ANL-6224
Physics and Mathematics
(TID-4500, 15th Ed.)
AEC Research and
Development Report

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9700 South Cass Avenue
Argonne, Illinois

THERMAL STRESSES IN CONCENTRICALLY
HEATED HOLLOW CYLINDERS

Supplemental Data on Coefficient
of Thermal Expansion, Modulus
of Elasticity and Poisson's Ratio
as Functions of Temperature

by

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October 1960

Operated by The University of Chicago
under
Contract W-31-109-eng-38

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ABSTRACT

The basic theoretical equations of thermal stress for a concentrically heated hollow cylinder having the coefficient of thermal expansion as a function of temperature have been developed. These expressions simulate those encountered in advanced courses on strengths of materials. From these basic unit-stress formulas, the axial, radial, and tangential unit-stress equations have been derived for cases in which the coefficient of thermal expansion is a linear function of the temperature and the transferred heat through the cylinder is constant. A table has been provided for the rapid computation of the cylindrical unit stresses for which the coefficient of thermal expansion is a linear function of the temperature or independent of the temperature.

Tables and graphs have been prepared for correlating the coefficient of thermal expansion, modulus of elasticity, Poisson's ratio, and product of coefficient of thermal expansion and modulus of elasticity at elevated temperatures by fitting curves to existing data. The stainless steels considered were 18% Cr-8% Ni - Type 304; 18% Cr-8% Ni, Ti - Type 321; 18% Cr-8% Ni, Cb - Type 347; and 25% Cr-20% Ni - Type 310.

Computations for a hypothetical example were used to illustrate the difference between using the mean coefficient of thermal expansion and using the coefficient of thermal expansion as a function of temperature.

NOMENCLATURE

<u>English Letters</u>	<u>Description</u>	<u>British Units</u>	<u>Metric Units</u>
A	Acceleration	in./sec ²	cm/sec ²
a	Inner radius of cylinder	in.	cm
b	Outer radius of cylinder	in.	cm
C ₁	Constant of integration		
C ₂	Constant of integration	in. ²	cm ²
E	Modulus of elasticity as a function of temperature	lb _f /in. ²	gm _f /cm ²
E _m	Mean modulus of elasticity based on mean temperature of body cross section	lb _f /in. ²	gm _f /cm ²
F	Force	lb _f	gm _f
g _c	Constant of proportionality	or 386.088 in. - lb _m /lb _f -sec ² 980.665 cm - gm/gm _f -sec ²	
k	Thermal conductivity of material	Btu/(sec)(F)	cal/(sec)(C)
L	Length of component at indicated temperature	in.	cm
l ₀	Initial length of component at initial temperature	in.	cm
M	Mass	lb _m	gm
q	Constant heat rate	Btu/sec	cal/sec
R _i	Residual of least squares method		
r	Radius of cylinder in general	in.	cm
t	Temperature in general, temperature at cylindrical radius	F	C
t _a	Temperature on the inner surface of the cylinder	F	C
t _b	Temperature on the outer surface of the cylinder	F	C
t _i	Indicated temperature	F	C

NOMENCLATURE

<u>English Letters</u>	<u>Description</u>	<u>British Units</u>	<u>Metric Units</u>
t_m	Mean temperature of body cross section	F	C
t_0	Initial temperature	F	C
u	Displacement in radial direction of cylinder	in.	cm
V	Unit volume expansion	in./in.	cm/cm
w	Displacement in axial direction of cylinder	in.	cm
z	Distance in axial direction of cylinder	in.	cm
 <u>Greek Letters</u>			
α	Coefficient of thermal expansion as a function of temperature	in./in.(F)	cm/(cm)(C)
α_c	Slope of coefficient of thermal expansion determined by straight line curve fitting	in./in.(F ²)	cm/(cm)(C ²)
α_m	Mean coefficient of thermal expansion based on mean temperature of body cross section	in./in.(F)	cm/(cm)(C)
α_0	Intercept coefficient of thermal expansion determined by straight line curve fitting	in./in.(F)	cm/(cm)(C)
α_u	Coefficient of thermal expansion based on uniform temperature of body cross section not producing any unit-stress	in./in.(F)	cm/(cm)(C)
Δl_0	Change of length from initial temperature to indicated temperature	in.	cm

NOMENCLATURE

<u>Greek Letters</u>	<u>Description</u>	<u>British Units</u>	<u>Metric Units</u>
Δt	Temperature difference across body section, temperature at inner surface of cylinder less temperature at outer surface of the cylinder	F	C
ϵ_a	Unit strain in the axial direction of the cylinder	in./in.	cm/cm
ϵ_r	Unit strain in the radial direction of the cylinder	in./in.	cm/cm
ϵ_t	Unit strain in the tangential direction of the cylinder	in./in.	cm/cm
θ	Angle	deg	deg
ν	Poisson's ratio as a function of temperature		
ν_m	Mean Poisson's ratio based on mean temperature of body cross section		
σ_a	Unit stress in the axial direction of the cylinder	lb _f /in. ²	gm _f /cm ²
σ_r	Unit stress in the radial direction of the cylinder	lb _f /in. ²	gm _f /cm ²
σ_t	Unit stress in the tangential direction of the cylinder	lb _f /in. ²	gm _f /cm ²
τ	Temperature difference in general; temperature at cylindrical radius less temperature on outer surface of the cylinder	F	C

THERMAL STRESSES IN CONCENTRICALLY HEATED HOLLOW CYLINDERS

Supplemental Data on Coefficient of Thermal Expansion,
Modulus of Elasticity and Poisson's Ratio as Functions
of Temperature

J. C. Heap

I. INTRODUCTION

A temperature gradient across the wall of a hollow circular cylinder concentrically heated causes a nonuniform expansion which sets up unit stresses known as thermal or temperature stresses. If the thermal stresses exceed the ultimate strength of the structural materials utilized, crushing and/or tearing asunder is probable, thus destroying the integrity of the structure.

Because of the anticipated extensive temperature ranges to be traversed during the Transient Reactor Test Facility experiments on the meltdown of enriched Experimental Breeder Reactor-II-type fuel elements in stagnant sodium, the thermal stresses developed in the long hollow cylindrically shaped autoclave must be investigated. For thermal stress problems normally encountered, the coefficient of thermal expansion, the modulus of elasticity and Poisson's ratio are assumed to be constant over the temperature gradient of the body under consideration. The large temperature differences conceived to be across the wall of the autoclave immediately raised the question whether the coefficient of thermal expansion, modulus of elasticity and Poisson's ratio can be considered constant throughout the body section at any instant during a reactor transient. If these terms are considered constant, do they still provide a realistic indication of the actual stresses?

In order to calculate thermal stresses in a body at an elevated temperature, it is imperative that the coefficient of thermal expansion, modulus of elasticity and Poisson's ratio be known. Existing data do not really permit correlating these quantities at the same elevated temperature. Because of the small amount of experimental data available and since the data are not tabulated at the same temperature for the preceding terms, this presentation attempts to fit curves to known data for certain stainless steels to overcome this difficulty.

In order to at least partially answer the aforementioned interrogation, the coefficient of thermal expansion is considered a function of temperature, whereas the modulus of elasticity and Poisson's ratio are regarded as constants based on the mean or average temperature of the body section being analyzed. The basis for assuming the modulus of elasticity and Poisson's ratio constant is twofold: (1) existing knowledge does not disclose a method

for integrating the differential equation in closed form where the temperature distribution for a steady state is a logarithmic function of the cylindrical radii, and (2) using only the coefficient of thermal expansion as a temperature function may provide a partial insight into the extent of constituents depending upon temperature.

II. SYSTEMS OF UNITS

In this presentation, the systems of units will be based upon a unit of force - the pound force lb_f or the gram force gm_f - since these bases are normally preferred by engineers and scientific personnel. The pound force or gram force is defined as the gravitational force on a mass of 1 pound or 1 gram at a selected standard location; thus, the weight of a body will have the same numerical value as its mass at the selected standard location. The selected standard location for establishing this standard will be on the earth's surface where the acceleration of gravity equals $386.088 \text{ in./sec}^2$ or 980.665 cm/sec^2 .

Expressing Newton's second law of motion as

$$F = MA/g_c \quad (\text{II.1})$$

where

F = force in pounds force (lb_f) or in grams force (gm_f)

M = mass in pounds mass (lb_m) or in grams mass (gm)

A = acceleration (in./sec^2) or (cm/sec^2)

g_c = constant of proportionality,

there is obtained

$$[1 \text{ lb}_f] = [1 \text{ lb}_m] [386.088 \text{ in./sec}^2]/g_c$$

or

$$[1 \text{ gm}_f] = [1 \text{ gm}] [980.665 \text{ cm/sec}^2] \quad (\text{II.2})$$

and

$$g_c = 386.088 (\text{in.})(\text{lb}_m)/(\text{lb}_f)(\text{sec}^2)$$

or

$$980.665 (\text{cm})(\text{gm})/(\text{gm}_f)(\text{sec}^2) \quad (\text{II.3})$$

To maintain dimensional consistency in all equations, the unit of time will be the second and the unit of length will be the inch or the centimeter, depending upon system of units selected.

III. SUPPOSITIONS

1. The temperature distribution is taken to be symmetrical with respect to the cylindrical axis and constant along this axis.
2. The deformation of the cylinder is symmetrical about the cylindrical axis.
3. The imaginary annulus formed from the cylinder by two cross sections perpendicular to the cylindrical axis and a unit distance apart remain plane during the thermal deformation.

Note: This assumption is valid for an annulus taken a sufficient distance from the ends where the cylinder is long compared to the diameter; hence, the unit axial elongations are constant.

4. The body under consideration is perfectly elastic, isotropic (modulus of elasticity and Poisson's ratio are the same in all directions), and homogeneous.
5. The coefficient of thermal conductivity is a constant.

IV. DEVELOPMENT OF BASIC DISPLACEMENT - STRESS FORMULAS

Referring to Fig. 1, the equation of equilibrium for an element of a long hollow cylinder by taking the summation of the forces in the radial direction of the bisector of the element angle, is

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_t}{r} = 0 \quad , \quad (IV.1)$$

where the mass of the element and terms of higher order are neglected.

The unit strains in the axial, radial and tangential directions are, as may be seen from Fig. 2,

$$\left. \begin{aligned} \epsilon_a &= \frac{dw}{dz} = \frac{1}{E_m} \left[\sigma_a - \nu_m (\sigma_r + \sigma_t) \right] + \int_{t_b}^t \alpha dt = \text{const.} \\ \epsilon_r &= \frac{du}{dr} = \frac{1}{E_m} \left[\sigma_r - \nu_m (\sigma_a + \sigma_t) \right] + \int_{t_b}^t \alpha dt \\ \epsilon_t &= \frac{u}{r} = \frac{1}{E_m} \left[\sigma_t - \nu_m (\sigma_a + \sigma_r) \right] + \int_{t_b}^t \alpha dt \end{aligned} \right\} \quad (IV.2)$$

FIG. 1
UNIT-STRESSES OF ELEMENT FOR
LONG HOLLOW CYLINDER

FIG. 2
DISPLACEMENTS OF ELEMENT FOR
LONG HOLLOW CYLINDER

The bracketed terms in the right side of Eq. IV.2 contain the general expressions of Hooke's law relating unit strains and unit stresses in the case of isotropic materials, and the integral terms are the components of unit strain produced by temperature difference. Denoting the unit volume expansion by the symbol V , there is obtained

$$V = \epsilon_a + \epsilon_r + \epsilon_t = \frac{1 - 2\nu_m}{E_m} (\sigma_a + \sigma_r + \sigma_t) + 3 \int_{t_b}^t \alpha dt \quad (IV.3)$$

Using Eqs. IV.2 and IV.3 and solving for the axial, radial and tangential stresses, there is found

$$\left. \begin{aligned} \sigma_a &= \frac{E_m}{1 + \nu_m} \left(\frac{dw}{dz} + \frac{\nu_m}{1 - 2\nu_m} V \right) - \frac{E_m}{1 - 2\nu_m} \int_{t_b}^t \alpha dt \\ \sigma_r &= \frac{E_m}{1 + \nu_m} \left(\frac{du}{dr} + \frac{\nu_m}{1 - 2\nu_m} V \right) - \frac{E_m}{1 - 2\nu_m} \int_{t_b}^t \alpha dt \\ \sigma_t &= \frac{E_m}{1 + \nu_m} \left(\frac{u}{r} + \frac{\nu_m}{1 - 2\nu_m} V \right) - \frac{E_m}{1 - 2\nu_m} \int_{t_b}^t \alpha dt \end{aligned} \right\} \quad (IV.4)$$

Taking the derivative of the second expression of Eq. IV.4 with respect to r , substituting this derivative and the second and third expressions of Eq. IV.4 into Eq. IV.1, there is obtained

$$\frac{d^2u}{dr^2} + \frac{1}{r} \frac{du}{dr} - \frac{u}{r^2} \equiv \frac{d}{dr} \left[\frac{1}{r} \frac{d}{dr} (ru) \right] = \frac{1 + \nu_m}{1 - \nu_m} \frac{d}{dr} \left[\int_{t_b}^t \alpha dt \right] \quad (IV.5)$$

Integration of Eq. IV.5 with respect to r and transposition yields

$$\frac{d}{dr} (ru) \equiv r \frac{du}{dr} + u = \frac{1 + \nu_m}{1 - \nu_m} \left[\int_{t_b}^t \alpha dt \right] r + 2 C_1 r \quad (IV.6)$$

Integration of Eq. IV.6 and transposition gives the displacement:

$$u = \frac{1}{r} \frac{1 + \nu_m}{1 - \nu_m} \int_a^r \left[\int_{t_b}^t \alpha dt \right] r dr + C_1 r + C_2 \frac{1}{r} \quad (IV.7)$$

where the constants of integration, C_1 and C_2 , are determined from the inner and outer boundary conditions of the cylinder.

Upon rearranging Eq. IV.6 in terms of du/dr , then substituting this rearrangement together with appropriate terms of Eq. IV.2, and Eq. IV.7 into the second of Eqs. IV.4 there is obtained the general solution for the radial unit stress, viz.,

$$\sigma_r = \frac{E_m}{1 + \nu_m} \left\{ - \frac{1 + \nu_m}{1 - \nu_m} \frac{1}{r^2} \int_a^r \left[\int_{t_b}^t \alpha dt \right] r dr + \frac{C_1}{1 - 2\nu_m} - \frac{C_2}{r^2} + \frac{\nu_m}{1 - 2\nu_m} \epsilon_a \right\} \quad (IV.8)$$

Since only thermal stresses are being considered, the inner and outer curved surfaces of the cylinder are void of external forces; thus,

$$(\sigma_r)_{r=a} = (\sigma_r)_{r=b} = 0 \quad . \quad (IV.9)$$

From the boundary conditions of Eq. IV.9, the constants of integration become

$$C_1 = \frac{(1 + \nu_m)(1 - 2\nu_m)}{1 - \nu} \frac{1}{b^2 - a^2} \int_a^b \left[\int_{t_b}^t \alpha dt \right] r dr - \nu_m \epsilon_a \quad (IV.10)$$

$$C_2 = \frac{1 + \nu_m}{1 - \nu_m} \frac{a^2}{b^2 - a^2} \int_a^b \left[\int_{t_b}^t \alpha dt \right] r dr \quad .$$

Substitution of the constants of integration determined with boundary conditions of no external forces applied to the synclastic surfaces gives

$$\sigma_r = \frac{E_m}{1 - \nu_m} \left\{ -\frac{1}{r^2} \int_a^r \left[\int_{t_b}^t \alpha dt \right] r dr + \frac{r^2 - a^2}{r^2 (b^2 - a^2)} \int_a^b \left[\int_{t_b}^t \alpha dt \right] r dr \right\} \quad (IV.11)$$

Differentiation of Eq. IV.11 and substitution of this differential and Eq. IV.11 into the equilibrium equation and transposition yields for the tangential unit stress

$$\sigma_t = \frac{E_m}{1 - \nu_m} \left\{ \frac{1}{r^2} \int_a^r \left[\int_{t_b}^t \alpha dt \right] r dr + \frac{r^2 + a^2}{r^2 (b^2 - a^2)} \int_a^b \left[\int_{t_b}^t \alpha dt \right] r dr - \int_{t_b}^t \alpha dt \right\} \quad (IV.12)$$

Rearranging the first expression of Eq. IV.2 and substituting in Eqs IV.11 and IV.12, there is ascertained that

$$\sigma_a = E_m \epsilon_a + \frac{E_m}{1 - \nu_m} \left\{ \frac{2\nu_m}{b^2 - a^2} \int_a^b \left[\int_{t_b}^t \alpha dt \right] r dr - \int_{t_b}^t \alpha dt \right\} \quad (IV.13)$$

Assuming free expansion of the cylinder, the magnitude of the axial unit strain is determined by summing up the normal forces over the cross section of the cylinder perpendicular to the cylindrical axis and equating to zero, i.e.,

$$2\pi \int_a^b \sigma_a r dr = 0 = 2\pi E_m \left\{ \epsilon_a \int_a^b r dr + \frac{2\nu_m}{1 - \nu_m} \frac{1}{b^2 - a^2} \int_a^b \left[\int_{t_b}^t \alpha dt \right] r dr \int_a^b r dr - \frac{1}{1 - \nu_m} \int_a^b \left[\int_{t_b}^t \alpha dt \right] r dr \right\} \quad (IV.14)$$

Solving for the axial unit strain,

$$\epsilon_a = \frac{2}{b^2 - a^2} \int_a^b \left[\int_{t_b}^t \alpha dt \right] r dr \quad . \quad (\text{IV.15})$$

The axial unit stress becomes, on substituting Eq. IV.15 into Eq. IV.13,

$$\sigma_a = \frac{E_m}{1 - \nu_m} \left\{ \frac{2}{b^2 - a^2} \int_a^b \left[\int_{t_b}^t \alpha dt \right] r dr - \int_{t_b}^t \alpha dt \right\} \quad . \quad (\text{IV.16})$$

From this development it is noted that the axial unit stress equals the summation of the radial unit stress and the tangential unit stress.

V. COEFFICIENT OF THERMAL EXPANSION, MODULUS OF ELASTICITY, AND POISSON'S RATIO VERSUS TEMPERATURE

From the data and graphs presented on the linear thermal expansion and the modulus of elasticity at elevated temperatures in Ref. 11 of the Bibliography, it is apparent that straight-line curve fitting provides an excellent fit to several of the stainless steels. Straight lines have been fitted to the existing data of four stainless steels by the least squares method over discrete ranges where the coefficient of thermal expansion, modulus of elasticity and Poisson's ratio are the ordinates and the temperature is the abscissa.

The stainless steels considered are 18% Cr-8% Ni Type 304; 18% Cr-8% Ni, Ti Type 321; 18% Cr-8% Ni, Cb Type 347; and 25% Cr-20% Ni Type 310.

The linear equations yielded excellent fits to the given data for the coefficient of thermal expansion and modulus of elasticity. Because of the variation in Poisson's ratio for data from temperature to temperature and even between test specimen and test specimen at the same temperature, the linear curve fitting provides only a good approximation for Poisson's ratio. However, the curve fitting yields data corresponding to the accuracy of the experimental data.

Data on the stainless steels investigated shows Poisson's ratio to be a function of temperature as the available data for these steels indicate, an increase in temperature usually causes an increase in Poisson's ratio until a level is reached where further temperature increase causes a decrease in Poisson's ratio. However, Poisson's ratio fluctuates from one temperature to the next.

Two tables and graphs are presented below for each stainless steel; for the correlative data; because of necessity, British units for the structural design and the metric units for the nuclear engineering have been employed. The temperature ranges for the correlative tables are 70-1200 F and 25-650 C. The largest uncertainty in using the fitted curves is for type 304 stainless steel, since the modulus of elasticity requires extrapolation above 480 F (248.9 C) for the data source used.

Figure 3 shows the coefficient of thermal expansion versus temperature, Fig. 4 the modulus of elasticity versus temperature, Fig. 5 the product of the coefficient of thermal expansion and modulus of elasticity versus temperature, and Fig. 6 Poisson's ratio versus temperature. Temperature, coefficient of thermal expansion, modulus of elasticity, product of coefficient of thermal expansion and modulus of elasticity, and Poisson's ratio for each of the four stainless steels are listed in Tables I to IV.

Tables V through VII contain the basic data used. The data of Tables V and VI were taken directly from Ref. 11. The data of Table VII were obtained from Ref. 2 and then the two given test specimen values were averaged for the numerical analysis. To use the data of Table V, the following method was developed:

The coefficient of thermal expansion, mathematically expressed, is

$$\alpha = \frac{1}{L} \frac{dL}{dt} \quad . \quad (V.1)$$

Assuming the coefficient of thermal expansion a linear function of temperature, i.e.,

$$\alpha = \alpha_0 + \alpha_c t \quad . \quad (V.2)$$

Then there is obtained

$$\frac{1}{L} \frac{dL}{dt} = \alpha_0 + \alpha_c t \quad . \quad (V.3)$$

Rearranging terms and integrating from the reference temperature and length to an indicated temperature and length gives

$$\ln \frac{L}{l_0} = \alpha_0 (t - t_0) + \frac{\alpha_c}{2} (t^2 - t_0^2) \quad . \quad (V.4)$$

Expressing the length at an indicated temperature in terms of the reference length and the change in length,

$$L = l_0 + \Delta l, \quad (V.5)$$

FIG. 3
 COEFFICIENT OF THERMAL EXPANSION VERSUS TEMPERATURE
 FOR TYPES 304, 310, 321 & 347 STAINLESS STEELS
 IN METRIC AND BRITISH UNITS

FIG. 4

MODULUS OF ELASTICITY VERSUS TEMPERATURE FOR TYPES 304, 310, 321
AND 347 STAINLESS STEELS IN METRIC AND BRITISH UNITS

FIG. 5
PRODUCT OF COEFFICIENT OF THERMAL EXPANSION AND MODULUS OF
ELASTICITY VERSUS TEMPERATURE FOR TYPES 304, 310, 321 AND 347
STAINLESS STEELS IN METRIC AND BRITISH UNITS

FIG. 6
POISSON'S RATIO VERSUS TEMPERATURE FOR TYPES 304, 310,
321 AND 347 STAINLESS STEELS IN METRIC AND BRITISH UNITS

Table I

CORRELATIVE DATA FOR TYPE 304 STAINLESS STEEL IN METRIC AND BRITISH UNITS

METRIC UNITS

$$\begin{aligned} \alpha &= 16.44 \times 10^{-6} + 6.979 \times 10^{-9} t \text{ cm}/(\text{cm})(\text{C}) \\ E &= 19.53 \times 10^8 - 8.574 \times 10^5 t \text{ gm}_f/\text{cm}^2 \\ \alpha E &= 32110 - 470 \times 10^{-3} t - 5980 \times 10^{-6} t^2 \text{ gm}_f/(\text{cm}^2)(\text{C}) \\ \nu &= 0.2843 - 0.00268 \times 10^{-3} t \end{aligned}$$

BRITISH UNITS

$$\begin{aligned} \alpha &= 9.063 \times 10^{-6} + 2.154 \times 10^{-9} t \text{ in.}/(\text{in.})(\text{F}) \\ E &= 27.99 \times 10^6 - 6.775 \times 10^3 t \text{ lb}_f/\text{in.}^2 \\ \alpha E &= 253.7 - 1.1 \times 10^{-3} t - 14.6 \times 10^{-6} t^2 \text{ lb}_f/(\text{in.}^2)(\text{F}) \\ \nu &= 0.2843 - 0.00149 \times 10^{-3} t \end{aligned}$$

t, C	α , cm/(cm)(C)	E, gm _f /cm ²	αE , gm _f /(cm ²)(C)	ν
25	16.61×10^{-6}	19.32×10^8	32090	0.284
50	16.79	19.10	32070	0.284
75	16.96	18.89	32040	0.284
100	17.14	18.67	32000	0.284
125	17.31	18.46	31960	0.284
150	17.49	18.24	31900	0.284
175	17.66	18.03	31840	0.284
200	17.84	17.82	31780	0.284
225	18.01	17.60	31700	0.284
250	18.18	17.39	31620	0.284
275	18.36	17.17	31530	0.284
300	18.53	16.96	31430	0.284
325	18.71	16.74	31330	0.283
350	18.88	16.53	31210	0.283
375	19.06	16.32	31100	0.283
400	19.23	16.10	30970	0.283
425	19.41	15.89	30830	0.283
450	19.58	15.67	30690	0.283
475	19.75	15.46	30540	0.283
500	19.93	15.25	30380	0.283
525	20.10	15.03	30220	0.283
550	20.28	14.82	30050	0.283
575	20.45	14.60	29870	0.283
600	20.63	14.39	29680	0.283
625	20.80	14.17	29480	0.283
650	20.98	13.96	29280	0.283

t, F	α , in./(in.)(F)	E, lb _f /in. ²	αE , lb _f /(in. ²)(F)	ν
70	9.21×10^{-6}	27.5×10^6	253.6	0.284
100	9.28	27.3	253.4	0.284
150	9.39	27.0	253.2	0.284
200	9.49	26.6	252.9	0.284
250	9.60	26.3	252.5	0.284
300	9.71	26.0	252.1	0.284
350	9.82	25.6	251.5	0.284
400	9.92	25.3	250.9	0.284
450	10.03	25.0	250.2	0.284
500	10.14	24.6	249.5	0.284
550	10.25	24.3	248.7	0.283
600	10.36	23.9	247.8	0.283
650	10.46	23.6	246.8	0.283
700	10.57	23.3	245.8	0.283
750	10.68	22.9	244.7	0.283
800	10.79	22.6	243.5	0.283
850	10.89	22.2	242.2	0.283
900	11.00	21.9	240.9	0.283
950	11.11	21.6	239.5	0.283
1000	11.22	21.2	238.0	0.283
1050	11.32	20.9	236.4	0.283
1100	11.43	20.6	234.8	0.283
1150	11.54	20.2	233.1	0.283
1200	11.65	19.9	231.4	0.283

Table II

CORRELATIVE DATA FOR TYPE 321 STAINLESS STEEL IN METRIC AND BRITISH UNITS

METRIC UNITS

$$\begin{aligned}\alpha &= 16.44 \times 10^{-6} + 6.979 \times 10^{-9}t \text{ cm}/(\text{cm})(\text{C}) \\ E &= 20.16 \times 10^8 - 9.016 \times 10^5 t \text{ gm}_f/\text{cm}^2 \\ \alpha E &= 33140 - 750 \times 10^{-3}t - 6290 \times 10^{-6}t^2 \text{ gm}_f/(\text{cm}^2)(\text{C}) \\ \nu &= 0.2762 - 0.0527 \times 10^{-3}t\end{aligned}$$

t, C	α , cm/(cm)(C)	E, gm _f /cm ²	αE , gm _f /(cm ²)(C)	ν
25	16.61×10^{-6}	19.93×10^8	33120	0.275
50	16.79	19.71	33090	0.274
75	16.96	19.48	33050	0.272
100	17.14	19.26	33010	0.271
125	17.31	19.03	32950	0.270
150	17.49	18.81	32890	0.268
175	17.66	18.58	32820	0.267
200	17.84	18.36	32740	0.266
225	18.01	18.13	32660	0.264
250	18.18	17.91	32560	0.263
275	18.36	17.68	32460	0.262
300	18.53	17.46	32350	0.260
325	18.71	17.23	32240	0.259
350	18.88	17.01	32110	0.258
375	19.06	16.78	31980	0.256
400	19.23	16.56	31840	0.255
425	19.41	16.33	31690	0.254
450	19.58	16.11	31540	0.253
475	19.75	15.88	31370	0.251
500	19.93	15.65	31200	0.250
525	20.10	15.43	31020	0.249
550	20.28	15.20	30830	0.247
575	20.45	14.98	30630	0.246
600	20.63	14.75	30430	0.245
625	20.80	14.53	30220	0.243
650	20.98	14.30	30000	0.242

BRITISH UNITS

$$\begin{aligned}\alpha &= 9.063 \times 10^{-6} + 2.154 \times 10^{-9}t \text{ in.}/(\text{in.})(\text{F}) \\ E &= 28.93 \times 10^6 - 7.124 \times 10^3 t \text{ lb}_f/\text{in.}^2 \\ \alpha E &= 262.2 - 2.2 \times 10^{-3}t - 15.3 \times 10^{-6}t^2 \text{ lb}_f/(\text{in.}^2)(\text{F}) \\ \nu &= 0.2771 - 0.0293 \times 10^{-3}t\end{aligned}$$

t, F	α , in./(in.)(F)	E, lb _f /in. ²	αE , lb _f /(in. ²)(F)	ν
70	9.21×10^{-6}	28.4×10^6	262.0	0.275
100	9.28	28.2	261.8	0.274
150	9.39	27.8	261.5	0.273
200	9.49	27.5	261.1	0.271
250	9.60	27.1	260.7	0.270
300	9.71	26.8	260.2	0.268
350	9.82	26.4	259.5	0.267
400	9.92	26.1	258.9	0.265
450	10.03	25.7	258.1	0.264
500	10.14	25.3	257.3	0.262
550	10.25	25.0	256.3	0.261
600	10.36	24.6	255.4	0.260
650	10.46	24.3	254.3	0.258
700	10.57	23.9	253.2	0.257
750	10.68	23.6	251.9	0.255
800	10.79	23.2	250.6	0.254
850	10.89	22.8	249.3	0.252
900	11.00	22.5	247.8	0.251
950	11.11	22.1	246.3	0.249
1000	11.22	21.8	244.7	0.248
1050	11.32	21.4	243.0	0.246
1100	11.43	21.1	241.3	0.245
1150	11.54	20.7	239.4	0.243
1200	11.65	20.4	237.5	0.242

Table III

CORRELATIVE DATA FOR TYPE 347 STAINLESS STEEL IN METRIC AND BRITISH UNITS

METRIC UNITS

$$\begin{aligned} \alpha &= 16.44 \times 10^{-6} + 6.979 \times 10^{-9} t \text{ cm}/(\text{cm})(\text{C}) \\ E &= 20.86 \times 10^8 - 9.157 \times 10^5 t \text{ gm}_f/\text{cm}^2 \\ \alpha E &= 34290 - 500 \times 10^{-3} t - 6390 \times 10^{-6} t^2 \text{ gm}_f/(\text{cm}^2)(\text{C}) \\ \nu &= 0.3186 - 0.0218 \times 10^{-3} t \end{aligned}$$

t, C	α , cm/(cm)(C)	E, gm _f /cm ²	αE , gm _f /(cm ²)(C)	ν
25	16.61×10^{-8}	20.63×10^8	34270	0.318
50	16.79	20.40	34250	0.318
75	16.96	20.17	34220	0.317
100	17.14	19.94	34180	0.316
125	17.31	19.71	34130	0.316
150	17.49	19.49	34070	0.315
175	17.66	19.26	34010	0.315
200	17.84	19.03	33930	0.314
225	18.01	18.80	33850	0.314
250	18.18	18.57	33770	0.313
275	18.36	18.34	33670	0.313
300	18.53	18.11	33560	0.312
325	18.71	17.88	33450	0.312
350	18.88	17.65	33330	0.311
375	19.06	17.42	33200	0.310
400	19.23	17.20	33070	0.310
425	19.41	16.97	32920	0.309
450	19.58	16.74	32770	0.309
475	19.75	16.51	32610	0.308
500	19.93	16.28	32440	0.308
525	20.10	16.05	32270	0.307
550	20.28	15.82	32080	0.307
575	20.45	15.59	31890	0.306
600	20.63	15.36	31690	0.306
625	20.80	15.13	31480	0.305
650	20.98	14.91	31270	0.305

BRITISH UNITS

$$\begin{aligned} \alpha &= 9.063 \times 10^{-6} + 2.154 \times 10^{-9} t \text{ in.}/(\text{in.})(\text{F}) \\ E &= 29.90 \times 10^6 - 7.236 \times 10^3 t \text{ lb}_f/\text{in.}^2 \\ \alpha E &= 271.0 - 1.2 \times 10^{-3} t - 15.6 \times 10^{-6} t^2 \text{ lb}_f/(\text{in.}^2)(\text{F}) \\ \nu &= 0.3190 - 0.0121 \times 10^{-3} t \end{aligned}$$

t, F	α , in./(in.)(F)	E, lb _f /in. ²	αE , lb _f /(in. ²)(F)	ν
70	9.21×10^{-6}	29.4×10^6	270.8	0.318
100	9.28	29.2	270.7	0.318
150	9.39	28.8	270.5	0.317
200	9.49	28.5	270.1	0.317
250	9.60	28.1	269.7	0.316
300	9.71	27.7	269.2	0.315
350	9.82	27.4	268.7	0.315
400	9.92	27.0	268.0	0.314
450	10.03	26.6	267.3	0.314
500	10.14	26.3	266.5	0.313
550	10.25	25.9	265.6	0.312
600	10.36	25.6	264.7	0.312
650	10.46	25.2	263.6	0.311
700	10.57	24.8	262.5	0.311
750	10.68	24.5	261.3	0.310
800	10.79	24.1	260.1	0.309
850	10.89	23.7	258.7	0.309
900	11.00	23.4	257.3	0.308
950	11.11	23.0	255.8	0.308
1000	11.22	22.7	254.2	0.307
1050	11.33	22.3	252.5	0.306
1100	11.43	21.9	250.8	0.306
1150	11.54	21.6	249.0	0.305
1200	11.65	21.2	247.1	0.305

Table IV

CORRELATIVE DATA FOR TYPE 310 STAINLESS STEEL IN METRIC AND BRITISH UNITS

METRIC UNITS

$$\alpha = 13.87 \times 10^{-6} + 7.326 \times 10^{-9} t \text{ cm}/(\text{cm})(\text{C})$$

$$E = 20.11 \times 10^8 - 8.096 \times 10^5 t \text{ gm}_f/\text{cm}^2$$

$$\alpha E = 27890 + 3500 \times 10^{-3} t - 5930 \times 10^{-6} t^2 \text{ gm}_f/(\text{cm}^2)(\text{C})$$

$$\nu = 0.3239 - 0.0049 \times 10^{-3} t$$

t, C	α , cm/(cm)(C)	E, gm _f /cm ²	αE , gm _f /(cm ²)(C)	ν
25	14.05×10^{-6}	19.91×10^8	27970	0.324
50	14.24	19.70	28050	0.324
75	14.42	19.50	28120	0.324
100	14.60	19.30	28180	0.323
125	14.79	19.10	28240	0.323
150	14.97	18.89	28280	0.323
175	15.15	18.69	28320	0.323
200	15.34	18.49	28360	0.323
225	15.52	18.29	28380	0.323
250	15.70	18.08	28390	0.323
275	15.88	17.88	28400	0.323
300	16.07	17.68	28410	0.322
325	16.25	17.48	28400	0.322
350	16.43	17.27	28390	0.322
375	16.62	17.07	28370	0.322
400	16.80	16.87	28340	0.322
425	16.98	16.67	28310	0.322
450	17.17	16.46	28260	0.322
475	17.35	16.26	28220	0.322
500	17.53	16.06	28160	0.321
525	17.72	15.86	28090	0.321
550	17.90	15.65	28020	0.321
575	18.08	15.45	27940	0.321
600	18.27	15.25	27860	0.321
625	18.45	15.05	27760	0.321
650	18.63	14.84	27660	0.321

BRITISH UNITS

$$\alpha = 7.633 \times 10^{-6} + 2.261 \times 10^{-9} t \text{ in.}/(\text{in.})(\text{F})$$

$$E = 28.80 \times 10^6 - 6.397 \times 10^3 t \text{ lb}_f/\text{in.}^2$$

$$\alpha E = 219.8 + 16.3 \times 10^{-3} t - 14.5 \times 10^{-6} t^2 \text{ lb}_f/(\text{in.}^2)(\text{F})$$

$$\nu = 0.3240 - 0.0027 \times 10^{-3} t$$

t, F	α , in./(in.)(F)	E, lb _f /in. ²	αE , lb _f /(in. ²)(F)	ν
70	7.79×10^{-6}	28.4×10^6	220.9	0.324
100	7.86	28.2	221.3	0.324
150	7.97	27.8	221.9	0.324
200	8.09	27.5	222.5	0.323
250	8.20	27.2	223.0	0.323
300	8.31	26.9	223.4	0.323
350	8.42	26.6	223.7	0.323
400	8.54	26.2	224.0	0.323
450	8.65	25.9	224.2	0.323
500	8.76	25.6	224.3	0.323
550	8.88	25.3	224.4	0.323
600	8.99	25.0	224.4	0.322
650	9.10	24.6	224.3	0.322
700	9.22	24.3	224.1	0.322
750	9.33	24.0	223.9	0.322
800	9.44	23.7	223.6	0.322
850	9.56	23.4	223.2	0.322
900	9.67	23.0	222.7	0.322
950	9.78	22.7	222.2	0.321
1000	9.89	22.4	221.6	0.321
1050	10.01	22.1	220.9	0.321
1100	10.12	21.8	220.2	0.321
1150	10.23	21.4	219.4	0.321
1200	10.35	21.1	218.5	0.321

Table V

LINEAR THERMAL EXPANSION DATA

(Data taken from Reference 11)

Linear Thermal Expansion in inches per foot in going
from 70 F to the indicated temperature

Type of Stainless Steel	Temperature t, F				
	400	600	800	1000	1200
304 } 321 } 347 }	0.038	0.062	0.088	0.115	0.142
310	0.032	0.053	0.076	0.100	0.123

Table VI

MODULUS OF ELASTICITY DATA

(Data taken from Reference 11)

Young's Modulus, lb_f/in^2

Type of Stainless Steel	Temperature t, F								
	70	210	300	392	480	752	1112	1472	1832
304	27.5×10^6	26.5×10^6	26.0×10^6	25.5×10^6	24.6×10^6	-	-	-	-
321	28.4	-	-	26.2	-	23.5×10^6	21.1×10^6	18.4×10^6	-
347	29.4	-	-	27.1	-	24.4	21.8	19.3	-
310	28.4	-	-	26.2	-	24.0	21.7	19.5	17.0×10^6

Table VII

POISSON'S RATIO DATA

(Data taken from Reference 2)

Type of Stainless Steel		Temperature t, F							
		75	300	500	700	900	1100	1300	1500
304	Specimen 1	0.25	0.27	0.32	0.32	0.27	0.29	0.27	0.23
	Specimen 2	0.24	0.30	0.29	0.32	0.29	0.30	0.30	0.27
321	Specimen 1	0.30	0.25	0.25	0.27	0.34	0.29	0.19	0.15
	Specimen 2	0.23	0.22	0.26	0.27	0.27	0.30	0.27	0.20
347	Specimen 1	0.33	0.33	0.34	0.29	0.37	0.33	0.37	0.32
	Specimen 2	0.27	0.26	0.28	0.30	0.29	0.29	0.33	0.25
310	Specimen 1	0.31	0.33	0.33	0.33	0.36	0.36	0.34	0.28
	Specimen 2	0.34	0.32	0.30	0.29	0.28	0.33	0.35	0.30

and substituting this length expression into Eq. V.4 there is obtained

$$\ln \left(1 + \frac{\Delta \ell}{\ell_0} \right) = \alpha_0 (t - t_0) + \frac{\alpha_c}{2} (t^2 - t_0^2) \quad (V.6)$$

The forementioned expression contains the linear thermal expansion, $\Delta \ell / \ell_0$, where the data of Table V can now be utilized by changing the units from inches per foot to inches per inch to maintain consistency with the established system of units.

Applying the least squares method over discrete range, the α 's are determined so that the sum of the residuals squared over a particular range is as small as possible, i.e.,

$$\sum_i R_i^2 = \min. = \sum_i \left[\ln \left(1 + \frac{\Delta \ell_i}{\ell_0} \right) - \alpha_0 (t_i - t_0) - \frac{\alpha_c}{2} (t_i^2 - t_0^2) \right]^2 \quad (V.7)$$

Taking the partial derivatives of the α 's and equating to zero, there is ascertained that

$$\frac{\partial \sum_i R_i^2}{\partial \alpha_0} = 0 = 2 \sum_i \left[\ln \left(1 + \frac{\Delta \ell_i}{\ell_0} \right) - \alpha_0 (t_i - t_0) - \frac{\alpha_c}{2} (t_i^2 - t_0^2) \right] (-)(t_i - t_0) \quad (V.8)$$

$$\frac{\partial \sum_i R_i^2}{\partial \alpha_c} = 0 = 2 \sum_i \left[\ln \left(1 + \frac{\Delta \ell_i}{\ell_0} \right) - \alpha_0 (t_i - t_0) - \frac{\alpha_c}{2} (t_i^2 - t_0^2) \right] (-)\left(\frac{1}{2}\right)(t_i^2 - t_0^2) \quad (V.9)$$

Factoring and rearranging Eqs. V.8 and V.9, the least squares method basic expressions for determining the α 's are

$$\alpha_0 \sum_i (t_i - t_0)^2 + \frac{\alpha_c}{2} \sum_i (t_i - t_0)^2 (t_i + t_0) = \sum_i (t_i - t_0) \ln \left(1 + \frac{\Delta \ell_i}{\ell_0} \right) \quad (V.10)$$

$$\alpha_0 \sum_i (t_i - t_0)^2 (t_i + t_0) + \frac{\alpha_c}{2} \sum_i (t_i - t_0)^2 (t_i + t_0)^2 = \sum_i (t_i - t_0)(t_i + t_0) \ln \left(1 + \frac{\Delta \ell_i}{\ell_0} \right) \quad (V.11)$$

Appendix A illustrates the utilization of this method in curve fitting.

For the determination of $\ln [1 + (\Delta \ell / \ell_0)]$ for values near one, the following series is used

$$\ln \left(1 + \frac{\Delta \ell}{\ell_0} \right) = \frac{\Delta \ell}{\ell_0} - 1/2 \left(\frac{\Delta \ell}{\ell_0} \right)^2 + 1/3 \left(\frac{\Delta \ell}{\ell_0} \right)^3 - 1/4 \left(\frac{\Delta \ell}{\ell_0} \right)^4 + \dots \quad (V.12)$$

VI. UNIT STRESSES FOR SPECIAL CONDITIONS

In order to completely resolve the unit stresses in the preceding development, the coefficient of thermal expansion, temperature, and cylindrical radius must be parameters of one another. The temperature change for conduction through a homogeneous cylinder wall in the steady state (see Appendix B) is

$$t - t_b = \tau = \frac{\Delta t}{\ln(b/a)} \ln \frac{b}{r} \quad (\text{VI.1})$$

The coefficient of thermal expansion for many materials can be closely approximated as a linear function of temperature (Section V and Appendix A), i.e.,

$$\alpha = \alpha_0 + \alpha_c t = \alpha_u + \alpha_c \tau \quad (\text{VI.2})$$

where

$$\alpha_u = \alpha_0 + \alpha_c t_b$$

Substituting these parametric equations into Eqs. IV.11, IV.12, and IV.16, integrating (see Appendix C), and regrouping, there are obtained

$$\begin{aligned} \sigma_r = \frac{E_m \Delta t}{2(1 - \nu_m) \ln(b/a)} & \left\{ \alpha_u \left[-\ln \frac{b}{r} - \frac{a^2}{b^2 - a^2} \left(1 - \frac{b^2}{r^2} \right) \ln \frac{b}{a} \right] \right. \\ & + \frac{\alpha_c \Delta t}{2 \ln(b/a)} \left[-\ln \frac{b}{r} - \frac{a^2}{b^2 - a^2} \left(1 - \frac{b^2}{r^2} \right) \ln \frac{b}{a} - \left(\ln \frac{b}{r} \right)^2 \right. \\ & \left. \left. - \frac{a^2}{b^2 - a^2} \left(1 - \frac{b^2}{r^2} \right) \left(\ln \frac{b}{a} \right)^2 \right] \right\} \quad (\text{VI.3}) \end{aligned}$$

$$\begin{aligned} \sigma_t = \frac{E_m \Delta t}{2(1 - \nu_m) \ln(b/a)} & \left\{ \alpha_u \left[1 - \ln \frac{b}{r} - \frac{a^2}{b^2 - a^2} \left(1 + \frac{b^2}{r^2} \right) \ln \frac{b}{a} \right] \right. \\ & + \frac{\alpha_c \Delta t}{2 \ln(b/a)} \left[1 + \ln \frac{b}{r} - \frac{a^2}{b^2 - a^2} \left(1 + \frac{b^2}{r^2} \right) \ln \frac{b}{a} - \left(\ln \frac{b}{r} \right)^2 \right. \\ & \left. \left. - \frac{a^2}{b^2 - a^2} \left(1 + \frac{b^2}{r^2} \right) \left(\ln \frac{b}{a} \right)^2 \right] \right\} \quad (\text{VI.4}) \end{aligned}$$

$$\begin{aligned} \sigma_a = \frac{E_m \Delta t}{2(1 - \nu_m) \ln(b/a)} & \left\{ \alpha_u \left[1 - 2 \left(\ln \frac{b}{r} \right) - \frac{2a^2}{b^2 - a^2} \left(\ln \frac{b}{a} \right) \right] \right. \\ & + \frac{\alpha_c \Delta t}{2 \ln(b/a)} \left[1 - \frac{2a^2}{b^2 - a^2} \left(\ln \frac{b}{a} \right) - 2 \left(\ln \frac{b}{r} \right)^2 - \frac{2a^2}{b^2 - a^2} \left(\ln \frac{b}{a} \right)^2 \right] \right\} \quad (\text{VI.5}) \end{aligned}$$

The maximum values of the tangential unit stresses and the axial unit stresses occur at the inner or the outer surface of the cylinder and are of the same magnitude. Placing $r = a$ and $r = b$ into Eqs. VI.4 and VI.5 gives

$$(\sigma_t)_{r=a} = (\sigma_a)_{r=a} = \frac{E_m \Delta t}{2(1-\nu_m) \ln(b/a)} \left\{ \alpha_u \left[1 - \frac{2b^2}{b^2-a^2} \left(\ln \frac{b}{a} \right) \right] + \frac{\alpha_c \Delta t}{2 \ln(b/a)} \left[1 - \frac{2a^2}{b^2-a^2} \left(\ln \frac{b}{a} \right) - \frac{2b^2}{b^2-a^2} \left(\ln \frac{b}{a} \right)^2 \right] \right\} \quad (VI.6)$$

and

$$(\sigma_t)_{r=b} = (\sigma_a)_{r=b} = \frac{E_m \Delta t}{2(1-\nu_m) \ln(b/a)} \left\{ \alpha_u \left[1 - \frac{2a^2}{b^2-a^2} \left(\ln \frac{b}{a} \right) \right] + \frac{\alpha_c \Delta t}{2 \ln(b/a)} \left[1 - \frac{2a^2}{b^2-a^2} \left(\ln \frac{b}{a} \right) - \frac{2a^2}{b^2-a^2} \left(\ln \frac{b}{a} \right)^2 \right] \right\} \quad (VI.7)$$

If α_c is set equal to zero and α_u is replaced by α_m , the coefficient of thermal expansion based on the mean temperature, then the developed equations reduce to the following integrated equations normally encountered in studying strengths of materials:

$$\sigma_r = \frac{E_m \alpha_m \Delta t}{2(1-\nu_m) \ln(b/a)} \left[-\ln \frac{b}{r} - \frac{a^2}{b^2-a^2} \left(1 - \frac{b^2}{r^2} \right) \ln \frac{b}{a} \right] \quad (VI.8)$$

$$\sigma_t = \frac{E_m \alpha_m \Delta t}{2(1-\nu_m) \ln(b/a)} \left[1 - \ln \frac{b}{r} - \frac{a^2}{b^2-a^2} \left(1 + \frac{b^2}{r^2} \right) \ln \frac{b}{a} \right] \quad (VI.9)$$

$$\sigma_a = \frac{E_m \alpha_m \Delta t}{2(1-\nu_m) \ln(b/a)} \left[1 - 2 \left(\ln \frac{b}{r} \right) - \frac{2a^2}{b^2-a^2} \left(\ln \frac{b}{a} \right) \right] \quad (VI.10)$$

$$(\sigma_t)_{r=a} = (\sigma_a)_{r=a} = \frac{E_m \alpha_m \Delta t}{2(1-\nu_m) \ln(b/a)} \left[1 - \frac{2b^2}{b^2-a^2} \left(\ln \frac{b}{a} \right) \right] \quad (VI.11)$$

$$(\sigma_t)_{r=b} = (\sigma_a)_{r=b} = \frac{E_m \alpha_m \Delta t}{2(1-\nu_m) \ln(b/a)} \left[1 - \frac{2a^2}{b^2-a^2} \left(\ln \frac{b}{a} \right) \right] \quad (VI.12)$$

To simplify the unit stress computations, various terms in the developed equations have been computed and are related in Table VIII.

Table VIII

STRESS COMPUTATION TERMS

$\frac{b}{r}$	$\left(\frac{b}{r}\right)^2$	$1 - \left(\frac{b}{r}\right)^2$	$\frac{r^2}{b^2 - r^2}$	$1 + \left(\frac{b}{r}\right)^2$	$\ln \frac{b}{r}$	$\left(\ln \frac{b}{r}\right)^2$	$\frac{r^2}{b^2 - r^2} \left(\ln \frac{b}{r}\right)$	$\frac{r^2}{b^2 - r^2} \left(\ln \frac{b}{r}\right)^2$
1.0	1.00	0.00	∞	2.00	0.00000	0.00000	0.00000	0.00000
1.1	1.21	-0.21	4.7619	2.21	0.09531	0.00908	0.45386	0.04326
1.2	1.44	-0.44	2.2727	2.44	0.18232	0.03324	0.41436	0.07555
1.3	1.69	-0.69	1.4493	2.69	0.26236	0.06883	0.38023	0.09975
1.4	1.96	-0.96	1.0417	2.96	0.33647	0.11321	0.35049	0.11793
1.5	2.25	-1.25	0.8000	3.25	0.40547	0.16441	0.32438	0.13153
1.6	2.56	-1.56	0.6410	3.56	0.47000	0.22090	0.30128	0.14160
1.7	2.89	-1.89	0.5291	3.89	0.53063	0.28157	0.28076	0.14898
1.8	3.24	-2.24	0.4464	4.24	0.58779	0.34550	0.26241	0.15424
1.9	3.61	-2.61	0.3831	4.61	0.64185	0.41197	0.24592	0.15784
2.0	4.00	-3.00	0.3333	5.00	0.69315	0.48046	0.23105	0.16015
2.1	4.41	-3.41	0.2933	5.41	0.74194	0.55047	0.21758	0.16143
2.2	4.84	-3.84	0.2604	5.84	0.78846	0.62167	0.20533	0.16189
2.3	5.29	-4.29	0.2331	6.29	0.83291	0.69374	0.19415	0.16171
2.4	5.76	-4.76	0.2101	6.76	0.87547	0.76645	0.18392	0.16102
2.5	6.25	-5.25	0.1905	7.25	0.91629	0.83959	0.17453	0.15992
2.6	6.76	-5.76	0.1736	7.76	0.95551	0.91300	0.16589	0.15851
2.7	7.29	-6.29	0.1590	8.29	0.99325	0.98655	0.15791	0.15684
2.8	7.84	-6.84	0.1462	8.84	1.02962	1.06012	0.15053	0.15499
2.9	8.41	-7.41	0.1350	9.41	1.06471	1.13361	0.14369	0.15298
3.0	9.00	-8.00	0.1250	10.00	1.09861	1.20694	0.13733	0.15087
3.1	9.61	-8.61	0.1161	10.61	1.13140	1.28007	0.13141	0.14867
3.2	10.24	-9.24	0.1082	11.24	1.16315	1.35292	0.12588	0.14642
3.3	10.89	-9.89	0.1011	11.89	1.19392	1.42544	0.12072	0.14413
3.4	11.56	-10.56	0.0947	12.56	1.22378	1.49764	0.11589	0.14182
3.5	12.25	-11.25	0.0889	13.25	1.25276	1.56941	0.11136	0.13950
3.6	12.96	-11.96	0.0836	13.96	1.28093	1.64078	0.10710	0.13719
3.7	13.69	-12.69	0.0788	14.69	1.30833	1.71173	0.10310	0.13489
3.8	14.44	-13.44	0.0744	15.44	1.33500	1.78223	0.09933	0.13261
3.9	15.21	-14.21	0.0704	16.21	1.36098	1.85227	0.09578	0.13035
4.0	16.00	-15.00	0.0667	17.00	1.38629	1.92180	0.09242	0.12812

VII. HYPOTHETICAL EXAMPLE

A hypothetical example having the coefficient of thermal expansion independent of the temperature at the mean temperature and the coefficient of thermal expansion as a linear function of the temperature is now considered. The imposed conditions and specifications on the cylinder follow: material: type 304 stainless steel; $t_b = 320$ F (160 C); $t_a = 820$ F (437.8 C); $a = 1.5$ in. (3.81 cm); $b = 4.5$ in. (11.43 cm); and a steady rate of transferred heat.

Computations with the prior data are given in Table IX. Figures 7 and 8 depict the temperature distribution and the unit stress distributions, respectively, across the wall of the cylinder when the coefficient of thermal expansion is a linear function of temperature.

For this example, the difference in unit stresses between using the coefficient of thermal expansion as a function of temperature and the mean coefficient of thermal expansion is not very large. The maximum surface unit stresses have a maximum disagreement of approximately 3%.

The maximum variation between the two methods of this example is normally not considered excessive for unit stress computations because of the inherent uncertainties, such as the suppositions and actual elastic constants, for a particular component. Using this example as a basis, sufficiently accurate thermal stress indications will be obtained for most unit stress computations if the coefficient of thermal expansion, modulus of elasticity, and Poisson's ratio are based on the mean temperature of the cross section. For extremely accurate thermal stress computations, especially when the temperature drop across the component cross section is large, it would be necessary to consider the elastic constants, coefficient of thermal expansion, and the thermal conductivity of the material as functions of temperature.

Table IX

HYPOTHETICAL EXAMPLE: NUMERICAL RESULTS

Material: Stainless Steel - Type 304

$$\Delta t = 500\text{F} (277.8\text{C})$$

$$t_m = 485.1\text{F} (251.7\text{C})$$

$$E_m = 24.7 \times 10^6 \text{ lb}_f/\text{in.}^2 (17.37 \times 10^8 \text{ gm}_f/\text{cm}^2)$$

$$\alpha_u = 9.76 \times 10^{-6} \text{ in.}/(\text{in.})(\text{F}) [17.56 \times 10^{-6} \text{ cm}/(\text{cm})(\text{C})]$$

$$\alpha_m = 10.11 \times 10^{-6} \text{ in.}/(\text{in.})(\text{F}) [18.20 \times 10^{-6} \text{ cm}/(\text{cm})(\text{C})]$$

$$\nu_m = 0.284$$

b/r		3.0	2.5	2.0	1.5	1.2	1.0
Linear Coefficient of Expansion	r (in.)	1.50	1.80	2.25	3.00	3.75	4.50
	(cm)	3.81	4.57	5.72	7.62	9.53	11.43
	t (F)	820.0	737.0	635.5	504.5	403.0	320.0
	(C)	437.8	391.7	335.3	262.5	206.1	160.0
	r (F)	500.0	417.0	315.5	184.5	83.0	0
	(C)	277.8	231.7	175.3	102.5	46.1	0
	$\sigma_a (\text{lb}_f/\text{in.}^2)$	-120 420	-89 660	-52 700	-6 190	29 010	57 200
	$\sigma_r (\text{lb}_f/\text{in.}^2)$	0	-15 900	-22 730	-18 720	-9 680	0
	$\sigma_t (\text{lb}_f/\text{in.}^2)$	-120 420	-73 760	-29 970	12 530	38 690	57 200
	$\sigma_a (\text{gm}_f/\text{cm}^2)$	-8 466 000	-6 303 000	-3 705 000	-435 000	2 039 000	4 021 000
	$\sigma_r (\text{gm}_f/\text{cm}^2)$	0	-1 118 000	-1 598 000	-1 316 000	-681 000	0
	$\sigma_t (\text{gm}_f/\text{cm}^2)$	-8 466 000	-5 185 000	-2 107 000	881 000	2 720 000	4 021 000
	$\sigma_a (\text{lb}_f/\text{in.}^2)$	-116 860	-87 910	-52 480	-6 800	28 640	57 590
	$\sigma_r (\text{lb}_f/\text{in.}^2)$	0	-15 500	-22 320	-18 560	-9 680	0
	$\sigma_t (\text{lb}_f/\text{in.}^2)$	-116 860	-72 410	-30 160	11 760	38 320	57 590
	$\sigma_a (\text{gm}_f/\text{cm}^2)$	-8 217 000	-6 181 000	-3 690 000	-478 000	2 014 000	4 049 000
	$\sigma_r (\text{gm}_f/\text{cm}^2)$	0	-1 090 000	-1 570 000	-1 305 000	-680 000	0
	$\sigma_t (\text{gm}_f/\text{cm}^2)$	-8 217 000	-5 091 000	-2 120 000	827 000	2 694 000	4 049 000

FIG. 7
TEMPERATURE DISTRIBUTION ACROSS WALL OF
CYLINDER IN METRIC AND BRITISH UNITS

FIG. 8
UNIT-STRESS DISTRIBUTION ACROSS WALL OF
CYLINDER IN METRIC AND BRITISH UNITS

APPENDIX A

Curve Fitting Computations for Types 304, 321 and 347 Stainless Steels for Coefficient of Thermal Expansion Versus Temperature Using a Straight Line Fit and Data from Table V

Solution: Required sums where $t_0 = 70$ F

t_i	$t_i - t_0$	$(t_i - t_0)^2$	$t_i + t_0$	$(t_i - t_0)^2(t_i + t_0)$	$(t_i - t_0)^2(t_i + t_0)^2$	$\ln\left(1 + \frac{\Delta\ell}{\ell_0}\right)$	$(t_i - t_0)\ln\left(1 + \frac{\Delta\ell}{\ell_0}\right)$	$(t_i - t_0)(t_i + t_0)\ln\left(1 + \frac{\Delta\ell}{\ell_0}\right)$
400	330	108 900	470	51 183 000	24 056 010 000	0.003 161 664	1.043 349 12	490. 374 09
600	530	280 900	670	188 203 000	126 096 010 000	0.005 153 366	2.731 283 98	1 829. 960 27
800	730	532 900	870	463 623 000	403 352 010 000	0.007 306 575	5.333 799 75	4 640. 405 78
1000	930	864 900	1070	925 443 000	990 224 010 000	0.009 537 706	8.870 066 58	9 490. 971 24
1200	1130	1 276 900	1270	1 621 663 000	2 059 512 010 000	0.011 763 871	13.293 174 23	16 882. 331 27
Σ		3 064 500		3 250 115 000	3 603 240 050 000		31.271 673 66	33 334. 042 65

The normal equations are

$$3\,064\,500\alpha_0 + \frac{1}{2}(3\,250\,115\,000)\alpha_c = 31.271\,673\,66$$

$$3\,250\,115\,000\alpha_0 + \frac{1}{2}(3\,603\,240\,050\,000)\alpha_c = 33\,334.042\,65$$

Solving for the constants α_0 and α_c gives

$$\alpha_0 = 9.06252 \times 10^{-6} \text{ and } \alpha_c = 2.1535 \times 10^{-9}$$

The fitted curve is

$$\ln\left(1 + \frac{\Delta\ell}{\ell_0}\right) = 9.06252 \times 10^{-6} (t - t_0) + 1.07675 \times 10^{-9} (t^2 - t_0^2)$$

Check on the given points to show exactness of fit.

t	400 F	600 F	800 F	1000 F	1200 F
$\ln\left(1 + \frac{\Delta\ell}{\ell_0}\right)$ Given	0.003 162	0.005 153	0.007 307	0.009 538	0.011 764
$\ln\left(1 + \frac{\Delta\ell}{\ell_0}\right)$ Calc	0.003 158	0.005 185	0.007 299	0.009 500	0.011 786
Difference	0.000 004	-0.000 032	0.000 008	0.000 038	-0.000 022

APPENDIX B

Temperature Distribution in the Steady State

For a steady-state temperature distribution in a cylindrical section, the rate of heat flowing radially through a differential volume of unit length becomes

$$q = -k (2\pi r l) \frac{dt}{dr} \quad . \quad (B.1)$$

The material being considered homogeneous and its thermal conductivity independent of the temperature, regrouping the terms and integrating gives

$$t = -\frac{q}{2\pi k} \ln r + \text{constant}. \quad (B.2)$$

Using the boundary conditions $r = a$, $t = t_a$ and $r = b$, $t = t_b$, there is obtained

$$t_a = -\frac{q}{2\pi k} \ln a + \text{constant} \quad (B.3)$$

and

$$t_b = -\frac{q}{2\pi k} \ln b + \text{constant}. \quad (B.4)$$

Subtracting Eq. B.4 from Eqs. B.2 and B.3, then dividing the first temperature difference equation by the second, and finally solving for the temperature difference in general, gives

$$\tau = \frac{\Delta t}{\ln(b/a)} \ln \frac{b}{r} \quad , \quad (B.5)$$

where $\Delta t = t_a - t_b$ and $\tau = t - t_b$.

The mean temperature of a body cross section based on this development is

$$t_m = \frac{\int_a^b \left[t_b + \frac{\Delta t}{\ln(b/a)} \ln \frac{b}{r} \right] 2\pi r dr}{\int_a^b 2\pi r dr} \quad (B.6)$$

upon integration and simplification,

$$t_m = t_b + \frac{\Delta t}{\ln(b/a)} \left[\frac{1}{2} - \frac{a^2}{b^2 - a^2} \ln \frac{b}{a} \right] \quad . \quad (B.7)$$

The thermal conductivity of types 304, 310, 321, and 347 stainless steels are functions of temperature and can be approximated over discrete ranges by linear functions. The thermal conductivity for these stainless steels increases with temperature as illustrated in Reference 11.

APPENDIX C

Special Integrals

The following integrals simplify the unit-stress determinations when the coefficient of thermal expansion is a linear function of the temperature. These are given here since these integrals are normally not presented in explicit form.

$$\int_a^r \left(\ln \frac{b}{r} \right) r \, dr = \frac{1}{4} (r^2 - a^2) + \frac{r^2}{2} \left(\ln \frac{b}{r} \right) - \frac{a^2}{2} \left(\ln \frac{b}{a} \right) \quad (C.1)$$

$$\int_a^b \left(\ln \frac{b}{r} \right) r \, dr = \frac{1}{4} (b^2 - a^2) - \frac{a^2}{2} \left(\ln \frac{b}{a} \right) \quad (C.2)$$

$$\begin{aligned} \int_a^r \left(\ln \frac{b}{r} \right)^2 r \, dr &= \frac{1}{4} (r^2 - a^2) + \frac{r^2}{2} \left(\ln \frac{b}{r} \right) - \frac{a^2}{2} \left(\ln \frac{b}{a} \right) \\ &\quad + \frac{r^2}{2} \left(\ln \frac{b}{r} \right)^2 - \frac{a^2}{2} \left(\ln \frac{b}{a} \right)^2 \end{aligned} \quad (C.3)$$

$$\int_a^b \left(\ln \frac{b}{r} \right)^2 r \, dr = \frac{1}{4} (b^2 - a^2) - \frac{a^2}{2} \left(\ln \frac{b}{a} \right) - \frac{a^2}{2} \left(\ln \frac{b}{a} \right)^2 \quad (C.4)$$

The integral for the unit strain produced by the temperature difference where Eq. VI.2 approximates the coefficient of thermal expansion is

$$\int_{t_b}^t \alpha \, dt = \alpha_u \tau + \frac{\alpha_c}{2} \tau^2 \quad (C.5)$$

ACKNOWLEDGEMENT

The author is indebted to Louise Taylor for her assistance in checking and computing the numerical portion of this presentation.

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