substitution of these equations into (16) gives

$$\begin{aligned} \Delta R &= (1 - u)(\cos \phi \cos \phi_a + \sin \phi \sin \phi_a)(D_g)_2 \\ \Delta Z &= (1 - u)(-\sin \phi \cos \phi_a + \cos \phi \sin \phi_a)(D_g)_2 \end{aligned}$$

so that

$$\Delta R = (D_g)_2 \quad \text{and} \quad \Delta Z = 0 \quad \text{at} \quad a$$

and

$$\Delta R = 0 \quad \text{and} \quad \Delta Z = 0 \quad \text{at} \quad b$$

$(D_g)_3$, Radial Deformation at b

For radial deformation at b, let

$$\begin{aligned} V &= \left[u \cos \phi_b \right] (D_g)_3 \\ W &= \left[-u \sin \phi_b \right] (D_g)_3 \end{aligned}$$

so that

$$\begin{aligned}(C_{vg})_{1,3} &= u \cos \phi_b \\ (C_{vg})_{2,3} &= -u \sin \phi_b\end{aligned}\tag{22}$$

Also

$$\begin{aligned}\Delta R &= (u)(\cos \phi \cos \phi_b + \sin \phi \sin \phi_b)(D_g)_3 \\ \Delta Z &= (u)(-\sin \phi \cos \phi_b + \cos \phi \sin \phi_b)(D_g)_3\end{aligned}$$

so that

$$\begin{aligned}\Delta R &= 0 \quad \text{and} \quad \Delta Z = 0 \quad \text{at} \quad a \\ \Delta R &= (D_g)_3 \quad \text{and} \quad \Delta Z = 0 \quad \text{at} \quad b\end{aligned}$$

$(D_g)_4$, Second Power Expansion in W

Let

$$\begin{aligned}V &= (0)(D_g)_4 \\ W &= (u - u^2)(D_g)_4\end{aligned}$$

so that $W = 0$ at a and b . Then

$$\begin{aligned}(C_{vg})_{1,4} &= 0 \\ (C_{vg})_{2,4} &= u - u^2\end{aligned}\tag{23}$$

$(D_g)_5$, Third Power Expansion in W

Let

$$\begin{aligned}V &= (0)(D_g)_5 \\ W &= (u - u^3)(D_g)_5\end{aligned}$$

so that $W = 0$ at a and b . Then

$$\begin{aligned}(C_{vg})_{1,5} &= 0 \\ (C_{vg})_{2,5} &= u - u^3\end{aligned}\tag{24}$$

(D_g)₆, Second Power Expansion in V

Let

$$V = (u - u^2)(D_g)_6$$

$$W = (0) (D_g)_6$$

so that $V = 0$ at a and b . Then

$$(C_{vg})_{1,6} = u - u^2 \quad (25)$$

$$(C_{vg})_{2,6} = 0$$

(D_g)₇, Third Power Expansion V

Let

$$V = (u - u^3)(D_g)_7$$

$$W = (0) (D_g)_7$$

so that $V = 0$ at a and b . Then

$$(C_{vg})_{1,7} = u - u^3 \quad (26)$$

$$(C_{vg})_{2,7} = 0$$

D. Displacement U vs Arbitrary Deformations

The remaining four variables $(D_g)_9$ to $(D_g)_{11}$ will be used to specify U without any changes in V and W . In order to accomplish this and thus keep to the rule that plane cross sections remain plane, relationships have to be developed for ϵ_y vs ϵ_x and ϵ_θ . First of all, the basic stress-strain equations are

$$\left. \begin{aligned} \epsilon_x &= \alpha T + (1/E_t)(S_x - \nu S_\theta - \nu S_y) \\ \epsilon_\theta &= \alpha T + (1/E_t)(S_\theta - \nu S_y - \nu S_x) \\ \epsilon_y &= \alpha T + (1/E_t)(S_y - \nu S_x - \nu S_\theta) \end{aligned} \right\} \quad (27)$$

where

α = coefficient of linear thermal expansion, (in./in.)/°F

T = temperature, °F

E_t = tensile modulus of elasticity, psi

S_x = stress in meridional direction

S_θ = stress in hoop direction

S_y = stress in thickness direction

ν = Poisson's ratio, $0 \leq \nu < .5$

Solving (27) for the stresses gives

$$\begin{aligned} S_x &= S_t + E'_t \left[(1 - \nu) \epsilon_x + \nu(\epsilon_\theta + \epsilon_y) \right] \\ S_\theta &= S_t + E'_t \left[(1 - \nu) \epsilon_\theta + \nu(\epsilon_y + \epsilon_x) \right] \\ S_y &= S_t + E'_t \left[(1 - \nu) \epsilon_y + \nu(\epsilon_x + \epsilon_\theta) \right] \end{aligned} \quad (28)$$

where

$$S_t = - E_t \alpha T / (1 - 2\nu)$$

$$E'_t = + E_t / \left[(1 + \nu)(1 - 2\nu) \right]$$

Then substituting the first two equations of (28) into the third equation of (27) and solving for ϵ_y , we obtain

$$\epsilon_y = \left(\frac{1 + \nu}{1 - \nu} \right) \alpha T + \left(\frac{1 - \nu - 2\nu^2}{1 - \nu} \right) \frac{\sigma_y}{E} - \left(\frac{\nu}{1 - \nu} \right) (\epsilon_x + \epsilon_\theta) \quad (29)$$

In order to express ϵ_y vs. (D_g) , the variables ϵ_x , ϵ_θ , and σ_y must be set up in terms of (D_g) .

The strains ϵ_x and ϵ_θ may be expressed in terms of (D_g) by substituting (18) into (17) where

$$(C_{vg})_{1,j} = (C_{vg})_{2,j} = 0 \quad \text{for } j = 8 \text{ to } 11$$

Thus

$$\begin{aligned}
\epsilon_x &= (C_{ev})_{1,1} \left[(C_{vg})_{1,1} (D_g)_1 + (C_{vg})_{1,2} (D_g)_2 + \dots + (C_{vg})_{1,11} (D_g)_{11} \right] \\
&\quad + (C_{ev})_{1,2} \left[(C_{vg})_{2,1} (D_g)_1 + (C_{vg})_{2,2} (D_g)_2 + \dots + (C_{vg})_{2,11} (D_g)_{11} \right] \\
&\quad + (C_{ev})_{1,3} U \\
&= \left[(C_{ev})_{1,1} (C_{vg})_{1,1} + (C_{ev})_{1,2} (C_{vg})_{2,1} \right] (D_g)_1 \\
&\quad + \left[(C_{ev})_{1,1} (C_{vg})_{1,2} + (C_{ev})_{1,2} (C_{vg})_{2,2} \right] (D_g)_2 \\
&\quad + \dots + \left[(C_{ev})_{1,1} (C_{vg})_{1,11} + (C_{ev})_{1,2} (C_{vg})_{2,11} \right] (D_g)_{11} \\
&\quad + (C_{ev})_{1,3} U
\end{aligned}$$

or

$$\epsilon_x = (C_{eg})_{1,1} (D_g)_1 + (C_{ev})_{1,2} (D_g)_2 + \dots + (C_{eg})_{1,11} (D_g)_{11} + (C_{ev})_{1,3} U$$

where

$$(C_{eg})_{1,i} = (C_{ev})_{1,1} (C_{vg})_{1,i} + (C_{ev})_{1,2} (C_{vg})_{2,i}$$

In summation notation,

$$\left. \begin{aligned}
\epsilon_x &= \sum_{i=1}^{11} i (C_{eg})_{1,i} (D_g)_i + (C_{ev})_{1,3} U \\
\epsilon_\theta &= \sum_{i=1}^{11} i (C_{eg})_{2,i} (D_g)_i + (C_{ev})_{2,3} U
\end{aligned} \right\} \quad (30)$$

where

$$(C_{eg})_{i,j} = \sum_{m=1}^2 (C_{ev})_{i,m} (C_{vg})_{m,j} \quad \text{for } i = 1 \text{ to } 2 \text{ and } j = 1 \text{ to } 7$$

and

$$(C_{eg})_{k,j} = 0 \quad \text{for } i = 1 \text{ to } 2 \text{ and } j = 8 \text{ to } 11$$

The thickness stress σ_y has the values $\sigma_y = -P_0$ at $Y = -H/2$ and $\sigma_y = -P_1$ at $Y = +H/2$, so that a simple power series for the effect of σ_y on ϵ_y is arbitrarily set up as follows:

$$\begin{aligned}
\left(\frac{1+\nu-2\nu^2}{1-\nu}\right) \frac{\sigma_y}{E} = & \left[(-.5 + Y/H)P_o + (-.5 - Y/H)P_i \right] (D_g)_8 \\
& + \left[(1 - 4Y^2/H^2) \right] (D_g)_9 \\
& + \left[(1 - 4Y^2/H^2)(Y/H) \right] (D_g)_{10} \\
& + \left[(1 - 4Y^2/H^2)(Y^2/H^2) \right] (D_g)_{11}
\end{aligned}$$

or

$$\left(\frac{1+\nu-2\nu^2}{1-\nu}\right) \frac{\sigma_y}{E} = \sum_{i=8}^{11} (C_{ev})_{3,i} (D_g)_i \quad (31)$$

where

$$\begin{aligned}
(C_{ev})_{3,8} &= (-.5 + Y/H)P_o + (-.5 - Y/H)P_i \\
(C_{ev})_{3,9} &= (1 - 4Y^2/H^2) \\
(C_{ev})_{3,10} &= (1 - 4Y^2/H^2)(Y/H) \\
(C_{ev})_{3,11} &= (1 - 4Y^2/H^2)(Y^2/H^2)
\end{aligned}$$

Now, substitution of (30) and (31) into (29) gives

$$\begin{aligned}
\epsilon_y = & \left(\frac{1+\nu}{1-\nu}\right) \alpha T + \left[\sum_{i=8}^{11} (C_{eg})_{3,i} (D_g)_i \right] \\
& - \left(\frac{\nu}{1-\nu}\right) \left\{ \sum_{i=1}^7 (C_{eg})_{1,i} + (C_{eg})_{2,i} \right\} (D_g)_i \\
& + \left[(C_{ev})_{1,3} + (C_{ev})_{2,3} \right] U \}
\end{aligned}$$

or

$$\epsilon_y = \left(\frac{1+\nu}{1-\nu}\right) \alpha T + \sum_{i=1}^{11} (C_{eg})_{3,i} (D_g)_i + (C_{ev})_{3,3} U \quad (32)$$

where

$$(C_{eg})_{3,i} = - \left(\frac{\nu}{1-\nu}\right) \left[(C_{eg})_{1,i} + (C_{eg})_{2,i} \right] \text{ for } i = 1 \text{ to } 7$$

and

$$(c_{ev})_{3,3} = - \left(\frac{\nu}{1-\nu} \right) \left[(c_{ev})_{1,3} + (c_{ev})_{2,3} \right]$$

From (17), $\epsilon_y = \partial U / \partial Y$; therefore, substituting this equation into (32) gives the differential equation

$$\partial U / \partial Y - (c_{ev})_{3,3} U = \left(\frac{1+\nu}{1-\nu} \right) \alpha T + \sum_{i=1}^{11} i (c_{eg})_{3,i} (D_g)_i$$

By definition, $U = 0$ at $Y = 0$, so that the solution to the differential equation is

$$U = c_{ut} + \sum_{i=1}^{11} i (c_{ug})_i (D_g)_i \quad (33)$$

where

$$c_{ut} = I_u(Y) \int_0^Y \left[\left(\frac{1+\nu}{1-\nu} \right) \alpha T / I_u(Y') \right] dY'$$

$$(c_{ug})_i = I_u(Y) \int_0^Y \left[(c_{eg})_{3,i} / I_u(Y') \right] dY' \quad \text{for } i = 1 \text{ to } 11$$

and

$$I_u(Y) = e^{\left(\int_0^Y (c_{ev})_{3,3} dY \right)}, \text{ the integrating factor}$$

The strains ϵ_x , ϵ_θ , and ϵ_y may now be found in terms of D_g and temperature by substituting (33) into (30) and (32):

$$\epsilon_x = \sum_{i=1}^{11} i (c_{eg})_{1,i} (D_g)_i + (c_{ev})_{1,3} \left[c_{ut} + \sum_{i=1}^{11} i (c_{ug})_i (D_g)_i \right]$$

$$\epsilon_\theta = \sum_{i=1}^{11} i (c_{eg})_{2,i} (D_g)_i + (c_{ev})_{2,3} \left[c_{ut} + \sum_{i=1}^{11} i (c_{ug})_i (D_g)_i \right]$$

$$\epsilon_y = \left(\frac{1+\nu}{1-\nu} \right) \alpha T + \sum_{i=1}^{11} i (c_{eg})_{3,i} (D_g)_i + (c_{ev})_{3,3} \left[c_{ut} + \sum_{i=1}^{11} i (c_{ug})_i (D_g)_i \right]$$

or

$$\left. \begin{aligned} \epsilon_x &= (C_{et}')_1 + \sum_{i=1}^{11} i (C_{eg}')_{1,i} (D_g)_i \\ \epsilon_\theta &= (C_{et}')_2 + \sum_{i=1}^{11} i (C_{eg}')_{2,i} (D_g)_i \\ \epsilon_y &= (C_{et}')_3 + \sum_{i=1}^{11} i (C_{eg}')_{3,i} (D_g)_i \end{aligned} \right\} \quad (34)$$

where

$$\begin{aligned} (C_{et}')_i &= (C_{ev})_{i,3} (C_{ut}) \quad \text{for } i = 1 \text{ to } 2 \\ &= (C_{ev})_{i,3} (C_{ut}) + \left(\frac{1+\nu}{1-\nu} \right) \alpha T \quad \text{for } i = 3 \end{aligned}$$

and

$$(C_{eg}')_{i,j} = (C_{eg})_{i,j} + (C_{ev})_{i,3} (C_{ug})_{3,j} \quad \begin{array}{l} \text{for } i = 1 \text{ to } 3 \\ \text{and } j = 1 \text{ to } 11 \end{array}$$

In order to express the stresses in terms of temperature and deformations (D_g) , (34) is substituted into (28). As a result,

$$\left. \begin{aligned} s_x &= (C_{st}')_1 + \sum_{i=1}^{11} i (C_{sg}')_{1,i} (D_g)_i \\ s_\theta &= (C_{st}')_2 + \sum_{i=1}^{11} i (C_{sg}')_{2,i} (D_g)_i \\ s_y &= (C_{st}')_3 + \sum_{i=1}^{11} i (C_{sg}')_{3,i} (D_g)_i \end{aligned} \right\} \quad (35)$$

where

$$\begin{aligned} (C_{st}') &= S_t + E'_t \left[(1-\nu)(C_{et}')_1 + \nu(C_{st}')_2 + \nu(C_{et}')_3 \right] \\ (C_{sg}')_{1,i} &= E'_t \left[(1-\nu)(C_{eg}')_{1,i} + \nu(C_{eg}')_{2,i} + \nu(C_{eg}')_{3,i} \right] \end{aligned}$$

etc.

E. Strain Energy

The principle of virtual work will be used next to determine the generalized forces (F_g) corresponding to the arbitrary deformations (D_g). Let:

$$\begin{aligned}\delta(D_g)_i &= \text{a virtual displacement of } (D_g)_i \\ \delta _ &= \text{a change in } _ \text{ produced by } \delta(D_g) \\ E_{in} &= \text{internal work on segment} \\ E_{ex} &= \text{external work on segment}\end{aligned}$$

By definition,^{*}

$$\delta E_{in} = \int (S_x \delta \epsilon_x + S_y \delta \epsilon_y + S_z \delta \epsilon_z) dVol \quad (36)$$

This may be put in terms of (D_g) by use of (34) and (35). Since there is no change in temperature during the virtual displacements $\delta(D_g)_i$, then from (34):

$$\left. \begin{aligned}\delta \epsilon_x &= \sum_{j=1}^{11} j (C_{eg}')_{1,j} \delta(D_g)_j \\ \delta \epsilon_\theta &= \sum_{j=1}^{11} j (C_{eg}')_{2,j} \delta(D_g)_j \\ \delta \epsilon_y &= \sum_{j=1}^{11} j (C_{eg}')_{3,j} \delta(D_g)_j\end{aligned}\right\} \quad (37)$$

Substitution of the strain changes from (37) and the stresses from (35) into the strain energy change of (36) gives

$$\begin{aligned}\delta E_{in} = \int \left\{ \left[(C_{st}')_1 + \sum_{i=1}^{11} i (C_{sg}')_{1,i} (D_g)_i \right] \left[\sum_{j=1}^{11} j (C_{eg}')_{1,j} (\delta D_g)_j \right] \right. \\ + \left[(C_{st}')_2 + \sum_{i=1}^{11} i (C_{sg}')_{2,i} (D_g)_i \right] \left[\sum_{j=1}^{11} j (C_{eg}')_{2,j} (\delta D_g)_j \right] \\ \left. + \left[(C_{st}')_3 + \sum_{i=1}^{11} i (C_{sg}')_{3,i} (D_g)_i \right] \left[\sum_{j=1}^{11} j (C_{eg}')_{3,j} (\delta D_g)_j \right] \right\} dVol\end{aligned}$$

^{*}In SEAL-SHELL-2, numerical iterations and differentiations for the X and Y directions in each segment K are performed with a five-point polynomial expansion of each variable. Thus, a total of 25 points are used in each segment.

or, after combining terms,

$$\delta E_{in} = \sum_{j=1}^{11} j \left[(B_{gt})_j + \sum_{i=1}^{11} i (B_{gg})_{j,i} (D_g)_i \right] (\delta D_g)_j \quad (38)$$

where

$$(B_{gt})_j = \sum_{m=1}^3 m (C_{st}')_m (C_{eg}')_{m,j}$$

$$(B_{gg})_{j,i} = \sum_{m=1}^3 m (C_{sg}')_{m,i} (C_{eg}')_{m,j}$$

By definition of generalized forces,

$$\delta E_{in} \equiv \sum_{j=1}^{11} (F_g)_j \delta (D_g)_j \quad (39)$$

and a comparison of (38) and (39) shows that

$$(F_g)_j = (B_{gt})_j + \sum_{i=1}^{11} i (B_{gg})_{j,i} (D_g)_i \quad (40)$$

for $j = 1$ to 11

The 11 simultaneous equations of (40) may be solved for (D_g) vs (F_g) to obtain

$$\underline{(D_g)_i = (A_{gt})_i + \sum_{j=1}^{11} i (A_{gg})_{i,j} (F_g)_j} \quad (41)$$

where, as a check on the solution:

$$\sum_{i=1}^{11} m (A_{gg})_{i,m} (B_{gg})_{m,j} = \begin{cases} 1 & \text{for } i = m \\ 0 & \text{for } i \neq m \end{cases}$$

and

$$(A_{gt})_i = - \sum_{j=1}^{11} j (A_{gg})_{i,j} (B_{gt})_j$$

for i, j , and $m = 1$ to 11

F. Segment Forces and Deflections

The forces acting on a general segment K are the following:

$(F_r)_{k-1}$ = radial force at position K-1 connecting segments K-1 and K over the circumference $2\pi R_a$

$(F_m)_{k-1}$ = bending moment at position K-1 connecting segments K-1 and K over the circumference $2\pi R_a$

$(F_r)_k + (W_r)_k$ = total radial force on segment K over the circumference $2\pi R_a$ at position K

$(F_m)_k + (W_m)_k$ = total bending moment on segment K over the circumference $2\pi R_b$ at position K

$(F_z)_k + (W_z)_k$ = total axial force on segment K over the circumference $2\pi R_b$ at position K

P_i, P_o = inside and outside pressures on segment K. These pressures do not act on the end surfaces $2\pi R_a H_a$ and $2\pi R_b H_b$

The axial force $(F_z)_{k-1}$ is not an independent load, because by vertical equilibrium on segment K,

$$(F_z)_{k-1} = (F_z)_k + (W_z)_k + \pi(R_{bo}^2 - R_{ao}^2) P_o - \pi(R_{bi}^2 - R_{ai}^2) P_i \quad (42)$$

where

$$R_{bo} = R_b - (-0.5H_b) \sin \phi_b$$

$$R_{bi} = R_b - (+0.5H_b) \sin \phi_b$$

$$R_{ao} = R_a - (-0.5H_a) \sin \phi_a$$

$$R_{ai} = R_a - (+0.5H_a) \sin \phi_a$$

on segment K.

An artificial numbering scheme is set up for the segment forces in order to keep track of the separate effects. As shown in Figs. 7 and 8:

$(F_k)_1$ = the generalized pressure effect with an arbitrary value of unity
This separates out the combined effect of P_i and P_o .

$(F_k)_2$ = the generalized applied load effect with an arbitrary value of unity
This separates out the combined effect of $(W_r)_k$, $(W_m)_k$, and $(W_z)_k$.

$(F_k)_3 = (F_z)_k$ = axial load connecting segments K and K+1

FIG. 7 LOADS ON SEGMENT K

FIG. 8 DEFLECTIONS OF SEGMENT K

$$\begin{aligned}
(F_k)_4 &= (F_r)_{k-1} = \text{radial load connecting segments } K-1 \text{ and } K \\
(F_k)_5 &= (F_m)_{k-1} = \text{bending moment connecting segments } K-1 \text{ and } K \\
(F_k)_6 &= (F_r)_k = \text{radial load connecting segments } K \text{ and } K+1 \\
(F_k)_7 &= (F_m)_k = \text{bending moment connecting segments } K \text{ and } K+1 \\
(D_k)_i &= \text{generalized displacement corresponding to the force } (F_k)_i \text{ for } i = 1 \text{ to } 7
\end{aligned}$$

In terms of the preceding forces, (42) becomes

$$(F_{k-1})_3 = \sum_{i=1}^7 (C_{k-1,k})_{3,i} (F_k)_i \quad (43)$$

where

$$\begin{aligned}
(C_{k-1,k})_{3,1} &= \pi(R_{bo}^2 - R_{ao}^2)P_o - \pi(R_{bi}^2 - R_{ai}^2)P_i \\
(C_{k-1,k})_{3,2} &= (W_z)_k \\
(C_{k-1,k})_{3,3} &= 1 \\
(C_{k-1,k})_{3,i} &= 0 \quad \text{for } i = 4 \text{ to } 7
\end{aligned}$$

The virtual work done by these forces on segment K is

$$\begin{aligned}
\delta E_{ex} &= \left[\int (2\pi R_i F_{xi} dX)(P_i)(\delta R_i \sin \phi_i + \delta Z_i \cos \phi_i) \right. \\
&\quad \left. - \int (2\pi R_o F_{xo} dX)(P_o)(\delta R_o \sin \phi_o + \delta Z_o \cos \phi_o) \right] (F_k)_1 \\
&\quad + \left[(W_r)_k \delta R_b - (W_m)_k \delta \phi_b - (W_z)_k \delta Z_b \right] (F_k)_2 \\
&\quad - (F_k)_3 \delta Z_b - (F_k)_4 \delta R_a + (F_k)_5 \delta \phi_a \\
&\quad + (F_k)_6 \delta R_b - (F_k)_7 \delta \phi_b
\end{aligned} \quad (44)$$

where

$$\delta\phi = W'(X) + \phi'(X)V, \quad \phi_y = \phi + \tan^{-1} \left[(Y/H)(H_{ab}/X_{ab}) \right]$$

$$R_y = R - Y \sin \phi, \quad \delta R_y = \delta V \cos \phi - \delta(W + U) \sin \phi - (Y \cos \phi) \delta\phi$$

$$Z_y = Z - Y \cos \phi, \quad \delta Z_y = -\delta V \sin \phi - \delta(W + U) \cos \phi + (Y \sin \phi) \delta\phi$$

$$R_i = R_y \text{ at } Y = +0.5H, \quad R_o = R_y \text{ at } Y = -0.5H$$

$$F_{xi} = F_x \text{ at } Y = +0.5H, \quad F_{xo} = F_x \text{ at } Y = -0.5H$$

$$\phi_i = \phi_y \text{ at } Y = +0.5H, \quad \phi_o = \phi_y \text{ at } Y = -0.5H$$

$$\delta R_i = \delta R_y \text{ at } Y = +0.5H, \quad \delta R_o = \delta R_y \text{ at } Y = -0.5H$$

$$\delta Z_i = \delta Z_y \text{ at } Y = +0.5H, \quad \delta Z_o = \delta Z_y \text{ at } Y = -0.5H$$

$$\delta R_b = \delta R_y \text{ at } Y = 0 \text{ and } u = 1$$

$$\delta\phi_b = \delta\phi_y \text{ at } Y = 0 \text{ and } u = 1$$

$$\delta Z_b = \delta Z_y \text{ at } Y = 0 \text{ and } u = 1$$

$$\delta R_a = \delta R_y \text{ at } Y = 0 \text{ and } u = 0$$

$$\delta Z_a = \delta Z_y \text{ at } Y = 0 \text{ and } u = 0$$

By definition,

$$\delta E_{ex} = \sum_{i=1}^7 \delta (F_k)_i \delta (D_k)_i \quad (45)$$

and a comparison of (44) and (45) shows that

$$\delta (D_k)_i = (C_{kv})_{i,1} \delta V + (C_{kv})_{i,2} \delta W + (C_{kv})_{i,3} \delta U \quad (46)$$

where

$$(C_{kv})_{1,1} = + \int (2\pi R_i F_{xi} dX)(P_i) \left\{ \left[(\cos \phi) - (0.5H \cos \phi) \phi'(X) \right] \sin \phi_i \right. \\ \left. + \left[(\cos \phi) - (0.5H \cos \phi) \phi'(X) \right] \cos \phi_i \right\} \\ - \int (2\pi R_o F_{xo} dX)(P_o) \left\{ \left[(\cos \phi) + (0.5H \cos \phi) \phi'(X) \right] \sin \phi_o \right. \\ \left. + \left[(\cos \phi) + (0.5H \cos \phi) \phi'(X) \right] \cos \phi_o \right\}$$

$$(C_{kv})_{2,1} = \left[(W_r)_k (\cos \phi) - (W_m)_k [\phi'(X)] - (W_z)_k (-\sin \phi) \right] \text{ at } u = 1 \\ = 0 \text{ for } u < 1$$

$$(C_{kv})_{3,1} = - (-\sin \phi) \text{ at } u = 1 \\ = 0 \text{ for } u < 1$$

etc.

Equations for $\delta(D_k)$ vs $\delta(D_g)$ are obtained by substituting (18) into (46). As a result,

$$\delta(D_k)_i = \sum_{j=1}^{11} j (C_{kg})_{i,j} \delta(D_g)_j \quad (47)$$

where

$$(C_{kg})_{i,j} = \sum_{s=1}^3 s (C_{kv})_{i,s} (C_{vg})_{s,j}$$

for i and $j = 1$ to 11

The generalized forces (F_g) may now be found in terms of segment forces (F_k) by substituting (47) into (45):

$$\delta E_{ex} = \sum_{i=1}^7 i (F_k)_i \sum_{j=1}^{11} j (C_{kg})_{i,j} \delta(D_g)_j$$

or

$$\delta E_{ex} = \sum_{j=1}^{11} j \left[\sum_{i=1}^7 i (C_{kg})_{i,j} (F_k)_i \right] \delta(D_g)_j \quad (48)$$

By principle of virtual work, $\delta E_{ex} \equiv \delta E_{in}$, and a comparison of (48) with (45) shows that

$$(F_g)_j = \sum_{i=1}^7 (C_{kg})_{i,j} (F_k)_i \quad (49)$$

for $j = 1$ to 11

The object of these manipulations is to obtain equations for segment deflections (D_k) vs segment forces (F_k) . Equation (47) gives (D_k) vs (D_g) , Eq (47) gives (D_g) vs (F_g) , and Eqs (49) give (F_g) vs (F_k) ; therefore, by a sequence of substitutions,

$$\begin{aligned} (D_k)_i &= \sum_{m=1}^{11} (C_{kg})_{i,m} (D_g)_m \\ &= \sum_{m=1}^{11} (C_{kg})_{i,m} \left[(A_{gt})_m + \sum_{n=1}^{11} (A_{gg})_{m,n} (F_g)_n \right] \\ &= \sum_{m=1}^{11} (C_{kg})_{i,m} \left[(A_{gt})_m + \sum_{n=1}^{11} (A_{gg})_{m,n} \sum_{j=1}^{11} (C_{kg})_{j,n} (F_k)_j \right] \end{aligned}$$

or

$$(D_k)_i = (A_{kt})_i + \sum_j (A_{kk})_{i,j} (F_k)_j \quad (50)$$

where

$$\begin{aligned} (A_{kt})_i &= \sum_{m=1}^{11} (C_{kg})_{i,m} (A_{gt})_m \\ (A_{kk})_{i,j} &= \sum_{m=1}^{11} \sum_{n=1}^{11} (C_{kg})_{i,m} (A_{gg})_{m,n} (C_{kg})_{j,n} \end{aligned}$$

By equating radial and angular deflections at the joints of segments, the radial and angular forces connecting the segments are determined. The axial forces^{*} are determined by successive application of (43). Boundary conditions at the end points O and N determine the unknown forces and deflections.

*The total axial deflection is the sum of the axial deflections of the individual segments.

Once the forces (F_k) are calculated, then Eq (49) determines (D_k) , Eq (47) determines deformations (D_g) , and Eq (35) determines stresses S_x , S_θ , and S_y .

G. Shear Correction

Simple beam theory is used to estimate the shear deflections. From Fig. 7 the shear forces per unit circumference at a and at b are given by

$$\left. \begin{aligned} 2\pi R_a Q_a &= (F_{k-1})_3 \cos \phi_a - (F_k)_4 \sin \phi_a \\ \text{and} \\ 2\pi R_b Q_b &= [(F_k)_3 + (W_z)_k (F_k)_2] \cos \phi_b \\ &\quad - [(F_k)_6 + (W_r)_k (F_k)_2] \sin \phi_b \end{aligned} \right\} \quad (51)$$

A simple estimation for shear force at a general position u is

$$Q_x = (1 - U) Q_a + (U) Q_b \quad (52)$$

so that substitution of (43) into (51) and then (51) into (52) gives

$$Q_x = \sum_{i=1}^7 c_i (C_{qk})_i (F_k)_i \quad (53)$$

where

$$\begin{aligned} (C_{qk})_1 &= (1 - U)(C_{k-1,k})_{3,1} \cos \phi_a / (2\pi R_a) \\ (C_{qk})_2 &= (1 - U)(C_{k-1,k})_{3,2} \cos \phi_a / (2\pi R_a) \\ &\quad + U[(W_z)_k \cos \phi_b - (W_r)_k \sin \phi_b] / (2\pi R_b) \end{aligned}$$

etc.

The basic beam assumption is that the shear stress distribution is parabolic; that is, let

$$\tau_{xy} \cong (1 - 4y^2/H^2)(1.5/H)Q_x \quad (54)$$

or, using (53)

$$\tau_{xy} = \sum_i^7 (C_{tk})_i (F_k)_i \quad (55)$$

where

$$(C_{tk})_i = (1 - 4Y^2/H^2)(1.5/H)(C_{qk})_i$$

The shear strain γ_{xy} is simply

$$\gamma_{xy} = (1/E_s) \tau_{xy} \quad (56)$$

or using (55)

$$\gamma_{xy} = \sum_i^7 (C_{ek})_i (F_k)_i \quad (57)$$

where

$$(C_{ek})_i = (1/E_s)(C_{tk})_i$$

Then, the energy change produced by shear is

$$(\delta E_{in})_{shear} = \int (\tau_{xy} \delta \gamma_{xy}) dVol$$

or, using (55),

$$(\delta E_{in})_{shear} = \int \left[\sum_i^7 (C_{tk})_i (F_k)_i \right] (\delta \gamma_{xy}) dVol$$

or

$$(\delta E_{in})_{shear} = \sum_i^7 (F_k)_i \left[\int (C_{tk})_i (\delta \gamma_{xy}) dVol \right] \quad (58)$$

By definition,

$$(\delta E_{in})_{shear} = \sum_i^7 (F_k)_i \left[\delta (D_k)_i \right]_{shear} \quad (59)$$

and comparing (58) and (59)

$$\left[\delta (D_k)_i \right]_{shear} = \int (C_{tk})_i (\delta \gamma_{xy}) dVol$$

or

$$\left[(D_k)_i \right]_{shear} = \int (C_{tk})_i (\gamma_{xy}) dVol \quad (60)$$

Then, substitution of (57) into (58) gives

$$\left[(D_k)_i \right]_{\text{shear}} = \sum_1^7 j \left[(A_{kk})_{i,j} \right]_{\text{shear}} (F_k)_j \quad (61)$$

where

$$\left[(A_{kk})_{i,j} \right]_{\text{shear}} = \int (C_{tk})_i (C_{ek})_j \, dVol$$

Finally, the total flexibility of the segment K is simply

$$\left[(A_{kk})_{i,j} \right]_{\text{total}} = \left[(A_{kk})_{i,j} \right]_{(50)} + \left[(A_{kk})_{i,j} \right]_{(61)}$$

H. Middle Surface Shift

As in SEAL-SHELL-1, there is in SEAL-SHELL-2 provision for a middle surface shift between two flexible segments. This shift is specified by a straight rigid segment of zero thickness between two flexible segments of finite positive thicknesses, as shown in Fig. 9.^{*}

In order to bridge the gap between the two flexible segments, force and deflection continuity equations are needed. For a rigid body motion of segment K:

$$\begin{aligned} (D_k)_6 &= (D_{k-1})_6 - (Z_{ab})(D_{k-1})_7, \quad + \rightarrow \\ (D_k)_7 &= (D_{k-1})_7, \quad + \curvearrowright \\ (D_k)_8 &= (D_{k-1})_8 - (R_{ab})(D_{k-1})_k, \quad + \downarrow \end{aligned} \quad (62)$$

Assuming no hoop forces on the rigid segment shown in Fig. 10 gives

$$\left. \begin{aligned} (F_{k-1})_6 &= (F_k)_6 + \left[(W_r)_k \right] (F_k)_2 - \left[(P_i - P_o) \int_{Z_a}^{Z_b} 2\pi R dZ \right] (F_k)_1 \\ (F_{k-1})_7 &= (F_k)_7 + \left[(W_m)_k \right] (F_k)_2 + \left[(P_i - P_o) \int_{X_a}^{X_b} 2\pi R (X - X_a) dX \right] (F_k)_1 \\ (F_{k-1})_8 &= (F_k)_8 + \left[(W_m)_k \right] (F_k)_2 - \left[(P_i - P_o) \int_{R_a}^{R_b} 2\pi R dR \right] (F_k)_1 \end{aligned} \right\} \quad (63)$$

^{*}The segment is allowed to expand thermally in SEAL-SHELL-2.

FIG.9 MIDDLE SURFACE SHIFT IN RIGID SEGMENT K

FIG.10 FORCES ON RIGID SEGMENT K

Thus, the forces and deflections across the rigid link have been related by (62) and (63) and the centerline shift has been accomplished.

IV. THE SEAL-SHELL-2 PROGRAM, MOLLO

A. Input Data

The input deck consists of an accounting card, one or more sets of problem cards, and a blank card, as shown in Table I. The sets of problem cards are not separated with blank cards; also, each set must start with the title and comment cards, and must end with card 1001. If a problem has some data cards the same as in the preceding problem, the corresponding cards may be omitted; however, it should be remembered that all data of the preceding problem apply to the present problem unless replaced by input cards in the present problem.

Only in the title card is there an integer, N_c . All other numbers are in floating exponential form with the exponent to the right in each field, and all blanks are treated as zeros. For example,

$$N = 20 = \begin{array}{|c|c|c|c|c|c|c|c|} \hline + & 2 & . & & & + & & 1 \\ \hline \end{array} \quad \text{or}$$
$$\begin{array}{|c|c|c|c|c|c|c|c|} \hline & 2 & 0 & . & & + & & 0 \\ \hline \end{array}$$

The user may put identification numbers in columns 73 to 80 in cards 110 to 1001, in order to keep track of the original problem sets.

The contents of the cards in Table 1 are explained next:

Accounting Card

Prob. No. = problem number

User = code name of program user

Charge No. = charge number

Supv. No. = supervisor's number, to be used if the problems are to run over ten minutes of machine time

TABLE 1 - INPUT CARDS FOR SEAL - SHELL - 2, MO110

ACCOUNTING CARD

1 To 8	9	10	11 To 15	16	17 To 21	22 To 24	25 To 32	33 To 37	38 To 42	43 To 45	46 To 80	COLUMNS
PROB. NO.	\$		MO110		USER		CHARGE NO.	PRODN		SUPER NO.		CONTENTS

A SET OF PROBLEM CARDS

1	2 To 10	11 To 20	21 To 30	31	32 To 40	41 To 50	51 To 60	61 To 70	71	72	73 To 80	COLUMNS
	TITLE (On each page of output)							110.+0	N _c	IDENTITY	TITLE AND COMMENTS	
	COMMENTS (On first page of output)							I.+0		"		
	N.+0	K _{ZN} .+0	K _{RO} .+0	K _{MO} .+0	K _{RN} .+0	K _{MN} .+0	101.+0				"	END
		G _{ZN}	G _{RO}	G _{MO}	G _{RN}	G _{MN}	102.+0				"	CONDITIONS
	O.+0	R _O	Z _O				201.+0				"	R AND Z
	K _b .+0	R _b	Z _b	R _c	Z _c	S _e	20I.+0				"	
	K _a .+0	K _b .+0	H _a	H _b			30I.+0				"	H
	K _a .+0	K _b .+0	α	E _t	E _s	ν	40I.+0				"	α, E, ν
	K _a .+0	T _a (-H/2)	T _a (-H/4)	T _a (0)	T _a (H/4)	T _a (H/2)	50I.+0				"	T
	K _b .+0	T _b (-H/2)	T _b (-H/4)	T _b (0)	T _b (H/4)	T _b (H/2)	50J.+0				"	
	K _a .+0	K _b .+0	P _i	P _O			60I.+0				"	P
	K _a .+0	K _b .+0	W _R	W _M	W _Z		70I.+0				"	W
	K _a .+0	K _b .+0	K _{FFD} .+0	K _a .+0	K _b .+0	K _{FFD} .+0	80I.+0				"	K _{FFD}
	K _a .+0	K _b .+0	K _{SS} .+0	K _a .+0	K _b .+0	K _{SS} .+0	90I.+0				"	K _{SS}
							100I.+0				"	LAST CARD OF SET

BLANK CARD (Blank in all 80 columns)

Title Card

110 = program number for SEAL-SHELL-2

N_c = number of comment cards, 0 to 20

Comment Cards

I = card number, 1 to N_c

If $N_c = 0$, there are no comment cards.

Cards 101 and 102

N = number of segments, 2 to 100

$K_{__}$ = $\begin{cases} 0 & \text{if the deflection } D_{__} \text{ is given} \\ 1. & \text{if the force } F_{__} \text{ is given} \end{cases}$

$G_{__}$ = magnitude of $D_{__}$ or $F_{__}$

Note the units of input:

<u>Item</u>	<u>Units</u>
D_{ro}, D_{rn}, D_{zn}	in.
D_{mo}, D_{mn}	radians
F_{ro}, F_{rn}, F_{zn}	lb
F_{mo}, F_{mn}	in.-lb
$R_{__}, Z_{__}, H_{__}$	in.
α	(in./in.)/ $^{\circ}F$
E_t, E_s	psi
$T_{__}$	$^{\circ}F$
P_i, P_o	psi
W_{rk}, W_{zk}	lb
W_{mk}	in.-lb

Positive directions of external loads are shown in Fig. 11. Deflections have the same positive directions as the corresponding forces, except that D_{zo} is arbitrarily set to zero.

FIG. II EXTERNAL LOADS, OTHER THAN PRESSURE

Cards 201, 202, etc.

R_o, Z_o = radius and elevation at position 0

K_b = segment number at the end of a group of segments

K_a = segment number at the beginning of the group

K_a equals 1 + K_b of previous group; that is $K_a = 1$ for card 202.

In general, $1 \leq K_a \leq K_b \leq N$ and K_b on the cards is increased card to card up to $K_b = N$.

R_b, Z_b = radius and elevation at position K_b

S_e = 0 for a straight group of segments

For this case, R_c and Z_c are ignored.

S_e = +1 for a curved group as in Fig. 4

R_c, Z_c = radius and elevation at the center of the ellipse or circle of the curved group

Curved segments in a group are spaced at equal increments of the parametric angle β in Eq (4). Each group fits within one quadrant of the ellipse.

Cards 301, 302, etc.

K_a = first segment of a group

K_b = last segment of a group

H_a = thickness on segment K_a at position $K_a - 1$

H_b = thickness on segment K_b at position K_b

The thickness increases linearly along each segment and at an inbetween position K , the thickness is

$$\left[(K_b - K) \times H_a + (K+1 - K_a) \times H_b \right] / (K_b+1 - K_a)$$

It is to be noted that each data card overrides the input of preceding cards. For example, if

card 301 has $K_a = 1$ and $K_b = 40$ and

card 302 has $K_a = 10$ and $K_b = 12$, then

segments 1 to 9 have the thicknesses of card 301, segments 10 to 12 have the thicknesses of card 302, and segments 13 to 40 have the thicknesses of card 301.

Cards 401, 402, etc.

K_a = first segment of a group
 K_b = last segment of a group

$$\left. \vphantom{\begin{matrix} K_a \\ K_b \end{matrix}} \right\} 1 \leq K_a \leq K_b \leq N$$

α = coefficient of linear thermal expansion

E_t = tensile modulus of elasticity

E_s = shear modulus of elasticity

ν = Poisson's ratio, $0 \leq \nu \leq .5$

Cards 501, 502, etc.

On each odd numbered card,

K_a = first segment of a group, $1 \leq K_a < N$

$T_a(y)$ = temperature at y for position $K_a - 1$ on segment K_a

Thus, $T_a(-H/2)$ is at the outside surface,

$T_a(0)$ is at the middle surface, and

$T_a(+H/2)$ is at the inside surface.

On the following even numbered card,

K_b = last segment of the group, $K_a \leq K_b \leq N$

$T_b(y)$ = temperature at y for position K_b on segment K_b

The temperature varies linearly in each segment from K_a to K_b , and each segment K has the same total variation from $K-1$ to K for $K = K_a$ to K_b .

Cards 601, 602, etc.

P_i = inside pressure on the group

P_o = outside pressure on the group

Positive directions are shown in Fig. 11. Again, each card overrides the data of preceding cards.

Cards 701, 702, etc.

W_r = radial load on each segment
 W_m = bending load on each segment
 W_z = axial load on each segment

Cards 801, 802, etc.

K_a = first segment of the group, $1 \leq K_a \leq N$
 K_b = last segment of the group, $K_a \leq K_b \leq N$
 K = a segment, $K_a \leq K \leq K_b$
 K_{fa} = 1 if $FK_6(K-1)$, $FK_7(K-1)$, and $FK_3(K-1)$ are to be printed for K
= 0 if not
 K_{fb} = 1 if $[FK_6(K) + FW_r(K)]$, $[FK_7(K) + FW_m(K)]$, and
 $[FK_3(K) + FW_z(K)]$ are to be printed for K
= 0 if not
 K_{db} = 1 if $DK_6(K)$, $DK_7(K)$, and $DK_3(K)$ are to be printed for K
 K_{ffd} = $100 K_{fa} + 10 K_{fb} + K_{db}$

There are two sets of data per card. Each set overrides the data of preceding sets.

Cards 901, 902, etc.

K_a = first segment of the group, $1 \leq K_a \leq N$
 K_b = last segment of the group, $K_a \leq K_b \leq N$
 K = a segment, $K_a \leq K \leq K_b$
 K_{oa} = 1 if stresses are to be printed at $y = -H/2$ for position $K-1$ on segment K
= 0 if not
 K_{ca} = 1 if stresses are to be printed at $y = 0$ for position $K-1$ on segment K
= 0 if not
 K_{ia} = 1 if stresses are to be printed at $y = +H/2$ for position $K-1$ on segment K

K_{sa} = stress signal at position K-1 on segment K

$$= 4K_{oa} + 2K_{ca} + K_{ia}$$

$K_{ob}, K_{cb}, K_{ib}, K_{sb}$ = corresponding stress printout signals at position K on segment K

$$K_{ss} = 10 K_{sa} + K_{sb}$$

There are two sets of data per card. Each set overrides the data of preceding sets, as in card groups 30K to 80K.

Card 1001

This is the last card of each problem.

Blank Card

There is only one blank card in the input deck, and this blank card is the last card of the deck.

B. Output Data

Input

The first part of the output is a listing of all the input data used in the problem. Also, the radius R, elevation Z, and thickness H are listed for all positions $K = 0$ to N.

End Loads (Fig. 7)

The forces $F_{zn}, F_{ro}, F_{mo}, F_{rn}$, and F_{zn} are listed in nine columns for the following nine load combinations:

<u>Column</u>	<u>Effect</u>
1	Thermal loading T
2	Pressure loading P_i and P_o
3	Force loading W_r, W_m , and W_z
4	Boundary effect G_{zn}
5	Boundary effect G_{ro}
6	Boundary effect G_{mo}
7	Boundary effect G_{rn}
8	Boundary effect G_{mn}
9	Superposition of all eight effects

Flexibility Matrix

The flexibility matrix gives the five deflections D_{zn} , D_{rs} , D_{mo} , D_{rn} , and D_{mn} produced by unit values of the first eight loads listed above. Note:

$$D_{zn} = \sum_1^n k (D_k)_3, \quad F_{zn} = (F_n)_3$$

$$D_{ro} = (D_1)_4, \quad F_{ro} = (F_1)_4$$

$$D_{mo} = (D_1)_5, \quad F_{mo} = (F_1)_5$$

$$D_{rn} = (D_n)_6, \quad F_{rn} = (F_n)_6$$

$$D_{mn} = (D_n)_7, \quad F_{mn} = (F_n)_7$$

End Deflections

The deflections D_{zn} , D_{ro} , D_{mo} , D_{rn} , and D_{mn} are listed for the nine load effects.

Distributed Forces and Deflections

The distributed force and deflection output is set by K_{ffd} for each segment, as discussed in the input.

Stresses

The stress output is controlled by K_{ss} for each segment, as discussed in the input. For each location specified, the six stresses listed are

τ_{xy} , S_x , S_θ , S_y , S_{max} , and $2\tau_{max}$, where:

$$S_{max} = \sqrt{S_1^2 + S_2^2 + S_3^2 - S_1S_2 - S_2S_3 - S_3S_1}$$

$$2\tau_{max} = \text{Max} (S_1 - S_2, S_2 - S_3, S_3 - S_1)$$

$$S_1 = S_{ave} + S_{rad}$$

$$S_2 = S_{ave} - S_{rad}$$

$$S_3 = S_\theta$$

$$S_{ave} = 0.5 (S_x + S_y)$$

$$S_{rad} = \sqrt{[0.5(S_x - S_y)]^2 + [\tau_{xy}]^2} \left. \vphantom{S_{rad}} \right\} \begin{array}{l} \text{Mohr's} \\ \text{Circle} \\ \text{Parameters} \end{array}$$

Thus, S_{max} is the Von-Mises generalized stress and $2\tau_{max}$ is twice the maximum shear stress.

V. SAMPLE PROBLEMS

A. Thick Shell Effect on Pressure Stresses

In thin shell theory, the stresses for a cylinder and sphere loaded by internal pressure are:

<u>Stress</u>	<u>Cylinder</u>	<u>Sphere</u>
S_x	0	$0.5 PR/H$
$S_\theta \equiv S_t$	PR/H	$0.5 PR/H$
S_y	0	0

where

P = inside pressure

R = middle surface radius

H = thickness

S_t = nominal stress in thin shell theory

In thick shell theory (Lame Effect), the stresses are:

<u>Stress</u>	<u>Cylinder</u>	<u>Sphere</u>
S_x	0	$\frac{P}{2} \cdot \frac{(R_o^3/R_y^3) + 2}{(R_o^3/R_i^3) - 1}$
S_θ	$\frac{P}{1} \cdot \frac{(R_o^2/R_y^2) + 1}{(R_o^2/R_i^2) - 1}$	$\frac{P}{2} \cdot \frac{(R_i^3/R_y^3) + 2}{(R_o^3/R_i^3) - 1}$
S_y	$\frac{P}{1} \cdot \frac{1 - (R_o^2/R_y^2)}{(R_o^2/R_i^2) - 1}$	$\frac{P}{1} \cdot \frac{1 - (R_o^3/R_y^3)}{(R_o^3/R_i^3) - 1}$

where

$$R_o = R + 0.5H = \text{outside radius}$$

$$R_i = R - 0.5H = \text{inside radius}$$

$$R_y = R + Y = \text{any radius}$$

Typical distributions of stress vs Y for a cylinder under internal pressure are shown in Fig. 12. The following table of stress errors compares thin shell theory vs SEAL-SHELL-2 calculations:

TABLE II - STRESS ERRORS FOR CYLINDER WITH $R_o/R_i = 1.286$

Method	Location =	$R_i/H = 3.5$	$R/H = 4.0$	$R_o/H = 4.5$
Thick Theory	$S_\theta/P =$	4.063	3.469	3.063
	$S_y/P =$	-1.000	-0.407	0
Thin Theory	$S_t/P=S_\theta/P =$	4	4	4
	$\Delta S_\theta/P =$	.063	.531	.937
	$\Delta S_\theta/S_t =$	1.6%	13.3%	23.4%
	$S_y/P =$	0	0	0
	$\Delta S_y/P =$	1.000	0.407	0
	$\Delta S_y/S_t =$	25.0%	10.2%	0.0%
SEAL-SHELL-2 Results	$S_\theta/P =$	4.072	3.493	3.062
	$\Delta S_\theta/P =$	.009	.024	.001
	$\Delta S_\theta/S_t =$	0.2%	0.6%	0.0%
	$S_y/P =$	- .979	- .351	0
	$\Delta S_y/P =$	.021	.056	0
	$\Delta S_y/S_t =$	0.5%	1.4%	0.0%

The stresses of thin shell theory and SEAL-SHELL-2 are compared with the stress of thick shell theory to determine stress errors. All errors are given as percentages of the nominal thin-shell stress S_t . For a cylinder with $R_o/R_i = 1.286$, the largest surface stress errors are 25.0% for thin shell theory and 0.5% for SEAL-SHELL-2. Plots of maximum surface stress error^{*} vs radius ratio R_o/R_i are shown in Fig. 13 for spheres and cylinders.

^{*}It is to be noted that 23.4% is used in Fig. 13 for thin-shell error, since it is not really fair to compare thickness stresses.

FIG. 12 CYLINDER UNDER INTERNAL PRESSURE

COMPARISON OF PRESSURE STRESSES IN SPHERE AND CYLINDER: SEAL-SHELL-2 VS THIN SHELL THEORY

Fig. 13

For radius ratios less than 1.7, the SEAL-SHELL-2 errors are less than 10% and the thin-shell theory errors are less than 60%.

B. Decay Length Effect

In the previous sample problem, there were no discontinuity stresses. To demonstrate the effect of segment length on discontinuity stresses, three problems were run with the SEAL-SHELL-2 program. In each problem, a cylinder has an end shear load, Q_0 , lb/in.; the dimensions are

$$R = 20, H = 1, X_n = 35$$

and the properties are

$$E = 30 \times 10^6, \nu = 0.3$$

for which the decay length is

$$1/\lambda = \sqrt[4]{\frac{R^2 H^2}{3(1 - \nu^2)}} = 3.5$$

The segment lengths of the three problems are:

$$0.5/\lambda = 1.75, 1/\lambda = 3.5, \text{ and } 2/\lambda = 7$$

Thin-shell theory gives the meridional bending stress as

$$S_x = \left(\frac{6Q_0}{\lambda H^2} \right) \left(e^{-\lambda x} \sin \lambda x \right)$$

and the resulting distributions are shown in Fig. 14.

It is interesting to note that even for the segment length X_{ab} equal to twice the decay length $1/\lambda$, the segment stress distribution is a good straight-line fit to the actual curve. An order of magnitude check of the stress error is given by the discontinuity in stress between adjacent segments. Wherever serious discontinuities occur, the SEAL-SHELL-2 problem should be rerun with shorter segments in the neighborhood of the discontinuities. In general, the segment length should be less than half of the decay length.

C. Model Pressure Vessel

A model pressure vessel was tested photoelastically by the Westinghouse Research Laboratories. The geometry and material properties are shown in

FIG.14 EFFECT OF SEGMENT LENGTH X_{ab} VS DECAY LENGTH $1/\lambda$ FOR CYLINDER WITH END SHEAR LOAD Q_0 LB/IN AND WITH $R=20H$

Fig. 15, along with the segment locations used in the SEAL-SHELL-2 problem. A comparison of the measured stress and calculated stresses is presented in Fig. 16.

The comparison of SEAL-SHELL-2 data and photoelastic data shows good agreement, probably with about the same accuracy as the test.

Input cards used for the pressure vessel problem are listed in Table III.

TABLE III - INPUT CARDS FOR PRESSURE VESSEL PROBLEM

10	20	30	40	50	60	70
PRESSURE VESSEL TEST						1.10+2
1.+2	1.+0	1.+0	0.+0	1.+0	0.+0	1.01+2
						1.02+2
	1.-3	8.52+0				2.01+2
2.+1	1.88+0	8.02+0	0.+0	4.77+0	1.+0	2.02+2
5.+1	2.75+0	6.5 +0	1.+0	6.5 +0	1.+0	2.03+2
6.+1	2.75+0	1.+0				2.04+2
9.+1	2.+0	0.+0	2.+0	1.+0	1.+0	2.05+2
1.+2	1.+3	0.+0				2.06+2
1.+0	6.+1	5.-1	5.-1			3.01+2
6.1+1	9.+1	5.-1	1.+0			3.02+2
9.1+1	1.+2	1.+0	1.+0			3.03+2
1.+0	1.+2	0.+0	4.3+3	1.5+3	4.5-1	4.01+2
1.+0	0.+0	0.+0	0.+0	0.+0	0.+0	5.01+2
1.+2	0.+0	0.+0	0.+0	0.+0	0.+0	5.02+2
1.+0	1.+2	1.+3				6.01+2
1.+0	1.+2	0.+0	0.+0	0.+0		7.01+2
1.+0	1.+2	0.+0				8.01+2
1.+0	1.+2	5.+0				9.01+2
						1.001+3
BLANK CARD						

D. Nozzle Design

Nozzles are usually analyzed with different geometries at the intersection of a cylinder and sphere. One such design is shown in Fig. 17. Photoelastic test results from Ref 4 are given in Fig. 18.

The geometry used in the SEAL-SHELL-2 analysis is also shown in Fig. 17, and the resulting stresses are put in Fig. 18 for direct comparison with the experimental data.

FIG. 15 PRESSURE VESSEL USED IN
PHOTOELASTIC TEST AND
SEAL-SHELL-2 PROBLEM

$E = 4300 \text{ PSI}$

$E_s = 1500 \text{ PSI}$

$V = 0.45$

SEAL-SHELL-2 GEOMETRY:

K	R	Z	ϕ	X	H
0	0.00	8.52	0	0	0.5
20	1.88	8.02	0.52	2.0	0.5
50	2.75	6.50	1.57	3.8	0.5
60	2.75	1.00	1.57	9.3	0.5
90	2.00	0.00	3.14	10.7	1.0
100	0.00	0.00	3.14	12.7	1.0

FIG.16 SEA-SHELL-2 VS PHOTO-ELASTIC DATA
(.....)
(——)

FIG.17 NOZZLE DESIGN

PHOTOELASTIC MODEL

(NOT TO SCALE)

SEAL-SHELL-2 MODEL

K	R	Z	R _c	Z _c	S _c
0	1.563	14.34			
10	1.563	9.34			
20	1.78	7.62			
30	2.19	7.127	2.38	7.70	-1.0
100	.001	-7.456	.001	0	+1.0

K _a	K _b	H _a	H _b
1	10	.25	.25
11	20	.25	.68
21	24	.68	.78
25	26	.78	.73
27	28	.73	.63
29	30	.63	.579
31	100	.579	.579

FIG.18 NOZZLE STRESSES FOR INSIDE PRESSURE P

STRESS	MERIDIONAL	HOOP
PHOTOELASTIC	—	- - -
SEAL-SHELL-2	x x x x	o o o

On the outside surface, the error in peak stress is

$$100\% \times |10.1 - 11.3| / 11.3 = 11\% ,$$

and on the inside surface, the error in peak stress is

$$100\% \times |11.5 - 11.3| / 11.3 = 2\% .$$

E. Pressure Vessel with Thick Ellipsoidal Head

In 1957, H. Fessler and R. T. Rose made a photoelastic study of stresses in the heads of thick pressure vessels. As reported in Ref 5, one geometry has

$$\begin{aligned} H_e &= H_c = 1 \\ R_c &= 5 \\ R_e/Z_e &= 1.37 \\ \nu &= 0.5 \end{aligned}$$

where

e = subscript for ellipse
c = subscript for cylinder

and the test results are shown in Fig. 19.

A SEAL-SHELL-2 problem was run with $\nu = .48$ and with the geometry of the following table:

K	P_b	Z_b	R_c	Z_c	S_c
0	5.0	13.65			
20	5.0	3.65			
60	.0005	0.00	.0005	3.65	+1

The resulting stresses are also shown in Fig. 19 in direct comparison with photoelastic data. The errors in peak stress for the outside and inside surfaces are

$$100\% \times |.87 - .91| / .91 = 4\%$$

and

$$100\% \times |1.01 - 1.02| / 1.02 = 1\%$$

respectively.

FIG.19 ELLIPSOIDAL HEAD ON THICK CYLINDER

STRESS	MERIDIONAL	HOOP
PHOTOELASTIC	————	-----
SEAL-SHELL-2	x x x x	o o o

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