COUPLED RESONATOR MODEL OF LINEAR ACCELERATOR TANKS

D. E. Nagle
Los Alamos Scientific Laboratory, University of California, Los Alamos, New Mexico

I. Introduction

This paper will review the coupled resonator model of the linear accelerator resonant tanks, in the form in which we have applied it for understanding our experimental results, both with the computers and with experimental models. The present form of the model has grown out of discussions with Ed Knapp, Bruce Knapp, and others. The steady state results will be discussed by E. Knapp in the following paper. R. Jameson in a talk to be given later in the week will discuss his theory and measurements of the transient behavior of the tanks and the design of a servo control system for rf amplitude and phase. In his talk you will see that many purely algebraic properties connected with the symmetry of the system are common to the transient and to the steady state theory.

The dispersion relations for the simple and for the doubly periodic ring or chain of coupled oscillators are familiar results of lattice vibration theory; see for example Brillouin's book.¹

Some results are perhaps novel, particularly those concerned with phase shifts, effects of perturbations, and locking phenomena. Also, the discussion of the relation between ring and chain has been used extensively in our work.


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Although pictorially we often refer to the familiar lumped circuits, which suggests a severe limitation on generality, this is only a convenience. The important equations refer to the resonant frequencies, their widths, amplitudes, and phases; and to the strength of coupling, quantities which at these frequencies are directly or indirectly easily measurable. The approach is useful so long as it is possible to describe the behavior using separated modes of a single cavity as they develop into bands of the coupled system. (We always refer to a tank as a chain of coupled cavities.) The agreement with measurement, as you will see in the following paper, is remarkably good.

Figure 1 illustrates the correspondence between coupled cavities, coupled circuits, and a linear lattice. The correspondence between circuits and cavities is direct as shown if we talk in terms of the amplitude and resonant frequency, a single mode of oscillation of the single cavity, and the coupling strength of this mode to the next cavity. $k$ can of course be given in terms of overlap integrals of field quantities. The analogy with the linear lattice with next-nearest-neighbor coupling is familiar. For most of the talk we consider only a single cavity mode at a time; this describes most of the experimental data.
II. Circuit Equations

The circuit equations are \( n = 1 \cdots 2N \)

\[
E_n = (2j \omega L + R + \frac{1}{j\omega C}) i_n + jk\omega L (i_{n-1} + i_{n+1})
\]  

\[
\frac{E_n}{2j\omega L} = I_n \approx (1 - \frac{\omega_o^2}{\omega^2}) i_n + \frac{k}{2} (i_{n-1} + i_{n+1})
\]

where \( \omega_o^{-2} = 2LC(1 + \frac{1}{jQ}) \quad Q = \frac{2\omega L}{R} \).

There are \( 2N \) solutions to homogeneous equations \( I_n = 0 \) of the form

\[
i_{n}^{(q)} = \text{const.} \times e^{2\pi j \frac{qn}{2N}}
\]

(q is the mode number, \( n \) is the circuit number). Provided that

\[
0 = (1 - \frac{\omega_o^2}{\omega_q^2}) e^{2\pi j \frac{qn}{2N}} + \frac{k}{2} (e^{2\pi j \frac{q(n-1)}{2N}} + e^{2\pi j \frac{q(n+1)}{2N}})
\]

as may be seen by substituting (4) into (2), or

\[
\omega_o^{-2} - \omega_q^{-2} + k \omega_o^{-2} \cos \frac{\pi q}{N} = 0.
\]

This is the dispersion relation for the simple ring of \( 2N \) circuits, or for the lattice with periodic boundary conditions. It is plotted as Fig. 2.

\[
\pi q/N = \varphi \quad \text{is the mode phase shift per cell; e.g., } q = N \quad \text{is called the } \pi-\text{mode.}
\]

III. Matrix Form at Circuit Equations

We had

\[
(\omega_o^{-2} - \omega^{-2}) i_n + \frac{k}{2} \omega_o^{-2} (i_{n-1} + i_{n+1}) = \omega_o^{-2} I_n.
\]

Let \( \vec{i} = \begin{pmatrix} i_1 \\ \vdots \\ i_N \end{pmatrix} \) \( \vec{I} = \begin{pmatrix} I_1 \\ \vdots \\ I_N \end{pmatrix} \)
Circuit equations are

\[ \omega_0^2 \mathcal{O} \mathbf{i} - \omega_0^2 \mathbf{i} = \omega_0^2 \mathbf{i}. \]  

(7)

The normal modes \( \mathbf{i} \) are the eigenvectors of the homogeneous problem, and the resonant frequencies of the modes are the eigenvalues.

Solution to the homogeneous problem \( I_s = \frac{i}{2} x, \omega_0^2 \) is the Green's Function

\[ G(r, s) = \sum_{\text{all modes}} \frac{\varphi^q(r) \varphi^q(s) w(q)}{\omega_q^2 - \omega_0^2}. \]

(8)

\( w(q) = 1/2 \) \( q = 0 \) or \( N \)

\( w(q) = 1 \) \( q = 1, 2, \ldots, (N - 1) \)

i.e., this is the response of the tank at the \( r \)th cavity to unit drive at cavity number \( s \).

IV. Relation Between Travelling Waves in a Ring and Standing Waves in a Tank

Figure 3 illustrates the relation between travelling waves on a ring and standing waves in a tank. The circumference of the circle is divided into \( 2N \) equal segments, representing \( 2N \) cavities connected in a ring. The travelling wave solutions, eq. (4) satisfy the periodic boundary condition
\( i_{2N+1} = i_1 \). They represent waves travelling counterclockwise around the ring. Now imagine the ring bisected by the line cutting cavities number 2N and number N in half, fold the figure along this line and straighten out the half circle to get a line segment. Number the line so that

\[
\begin{align*}
2N & \rightarrow 0 \\
1 & \rightarrow 1 \\
2N - 1 & \rightarrow 1 \\
2 & \rightarrow 2 \\
2N - 2 & \rightarrow 2 \\
\end{align*}
\]

etc.

The line represents the tank, with cavities of 1/2 length at each end. The solutions (eq. (4)) map into \( (i^q + i^q^*) = \frac{1}{p} \cos \frac{\pi N}{N/2} \quad (8) \)

The running wave solutions (4) satisfy

\[
\sum_{p=1}^{2N} i^q^* i^r_p = \oint_{2N} (r - q) \quad (4')
\]

where \( \oint_{2N} (x) = 1 \quad x = 0, 2N, 4N \cdots \)

\( = 0 \) otherwise.

and the standing wave solutions satisfy

\[
2 \sum_{p=0}^{N} w_p \cos \frac{\pi N}{N} \quad \cos \frac{\pi r p}{N} = \oint_{2N} (r - q) + \oint_{2N} (r + q) \quad (4'')
\]

where \( w_p = 1 \quad p = 1 \cdots N - 1 \)

\[
\begin{align*}
w_p &= 1/2 \\
p &= 0 \text{ or } N.
\end{align*}
\]

For the chain the circuit matrix is
V. Deviations from the Floquet Law

\[ \varphi = q \cdot \pi / N \]

In the presence of losses in the cavities, above wave functions (8) are not exact. For example, in the \( \pi \)-mode, the first-order correction requires

\[ \varphi_n = \frac{\pi q}{N} + \Delta \varphi_n \]

where

\[ \Delta \varphi_{n+1} - 2 \Delta \varphi_n + \Delta \varphi_{n-1} = \frac{2}{k Q \pi} \cdot \]

This is plotted in Fig. 4. With the drive in the center, the cell-to-cell phase shift is a maximum in the center, and decreases linearly. For \( k = 1 \), \( Q = 20,000 \) and a tank of 65 cells, the center-to-end phase shift is \( 28^\circ \). By cutting the cavities halfway between beam loaded and unloaded, the over-all change in phase shift would be \( 28^\circ \). Visscher's program shows \( 10^\circ \) is quite all right.

VI. Perturbation Theory

We can use the ordinary first-order perturbation theory to calculate the effect of deviations in cavity geometry:

Unperturbed Circuit Eq

\[ \omega_o^{-2} \sum_i \omega^{-2} i = \omega_o^{-2} \cdot \] (7)
Now perturb resonant frequencies of cavities
\[ \omega_o^{-2} (\omega + \varepsilon_i ) (i + \varepsilon_i) = (\omega^2 + \varepsilon \omega^{-2}) (i + \varepsilon_i) \]
define
\[ \omega_n^{-2} = \omega_o^{-2} (1 + \varepsilon_n). \]

First-Order Solutions
\[ \omega_n^{-2} = \omega^{-2} (\frac{1}{N} \sum_{r=0}^{N} \varepsilon_r). \]

\[ \varepsilon_r = \sum_{i=1}^{N} \frac{\cos \frac{2\pi i r}{N} - \cos \frac{2\pi i r}{N}}{k (\cos \frac{2\pi i l}{N} - \cos \frac{2\pi i r}{N})} \]

where \( \varepsilon_r \) is the \( r \)th Fourier Transform of the error. This is written for a ring but the application to the line is straightforward. One sees \( k \) must be kept large.

VII. Circuits of Two Kinds

Many of the structures of interest, cloverleaf, crossbar, side coupled iris, loop coupled iris, etc., are of a doubly resonant character. The cavities useful mode resonates at frequency \( \omega_1 \), say, and the coupling region itself resonates at frequency \( \omega_2 \). Thus in the cloverleaf the slots themselves have a resonance at the frequency \( \omega_2 \). We may represent approximately the actual behavior by a very simple model, namely of circuits of two kinds, as shown in Fig. 5. The equations now become
There are solutions to $I = 0$ of the form
\begin{equation}
\begin{aligned}
i_{2n} &= i_{2} e^{2nj\varphi} \\
i_{2n+1} &= i_{1} e^{(2n+1)j\varphi}
\end{aligned}
\end{equation}
if
\begin{equation}
(\omega_{1}^{-2} - \omega_{2}^{-2})(\omega_{2}^{-2} - \omega_{1}^{-2}) = k^{2} \omega_{1}^{-2} \omega_{2}^{-2} \cos^{2} \varphi
\end{equation}
This is the dispersion relation for the ring or chain of such resonators. 
It is shown in Fig. 6. It has two branches, known in lattice dynamics as the acoustical and optical branches.

There is a forbidden band of frequencies between the two branches.
For $\omega_{1} < \omega_{2}$ the lower $\pi/2$ mode corresponds to excitation of the man cavities $+, -, +, -, \ldots$, and zero energy in the coupling cells.
This mode is called the $\pi$-mode if we use the simple model. The upper $\pi/2$ mode (energy in the coupling cells and none in the cavities) is missing for $\omega_{1} < \omega_{2}$ because of the boundary conditions at the ends. The top of the acoustical band $\omega = \omega_{1}$, $\varphi = \pi/2$ turns out to be a useful operating point, as will be explained in the following paper. The fit of the measured points to this very simple model is remarkably close.
COPLED CAVITIES

COPLED CIRCUITS

LINEAR LATTICE

NEAREST NEIGHBOR COUPLING

ANALOGOUS PERIODIC STRUCTURES

Fig. 1
\[ 2\pi \times \text{FREQUENCY} \]

\[ \omega_0 \]

\[ \omega^2 - \omega_0^2 = \omega_0^2 k \cos \phi \]

PHASE SHIFT \( \phi \)

\[ 0 \]

\[ \frac{\pi}{2} \]

\[ \pi \]

DISPERSION CURVE FOR SIMPLE CHAIN

Fig. 2
MAPPING OF THE RING
ONTO THE CHAIN

Fig. 3
\[ \Delta \varphi = \frac{1 - \frac{k_2}{kQ_0}}{kQ_0} \left\{ (2n+1) \beta - \beta (\beta + 1) \right\} \]
CAVITIES OF TWO KINDS

CIRCUITS OF TWO KINDS

DIATOMIC LINEAR LATTICE
\[ (w^2 - w_1^2)(w^2 - w_2^2) + \text{const} \phi \]

**Acoustical Band**

**Dispersion Relation for Cavities of Two Kinds**