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The Analytic Structure of the Off-Mass-Shell S-Wave
 $\pi^+\pi^-$ Scattering Amplitude.

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ABSTRACT

Using rather general assumptions, it is shown that the on-mass-shell P-Wave determines the off-mass-shell dependence of the S-Wave $\pi^+\pi^-$ amplitude. This is used to determine $(2a_0 + a_2)/a_1$ experimentally.

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In two earlier papers we studied the off-mass-shell dependence of the $\pi\pi$ scattering amplitude.^{1,2} Following Lovelace³ and Wagner⁴ we assume that the off-mass-shell continuation of the $\pi\pi$ scattering amplitude can be effected via the off-mass-shell continuation of the Mandelstam variables t and u , that is

$$t(\Delta^2, z) = -\frac{1}{2} (s + \Delta^2 - 3\mu^2 - 4qq_{\text{off}}^2 z) \quad (1)$$

$$u(\Delta^2, z) = t(\Delta^2, -z) \quad (2)$$

We denote the momentum, mass squared and cosine of the scattering angle of the incident off-mass-shell pion by q_{off} , Δ^2 , and z . The dipion effective mass, the pion mass and the magnitude of the final state momenta are indicated by $\sqrt{s}$, μ and q . Any quantities above which are not invariant, are defined in the di-pion rest frame. If the amplitude is known as a function of s , t , u then the off-mass-shell continuation of the amplitude as given by equations 1, and 2, is unambiguous. In previous work, a specific form was assumed for the amplitude, either Weinberg's current algebra model⁵ or the Veneziano model.^{3,6} One problem is that in both models the amplitudes are assumed to be real, although Lovelace³ has suggested a unitarization procedure. However, it may well be that the relationship between the on and off-mass-shell amplitude given by the above models can be generalized to obtain a unitary (and hence complex) description. Thus we would like to generalize our previous work^{1,2} to minimize the dependence of the results on a specific model.

Although we do not attempt to calculate theoretically the scattering amplitude either on or off-mass-shell, we will show that under very weak assumptions there exists a relation between the off-mass-shell S-wave and the unitary, on-mass-shell P-wave in certain kinematic regions. This may prove very useful experimentally.

It was pointed out by Chew and Mandelstam⁷ that the on-mass-shell $\pi\pi$ scattering amplitude, $T^I(s,t,u)$, in any isotopic spin state can be written in terms of one function as

$$T^0(s,t,u) = 3A(s,t,u) + A(t,u,s) + A(u,t,s) \quad (3)$$

$$T^1(s,t,u) = A(t,u,s) - A(u,t,s) \quad (4)$$

$$T^2(s,t,u) = A(t,u,s) + A(u,t,s) \quad (5)$$

We work with amplitudes in which only one pion is off the mass shell. In this case, Eq. 3-5 still hold, with s,t,u independent variables. (With two or more pions off the mass shell, there are more than three variables and this simplicity is lost, unless further assumptions are made).

Consequently

$$T_{\ell=\text{even}}^{\pi^+\pi^-}(s,t,u) = A(s,t,u) + \frac{A(t,u,s) + A(u,t,s)}{2} \quad (6)$$

$$T_{\ell=\text{odd}}^{\pi^+\pi^-}(s,t,u) = \frac{A(t,u,s) - A(u,t,s)}{2} \quad (7)$$

where ℓ denotes the angular momentum of the dipion system. It is an experimental fact that for $\sqrt{s} \leq 1$ GeV only $\ell=0$ and $\ell=1$ states contribute on or off-mass-shell. Thus a linear expansion of Eq. 4 in z around $z=0$ is

$$T_{\ell=\text{odd}}^{\pi^+\pi^-}(s,t,u) = \left\{ A_1[t(\Delta^2,0),t(\Delta^2,0),s] - A_2[t(\Delta^2,0),t(\Delta^2,0),s] \right\} \left(\frac{\partial t}{\partial z} \right)_z \quad (8)$$

where

$$A_1(t,u,s) = \frac{\partial A(t,u,s)}{\partial t} \quad (9)$$

and

$$A_2(t,u,s) = \frac{\partial A(t,u,s)}{\partial u}$$

Absence of the $\ell = 2$ and higher partial waves allow us to omit the z dependence of the $\ell = \text{even}$ amplitude and obtain

$$T^{\pi^+\pi^-}(s,t,u)_{\ell=\text{even}} = A[s,t(\Delta^2,0),t(\Delta^2,0)] + \frac{A[t(\Delta^2,0),t(\Delta^2,0),s] + A[t(\Delta^2,0),t(\Delta^2,0),s]}{2} \quad (10)$$

Further, a power series expansion of this amplitude in $\Delta^2 + \mu^2$ can be terminated with linear terms. The quadratic terms are related to the D-waves through the Δ^2 and z dependence of t and u , and hence are expected to be small. Thus the off-mass-shell amplitude is

$$T_{\ell=\text{even}}^{\pi^+\pi^-}(s,t,u) = \left[T_{\ell=\text{even}}^{\pi^+\pi^-}(s,t,u) \right]_{\Delta^2=-\mu^2} + \left\{ 2A_2[s,t(-\mu^2,0),t(-\mu^2,0)] \right. \\ \left. + A_1[t(-\mu^2,0),t(-\mu^2,0),s] + A_3[t(-\mu^2,0),t(-\mu^2,0),s] \right\} \left(\frac{\partial t}{\partial \Delta^2} \right)_{(-\mu^2,0)} \quad (11)$$

So far the analysis has been straight forward. The key assumption is made here, namely,

$$A_2[s,t(-\mu^2,0),t(-\mu^2,0)] = -A_2[t(-\mu^2,0),t(-\mu^2,0),s] \quad (12)$$

The motivation for Eq. 12 will be discussed in detail below. Observe at this point, however, that it has the following consequence. Comparison of Eqs. 8, 11 and 12 shows

$$T_{\ell=\text{even}}^{\pi^+\pi^-}(s,t,u) = T_{\ell=\text{even}}^{\pi^+\pi^-}(s,t,u)_{\Delta^2=-\mu^2} - (\Delta^2 + \mu^2) \left[\frac{T_{\ell=\text{odd}}^{\pi^+\pi^-}(s,t,u)}{4zq\epsilon_{\text{off}}} \right]_{\Delta^2=-\mu^2} \quad (13)$$

In terms of the on-mass-shell phase shift

$$T_{\ell=\text{even}}^{\pi^+\pi^-}(s,t,u) = \frac{1}{3q} \left[e^{i\delta_0^2} \sin \delta_0^2 + 2e^{i\delta_0} \sin \delta_0 \frac{\Delta^2 + \mu^2}{s - 4\mu^2} 9e^{i\delta_1} \sin \delta_1 \right] \quad (14)$$

Next we turn to the motivation of Eq. 11. Suppose $A(s,t,u)$ is known to be the sum of functions of only two variables. Then Bose statistics implies that

$$A(s,t,u) = F(s,t) + F(s,u) + G(t,u) \quad (15)$$

where F and G are arbitrary except that G is symmetric. This is characteristic of a wide class of theories and in fact is hard to avoid in a natural way. Further suppose that the absence of $I=2$ resonances means that there is no explicit s dependence of the $I=2$ amplitude (there is an s dependence as a function of s and z through the t,u variables).

then

$$F(t,u) = -G(t,u). \quad (16)$$

This must be true for any model containing only poles and is one of the fundamental features of the Veneziano model.^{3,4} Strictly speaking, there may be an explicit dependence of the $I = 2$ amplitude in the form of a small, non-singular function, which would give small corrections to our results. Combining Eq. 15 and 16, it is simple algebra to derive our basic assumption Eq. 12.

The current algebra expansion^{1,7} of the $\pi\pi$ amplitude fits our assumptions and it is simple to check that the relation between S and P waves given by Eq. 13 and 14 holds. An important difference is that in our case $[T^{\pi^+\pi^-}(s,t,u)]_{\Delta^2=\mu^2}$ is a unitary, complex amplitude for $\ell=0, 1$. Thus, $T^{\pi^+\pi^-}_{\ell=\text{even}}(s,t,u)$ can vanish only for isolated values of s where the phase of the even and odd on-mass-shell amplitudes are equal. However, the S - P wave interference term is proportional to $\text{Re}(T^{\pi^+\pi^-}_{\text{even}} T^{\pi^+\pi^-*}_{\text{odd}})$ and will vanish at values of $\Delta^2 = \Delta_0^2(s)$ given by

$$\Delta_0^2 + \mu^2 = \frac{s-\mu^2}{9} \cotan \delta_1^+ [\sin 2\delta_0^+ + \frac{1}{2} \sin 2\delta_0^+] + \frac{s-4\mu^2}{9} [2 \sin^2 \delta_0^+ + \sin^2 \delta_0^2] \quad (17)$$

Note that the interference term contains $T^{\pi^+\pi^-}_{\text{odd}}$ off-mass-shell as given by Eq. 8. The Δ^2 dependence of $\frac{\partial t}{\partial z}$ in this expression does not influence the zeros. The Δ^2 dependence of the rest of the expression involves second derivative of $A(s,t,u)$ and hence D-waves. This is assumed to be small in deriving Eq. 17.

At threshold Eq. 17 reduces to

$$\Delta_0^2 + \mu^2 = \frac{4}{9} \frac{2a_0 + a_2}{a_1} \quad (18)$$

where a_0 , a_1 , and a_2 denote the isotopic spin 0, 1, and 2 scattering lengths.

To obtain the experimental value of Δ_0^2 we fitted simultaneously the s and Δ^2 dependence of the forward-backward asymmetry to the form¹⁰

$$\alpha(s, \Delta^2) = \left[\sum_{n=1}^{\infty} b_n (s-4\mu^2)^n \right] \left\{ 1 - (\Delta^2 + \mu^2) / m(s-4\mu^2) + \frac{4}{9} d \right\} \quad (19)$$

for $\sqrt{s} < 600$ MeV. After the constants b_n , m and d are determined we find that the zeros are given by the equation $\Delta_0^2 = m(s-4\mu^2) + \frac{4}{9}d - \mu^2$.

Thus,

$$\frac{2a_0 + a_s}{a_s} = d = 9.0^{+4.5}_{-1.1} \quad (20)$$

This is the principal experimental result of this investigation to date. However, the relationship between the off-mass-shell S-wave and the on-mass-shell P-wave summarized by Eqs. 13 and 14 may have far reaching consequences.

It is tempting to study the s dependence of the off-mass-shell zeros with the help of Eq. 14. The fact that the off-mass-shell correction is proportional to $\Delta^2 + \mu^2$ makes it extremely sensitive to absorptive corrections. For example, the procedures of Jackson et al.,⁸ predicts that absorption will remove this correction term completely. This is because the $\Delta^2 + \mu^2$ term in Eq. 14 cancels the pion propagator in the Born term for the full production amplitude. This leaves only a $J=\frac{1}{2}$ term in the partial wave expansion which Jackson et al.,⁸ multiply by a factor $\sqrt{(1-C_+)(1-C_-)}$, the C 's representing absorption effects. In the ρ region $C_+=1$, thus eliminating this term. The observation of off-mass-shell zeros in the threshold region suggests that C_- is a function of s . This unexpected importance and possible s dependence of the absorption effects has not been suggested before, and may require both the modification of absorption correction and re-interpretation of the off-mass-shell zeros in the ρ region¹⁰. We are still investigating this problem.

The importance of finding a general relation for the off-mass-shell dependence, such as Eq. 14, is exemplified by the fact that attempts to include explicit Δ^2 dependence of the $\pi\pi$ amplitude in absorption model calculations were begun immediately after the observation of the off-mass-shell zeros^{1,2} which follow from the Weinberg³ and Veneziano^{3,4} models. These were frustrated by the ambiguities of whether to interpret these as the real part of the amplitude, the phase shift, etc. Eq. 14 makes a prediction for the complete amplitude which can be used in the absorption model following the procedure outlined by Gutay, et al.¹¹

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9. J. H. Scharenguivel, L. J. Gutay, D. H. Miller, F. T. Meiere, S. Marateck, Nuclear Physics B22, (1970) 16. Eq. 19 in this paper is a generalization of Eq. 9 in the above reference. In Eq. 19 the zeros of $\alpha(s, \Delta^2)$ lie on a straight line but the intercept is not constrained to be zero at $s = \mu^2$.
10. We have been informed by Dr. J. D. Kimel that absorption effects can generate off-mass-shell zeros with no explicit Δ^2 dependence of the $\pi\pi$ amplitude by using an s dependent C . This possibility is discussed in Reference 2 without the s dependence of C . The present paper suggests that a combination of explicit Δ^2 dependence and absorption effects may be the proper approach.
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