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CHARACTERISTICS AND STABILITY ANALYSES OF TRANSIENT ONE-DIMENSIONAL TWO-PHASE FLOW EQUATIONS AND THEIR FINITE DIFFERENCE APPROXIMATIONS

BY

R. W. Lyczkowski, Dimitri Gidaspow*,

C. W. Solbrig and E. D. Hughes

Aerojet Nuclear Company**

550 Second St.

Idaho Falls, Idaho 83401

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*Consultant to Aerojet Nuclear Company with the Institute of Gas Technology, Chicago, Illinois 60610

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ABSTRACT

Equation systems describing one-dimensional, transient, two-phase flow with separate continuity, momentum, and energy equations for each phase are classified by use of the method of characteristics. Little attempt is made to justify the physics of these equations. Many of the equation systems possess complex-valued characteristics and hence, according to well-known mathematical theorems, are not well-posed as initial-value problems (IVPs). Real-valued characteristics are necessary but not sufficient to insure well-posedness. In the absence of lower order source or sink terms (potential type flows), which can affect the well-posedness of IVPs, the complex characteristics associated with these two-phase flow equations imply unbounded exponential growth for disturbances of all wavelengths.

Analytical and numerical examples show that the ill-posedness of IVPs for the two-phase flow partial differential equations which possess complex characteristics produce unstable numerical schemes.

These unstable numerical schemes can produce apparently stable and even accurate results if the growth rate resulting from the complex characteristics remains small throughout the time span of the numerical experiment or if sufficient numerical damping is present for the increment size used. Other examples show that clearly nonphysical numerical instabilities resulting from the complex characteristics can be produced. These latter types of numerical instabilities are shown to be removed by the addition of physically motivated differential terms which eliminate the complex characterisitics.

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1. INTRODUCTION

Transient two-phase flow analysis is of importance in many engineering fields, for example, in the field of safety analysis of pressurized light water nuclear reactors. During the hypothetical loss-of-coolant accident (LOCA) in a pressurized water reactor, the entire spectrum from subcooled water to superheated steam is possible (Ybarrondo, Solbrig, and Isbin, 1972). Therefore, the fluid-solids two-phase models which consider the pressure drop to take place only in the fluid appear inadequate to describe all flow regimes. Examples of such flow descriptions are the nonsteady one-dimensional flow analysis of Rudinger and Chang (1964) and those of steady supersonic two-dimensional flow presented by Hoffman (1963) and Saltanov (1972). Murray (1965) in his fluidization analysis considers pressure drops in the solid and gas phases but, on the basis of a physical argument, drops the pressure gradient in the solid phase. These equation sets have real characteristics and hence satisfy the necessary condition required for well-posed initial-value problems (IVPs).

Of the equation sets which consider pressure drops in both phases, disagreement exists as to the handling of the pressure gradient term. Delhaye (1969), Kalinin (1970), Nigmatulin (1967), Panton (1968), Pai (1973), and Soo (1967) present momentum equations with the volume fraction of each phase in the gradient of pressure. Harlow and Amsden (1975) consider this treatment as unrealistic even though this paper shows that the characteristics are real[a]. In contrast, Jarvis (1965), Wallis (1969), Boure et al. (1971), Mecredy and Hamilton (1972), and Boure, Bergles and Tong (1973) represent the volume fraction as multiplier on the pressure gradient. This paper[a] shows

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[a] Results of a preliminary nature were presented by Gidaspow et al. (1973)

that the characteristics of the equations presented by these latter authors for compressible transient one-dimensional flow are complex-valued for subsonic two-phase flow except for equal phase velocities[b]. Mathematically such equations are ill-posed as initial-value problems according to the theorems summarized in Appendix A. Ill-posed initial-value problems are unsatisfactory because all finite difference schemes consistent with the differential equations are unstable (Richtmyer and Morton, p 59, 1967). Jarvis (1965) attempted to solve his equation set using the Lax method. He attributed the instabilities he encountered to the fact that the system was found to be nonhyperbolic, that is, the system may have complex characteristics. Siegmann (1971) did not even attempt to solve such similarly ill-posed problems and Boure (1973) appears to have encountered severe stability difficulties. Analytical and numerical examples are presented in this paper to illustrate that the ill-posedness of IVPs for the two-phase flow partial differential equations which possess complex characteristics can produce numerical instabilities which are clearly nonphysical in nature. These numerical instabilities can be removed by the addition of physically motivated differential terms which eliminate the complex characteristics.

[b] Boure (1973) working independently also found complex characteristics. Gidaspow (1974) showed analytically that when both phases are incompressible, two of the characteristics are always complex for a two-phase mixture.

2. BASIC TWO-PHASE FLOW EQUATIONS

The continuity equation for phase a is:

$$\frac{\partial \rho^a}{\partial t} + \frac{\partial}{\partial x} (\rho^a v^a) = \dot{m}^a, \; a = \ell, g \tag{1}$$

where ℓ refers to the liquid and g refers to the gas and where $\sum \dot{m}^a = 0$ due to overall conservation of mass for both phases. In Equation (1) ρ^a is the partial density of phase a which equals the volume fraction, α^a, times the thermodynamic density, ρ_a. The phase velocities are denoted by v^a and the rates of formation of phases are denoted by $\dot{m}^a$. The spatial coordinate is denoted by x and the time is denoted by t.

The conservation of momentum in one-dimension, with friction and α^a inside the pressure gradient, has been written as (for example, Kalinin, 1970)

$$\frac{\partial(\rho^a v^a)}{\partial t} + \frac{\partial(\rho^a v^a v^a)}{\partial x} + \frac{\partial(\alpha^a P)}{\partial x} = -\rho^a g + \bar{A}_{ab} D_{ab} \cdot (v^a - v^b) + \dot{m}^a \hat{v}^a + F_x^a . \tag{2}$$

In Equation (2), P is the pressure, $\bar{A}_{ab}$ is the interphase area per unit volume, D_{ab} is the drag coefficient, and $\hat{v}^a$ is the intrinsic velocity associated with the source term. F_x^a is the wall friction for phase a, and g is the acceleration of gravity. As in the Mecredy and Hamilton article (1972) $\bar{A}_{\ell g} D_{\ell g} = \bar{A}_{g\ell} D_{g\ell}$. The total wall friction is the sum of F_x^a. For potential type two-phase flow, the right hand side of Equation (2) remains the same whether friction is treated through the use of drag coefficients or by means of Reynolds stresses. The momentum equations have been written as in Equation (2) for classification purposes only. We found that the characteristics for the case of α^a outside the gradient of pressure in the phase momentum equations could always be obtained by setting P = 0 in the characteristic determinants which were obtained.

The energy equation for phase a in terms of the entropy per unit mass, S_a, can be written as

$$\frac{\partial(\rho^a S_a)}{\partial t} + \frac{\partial}{\partial x} (\rho^a v^a S_a) = \frac{\omega_a}{T_a} \tag{3}$$

where T_a is the phase temperature and ω_a includes disspation due to friction and mass transfer, as well as external and interphase heat transfer. Equation (3) is similar to Kalinin's (1970) Equations (32) and (33), except that his equations are based on the assumption of no change in phase volumes. In place of the entropy equations a uniform temperature could have been specified, as is done for gas transmission in pipelines (Wilkinson, et al., 1965).

The equation of state for phase a is

$$\rho_a = \rho_a (S_a, P) \tag{4}$$

where the usual (Boure et al. p 75, 1971) hydrostatic type local equilibrium assumption has been applied by postulating that

$$P = P_\ell = P_g \quad . \tag{5}$$

Closure is obtained by Equations (1) through (4) for $a = \ell, g$ and with $\alpha^\ell + \alpha^g = 1$. For rapid interfacial rates of heat transfer the phase temperatures become equal and then Equation (4) may be replaced by

$$\rho_a = \rho_a (T, P). \tag{5a}$$

3. CHARACTERISTICS ANALYSIS OF THE BASIC EQUATIONS

Equations (1) and (3) can be combined to obtain

$$\frac{d\,S_a}{dt^a} = \frac{\omega_a}{\rho^a\,T_a} - \frac{S_a\,\dot{m}^a}{\rho^a} \qquad (6)$$

where $d/dt^a = \partial/\partial t + v^a\,\partial/\partial x$ is the convective derivative along v^a.

The chain rule may be applied to Equation (4) to eliminate the density of phase a in Equation (1) with the expression

$$\frac{d\,\rho_a}{dt^a} = \left(\frac{\partial\,\rho_a}{\partial P}\right)_{S_a}\frac{dP}{dt^a} + \left(\frac{\partial\,\rho_a}{\partial S_a}\right)_P\frac{d\,S_a}{dt^a} \qquad (7)$$

The adiabatic speed of sound for each phase is defined by

$$\left(\frac{\partial\rho_a}{\partial P}\right)_{S_a} = C_a^{-2} \qquad (8)$$

where $C_a^{-2} > 0$ according to the Second Law of Thermostatics.

After Equation (6) is substituted into Equation (7), the convective derivative of density is seen to be a function of the derivative of pressure and the nonhomogeneous terms from Equation (6). The nonhomogeneous terms which correlate heat and mass transfer and friction are assumed not to involve partial derivatives. Therefore, the characteristics will not be affected by them[a]. These nonhomogeneous terms consequently are now dropped from further considerations.

The fundamental set $\vec{U} = (\alpha^g, P, v^\ell, v^g, S_g, S_\ell)$ is used. Equations (6), (7), and (8) are combined through use of Equation (1), and Equations (1), (2), and (6) are written in matrix form as

$$A \frac{\partial \vec{U}^T}{\partial t} + B \frac{\partial \vec{U}^T}{\partial x} = 0 \tag{9}$$

where

$$A = \begin{bmatrix} \rho_g & \frac{\alpha^g}{C_g^2} & 0 & 0 & 0 & 0 \\ -\rho_\ell & \frac{\alpha^\ell}{C_\ell^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho^g & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho^\ell & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{10}$$

[a] However, well-posedness and stability are generally affected by these lower-order nonhomogeneous terms (Ramshaw and Trapp, 1975)

and

$$B = \begin{bmatrix} \rho_g v^g & \frac{\alpha^g v^g}{C_g^2} & \rho^g & 0 & 0 & 0 \\ -\rho_\ell v^\ell & \frac{\alpha^\ell v^\ell}{C_\ell^2} & 0 & \rho^\ell & 0 & 0 \\ P & \alpha^g & \rho^g v^g & 0 & 0 & 0 \\ -P & \alpha^\ell & 0 & \rho^\ell v^\ell & 0 & 0 \\ 0 & 0 & 0 & 0 & v^g & 0 \\ 0 & 0 & 0 & 0 & 0 & v^\ell \end{bmatrix} \tag{11}$$

The characteristic polynomial resulting from det $(A\lambda + B) = 0$ was algebraically evaluated through use of an IBM FORMAC computer program (Xenakis, 1969) and expressed in terms of the dimensionless relative velocity $\bar{v}_{\ell g} = (v^\ell - v^g)/C_m$ as

$$\hat{\lambda}(\bar{v}_{\ell g} + \hat{\lambda})[\hat{\lambda}^2 (\bar{v}_{\ell g} + \hat{\lambda})^2 - \hat{\lambda}^2 \frac{\rho_m}{\rho_\ell \rho_g} (\alpha^\ell \rho_g + \frac{P\alpha^g}{C_g^2}) - (\bar{v}_{\ell g} + \hat{\lambda})^2 \frac{\rho_m}{\rho_g \rho_\ell} (\alpha^g \rho_\ell + \frac{P\alpha^\ell}{C_\ell^2}) + \frac{P\rho_m}{\rho_g \rho_\ell C_m^2}] = 0 \tag{12a}$$

where ρ_m is the usual mixture density defined by Equation (16) and $\hat{\lambda} = (v^g + \lambda)/C_m$.

The natural scale factor, C_m, is the usual (Wood, 1946, Wallis, 1969) speed of sound in the mixture obtained for the homogeneous (equal phase velocity) model. This factor is derived in Appendix B for the case $v^g = v^\ell$ by the method of characteristics. The homogeneous speed of sound is distinctly different from the so-called homogeneous equilibrium speed of sound, derived similarly in Appendix C. The former sound speed is frequently referred to as the "frozen" sound speed, whereas the latter sound speed is usually referred to simply as the "equilibrium" sound speed for mixtures.

From Equation (12a), $\lambda = -v^g$ and $\lambda = -v^\ell$ are characteristics as they must be, since from Equation (6) the phase entropies S_g and S_ℓ propagate along with the phase velocities v^g and v^ℓ. Hence, Equation (6) is a compatibility condition for the invariants S_a along $dx/dt_a = -\lambda$. The propagation of S_a along v^a is analogous to the propagation of the single phase entropy, S, along with the fluid velocity, v.

The characteristic determinant obtained for the case of α^a outside the pressure gradient in the momentum equations is

$$\hat{\lambda}\,(\bar{v}_{\ell g} + \hat{\lambda})\,[\hat{\lambda}^2(\bar{v}_{\ell g} + \hat{\lambda})^2 - \hat{\lambda}^2 \frac{\alpha^\ell \rho_m}{\rho_\ell} - (\bar{v}_{\ell g} + \hat{\lambda})^2 \frac{\alpha^g \rho_m}{\rho_g}] = 0. \qquad (12b)$$

Of interest is the fact that Equation (12b) is Equation (12a) with P = 0. The quartic factor in this equation is the same characteristic determinant obtained by Boure (1973). For equal phase velocity ($\bar{v}_{\ell g} = 0$) in Equation (12b) for the case with α outside the pressure gradient, four real characteristics are obtained, two of which were reported as Equations 6.105 and 6.106 by Wallis (1969). The four characteristics are

$$\lambda = -v \pm C_s$$

and (13)

$$\lambda = -v^{\ell} = -v^{g} = -v$$

where C_s is the so-called stratified flow sound speed given by

$$C_s^2 = \frac{\rho_p \rho_m C_m^2}{(\rho_g \rho_\ell)} \tag{14}$$

with a pseudo density defined by

$$\rho_p = \rho_g \alpha^{\ell} + \rho_\ell \alpha^{g} \tag{15}$$

which contrasts with the mixture density given by

$$\rho_m = \rho_g \alpha^g + \rho_\ell \alpha^\ell \tag{16}$$

For the case of any finite nonzero relative velocity, and as shown in Figure 1a[a], two characteristics from Equation (12b) were found to possess complex conjugates for compressible subsonic flow, except for equal velocities and single phase flow. The regions of real and complex roots of the quartic factor in Equations (12a) and (12b) are indicated on the figure.

With the volume fractions inside the gradient, as in Equation (2), the characteristics are real in a large region of interest, as shown in Figure 1b. Therefore, this alternate set of equations satisfies the necessary condition for well-posed IVPs. However, Figure 2 shows that an imaginary region can appear in this case for equal phase velocities and high pressures for other values of phase densities and sound speeds. Harlow and Amsden (1975) consider the set of momentum equations with α^a inside the gradient of pressure to be unrealistic. Pai (1973) on the other hand considers the terms $P \frac{\partial \alpha^a}{\partial x}$ as "new" interaction terms "due to the pressure".

[a] Gidaspow (1974) showed that two of the characteristics are always complex for all nonzero relative velocities when the phases are both incompressible.

The accuracy of the computer program used to calculate the eigenvalues was checked by comparing the results to analytical expressions for the eigenvalues in simplified cases. The analytical solutions used are given in the following.

The boundary between the real and complex regions can be predicted analytically for the case of α^a inside the gradient for $v^\ell=v^g=0$. It is given by

$$P = \frac{2\rho_p C_p^4}{C_s^2}\left[1 \pm \sqrt{1-\left(\frac{C_s}{C_p}\right)^2}\,\right] - \rho_p C_p^2 \qquad (17)$$

where C_p is a pseudo-two-phase sound speed given by

$$C_p^{-2} = \alpha^g C_g^{-2} + \alpha^\ell C_\ell^{-2} \qquad (18)$$

For zero relative velocities Determinant (12a) gives

$$\sqrt{2}\,\hat{\lambda} = \pm\left\{\frac{\rho_m}{\rho_g\rho_\ell}\left[\left(\rho_p+\frac{P}{C_p^2}\right) \pm \sqrt{\left(\rho_p+\frac{P}{C_p^2}\right)^2 - \frac{4P\rho_p}{C_s^2}}\,\right]\right\}^{1/2}. \qquad (19)$$

and $\hat{\lambda}$ is real if

$$\frac{C_p}{C_s} \leq 1 \qquad (20)$$

otherwise as many as four complex roots may be obtained. Then the system of continuity and momentum equations will be elliptic. Equation (19) shows that λ is given by $\pm C_{fast}$ and $\pm C_{slow}$ where C_{fast} and C_{slow} are "fast" and "slow" sound speeds analogous to those found in magnetoacoustics (Jeffrey and Taniuti, 1964).

For single-phase flow, two additional limiting subcases of the general Determinant (12) can be obtained. The characteristics in this case are (in addition to the phase velocities)

Case I: $\alpha^g = 0$

$$\hat{\lambda} = -\bar{v}_{\ell g} \pm 1 \tag{21}$$

which implies that

$$\lambda = -v^{\ell} \pm c_{\ell} . \tag{21a}$$

The extraneous roots are

$$\hat{\lambda} = \pm \sqrt{\frac{P}{\rho_g \, c_{\ell}^2}} \tag{22}$$

which implies that

$$\lambda = -v^g \pm \sqrt{P/\rho_g} . \tag{22a}$$

Case II: $\alpha^{\ell} = 0$

$$\hat{\lambda} = \pm 1 \tag{23}$$

which implies that

$$\lambda = -v^g \pm c_g . \tag{23a}$$

The extraneous roots are

$$\hat{\lambda} = -\bar{v}_{\ell g} \pm \sqrt{\frac{P}{\rho_{\ell} \, c_g^2}} . \tag{24}$$

which implies that

$$\lambda = v^{\ell} \pm \sqrt{P/\rho_{\ell}}\,. \tag{24a}$$

4. CHARACTERISTICS ANALYSIS OF THE BASIC EQUATIONS WITH TRANSIENT FLOW FORCES ADDED

Addition of transient flow force terms to the momentum equations has been found to eliminate large portions of imaginary regions if the terms are of the proper form. The transient flow forces depend mainly on the relative acceleration of the two phases. The exact form of the relative acceleration is conjecture (Anderson and Jackson, 1967, Hinze, 1959, Murray, 1965, Tchen, 1947). Equation (2) may be rewritten as

$$\frac{\partial(\rho^a v^a)}{\partial t} + \frac{\partial(\rho^a v^a v^a)}{\partial x} + \frac{\partial(\alpha^a P)}{\partial x} = -\rho^a g + \bar{A}_{ab} D_{ab} (v^a - v^b) + \dot{m}^a \hat{v}^a + A_m \frac{d}{dt} (v^b - v^a) + F_x{}^a \tag{25}$$

to include a general form of this transient flow force. α^a has been included inside the gradient of pressure for the reasons given earlier.

Two forms of the relative acceleration $\frac{d}{dt}(v^b - v^a)$ are investigated here although more have been proposed in the literature (Anderson and Jackson, 1967, Mecredy and Hamilton, 1972, for example).

A form of the coefficient A_m was proposed by Zuber (1964). However, the form suggested by Mecredy and Hamilton (1972) was used throughout this study and is

$$A_m = \begin{cases} \frac{1}{2} \rho_\ell \alpha^g \dfrac{(1 + 2\alpha^g)}{\alpha^\ell} & 0 \le \alpha^g < \frac{1}{2} \\ \frac{1}{2} \rho_g \alpha^\ell \dfrac{(3 - 2\alpha^g)}{\alpha^g} & \frac{1}{2} \le \alpha^g \le 1 \end{cases} \tag{26}$$

The transition in Equation (26) was not given in Mecredy and Hamilton (1972) but is assumed here as reasonable. A_m is zero when α^g equals 0 or 1 and has its maximum value for α_g equal to 0.5 at which it jumps in value from ρ_ℓ to ρ_g.

One form of the relative acceleration proposed by Anderson and Jackson (1967) is

$$\frac{d}{dt}(v^\ell - v^g) = \frac{\partial v^\ell}{\partial t} + v^\ell \frac{\partial v^\ell}{\partial x} - \left(\frac{\partial v^g}{\partial t} + v^g \frac{\partial v^g}{\partial x}\right). \tag{27}$$

The characteristic determinant for the system of Equations (1), (3), (25), and (27) was evaluated by the FORMAC program. The factored quartic is, in dimensionless form,

$$\hat{\lambda}^2 (\bar{v}_{\ell g} + \hat{\lambda})^2 - \frac{\hat{\lambda}\,\rho_m}{\rho_g \rho_\ell} \frac{\left(\alpha^\ell \rho_g + \frac{P\alpha^g}{C_g^2} + \frac{A_m}{\alpha^g}\right)}{\text{DENOM}} - (\bar{v}_{\ell g} + \hat{\lambda})^2 \frac{\rho_m}{\rho_g \rho_\ell} \frac{\left(\alpha^g \rho_\ell + \frac{P\alpha^\ell}{C_\ell^2} + \frac{A_m}{\alpha^\ell}\right)}{\text{DENOM}} = \frac{-P\,\rho_m}{\rho_g \rho_\ell\, C_m^2\,(\text{DENOM})} \tag{28}$$

where

$$\text{DENOM} = 1 + \frac{A_m C_m^2 \rho_m}{\rho_g \rho_\ell} \left[\left(\frac{\alpha^g}{\alpha^\ell C_g^2} + \frac{\alpha^\ell}{\alpha^g C_\ell^2}\right) + \left(\frac{\rho_\ell}{\rho_g C_g^2} + \frac{\rho_g}{\rho_\ell C_\ell^2}\right)\right]$$

When $A_m = 0$, the previous determinant, Equation (12a), results and when in addition $P = 0$, Equation (12b) results. When $P = 0$, the characteristic determinant for the case of α^a outside the pressure gradient is obtained.

use of Equation (2[illegible]) for the relative acceleration eliminates the imaginary region shown in Figure 1 for $F_{[illegible]} = 1$.

Characteristics data corresponding to Figure 1 for [illegible] relative velocities and sound speeds are shown in Figure 2. The imaginary regions are smaller for Figure 2a than for Figure 1a. However, the imaginary region of Figure 2b is larger than that of Figure 1b. In addition, it is substantially shifted toward the zero relative velocity axis. Similar trends resulted for steam-water at 1000 and 14.7 psia.

The second form of the relative acceleration used in this study is ([illegible]),

$$\frac{d}{dt}\left(v^{\ell} - v^{g}\right) = \frac{\partial v^{\ell}}{\partial t} - \frac{\partial v^{g}}{\partial t} \qquad (28)$$

The characteristic determinant for Equations (1), (3), (28), and ([illegible]), was then evaluated by the FORMAC program. The quartic factor is:

$$(\lambda + v^{g})^2 (\lambda + v^{\ell})^2 \left[(\alpha^{g} \rho_{g} \alpha^{\ell} \rho_{\ell}) \left(\frac{\alpha^{g} \rho_{\ell}}{C_{\ell}^2} + \frac{\alpha^{\ell} \rho_{g}}{C_{g}^{2}} \right)\right]$$

$$-(\lambda + v^{g})^2 \left[(\alpha^{g} \rho_{g} \alpha^{\ell} \rho_{\ell}) \left(\frac{\alpha^{g} P}{C_{g}^2} + \alpha^{\ell} \rho_{g} \right)\right]$$

$$-(\lambda + v^{\ell})^2 \left[(\alpha^{g} \rho_{g} \alpha^{\ell} \rho_{\ell}) \left(\frac{\alpha^{\ell} P}{C_{\ell}^2} + \alpha^{g} \rho_{\ell} \right)\right]$$

$$+ A_m \lambda [(\lambda + v^g)^2 (\lambda + v^\ell) (\alpha^g \rho_g) \left(\frac{\alpha^g \rho_\ell}{C_g^2} + \frac{\alpha^\ell \rho_g}{C_\ell^2} \right) \tag{30}$$

$$+ (\lambda + v^g) (\lambda + v^\ell)^2 (\alpha^\ell \rho_\ell) \left(\frac{\alpha^g \rho_\ell}{C_\ell^2} + \frac{\alpha^\ell \rho_g}{C_g^{\ 2}} \right)$$

$$- (\lambda + v^g) (\alpha^\ell \rho_g \rho_\ell) - (\lambda + v^\ell) (\alpha^g \rho_g \rho_\ell)]$$

$$= -\alpha^g \rho_g \alpha^\ell \rho_\ell P .$$

Since the relative velocity is no longer a natural group, the characteristics maps were plotted using the actual velocity (v^ℓ with v^g as a parameter) rather than the dimensionless relative velocity. In order to conveniently note any improvement over the zero added mass cases, Figure 1 has been replotted dimensionally as Figure 4. The characteristics maps for $v^g = 0$ ft/sec, $3500 \geq v^\ell \geq 0$ ft/sec are plotted in Figure 5 for steam-water at 2200 psia. Use of the Mecredy and Hamilton form of the coefficient, A_m, together with Equation (29), results in the imaginary regions for both forms of the pressure force terms being shrunk, unlike the case for the relative acceleration of the form of Equation (27). Figure 5a which corresponds to Figure 4a shows that the real region extends nearly to zero liquid velocity for $\alpha^g = 0.5$ which is where the added mass has its maximum value. Figure 5b which corresponds to Figure 4b shows that the imaginary regions have broken up and shrunk to almost zero area. For $v^g = 2000$ ft/sec, no imaginary region exists from $0 \leq v^\ell \leq 3500$. For $v^g = -2000$ ft/sec, the imaginary region is larger than for $v^g = 0$, it is smaller than for no added mass, continuous, and no imaginary region exists for $v^\ell - v^g \leq 2300$ ft/sec.

Additional studies at 1000 and 14.696 psia led to the conclusion that use of the Mecredy and Hamilton form of the added mass coefficient A_m and relative acceleration given by Equation (29) is not sufficient to completely eliminate the region of complex characteristics shown in Figure 1 (or Figure 4). However, these additional studies show that if the coefficient is sufficiently large, the characteristics are made real everywhere. Realistic values for the coefficient requires evaluation of experimental data.

5. CHARACTERISTICS ANALYSIS FOR ANOTHER TWO-PHASE FLOW SYSTEM

Abbott (1966) extended the classical theory of one-dimensional long wave motions to a two-layer stratified fluid. He obtained a quartic very similar to those obtained in this paper. However, in his limited investigation he obtained and interpreted only real characteristics. Further study of his work showed that if he had investigated a larger range of parameters, he would have obtained complex roots as shown in Figure 6. This figure shows a plot of his results presented in his Figure 4.14. Also plotted on the graph are results for other values of velocities, u, for the lighter of the two fluids. The characteristics for the case of the fluids flowing countercurrently at one foot per second (velocity difference of two feet per second) are still real but when the lighter of the two fluids flows countercurrently at ten feet per second, two complex characteristics result as indicated in Figure 6 by the curve for $u = -10$ ft/sec which intersects the zero axis only twice.

6. STABILITY ANALYSIS OF A FULLY IMPLICIT NUMERICAL SCHEME

We will now illustrate by an example which shows that when the basic two-phase flow equation set is ill-posed as an initial-value problem because of the existence of complex characteristics, that a fully implicit numerical method which is unconditionally stable for hyperbolic systems becomes unstable for nonhyperbolic systems.

The centrally differenced completely implicit finite difference approximation to Equation (1) at node j is

$$A \frac{\left(\vec{U}_j^{n+1} - \vec{U}_j^{n}\right)}{\Delta t} + A_1 \frac{\left(\vec{U}_{j+\frac{1}{2}} - \vec{U}_{j-\frac{1}{2}}\right)^{n+1}}{\Delta x} = 0 . \tag{31}$$

The standard von-Neumann stability analysis for the amplification factor ξ results in an eigenvalue problem for determination of that factor. The observation that the determination of the characteristics of the partial differential equations involves solving a similar eigenvalue problem for the characteristics, λ, results in a relationship between ε and λ given by

$$\xi = \left(1 - \lambda\, i\, \frac{\Delta t}{\Delta x} \sin \omega\right)^{-1} . \tag{32}$$

If λ takes the form, $a \pm ib$, the magnitude of the growth factor is

$$|\xi| = \left[1 \pm 4b \left(\frac{\Delta t}{\Delta x}\right) \sin \omega + 4 \left(a^2 + b^2\right)\left(\frac{\Delta t}{\Delta x}\right)^2 \sin^2 \omega\right]^{-\frac{1}{2}} \tag{33}$$

where $\omega = \pi\Delta x/2L$ for a disturbance of wavelength 2L, where L is the characteristic length of the system.

From Equation (33) $|\xi|$ can be shown to be ≤ 1 whenever

$$\frac{\Delta t}{\Delta x} \geq \frac{\pm\ b}{(a^2 + b^2)\ \sin\omega} \ . \tag{34}$$

If λ is real, that is $b = 0$, the well known result that stability is unconditional is obtained. For the case $\lambda = a - ib$, even if the characteristic eigenvalues are complex, Equation (36) predicts time-step stability above some $(\frac{\Delta t}{\Delta x})$ for a given ω. Obviously, such a stability restriction does not allow convergence of the finite difference scheme to be tested by decreasing the time-step size. Since stability cannot be obtained for all ω, the numerical scheme is, therefore, defined to be unstable because "the initial data seldom have the required properties for stability and even if they do, round off errors are likely to perturb the calculation into a neighboring diverged situation" (Lax and Richtmyer, 1956).

Computations of $|\xi|$ were made with the complex λ from the basic equation set with α^a outside the pressure gradient for steam-water with $\omega = \pi\Delta x/2L$. Figures 7 and 8 show $|\xi|_{max}$ (the magnitude of the maximum time-step eigenvalue).resulting from the complex characteristics having negative b at 500 psia for a large range of relative velocities, $|v^\ell - v^g|$, and α^g, where α^g is the volume fraction of steam. For $v^\ell = v^g = 0$ and $\alpha^g = 0$ and 1, real characteristics result, $|\xi|_{max}$ becomes ≤ 1, and stability results.

Examination of Figures 7 and 8 leads to the following. The basic set of two phase flow equations with α^a outside the pressure gradient even though ill-posed as an initial-value problem can theoretically yield stable solutions if the time-step is large enough. The time-step required for time-step stability is a function of the relative velocity and volume fraction. This time-step size decreases for increasing relative velocity for fixed volume fractions. As the volume fraction increases at fixed relative velocity, this time-step increases. Accuracy of the numerical solutions might be very poor, however, because the time-steps are quite large. Clearly, a convergence study would be difficult to perform because $|\xi|$ always has a maximum value significancy greater than unity. Possibly, some transient calculations could be performed until significant error growth accumlates. For Δt of 0.1 to 2 msec needed to study fast transients, the error accumulates faster in the region of low volume fractions and high relative velocities. Similar results with smaller $|\xi|$ were observed at 1000 psia for the liquid-vapor water system.

7. A SAMPLE CALCULATION ILLUSTRATING ERROR GROWTH CAUSED BY COMPLEX CHARACTERISTICS

A sample calculation was made for the basic two-phase equation set with α^a outside the pressure gradient for the approximate conditions shown in Figures 7 and 8 to obtain an indication of the nature of the predicted instability. The calculations were made with the code UVUT (Unequal Velocity Unequal Temperature) under development by Aerojet Nuclear Company. The completely implicit equations were solved using a stable iteration scheme based on the modified ICE method developed at Los Alamos Scientific Laboratory (Harlow and Amsden, 1971).

A horizontal pipe 13.44 feet long is initially filled with a mixture of 80 volume percent steam and 20 volume percent water at 1000 psia. At $t = 0+$, one end of the pipe is opened to a reservoir maintained at 750 psia and the other end is opened to a reservoir maintained at the initial pressure, phase energies, and volume fraction. The momentum and mass transfer mechanisms (f_i in Wallis' book, 1969) were set equal to zero so that the equations solved had no sources or sinks. Nine finite difference nodes and a time-step size of 0.5 msec were used. The expected behavior for this sample problem would be a steady state situation having smoothly varying volume fraction, pressure, and velocity profiles. However, as Figures 9 and 10 show, massive instabilities develop in the profiles for pressure and volume fraction. At 68 msec, the pressure is actually below ambient pressure over most of the pipe length. Clearly, these instabilities are nonphysical in nature and are caused by significant error growth directly attributable to the complex characteristics because $|\xi|_{max}$ is much greater than unity for the relative velocities developed (greater than 2000 ft/sec).

Stable and accurate code calculations all the way to steady state were possible for the parabolic decrease in area for a liquid flowing downward in gravity dominated flow predictable from Bernoulli's equation as shown in Figure 11. These calculations compare favorably with the analytical solution for this case. No instabilities were noted because the predicted value of $|\xi|_{max}$ was only very slightly above unity at such low relative velocities. Steady state conditions were achieved before considerable error could accumulate. Other problems run with single phase steam or water have been run successfully with the same code with no signs of instability.

In order to show that the addition of terms which render the characteristics real could stabilize the computations, the following numerical experiment was performed.

The bubble expansion work terms found in Milne-Thompson, p 490 (1965) were generalized for phases flowing at unequal velocities as

$$C_\alpha \rho_m \, (v^g - v^\ell) \left[\frac{\partial \alpha^g}{\partial t} + \tfrac{1}{2} (v^g + v^\ell) \, \frac{\partial \alpha^g}{\partial x} \right] . \qquad (35)$$

Equation (35) with $C_\alpha = \frac{1}{2}$ was added to the right side of the vapor momentum equation and subtracted from the left side of the liquid momentum equation. Analysis of the characteristics showed that the addition of this term rendered the nonhyperbolic set hyperbolic over a large region of physical interest. When the sample problem was rerun, the results were stabilized as Figures 9 and 10 show.

Several other stabilizing terms were investigated. Criteria for such terms were that the terms render the characteristics real and the steady-state results for the gravity dominated flow sample problem agree closely with the steady-state analytical solution without the terms. Although the addition of Equation (35) to the phase momentum equations did not greatly affect this steady-state result, another term was found which not only had a greater region of real characteristics but which was numerically smaller. It is given by

$$C \frac{\alpha^{\ell}\alpha^{g}\rho_{\ell}\rho_{g}}{(\alpha^{\ell}\rho_{g}+\alpha^{g}\rho_{\ell})} (v^{g} - v^{\ell})^{2} \frac{\partial \alpha^{g}}{\partial x} \tag{36}$$

Equation (36) was added and subtracted from the right side of the momentum equations with C = 2. The existence of such a term has been hypothesized by Wallis (1969, p 135) to account for forces produced by concentration or void gradients. This term also stabilized the horizontal pipe sample problem.

8. CONCLUSIONS

Equation systems which describe transient one-dimensional two-phase flow have been classified using the method of characteristics. The analysis showed that most two-phase flow models proposed in the literature yield complex-valued characteristics in the practical regions of interest for the two-phase steam-water system.

Well-known mathematical theorems led to the conclusion that if a set of quasilinear first order partial differential equations has complex characteristics, it is ill-posed as an initial value problem[a]. This ill-posedness manifests itself as instability of the differential equations, with no sources or sinks, to disturbances of all wavelengths and as numerical instability for all finite difference schemes consistent with these differential equations. Apparently stable and even accurate computations can be performed using a completely implicit, centered difference numerical scheme (which must be iteration stable). The class of problems which can be solved cannot yet be predicted a priori.

Computer experiments verified predicted time-step instability for an ill-posed set of equations from the separated one-dimensional potential two-phase flow literature. Addition of physically motivated terms rendered the set hyperbolic (well-posed) as an initial-value problem and numerically stable.

[a] These theorems are summarized in Appendix A.

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APPENDIX A

SUMMARY OF MATHEMATICAL DEFINITIONS AND THEOREMS ON HYPERBOLICITY AND WELL-POSEDNESS

This appendix summarizes the definitions and theorems pertaining to hyperbolicity and well-posedness of systems of first order partial differential equations (FOPDEs). This summary is considered necessary because, although these theorems and definitions are well-known to mathematicians and mathematical scientists, they are not that well-known to the community of two-phase flow engineers and scientists. These theorems, definitions and additional clarifying comments from the mathematical literature show that real-valued characteristics are necessary but not sufficient to insure the well-posedness of FOPDEs. Lack of well-posedness caused by the existence of complex characteristics implies growth of disturbances of <u>all</u> wavelengths in the absence of stabilizing sources or sinks. The complex characteristics in addition cause numerical instabilities when these FOPDEs are solved by finite differences.

A-1. DEFINITION OF THE PROBLEM UNDER CONSIDERATION

This section defines the problem to which a solution is desired. This problem is the same problem solved by all thermal hydraulic codes.

A system of quasilinear FOPDEs

$$A \frac{\partial U}{\partial t} + \sum_{j=1}^{m} \tilde{A}_j \frac{\partial U}{\partial x_j} = \tilde{B} , \quad 1 \le j \le m \tag{A-1}$$

is under consideration here where the square n x n matrices A, $\tilde{A}_j$ and $\tilde{B}$ depend only on $(U, t, x_1, x_2, \ldots x_m)$ and U is the column vector of n dependent variables. If A is nonsingular, the system may be conveniently written as

$$\frac{\partial U}{\partial t} + \sum_{j=1}^{m} A_j \frac{\partial U}{\partial x_j} = B \tag{A-2a}$$

where

$$A_j = A^{-1} \tilde{A}_j \tag{A-2b}$$

and

$$B = A^{-1} \tilde{B} \tag{A-2c}$$

The initial-value problem under consideration is to find a solution of System (A-2a) in some region

$$a_i \le x_i \le b_i \qquad (1 \le i \le m)$$

$$t \ge 0$$

subject to the initial condition

$$U(0,\vec{x}) = G(\vec{x}) \,, \tag{A-3}$$

and the value of U prescribed on the boundaries

$$\left.\begin{array}{l} x_i = a_i \\ x_i = b_i \,. \end{array}\right. \quad (1 \le i \le m) \tag{A-4}$$

A-2. CHARACTERISTICS, WELL-POSEDNESS AND STABILITY OF THE DIFFERENTIAL EQUATIONS

This section will establish by definitions and theorems that unless the system of FOPDEs has real characteristics, the system is not well-posed as an initial-value problem.

Definition 1. (Lax, 1958)

The initial-value problem is said to be well-posed (or properly posed) if Equations (A-2), (A-3), and (A-4) have a unique solution for all sufficiently (say j times) differentiable data G (x).

Pal (1969) has pointed out that the B matrix is known not to influence convergence for any numerical procedure[a]. That matrix is dropped in all further discussions.

Theorem 1. (Lax, 1958)

The initial-value problem, Equations (A-2), (A-3), and (A-4) are well-posed if and only if all linear combinations $\sum A_j\mu_j$ of the coefficient matrices A_j with real coefficients μ_j have only real eigenvalues. The solution depends continuously on the data, that is, the value of the solution at any point is a continuous linear functional of the data.

Theorem 1 essentially defines complete hyperbolicity for a system of FOPDEs.[b]

Definition 2. (Pal, 1969 adapted from Richtmyer and Morton, 1967, and Lax, 1974).

System (A-2a) is hyperbolic if all linear combinations $\sum A_j\mu_j$ of the coefficient matrices with real numbers μ_j have only real eigenvalues

[a] Lower order terms can influence well-posedness, however.

[b] Distinct characteristics (strict hyperbolicity) are sufficient for well-posedness, but not necessary (Lax, 1974).

$\lambda_1, \lambda_2 \dots \lambda_n$ and n linearly independent eigenvectors so that a nonsingular matrix $T(\mu)$ exists such that

$$T(\Sigma A_j\mu_j)T^{-1} = \Lambda = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & 0 \\ 0 & & \ddots & \\ & & & \lambda_n \end{pmatrix} \tag{A-5}$$

is symmetric and T depends smoothly on μ. Sedney (1970) has stated that if the system is not completely hyperbolic many possibilities of system classification exist and that not all types have even been given names.

Garabedian (1964) gives a heuristic motivation for hyperbolic classification by considering the case for which the A_j matrices are evaluated at some reference condition. In this manner a separation of variables solution yields

$$U = U_o e^{i(\mu_1 x_1 + \dots \mu_m x_m)} e^{i\lambda t} , \tag{A-6}$$

where $\mu_1, \dots \mu_m, \lambda$ are such that

$$\left| \lambda + \sum_{j=1}^{m} \mu_j A_j \right| = 0 , \tag{A-7}$$

and U_o is a column vector (nontrivial) with the property

$$\lambda U_o + \Sigma A_j \mu_j U_o = 0 \tag{A-8}$$

Garabedian further states that when $\mu_1, \dots \mu_m$, and λ are real, Equation (A-6) is the general term in a Fourier expansion. He then asks that for all real choices of the parameters $\mu_1, \dots \mu_m$, every root λ of the characteristic Equation (A-7) be real too. Then an oscillatory dependence of

Equation (A-6) on the space variables x_1, ... x_m results in a function U which neither grows nor decays exponentially with time t. These eigenvalues also satisfy

$$|T \Sigma A_j \mu_j T^{-1} - \lambda| = 0 \quad , \qquad \text{(A-9)}$$

which is the determinant of Equation (A-5). All λ real defines hyperbolicity. All λ distinct defines strict hyperbolicity. Ill-posedness (complex λ) would, therefore, imply the exponential time growth of perturbations introduced at zero time even in the absence of sources and sinks.

A-3. CHARACTERISTICS AND STABILITY OF THE FINITE DIFFERENCE EQUATIONS

This section shows that the initial-value problem for the FOPDEs must be well-posed in order for general explicit numerical schemes to be stable in the Von Neumann sense.

A general explicit finite difference form of Equation (A-2) is given by

$$U^{n+1}(x) = \Sigma\, C_j\, U^n\, (x + \ell_j) \quad . \tag{A-10}$$

Lax's Equivalence Theorem (Lax and Richtmyer, 1956)

Given the properly posed initial-value problem (Equations A-2, A-3, and A-4) and a finite difference approximation to it that satisfies the consistency condition, stability is a necessary and sufficient condition that it be a convergent approximation.

Lax's Condition[a] (Lax, 1958)

If all linear combinations $\sum \mu_j C_j$ of the coefficient matrices C_j in a difference scheme have only real eigenvalues and if each C_j is nonnegative, that is, all eigenvalues of the C_j are nonnegative, then the finite difference scheme is convergent.

Pal (1969) points out that in most difference schemes, the coefficient matrices C_j are linear combinations of the coefficient matrices[b] of the differential system, and so for a hyperbolic system, the first condition is satisfied. The second condition is achieved by restricting the bounds on $\Delta t/\Delta x_i$. By Lax's equivalence theorem, then, the difference scheme is stable.

[a] This condition is due to Friedrichs when the C_j are symmetric. Lax showed this condition implies the Von Neumann condition (Lax, 1974).

[b] Lax believes this statement is not generally true, for example, the Lax-Wendroff scheme is nonlinear in the A matrix (Lax, 1974).

A-4. SOLVABILITY OF ILL-POSED PROBLEMS

The theorems cited in Sections A-2 and A-3 assure that real characteristics result in hyperbolic equations which will yield stable, unique solutions when the equations are solved by consistent numerical methods. But what happens when some of the characteristics become complex warrants investigation. John (1955) states that typically for improper problems the solution does not depend continuously on the data (Lax and Richtmyer, 1956). An illustration is given by the following. If some of the characteristics are complex, they occur as complex conjugates of the form

$$\lambda = a \pm bi \ . \tag{A-11}$$

Then Equation (A-6) assumes the form

$$U = U_o \, \ell^{i(at + \mu_1 x_1 + \mu_m x_m)} \, \ell^{\pm bt} \tag{A-12}$$

The characteristic root with imaginary part, $\lambda = a - bi$, would cause growth in time on the order $U_o \, \ell^{bt}$ just as Garabedian stated. This growth would occur for disturbances of all wavelengths. If all the characteristics are complex, then the equations must be solved as a boundary value problem (Courant and Lax, 1949). If some of these characteristics are complex and some are real, then as many data are prescribed on the initial curve as there are real characteristics and as many data on the whole boundary as there are complex conjugate pairs (Courant and Lax, 1949). In either case, information must be supplied at some future time. This prescription is clearly physically impossible and is conjectured to possibly even violate the Second Law of Thermodynamics.

The literature devoted to solving other ill-posed initial-value problems for Laplace's or Chaplygin's equation offers some other possible approaches (Payne, 1960, Schaefer, 1967, and 1973, Payne and Sather 1967, and 1967a for example), to render them well-posed. These solutions include:

a) Restricting the class of initial data,

b) Imposing hypotheses on general coefficients in the differential equations,

c) Removing the unstable or physically unacceptable solutions from the solution space.

The restrictions on the initial data may not allow physical data to be admissible according to John (1955). Imposing hypotheses on the equation coefficients may result in problems that have no physical meaning according to Payne and Sather (1967). The solution space might possibly not contain the solution to the problem under consideration. The only resolution appears to be that the original set of equations be restructured to hyperbolic ones. This restructuring is necessary from a computational standpoint because Richtmyer and Morten (1967) show that "if the initial-value problem is improperly posed... then no difference scheme that is consistent with the problem can be stable".

APPENDIX B: DERIVATION OF HOMOGENEOUS MIXTURE SOUND SPEED USING METHOD OF CHARACTERISTICS

In the homogeneous model, the phases are assumed to move at the same velocity that is v^{ℓ} v^{g} = v. Summation of Equation (2) over index a yields the overall momentum equation as

$$\rho_m \frac{dv}{dt} + \frac{\partial P}{\partial x} = 0 \tag{B-1}$$

with friction, mass transfer, and body forces neglected, where $\frac{d}{dt}$ denotes the convective derivative, and where ρ_m, the mixture density, is given by

$$\rho_m = \rho^g + \rho^\ell = \alpha^g \rho_g + \alpha^\ell \rho_\ell \tag{B-2}$$

The continuity equation for phase g with no interphase heat or mass transfer can be written in convective form by use of the definition of sound given by Equation (8) as

$$\frac{d\alpha^g}{dt} + \frac{\alpha^g}{\rho_g C_g^2} \frac{dP}{dt} + \alpha^g \frac{\partial v}{\partial x} = 0 \tag{B-3}$$

Use of the relation

$$\alpha^g + \alpha^\ell = 1 \tag{B-4}$$

in the second continuity equation results in

$$-\frac{d\alpha^g}{dt} + \frac{\alpha^\ell}{\rho_\ell C_\ell^2} \frac{dP}{dt} + \alpha^\ell \frac{\partial v}{\partial x} = 0 \tag{B-5}$$

Equations (B-1), (B-3), and (B-5) in matrix form are

$$\begin{bmatrix} \rho_g & \frac{\alpha^g}{C_g^2} & 0 \\ -\rho_\ell & \frac{\alpha^\ell}{C_\ell^2} & 0 \\ 0 & 0 & \rho_m \end{bmatrix} \begin{bmatrix} \frac{\partial \alpha^g}{\partial t} \\ \frac{\partial P}{\partial t} \\ \frac{\partial v}{\partial t} \end{bmatrix} + \begin{bmatrix} \rho_g v & \frac{\alpha^g v}{C_g^2} & \rho^g \\ -\rho_\ell v & \frac{\alpha^\ell v}{C_\ell^2} & \rho^\ell \\ 0 & 1 & \rho_m v \end{bmatrix} \begin{bmatrix} \frac{\partial \alpha^g}{\partial x} \\ \frac{\partial P}{\partial x} \\ \frac{\partial v}{\partial x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{(B-6)}$$

Equation (B-6) is in the standard form

$$A \frac{\partial U}{\partial t} + B \frac{\partial U}{\partial x} = 0 \quad \text{(B-7)}$$

The characteristic determinant

$$| A \lambda + B | = 0 \quad \text{(B-8)}$$

for Equation (B-6) is

$$(v + \lambda)^3 - (v + \lambda) C_m^2 = 0 \quad \text{(B-9)}$$

where the group C_m is the usual sound speed for a homogeneous mixture of isentropic phases rather than an isentropic mixture of phases at equal temperature. The sound speed, C_m, is given by

$$\frac{1}{C_m^2} = \rho_m \left(\frac{\alpha^g}{\rho_g C_g^2} + \frac{\alpha^\ell}{\rho_\ell C_\ell^2} \right) \quad \text{(B-10)}$$

Equation (B-9) shows that the characteristics for the homogeneous system are

$$\lambda = -v \quad \text{and} \quad \lambda = -v \pm C_m \tag{B-11}$$

analogous to the well-known problem in gas dynamics (von Mises, 1958).

A partial differential equation for the pressure can be obtained as follows. Differentiation of Equation (B-1) with respect to x and a reversal of cross partials yields:

$$\frac{d}{dt} \left(\frac{\partial v}{\partial x} \right) = - \frac{\partial}{\partial x} \left(\frac{1}{\rho_m} \frac{\partial P}{\partial x} \right) \tag{B-12}$$

Summation of continuity Equations (B-3) and (B-5) gives

$$\sum_a \frac{\alpha^a}{\rho_a C_a^2} \frac{dP}{dt} + \frac{\partial v}{\partial x} = 0. \tag{B-13}$$

Differentiation of Equation (B-13) with respect to the time gives

$$\frac{d}{dt} \left(\frac{\partial v}{\partial x} \right) = - \frac{d}{dt} \left(\sum_a \frac{\alpha^a}{\rho_a C_a^2} \frac{dP}{dt} \right) \tag{B-14}$$

Since the left hand sides of Equations (B-12) and (B-14) are the same

$$\frac{\partial}{\partial x} \left(\frac{1}{\rho_m} \frac{\partial P}{\partial x} \right) = \frac{d}{dt} \left(\sum_a \frac{\alpha^a}{\rho_a C_a^2} \frac{dP}{dt} \right). \tag{B-15}$$

In acoustics the convective derivative

$$\frac{d}{dt} = \frac{\partial}{\partial t} + v \frac{\partial}{\partial x} \tag{B-16}$$

becomes a partial with respect to time only after average and fluctuating quantities are substituted. Equation (B-15) becomes

$$\frac{\partial}{\partial x} (\rho_m \frac{\partial P}{\partial x}) = \frac{\partial}{\partial t} (\sum_a \frac{\alpha^a}{\rho_a C_a^2} \frac{\partial P}{\partial t}) \tag{B-17}$$

Clearly, the pressure propagates with the mixture sound speed velocity C_m as defined by Equation (B-10). The value of this quantity has been verified experimentally for bubbly flow (Karplus, 1961, Henry, Grolmes and Fauske, 1969). In general the individual sound speeds in C_m are evaluated at constant entropies. However, for a nearly isothermal mixture, as in many experimental investigations, they should probably be evaluated at the constant temperature of the experiment, because heat conduction through the liquid is fast.

APPENDIX C. HOMOGENEOUS EQUILIBRIUM MODEL

The homogeneous sound speed was derived in Appendix B. The equilibrium homogeneous model is the most common one used in modeling two-phase, single component flow. It is used, for example, in the RELAP4 computer program for transient thermal-hydraulic analysis (Moore and Rettig, 1973). However, the derivation of two-phase sound speeds and characteristics usually involves laborious calculations and unnecessary simplifying assumptions. For example, Fischer and Häfele (1967) assume the liquid to be incompressible and neglect the volume of the gas compared to that of the liquid. This assumption is not good for pressurized water reactor safety analysis. Siegmann (1971) deletes work terms in the energy equation. To present a rigorous simple derivation is therefore useful.

The conservation equations with no friction or external heat input, with phases moving at the same velocity, v, are:

Continuity:

$$\frac{\partial \rho_m}{\partial t} + \frac{\partial(\rho_m v)}{\partial x} = 0 \qquad \text{(C-1)}$$

Momentum:

$$\frac{\partial(\rho_m v)}{\partial t} + \frac{\partial(\rho_m vv)}{\partial x} + \frac{\partial P}{\partial x} = 0 \qquad \text{(C-2)}$$

Energy:

$$\frac{\partial(\rho_m S_m)}{\partial t} + \frac{\partial(\rho_m v S_m)}{\partial x} = 0 \qquad \text{(C-3)}$$

where the mixture density is

$$\rho_m = \alpha^\ell \rho_\ell + \alpha^g \rho_g \qquad \text{(C-4)}$$

and where a mean mixture entropy can be defined by

$$\rho_m S_m = \alpha^\ell \rho_\ell S_\ell + \alpha^g \rho_g S_g \qquad \text{(C-5)}$$

This definition of mean mixture entropy leads to the important result that

$$\frac{dS_m}{dt} = 0 \tag{C-6}$$

which is obtained by combining the energy equation with the continuity equation.

Along the two-phase envelope the density of the liquid and of the gas is a function of pressure only. Thus, ρ_m in Equation (C-4) is a function of pressure and the void fraction α^g. But Equation (C-5) fixes α^g. Therefore,

$$\rho_m = \rho_m (P, S_m) \tag{C-7}$$

or if the inverse function is assumed to exist

$$P = P (\rho_m, S_m). \tag{C-8}$$

Therefore, in Equation (C-2), the pressure gradient can be written as

$$\left(\frac{\partial P}{\partial x}\right)_{S_m} = \left(\frac{\partial P}{\partial \rho_m}\right)_{S_m} \frac{\partial \rho_m}{\partial x} \tag{C-9}$$

A lengthy thermodynamic evaluation given by Bridgman (1961), page 223 or Landau and Lifshitz (1959), p 248 shows that

$$\left(\frac{\partial P}{\partial \rho_m}\right)_{S_m} = \frac{TP'^2}{\rho_m^{\,2} C_v} \tag{C-10}$$

where

$$P' = \frac{dP}{dT} = \frac{1}{T} \frac{h_g - h_\ell}{V_g - V_\ell} \tag{C-11}$$

T is the temperature, C_v is the heat capacity of the mixture at constant volume, h_g and h_ℓ are the enthalpies and V_g and V_ℓ are the specific volumes of the gas and the liquid, respectively. The specific heat in Equation (C-10) is always a positive quantity according to the second law of thermostatics. Its expression is given in the references cited.

The system of equations (C-1) and (C-2) can; therefore, be written as

$$\begin{bmatrix} \frac{\partial \rho_m}{\partial t} \\ \frac{\partial v}{\partial t} \end{bmatrix} + \begin{bmatrix} v & \rho_m \\ \frac{1}{\rho_m}\left(\frac{\partial P}{\partial \rho_m}\right)_{S_m} & v \end{bmatrix} \begin{bmatrix} \frac{\partial \rho_m}{\partial x} \\ \frac{\partial v}{\partial x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{C-12}$$

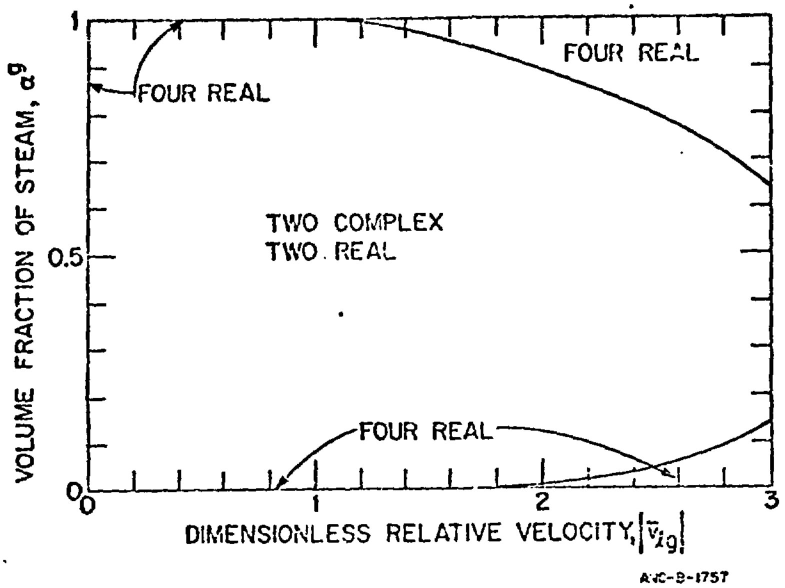
The characteristic determinant is simply

$$\begin{vmatrix} v + \lambda & \rho_m \\ \frac{1}{\rho_m}\left(\frac{\partial P}{\partial \rho_m}\right)_{S_m} & v + \lambda \end{vmatrix} = 0 \tag{C-13}$$

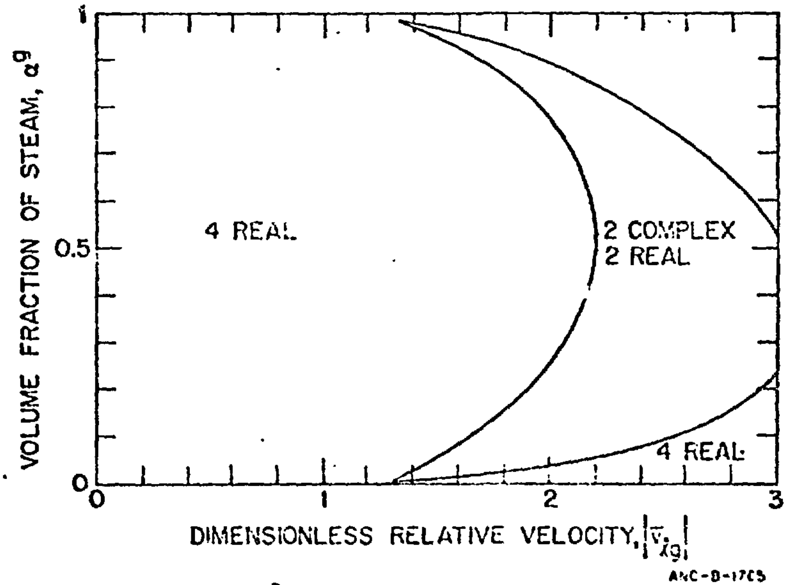
The characteristic directions are then given by

$$\lambda = -v \text{ and } -v \pm \sqrt{\left(\frac{\partial P}{\partial \rho_m}\right)_{S_m}} \tag{C-14}$$

Therefore, a complete analogue exists between gas dynamics and the equilibrium homogeneous flow. Also, the equilibrium sound speed is defined even if both phases are incompressible.



(a) α^a Outside the Pressure Gradient



(b) α^a Inside the Pressure Gradient

Figure 1

Characteristics maps for various relative velocities for steam-water. Pressure = 2200 psia, ρ_g = 3.332 lb_m/cu ft, ρ_ℓ = 47.529 lb_m/cu ft, C_g = 1988 ft/sec, C_ℓ = 3478 ft/sec, steam temperature = 861.7°F, water temperature = 538.9°F.

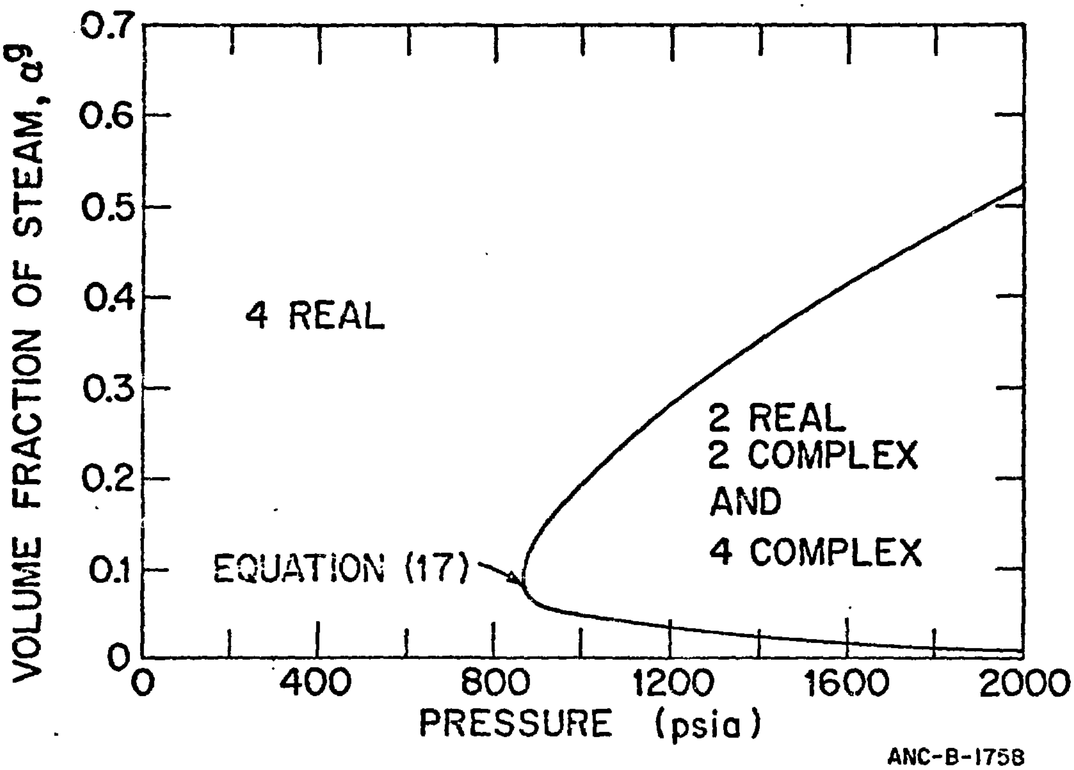
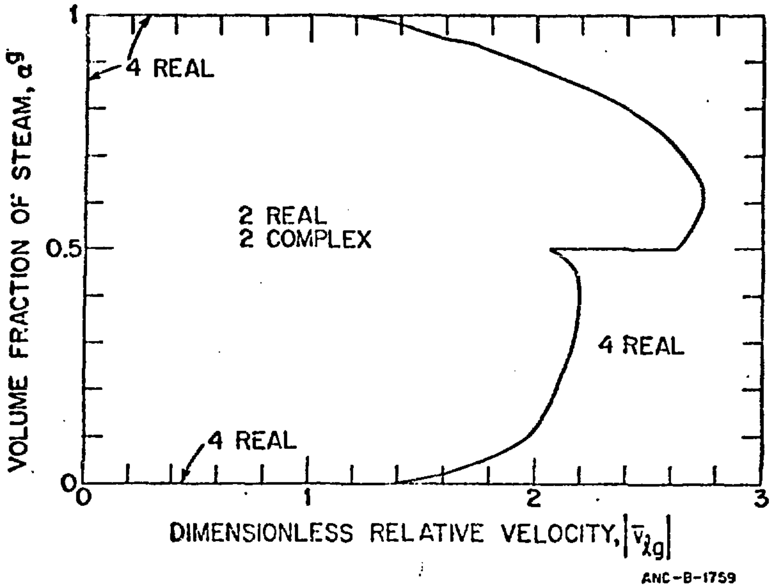
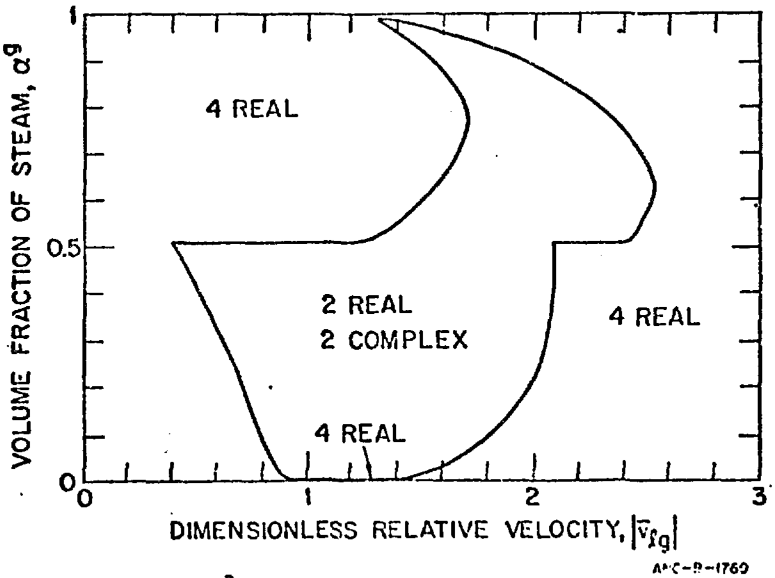


Figure 2

Characteristics map for various pressures for α^a inside the pressure gradient. $v^g = v^\ell = 0$. $\rho_g = 4.0\ lb_m/cu\ ft$, $\rho_\ell = 40.0\ lb_m/cu\ ft$, $C_g = 1000$ ft/sec, $C_\ell = 3500$ ft/sec.



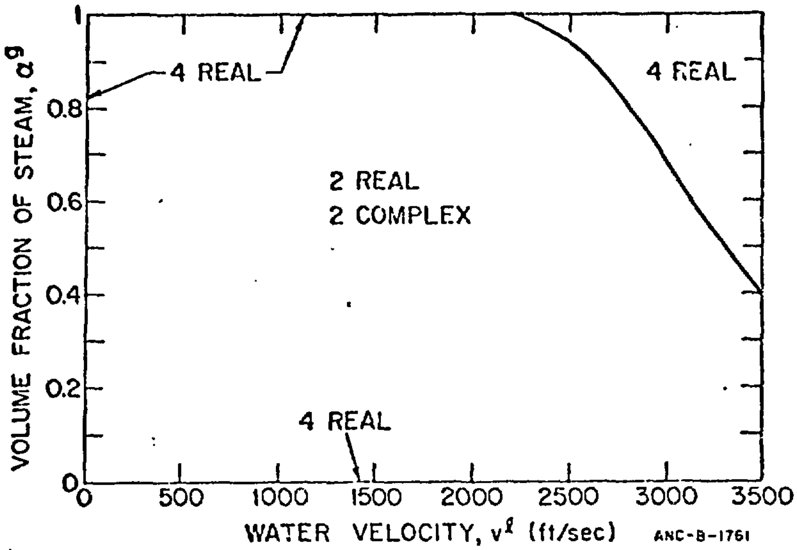
(a) α^a Outside the Pressure Gradient



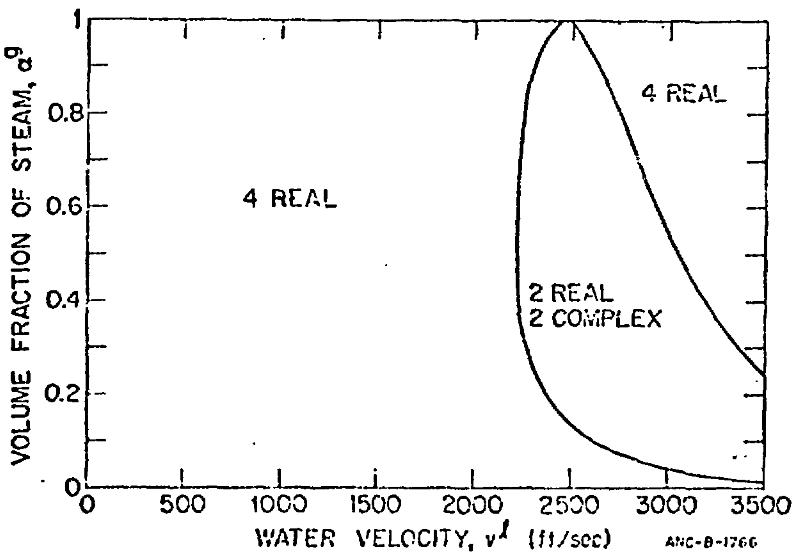
(b) α^a Inside the Pressure Gradient

Figure 3

Characteristics maps for various relative velocities for steam-water. Same conditions as Fig. 1 with Hamilton and Mecredy's form of the coefficient A_m and Equation (27) for relative acceleration.



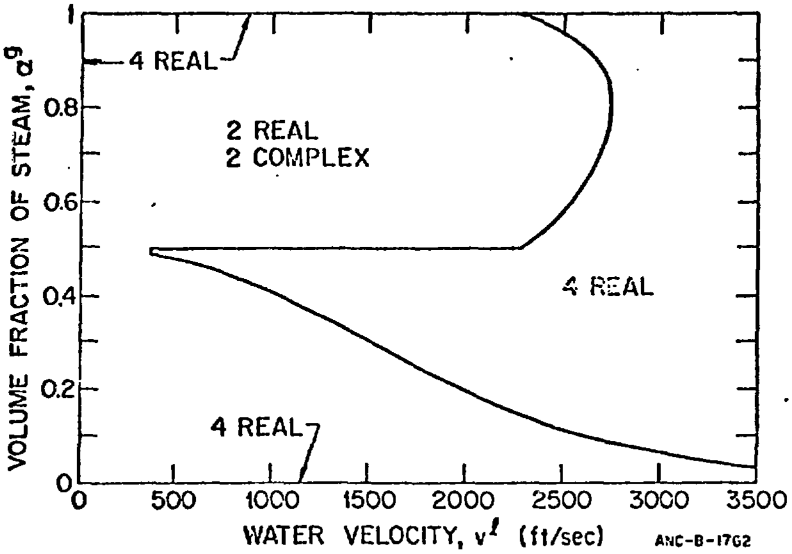
(a) α^a Outside the Pressure Gradient



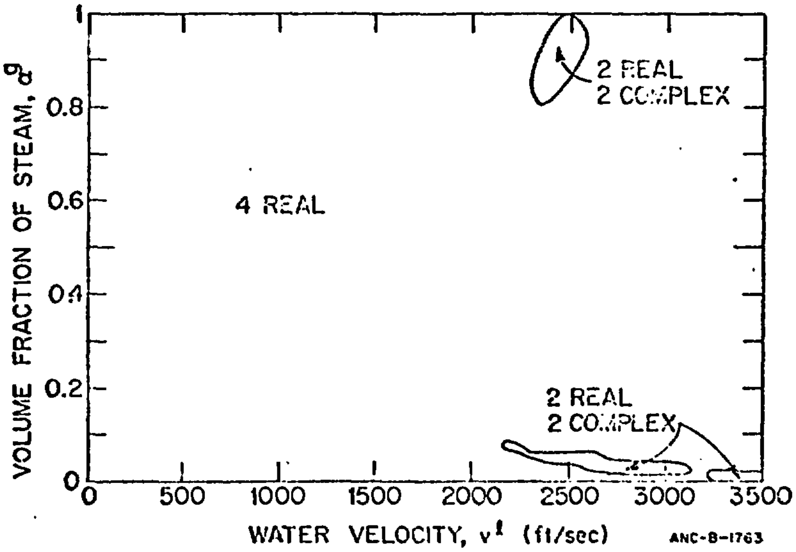
(b) α^a Inside the Pressure Gradient

Figure 4

Characteristics maps for various relative velocities for steam-water. Fig. 1 replotted using dimensional liquid velocity as a variable. v^g = 0. ft/sec.



(a) α^a Outside the Pressure Gradient



(b) α^a Inside the Pressure Gradient

Figure 5

Characteristics maps for various water velocities. Steam velocity = 0.0 ft/sec. Same conditions as Fig. 1 with Hamilton and Mecredy's form of the coefficient A_m and Equation (29) for relative acceleration.

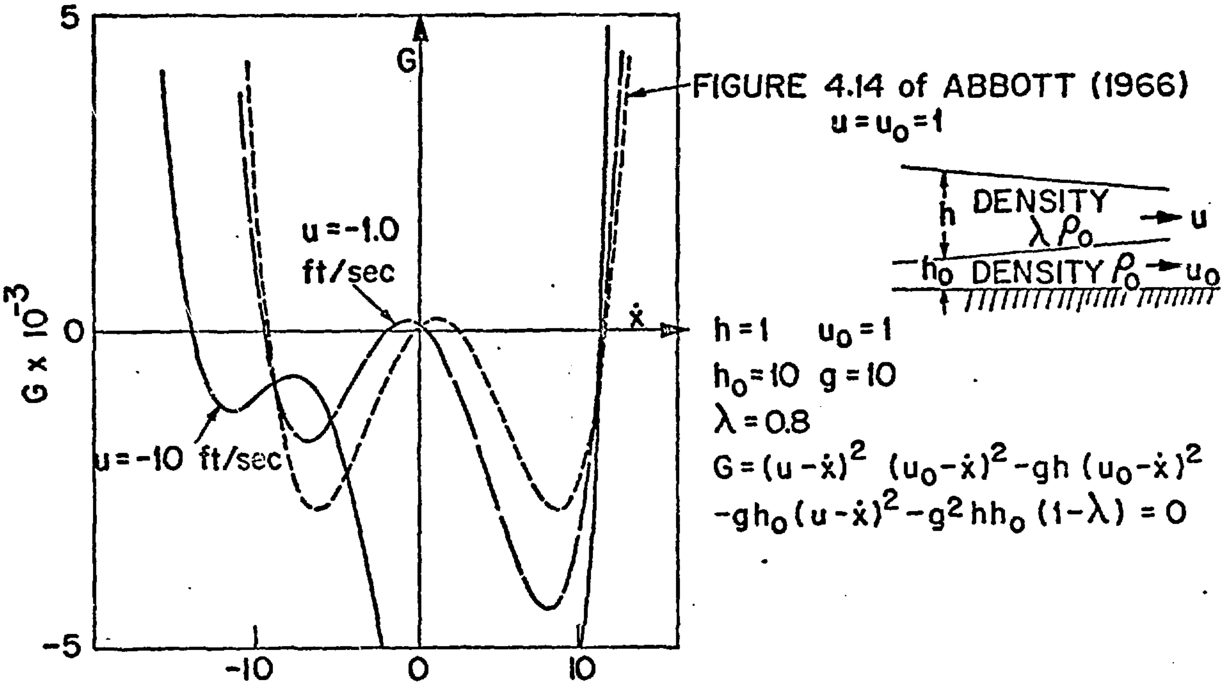


Figure 6

Characteristics for one-dimensional long waves in a stratified fluid. Plot of Equation 4.57 of Abbott (1966).

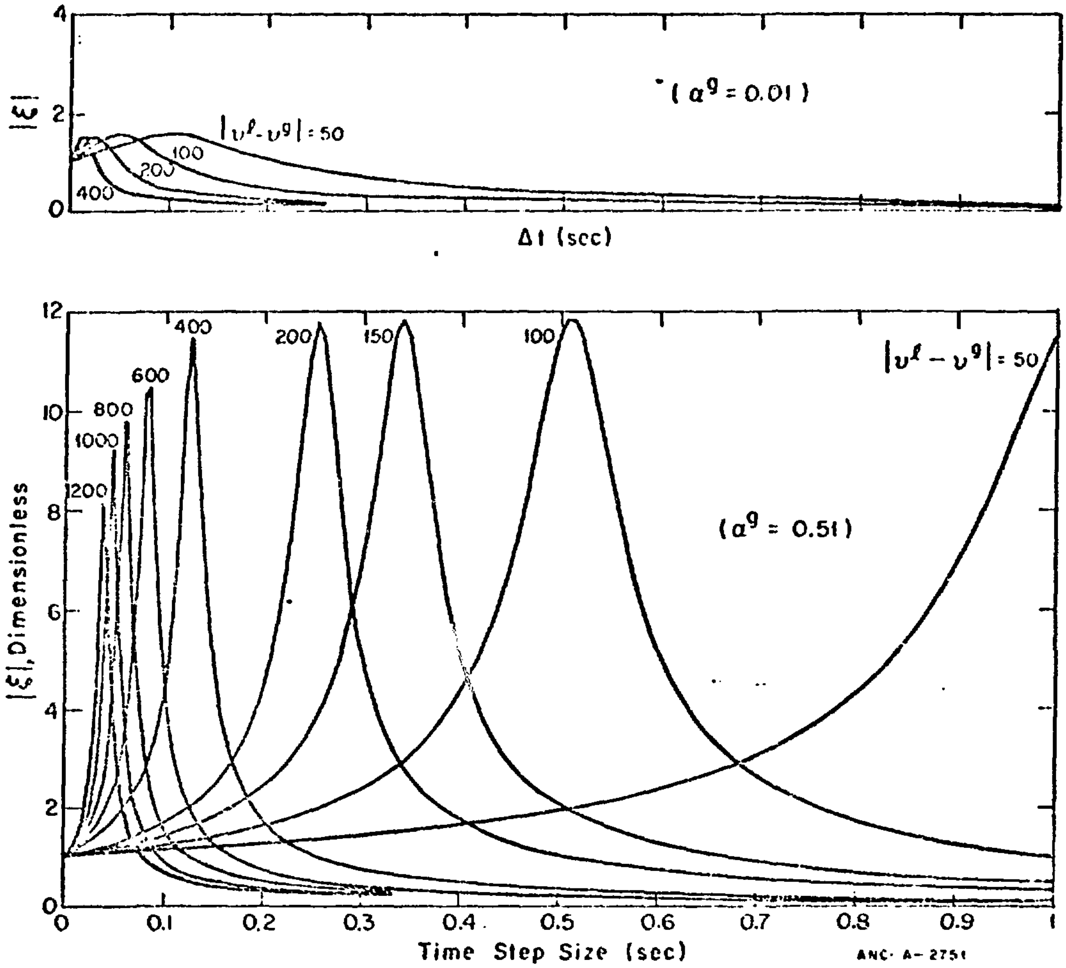


Figure 7

Maximum time step eigenvalue as a function of relative velocity at 500 psia, Δx = 1.4933 ft., $\Delta x/L = 1/9$, T_ℓ = 99.8°F, T_g = 1386.°F.

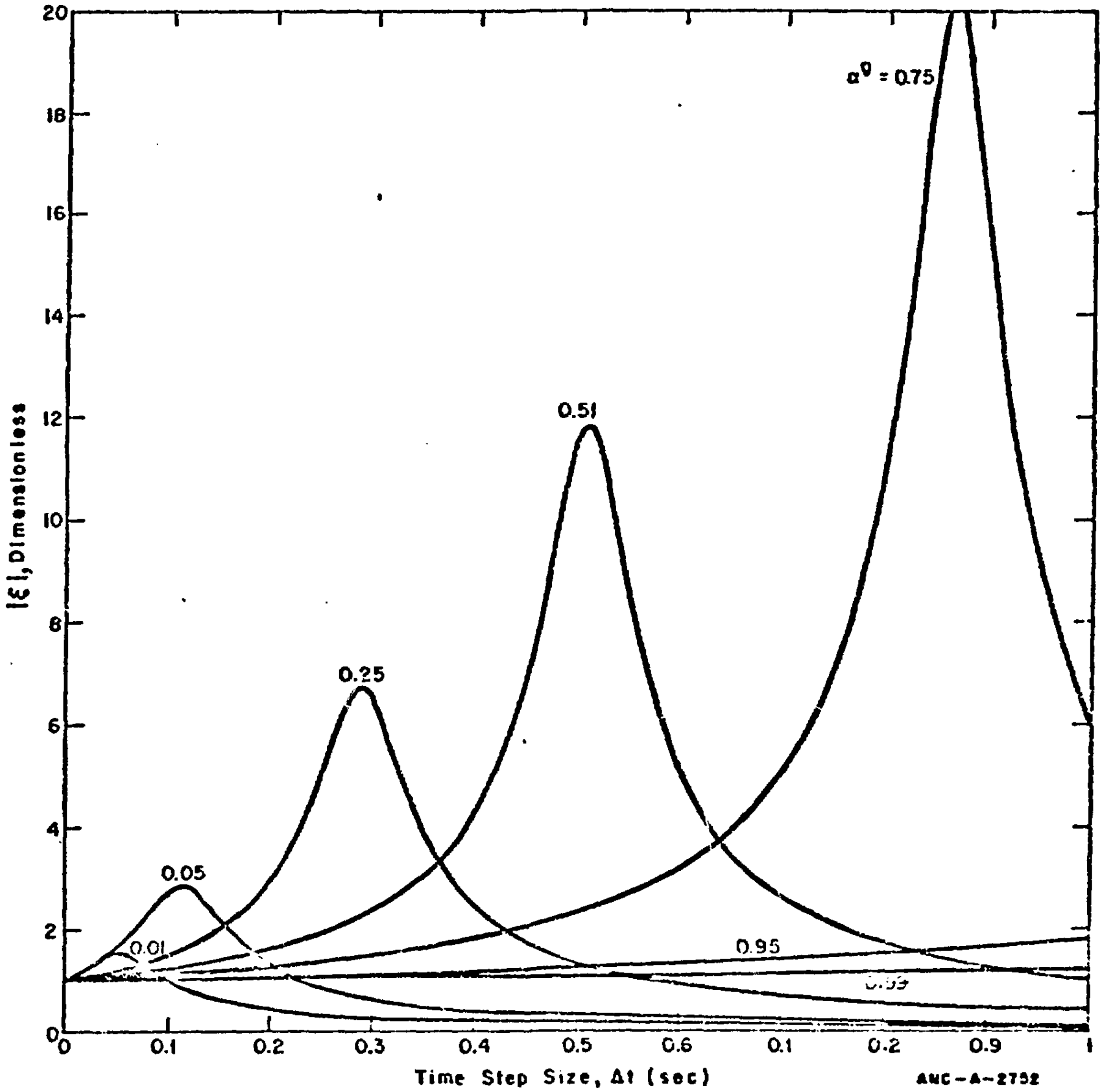


Figure 8

Magnitude of time step eigenvalues as a function of steam volume fraction at 500 psia. Δx = 1.4933 ft, $\Delta x/L$ = 1/9, T_ℓ = 99.3°F, T_g = 1386.2°F, $|v^\ell - v^g|$=100 ft/sec.

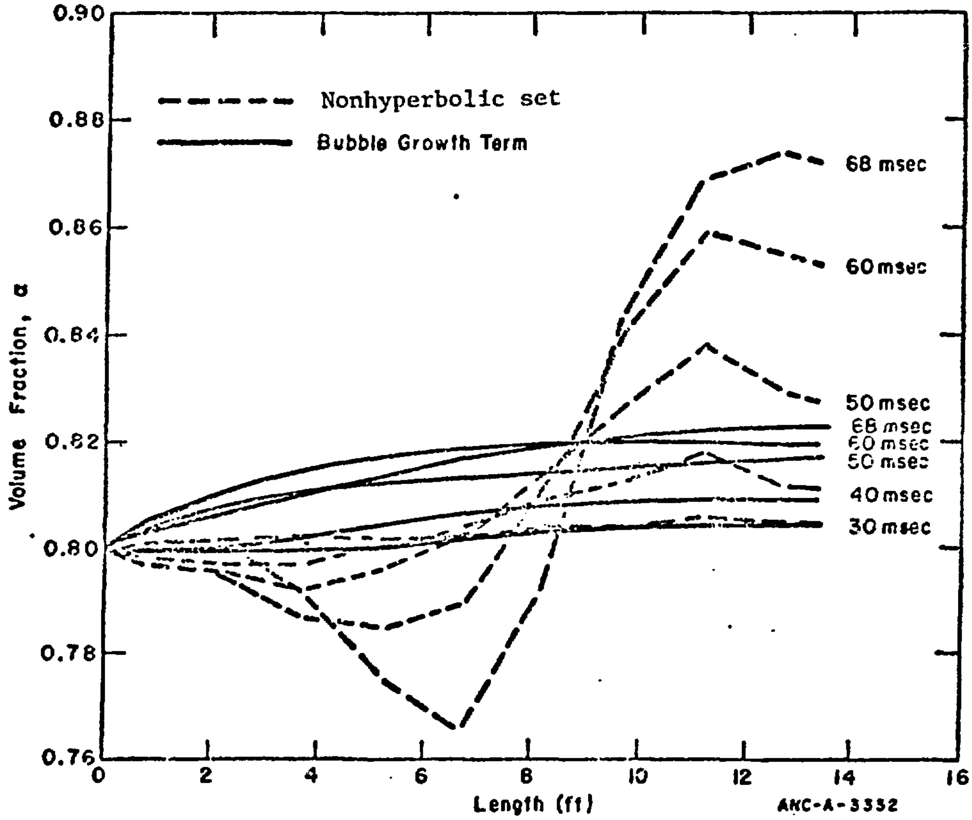


Figure 9

Two-phase open pipe blowdown - comparison of volume fraction transients for $C_\alpha=0$ and $C_\alpha=1/2$. Initial volume fraction = 0.8, ΔP = 250 psia, Δt = 0.5 msec, 9 volumes.

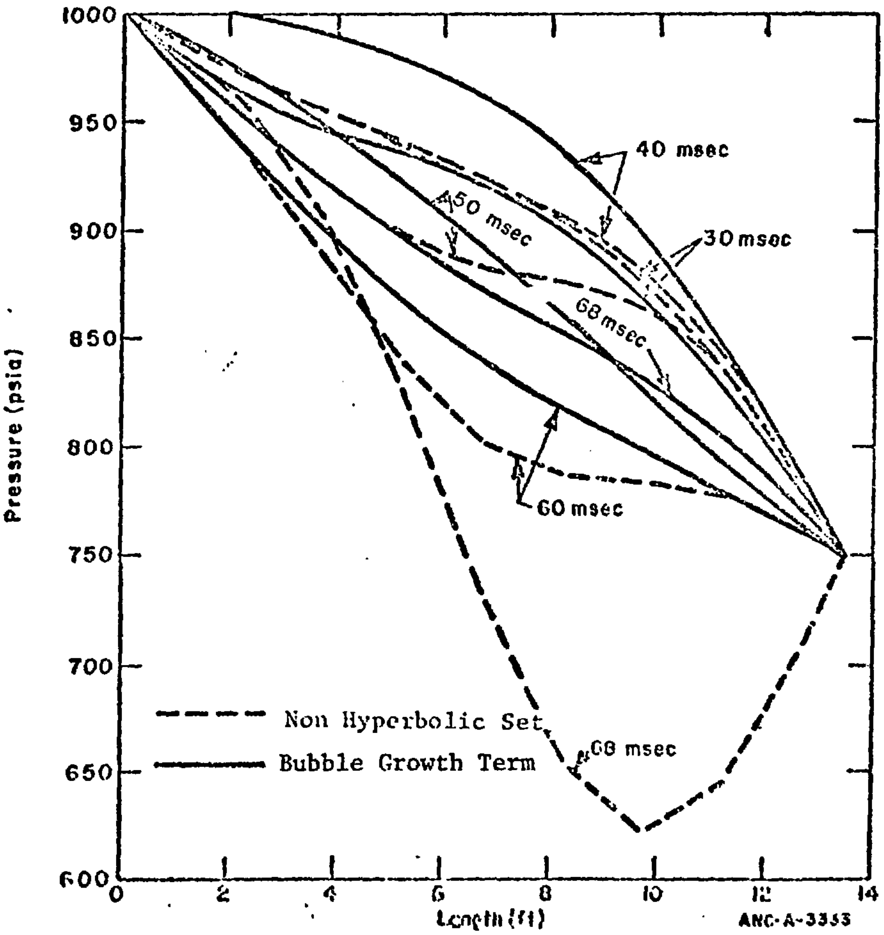


Figure 10

Two-Phase open pipe blowdown - comparison of volume fraction transients computed for ill-posed and well-posed equations. Initial volume fraction = 0.8, ΔP = 250 psia, Δt = 0.5 msec, 9 volumes.

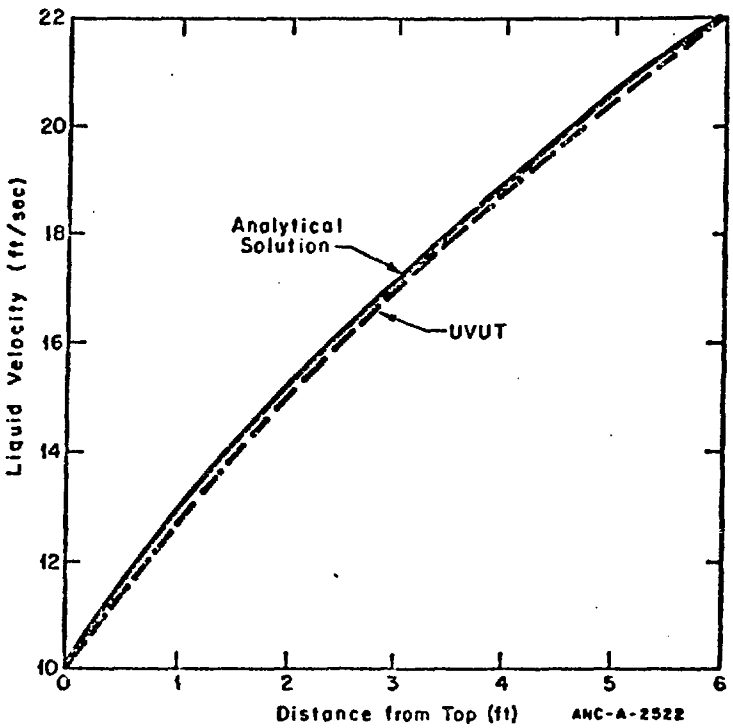


Figure 11

Comparison of steady state analytical solution with calculations of UVUT code for well-posed equations. Necking down of a water column in gravity dominated flow.