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SCATTERING OF POSITIVE PIONS ON PROTONS AT 310 Mev:
RECOIL-NUCLEON POLARIZATION AND PHASE-SHIFT ANALYSIS

James Herbert Foote
(Thesis)

September 16, 1960

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ABSTRACT

The recoil-proton polarization in π^+ -p elastic scattering at 310-Mev incident-pion laboratory kinetic energy has been experimentally measured at four scattering angles with plastic scintillation counters. These polarization results have been combined with accurate differential- and total-cross-section data at 310 Mev, and a comprehensive phase-shift analysis performed. The D-wave phase shifts were found to be definitely needed in order to attain an adequate fit to the data. A general search for phase-shift solutions was carried out, using S-, P-, and D-wave phase shifts. The solution that best fits the data is of the Fermi type. The calculated errors in the phase shifts of this set vary from 0.4 to 0.6 deg. Because it was felt that these errors might be deceptively restrictive, the effects of small nuclear F-wave phase shifts on the results of the analysis were investigated and were found to be large; not only are the uncertainties in the original Fermi-type solution increased, but additional sets of phase shifts that fit the data well arise. One of these new solutions is similar to the original Fermi set except that the magnitudes of the phase shifts in this new fit are in general larger than those in the initial solution and the signs of the D-wave phase shifts are reversed. The nuclear phase shifts in the original Fermi solution and their rms errors are (when

F-wave phase shifts are allowed): $S_{3,1} = -17.2 \pm 2.6$ deg, $P_{3,1} = -2.9 \pm 4.0$ deg, $P_{3,3} = 135.0 \pm 0.6$ deg, $D_{3,3} = 3.1 \pm 2.6$ deg, $D_{3,5} = -4.9 \pm 2.1$ deg, $F_{3,5} = 0.5 \pm 0.6$ deg, $F_{3,7} = -0.6 \pm 1.4$ deg. Although theory appears to favor this set, further theoretical and experimental evidence is desirable. Inelastic-scattering processes were neglected during the phase-shift analysis; calculations indicate that, if these processes could properly be taken into account, any changes in the quoted values of the phase shifts would probably be well within the corresponding errors given here. Extension of the phase-shift inquiries to include G waves was attempted, but it was observed that the available data and theory do not allow the G-wave interaction to be significantly incorporated into the analysis.

Results obtained for the recoil-proton polarization, related rms experimental errors, and mean center-of-mass recoil angles are: $+0.044 \pm 0.062$ at 114.2 deg, -0.164 ± 0.057 at 124.5 deg, -0.155 ± 0.044 at 133.8 deg, and -0.162 ± 0.037 at 145.2 deg. The sign of the recoil-proton polarization is defined to be positive when a preponderance of the protons recoiling to the right side of the incident pi-meson beam had their spin vectors pointing up. A beam of 1×10^6 pions per sec incident upon a 1.0-g/cm^2 thick liquid-hydrogen target produced the recoil protons, which were then scattered by a carbon target at a mean energy varying with recoil angle from 113 to 141 Mev. The polarization of the recoil protons was analyzed by measuring the asymmetry produced in the carbon scattering. A proton beam of known polarization was used in order to determine the analyzing ability (measured asymmetry divided by the polarization of the incident protons) of the system at each recoil angle. Values obtained for the analyzing ability vary from 0.41 to 0.57.

I. INTRODUCTION

The investigation of pi mesons and their interactions is of fundamental importance to the understanding of nuclear phenomena. For example, we think of nuclear forces as resulting principally from the interchange of pions (positive, negative, and neutral pi mesons) between nuclear particles. Because of the basic nature of nucleons (neutrons and protons), their interactions with pions are of particular significance. Present theories concerning pion-nucleon processes do not completely explain the experimental data. The improvement of the quantity and quality of the knowledge obtained from pion experiments will aid the theoretical interpretation of the processes involved. Eventually, it is hoped that sufficient data will be available so that one can accurately check any proposed theory describing the pion-nucleon interactions and its associated phenomena.

An important class of pion-nucleon interactions includes π^+ -p and π^- -p elastic scattering. In order to investigate these processes, we can measure the differential cross section (DCS), the total cross section, and the polarization of the recoil protons as a function of scattering angle. Cross sections have been measured by many experimenters at many energies and with varying degrees of accuracy. However, very few measurements exist of the recoil-proton polarization in elastic pion-proton scattering because of the difficulty of obtaining pion beams of high energy and, in addition, high intensity. Beams with both of these characteristics are needed in order that the polarization of the recoil protons can be satisfactorily analyzed. If the flux of these protons were not adequate or if their energy were too low, we would not be able to determine their polarization with the desired accuracy.

The data obtained in $\pi^\pm$ -p scattering experiments can be analyzed by the method of partial waves. In this type of analysis, the quantum-mechanical wave function representing the amplitude for the scattering process is expanded in a series of terms. Each term is

related to a definite orbital-angular-momentum state of the $\pi^\pm$ -p system and is the "partial-wave" function corresponding to that particular state. The states with the orbital-angular-momentum quantum number $L = 0, 1, 2, 3, 4, \dots$ are referred to as S, P, D, F, G, $\dots$ states. The successive terms in the partial-wave expansion are therefore known as S, P, D, F, G, $\dots$ waves. Owing to the proton spin, more than one total-angular-momentum state is usually contained in each orbital-angular-momentum term.

The partial-wave expansion contains parameters called the phase shifts, which depend upon the characteristics of the interaction and which have zero magnitude if there is no interaction. The pion-nucleon scattering process is generally different for the different orbital- and total-angular-momentum states involved. Thus the related phase shifts vary with L and J , where J is the total-angular-momentum quantum number. The phase shifts are obtainable from such experimental results as cross-section and polarization measurements. The amount of success with which an analysis in terms of phase shifts can be performed is a measure of the completeness of the experimental data at the energy being considered. A satisfactory comprehensive theory must predict the behavior and magnitude of the phase shifts. These parameters therefore provide a meeting place for theory and experiment. The more accurately the phase shifts are known, the more severely is an acceptable theory limited.

Although the partial-wave expansion contains an infinite number of terms, we neglect all but the first few when considering the pion-nucleon nuclear interaction at moderate energies. In so doing, we assume that the orbital-angular-momentum states of higher order (and their related phase shifts) become important only gradually as the energy of the incident pion increases. This assumption appears reasonable from a classical viewpoint. Classically, an increase in the energy of the pion is related to an increase in the maximum value of angular momentum that can lie within the reach of the nuclear force. From

this point of view, only the S wave is important at very low energies. As the incident pi meson obtains more energy, the P-wave interaction begins to enter. At still greater energies, we expect to see the effects of D waves and even higher-order angular-momentum states.

Many phase-shift analyses of $\pi^\pm$ -p cross-section data have been performed in the past. At pion laboratory kinetic energies below about 200 Mev, the experimental data have been fitted satisfactorily by using only the first two terms of the partial-wave expansion, that is, S and P waves. Above the 200-Mev energy region, the possible participation of D waves in the pion-proton interaction has made the results of the data analyses uncertain. It has been difficult to determine the values of the D-wave phase shifts because of the insensitive manner in which these parameters enter into the cross-section equations and the relatively large errors in many of the cross-section measurements. The indefiniteness of the D-wave phase shifts has introduced uncertainties in other phase shifts. In these earlier analyses, not only have the values and signs of some of the phase shifts in a solution been uncertain, but several different types of solutions have been obtained. These dissimilar sets of phase shifts are all good fits to the data.

Measurements of the recoil-proton polarization can be very useful in removing these uncertainties. Different variations of the polarization with scattering angle are predicted by the various types of phase-shift solutions obtained when only the cross-section data is available. On the basis of polarization measurements, one may therefore be able to decide which type of phase-shift set is the physically valid one. These measurements also improve our knowledge of the individual parameters in a solution because many of the phase shifts are sensitive to the recoil-proton polarization data. The phase shifts related to D waves are especially sensitive to the results of polarization measurements.

There now exists a limited amount of experimental information on the polarization of the recoil protons in $\pi^\pm$ -p elastic scattering. Kunze, Romanowski, Ashkin, and Burger used a counter-controlled expansion cloud chamber in which a carbon analyzing target was placed.¹ They investigated π^- -p scattering at 225-Mev incident pion energy. (All energies mentioned in this report are in the laboratory system unless otherwise specified.) The recoil protons, which were produced by the pi-meson beam, entered the cloud chamber and were scattered by the carbon target. These scattering events were photographed, and the resultant tracks examined. The polarization of the protons entering the chamber could then be determined. In this way, data were obtained at two recoil angles.

In another polarization experiment, Grigor'ev and Mitin investigated π^+ -p scattering at 307 Mev with the aid of photographic emulsions.² The emulsions served as both analyzer of the recoil-proton polarization and detector of the asymmetry thus produced. Through use of this method, they obtained the polarization of the protons at one recoil angle.

Vasilevsky and Vishnyakov report tentative results at three angles on the polarization of the recoil protons in π^- -p scattering at about 300 Mev.³ This is a counter experiment and employs approximately 900 Geiger counters to detect the desired events.

There are large experimental errors in all the recoil-proton polarization results just discussed. Nevertheless, these data have been useful in the analysis of pion-proton scattering. The polarization results have indicated that certain sets of phase shifts are not physically acceptable. (The advent and development of the dispersion relations have also aided in eliminating certain ambiguities.) Worthwhile information has been obtained concerning the magnitudes and signs of the π^+ -p D-wave phase shifts; there are, however, still sizable errors associated with these parameters. Considerable uncertainties also exist in the values of other phase shifts.

Before a precise set of phase shifts can be obtained, accurate polarization experiments are needed. In deciding to perform this type of experiment, we have had to consider carefully the problem of obtaining a high-energy, high-intensity pion beam. We have fortunately been able to produce a beam with the desired characteristics at the 184-inch synchrocyclotron. The resulting beam contains positive pions and has a maximum intensity of approximately 2×10^6 pions per sec at about 300 Mev. This energy is adequately high so that D-waves should definitely be present, but yet sufficiently low so that only a minimum of inelastic scattering should occur. Inelastic scattering is undesirable because it can complicate the measurements and subsequent analysis.

Our pion beam has now been used to detect the polarization of the recoil protons in π^+ -p elastic scattering at 310 Mev. Plastic scintillation counters were used for this purpose. We have obtained data at four recoil angles and with an accuracy considerably exceeding any previous measurement of this kind. The polarization data have been combined with accurate cross-section data at 310 Mev, and a comprehensive phase-shift analysis performed. This report discusses the polarization measurements, the phase-shift analysis, and the results of these endeavors.

II. THEORETICAL BACKGROUND

Before we consider the polarization measurements and phase-shift analysis, a discussion of the relevant theory is in order. Let us first discuss the quantities pertinent to the experiment, and present the theoretical basis for expecting the recoil protons to be polarized. The necessary formulas for the phase-shift analysis will then be developed. Finally, we will explain the various types of phase-shift ambiguities that can arise in this kind of analysis.

A. Polarization

1. Polarization and Related Quantities

In order to define polarization and its related quantities, let us employ an x-y-z Cartesian-coordinate system. The associated spherical angular coordinates θ_i and ϕ_i are defined in the customary manner.* We consider a beam of protons moving along the z axis in the +z direction, with a scattering target placed at the origin. Let the x and z axes lie in the horizontal plane and allow the +y direction to be up. The polarization of the incident proton beam in the direction perpendicular to the horizontal plane can be defined as $P = (N_U - N_D)/(N_U + N_D)$ where N_U and N_D are the numbers of incident protons per unit beam with their spin vectors pointing up and down, respectively. (When we speak of protons with their spin vectors pointing parallel and antiparallel to a specified direction, we are referring to the two distinct groups of protons obtained in a Stern-Gerlach experiment in which the inhomogeneity in the magnetic field is along the specific direction being considered.)

*The angle θ_i is measured with respect to the +z axis, and ϕ_i is measured in the x-y plane with respect to the +x axis, the +y axis lying at $\phi=90$ deg. In this report, we designate general laboratory scattering angles by θ_i and ϕ_i , and laboratory angles at the center of the particle detectors by Θ_i and $\bar{\Phi}_i$, where i is an identifying subscript.

If a beam of protons is polarized in the direction perpendicular to the horizontal (x-z) plane and is incident upon a target composed of spin-zero nuclei, one can write [Eq. (6) of Chamberlain et al.⁴]

$$I(\theta_i, \phi_i) = I_0(\theta_i) [1 + P_{inc} P(\theta_i) \cos (\phi_i)] , \quad (II-1)$$

where

$I(\theta_i, \phi_i)$ is the value of the DCS for elastic scattering of polarized protons in the direction defined by the laboratory angles θ_i and ϕ_i , P_{inc} is the polarization (in the y direction) of the incident proton beam, $I_0(\theta_i)$ is the DCS for the elastic scattering of an unpolarized proton beam under conditions identical to those for the scattering of the polarized beam, and $P(\theta_i)$ is the polarization that would be produced by the previously mentioned elastic scattering of an unpolarized beam (this polarization is perpendicular to the plane of the scattering). The quantities $I_0(\theta_i)$ and $P(\theta_i)$ depend upon the angle and energy of scattering and the target material.

Let us consider the elastic scattering of protons in two directions defined by the same value of θ_i but with values of ϕ_i differing by 180 deg. For $P_{inc} \neq 0$, Eq. (II-1) states that, in general, there will be a difference in the number of protons scattering in these two directions. This difference is quantitatively described by the asymmetry, e , which can be defined as

$$e(\theta_i, \phi_i) = \frac{N(\theta_i, \phi_i) - N(\theta_i, \phi_i + \pi)}{N(\theta_i, \phi_i) + N(\theta_i, \phi_i + \pi)} . \quad (II-2)$$

The quantities $N(\theta_i, \phi_i)$ and $N(\theta_i, \phi_i + \pi)$ are the intensities of elastically scattered protons at the designated angles. Combining Eqs. (II-1) and (II-2), one obtains

$$e(\theta_i, \phi_i) = P_{inc} P(\theta_i) \cos (\phi_i) . \quad (II-3)$$

The asymmetry can be experimentally determined by observing, with a proton detector, the intensities $N(\theta_i, \phi_i)$ and $N(\theta_i, \phi_i + \pi)$. The center of the detector is customarily placed at ϕ_i values of 0 and 180 deg, so that the $\cos(\phi_i)$ factor of Eq. (II-3) is unity. Then one measures the largest possible asymmetry for a given P_{inc} and θ_i . A large asymmetry is desirable in order to minimize the effects of certain types of experimental errors. Owing to the finite size of the detector, an asymmetry averaged over a region of θ_i and ϕ_i values is obtained. We therefore rewrite Eq. (II-3) as

$$\bar{e}(\Theta_i) = \bar{P}_{inc} \bar{P}(\Theta_i), \quad (\text{II-4})$$

where the bars indicate that the quantities are average values and Θ_i is the value of θ_i at the center of the particle detector. In obtaining Eq. (II-4), we have assumed $\phi_i = 0$; any significant variation from +1 of the $\cos(\phi_i)$ factor in Eq. (II-3) is included in $\bar{P}(\Theta_i)$. The bar over P_{inc} allows for averaging that may take place when obtaining the polarized protons.

The scattering of a polarized beam in order to determine its polarization is referred to as an "analyzing" scattering. A proton that has been scattered and then detected is designated an "analyzed" proton. The factor $\bar{P}(\Theta_i)$ in Eq. (II-4) is called the "analyzing ability" of the arrangement. This is not to be confused with the "analyzing efficiency," which is defined later in Table V.

We have discussed only elastic scattering in this section. When protons are incident upon an analyzing target such as carbon, inelastic scattering can also occur. Although some kinds of inelastic processes may produce as large an asymmetry as the elastic scattering, other types do not. Thus the inelastic reactions tend to lower the average measurable asymmetry. We therefore try to arrange the experimental conditions in an asymmetry measurement so as to discriminate against as many of the inelastic processes as possible.

2. Recoil-Proton Polarization

Fermi examined theoretically the elastic scattering of pions on unpolarized protons and showed that one can in general expect the recoiling protons to be polarized.⁵ The polarization will be perpendicular to the plane of the scattering. We now discuss a few aspects of pion-proton elastic scattering and obtain an expression for the recoil-proton polarization in terms of scattering amplitudes.

It is convenient to discuss the pion-proton scattering in the center-of-mass (c.m.) system. Let the scattering take place in the horizontal plane, which is experimentally the simplest plane to treat. We can consider either the pion or the proton to be moving in the $+z$ direction before the collision. The scattering is assumed to occur at the origin and the $+y$ direction is again up, perpendicular to the plane of the scattering. The spherical angular coordinates in the center of mass will be denoted θ and ϕ (with no subscript). We will use the symbol $\theta_{\text{c.m.}}$ to represent the angle in the c.m. system between the direction of scattering and the initial direction of motion of either particle. This angle will be referred to as the c.m. scattering angle.

Quantities that remain constant during the pion-proton elastic-scattering process are the quantum numbers J (total angular momentum), L (orbital angular momentum), s (proton spin), and M_J (component of the total angular momentum in the z direction). The quantum number L remains constant because it can only change by one unit and still combine with s to conserve total angular momentum. Such an alteration in L would violate a basic assumption in this discussion, the conservation of parity. Although L and s cannot change during the scattering process, the components of the orbital angular momentum and proton spin in the z direction can change. This situation gives rise to "spin-flip" scattering, in which the component of the proton spin in the direction of the incident beam is reversed.

In order mathematically to express Fermi's conclusion that the recoiling protons will be polarized, let α and β represent the proton spin states corresponding to the spin in the $+z$ and $-z$ directions, respectively. We denote the c.m. elastic-scattering amplitudes by the symbol "S" with two subscripts, where the first subscript refers to the final spin state and the second to the initial spin state (the reverse of Fermi's subscript notation). Thus the non-spin-flip scattering amplitudes will be denoted $S_{\alpha\alpha}$ and $S_{\beta\beta}$, and the spin-flip scattering amplitudes, $S_{\beta\alpha}$ and $S_{\alpha\beta}$. These four scattering amplitudes are functions of the c.m. scattering angle (see Section II-B-1).

The elastic scattering of a beam of pions on unpolarized protons can be described in terms of the scattering amplitudes just presented. If N_U and N_D are the numbers of the recoiling protons per unit incident pion beam with their spin vectors pointing in the $+y$ and $-y$ directions, respectively, then Fermi has shown that one can write (see Appendix A for derivation)

$$\left. \begin{aligned} N_U &\propto |S_{\alpha\alpha} - i S_{\beta\alpha}|^2 \\ N_D &\propto |S_{\alpha\alpha} + i S_{\beta\alpha}|^2 \end{aligned} \right\} \quad (II-5)$$

where both expressions contain the same constant of proportionality. In his derivation, Fermi used the equalities $S_{\alpha\alpha} = S_{\beta\beta}$ and $S_{\beta\alpha} = -S_{\alpha\beta}$. We will discuss these equalities in Section II-B-1.

Let I denote the DCS in the c.m. system for the pion-proton elastic scattering, and let P be the polarization of the recoiling protons. Using Eqs. (II-5), one obtains

$$I \propto N_U + N_D \propto |S_{\alpha\alpha}|^2 + |S_{\beta\alpha}|^2, \quad (II-6)$$

and

$$\begin{aligned} P &= \frac{N_U - N_D}{N_U + N_D} = \frac{i(S_{\alpha\alpha} S_{\beta\alpha}^* - S_{\beta\alpha} S_{\alpha\alpha}^*)}{|S_{\alpha\alpha}|^2 + |S_{\beta\alpha}|^2} \\ &= \frac{2 \operatorname{Im}(S_{\alpha\alpha}^* S_{\beta\alpha})}{|S_{\alpha\alpha}|^2 + |S_{\beta\alpha}|^2}. \end{aligned} \quad (II-7)$$

Because the scattering amplitudes are complex and vary with scattering angle (see Section II-B-1), the polarization will be different in general from zero and will vary with angle. Equation (II-7) shows that there can be no polarization unless both spin-flip and non-spin-flip scattering occur.

We can determine the polarization of the protons recoiling at a specific laboratory angle by performing an analyzing scattering as described in Section II-A-1. Let us rewrite Eq. (II-4) as

$$\bar{e}(\Theta_2) = \bar{P}_1 \bar{P}_2(\Theta_2), \quad (\text{II-8})$$

where 1 refers to the pion-proton scattering, and 2 to the analyzing scattering. According to Eq. (II-8), we can ascertain the recoil-proton polarization, $\bar{P}_1$, by measuring $\bar{e}(\Theta_2)$ and $\bar{P}_2(\Theta_2)$. Our asymmetry measurements will be described in Sections III-B and III-C. The determination of $\bar{P}_2(\Theta_2)$ will be discussed in Section III-D. Once the polarization as a function of laboratory scattering angle has been measured, a relativistic kinematical transformation yields the polarization as a function of c.m. scattering angle. Because the polarization is perpendicular to the plane of the scattering, its values do not change in the transformation; only the scattering angles change.

B. Phase-Shift Formulas

In this section and its related appendices, we apply the method of partial waves to pion-proton scattering. Expressions for the non-spin-flip and spin-flip elastic-scattering amplitudes are derived. Both nuclear and Coulomb effects are taken into account. Also, we incorporate into the equations first-order relativistic corrections to the Coulomb-scattering amplitudes. Finally, we discuss our phase-shift notation and the utilization of the general phase-shift equations.

1. Equations Describing Pion-Proton Nuclear Elastic Scattering

The wave function describing the scattered particles in pion-proton nuclear elastic scattering can be separated into a non-spin-flip (NF) and a spin-flip (F) part:

$$\psi_{SC} = S_{NF} \chi_{1/2}^{M_S^I} + S_F \chi_{1/2}^{-M_S^I}, \quad (II-9)$$

where

$$S_{NF} = g(\theta) \frac{e^{ikr}}{r}, \quad (II-10)$$

and

$$S_F = h(\theta, \phi) \frac{e^{ikr}}{r}, \quad (II-11)$$

$$g(\theta) = \kappa \sum_{L=0}^{\infty} \left[(L+1) \left(\frac{b_L^+ \exp(2i\delta_L^+) - 1}{2i} \right) + L \left(\frac{b_L^- \exp(2i\delta_L^-) - 1}{2i} \right) \right] P_L(\cos \theta), \quad (II-12)$$

and

$$h(\theta, \phi) = \kappa \sum_{L=1}^{\infty} \left[\frac{b_L^+ \exp(2i\delta_L^+) - 1}{2i} - \frac{b_L^- \exp(2i\delta_L^-) - 1}{2i} \right] D_L Y_L^{\pm 1}(\theta, \phi). \quad (II-13)$$

These results are derived in Appendix B.

Definitions of quantities found in Eqs. (II-9) through (II-13) are:

L = orbital-angular-momentum quantum number,

r = distance between the two particles involved in the collision,

θ, ϕ = spherical angular coordinates defining the direction of scattering of the particle (either pion or proton) considered to move in the $+z$ direction before the collision,

λ = wavelength of either particle in the c. m. system ($k=1/\lambda$),

$\delta_L^\pm$ = phase shifts related to states with a specified L and with $J=L\pm 1/2$, where J is the total-angular-momentum quantum number (the definition of $b_L^\pm$ is given in the next paragraph),

$\chi_{1/2}^{M_S}$ = proton-spin wave function,

M_S = z component of the proton spin (if the proton is in the spin state α , $M_S=+1/2$; if the proton is in the state β , $M_S=-1/2$),

M_S^I = value of M_S before the collision (I =initial),

$P_L(\cos \theta)$ = Legendre polynomial,

$$D_L = \left(\frac{4\pi L(L+1)}{2L+1} \right)^{1/2}, \quad (\text{II-14})$$

$Y_L^{\pm 1}(\theta, \phi)$ = spherical harmonics

$$= \mp \left(\frac{2L+1}{4\pi L(L+1)} \right)^{1/2} \sin \theta \frac{d}{d(\cos \theta)} \left[P_L(\cos \theta) \right] e^{\pm i\phi}. \quad (\text{II-15})$$

The upper sign in the superscript on the spherical harmonic is to be used for $M_S^I = +1/2$; the lower sign, for $M_S^I = -1/2$.

The $b_L^\pm$ factors in Eqs. (II-12) and (II-13) have magnitudes less than or equal to unity, and take into account inelastic reactions. By setting $b_L^\pm = \exp(-2\beta_L^\pm)$, the expression $b_L^\pm \exp(2i\delta_L^\pm)$ can be written $\exp[2i(\delta_L^\pm + i\beta_L^\pm)]$. Therefore inelastic scattering can be represented mathematically by complex phase shifts. In this report, the term

phase shift will refer to just the real part, $\delta_L^\pm$. The $b_L^\pm$ factors will be called "inelastic parameters". Often we will set all the inelastic parameters equal to unity, thus neglecting inelastic processes.

The phase shifts always enter into the equations in the form $2\delta_L^\pm$. Thus multiples of 180 deg can be added to or subtracted from the phase shifts without changing any function of these parameters. Before quoting phase-shift values, we will frequently make changes of 180 deg in order to reach a desired angular region.

The quantities S_{NF} and S_F in Eq. (II-9) are the non-spin-flip and spin-flip scattering amplitudes introduced in Section II-A-2. The equalities mentioned in that section can now be seen to be true. From Eqs. (II-10) and (II-12) we see that S_{NF} is independent of M_S^I , and therefore $S_{\alpha\alpha} = S_{\beta\beta}$. Setting ϕ equal to 0 or 180 deg in Eq. (II-15), and using Eqs. (II-11) and (II-13), we obtain the relationship $S_{\beta\alpha} = -S_{\alpha\beta}$. This specification of the ϕ value is actually no restriction because one may choose the x - z plane, which contains $\phi=0$ and 180 deg, to coincide with any scattering plane of interest.

We now write the equations for the DCS and the recoil-nucleon polarization in elastic pion-proton scattering in terms of $g(\theta)$ and $h(\theta, \phi)$, to which we apply the α and β subscript notation. (In the rest of this report we will call $g(\theta)$ and $h(\theta, \phi)$ the non-spin-flip and spin-flip scattering amplitudes.) For $\phi=0$ or 180 deg, we can write

$$I(\theta_{c.m.}) = |g_{\alpha\alpha}|^2 + |h_{\beta\alpha}|^2, \quad (II-16)$$

and

$$P(\theta_{c.m.}) = \frac{2\text{Im}(g_{\alpha\alpha}^* h_{\beta\alpha})}{I(\theta_{c.m.})}. \quad (II-17)$$

These expressions have been obtained from Eq. (II-6) in conjunction with Bethe and de Hoffmann⁶ and Eq. (II-7). With ϕ specified, $g_{\alpha\alpha}$ and $h_{\beta\alpha}$ depend only on the one angular coordinate θ . Because θ can refer to the angle between the direction of scattering and the initial direction of motion

of either particle, we have used the symbol $\theta_{c.m.}$ in Eqs. (II-16) and (II-17) following the definition in Section II-A-2.

If we change the angle ϕ in Eq. (II-15) from 0 to 180 deg, $h_{\beta\alpha}$, and therefore the polarization, changes sign. Thus protons recoiling at the same angle with respect to the incident pion beam, but on opposite sides of the beam, will have polarizations of the same magnitude but of opposite sign.

2. Inclusion of Coulomb Scattering

a. Scattering amplitudes with the Coulomb interaction present. A localized nuclear interaction was assumed in the development of Eqs. (II-12) and (II-13). When the infinite-range Coulomb interaction is also included in the analysis, the situation becomes more complicated. According to Mott and Massey,⁷ Eqs. (II-12) and (II-13) can also be employed to describe the scattering of a spin-1/2 particle by a pure Coulomb potential. If we now add a local nuclear interaction to the Coulomb scattering, the outgoing parts of the wave function describing the interaction are expected to differ only in phase and amplitude from those in the pure Coulomb case, in analogy to the partial-wave treatment in Halliday.⁸ (This idea is discussed on pp. 119-120 of Schiff.⁹) Therefore expressions in the form of Eqs. (II-12) and (II-13) can also describe the elastic scattering arising from a combination of Coulomb and nuclear interactions. With Coulomb scattering included, the phase ikr in Eqs. (II-10) and (II-11) is replaced by $i[kr - n \ln(2kr)]$, where n will be defined later. The additional phase factor arises from the fact that the wave function can be distorted by the Coulomb interaction even at large distances from the scattering center.

With both nuclear and Coulomb scattering present, we can write the non-spin-flip and spin-flip elastic-scattering amplitudes as

$$g_T(\theta) = - \frac{\lambda n}{2 \sin^2(\theta/2)} \exp \left\{ -i n \ln [\sin^2(\theta/2)] \right\} \\ + \lambda \sum_{L=0}^{\infty} \left[(L+1) \left(\frac{b_L^+ \exp(2i\delta_L^+) - \exp(2i\Phi_L)}{2i} \right) + L \left(\frac{b_L^- \exp(2i\delta_L^-) - \exp(2i\Phi_L)}{2i} \right) \right] \\ \times P_L(\cos \theta), \quad (\text{II-18})$$

$$h_T(\theta, \phi) = \lambda \sum_{L=1}^{\infty} \left[\frac{b_L^+ \exp(2i\delta_L^+) - b_L^- \exp(2i\delta_L^-)}{2i} \right] D_L Y_L^{\pm 1}(\theta, \phi), \quad (\text{II-19})$$

where T denotes the total (nuclear plus Coulomb) scattering. In Appendix C, we obtain these results from expressions in the form of Eqs. (II-12) and (II-13). The phase shifts $\delta_L^{\pm}$ and the inelastic parameters $b_L^{\pm}$ now describe the total interaction. The quantity Φ_L is zero for $L=0$ and is given by

$$\Phi_L = \sum_{x=1}^L \tan^{-1}(n/x) \quad (\text{II-20})$$

for $L \geq 1$, with

$$n = ZZ' e^2 / \hbar v,$$

where Ze and $Z'e$ are the charges of the interacting particles, and v is the laboratory velocity of the incident pion. Although we will refer to Φ_L as the nonrelativistic Coulomb phase shift of order L , it is actually the difference between the nonrelativistic Coulomb phase shifts of order L and of order zero. Equations (II-18) and (II-19), in a slightly different form and with the inelastic parameters set equal to unity, can be found in the work of Critchfield and Dodder.¹⁰

The first term in Eq. (II-18) is the nonrelativistic Coulomb-scattering amplitude, which approaches infinity as the scattering angle approaches 0 deg. Because of this singular behavior, we will find the

form of Eq. (II-18) an advantageous one. The summation in this expression for $g_T(\theta)$ contains just the difference between the total and the nonrelativistic Coulomb-scattering amplitudes, and is expected to converge more rapidly than an expansion in the form of Eq. (II-12).

Let us divide the phase shifts describing the total interaction into a pure Coulomb part and an additional portion that arises only when the nuclear interaction is added to the Coulomb interaction. We then can write the total phase shifts as $\delta_L^\pm = \Phi_L^\pm + \delta_{L,N}^\pm$, where the symbols $\Phi_L^\pm$ represent the complete Coulomb phase shifts of order L and are set equal to $\bar{\Phi}_L + \Delta\Phi_L^\pm$.^{*} The quantities $\Delta\Phi_L^\pm$ are corrections to $\bar{\Phi}_L$ due to modifications of the nonrelativistic Coulomb scattering. The modifications that we will discuss are the relativistic corrections given by Solmitz.¹¹ The quantities $\delta_{L,N}^\pm$ approximate the pion-proton nuclear phase shifts of order L . It is to be stressed that they are only approximations; the quantities obtained when the pure Coulomb phase shifts are subtracted from the total phase shifts still contain remnants of the Coulomb interaction. We assume that the additional corrections needed to obtain the true nuclear phase shifts are small.

^{*}As in the case of $\bar{\Phi}_L$, the phase shifts $\delta_L^\pm$ (and $\Phi_L^\pm$) are actually the differences between the total phase shifts (and the complete Coulomb phase shifts) of order L and the nonrelativistic Coulomb phase shift of order zero. A few remarks summarizing the notation and phase-shift transformations of this section and Appendix C may be of value at this time. In terms of the notation in Appendix C, the phase shifts for the total interaction can be written $\delta_{L'}^\pm = \eta_L^\pm + \delta_{L,N}^\pm$ where $\eta_L^\pm \approx \eta_L + \Delta\eta_L^\pm$. Subtracting η_0 from $\delta_{L'}^\pm$, $\eta_L^\pm$, and η_L , and denoting $\Delta\eta_L^\pm$ by $\Delta\Phi_L^\pm$, we obtain $\delta_L^\pm = \Phi_L^\pm + \delta_{L,N}^\pm$ where $\Phi_L^\pm \approx \bar{\Phi}_L + \Delta\Phi_L^\pm$. These last equations are those presented above.

b. First-order relativistic Coulomb corrections. First-order relativistic corrections to the nonrelativistic Coulomb-scattering amplitudes can be written

$$\Delta g_C = \lambda n A \quad (\text{non-spin-flip correction}), \quad (\text{II-21})$$

and

$$\Delta h_C = \mp \frac{\lambda n B \sin \theta}{2 \sin^2(\theta/2)} e^{\pm i\phi} \quad (\text{spin-flip correction}), \quad (\text{II-22})$$

where

$$A = \frac{(\beta_\pi \beta_P)/2 + (2\mu_P - 1)\beta_P^2/4}{1 + \beta_\pi \beta_P},$$

$$B = \frac{(\mu_P \beta_\pi \beta_P)/2 + (2\mu_P - 1)\beta_P^2/4}{1 + \beta_\pi \beta_P},$$

β_P, β_π = c. m. velocities of the proton and pion, divided by the velocity of light,

μ_P = magnetic moment of the proton in nuclear magnetons,

and

$n = \pm e^2/\hbar v$ (the + sign is for π^+ -p scattering, the - sign is for π^- -p).

These formulas were obtained from Eqs. (2) and (3) of Solmitz;¹¹ we used the relationship $v/c = (\beta_\pi + \beta_P)/(1 + \beta_\pi \beta_P)$, where (as in the expression for n) v is the laboratory velocity of the incident pion. The effect of the magnetic moment of the proton is included in these corrections. The double sign before the expression for Δh_C , and the $e^{\pm i\phi}$ factor after, are necessary to account for the two possible initial spin states. Once again, the upper signs refer to $M_S^I = +1/2$, and the lower signs to $M_S^I = -1/2$. The order of these signs has been chosen so that the relative phase of the nuclear and Coulomb spin-flip scattering amplitudes in Eq. (1) of reference 11 agrees with the corresponding relative phase in our Eq. (II-31).

To incorporate these corrections into our analysis, we shall decompose them into partial waves. This will allow them to be separated into two parts, one corresponding to states with $L \leq L_{\text{MAX}}$ and the second part containing the remainder. The quantity L_{MAX} is the maximum value of the quantum number L whose related partial wave is affected by the nuclear interaction. For $L \leq L_{\text{MAX}}$, unitarity will be maintained by employing the usual partial-wave expressions but now interpreting part of each phase shift as arising from the correction terms. These phase-shift corrections will be estimated by comparing the first-order Solmitz corrections with Eqs. (II-18) and (II-19), these latter also taken to lowest order. Our basic assumption will be that these corrections to the Coulomb phase shifts are not altered by the other interactions. We will subtract them, along with the nonrelativistic Coulomb phase shifts, from the total phase shifts, in order to obtain estimates of the nuclear phase shifts. In contrast to the method for $L \leq L_{\text{MAX}}$, the part of the correction Δh_C for $L > L_{\text{MAX}}$ will simply be added to the rest of the spin-flip scattering amplitude, with no attempt being made to preserve unitarity in the higher-order states (see Section II-B-2-c).

If only the Coulomb interaction were present, we could write the scattering amplitudes, to the accuracy used by Solmitz, as

$$\begin{aligned} g_{C,S}(\theta) &= \frac{-\lambda n}{2 \sin^2(\theta/2)} + \Delta g_C \\ &= \frac{-\lambda n}{2 \sin^2(\theta/2)} + \sum_{L=0}^{\infty} g_L P_L(\cos \theta), \end{aligned} \quad (\text{II-23})$$

and

$$h_{C,S}(\theta, \phi) = \Delta h_C = \sum_{L=1}^{\infty} h_L Y_L^{\pm 1}(\theta, \phi), \quad (\text{II-24})$$

where C and S refer to Coulomb and Solmitz, respectively. Using the orthogonality of the Legendre polynomials and spherical harmonics, we find

$$g_0 = \Delta g_C = \lambda n A, \quad (II-25)$$

$$g_L = 0 \quad \text{for } L \geq 1, \quad (II-26)$$

and

$$h_L = \lambda n B \left[\frac{4\pi(2L+1)}{L(L+1)} \right]^{1/2} \quad \text{for } L \geq 1. \quad (II-27)$$

We now wish to compare the Solmitz first-order expressions for the scattering amplitudes with Eqs. (II-18) and (II-19), which are written in terms of phase shifts. With no nuclear interaction present, $\delta_L^\pm = \Phi_L^\pm \approx \bar{\Phi}_L + \Delta\Phi_L^\pm$ and the inelastic parameters are unity. Comparing Eqs. (II-23) and (II-24) with Eqs. (II-18) and (II-19) when the exponentials in the latter two equations are expanded just to first order, and allowing only the Coulomb interaction, we obtain

$$\left. \begin{aligned} \lambda [(L+1) \Delta\Phi_L^+ + L \Delta\Phi_L^-] &\approx \Delta g_C \quad \text{for } L = 0 \\ &\approx 0 \quad \text{for } L \geq 1, \end{aligned} \right\} \quad (II-28)$$

and

$$\lambda(\Delta\Phi_L^+ - \Delta\Phi_L^-) \approx \lambda n B \left[\frac{2L+1}{L(L+1)} \right] \quad \text{for } L \geq 1. \quad (II-29)$$

Equations (II-25) through (II-27) were used in deriving these results.

From Eqs. (II-28) and (II-29), one obtains the following expressions for the corrections to the nonrelativistic Coulomb phase shifts:

$$\Delta\Phi_0 (= \Delta\Phi_0^+) \approx \Delta g_C / \lambda,$$

$$\Delta\Phi_L^+ \approx \frac{n B}{L+1} \quad \text{for } L \geq 1,$$

$$\Delta\Phi_L^- \approx -\frac{n B}{L} \quad \text{for } L \geq 1.$$

Using these results and Eq. (II-20), we can compute the numbers presented in Table I. It is observed that the quantities $\Delta\Phi_L^\pm$ are small and,

Table I. Nonrelativistic Coulomb phase shifts, first-order relativistic corrections, and corrected Coulomb phase shifts (deg). The signs given here apply to π^+ -p scattering and are reversed for π^- -p. The incident pion laboratory kinetic energy is 310 Mev.

L	Φ_L	$\Delta\Phi_L^+$	$\Delta\Phi_L^-$	Φ_L^+	Φ_L^-
0	0.00	0.09	-	0.09	-
1	0.44	0.09	-0.17	0.53	0.27
2	0.66	0.06	-0.09	0.72	0.57
3	0.81	0.04	-0.06	0.85	0.75
4	0.92	0.03	-0.04	0.95	0.88

for low L , Φ_L is also small. Thus, for low L and θ not too near 0 deg, the approximations made in expanding Eqs. (II-18) and (II-19) to first order (with only the Coulomb interaction allowed) are justified.

c. Final form of the scattering amplitudes. If we are advantageously to apply the method of partial waves to our data-analysis problem, we must limit to a finite number the terms in the summation expressions representing the elastic-scattering amplitudes. When considering the nuclear interaction only, we assume that the phase shifts related to L values above a certain maximum value (L_{MAX}) are zero. In other words, the nuclear short-range interaction is expected to appreciably influence only a finite number of the incident partial waves. On the other hand, the infinite-range Coulomb potential affects the partial waves related to all values of L . In order eventually to limit the number of terms in the summation in Eq. (II-18), we separated out the nonrelativistic Coulomb-scattering amplitude. We can restrict the number of terms in Eq. (II-19) by treating Δh_C in a similar manner. The correction Δg_C poses no problem because it is a constant, independent of θ .

Let us divide the summations in Eqs. (II-18) and (II-19) into two parts, the first corresponding to states with $L \leq L_{MAX}$ and the second containing all other states. The contributions to $g_T(\theta)$ and $h_T(\theta, \phi)$ for $L > L_{MAX}$ arise from the Coulomb corrections (if $\Delta \Phi_L^\pm = 0$, these higher-order angular-momentum contributions are also zero). We will replace these higher-order summations by the $L > L_{MAX}$ parts of the Solmitz amplitude corrections. All of Δg_C has been included in the $L = 0$ term of $g_T(\theta)$, so we set equal to zero the $L > L_{MAX}$ portion of $g_T(\theta)$. We include the $L > L_{MAX}$ part of Δh_C by adding the entire Δh_C and then subtracting off the $L \leq L_{MAX}$ portion. We therefore replace the $L > L_{MAX}$ summation in $h_T(\theta, \phi)$ by

$$\Delta h_C - \kappa B \sum_{L=1}^{L_{MAX}} \left(\frac{2L+1}{L(L+1)} \right) D_L Y_L^{\pm 1}(\theta, \phi).$$

Equations (II-24) and (II-27) were used in obtaining this last expression.

Our general forms of the non-spin-flip and spin-flip elastic-scattering amplitudes can now be written

$$g_T(\theta) = - \frac{\lambda n}{2 \sin^2(\theta/2)} \exp \left\{ -in \ln [\sin^2(\theta/2)] \right\} + \lambda \sum_{L=0}^{L_{MAX}} \left[(L+1) \left(\frac{b_L^+ \exp(2i\delta_L^+) - \exp(2i\Phi_L)}{2i} \right) + L \left(\frac{b_L^- \exp(2i\delta_L^-) - \exp(2i\Phi_L)}{2i} \right) \right] \times P_L(\cos \theta), \quad (II-30)$$

and

$$h_T(\theta, \phi) = \mp \frac{\lambda n B \sin \theta}{2 \sin^2(\theta/2)} e^{\pm i\phi} + \lambda \sum_{L=1}^L \left[\frac{b_L^+ \exp(2i\delta_L^+) - b_L^- \exp(2i\delta_L^-)}{2i} - n B \left(\frac{2L+1}{L(L+1)} \right) \right] D_L Y_L^{\pm 1}(\theta, \phi). \quad (II-31)$$

In obtaining these expressions, we used Eqs. (II-18) and (II-19) in conjunction with the results of the last paragraph, and Eq. (II-22).

We summarize the sign conventions employed in Eqs. (II-30) and (II-31):

(a) In each place where double signs occur in the expression for $h_T(\theta, \phi)$, the upper sign is to be used when $M_S^I = +1/2$; the lower sign, when $M_S^I = -1/2$.

(b) The $\pm$ superscripts on δ_L and b_L refer to states with $J = L \pm 1/2$.

(c) The quantity n is positive for π^+ -p scattering, and negative for π^- -p.

Equations (II-30) and (II-31) are similar to expressions that are obtained if one simply adds the nuclear and Coulomb scattering amplitudes. However, differences exist because the method presented here adds nuclear and Coulomb phase shifts rather than amplitudes for $L \leq L_{MAX}$.

Except for the modifications due to the Solmitz corrections, our approach is essentially that used by Stapp, Ypsilantis, and Metropolis.¹²

3. Utilization of the General Equations; Phase-Shift Notation

In order to obtain equations for the DCS and recoil-proton polarization in terms of phase shifts, when both nuclear and Coulomb effects are present, we substitute the scattering amplitudes given in Eqs. (II-30) and (II-31) into Eqs. (II-16) and (II-17). These results are general in that they can be applied to either π^+ -p or π^- -p elastic scattering. However, it is not advantageous to apply Eqs. (II-30) and (II-31) directly to π^- -p elastic scattering because of the sizable amount of charge-exchange scattering that must be taken into account by the inelastic parameters. One customarily assumes the conservation of isotopic spin. Then, with appropriate modification, Eqs. (II-30) and (II-31) can be used to describe π^- -p elastic scattering. In the remainder of this report, we will usually restrict our considerations to the scattering of π^+ mesons off protons.

The phase-shift notation that we will employ is given in Table II. The conventional symbols for the S-, P-, and D-wave phase shifts have been modified in order to present a consistent notation when F waves are included in the analysis. As before, the first subscript is twice the total isotopic spin, and the second is twice the total angular momentum. Because we are dealing with π^+ -p scattering, only the state with isotopic spin of 3/2 enters into the interaction.

The application of our general phase-shift equations to the analysis of scattering data by an electronic computer is discussed in Appendix D. Formulas are presented that enable one to change a phase shift and recalculate values of the polarization and DCS without being required to recompute any trigonometric functions.

Table II. Phase-shift notation for π^+ -p scattering

L	J	Phase-shift symbol
0	1/2	$S_{3,1}$
1	1/2	$P_{3,1}$
1	3/2	$P_{3,3}$
2	3/2	$D_{3,3}$
2	5/2	$D_{3,5}$
3	5/2	$F_{3,5}$
3	7/2	$F_{3,7}$

In Appendix E we present several useful phase-shift equations, which can be obtained from the more-general formulas of Section II-B. Included are expressions for the total nuclear cross section, the total nuclear elastic- and inelastic-scattering cross sections, and the real and imaginary parts of the nuclear forward elastic-scattering amplitudes.

C. Types of Phase-Shift Solutions

Owing to the nature of the equations, more than one set of phase shifts have arisen in the analysis of pion-proton scattering data. Each set has distinct characteristics and, within certain limitations, yields a satisfactory fit to the experimental data. It is important to determine which of the several possible solutions corresponds to the true solution. The various uncertainties in the π^+ -p phase shifts may be classed as the Fermi-Yang-Minami ambiguity, the D-wave phase-shift ambiguity, and the uncertainty in the absolute sign of a given set of phase shifts. (In the ambiguity discussions to follow, we neglect inelastic scattering.)

1. Fermi and Yang Solutions

The phase-shift set known as the Fermi type is the most universally accepted solution. Its principal characteristic is a large $P_{3,3}$, which passes through 90 deg in the resonance-energy region near 200-Mev pion laboratory kinetic energy. Other characteristics of the Fermi set include a small $P_{3,1}$ and a relatively small $S_{3,1}$, even at energies as high as 300 Mev.

The Yang-type solution originally arose in connection with analyses that assumed all nuclear phase shifts to be negligible except those related to S and P waves (hereafter referred to as SP analyses). When fitting DCS data alone, only the absolute squares of the non-spin-flip and spin-flip scattering amplitudes enter into the equations. The relative sign of these amplitudes is unimportant because there is no interference term involving them. Under these circumstances, the analysis of the partial-wave amplitudes leads to the result that, besides

the Fermi set, there is another set of phase shifts that will fit the data well. Let us denote the Fermi phase shifts by symbols without primes, and the second (Yang) set by symbols with primes. Then, neglecting Coulomb effects and considering only the π^+ -p phase shifts, we can write equations relating the two solutions:

$$S'_{3,1} = S_{3,1}$$

$$P'_{3,1} = \omega - P_{3,1}$$

$$P'_{3,3} = \omega - P_{3,3}$$

where

$$\tan \omega = \frac{2 \sin(2 P_{3,3}) + \sin(2 P_{3,1})}{2 \cos(2 P_{3,3}) + \cos(2 P_{3,1})}$$

These expressions lead to the result that $P_{3,3} - P_{3,1} = -(P'_{3,3} - P'_{3,1})$. (The preceding equations can be understood by an appropriate plotting of the relevant partial-wave amplitudes in the complex plane.) Because the relative sign of the non-spin-flip and spin-flip amplitudes enters into the polarization expression, the Fermi and Yang solutions predict different variations of the polarization with angle.

2. Minami and Other Solutions

Another type of phase-shift set that occurs is the Minami solution.¹³ In order to obtain it from the Fermi solution, one separates the phase shifts into groups of two, each group containing the phase shifts related to states of the same total isotopic spin and total angular momentum. If we then interchange the values of the phase shifts in each pair, and we do this simultaneously for all pairs, we obtain the Minami solution. As applied to π^+ -p scattering, the Minami transformation means the interchanging of the values of the phase shifts $S_{3,1}$ and $P_{3,1}$, $P_{3,3}$ and $D_{3,3}$, $D_{3,5}$ and $F_{3,5}$, and so forth. Because $P_{3,3}$ of the Fermi solution is large, $D_{3,3}$ of the Minami solution will be large. If the Fermi solution is a good fit to the existing DCS data,

the corresponding Minami solution will also fit the data well. On the other hand, the two solutions predict different behaviors for the polarization because the Minami transformation causes the sign of this quantity to reverse.¹⁴

The transformation just described can also be applied to the Yang solution.¹⁵ We shall refer to the resulting set of phase shifts as the "Minami-Yang" type, as opposed to the "Minami" type, which is obtained from the Fermi set.

For every solution obtainable in an SP analysis, two solutions can occur when the pion energy is sufficiently high so that the D-wave nuclear phase shifts are not negligible.¹⁶ We again assume that only DCS data are being fitted. The D-wave phase shifts in the two competing sets have different characteristics. The polarization is especially sensitive to these phase shifts and will behave differently for the two solutions.

Although Coulomb effects are not helpful to us in attempting to resolve the ambiguities previously discussed, Coulomb scattering can be very useful in determining the absolute sign of a set of phase shifts. Assume that only DCS data exist and only at c.m. scattering angles sufficiently large so that Coulomb-nuclear interference effects are small. Then, for each set of phase shifts that satisfactorily fits the data, another set that also fits can be found by changing the signs of all the phase shifts in the first set. That is, only the relative sign of the phase shifts can be determined from the assumed data. On the other hand, if adequate DCS measurements are available at sufficiently small angles, the Coulomb-nuclear interference effects can reveal which of the sets of signs is correct: one set predicts destructive interference at small angles; the other set, constructive. Besides reversing the Coulomb-interference effects, the process of inverting the sign of every phase shift also inverts the sign of the polarization at all angles.

We have seen how the different possible solutions predict varied behaviors for the polarization as a function of angle. Measurements of the recoil-proton polarization are therefore useful in the investigation of the pion-proton phase-shift ambiguities.

In Section IV, we will discuss our phase-shift analysis of π^+ -p scattering data at 310 Mev. The various types of solutions just explained will appear in this analysis, and the effects of the available polarization data will be observed.

III. POLARIZATION MEASUREMENTS

Our experimental measurements of the recoil-proton polarization in elastic π^+ -p scattering at 310 Mev will now be discussed. Let us first explain the manner in which the required high-intensity π^+ beam was obtained and the characteristics of this beam. We will then describe the method, apparatus, and procedures used to determine the polarization of the recoil protons. The calibration of the apparatus will be included in this discussion. Finally, we will present the results of the polarization measurements and discuss uncertainties in these results.

A. Positive-Pion Beam

1. Positive-Pion Production

The external proton beam of the 184-in. synchrocyclotron at Berkeley produced the desired positive pi mesons. The mesons were created in the experimental area known as the Physics Cave. Before reaching this region, the proton beam passed through a bending magnet, a focusing magnet, and through a hole in the concrete shielding surrounding the cyclotron. At the point where it entered the cave, the beam was about 2.5 in. wide and 1.5 in. high. These dimensions were determined by exposing x-ray film in the beam. The proton beam had an energy of approximately 743 Mev, an rms energy spread of about ± 8 Mev, and a maximum intensity of $(2 \pm 1) \times 10^{11}$ particles per sec.

Positive pi mesons can be produced by protons in various nuclear interactions. In producing our pi-meson beam, we made particular use of the reaction $p + p \rightarrow \pi^+ + d$ (to be denoted pp- π d). Because there are only two particles in the final state, the mesons have a unique energy at any given angle in the c.m. system. There are also other reactions that give π^+ mesons. They yield, however, more than two particles in the final state. Kinematics allows the positive pions from these reactions to have a spectrum of energies at each angle, rather than a single energy as in the pp- π d case. We were able to obtain an optimum number of 310-Mev pi mesons by taking maximum advantage of the pp- π d reaction.

A polyethylene (CH_2) target was placed in the external proton beam near the point at which the beam entered the cave (see Fig. 1). This type of material was selected principally on the basis of its free-proton constituent (H_2), which can enter into the $\text{pp}-\pi$ d process. We utilized the positive pions that were produced in the forward direction because the DCS for the $\text{pp}-\pi$ d reaction is greatest at 0 deg. The thickness of the CH_2 target was experimentally determined to give the maximum number of positive pi mesons leaving the target in the forward direction with the desired energy. The optimum target thickness was about 19 in.

The high intensity of our resulting positive-pion beam was partially due to a fortunate characteristic of the experimental arrangement. Let us consider the π^+ mesons that were produced in the forward direction by the $\text{pp}-\pi$ d reaction. The various energy losses in the CH_2 were such that the energy of these pions as they emerged from the target was approximately independent of the position in the CH_2 at which they were produced. Thus by choosing the correct thickness of polyethylene, all the pions of the type mentioned here possessed energies in a small interval centered at the desired value.

The other π^+ -producing reactions contributed mesons of a large range of energies. Positive pions of this origin, with energies approximately equal to the required value, were also utilized in our beam.

2. Pi-Meson Beam

The positive pions with the requisite energy, after leaving the polyethylene target, were momentum-analyzed and focused by a series of two bending and three quadrupole focusing magnets (Fig. 1). The first focus of the system was within the center quadrupole magnet. This magnet acted on the off-axis particles in such a manner as to increase the number reaching the final focus, which was at the liquid-hydrogen target shown in Fig. 1. In order to obtain the desired physical arrangement, the second bending magnet was built into the concrete shielding surrounding the cave.

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Fig. 1. Scale drawing of the magnet system for the π^+ beam. The bending magnets are designated M_1 and M_2 ; Q_1 , Q_2 , and Q_3 are the quadrupole focusing magnets. Magnets Q_1 and Q_3 have 8-in. apertures, and Q_2 has a 4-in. aperture. Also shown is the counter arrangement used to detect the recoil-proton polarization. The dimensions of the counters and carbon target are not to scale.

Only positive particles in a definite momentum range were able to pass through the entire series of magnets. Other particles were lost at various points along the beam path. The iron structure of a magnet or a wall of the cave usually intercepted the positive particles that were rejected by the system. Particles of negative charge were bent away from the beam direction by the first bending magnet. The external proton beam of the cyclotron, after passing through the CH_2 and the first focusing magnet, was changed in direction slightly by the first bending magnet. It then struck the rear wall of the cave.

Besides π^+ mesons, there were other positive particles leaving the CH_2 target in the forward direction with momentum acceptable to the magnet system. Lower-energy particles with the proper momentum, such as protons, were removed from the beam by placing a 2-in. -thick piece of carbon absorber directly after the central focusing magnet. Higher-energy particles with the required momentum, such as mu mesons, were not affected greatly by the carbon, and remained in the beam. The pi mesons of interest lost only a small amount of energy in the carbon.

A well-defined beam was obtained at the final focus of the magnet system. The symmetry of the magnet arrangement enabled the second half of the system to approximately cancel the momentum dispersion created by the first half. Thus a distinct focus was obtained in which there was little correlation between momentum and position across the beam. In order to investigate the shape of the beam, the intensity was lowered considerably and a counter telescope was moved through the focus. The telescope consisted of two square scintillation counters of 0.50 in. and 0.25 in. on a side. The beam was found to be nearly symmetric in both the horizontal and vertical directions. Its full width and height at half maximum intensity were about 3 in. and 2 in., respectively.

At the final focus, the mean energy of our beam was 310 Mev (momentum of 427 Mev/c), and the maximum intensity was about $2 \times 10^6 \pi^+$ mesons per sec. (The beam intensity actually employed in the polarization measurements is given in Section III-C-1.) Owing to their natural decay, about one-third of the acceptable pi mesons were lost between the CH_2 target and the final focus. The rms uncertainty in the mean energy of the beam was approximately ± 3 Mev, and the rms energy spread in the beam was ± 9 Mev, corresponding to a momentum spread of $\pm 2\%$. The stated value of 310 Mev is the meson energy at the center of the liquid-hydrogen target. Because of energy loss by ionization, the mesons had a slightly higher energy before the target, and a still higher energy before the 2-in. -thick piece of carbon absorber. The energy of the mesons was measured by determining their range in copper, and also by the suspended-wire technique.

B. Method and Apparatus

1. Method

The π^+ beam passed through a liquid-hydrogen target, which was placed at the final focus of the magnet system. A small fraction of the beam particles interacted with protons in the liquid hydrogen. Protons were knocked out of the target in many directions. In terms of the nomenclature in Fig. 2, counters A and B selected the recoil protons that left the target at angles approximating Θ_1 . Counter C was placed at the appropriate angle (Θ_C) to count the elastically scattered pi mesons that had knocked protons in the AB direction. This counter placed a severe restriction on the type of scattering event that could be detected by the system. In general, events other than elastic π^+ -p scattering could not produce a count in C as well as a particle through A and B. Iron of 0.125-in. thickness surrounded counter C and helped guard against low-energy charged particles.

A portion of the recoil protons, after passing through counters A and B, were scattered by the carbon analyzing target placed immediately following B. We chose carbon as the material for this target because of its ability to analyze the polarization of protons in the energy region of our recoil protons (110 to 140 Mev).¹⁷ Counter B played a dual role in that it also served as part of the analyzing target. Carbon being one of its principal constituents, counter B produced about the same asymmetry as did the actual carbon target.

The two counter telescopes shown in Fig. 2 detected protons that were scattered by the analyzing target. Copper absorber was placed between the counters in each telescope to help prevent unwanted particles from counting in D_O or D_E . The counter telescopes were interchangeable in position. In this way, each independently measured the asymmetry produced by the analyzing scattering. The second telescope increased our counting rate and served as a check on the first set of counters. The size of D_O and D_E was chosen so that these counters accepted almost all the scattered protons detected by counters III and IV.

Because of the low counting rates expected, counters with large areas were used. We had to reach a compromise, however, between counting rate and angular resolution. The sizes of the counters in the analyzing telescopes were limited because of the undesirability of excessively lowering the average measurable asymmetry. One wishes to measure as large an asymmetry as possible, consistent with a satisfactory counting rate, in order to minimize the influence of errors that affect the asymmetry by a fixed amount. Immoderately large counters would extend over an excessively great range of the analyzing angles θ_2 and ϕ_2 . Only over certain regions of values of these angles are both the asymmetry and counting rate satisfactory. As ϕ_2 approaches 90 and 270 deg, the asymmetry disappears, according to Eq. (II-3). If θ_2 is too small, the asymmetry due to nuclear scattering is considerably lower than the maximum obtainable value,¹⁷ and also the unpolarized Coulomb scattering can enter. At large values of θ_2 the intensity of the scattered protons decreases greatly,¹⁷ and the effects of inelastic scattering increase. It is hoped that a reasonable compromise was reached in our experimental arrangement.

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Fig. 2. Scale drawing (plan view) of counter and target arrangement used to measure the polarization of the recoil protons.

In order to limit the spread of recoil angles accepted by the system, and to aid the θ_2 angular resolution, counters A and B were made smaller than those employed in the analyzing telescopes. The estimated rms spread in the θ_1 values of the accepted recoil protons was ± 2.4 deg (corresponding to ± 4.8 deg in $\theta_{c.m.}$). This number did not vary appreciably over the range of recoil angles investigated. Principal sources of the spread in θ_1 were (estimated rms values are given):

(a) counter size	± 0.8 deg
(b) pi-meson beam convergence	± 1.8 deg
(c) beam width and liquid-hydrogen-target length	± 1.3 deg

The rms sum of these numbers is the value of 2.4 deg just presented.

2. Counters and Electronics

Each counter was composed of polystyrene plastic scintillator and was viewed by one RCA-6810 photomultiplier tube. A solid lucite light pipe connected each photomultiplier to its corresponding scintillator. The dimensions of the scintillating regions of the counters (all rectangular in area) are given in Table III.

Our electronics arrangement employed fast coincidence circuits of the Wenzel type¹⁸ to detect the scattering events of interest. Output pulses from each of the counters were delayed and amplified when necessary, and fed into the coincidence circuits. A coincidence between pulses from counters A, B, and C detected π^+ -p scattering events at the liquid-hydrogen target. The output pulse from the ABC coincidence was amplified, split, and fed into two additional coincidence circuits. One of these circuits accepted pulses from counters III and D_O ; the other received pulses from IV and D_E . In this manner, coincidences were formed of the types ABC III D_O and ABC IV D_E . The output pulses representing the five-fold coincidences, and also an ABC output pulse, were amplified, passed through amplitude discriminators, and finally were fed into scaling units.

Table III. Dimensions of the scintillation counters used to measure the polarization of the recoil protons

Counter	Dimensions of counter (width X height X thickness) (in.)
A	2 X 6 X 1/4
B	2 X 8 X 1/4
C	12 X 12 X 1
III, IV	4 X 20 X 3/4
D _O , D _E	6 X 22 X 3/4

3. Scattering Apparatus

The liquid-hydrogen target, with slight modification, was that described by Garrison.¹⁹ Enclosing the hydrogen was a stainless-steel cylindrical can with a wall thickness of 0.004 in. and a diameter of 5.6 in. The amount of liquid hydrogen in the scattering plane was 1.0 g/cm². A 0.125-in. -thick dural vacuum jacket surrounded the liquid-hydrogen container. The pion beam passed through 3-in. -diam. holes cut in the vacuum jacket. Mylar of 0.015-in. thickness covered these holes. In order to determine the portion of our final counting rate not due to the liquid hydrogen, a second target assembly was also employed. This "blank" was similar in construction to the liquid-hydrogen target assembly but contained no hydrogen. When desired, the actual target was moved out of position and the evacuated blank placed on the beam line.

Our counters, targets, and principal supporting frameworks are shown in Fig. 3. Not indicated is the manner in which counters A and B and the carbon target were attached to the scattering arm. Counter C and its support are also not included in the drawing. In Fig. 3, as in Fig. 2, it is the scintillating region of each counter that is shown. Distances between counters and targets are given in Table IV. As indicated in Fig. 3, the analyzing angles were measured by means of a plumb bob attached to each counter telescope.

C. Experimental Procedures

1. General Procedures

We optimized the pion beam by making a series of variations in the magnet currents and the thickness of the meson-producing target. The currents initially selected in this investigation were obtained through use of the suspended-wire technique. The small-counter telescope, which was described in Section III-A-2, was placed at the final focus. Its counting rate was observed while the various parameters of the system were changed. In this way, we optimized the beam in position and intensity, and obtained the desired energy.

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Fig. 3. Scale drawing (elevation view) of counters, targets, and principal supporting frameworks used to measure the polarization of the recoil protons. The angles Θ_1 and Θ_2 have been set equal to 0 deg in this figure.

Table IV. Distances between centers of components of the apparatus
used to measure the polarization of the recoil protons

From	To	Distance (in.)
Liquid-hydrogen target	Counter C	16.5 - 19.25 (depending on Θ_1)
Liquid-hydrogen target	Counter A	24
Counter A	Carbon target	24
Carbon target	Counters III or IV	37.5
Counters III or IV	Counters D_O or D_E	55.5

The appropriate voltages at which to set our counters and the proper amounts by which to delay the pulses from the counters were determined by observing coincidence counting rates as a function of these parameters. In ascertaining the voltage and delay settings, we examined particles that were of the same type and energy as those to be investigated in the asymmetry measurements. We therefore adjusted the system to count the desired particles and to discriminate against unwanted particles. After selecting the final voltages, time delays, and amplifier settings, a simultaneous change of ± 50 v in all the counter voltages did not significantly alter the counting rates. On many occasions during the data-accumulating period, this test was performed as a check on the stability of the electronics.

Background particles posed a considerable problem at the beginning of the experiment. Much of the background was produced by the external proton beam of the cyclotron stopping in the rear wall of the cave. In anticipation of difficulty, we solidly embedded the second bending magnet in the cave wall, placed concrete roof blocks on the cave, and put concrete above, below, and on both sides of the last focusing magnet. These precautions were not sufficient. We were able to further reduce the accidental counting rate by using the fast electronics already described and by employing as long a cyclotron beam spill as possible. We finally were forced to lower the intensity of the external proton beam, and therefore the pion beam, by a factor of two (the resulting π^+ intensity was 1×10^6 per sec).

To determine our accidental counting rate, we delayed the ABC coincidence output pulse by 5.2×10^{-8} sec before it entered into a coincidence of the type ABC III D_O or ABC IV D_E . This amount of delay represented the time difference between two radio-frequency fine-structure pulses of the cyclotron. We investigated singles rates and various coincidence rates, and concluded that our principal source of accidentals was a valid ABC event forming a coincidence with a second particle

that passed through one of the sets of analyzing counters. The accidentals were reduced by piling lead bricks near counter B, as shown in Fig. 2. This lead shielding extended approximately 1 ft. above and below the beam line. It limited the number of particles that could pass through the analyzing counters without also passing through A and B. At our smaller recoil angles, the lead wall nearer the pion beam was extended until it almost completely shielded the analyzing counters from the beam. We placed additional shielding, at all recoil angles, just before the liquid-hydrogen target; it was put on the same side of the pion beam as the scattering arm. This lead shielding eliminated many particles that scattered off or near the last focusing magnet.

The region of laboratory recoil angles investigated was 17 to 32 deg, and the range of analyzing-target thicknesses was 0.5 to 2.0 in. The recoil angle Θ_1 could not be made excessively small or the set of analyzing counters nearer the pion beam would extend into the beam. The carbon target could not be made too thin, or our counting rate would become prohibitively low. We were limited at the other extremes by the desirability of obtaining a relatively high average energy at the analyzing scattering. As explained in Section III-B-1, it was advantageous to measure as large an asymmetry as feasible. For a given incident proton polarization, the asymmetry that can be produced by carbon decreases rapidly below 135 Mev.¹⁷ We therefore did not allow our average scattering energy at the carbon target to fall below this value any further than necessary. Our recoil angles were thus restricted to the forward direction in the laboratory, corresponding to large angles of scattering in the center of mass. We used thinner carbon targets at the larger recoil angles in order to compensate at least partially for the decrease in energy of the recoil protons.

The range of Θ_2 values (analyzing-telescope angles) used in the asymmetry measurements was 15.5 to 17.0 deg. In deciding upon these settings, we compromised between various factors. These factors, which were discussed in Section III-B-1, included inelastic scattering, counting rate, and magnitude of the asymmetry.

On at least one occasion during the experiment, we observed the ABC counting rate with no liquid hydrogen in the target. We compared the counting rate when the evacuated target assembly was on the beam line with the corresponding rate when the blank was in position. The agreement was found to be satisfactory for the polarization measurements, and therefore the blank was considered a reliable facsimile of the actual target assembly.

On another occasion during the experiment, we removed the carbon analyzer and left only counter B to scatter the recoil protons. The rate of analyzed protons decreased by approximately the predicted amount, thereby increasing our confidence in the experimental method.

A few more comments about our general experimental procedures are in order before we discuss specific procedures at each recoil angle. An argon-filled ionization chamber was placed in the pion beam before the liquid-hydrogen target in order to monitor the beam intensity. Our counting rates were normalized to a standard amount of beam through the ionization chamber. Because the polarization measurements did not require a knowledge of the absolute intensity of π^+ mesons striking the target, no corrections were made for beam contamination. For each of four values of Θ_1 , we analyzed, under the same conditions, the polarization of the protons recoiling to both the left and right sides of the pion beam (in the horizontal plane). The two resulting asymmetries at each Θ_1 were then compared. According to Section II-B-1, these two asymmetries should have the same magnitude but opposite signs. The agreement generally obtained served as a check on the experimental proceedings.

2. Procedures at Each Recoil Angle

We began the data collecting at each recoil angle by determining the range of the recoil protons. During these measurements, the angle Θ_2 of the selected analyzing telescope was set near 0 deg and the carbon target to be used in the asymmetry determination was in

its position immediately after counter B. One of our range curves is shown in Fig. 4. At the early recoil angles, range curves for both sets of analyzing counters were obtained. We found satisfactory agreement between the two telescopes, and eventually investigated only one range curve at each recoil angle. Equal ranges were also observed for protons recoiling to the left and right sides of the pion beam at a given value of Θ_1 . The mean energies of the protons, as determined from the range curves, agreed well with the predictions of kinematics. An examination of the tails on the range curves indicated that about 97% of the detected particles were the desired recoil protons.

The "running point", indicated by an arrow in Fig. 4, refers to the amount of copper absorber that was placed between the counters in each analyzing telescope during the asymmetry measurements. The copper partially guarded against particles associated with inelastic-scattering processes in the liquid-hydrogen and carbon targets and stopped a portion of the stray background particles. At the same time, the absorber allowed the detection of the recoil protons that were elastically scattered at the analyzing target. (Further information on experimental procedures in polarization experiments is available in the work of Ypsilantis²⁰ and of Tripp.²¹)

Following the range-curve measurements, we next obtained the profile of the recoil-proton beam defined by the ABC coincidences. Each analyzing telescope was individually moved through this beam and counting rates determined at various angular settings. The profile and subsequent asymmetry measurements were made under as identical conditions as possible. In particular, both series of measurements used the same analyzing target and the same amount of copper before D_O and D_E . A beam profile is shown in Fig. 5. The center line was determined from the experimental data and represents the center, horizontally, of the beam of detected recoil protons.

Fig. 4. Range curve of the recoil-proton beam at $\Theta_1 = 16.9$ deg right.

Fig. 5. Beam profile of the recoil-proton beam at $\Theta_1 = 16.9 \text{ deg left}$. The angular reading of the profile center line lies near 8 deg rather than 0 deg because the point from which the plumb bob hung was not at the center of the counter telescope.

After aligning the apparatus at a selected recoil angle, and after obtaining a range curve and two beam profiles, we measured the asymmetry of the recoil protons that scattered off the carbon target. No variation of asymmetry with beam intensity was found as long as the pion intensity did not exceed 1×10^6 particles per sec. The analyzing telescopes were regularly interchanged in order to allow each set of counters independently to measure the asymmetry. By alternating the telescopes frequently, we reduced the adverse effect of slow time variations in the equipment on the asymmetry measurements. The left and right analyzing angles for each telescope were set with respect to the center line of the profile obtained with that telescope. In this way, we minimized the influence of differences in the two counter arrangements on the measured asymmetries. Systematic errors in the asymmetries were lessened by accurately determining with each telescope the center line of the recoil-proton beam, and by precisely setting the analyzing angles. The profiles were checked frequently during the asymmetry measurements by repeating two observations on each side of the center line.

With the telescopes positioned at the appropriate analyzing angles, a series of counting rates was determined. The ABC III D_O and ABC IV D_E rates were obtained for the following experimental arrangements:

- (a) liquid-hydrogen target on the pion-beam line, and normal time delays
- (b) liquid-hydrogen target on the pion-beam line, and the ABC pulse delayed by 5.2×10^{-8} sec (accidental rate)
- (c) blank on the pion-beam line, and normal time delays.

The accidental rate with the blank on the beam line was found to be negligible and was therefore not measured regularly. We obtained the rate of analyzed recoil protons by subtracting the rates in (b) and (c) from that in (a), and by combining the statistical counting errors in the appropriate manner.

The types of particles that we wished to detect in measurement (c) may have passed through the liquid hydrogen during the (a) measurement. If this were the case, rate (c) should have been determined with additional copper absorber before D_O and D_E in order to compensate for the ionization energy loss in the absent liquid hydrogen. The rate in (c) was observed with and without the added absorber, and no difference was detected. Therefore we generally neglected this copper correction.

Significant experimental quantities are listed in Table V. Included are pertinent angles and energies, analyzing-target thicknesses, five-fold coincidence counting rates, and analyzing efficiencies. Our final five-fold counting rates were limited by the number of ABC coincidences. The ABC rate, in turn, was restricted by counter B and to a smaller extent by counters A and C. The accidental and blank corrections each averaged about 5% of the corresponding corrected analyzed-proton rate. The rms energy spread of the recoil protons, as determined from the range curves, did not vary greatly with angle and was typically about ± 10 Mev.

Table V. Significant experimental quantities--angles, analyzing-target thicknesses, energies, five-fold coincidence counting rates, and analyzing efficiencies--for the four mean laboratory angles of detected recoil protons.

Experimental quantity	Mean laboratory angle of detected recoil protons ^a (deg)			
	16.6	22.1	26.6	31.6
Laboratory angle of conjugate pi mesons (deg)	131.6	117.2	106.2	94.7
Center-of-mass scattering angle (deg)	145.2	133.8	124.5	114.2
Analyzing-telescope angle, Θ_2 (deg)	15.5	15.5	17.0	17.0
Thickness of carbon analyzing target (in.)	2.0	1.0	0.5	0.5
Mean kinetic energy of recoil protons at center of liquid-hydrogen target (Mev)	178	167	154	139
Mean kinetic energy of conjugate pi mesons at center of liquid-hydrogen target (Mev)	132	143	156	171
Mean kinetic energy of recoil protons at center of carbon analyzing target (Mev)	141	140	128	113
Approximate average ABC III D_O or ABC IV D_E coincidence rate per minute ^b	5	2	1	1
Approximate analyzing efficiency of each telescope ^c	1/300	1/600	1/1100	1/700

^a Because of the angular variation in the DCS, each mean laboratory angle is about 0.3 deg smaller than Θ_1 , the corresponding angle at the center of counters A and B.

^b Corrected for accidental and blank counts.

^c Analyzing efficiency = (five-fold rate)/(ABC rate).

D. Calibration and Initial Polarization Measurements

1. Calibration

As explained in Section II-A-2, the formula $\bar{e} = \bar{P}_1 \bar{P}_2$ is applicable to the experiment discussed in this report. The bars indicate that we are concerned with average values of these quantities because our pion beam, counters, and targets all have extended dimensions. In order to obtain $\bar{P}_1$ at various recoil angles, we measured $\bar{e}$ and $\bar{P}_2$. We have described how $\bar{e}$ was determined. The "calibration" portion of the experiment, in which we measured $\bar{P}_2$, will now be discussed.

The analyzing ability of an experimental arrangement depends on the characteristics of the polarized beam, analyzing target, and detecting counters. Examples of quantities affecting $\bar{P}_2$ are the energy of the polarized protons at the analyzing target, the type and thickness of material composing the target, the angles subtended by the counters measuring the asymmetry, and the amount of copper absorber in the analyzing telescopes. If all components and characteristics of the system are identical for two different asymmetry measurements, then the analyzing abilities are the same.

In order to determine the analyzing ability of our system for each measured recoil-proton asymmetry, we employed a proton beam of known polarization. The polarized protons passed through counters A and B, scattered off the analyzing target, and were detected by the same analyzing telescopes as those employed in the recoil-proton measurements. Corresponding to the recoil-proton investigations, the analyzing scattering took place in the horizontal plane and the incident protons were polarized in a direction perpendicular to this plane.

Equation (II-8) can be rewritten for the calibration portion of the experiment as $\bar{e}^{(C)} = \bar{P}_1^{(C)} \bar{P}_2^{(C)}$. By knowing $\bar{P}_1^{(C)}$ and by measuring $\bar{e}^{(C)}$, we could experimentally determine $\bar{P}_2^{(C)}$. If the conditions under which we obtained $\bar{P}_2^{(C)}$ were the same as those in the measurement of a recoil-proton asymmetry, then the equality $\bar{P}_2^{(C)} = \bar{P}_2$ is valid, where $\bar{P}_2$ is the

analyzing ability that we wished to ascertain. Because the characteristics of the analyzing scattering were different for each recoil angle (see Table V), four separate analyzing abilities had to be determined. This method of obtaining the values of $\overline{P}_2$ took into account the small portion of the analyzed recoil protons that had been inelastically scattered at the carbon target.

We produced the proton beam of known polarization by passing unpolarized protons through the magnet system shown in Fig. 1 and scattering them off a carbon target placed at the final focus. The protons were obtained by degrading the external proton beam of the cyclotron as it entered the Physics Cave. With the 2-in. -thick carbon absorber removed from its position after the central focusing magnet, the degrader thickness and the magnet currents were adjusted to give an unpolarized proton beam of the desired energy. The proton-beam size at the final focus of the magnet system was nearly the same as that of the π^+ beam. The liquid-hydrogen target used in the recoil-proton measurements was replaced by a carbon target measuring 0.25-in. thick by 6-in. wide and 8-in. high, which was centered on the beam line. A range curve of the unpolarized proton beam showed the fraction of mesons in the beam to be negligible and the mean energy of scattering in the carbon to be 173 Mev.

The scattering arm was placed so that counters A and B accepted a mean scattering angle of about 13.8 deg (left). By using data from Dickson and Salter,¹⁷ Tyrén et al. and Alphonse et al.,²² and Hafner,²³ we calculated the mean polarization of the scattered protons detected by counters A and B to be 0.71 ± 0.05 (in the direction perpendicular to the plane of scattering). We included the effects of inelastic scattering in this calculation. Although a higher elastic-scattering polarization could have been obtained at a larger angle, the relative importance of the less-desirable inelastic scattering would have been increased. The rms error of ± 0.05 in the polarization is based on uncertainties in the elastic and inelastic experimental data employed in the calculation of the polarization, and uncertainties in the distribution and values of the scattering angles accepted by counters A and B.

Using the polarized-proton beam defined by counters A and B, we reproduced the different sets of recoil-proton analyzing conditions as closely as possible and measured the four resulting asymmetries. In order to obtain the required mean scattering energies at the analyzing targets, sufficient amounts of degrader were placed just before counter A. The thickness of degrader was different for each of the four measurements. Range curves showed that we had attained the same mean scattering energies as in the recoil-proton observations to within about 2 Mev. The rms energy spread in the polarized-proton beam was ± 8 Mev, slightly less than the ± 10 -Mev energy spread of the recoil protons. For each of the four calibration measurements, a beam profile was obtained with each analyzing telescope and the appropriate analyzing angles were set with respect to the observed center lines. The positions of these profile center lines were not the same as in the recoil-proton measurements, owing to the differences in the angular distributions of the protons from p-C and π^+ -p scattering.

Data were obtained in the calibration measurements by observing the AB III D_O and AB IV D_E coincidence rates. Counter C could not be employed in the calibration procedures because the conjugate particles (carbon nuclei) received too little energy to be counted. We determined the "blank" rate by removing the 0.25-in. -thick carbon target from its position in the unpolarized-proton beam. The calibration counting rates, after correcting for accidental and blank counts, were approximately ten times the rates in the recoil-proton measurements. Our accidental coincidences averaged about 5% of the corresponding corrected analyzed-proton rate, and the target-out (blank) coincidences averaged about 14%. Much higher counting rates could have been obtained by raising the intensity of the external proton beam of the cyclotron. We restricted our net counting rate in order to limit the accidental and blank coincidences to reasonable levels. The effect of background particles was reduced by stacking lead bricks at the same positions as in the recoil-proton measurements.

2. Initial Polarization Measurements

Our data on the polarization of the recoil protons were obtained during two different running periods at the cyclotron. In general, the procedures and the apparatus were the same in both runs. Where differences existed we have referred to the Run-2 arrangement, as a preponderance of our data was acquired during the second period. Owing principally to the larger-area telescope counters employed in the first run, the analyzing abilities measured then were smaller than those later obtained. The polarized proton beam used in the calibration portion of Run 1 had a polarization of 0.58 ± 0.09 . Only one analyzing telescope was employed in the initial polarization measurements.

During the recoil-proton measurements in the first run, we photographed the pulses from the counters as a check on the performance of the electronics. Signals from the counters were displayed on a four-beam oscilloscope. Whenever the electronics detected a five-fold coincidence, the oscilloscope was triggered and the pulses appearing on the four sweeps were recorded on 35-mm film. The film was later developed and projected on a viewer. We measured and plotted the heights and relative positions of the pulses from each counter.

The resulting distributions enabled us to select restrictive criteria for the validity of an event. We rejected a set of pulses if the position or height of any individual pulse did not closely conform to the appropriate normal value. The acceptable film events determined an asymmetry at each recoil angle. There was no blank counting rate to be subtracted; blank coincidences were negligible during the early measurements owing to the relatively low intensity of the pion beam. Accidentals that could deceive the electronics were presumably eliminated in the film analysis because of the restrictive criteria. Values of the asymmetries calculated from the film data agreed well with the electronic asymmetries and increased our confidence in the electronic method.

E. Errors and Results

1. Experimental Errors

Principal sources of experimental error in the asymmetry measurements were counting statistics and uncertainty in the center line of the recoil-proton beam. Uncertainty in the position of the center line can arise, for example, from variations in the direction of the π^+ beam due to magnet-current fluctuations. Another source of this type of error is in the determination of the beam-profile center line from the observed profile counting rates.

In order to estimate the error caused by the center-line uncertainty, we employed the formula $de/d\theta_2 \approx d(\ln I_0)/d\theta_2$ (p. 71 of Ypsilantis²⁰). The symbol I_0 denotes the DCS for the analyzing scattering at the laboratory angle θ_2 , averaged over left and right scatterings. This equation is valid for small e^2 and gives the uncertainty in the measured asymmetry due to the uncertainty in the position of the center line of the polarized-proton beam. When applied to our analyzing arrangements, the above equation gives $de/d\theta_2 \approx 0.2/\text{deg}$. This result reflects the rapid variation of the DCS with angle and indicates that care must be taken when determining the beam center line and when setting the analyzing angles.

We obtained an estimate of the uncertainty in the position of the recoil-proton-beam center line by examining the variation at each recoil angle of the observed beam-profile center lines. It was assumed that these fluctuations reflected the various sources of error and therefore gave an approximate experimental determination of the composite uncertainty. This investigation yielded an rms error in the profile center line of ± 0.10 deg for Run 1 and ± 0.06 deg for Run 2. In terms of uncertainty in e , the $de/d\theta_2$ equation gives rms values of about ± 0.020 for Run 1 and ± 0.012 for Run 2. These numbers are based on the recoil-proton observations but appear approximately valid for the calibration portions of the experiment also.

A thorough error analysis was performed on the scattering apparatus during the period between the two runs. We devoted particular attention to the reproducibility in aligning and positioning the equipment, and to the accuracy of the scale by which we set the analyzing angles. The reproducibility was found to be quite satisfactory. The inaccuracy in setting the plumb bobs at the required angles was negligible. Although small errors were observed in the analyzing-angle scale, their effect on the measured asymmetries was slight compared with the two principal uncertainties previously mentioned.

We estimate an rms uncertainty of ± 0.45 deg in each mean laboratory recoil angle given in Table V. This corresponds to an error of about ± 0.90 deg in each c.m. scattering angle. Principal sources of this error are uncertainties in the position and direction of the pi-meson beam at the liquid-hydrogen target, the position of counter B, the position of the liquid-hydrogen target along the beam line, and the correction applied in order to obtain the mean recoil angle from the angle at the geometric center of counter B. In the calibration for Run 2, these sources of error yield an rms uncertainty of ± 0.6 deg in the mean laboratory scattering angle accepted by counters A and B.

2. Experimental Results

Tables VI and VII present the experimental results of both runs. The satisfactory agreement that was obtained between the two sets of analyzing counters in Run 2 is not shown; only the combined results are presented. When combining two asymmetry or polarization measurements, the individual quantities have been weighted by the inverse of the square of their errors.²⁴ The rms error in the result is $[(\Delta x_1)^{-2} + (\Delta x_2)^{-2}]^{-1/2}$, where Δx_1 and Δx_2 are the rms errors in the individual quantities.

Table VI. Experimentally measured asymmetries of the analyzed recoil protons^a

Mean c. m. scattering angle (deg)	Run 1 ^b		Run 2 ^c	
	Left ^d	Right ^d	Left	Right
114.2	-0.132±0.089	-0.074±0.066	+0.005±0.039	+0.039±0.033
124.5	_____	_____	+0.099±0.054	-0.091±0.038
133.8	+0.130±0.064	-0.212±0.053	+0.068±0.031	-0.039±0.031
145.2	+0.045±0.053	-0.073±0.038	+0.046±0.031	-0.123±0.028

^aThe errors given are standard deviations and are due to counting statistics only.

^bAll Run-1 asymmetries are based on the results of the film analysis, except the 133.8-deg (left) asymmetry, for which only electronic data exist.

^cThe asymmetries measured with each analyzing telescope were combined in order to obtain the Run-2 asymmetries given here. A total of 800 to 2000 analyzed recoil protons determined each Run-2 asymmetry listed.

^dThe "Left" and "Right" column headings refer to the side of the incident pi-meson beam on which the recoil protons were observed.

Table VII. Summary of experimental results

Experimental quantity	Run No.	Mean c. m. scattering angle (deg)			
		114.2	124.5	133.8	145.2
Recoil-proton asymmetry ($\bar{e}$) ^a	1	+0.002±0.055	—	-0.178±0.043	-0.063±0.034
	2	+0.020±0.027	-0.094±0.032	-0.054±0.023	-0.088±0.022
Analyzing ability ($\bar{P}_2$) ^b	1	+0.276±0.047	—	+0.407±0.043	+0.452±0.041
	2	+0.413±0.048	+0.573±0.046	+0.500±0.047	+0.517±0.023
Recoil-proton polarization ($\bar{P}_1 = \bar{e}/\bar{P}_2$)	1	+0.007±0.199	—	-0.438±0.116	-0.139±0.076
	2	+0.048±0.065	-0.164±0.057	-0.108±0.047	-0.170±0.043
Recoil-proton polarization ^c	1 and 2	+0.044±0.062	-0.164±0.057	-0.155±0.044	-0.162±0.037

^aThese results were obtained by combining the "Left" and "Right" asymmetries of Table VI at each scattering angle, after reversing the sign of the "Left" asymmetry and after adding (in rms fashion) to each statistical counting error in Table VI the beam-center-line uncertainty discussed in Section III-E-1.

^bWe determined each analyzing ability by computing $\bar{P}_2 = \bar{P}_2^{(C)} = \bar{e}^{(C)}/\bar{P}_1^{(C)}$, where $\bar{e}^{(C)}$ is the appropriate asymmetry that was measured during the calibration portion of the experiment, and $\bar{P}_1^{(C)}$ is the polarization of the proton beam used in the calibration measurement. The errors presented here arise from the experimental uncertainties in the calibration asymmetries (counting statistics and beam-center-line uncertainty). The error in $\bar{P}_1^{(C)}$ is not included. The results of both analyzing telescopes in Run 2 have been combined.

^cThese final polarization values were obtained by combining the results of Runs 1 and 2. A plot of these values is given later in Fig. 7. The errors are assumed to be independent.

The uncertainty in the polarization of each calibration proton beam is not included in the errors given in Table VII. Thus there is an additional rms error of $\pm 15.5\%$ in all Run-1 values of $\bar{P}_2$ and $\bar{P}_1$, and of $\pm 7\%$ in all Run-2 values. When combining the polarization results of the two runs, we neglected this type of uncertainty. The 15.5% error in Run 1 and 7% error in Run 2 are partially correlated because they are based to a certain extent on the same experimental scattering data. Even if these errors were completely correlated, which is not the situation, the maximum possible effect on any of our final (combined) polarization values would be an additional rms uncertainty of only $\pm 11\%$. This is small compared with the final errors given.

Our sign conventions will now be summarized. In Table VI, the sign of the asymmetry is considered positive if more of the recoil protons scattered to the left than to the right at the carbon target. A positive analyzing ability in Table VII signifies that a majority of the protons scattered to the left at the analyzing target when a preponderance of the incident protons had their spin vectors pointing up (out of the plane of Fig. 2). The sign of the recoil-proton polarization is defined to be positive when a majority of the protons recoiling to the right side of the incident pi-meson beam had their spin vectors pointing up. This definition corresponds to setting $\phi = 0$ deg in the phase-shift equations of Section II-B.

IV. PHASE-SHIFT ANALYSIS

We have completed a series of measurements of the recoil-proton polarization in elastic π^+ -p scattering at 310 Mev. Data have been obtained at four angles of observation. Complementing these results, Ernest H. Rogers has measured the elastic DCS and the total cross section in π^+ -p scattering at the same energy.²⁵ The polarization and cross-section data are noteworthy because of the relatively high accuracy that has been obtained. We will now describe our phase-shift analysis of these experimental measurements. The quantity, variety, and quality of the available data are reflected in the results of the analysis.

A. Search Program

1. General Method

In the analysis of our experimental cross-section and polarization data, we use the formulas developed in Section II-B. Because of the complicated equations involved, a trial-and-error procedure is employed in order to solve for the phase shifts. We assign definite values to the phase shifts, substitute them into the relevant formulas, and calculate the observable quantities. The results of these calculations can then be compared with the available experimental data. If the comparison does not yield satisfactory agreement, other values of the phase shifts can be selected and the calculations performed again. An electronic computer can readily perform the many repetitions of this cycle necessary in order to obtain an adequate fit to the data.

We have written an IBM-704 program that incorporates the trial-and-error procedure just described. The program instructs the computer to search for a set of phase shifts that will fit the data, beginning at a given set of values. The grid search procedure is employed, in which the phase shifts are varied in cycles.²⁶ Successively smaller changes are made in the phase shifts until a fit is obtained with the de-

sired accuracy. When varying a phase shift, the program uses the procedure described in Appendix D, thus avoiding iterative calculation of trigonometric functions.

The program is arranged so that, in the search for a fit to the data, the computer varies the phase shifts but not the inelastic parameters. In the major portion of our phase-shift investigations, and unless otherwise stated, the inelastic parameters are assumed to be unity; that is, only elastic scattering is allowed. This assumption is reasonable owing to the apparently small amount of inelastic scattering at 310 Mev (see Section V-A). If there were substantial inelastic scattering, the inelastic parameters could be considerably less than unity. We might then have to vary both the inelastic parameters and the phase shifts in the search for the true solution, and the analysis would become more complicated.

Although we generally disregard inelastic scattering, we will eventually want to investigate its influence on the results of the phase-shift analysis. Our program enables the computer to accept selected values of the inelastic parameters and employ these initial values throughout the search procedure. Various combinations of these parameters can be chosen, the solution of interest can be redetermined, and the resultant phase-shift changes can be examined. In this way, one is able to obtain estimates of the errors introduced into the analysis by the assumption that all the inelastic parameters are unity.

2. The Least-Squares Quantity M; Fit Criterion

The predictions of a given set of phase shifts are compared with the available experimental data by computing the quantity M, where

$$M = \sum_i \left[\frac{X_i^{(c)} - X_i^{(e)}}{E_i} \right]^2$$

Here $X_i^{(e)}$ is the quantity X_i as obtained from experiment, E_i is the experimental error (standard deviation) in $X_i^{(e)}$, and $X_i^{(c)}$ is the quantity X_i as calculated by the computer from a given set of phase shifts. We sum over all the experimental measurements.

Expressing M in terms of quantities for which we have experimental data, we write

$$M = \sum_j \left[\frac{P_j^{(c)} - P_j^{(e)}}{E_j^{(P)}} \right]^2 + \sum_k \left[\frac{I_k^{(c)} - (1+\epsilon) I_k^{(e)}}{E_k^{(I)}} \right]^2 + \left[\frac{\epsilon}{E^{(\epsilon)}} \right]^2 + \left[\frac{I_T^{(c)} - I_T^{(e)}}{E^{(T)}} \right]^2, \quad (\text{IV-1})$$

where P_j is the polarization of the recoil protons at the c.m. scattering angle $\theta_{\text{c.m.}}^{(j)}$, $E_j^{(P)}$ is the experimental error in $P_j^{(e)}$, I_k is the elastic DCS for scattering at the c.m. angle $\theta_{\text{c.m.}}^{(k)}$, $E_k^{(I)}$ is the experimental error in $I_k^{(e)}$, ϵ is the variable normalization parameter for the DCS, $E^{(\epsilon)}$ is the experimental error in ϵ (the experimental value of ϵ is $0 \pm E^{(\epsilon)}$), I_T is the total cross section (elastic plus inelastic) between the cutoff angles $\theta_{\text{c.m.}}^{(1)}$ and $\theta_{\text{c.m.}}^{(2)}$, and $E^{(T)}$ is the experimental error in $I_T^{(\epsilon)}$. The first summation in the expression for M extends over all angles for which polarization data exist; the second summation, over all angles for which DCS data were obtained. We assume that the experimental errors entering into M are independent, normally distributed, and realistically estimated.

The search program requires the computer to find a set of phase shifts for which M has a minimum value. In this way, a "least-squares" fit to the data is attained. Such a fit corresponds to a minimum point in the sense that a change of $\pm \Delta_{\text{FINAL}}$ in any one of the phase shifts gives a larger value of M than the value calculated at the minimum. Here Δ_{FINAL} is the smallest increment employed when the phase shifts are varied. The resulting value of M may not have the absolute minimum magnitude obtainable because the computer stops at the first relative minimum that it notices. Different initial sets of phase shifts can lead to different minima, some of which may have even lower M values.

During the search procedure, the computer varies ϵ in the same manner that it varies the phase shifts. Thus the computer is able to modify the absolute scale of the DCS in order to improve the fit to the data. The experimental error in ϵ , $E^{(\epsilon)}$, is comprised of the uncertainties in the DCS absolute scale. Errors of this type include uncertainties in the intensity and contamination of the incident pi-meson beam and in the thickness of the liquid-hydrogen target. Independent errors, such as statistical counting uncertainties, are attached to each DCS measurement individually and are denoted $E_k^{(I)}$. These independent errors indicate the accuracy with which the various measurements are known with respect to one another (effects of systematic uncertainties in the shape of the DCS are discussed in Section IV-C-1). The use of the variable ϵ enables the phase-shift analysis to keep the independent errors in the individual DCS measurements separate from the uncertainties in the absolute scale, thus allowing an optimum amount of information to be obtained from the DCS data and permitting independent errors in the expression for M . Although we will generally disregard ϵ in our further discussions of the program and when quoting results, it will always be present in our analysis.

Owing to the influence of the small relative error in $I_T^{(e)}$, the principal effect of ϵ was to enable the elastic DCS curve to be normalized to the total-cross-section measurement. In performing this normalization, we usually assumed that we could neglect the inelastic-scattering contribution to the total cross section. Because the amount of inelastic scattering at 310 Mev is apparently not appreciable, the error introduced by its disregard in the normalization procedure appears to be small compared with the error in the total-cross-section measurement.

It is illuminating to visualize the hypersurface that would be obtained if M could be plotted as a function of the phase shifts. The region around a point where M has a minimum value corresponds to a depression in the hypersurface. In the phase-shift discussions to follow, we will often refer to this visual representation.

3. Rounding-Error Check

A possible source of trouble in the search procedure arises from the fact that the computer can only work to a limited number of figures of accuracy. The IBM-704 computer generally rounds off the results of its numerical operations to 27 binary digits (about eight decimal digits) of precision. If Δ_{FINAL} is made too small, the corresponding changes in M might become of the same order as the errors in M created by the rounding process. The search method would then lose its utility because the true minimum could be confused with false minima created by the rounding errors. In terms of our hypersurface model, the exploratory steps taken along the M surface would be sufficiently small so that the computer would notice the unevenness caused by the approximations inherent in the rounding procedure.

In order to investigate the influence of rounding error on our results, a special subroutine has been written into the program. This subroutine is employed, when desired, after the regular minimization procedure has been completed. The computer makes several successive changes of the magnitude Δ_{FINAL} in each phase shift, beginning at the minimum point under consideration. After a phase shift is changed in the positive direction the desired number of times, it is returned to its value at the minimum and several variations in the negative direction are made. Only one phase shift is varied at a time, and its initial value is restored after the series of changes is completed.

After each modification in a phase shift, the computer calculates and prints out (via magnetic tape) the new value of M . These results can be examined for erratic behavior, which might indicate rounding-error trouble. Our investigations have shown that any error due to rounding is less than the accuracy for which we are striving.

B. Error Analysis

1. Error-Matrix Method

The usefulness of any possibly acceptable phase-shift fit will be increased if we are able to ascertain the accuracy with which the experimental data determine the individual phase shifts. A method involving the error matrix is customarily used to obtain the uncertainty in each phase shift. This method will be examined briefly here. Additional information can be found in the phase-shift-analysis discussion of Anderson, Davidson, Glicksman, and Kruse.²⁷

To obtain the error in each phase shift, we investigate the region on the M hypersurface near the bottom of the depression whose lowest point corresponds to the solution under consideration. We wish to study the behavior of M when the phase shifts are varied from their values at the minimum point to other nearby values. Let us expand M in a Taylor series about the minimum and retain terms only up through second order in the phase-shift differences. The first derivatives of M with respect to the phase shifts are zero because they are evaluated at the minimum. The resulting equation is therefore

$$M(\delta_1^M + \Delta\delta_1, \delta_2^M + \Delta\delta_2, \dots, \delta_N^M + \Delta\delta_N) \\ \approx M_0(\delta_1^M, \delta_2^M, \dots, \delta_N^M) + \sum_{i=1}^N \sum_{j=1}^N G_{ij} \Delta\delta_i \Delta\delta_j, \quad (\text{IV-2})$$

where N is the number of phase shifts assumed; the quantities $\delta_1^M, \delta_2^M, \dots, \delta_N^M$ are the values of the N phase shifts at the minimum point on the hypersurface; $\Delta\delta_1, \Delta\delta_2, \dots, \Delta\delta_N$ are the changes in the phase shifts from their values at the minimum; $M_0(\delta_1^M, \delta_2^M, \dots, \delta_N^M)$ is the value of M at the minimum; and G_{ij} is $(1/2)(\partial^2 M / \partial \delta_i \partial \delta_j)$ evaluated at the minimum. The i and j summations extend over all

the phase shifts employed. To the extent that Eq. (IV-2) is valid, M varies quadratically with the changes in the phase shifts from their values at the minimum. In the discussions in the rest of this section (IV-B) and in its related appendices, we assume that this quadratic behavior is approximately correct.

The error matrix, G^{-1} , is the array of numbers obtained by inverting the matrix G , which consists of the symmetric array of quantities G_{ij} . An extension of our IBM-704 program enables the computer to calculate G and G^{-1} . The method by which the elements of G are obtained is explained in Appendix F.

The elements of G^{-1} are related to the uncertainties in the phase shifts. According to statistical theory, we can write (to within the accuracy of the assumptions made):²⁷

$$\sqrt{(G^{-1})_{ii}} = (\Delta\delta_i)_{\text{rms}},$$

which is the root-mean-square error (standard deviation) in δ_i^M , and

$$(G^{-1})_{ij} = C_{ij} \times (\Delta\delta_i)_{\text{rms}} \times (\Delta\delta_j)_{\text{rms}}, \text{ for } i \neq j, \quad (\text{IV-3})$$

where C_{ij} is the ij th correlation coefficient (with a value between +1 and -1), and $(G^{-1})_{ii}$ and $(G^{-1})_{ij}$ are elements of G^{-1} . The correlation coefficients indicate the degree to which the phase shifts are related. For $C_{ij} = 0$, δ_i and δ_j are independent; for $C_{ij} = \pm 1$, there is maximum dependency. A geometrical interpretation of the correlation coefficients is given in Appendix G. If F is any function of the phase shifts, then the error (standard deviation) in F is calculated from the matrix G^{-1} by using the formula²⁸

$$(\Delta F)_{\text{rms}} = \sqrt{\sum_{i=1}^N \sum_{j=1}^N \frac{\partial F}{\partial \delta_i} \frac{\partial F}{\partial \delta_j} (G^{-1})_{ij}}, \quad (\text{IV-4})$$

where the elements $(G^{-1})_{ij}$ are in (radians)².

2. Auxiliary Method of Error Determination

The rms error in each phase shift can also be determined by a procedure which we will call AUX (auxiliary). * An important function of AUX is to check the results obtained through use of the error-matrix method. It can also be employed to investigate the general shape of the hypersurface in the vicinity of a minimum. In order to obtain the error in every phase shift, AUX requires considerably more computer time than is needed by the error-matrix method. The reason for this is that AUX employs the time-consuming minimization procedure of our search program, and the other error method does not.

In order to understand AUX, let us consider the situation in which only two phase shifts exist. Then, a plot of M as a function of its two parameters is a three-dimensional surface and is easily visualized. We wish to obtain the errors in a set of phase shifts. The region around the corresponding minimum on the M surface must therefore be investigated, as in the error-matrix method. Let δ_1 and δ_2 represent the two phase shifts and let δ_1^M and δ_2^M be their values at the minimum. We define

$$x = \delta_1 - \delta_1^M,$$

$$y = \delta_2 - \delta_2^M,$$

$$z = \Delta M = M(\delta_1, \delta_2) - M_0(\delta_1^M, \delta_2^M).$$

The origin of the x - y - ΔM coordinate system is therefore at the minimum of the depression that is to be inspected.

With only two phase shifts present, Eq. (IV-2) can be rewritten (in the x - y notation) as

$$\Delta M \approx G_{xx} x^2 + 2 G_{xy} xy + G_{yy} y^2. \quad (\text{IV-5})$$

* This procedure was suggested for use in our analysis by Professor Owen Chamberlain.

For a fixed ΔM , Eq. (IV-5) represents an ellipse. Different values of ΔM give ellipses of different sizes. The ellipses represent the intersections with the M surface of planes parallel to the x - y plane. The projections on the x - y plane of three of these intersections are shown in Fig. 6.

Curve C in Fig. 6 is the locus of points at which lines, perpendicular to the x axis, are tangent to the family of ellipses represented by Eq. (IV-5). In Appendix G, we show that the points on curve C (denoted by the subscript C) possess coordinates that follow a $\Delta M_C \propto x_C^2$ behavior. We also prove there that the value of x_C at point P in Fig. 6 is the rms error in the phase shift δ_1 . Point P is the intersection of curve C and the $\Delta M = 1$ ellipse.

We now have the necessary information to understand the manner in which AUX determines the error in each phase shift. For the purpose of ascertaining the error in δ_1 , the value of this phase shift is changed from δ_1^M to $\delta_1^M + x'$. The search procedure that was originally employed to locate the minimum is now used again. However, δ_1 is now never varied but is held constant at the value $\delta_1^M + x'$. In our three-dimensional example, only δ_2 is changed. This corresponds to a movement along the line that is perpendicular to the x axis and passes through x' , the selected value of x . The lowest value of M obtainable along this line is the value associated with the ellipse to which the line is tangent. Therefore the minimum point found by our modified search procedure is on curve C .

Knowing the values of x and ΔM at this minimum point, we can employ the $\Delta M_C \propto x_C^2$ formula and obtain the value of x at point P (corresponding to $\Delta M = 1$). In an actual application of AUX, the formula $\Delta M_C \propto x_C^2$ may only be approximately true because the minimum may not be perfectly quadratic in shape. We therefore try to select the initial increment, x' , so that the resultant ΔM will be close to unity. In this way, we need only assume the validity of $\Delta M_C \propto x_C^2$

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Fig. 6. The ellipses represent curves of constant M for the case of only two phase shifts. The eccentricity of the ellipses and their orientation with respect to the x and y axes are arbitrary. The significance of curve C and point P is explained in Section IV-B-2.

over a small range of x values in order to obtain the x coordinate at point P. The resulting value of x is the rms error in δ_1 , at least to the extent that the error theory, based on the second-order Taylor-series expansion, is valid. We obtain the value of y representing the rms error in δ_2 by using a procedure analogous to that employed to find the error in δ_1 .

The AUX method can be generalized to the problem where there are more than two phase shifts. The points on the M hypersurface that correspond to a fixed ΔM then form an ellipsoid or hyper-ellipsoid [assuming that Eq. (IV-2) is valid]. An extension of our IBM-704 program permits the calculation, through the use of AUX, of the error in each of the phase shifts. One phase shift, δ_k , is changed by a preselected amount and then held fixed while the other parameters are varied, in analogy to the example employing only two phase shifts. From the smallest value of M obtained in this manner, one can estimate the rms error in δ_k by employing the $\Delta M_C \propto x_C^2$ formula and calculating x_C for $\Delta M_C = 1$.

In order to approximately take into account any asymmetry in the shape of the M depression, we determine both the positive and negative changes in each phase shift required to give $\Delta M = 1$. An estimate of the rms error, averaged over the two directions of variation, is thereby obtained.

We found the rms phase-shift errors obtained by AUX to generally agree with those derived from the error matrix. In certain of our investigations, the lack of sufficient data or the existence of many variable parameters caused the shape of an examined minimum to deviate noticeably from the desired quadratic behavior. Although disparities were then found between the results of the two error methods, satisfactory agreement could often be obtained by limiting the error investigations to the region on the M hypersurface corresponding to $\Delta M \lesssim 1$.

C. Phase-Shift Investigations²⁹

We next describe our phase-shift analysis, which uses the search program and error procedures explained in Sections IV-A and IV-B. The information obtained in these investigations will be presented. First of all, we discuss the analysis involving S, P, and D waves and the evidence that the D-wave phase shifts are needed in order to attain a satisfactory fit to the data. The ambiguity in the D-wave phase shifts is examined. We investigate the sensitivity of the various phase shifts to the different types of experimental data. The inclusion of F waves in the analysis is discussed, and also described is the attempt to add G waves.

1. The SPD Random Search^{*}

The phase-shift investigations were begun with a random search involving S-, P-, and D-wave phase shifts. In order to find every minimum that might lie in the neighborhood of the true solution, the computer was asked to begin searching at a large number of random points scattered over the ϕM hypersurface. A total of 244 random sets of phase shifts were fed into the computer. The values of all five phase shifts ($S_{3,1}$, $P_{3,1}$, $P_{3,3}$, $D_{3,3}$, $D_{3,5}$) in every set were randomly selected. The initial value of ϵ was always zero. From these 244 random positions on the hypersurface, the computer searched and found 27 distinct clusters of solutions (phase-shift fits). The solutions in each cluster agree with one another to within a few tenths of a degree in every phase shift. The different clusters apparently correspond to various relative minima. Each of the ten relative minima in the group with the lowest values of M was detected by the computer at least five times. If one assumes that the relative minima are randomly spaced on the ϕM hypersurface and can be entered with equal ease, then the probability of having overlooked a set of phase shifts with a low M value is less than 1%.

^{*}The notation SPD will refer to our analysis involving S-, P-, and D-wave nuclear phase shifts only. We will also use the abbreviation SPDF, which is a straightforward extension of this notation.

Since the completion of our SPD random search, both the computer program and the input data have been revised and extended. The most important changes were the addition of a total-cross-section measurement and the inclusion of DCS data at angles sufficiently small so that Coulomb-nuclear interference effects are noticeable. It is assumed that no new minima with low values of M were created by the changes made. (The validity of this assumption is supported by the results of the SPDF random search to be described in Section IV-C-5.) In general, the changes in the data and program produced only small alterations in the phase-shift values related to each minimum. The presence of the DCS data at small angles caused the M values of several of the original minima to increase considerably. These minima correspond to sets of phase shifts that give the incorrect sign for the Coulomb-nuclear interference effects.

In all results to follow, we employ the revised and extended data and program. The data used include our four polarization measurements, values of the DCS at 23 angles of observation, and a total-cross-section measurement of 56.4 ± 1.4 mb (between the c.m. cutoff angles 14.7 and 158.0 deg). All the cross-section data were obtained by Ernest H. Rogers.²⁵ The polarization and DCS data are given in Tables VII and VIII, and are plotted in Figs. 7 and 8.

Of the 27 distinct sets of phase shifts found in the SPD random search, all but three have negligible probabilities of lying in the vicinity of the true solution. We base this statement on the χ^2 distribution of statistical theory, which can be applied at least approximately to our results.³⁰ The χ^2 distribution for 23 degrees of freedom is used here because we are endeavoring to fit 29 pieces of experimental information (including $\epsilon = 0.00 \pm 0.06$) with five phase shifts and the parameter ϵ . The 24 solutions that were discarded on the basis of statistical theory have values of M in the range 86 to 1100, and are therefore highly improbable (the mean M value expected is equal to the number of degrees of freedom). If the polarization data had not been present in the

Table VIII. Experimental DCS measurements (in the center-of-mass system) used in the phase-shift analysis. The errors given are standard deviations and are independent. Not included is an rms error of $\pm 6\%$ in the absolute DCS scale. These data were obtained by Ernest H. Rogers.²⁵

C. m. scattering angle (deg)	$I(\theta_{\text{c.m.}})$ (mb/sterad)
14.0	18.71 $\pm$ 0.60
19.6	16.05 $\pm$ 0.46
25.2	13.82 $\pm$ 0.31
30.6	12.99 $\pm$ 0.25
34.6	12.28 $\pm$ 0.27
36.2	11.65 $\pm$ 0.27
44.0	9.82 $\pm$ 0.15
51.8	8.59 $\pm$ 0.26
56.8	7.54 $\pm$ 0.28
60.0	6.58 $\pm$ 0.22
69.6	4.73 $\pm$ 0.10
75.3	3.62 $\pm$ 0.09
81.6	2.77 $\pm$ 0.08
97.8	1.66 $\pm$ 0.07
105.0	1.51 $\pm$ 0.06
108.1	1.62 $\pm$ 0.07
120.9	2.08 $\pm$ 0.08
135.2	2.93 $\pm$ 0.14
140.6	3.36 $\pm$ 0.12
144.7	3.76 $\pm$ 0.15
152.2	4.10 $\pm$ 0.21
156.4	4.51 $\pm$ 0.17
165.0	4.88 $\pm$ 0.12

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Fig. 7. Experimental recoil-proton polarization measurements given in Table VII. The solid curves represent the fits to the data predicted by the SPD solutions in Table IX. The SP fit, which is discussed in Section IV-C-2, is indicated by the dashed curve.

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Fig. 8. The experimental c.m. DCS measurements given in Table VIII have been multiplied by $1 + \epsilon$ in order to normalize them to the total cross section. The value of ϵ used (-0.018) is that giving the minimum magnitude of M for both the SPD and SP Fermi-type solutions. Independent errors only are shown. The solid curve, which represents the Fermi SPD solution, fits the data well. The dashed curve at small angles shows the behavior of the SPD Fermi and Yang solutions that possess phase-shift signs opposite to those given in Table IX. The curve with short dashes, shown only at large angles, is the Fermi SP fit discussed in Section IV-C-2. It is given only where it deviates sufficiently from the SPD fit to be easily drawn.

analysis, some of these improbable sets of phase shifts would have had low M values and therefore could not have been discarded on the statistical basis alone.

Our three possibly acceptable solutions are presented in Table IX. The phase shifts given there are of the nuclear type. They were acquired by subtracting the Coulomb phase shifts δ_L^+ , which are listed in Table I, from the total phase shifts obtained by the search program. The three solutions in Table IX are of the Fermi type, Minami type, and Yang type, in order of increasing M . The connections between these sets of phase shifts are not precisely the relationships explained in Section II-C because of the additional constraints created by the polarization data. However, the features that characterize these solutions can be noted.

Two other sets of phase shifts are good fits to all but the DCS data at small angles. These solutions are similar to the Fermi and Yang fits in Table IX except that the signs of most of the phase shifts are opposite to the signs of the corresponding quantities in the table. Because these two solutions give destructive Coulomb-nuclear interference in the forward direction of scattering, we can definitely exclude them by using the DCS data at small angles (see Fig. 8).

Figures 7 and 8 show the manner in which the SPD solutions in Table IX fit the data. The DCS curves calculated from the Minami and Yang sets of phase shifts are not shown; they closely resemble the Fermi plot. All three phase-shift sets give values for the total cross section that are in good agreement with the experimental measurement.

We present in Table X the error matrix that is associated with our SPD Fermi solution. The square root of each diagonal element of this matrix is given later in Table XVI. In order to make the problem manageable, we have neglected the systematic uncertainties in the shape of the DCS and have used only the independent uncertainties referred to in Section IV-A-2. It is these independent errors that are given in

Table IX. The solutions found in the SPD random search that best fit the experimental data.

Type of solution	M	Nuclear phase shift (deg)				
		$S_{3,1}$	$P_{3,1}$	$P_{3,3}$	$D_{3,3}$	$D_{3,5}$
Fermi	15.8	-18.5	-4.7	134.8	1.9	-4.0
Minami	32.0	-7.1	-22.3	-1.9	135.6	0.8
Yang	37.7	-23.2	126.2	159.0	7.5	-4.6

Table X. Error matrix for the SPD Fermi solution. The matrix elements are in $(\text{deg})^2$.

	$S_{3,1}$	$P_{3,1}$	$P_{3,3}$	$D_{3,3}$	$D_{3,5}$
$S_{3,1}$	0.41	0.26	0.17	0.11	-0.20
$P_{3,1}$		0.32	0.05	0.11	-0.18
$P_{3,3}$			0.42	-0.01	0.05
$D_{3,3}$				0.13	-0.10
$D_{3,5}$					0.19

Table VIII and shown in Fig. 8. We investigated the influence on the phase shifts of the systematic uncertainties just mentioned, and found the effects to be small compared with the rms errors obtained from the error matrix for the SPD Fermi solution.

In the remainder of this section, our attention will often be concentrated on the Fermi solution given in Table IX. The reasons for disregarding the Minami and Yang sets of phase shifts will be discussed in Section V-A.

2. Inadequate SP Fit

Is the inclusion of D waves in the analysis necessary in order to obtain a good fit to the data, or will S and P waves alone suffice? Besides our SPD analysis, we have also analyzed the data assuming that the pion-nucleon nuclear interaction affects only the S and P waves. The best SP fit that we obtained is given in Table XI; the corresponding polarization and DCS curves are shown in Figs. 7 and 8. This solution is of the Fermi type and is obviously an inadequate fit to the experimental data. The poor fit is shown numerically in the large M value of 92.5. Although the D-wave nuclear phase shifts are small in our SPD Fermi set, they are definitely needed in order to obtain a satisfactory fit.

By comparing the SP and SPD Fermi solutions, we observe that the inclusion of D waves in the analysis has a noticeable effect on $S_{3,1}$ and $P_{3,1}$. Each is reduced in absolute magnitude when the D-wave nuclear phase shifts are allowed to have values other than zero. Only the phase shift $P_{3,3}$ is rather insensitive to the number of partial waves included in the analysis.

3. Ambiguity in the D-Wave Phase Shifts

When our four polarization measurements are excluded from the SPD analysis, an uncertainty appears in the D-wave phase shifts. This ambiguity was mentioned in Section II-C-2. It gives rise to two Fermi-type solutions yielding low values of M, instead of just the one pre-

viously discussed. The two Fermi phase-shift sets, obtained when only the cross-section data are utilized, are given in Table XI. (They possess lower M values than the Fermi solution in Table IX because there are fewer experimental measurements to fit.) A principal difference between these two solutions is that the D -wave phase shifts in one set have signs reversed compared with those in the other set. The usefulness of the polarization data in differentiating between these two SPD phase-shift solutions is demonstrated in Fig. 9.

The utility of the polarization measurements in eliminating the D -wave phase-shift ambiguity from our SPD analysis is also indicated in Figs. 10 and 11. The AUX error procedure was employed in order to obtain these plots. In terms of the three-dimensional example in Fig. 6, these curves give ΔM as a function of x for points along curve C. When many phase shifts are present as in our SPD analysis, an analogous plot can be obtained for each parameter. The differences between Figs. 10 and 11 arise from the fact that only the cross-section data are used in the first figure, and all data including the polarization measurements are employed in the second. The minima depicted in Figs. 10 and 11 correspond to the Fermi-I solution in Table XI and the Fermi solution in Table IX, respectively, and therefore represent the same solution fitted to different amounts of experimental data. Several of the curves in Fig. 10 deviate greatly from a quadratic behavior owing to the existence of the nearby SPD Fermi-II solution. A comparison of Figs. 10 and 11 therefore shows the effectiveness with which the polarization data is able to dispose of the SPD Fermi-II solution, whose M value changes from 14 to 195 when the polarization data is included in the analysis.

4. Sensitivity of the Phase Shifts

We examined the SPD Fermi fit in Table IX and investigated the sensitivity of its individual phase shifts to the various types of experimental measurements. Different combinations of the 310-Mev data were used, and approximate values for the errors in the phase

Table XI. The "SP Fermi" solution is our best SP fit to the experimental data. "Fermi I" and "Fermi II" are the two SPD Fermi solutions with low M values that are obtained when the computer is required to fit only the cross-section data (these solutions exhibit the ambiguity in the D-wave phase shifts).

Type of solution	M	Nuclear phase shift (deg)				
		$S_{3,1}$	$P_{3,1}$	$P_{3,3}$	$D_{3,3}$	$D_{3,5}$
SP Fermi	92.5	-22.3	-8.1	136.1	0	0
Fermi I	13.9	-16.8	-4.0	134.8	3.3	-5.4
Fermi II	14.1	-24.0	-8.8	137.3	-3.5	2.4

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Fig. 9. Variation of polarization with angle predicted by the two SPD Fermi solutions with low M values that are obtained when the computer fits only the cross-section data. These solutions exhibit the ambiguity in the D-wave phase shifts. The values of the phase shifts for these fits are given in Table XI. The experimental recoil-proton polarization measurements are also shown above.

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Fig. 10. In terms of the three-dimensional example in Fig. 6, we have plotted ΔM as a function of x , now denoted $\Delta \delta_i$, for points along curve C. With five phase shifts present (SPD fit), an analogous plot is obtained for each. The variable $\Delta \delta_i$ represents the change in any one of the phase shifts from its value at the minimum point. The origin of the coordinate system corresponds to the SPD Fermi-I solution given in Table XI. Only the cross-section data are utilized in the calculations summarized here.

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Fig. 11. The curves are the same as those in Fig. 10 except that now the polarization measurements are also included in the analysis. The origin of the coordinate system corresponds to the SPD Fermi solution in Table IX. In order to simplify the figure, the behavior near the origin of certain curves is not shown.

shifts were calculated for each combination. A comparison of the resulting sets of errors should indicate the types of data to which each phase shift is sensitive. In order to obtain the desired estimates of the errors, the error-matrix method was employed. The results of these calculations are summarized in Table XII.

Certain qualitative conclusions can be drawn from the relative magnitudes of the errors listed in Table XII:

(a) The phase shift $P_{3,3}$ is essentially sensitive only to the total cross section; $S_{3,1}$ is also sensitive to this type of experimental measurement, but to a lesser extent.

(b) The phase shifts $S_{3,1}$, $D_{3,3}$, and $D_{3,5}$ are strongly sensitive to the polarization data.

(c) All the phase shifts are dependent to various degrees on the DCS measurements; $P_{3,3}$ is relatively insensitive to these data. The three preceding conclusions depend, in part, on the magnitudes of the phase shifts. Therefore care must be exercised when applying these results to energies other than those in the region around 310 Mev.

The ability of the polarization measurements to reduce the magnitudes of the phase-shift errors can also be observed by comparing Figs. 10 and 11. These figures indicate that the addition of the polarization data not only eliminates one of the two SPD Fermi solutions but also causes the minimum point of the remaining depression to become more sharply defined, thus reducing the errors in certain of the phase shifts.

5. Inclusion of F Waves

Because of the relatively high accuracy with which the phase shifts in our SPD Fermi fit are determined, we felt it necessary to extend the analysis to include F waves. It appeared quite possible that the addition of small F-wave phase shifts might cause changes in the other phase shifts larger than the quoted errors. This indeed turned out to be true. We found that the inclusion of a small F-wave nuclear interaction not only alters the values of almost all the S-, P-, and D-wave phase shifts but also causes their errors to increase considerably. Also, new solutions appear that fit the data well.

Table XII. Investigation of the sensitivity of the phase shifts at 310 Mev to the different kinds of experimental data. The type of solution examined is the SPD Fermi fit in Table IX. Given below are the data utilized in each set of calculations and the estimates obtained for the rms errors.

Number of experimental measurements utilized			Estimated rms error in				
Total cross section	Differential cross section	Recoil-proton polarization	$S_{3,1}$	$P_{3,1}$	$P_{3,3}$	$D_{3,3}$	$D_{3,5}$
			(deg)				
0	23	0	1.9	0.8	1.6	1.3	1.3
1	23	0	1.6	0.8	0.6	1.3	1.3
0	6 ^a	4	1.4	1.1	1.7	0.5	0.7
1	6 ^a	4	1.2	1.1	0.7	0.5	0.7
0	23	4	1.0	0.7	1.6	0.4	0.4
1	23	4	0.6	0.6	0.6	0.4	0.4

^aThese measurements are spread throughout the angular range for which DCS data exist.

With the F-wave nuclear phase shifts allowed to be different from zero, another random search for solutions was conducted. New random initial values were picked for the phase shifts related to the S, P, and D waves. The initial F-wave phase shifts were also chosen at random, but were restricted to the interval 0 ± 9 deg because we assumed these parameters to be small. The number of random sets used was 260, and about twice as many minima were found as in the SPD random search. Every solution with an M value of less than 40 was obtained at least five times. According to the χ^2 distribution, now for 21 degrees of freedom, the probability is less than 1% that the M value of the true solution is greater than 40.

As a check on the SPD random-search results, we made SPD fits to the data using as starting points the first five phase shifts in the various SPDF solutions. All the original SPD solutions appeared. In addition, only two new minima were found and these possess extremely high M values. Therefore, we had apparently obtained all the existing SPD solutions with low M values in our original random search.

Every discovered SPDF solution with a value of M less than 40 is listed in Table XIII. The Fermi-I, Minami-I, and Yang-I solutions correspond to the three SPD fits given in Table IX. The designation "Minami-Yang" refers to the type of fit of that name discussed in Section II-C-2. Many of the phase-shift values in the various solutions denoted "I" in Table XIII are approximately connected by the ambiguity relationships discussed in Section II-C. Similarly interrelated are the three fits denoted "II". We will disregard solution 6 because of its excessively large $F_{3,7}$. When SPD fits to the cross-section data only are obtained, the SPDF Fermi-I and -II solutions reduce to the solutions of the same names given in Table XI and therefore appear to be manifestations of the ambiguity in the D-wave phase shifts. The error matrices for these two sets of phase shifts are presented in Tables XIV and XV. The square root of each diagonal element of these matrices is given later in Table XVI.

Table XIII. Solutions found in the SPDF random search that possess values of M less than 40.

No.	Type of solution	M	Nuclear phase shift (deg)						
			$S_{3,1}$	$P_{3,1}$	$P_{3,3}$	$D_{3,3}$	$D_{3,5}$	$F_{3,5}$	$F_{3,7}$
1	Fermi I	14.1	-17.2	-2.9	135.0	3.1	-4.9	0.5	-0.6
2	Minami-Yang I	17.6	123.1	-22.4	3.1	158.6	0.2	-2.8	-0.1
3	Fermi II	18.3	-35.5	-16.1	151.4	-11.4	13.1	-1.1	-1.8
4	Yang II	26.6	-32.0	142.2	160.4	17.8	-6.4	-1.7	-1.3
5	Minami-Yang II	26.9	139.9	-39.0	13.1	164.0	-4.9	-5.7	2.0
6	_____	27.8	-19.2	-7.6	153.8	2.0	-21.1	-2.7	13.0
7	Minami I	31.7	-7.2	-22.4	-2.0	136.8	0.8	0.2	0.1
8	Yang I	34.2	-23.6	124.7	159.5	5.8	-4.1	-1.5	0.7

Table XIV. Error matrix for the SPDF Fermi-I solution.
The matrix elements are in $(\text{deg})^2$.

	$S_{3,1}$	$P_{3,1}$	$P_{3,3}$	$D_{3,3}$	$D_{3,5}$	$F_{3,5}$	$F_{3,7}$
$S_{3,1}$	6.93	10.38	-0.08	6.65	-5.56	1.27	-3.61
$P_{3,1}$		16.14	-0.36	10.34	-8.54	1.96	-5.66
$P_{3,3}$			0.42	-0.28	0.27	-0.05	0.16
$D_{3,3}$				6.76	-5.51	1.28	-3.67
$D_{3,5}$					4.61	-1.04	3.00
$F_{3,5}$						0.31	-0.70
$F_{3,7}$							2.03

Table XV. Error matrix for the SPDF solution Fermi-II.

The matrix elements are in $(\text{deg})^2$.

	$S_{3,1}$	$P_{3,1}$	$P_{3,3}$	$D_{3,3}$	$D_{3,5}$	$F_{3,5}$	$F_{3,7}$
$S_{3,1}$	0.50	-0.11	0.30	-0.08	0.08	-0.08	0.13
$P_{3,1}$		0.43	-0.37	0.24	-0.30	0.13	-0.11
$P_{3,3}$			0.70	-0.25	0.26	-0.13	0.12
$D_{3,3}$				0.22	-0.22	0.08	-0.08
$D_{3,5}$					0.29	-0.11	0.11
$F_{3,5}$						0.08	-0.06
$F_{3,7}$							0.09

The Fermi-II solution and the two Minami-Yang fits were also found in the SPDF random search but then had improbably large M values because of their inability to fit the polarization data. The presence of small F-wave phase shifts has enabled these three previously unacceptable solutions to become good fits to the polarization measurements. We present in Fig. 12 the variation of the polarization with c.m. scattering angle predicted by the first four SPDF solutions in Table XIII. The analogous curve for solution 5 is intermediate between those for 2 and 3. We do not show the polarization plots for the SPDF solutions Minami I and Yang I, but they are essentially the same as the corresponding curves in Fig. 7.

6. Addition of G Waves

An attempt was made to observe the effects of G waves on the SPDF analysis, again with the aid of the IBM-704 computer. When no restrictions are placed on the size of the G-wave phase shifts, we found that our former solutions become poorly defined, and additional sets of phase shifts that fit the data well appear. The SPDF Fermi-I and Fermi-II solutions are altered in character considerably when the nuclear G-wave interaction is allowed because the computer is best able to fit the data by changing some of the phase shifts in these solutions by as much as 10 to 20 deg (the M values dropping to about 10 and 16, respectively). Even if the magnitudes of the nuclear G-wave phase shifts are held to within the arbitrary limit of 0.2 deg, the uncertainties in many of the other phase shifts in the two Fermi solutions increase to one and one-half to two times their former values. With the nuclear G-wave interaction allowed, we reinvestigated all the minima obtained in the SPDF random search. The magnitudes of the nuclear G-wave phase shifts in a given fit were arbitrarily restricted to be less than one-fifth the magnitude of the larger nuclear F-wave phase shift in the same fit. Even this constraint did not prevent new solutions with low M values from arising. With our present data and the limited amount of available

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Fig. 12. Variation of the polarization with c.m. scattering angle predicted by the first four SPDF solutions in Table XIII. For reasons of clarity, the large-angle behavior of two of the curves is not shown. All curves satisfactorily fit the three negative polarization measurements.

theoretical information concerning the phase shifts related to angular-momentum states of higher order, we conclude that we cannot meaningfully include G waves in the analysis.

V. DISCUSSION OF RESULTS

A. Phase-Shift Analysis

The results of the polarization experiment have been combined with the recently obtained accurate cross-section data and a comprehensive phase-shift analysis performed. The D-wave phase shifts were found to be definitely needed in order to attain an adequate fit to the data. We investigated the influence on the analysis of the presence of small F-wave phase shifts: not only are the errors in our original Fermi-type solution increased, but additional solutions that fit the data well arise. Although the introduction of a small F-wave interaction does not greatly improve the best obtainable fit to the data, no justification can be found for completely neglecting $F_{3,5}$ and $F_{3,7}$. We attempted to extend the phase-shift inquiries to include G waves but found that the available data and theory do not allow the G-wave interaction to be significantly incorporated into the analysis.

Our investigations indicate that it is difficult to obtain a completely meaningful set of phase shifts from pion-nucleon experimental data by using the partial-wave treatment alone. Further assistance from theory may be required before one can handle with confidence all the angular-momentum states measurably affected by the interaction. The discussions to follow will principally be limited to the results of our SPDF investigation.

Let us begin the discussion of the various phase-shift solutions by discarding all those that are of the Yang, Minami, or Minami-Yang type. A principal reason for rejecting these sets of phase shifts is that they appear to disagree with the requirements of the dispersion relations for the spin-flip amplitude of the pion-nucleon scattering in the forward direction.^{15, 31, 32} The Minami-type solution is also unreasonable because of its large $D_{3,3}$ and the implausible behavior of its phase shifts at low energy.^{15, 33}

Of the phase-shift solutions listed in Table XIII, only the Fermi-I and Fermi-II sets remain to be considered (we earlier rejected set 6 because of its excessively large $F_{3,7}$). In Table XVI, we summarize the characteristics of these two SPDF Fermi-type fits. The SPD Fermi set is also included for comparison. The rms errors listed are the square roots of the diagonal elements of the respective error matrices. In comparing the closely related SPD Fermi and SPDF Fermi-I solutions, we notice that only $P_{3,3}$ is essentially unaffected by the addition of the F-wave interaction (owing to the strong dependence of this phase shift on only the total cross section). Although $F_{3,5}$ and $F_{3,7}$ in the SPDF Fermi-I solution are small and their errors overlap 0 deg, the effect of their presence is considerable.

Table XVI shows the drastic increases in the phase-shift errors that occur when F waves are added to the SPD Fermi solution and the SPDF Fermi-I set is thereby obtained. This would seem, at first glance, to indicate that much less information can be derived from this type of solution now that F waves are allowed. Actually this is not true because many of the correlation coefficients are large in the SPDF Fermi-I solution. Large correlation coefficients signify strong relationships between the phase shifts, and thus information about one phase shift will, in general, give useful information about other phase shifts. In any comparison of theory with the SPDF Fermi-I set, it will be important to use the entire error matrix (Table XIV).

In order to facilitate the phase-shift analysis, we neglected inelastic scattering. Additional uncertainties in the solutions of Table XVI exist because of this disregard of all but the elastic-scattering reaction. There is little experimental information available on inelastic processes in π^+ -p scattering at 310 Mev. However, estimates can be made of the magnitude of the total inelastic cross section at this energy by combining

Table XVI. Phase shifts for solutions of the Fermi type arising in the SPD and SPDF analyses of π^+ -p scattering data at 310 Mev. The units are degrees. The errors are standard deviations and are obtained from the error matrices presented in Tables X, XIV, and XV.

Nuclear phase shift	Solution		
	SPD (M = 15.8	SPDF Fermi I 14.1	SPDF Fermi II 18.3)
$S_{3,1}^2$	-18.5 ± 0.6	-17.2 ± 2.6	-35.5 ± 0.7
$P_{3,1}$	-4.7 ± 0.6	-2.9 ± 4.0	-16.1 ± 0.7
$P_{3,3}$	134.8 ± 0.6	135.0 ± 0.6	151.4 ± 0.8
$D_{3,3}$	1.9 ± 0.4	3.1 ± 2.6	-11.4 ± 0.5
$D_{3,5}$	-4.0 ± 0.4	-4.9 ± 2.1	13.1 ± 0.5
$F_{3,5}$	————	0.5 ± 0.6	-1.1 ± 0.3
$F_{3,7}$	————	-0.6 ± 1.4	-1.8 ± 0.3

the experimental measurements of Willis³⁴ at 500 Mev with theories such as those by Rodberg,³⁵ Franklin,³⁶ and Kazes.³⁷ The results indicate that the π^+ -p total inelastic cross section is less than 1 mb at 310 Mev.

The inclusion in our analysis of even this small amount of inelastic scattering can cause changes in the phase shifts. We have observed the alterations in the solutions given in Table XVI when a total inelastic cross section of 1 mb is allowed. Various extreme assumptions were made about the manner in which this amount of inelastic scattering might be distributed among the different angular-momentum states of the interaction. Each inelastic parameter was assumed, in turn, to have a value sufficiently less than unity so as to account for the entire 1-mb cross section (all the other inelastic parameters remaining at unity). Equation (E-4) was used in order to calculate these values. For each assumed set of inelastic parameters and for each solution considered, the computer redetermined the values of the phase shifts, yielding the minimum magnitude of M (this general procedure was discussed briefly in Section IV-A-1). We conclude from the results of this investigation that, if inelastic-scattering processes could properly be taken into account, any changes in the quoted values of the phase shifts would probably be well within the corresponding errors given in Table XVI.

B. Comparison of the SPDF Fermi-Type Solutions

Let us examine more closely the two SPDF Fermi-type solutions, both of which are excellent fits to the data. Both sets are reasonable from the point of view that the F-wave phase shifts are small compared with those related to the D wave. We are unwilling to discard the Fermi-II solution on the basis of lack of continuity with results of phase-shift analyses at other energies because we believe these other analyses may suffer the same uncertainties as our SPD results. In the remainder of this section, comparisons between the two SPDF Fermi solutions will be made in an attempt to eliminate one of these two sets of phase shifts.

Both solutions give $\text{Re}[f(0^\circ)] = -0.686 \pm 0.012$ in units of $\hbar/\mu c$ (μ denotes the pi-meson rest mass) where $\text{Re}[f(0^\circ)]$ is the real part of the forward-scattering amplitude, for π^+ -p nuclear elastic scattering, in the c.m. system. The value -0.686 was calculated by inserting the nuclear phase shifts of Table XVI into Eq. (E-2). We obtained the error by using Eq. (IV-4) and the error matrices in Tables XIV and XV. The sign of $\text{Re}[f(0^\circ)]$ is determined by the absolute sign of the set of phase shifts used, which in turn is determined by the sign of the Coulomb-nuclear interference contribution to the DCS. We neglect a small correction (apparently less than 1%) to $\text{Re}[f(0^\circ)]$ arising from the disregard of possible inelastic contributions to the total cross section when the computer normalizes the experimental elastic DCS to the experimental value of the total cross section. If inelastic scattering takes place but is neglected in the phase-shift analysis, DCS values calculated from the resulting sets of phase shifts will be too large. Because of the close relationship between $\text{Re}[f(0^\circ)]$ and the value of the DCS for nuclear scattering at $\theta_{\text{c.m.}} = 0^\circ$, the disregard of inelastic scattering causes the magnitude quoted for $\text{Re}[f(0^\circ)]$ to be slightly too great.

Our result for $\text{Re}[f(0^0)]$ agrees well with values predicted by the dispersion relations and based on other experimental data. *

Figure 13 shows the accord obtained between our value and the curve calculated by Spearman for $f^2 = 0.08$, where f^2 is the renormalized, unrationalized, pion-nucleon coupling constant.³⁸ Another recent analysis is that by Cronin, who predicts -1.35×10^{-13} cm at 310 Mev for the real part of the forward-scattering amplitude in the laboratory system (for $f^2 = 0.08$).³⁹ When transformed to the laboratory system, our result becomes $(-1.36 \pm 0.02) \times 10^{-13}$ cm, again in good agreement with the dispersion relations.

When the two SPDF Fermi-type solutions are compared with the predictions of the phase-shift formulas of Chew, Goldberger, Low, and Nambu,⁴⁰ we find that Fermi I is in better agreement. The P-wave phase shifts of Fermi I are more in accord with the effective-range formulas of Chew et al. than are the corresponding phase shifts of Fermi II. The effective-range equations predict approximately -5 deg for $P_{3,1}$ and 127 deg for $P_{3,3}$ at 310 Mev. We obtained these results by assuming $f^2 = 0.08$ and $\omega_r = 2.1$. The quantity ω_r is the value of ω at the resonance, where ω denotes the total energy in the c.m. system, exclusive of the nucleon rest energy, in units of μc^2 . The effective-range formulas are expected to be valid only at low energies. Therefore the fact that the Fermi-II set disagrees more noticeably with these equations than does the Fermi-I solution is not sufficient reason by itself for discarding the former set of phase shifts. One often compares experimentally obtained values of $P_{3,3}$ with the effective-range theory by means of the Chew-Low plot⁴¹ [i. e. $(q^3 \cot P_{3,3})/\omega$ versus ω , where q is the momentum of the pi meson in the c.m. system, in units of μc]. The values of $P_{3,3}$ in both Fermi I and Fermi II give results that fall below the straight line

* We acknowledge informative discussions with Dr. H. P. Noyes concerning the dispersion relations.

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Fig. 13. The quantity $\text{Re}[f(0^\circ)]$ is the real part of the forward-scattering amplitude, for π^+ -p nuclear elastic scattering, in the c.m. system. It is plotted as a function of incident pion laboratory kinetic energy. The curves were calculated by Spearman,³⁸ using the dispersion relations and the indicated values of f^2 . Only the higher-energy portion of the curve corresponding to $f^2 = 0.085$ is shown. The experimental value at 310 Mev is that obtained from the SPDF Fermi-I and Fermi-II solutions discussed in this report.

passing through the low-energy points on this type of plot, in accord with the results of other experiments at energies near or above 300 Mev. The D-wave phase shifts in the SPDF Fermi-I solution agree in sign and reasonably well in magnitude with the theoretical formulas of Chew et al., which predict $D_{3,3} = +0.3$ deg and $D_{3,5} = -2.5$ deg at 310 Mev; the D-wave phase shifts in Fermi II disagree in both sign and magnitude. However, these formulas do not include the effects of the pion-pion interaction and thus may not give accurate predictions.

The straight-line plot⁴² at low energies of $S_{3,1}$ as a function of q can be linearly extrapolated to 310 Mev and compared with the values of this phase shift in our two SPDF Fermi solutions. The extrapolated value obtained is near -13 deg, and therefore the comparison yields the better agreement for Fermi I. Once again, this alone is not adequate evidence against Fermi II because the linear relationship between $S_{3,1}$ and q probably does not extend to energies as high as 310 Mev.

Although both the SPDF Fermi-I and Fermi-II solutions give results that agree with the dispersion relations predicting $\text{Re}[f(0^\circ)]$, these two sets of phase shifts yield contrasting results when compared with the dispersion relations for the spin-flip forward-scattering amplitude, following the method of Davidon and Goldberger.^{31, *} Dispersion-relation theory predicts that $y = f^2 + Cx$, where f^2 is again the pion-nucleon coupling constant, C is a constant, x is a given function of the energy, and y depends in a stated way on the phase shifts and the energy. As shown in Reference 31, Fermi-type phase shifts that are based on SP analyses over a range of energies lower than 310 Mev exhibit approximately the predicted y - x linear behavior

*We wish to thank Professor J. Ashkin of Carnegie Institute of Technology for suggesting the use of the spin-flip dispersion relations as a possible means of discriminating between the two SPDF Fermi solutions.

and extrapolate to a reasonable value of f^2 . (At sufficiently low energies, we would expect the SP-type analysis to be adequate.) Strictly speaking, the function y depends on the phase shifts at all energies. However, for Fermi-type solutions and for the region of energies considered in the Davidon and Goldberger article, y depends principally on the values of the phase shifts at the energy at which it is being evaluated and on the behavior of $P_{3,3}$ at other energies, about which reasonable assumptions can be made when necessary.

Approximate calculations using the Fermi-I solution give

$y \approx +0.03 \pm 0.08$; when Fermi II is considered, $y \approx +0.33 \pm 0.02$. We have included in the errors quoted only the error arising from the term $\text{Re}(a_3)$ in Eq. (2.6) of Reference 31. The entire error matrices (Tables XIV and XV) were used when calculating these errors. Assuming that the other uncertainties in the calculation do not greatly change the general features of these results for y , we find that the Fermi-I solution is in moderately good agreement with the straight line of Reference 31 (which yields about 0.15 for y at 310 Mev) but that Fermi II disagrees. Relying on the Davidon and Goldberger analysis, then, we apparently may say that only the Fermi-I solution is admissible.

C. Concluding Remarks

Although theory appears to favor the Fermi-I set over the Fermi-II, further theoretical evidence and, in addition, experimental justification are desirable. Useful experimental information could probably be obtained by performing supplemental polarization measurements at sufficiently small angles. We note in Fig. 12 that appreciably different values of the polarization are predicted by the two Fermi solutions at c.m. scattering angles in the vicinity of 60 deg. If a practicable method could be developed for determining the polarization of protons with energies approximating 50 Mev, one could perform recoil-proton polarization measurements that might distinguish between the two SPDF Fermi solutions. The same data might also provide experimental evidence against the SPDF Minami, Yang, and Minami-Yang solutions.

In conclusion, the success of the SPD analysis was so striking that an investigation of the effects of F waves was in order. The inclusion of F waves has given a good fit to the data, but not an appreciably better fit than in the SPD analysis. The errors in the phase shifts of the Fermi-I type have become very much larger than they were before the F waves were added, but because many of the correlation coefficients are quite large there is still a great deal of information contained in the SPDF analysis. It is hoped that this work constitutes a significant step in the quantitative study of pion-nucleon scattering.

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APPENDIXES

A. Derivation of the Expression for the Recoil-Proton Polarization

We present here Fermi's derivation leading to the equation for the polarization of the recoil protons.⁵ If the proton is initially in the spin state α , the scattered wave can be written

$$\psi_{SC} = S_{\alpha\alpha} \alpha + S_{\beta\alpha} \beta. \quad (A-1)$$

If the proton is initially in the spin state β , the expression for the scattered wave is

$$\psi_{SC} = S_{\alpha\beta} \alpha + S_{\beta\beta} \beta = -S_{\beta\alpha} \alpha + S_{\alpha\alpha} \beta. \quad (A-2)$$

The equalities $S_{\alpha\alpha} = S_{\beta\beta}$ and $S_{\beta\alpha} = -S_{\alpha\beta}$, which are discussed in Section II-B-1, have been used here. Let γ and δ represent the spin eigenfunctions corresponding to the proton spin pointing in the $+y$ and $-y$ directions, respectively. Expressing these eigenfunctions in terms of the spin wave functions α and β , we write⁴³

$$\gamma = 2^{-1/2} (\alpha + i\beta), \quad \delta = 2^{-1/2} (\alpha - i\beta). \quad (A-3)$$

(Although we now discuss the y direction, the z axis is nevertheless a convenient one with which to begin, being the direction along which the incident beam moves.) Solving Eqs. (A-3) for α and β in terms of γ and δ , substituting the results into Eqs. (A-1) and (A-2), and rearranging, one obtains

<u>Initial spin state</u>	<u>Scattered wave</u>
α	$2^{-1/2} (S_{\alpha\alpha} - iS_{\beta\alpha}) \gamma + 2^{-1/2} (S_{\alpha\alpha} + iS_{\beta\alpha}) \delta$
β	$-i 2^{-1/2} (S_{\alpha\alpha} - iS_{\beta\alpha}) \gamma + i 2^{-1/2} (S_{\alpha\alpha} + iS_{\beta\alpha}) \delta.$

According to either of these expressions for the scattered wave, the probability for the spin of the scattered proton to be parallel or antiparallel to the +y direction is proportional to

$$\begin{array}{l} \text{and} \quad \left| S_{\alpha\alpha} - i S_{\beta\alpha} \right|^2 \quad (\text{parallel}) , \\ \left| S_{\alpha\alpha} + i S_{\beta\alpha} \right|^2 \quad (\text{antiparallel}) . \end{array}$$

Thus Eqs. (II-5) are proven to be valid. The expression for the recoil-proton polarization is easily obtained from Eqs. (II-5), as is shown in Section II-A-2.

B. Method of Partial Waves Applied to Pion-Proton Scattering

We will now use the method of partial waves in order to obtain the basic phase-shift equations presented in Section II-B-1. Coulomb scattering is neglected here; it will be considered in Section II-B-2 and Appendix C. For definition of quantities used in this appendix, refer to Section II-B-1 and the references to be cited here.

In the method of partial waves, the interaction of a beam of particles with a localized scattering center is represented by the scattering of a plane wave. If the interacting particles have no spin, the initial wave function can be written [Eq. (1) of Halliday⁸]

$$\psi_I = A e^{ikz} = \sum_{L=0}^{\infty} B_L(r) P_L(\cos \theta). \quad (B-1)$$

Extending the partial-wave treatment to pion-proton scattering, in which the proton has a spin of 1/2, we rewrite Eq. (B-1) as

$$\psi_I = \sum_{L=0}^{\infty} B_L(r) N_L Y_L^0(\cos \theta) \chi_{1/2}^{M_S^I}. \quad (B-2)$$

We have included the proton-spin wave function, $\chi_{1/2}^{M_S^I}$, in the initial wave function because the proton spin can also enter into the reaction. The wave function ψ_I describes a proton in a definite spin state. For convenience, the Legendre polynomials have been changed to spherical harmonics through use of the relation [Eq. (2.5) of Reference 44]

$$P_L(\cos \theta) = N_L Y_L^0(\cos \theta), \quad (B-3)$$

where

$$N_L = [4\pi/(2L+1)]^{1/2}.$$

The superscript on each spherical harmonic represents the z component (M_L) of the orbital angular momentum, which is zero for the incident pion beam.

Let us consider a pion-proton system in an orbital-angular-momentum state with quantum number L . This system contains total-angular-momentum states with quantum numbers $J = L + 1/2$ and $J = L - 1/2$. Utilizing Eq. (5.5) and Table 5.1 of Blatt and Weisskopf,⁴⁴ we can write [denoting $Y_L^0(\cos \theta)$ by Y_L^0]

$$Y_L^0 \chi_{1/2}^{M_S^I} = \mp \left(\frac{L}{2L+1} \right)^{1/2} y_{L-1/2, L, 1/2}^{M_S^I} + \left(\frac{L+1}{2L+1} \right)^{1/2} y_{L+1/2, L, 1/2}^{M_S^I} \quad (B-4)$$

where the functions $y_{J, L, 1/2}^{M_J}$ are the total-angular-momentum wave functions. Whenever a choice of algebraic sign appears in the equations of this appendix, Appendix C, or Section II-B, the upper sign refers to the scattering in which $M_S^I = +1/2$; the lower sign is for $M_S^I = -1/2$.

When the method of partial waves is applied to pion-proton nuclear scattering, one utilizes total-angular-momentum wave functions because J and M_J are conserved and hence the amplitude in each total-angular-momentum state can only change in phase (unless inelastic scattering occurs, in which case the magnitude of the amplitude can also change). In order to write the initial wave function, ψ_I , in terms of the $y_{J, L, 1/2}^{M_J}$ functions, we insert Eq. (B-4) into Eq. (B-2). Then, generalizing the development in Halliday⁸ to pion-proton scattering, we rewrite Eq. (5) of that reference as

$$\begin{aligned} \psi_{SC} = \lambda \sum_{L=0}^{\infty} (2L+1) N_L \left[\mp \left(\frac{L}{2L+1} \right)^{1/2} \left(\frac{b_L^- \exp(2i\delta_L^-) - 1}{2i} \right) y_{L-1/2, L, 1/2}^{M_S^I} \right. \\ \left. + \left(\frac{L+1}{2L+1} \right)^{1/2} \left(\frac{b_L^+ \exp(2i\delta_L^+) - 1}{2i} \right) y_{L+1/2, L, 1/2}^{M_S^I} \right] \frac{e^{ikr}}{r} \end{aligned} \quad (B-5)$$

(In pion-proton nuclear scattering, one discusses a localized "interaction" rather than the "potential" used in Halliday.) The wave function ψ_{SC} represents the elastically scattered particles. In writing Eq. (B-5), we have allowed the phase shifts, and therefore the interaction, to depend on both the L and J quantum numbers of the system. For $L = 0$, there is only one total-angular-momentum state ($J = 1/2$) and therefore only one phase shift.

Let us now express the total-angular-momentum wave functions of Eq. (B-5) in terms of orbital-angular-momentum and spin wave functions. Utilizing Eqs. (5.1) and Table 5.1 of Blatt and Weisskopf,⁴⁴ we can write

$$y_{L-1/2, L, 1/2}^{M_S^I} = \mp \left(\frac{L}{2L+1} \right)^{1/2} y_L^0 x_{1/2}^{M_S^I} \pm \left(\frac{L+1}{2L+1} \right)^{1/2} y_L^{2M_S^I} x_{1/2}^{-M_S^I} \quad (B-6)$$

and

$$y_{L+1/2, L, 1/2}^{M_S^I} = + \left(\frac{L+1}{2L+1} \right)^{1/2} y_L^0 x_{1/2}^{M_S^I} + \left(\frac{L}{2L+1} \right)^{1/2} y_L^{2M_S^I} x_{1/2}^{-M_S^I} \quad (B-7)$$

The spherical harmonics in Eqs. (B-6) and (B-7) can be written in terms of Legendre polynomials by employing Eqs. (B-3) and (II-15). The validity of Eq. (II-15) can be seen by applying the formula of Rodrigues⁴⁵ to Eq. (2.4) of Blatt and Weisskopf,⁴⁴ and by using their relationship

$$\left(y_L^{M_L} \right)^* = (-)^{M_L} y_L^{-M_L}.$$

Substituting Eqs. (B-6) and (B-7) into Eq. (B-5), and using Eqs. (B-3), one obtains Eqs. (II-9) through (II-13). These derived expressions give the wave function describing the scattered particles in pion-proton nuclear elastic scattering.

C. Inclusion of Coulomb Effects in the Scattering Amplitudes

In Section II-B-2-a, we discussed the fact that expressions in the form of Eqs. (II-12) and (II-13) can describe pion-proton scattering even when the Coulomb interaction is also present. We will use this result in order to obtain Eqs. (II-18) and (II-19).

Rewriting Eqs. (II-12) and (II-13) for the case in which both nuclear and Coulomb scattering are present, we obtain

$$g'_T(\theta) = \lambda \sum_{L=0}^{\infty} \left[(L+1) \left(\frac{b_L^+ \exp(2i\delta_L^{+'}) - 1}{2i} \right) + L \left(\frac{b_L^- \exp(2i\delta_L^{-'}) - 1}{2i} \right) \right] P_L(\cos \theta), \quad (C-1)$$

and

$$h'_T(\theta, \phi) = \lambda \sum_{L=1}^{\infty} \left[\frac{b_L^+ \exp(2i\delta_L^{+'}) - b_L^- \exp(2i\delta_L^{-'})}{2i} \right] D_L Y_L^{\pm 1}(\theta, \phi), \quad (C-2)$$

where T denotes the total scattering. The prime superscript will be dropped after a later transformation. The phase shifts and inelastic parameters now describe the total (nuclear plus Coulomb) interaction. In the next few paragraphs, we will manipulate Eqs. (C-1) and (C-2) to obtain expressions more convenient for our use.

For the case of nonrelativistic Coulomb scattering of spinless particles, Eqs. (C-1) and (C-2) reduce to (by setting $\delta_L^{+'} = \delta_L^{-'} = \eta_L$ and $b_L^{\pm} = 1$)

$$g'_C(\theta) = \lambda \sum_{L=0}^{\infty} (2L+1) \left(\frac{\exp(2i\eta_L) - 1}{2i} \right) P_L(\cos \theta), \quad (C-3)$$

and

$$h'_C(\theta, \phi) = 0,$$

where the subscript C signifies Coulomb scattering. The symbol η_L denotes the nonrelativistic Coulomb phase shift of order L, and can be written $\eta_L = \arg \Gamma(L+1+in)$, where n is defined in Section II-B-2-a.* We can also write $g'_C(\theta)$ in closed form [Eq. (20.10) of Schiff⁹]:

$$g'_C(\theta) = \frac{\lambda n}{2 \sin^2(\theta/2)} \exp\{-in \ln[\sin^2(\theta/2)] + i\pi + 2i\eta_0\}, \quad (C-4)$$

where

$$\eta_0 = \arg \Gamma(1+in).$$

It will be observed shortly that the choice of the phase of Eq. (C-4) agrees with Eq. (20.24) of Schiff.

Let us now write

$$g'_T(\theta) = g'_C(\theta) + [g'_T(\theta) - g'_C(\theta)].$$

Upon substitution of Eqs. (C-1), (C-3), and (C-4) into this last equation, one obtains

$$g'_T(\theta) = \frac{\lambda n}{2 \sin^2(\theta/2)} \exp\{-in \ln[\sin^2(\theta/2)] + i\pi + 2i\eta_0\} + \lambda \sum_{L=0}^{\infty} \left[(L+1) \left(\frac{b_L^+ \exp(2i\delta_L^+) - \exp(2i\eta_L)}{2i} \right) + L \left(\frac{b_L^- \exp(2i\delta_L^-) - \exp(2i\eta_L)}{2i} \right) \right] \times P_L(\cos \theta). \quad (C-5)$$

Equation (C-5) is seen to reduce to Eq. (20.24) of Schiff for the case of spin-zero particles ($\delta_L^+ = \delta_L^- = \eta_L + \delta_L$, where δ_L is defined in the reference) and no inelastic scattering.

* This expression for η_L can be found just below Eq. (20.19) of Schiff.

It is convenient to multiply Eq. (C-5) by the phase factor $\exp(-2i\eta_0)$. In order not to change any physically observable quantity, we must also multiply the spin-flip amplitude given in Eq. (C-2) by the same phase. The resulting expressions are Eqs. (II-18) and (II-19), where we have defined

$$\delta_L^\pm = \delta_L^{\pm'} - \eta_0 ,$$

$$\bar{\Phi}_L = \eta_L - \eta_0 = 0 \quad \text{for } L = 0$$

$$= \sum_{x=1}^L \tan^{-1}(n/x) \quad \text{for } L \geq 1 ,$$

and

$$g_T(\theta) = g_T^*(\theta) \exp(-2i\eta_0) ,$$

$$h_T(\theta, \phi) = h_T^*(\theta, \phi) \exp(-2i\eta_0) .$$

In obtaining Eq. (II-18), we have used the fact that $e^{i\pi} = -1$. The formula for $\bar{\Phi}_L$ can be derived from the definitions of η_L and η_0 given earlier. In Section II-B-2, we refer to $\bar{\Phi}_L$ rather than η_L as the nonrelativistic Coulomb phase shift of order L . We also refer to $\delta_L^\pm$ as the total phase shifts of order L .

D. Form of Phase-Shift Equations Suitable for Computer Calculations

In Section II-B, general phase-shift equations were developed. We will now discuss the application of these equations to the analysis of pion-proton scattering data by an electronic computer.

It is advantageous to rewrite Eqs. (II-30) and (II-31) in the form

$$g_T(\theta) = g_1(\theta) + g_2(\theta)$$

and

$$h_T(\theta, \phi) = h_1(\theta, \phi) + h_2(\theta, \phi),$$

where

$$g_1(\theta) = \lambda \sum_{L=0}^{L_{MAX}} [(L+1) a_L^+ + L a_L^-] P_L(\cos \theta), \quad (D-1)$$

$$g_2(\theta) = - \frac{\lambda n}{2 \sin^2(\theta/2)} \exp\{-in \ln[\sin^2(\theta/2)]\}$$

$$h_1(\theta, \phi) = \lambda \sum_{L=0}^{L_{MAX}} (2L+1) \left(\frac{\exp(2i\phi_L) - 1}{2i} \right) P_L(\cos \theta) \quad (D-2)$$

$$h_1(\theta, \phi) = \lambda \sum_{L=1}^{L_{MAX}} (a_L^+ - a_L^-) D_L Y_L^{\pm 1}(\theta, \phi), \quad (D-3)$$

$$h_2(\theta, \phi) = \mp \frac{\lambda n B \sin \theta}{2 \sin^2(\theta/2)} e^{\pm i\phi} - \lambda n B \sum_{L=1}^{L_{MAX}} \left(\frac{2L+1}{L(L+1)} \right) D_L Y_L^{\pm 1}(\theta, \phi), \quad (D-4)$$

$$a_L^{\pm} = \frac{b_L^{\pm} \exp(2i\delta_L^{\pm}) - 1}{2i}$$

Let us consider the scattering process at $\phi = 0$ deg so that the $e^{\pm i\phi}$ factors drop out. We can separate Eqs. (D-1) through (D-4) into real and imaginary parts. For a given set of phase shifts, there are then seven quantities to calculate at each desired c.m. angle [four real, but only three imaginary because $h_2(\theta, \phi)$ is real for $\phi = 0$ deg]. These quantities can be combined according to the rules dictated by Eqs. (II-16) and (II-17) in order to give values of the polarization and DCS.

In the search for a set of phase shifts that fits the data (see Section IV-A for details), the total phase shifts, $\delta_L^{\pm}$, may be varied many times. The quantity $h_2(\theta, \phi)$ and the real and imaginary parts of $g_2(\theta)$ do not contain the total phase shifts and therefore need be computed only once. However, the amplitudes $g_1(\theta)$ and $h_1(\theta, \phi)$ must be recalculated each time a phase shift is varied. An expedient way to perform these iterated calculations involves the use of the equalities

$$a_L^{\text{NEW}} = \frac{b_L \exp[2i(\delta_L^{\text{OLD}} + \Delta)] - 1}{2i} = \frac{b_L \exp(2i\Delta) \exp(2i\delta_L^{\text{OLD}}) - 1}{2i},$$

where Δ is the change in the phase shift δ_L , and δ_L^{OLD} is the value of the phase shift before the change. When separated into real and imaginary parts, these last equations yield

$$\bar{a}_{L, \text{RE}}^{\text{NEW}} = \bar{a}_{L, \text{RE}}^{\text{OLD}} \cos(2\Delta) - \bar{a}_{L, \text{IM}}^{\text{OLD}} \sin(2\Delta), \quad (\text{D-5})$$

and

$$\bar{a}_{L, \text{IM}}^{\text{NEW}} = \bar{a}_{L, \text{IM}}^{\text{OLD}} \cos(2\Delta) + \bar{a}_{L, \text{RE}}^{\text{OLD}} \sin(2\Delta), \quad (\text{D-6})$$

where (for both OLD and NEW quantities)

$$\bar{a}_{L, \text{RE}} = a_{L, \text{RE}} = \frac{b_L \sin(2\delta_L)}{2},$$

and

$$\bar{a}_{L, \text{IM}} = a_{L, \text{IM}} - 1/2 = -\frac{b_L \cos(2\delta_L)}{2}.$$

Through the use of Eqs. (D-5) and (D-6), a phase shift can be changed and values of the polarization and DCS recalculated without the computation of any new trigonometric functions. * Of course, $\sin(2\Delta)$ and $\cos(2\Delta)$ must be initially calculated.

In a data analysis by an electronic computer, one wants to avoid the repeated calculation of trigonometric functions owing to the relatively long computing time involved. Thus the approach suggested here is an advantageous one.

We may wish to calculate a value of the total cross section and compare it with an experimental value that has been measured between the c.m. cutoff angles $\theta_{c.m.}^{(1)}$ and $\theta_{c.m.}^{(2)}$. The contribution to the total cross section (between $\theta_{c.m.}^{(1)}$ and $\theta_{c.m.}^{(2)}$) due to $g_1(\theta)$ and $h_1(\theta, \phi)$ alone can be simply expressed in terms of the cutoff angles and certain coefficients calculable from the phase shifts. The remaining elastic-scattering contribution, arising from $g_2(\theta)$, $h_2(\theta, \phi)$, and their cross terms with $g_1(\theta)$ and $h_1(\theta, \phi)$, is not expressible in terms of a simple equation. However, this contribution is appreciable only at small angles and a straightforward numerical integration can be performed in order to obtain it. If the inelastic parameters are allowed to differ from unity, then there is an additional term [involving Eq. (E-4)] in the expression for the total cross section. It must also be taken into account.

* We wish to thank Mr. Kent K. Curtis of the Mathematical and Computing Section of the Theoretical Group for suggesting this procedure. Appreciation is also due Mr. Edwin M. Towster of the same department for other useful programming ideas.

E. Several Useful Phase-Shift Equations

We present here several useful formulas, which can be obtained from the more-general expressions of Section II-B. These equations are directly applicable to π^+ -p scattering, in which there is only one isotopic-spin state to consider. If only the nuclear interaction is present, we can use Eqs. (II-12), (II-13), and (II-16), and write:

$$I_T(\text{nucl.}, \text{el.}) = \pi \lambda^2 \sum_{L=0}^{L_{\text{MAX}}} \left\{ (L+1) [1 + (b_L^+)^2 - 2 b_L^+ \cos(2\delta_L^+)] \right. \\ \left. + L [1 + (b_L^-)^2 - 2 b_L^- \cos(2\delta_L^-)] \right\}, \quad (\text{E-1})$$

where $I_T(\text{nucl.}, \text{el.})$ is the total nuclear elastic-scattering cross section;

$$\text{Re}[f(0^\circ)] = (\lambda/2) \sum_{L=0}^{L_{\text{MAX}}} [(L+1) b_L^+ \sin(2\delta_L^+) + L b_L^- \sin(2\delta_L^-)], \quad (\text{E-2})$$

where $\text{Re}[f(0^\circ)]$ is the real part of the forward-scattering amplitude for pion-nucleon nuclear elastic scattering;

$$\text{Im}[f(0^\circ)] = (\lambda/2) \sum_{L=0}^{L_{\text{MAX}}} \left\{ (L+1) [1 - b_L^+ \cos(2\delta_L^+)] \right. \\ \left. + L [1 - b_L^- \cos(2\delta_L^-)] \right\}, \quad (\text{E-3})$$

where $\text{Im}[f(0^\circ)]$ is the imaginary part of the forward-scattering amplitude for pion-nucleon nuclear elastic scattering. The total nuclear cross section (elastic plus inelastic) can be obtained from $\text{Im}[f(0^\circ)]$ by using the Optical Theorem, which states

$$I_T(\text{nucl.}, \text{el. plus inel.}) = 4 \pi \lambda \text{Im}[f(0^\circ)] .$$

Finally, we can write an equation for the total inelastic-scattering cross section:

$$I_T(\text{nucl.}, \text{inel.}) = \pi \lambda^2 \sum_{L=0}^{L_{\text{MAX}}} \left\{ (L+1) [1 - (b_L^+)^2] + L [1 - (b_L^-)^2] \right\} . \quad (\text{E-4})$$

In the series of expressions just presented, the phase shifts and inelastic parameters are those that describe the nuclear interaction alone. Equation (E-4) is also valid when Coulomb effects are present if the inelastic parameters are redefined as those for the total interaction. For the case of no inelastic scattering ($b_L^\pm = 1$), Eq. (E-1) reduces to the expression

$$I_T(\text{nucl.}, \text{el.}) = 4 \pi \lambda^2 \sum_{L=0}^{L_{\text{MAX}}} [(L+1) \sin^2(\delta_L^+) + L \sin^2(\delta_L^-)] . \quad (\text{E-5})$$

The equations just given for the various total cross sections cover the entire range of angles; that is, no cutoff angles are used.

F. Calculation of the G Matrix

In the process of obtaining the error matrix, G^{-1} , the elements of G must be calculated. The method by which our IBM-704 program accomplishes this computation will now be discussed.

Equation (IV-2) can be rewritten as

$$\Delta M \approx \sum_{i=1}^N \sum_{j=1}^N G_{ij} \Delta \delta_i \Delta \delta_j, \quad (F-1)$$

where $\Delta \delta_i$ and $\Delta \delta_j$ can have either positive or negative values. If only the one phase shift δ_k is changed from its value at the minimum, then Eq. (F-1) becomes

$$\Delta M \approx G_{kk} (\Delta \delta_k)^2. \quad (F-2)$$

If two phase shifts δ_k and δ_l are varied, Eq. (F-1) then gives

$$\Delta M \approx G_{kk} (\Delta \delta_k)^2 + 2G_{kl} \Delta \delta_k \Delta \delta_l + G_{ll} (\Delta \delta_l)^2. \quad (F-3)$$

The equality $G_{kl} = G_{lk}$ has been utilized.

We calculate the elements of G by varying the phase shifts individually and in pairs, and by making both positive and negative variations. Equations (F-2) and (F-3) show how the resulting changes in M are related to the phase-shift changes and to the elements of G . The magnitude of the alteration in any specific phase shift of a given set is always the same in our program, although different phase shifts may be allowed to vary by different amounts. Both positive and negative phase-shift changes are made because M will not always vary symmetrically about the minimum point. We therefore are able to find an approximate value for each G_{kk} and G_{kl} that is an average over the region near the minimum.

In order to calculate the average value of G_{kk} , let a phase shift δ_k be changed in the positive and negative directions (from its value at the minimum) by the amount $\Delta\delta_k$ and let the resultant changes in M be designated ΔM_k^+ and ΔM_k^- . Then, employing Eq. (F-2), we can write

$$(G_{kk})_{av} \approx \frac{\Delta M_k^+ + \Delta M_k^-}{2(\Delta\delta_k)^2} \quad (F-4)$$

The advantage of this form for $(G_{kk})_{av}$ is that it is independent, to first order, of a small error in the location of the minimum. When two phase shifts are varied, the changes in M corresponding to the four possible directions of variation are denoted ΔM_{kl}^{++} , ΔM_{kl}^{--} , ΔM_{kl}^{+-} , and ΔM_{kl}^{-+} . For example, ΔM_{kl}^{+-} signifies that $\Delta\delta_k$ is greater than 0 and $\Delta\delta_l$ is less than 0. We now rewrite Eq. (F-3) for each of the four sets of changes:

$$\Delta M_{kl}^{++} \approx \Delta M_{kl}^{--} \approx G_{kk} (\Delta\delta_k)^2 + 2 G_{kl} |\Delta\delta_k \Delta\delta_l| + G_{ll} (\Delta\delta_l)^2$$

and

$$\Delta M_{kl}^{+-} \approx \Delta M_{kl}^{-+} \approx G_{kk} (\Delta\delta_k)^2 - 2 G_{kl} |\Delta\delta_k \Delta\delta_l| + G_{ll} (\Delta\delta_l)^2$$

From these equations, one obtains

$$(G_{kl})_{av} \approx \frac{\Delta M_{kl}^{++} + \Delta M_{kl}^{--} - \Delta M_{kl}^{+-} - \Delta M_{kl}^{-+}}{8 |\Delta\delta_k \Delta\delta_l|} \quad (F-5)$$

Again, if the point considered to be the minimum is displaced by a small amount from the true minimum, the value obtained for $(G_{kl})_{av}$ is not affected significantly.

Our program calculates the elements of G by utilizing Eqs. (F-4) and (F-5). The increments $\Delta\delta_1, \dots, \Delta\delta_N$ are given the computer at the start of the calculation. The phase shifts are then altered in the manner already discussed, and the resulting changes in M are calculated. After all necessary variations have been made, the elements of G can be computed. A subroutine in the program then inverts G to obtain the error matrix. The method employed here to calculate the elements of G is slightly different from that discussed by Anderson et al.²⁷

An attempt is made to choose the quantities $\Delta\delta_k$ so that a change in any one phase shift alone results in a variation in M of about unity. In Appendix G, it is shown that the $\Delta M = 1$ region on the M hypersurface is associated with the rms error in each phase shift. Thus an investigation centered around this region will probably give the most accurate estimation of the errors.

G. Derivations Pertaining to Auxiliary Error Method and Correlation Coefficients

With reference to Section IV-B-2 and Fig. 6, we will prove here that the points along curve C are related by the formula $\Delta M_C \propto x_C^2$. It will be shown that the value of the x coordinate at point P in Fig. 6 is the rms error in the phase shift δ_1 . A geometrical interpretation of the correlation coefficients will also be given.

According to Eq. (IV-5), we can write

$$G_{xx} x^2 + 2 G_{xy} xy + G_{yy} y^2 \approx M - M_0. \quad (G-1)$$

Differentiation with respect to y yields

$$2 G_{xy} x + 2 G_{yy} y \approx \frac{\partial M}{\partial y}.$$

At a point on curve C, we have $\partial M / \partial y = 0$, and therefore this last equation gives $y_C \approx - (G_{xy} / G_{yy}) x_C$. Substituting this result into Eq. (G-1), we obtain

$$x_C^2 \approx \frac{G_{yy} \Delta M_C}{G_{xx} G_{yy} - G_{xy}^2}. \quad (G-2)$$

Equation (G-2) relates the values of x and ΔM at points along curve C and is of the form $\Delta M_C \propto x_C^2$.

For $\Delta M_C = 1$ (point P), Eq. (G-2) can be rewritten

$$x_P^2 \approx \frac{G_{yy}}{G_{xx} G_{yy} - G_{xy}^2}.$$

The expression on the right of this last equation is also the expression obtained for the first diagonal element of the error matrix, G^{-1} , when M depends upon only two phase shifts. According to statistical theory, the square root of this first diagonal element is the rms error in the first phase shift. Therefore the value of x at point P in Fig. 6 is the rms error in δ_1 . The above derivation can be generalized in a straightforward manner to the case of more than two phase shifts.

We now extend our discussion to yield a geometrical interpretation of the correlation coefficients. In our example employing only two phase shifts, the off-diagonal elements of the error matrix can be written

$$(G^{-1})_{xy} = (G^{-1})_{yx} = \frac{-G_{xy}}{G_{xx}G_{yy} - G_{xy}^2}.$$

Using this result and equations in the present appendix, we can show that $(G^{-1})_{xy} = x_p y_p$, where x_p and y_p are the coordinates of point P. Comparing this expression with Eq. (IV-3) and remembering that $x_p = (\Delta\delta_1)_{rms}$, we obtain

$$C_{xy} = y_p / (\Delta\delta_2)_{rms}. \quad (G-3)$$

Thus the correlation coefficient C_{xy} is the ratio of the y values of the points at which lines perpendicular to the x and y axes, respectively, are tangent to the $\Delta M = 1$ ellipse. Similarly, C_{xy} is also the inverse ratio of the x values of these points. In order to obtain the correct sign for C_{xy} when using Eq. (G-3), x_p must be positive. This geometrical interpretation can be generalized to the case of more than two phase shifts.

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