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MODE COUPLING OF ELECTRON PLASMA WAVES

Judith Alice Harte

(Ph. D. Thesis)

October 31, 1975

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MODE COUPLING OF ELECTRON PLASMA WAVES

By

Judith Alice Harte*

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ABSTRACT

The driven coupled mode equations are derived for a two fluid, unequal temperature ($T_e \gg T_i$) plasma in the one-dimensional, electrostatic model and applied to the coupling of electron plasma waves. We assume that the electron to ion mass ratio $m_e/M_i \ll 1$ and $\eta_{ko}^2 k \lambda_{De} < 1$ where η_{ko}^2 is the pump wave's power normalized to the plasma thermal energy, k the mode wave number and λ_{De} the electron Debye length. Terms up to quadratic in pump power are retained. These equations describe the linear plasma modes oscillating at the wave number k and at ω_{ek} , the Bohn Gross frequency, and at Ω_k , the ion acoustic frequency, subject to the damping rates ν_{ek} and ν_{ik} for electrons and ions and their interactions due to intense high frequency waves E_k^z . n_0 is the background density, n_{ik} the fluctuating ion density, ω_{pe} the plasma frequency.

$$\left(\frac{\delta^2}{\delta t^2} + \nu_{ek} \frac{\delta}{\delta t} + \omega_{ek}^2 \right) E_k^z = \frac{-\omega_{pe}^2}{n_0} \sum_{k'} n_{ik}, E_{k-k'}^z$$

and

$$\left(\frac{\delta^2}{\delta t^2} + \nu_{ik} \frac{\delta}{\delta t} + \Omega_k^2 \right) n_{ik} = \frac{-k}{4\pi M_i} \sum_{k'} k' E_k^z E_{k-k'}^z$$

Ignoring the nonlinear terms, these equations describe parametric instabilities which are of interest to laser plasma fusion studies. The instability analysis is extended to allow a finite wavelength pump field. When a finite wave number electron plasma wave acts as the pump a reduction of the threshold pump power necessary for onset of the instability can occur. The dispersion of electron plasma waves allows

the resonance conditions to be met. We derive the wave number and pump power range of this threshold reduction. Asymmetry of the unstable modes also results. In addition a one dimensional electrostatic plasma simulation code was used to study the nonlinearities of the coupled mode equations. The results of the simulations of finite amplitude electron plasma waves demonstrate the dominance of a two step, non resonant direct driving mechanism. These simulations are explained using the coupled mode equations one at a time. First, the standing finite amplitude electron plasma modes directly drive standing ion acoustic waves at half each electron mode wavelength. Then the electron modes couple with the ion waves to drive other electron plasma waves directly. This process very rapidly results in a turbulent spectrum of electron modes. It is much more rapid than the resonant processes such as the parametric instabilities for electron plasma waves with wave numbers $k\lambda_{De} > 0.03$ assuming that the pump wave amplitude is two orders of magnitude or more above background and that the fastest growing mode has $k\lambda_{De} \leq 0.15$. Since electron modes with wave numbers typically $k\lambda_{De} \sim 0.1$ are driven unstable in the parametric decay of electromagnetic waves, this two step process can be important for transfer of energy among electron plasma waves.

TABLE OF CONTENTS

- I. Introduction
- II. The Coupled Mode Equations
- III. Parametric Excitation of Coupled Waves by Finite Wavelength Pump:
Driven Parametric Plasmon-Phonon Interaction
 - A. Introduction
 - B. The Dispersion Equation
 1. Formulation of the Problem
 2. Review of $k_0=0$ Solution
 3. Finite k_0 Dispersion Equation
 - a. Uncoupled Brillouin Diagrams
 - b. Effective Coupling Coefficient
 - c. Damping Coefficients
 - C. Solution to Finite k_0 Dispersion Equation
 1. The Computer Solution
 2. Threshold Reduction for the Oscillating Two Stream Instability in the Weak Damping Domain
 3. Asymmetry of Unstable Wave Numbers for the Parametric Decay Branch
 - D. Summary
 - E. Appendix on Computer Program, DISREL
- IV. Mode Coupling of Finite Amplitude Electron Plasma Waves
 - A. Introduction
 - B. Coupled Mode Equations and Direct Driving
 - C. Simulation Calculations of the Mode Coupling
 - D. Summary
- Appendix: Plasma Simulation using Digital Computer; Treating Specifically a One-Dimensional Electrostatic Code, CYCLOPS.
 - A. Introduction

B. General Operation

- 1. Main Loop**
- 2. Phase Space Initialization**
- 3. Diagnostics**

C. General Evaluation

Chapter I

Introduction

The use of controlled thermonuclear fusion as a power source has been the primary goal of plasma physics for the last twenty-five years. The problem of confining a hot gas of charged particles for times long enough so that the thermonuclear reactions which occur produce more energy than was consumed in initiating them (break even power) has been found to be an extremely difficult problem. At each stage of investigation, plasmas demonstrate more and more complex phenomena, inhibiting well-planned confinement schemes. (Plasmas always seem to exhibit properties which are quite the antithesis of the property for which they were named. The root plasma means "to mold". In 1929 Langmuir proposed the name "plasma" for the positive column of a gas discharge based on its behavior of molding itself to the shape of the containing vessel.¹⁾ In particular, these hot gases generally exhibit instabilities of one form or another. The instabilities which are characterized, in the linear analysis, by the exponential growth of plasma parameters such as densities, velocities, currents, or fields lead to the destruction of the confined plasma and to energy losses making the thermonuclear reactions no longer possible. In considering just the temperature differences necessary to achieve fusion, it seems natural that particles hotter than their surroundings by a factor of $\sim 10^5$ or more should exhibit properties which are orders of magnitude more complex. The presence of an electrical charge on the particles introduces additional phenomena compounding still further complexities of plasma behavior. It is inevitable, therefore, that instabilities which are essentially the amplification of random background noise due to the rapid transfer of energy from another source

are an inescapable tendency of confined plasmas. A practical energy producing system based on thermonuclear reactions in plasmas must recognize this tendency. It should be rooted in a thorough understanding of the plasma's natural tendencies. Many fusion schemes are based on the suppression of instabilities. Others plan on making use of these (plasma instabilities). Laser plasma fusion is of the latter type.

Laser plasma fusion proposes to obtain fusion power from solid targets irradiated by very powerful pulsed lasers on such fast time scales (10's to 100's of pico seconds) that the plasma will just not have time to displace significantly destroying the confinement before fusion occurs. This is called inertial confinement. The plasma instabilities, which do occur on these fast time scales, are being studied as a means of heating plasmas. (This process is referred to as "anomalous heating".) The targets must be compressed by tremendous amounts to densities of 10^{26} particles/cc to satisfy break even fusion conditions (Lawson Criterion) in these time scales.

The interaction of intense electromagnetic radiation with a plasma is a major topic of concern for laser plasma fusion schemes. This complex problem is tractable in various limiting cases. A one dimensional electrostatic treatment models an important regime, namely the interaction of an electromagnetic wave with a plasma occurring in the plane of the ξ vector before the particles have been accelerated to relativistic speeds. (See Figure I-I.) In the linear approximation (derived by assuming that the fluctuating amplitudes are small compared with their constant background values) plasmas are known to possess normal modes in the electrostatic approximation. If an electron is perturbed from its equilibrium position, it will oscillate about that position. The electrostatic forces due to charge separation act as the restoring force. The frequency of oscillation, often referred to as the Bohm-Gross frequency is given by

(1.1)

$$\omega_{ek}^2 = \omega_{pe}^2 \left(1 + 3k^2 \lambda_{DE}^2 \right)$$

where $\omega_{pe}^2 = \left(4\pi n_0 e^2 / M_e \right)$ electron plasma frequency

$$\lambda_{De}^2 = \left[\theta_e / \left(4\pi n_0 e^2 \right) \right] = \text{electron Debye length}$$

and n_0 , e , m_e , θ_e , k are the plasma background density, electron charge, electron mass, electron temperature in electron volts where $\theta_e = kT_e$, k = Boltzmann constant, T_e = electron temperature, and electron plasma wave number. The oscillation frequency of these electron plasma waves is too high for the massive ions to respond. The ions are therefore assumed to be stationary and are treated as just as neutralizing background.

When the electron temperature is significantly larger than the ion temperature, a plasma also exhibits normal modes with much lower frequencies called ion acoustic waves. The difference between the inertia of the ions and electrons gives rise to an electrostatic restoring force which drives these oscillations. The frequency Ω_k is given by

$$\Omega_k^2 = \left(\frac{M_e}{M_i} \right) \omega_{pe}^2 k^2 \lambda_{De}^2 / \left(1 + k^2 \lambda_{De}^2 \right) \quad (1.2)$$

where M_i = ion mass, k = wave number.

A one-dimensional, unequal temperature plasma therefore will resonantly oscillate at these normal modes. Equations (1.1) and (1.2) describe the waves' dispersion and yield the relevant ω versus k curves often referred to as Brillouin diagrams. The evolution and interaction of these waves in one-dimensional, electrostatic, uniform density, unbounded plasma model is the subject of this thesis.

In the next chapter these normal modes of the linear system will be derived from the two species fluid equations. In addition, the nonlinearities up to products of small fluctuating quantities will be retained. These equations describe the linear modes and their coupling due to the nonlinearities. In the spirit of this expansion at least one field term, usually called the pump, must be at high level for these nonlinear processes to be important. This is certainly the case for laser plasma fusion schemes in which laser light is focussed to very high intensities (hopefully up to 10^{17} watts/cm²).

In Chapter III the coupled mode equations will be treated in the resonant approximation assuming a high frequency pump wave. A dispersion relation for the resulting coupled waves is obtained. For an infinite wave length ($k_{\text{pump}} = k_0 = 0$) pump this reduces to the dispersion relation obtained by Nishikawa³ for parametric plasmon-photon interaction. His solution is reviewed and the more general equation including the effects of a spatially inhomogeneous pump is solved by computer and analytically in various approximations. The results of computer root solver for the complex frequency solution corresponding to real wave numbers are presented. Laser-plasma parameters are used. These are compared and contrasted with the $k_0 = 0$ solution. The new effects due to the spatially non-uniform pump are threshold reduction for instability when electron plasma waves act as pump and asymmetric unstable wave numbers for the resonant instability.

Chapter IV treats the nonlinear mode conversion of plasma waves driven unstable by the parametric instability for an infinite wavelength high frequency pump in the one dimensional, electrostatic, unequal temperature plasma model. This problem was the main impetus for the work on mode coupling. An understanding of the mechanism by which the unstable modes couple with each other and with the plasma particles is of great importance to plasma fusion schemes. For this reason the problem has been studied in detail using a plasma simulation code and a simple theoretical treatment based on the model equations. In this investigation non-resonant direct driving processes were seen to dominate over the resonant processes for pump waves with $k_0 \lambda_{DE} > 0.03$ assuming the pump waves amplitude is a least two orders of magnitude above background and the fastest growing mode has $k \lambda_{DE} < 0.15$. This is the case for waves driven unstable by the parametric instabilities which suggests that the direct driving process is an important mechanism leading to the turbulent spectrum of plasma waves, resulting from parametric instabilities.

Finally, the Appendix gives a brief description of the one dimensional plasma simulation code which was used to study the mode coupling. Non-physical nonlinear mode coupling due to the quiet start⁴ initial phase space loading is discussed.

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Figure 1-1

This schematic illustrates a plane polarized electromagnetic wave incident from the left with $k_{\text{light}} = k \begin{vmatrix} \Lambda \\ y \end{vmatrix}$ upon a vacuum plasma interface with plasma density, $n(y)$, increasing linearly with y .

Chapter II

The Coupled Mode Equations

In this chapter coupled equations for a one dimensional electrostatic plasma in the presence of a high frequency external field are derived. These equations describe the linear oscillations, namely, electron plasma waves and ion acoustic waves, as well as the interaction among these modes. Here the nonlinearities only up to second order in fluctuating quantities are retained. The equations are derived using assumptions similar to Nishikawa¹ and Thomson². The electric field energy for the external and self-consistent fields is assumed small compared with the plasma thermal energy (i.e. $\eta \frac{2E^2}{4\pi n_0 \theta} < 1$). The frequency of the external field is taken to be almost equal to the plasma frequency ($\omega_0 = \omega_{pe} = \left[\frac{4\pi n_0 e^2}{m_e} \right]^{1/2}$). The two species fluid equations are necessary. They are

$$\frac{\partial n_\sigma}{\partial t} + \frac{\partial}{\partial x} (n_\sigma v_\sigma) = 0 \quad (2.1)$$

$$n_\sigma \left(\frac{\partial v_\sigma}{\partial t} + v_\sigma \frac{\partial v_\sigma}{\partial x} \right) = \frac{\gamma_\sigma \theta_\sigma}{M_\sigma} \frac{\partial n_\sigma}{\partial x} = \frac{e_\sigma}{M_\sigma} n_\sigma E - v_\sigma n_\sigma \gamma_\sigma \quad (2.2)$$

where σ denotes the species (either electrons or ions) ($\sigma = e$ or i). The Poisson equation relates the plasma's self field, E' , to the plasma densities:

$$\frac{\partial E'}{\partial x} = 4 \pi \sum_\sigma e_\sigma n_\sigma \quad (2.3)$$

where n , v , γ , θ , M , e , v , E are the density, fluid velocity, gas constant, temperature, mass, charge, damping frequency and total field. It is assumed that the collision frequencies are constant, that the pressure gradient $v \rho_0$ can be expressed

as a density gradient in the usual form

$$\frac{\partial P_{\alpha}}{\partial x} = \gamma_{\alpha} \theta_{\alpha} \frac{\partial n_{\alpha}}{\partial x} \quad (2.4)$$

and that the ions are singly ionized.

The variables are separated into their zero order spatially homogeneous contributions (denoted by 0) and the smaller fluctuating quantities due to wave motions in the plasma. Combining the partial derivative with respect to time of equation (2.1) with the spatial gradient of equation (2.2) yields

$$\frac{\partial^2 n_{\alpha}}{\partial t^2} + v_{\alpha} \frac{\partial n_{\alpha}}{\partial t} - \frac{\gamma_{\alpha} \theta_{\alpha}}{M_{\alpha}} \frac{\partial^2 n_{\alpha}}{\partial x^2} + \frac{e_{\alpha}}{M_{\alpha}} \frac{\partial}{\partial x} (n_{\alpha} E) - \frac{\partial^2}{\partial x^2} (n_{\alpha} v_{\alpha}^2) = 0 \quad (2.5)$$

For the electrons, each fluctuating quantity has a high frequency part (denoted by λ [for Langmuir]) representing its response near the electron plasma frequency and a low frequency contribution (denoted by s [for sound]) corresponding to its response near the ion acoustic frequency. The ions restricted by their inertia can respond only on the slow time scale of the ion acoustic frequency. Equation (2.5) therefore contains a high and low frequency electron equation and a low frequency ion equation. We neglect the nonlinear terms in equation (2.5) which are higher than quadratic in the first order fluctuating quantities. To do this systematically a "smallness" parameter is used. The linearized equation of motion yields

$$k v_{e\lambda} v_{k\lambda} \frac{\partial}{\partial t} n_{DE} n_{pe} \sim \epsilon \omega_{pe}$$

where $k \lambda_{DE} n_{\epsilon} \ll 1$. ϵ then is the small expansion parameter such that any terms of order ϵ^2 or higher will be ignored. Also $\frac{m_e}{M_i} \ll 1$; thus terms down by a mass ratio will also be ignored. Finally notice that a product of two high frequency terms does not contain a relevant high frequency contribution because it would oscillate at $2\omega_{pe}$ which is nonresonant.

The high frequency electron form of equation (2.5) yields the first coupled mode equation. This is derived using the high frequency Poisson equation and the fact that $n_e^s = n_i + 0(\epsilon)$. One obtains upon Fourier transforming in space that

$$\left(\frac{\partial^2}{\partial t^2} + v_e \frac{\partial}{\partial t} + \omega_{ek}^2 \right) n_{ek}^{\ell} = \frac{iek}{m_e} \sum_{k'} n_{ik'} E_{k-k'}^{\ell} \quad (2.6)$$

or in terms of high frequency fields

$$\left(\frac{\partial^2}{\partial t^2} + v_e \frac{\partial}{\partial t} + \omega_{ek}^2 \right) E_k^{\ell} = \frac{-\omega_{pe}^2}{n_0} \sum_{k'} n_{ik'} E_{k-k'}^{\ell} \quad (2.6')$$

$$\text{where } \omega_{ek}^2 \equiv \omega_{pe}^2 \left(1 + 3k^2 \lambda_{De}^2 \right)$$

$$\omega_{pe}^2 \equiv \frac{4\pi n_0 e^2}{M_e}, \quad \lambda_{De}^2 \equiv \frac{\theta}{4n_0 e^2}$$

For the low frequency electron motion equation (2.5) up to order ϵ yields:

$$\begin{aligned} & \frac{\partial^2}{\partial t^2} n_e^s + v_e \frac{\partial n_e^s}{\partial t} - \frac{\gamma_e \theta_e}{M_e} \frac{\partial^2 n_e^s}{\partial x^2} - \frac{e}{M_e} \frac{\partial}{\partial x} \left(n_e^s E \right)^s \\ & - \frac{\partial^2}{\partial x^2} \left[n_0 \left(v_e^{\ell} \right)^2 \right] = 0 \end{aligned} \quad (2.7)$$

The expression from the linearized equation of motion for v_e^{ℓ}

$$v_e^{\ell} = - \frac{e}{M_e} \frac{iE^{\ell}}{\omega} \quad (2.8)$$

can be used in equation (2.7). The low frequency ion equation again from equation (2.5) up to order ϵ is

$$\frac{\partial^2}{\partial t^2} n_i + v_i \frac{\partial n_i}{\partial t} - \frac{\gamma_i \theta_i}{M_i} \frac{\partial^2 n_i}{\partial x^2} = 0 \quad (2.9)$$

If equation (2.9) is added to $\frac{M_e}{M_i} \times$ equation (2.7) using $n_e^s \approx n_i$ and ignoring terms down by a mass ratio, one finds³

$$\begin{aligned} & \frac{\partial^2}{\partial t^2} n_i + v_i \frac{\partial n_i}{\partial t} - \frac{\gamma_e \theta_e + \gamma_i \theta_i}{M_i} \frac{\partial^2 n_i}{\partial x^2} - \frac{e}{M_i} \frac{\partial}{\partial x} \left(n_e^s E \right)^s \\ & - \frac{M_e}{M_i} n_0 \frac{\partial^2}{\partial x^2} \left(v_e^{\ell} v_e^{\ell *} \right) = 0 \end{aligned} \quad (2.10)$$

Again using the high frequency Poisson equation $\frac{\partial E^{\ell}}{\partial x} = -4\pi e n_e^{\ell}$ and equation (2.8), equation (2.10) becomes after Fourier transforming in space

$$\left(\frac{\partial^2}{\partial t^2} + \nu_i \frac{\partial}{\partial t} + \frac{\gamma e^0 e^{\gamma i} i}{M_i} k^2 \right) n_{ik} = \frac{k}{4\pi M_i} \left[(-k) \sum_{k'} E_{k'}^{\ell} \cdot x \right. \\ \left. E_{k-k'}^{\ell} + \sum_{k'} k' E_{k'}^{\ell} \cdot E_{k-k'}^{\ell} \right] \quad (2.11)$$

By simply changing the arbitrary index of summation from m' to $m-m'$ one can derive the following identity⁴

$$2 \sum_{m'} m' A_{m'} A_{m-m'} = m \sum_{m'} A_{m'} A_{m-m'} \quad (2.12)$$

Using equation (2.12), equation (2.11) becomes the second coupled mode equation

$$\left(\frac{\partial^2}{\partial t^2} + \nu_i \frac{\partial}{\partial t} + \Omega_k^2 \right) \frac{n_{ik}}{n_0} = \frac{-k}{4\pi M_i n_0} \sum_{k'} k' E_{k'}^{\ell} \cdot E_{k-k'}^{\ell} \quad (2.13)$$

where $\Omega_k^2 = k^2 \lambda_{DE}^2 \omega_{pe}^2 \left(\frac{M_e}{M_i} \right)$

assuming $\theta_e \gg \theta_i$

It is interesting to note that B. B. Kadomtsev⁵ derives a similar set of coupled mode equations for the interaction between Langmuir waves and ion sound waves. In evaluating the low frequency coupling term, he linearizes the electric field force term in the electron equation of motion. His driving term is therefore different by a factor of 2, increasing the effective pump power.

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Chapter III

Parametric Excitation of Coupled Waves by Finite Wavelength Pump: Driven Plasmon*-Phonon* Interaction

A. Introduction

Parametric excitation of coupled waves refers to the process in which two modes, which correspond to normal modes in the linear system, are amplified due to a third wave called the pump. The beat wave of the pump and one of the excited modes oscillates near the resonant frequency of the third mode and vice-versa. Thus the forced oscillations resonate with the natural oscillations leading to strong coupling among the waves and to efficient energy conversion between the pump wave and the coupled waves. This has been shown to be a prevalent and important mechanism for nonlinear mode conversion. (References: Laser-plasma,^{1,14} microwave plasma,^{2,9} stimulated Raman³, Brillouin scattering⁴, and radio waves with the ionosphere⁵). The coupled modes are driven as an instability in this manner when the pump exceeds some threshold value and when the phase matching conditions are (at least approximately) fulfilled. In this chapter the parametric coupling which occurs among the electrostatic modes of an unequal temperature ($\theta_e \gg \theta_i$) plasma due to a high frequency (ω_{pe}) electromagnetic or electrostatic pump wave is studied. The wavelength of the pump field will not be assumed infinite in the analysis as opposed to most previous treatments. The spatial non-uniformity of the pump wave is shown to bring about a reduction in the electric field strength necessary for threshold of the instability as well as a shifting in k and ω space of the growing modes.

This problem has direct applications to the understanding of laser-plasma inter-

*The word "plasmon" refers to an electron plasma wave or Langmuir wave.

*The word "phonon" refers to an ion acoustic wave.

actions. Parametric instabilities which occur in the plasma at the "critical" surface at which the plasma frequency equals the frequency of the laser light have been used to explain the excitation of large internal self fields in a plasma irradiated by an intense electromagnetic wave.⁶ These fields in turn lead to turbulent (sometimes called anomalous) electron heating by particle trapping in the waves^{7,24,25} or by diffusion in velocity space due to the turbulent electric field spectrum.⁸ In the theoretical treatment of these laser plasma interactions one usually theorizes an infinite wavelength oscillating electrostatic field (see Figure 1-1) driving ion acoustic and electron plasma waves unstable. Both simulation calculations and solutions of the fluid equations including mode coupling terms have substantiated this theory.⁸ The electric field spectrum obtained in the solution to the fluid coupled mode equations (derived in Chapter II) is shown in Figure 3-1. This figure is copied directly from reference 8. The line indicates the $1/k^2$ dependence. The evolution of this spectrum has been shown to be caused by plasma mode coupling. That is, the coupling of the external laser field as well as the internal plasma fields leads to the nonlinear saturation obtained here. The energy of the internal fields is seen to be of the same order of magnitude as the laser field. These fields could also exceed the critical pump power necessary for the threshold of the parametric instabilities and cause exponentiation of other electron plasma waves and ion acoustic waves. Thus the analysis of the parametric instability which allows a finite wave number pump wave (i.e. $k_0 \neq 0$) is necessary. This analysis can also be applied to evaluate the effect of a spatially inhomogeneous pump wave due to a finite spot size or to parametric coupling of the resonantly driven mode or any electric field varying in space along the longitudinal direction (cf. Reference 26).

The first systematic study of parametric plasmon-photon interaction was done by Silin.⁹ He used an expansion in Bessel functions to solve the appropriate equations under the influence of a time oscillating electric field, assuming as a small quantity the electron to ion mass ratio. His results are valid only well above threshold as he ignored plasma damping. DuBois and Goldman¹⁰ included Landau damping in their

analysis of this parametric process for weak pump field intensities. Lee and Su¹¹ obtained similar results with a fluid model. Jackson¹² obtained a general formulation of this situation using a Vlasov model which allowed for arbitrary pump power and wavenumber. Nishikawa¹³ treated the general problem of the parametric excitation of coupled waves obtaining general results for any system satisfying his model equations. He then applied these results to a parametric plasmon-photon* deriving a set of coupled mode equations in the form of his model equations from the two species fluid equations linearized with respect to the pump power. Nishikawa's results included simple, useful, and physically intuitive expressions for the pump power necessary for threshold of the instability called "thresholds", the positive imaginary frequency solutions called "growth rates", and their corresponding real frequencies in various limiting cases. In addition he was the first to recognize the existence of a purely growing unstable mode (commonly called the oscillating two stream instability [OTSI]). Because of the straight-forward approach and the useful results of Nishikawa's analysis¹³, we have chosen to treat the problem of including the pump waves' spatial inhomogeneities in the instability analysis in a similar way.

The general normal mode form which Nishikawa¹³ used as a model clearly relates this parametric process to the parametric processes which have been studied by electrical engineers.^{14,15, 16} These references treated the coupling in the linear approximation. Only a finite number of coupling frequencies were allowed. The usefulness of viewing this coupled mode process in terms of the linear dispersion relation as the electrical engineers do is re-emphasized in a recent paper by Cassedy¹⁷. In this paper Cassedy finds the unstable roots by expanding about the possible unstable (i.e. backward wave parametric interactions) interactions of the uncoupled dispersion curves. He also applies the saddle point¹⁸ or equivalently the merging pole¹⁹ criteria to investigate the absolute stability of the plasmon-photon interaction. He therefore considers complex wave numbers as well as complex frequencies, whereas in the analysis of this thesis as in Nishikawa's¹³ complex frequencies are obtained for real wave numbers only.

* The word "photon" refers to a light wave.

The assumption of an infinite wave length ($k_0=0$) pump wave was not made in Jackson's paper¹². However no explicit results were obtained. Lee and Su¹¹ considered the finite wavelength longitudinal pump only to demonstrate that it was equivalent to a spatially homogeneous driver in the limit $k_0 \rightarrow 0$. Then explicit results were obtained for $k_0=0$. Ott²⁰ obtained the dielectric function including the effect of a finite wavelength pump. He recommended further, investigation of this situation, particularly on the decay type (PD) instability with large frequency shifts. Nishikawa²¹ included the possibility of a spatially inhomogeneous pump in his parametric instability review paper. He obtained a threshold suppression in the regime predicted by Ott²⁰. Here we treat a different situation in which the damping is small compared with the ion acoustic frequency. This case generally applies to laser plasma interactions. We consider the effect of the spatial nonuniformity of the pump wave in detail solving the complete dispersion relation by computer and obtaining approximate analytical results in the interesting cases.

In the next section the relevant dispersion equation for the coupled system is derived in a way analogous to Nishikawa's¹³ method. The known solutions in the $k_0 \rightarrow 0$ limit are reviewed. With these in mind the equation allowing a nonzero k_0 is examined.

We describe the uncoupled ω vs. k graphs (Brilouin diagrams¹⁴) looking in particular at their intersections. Important among these intersections are those occurring at the point of perfect phase matching when

$$\omega_0 = \omega_1 + \omega_2 \quad \text{and} \quad (3.1)$$

$$k_0 = k_1 + k_2$$

where ω_j , k_j represent the frequency and wave number of the pump ($j=0$), low frequency ($j=1$) and ($j=2$) high frequency waves. Also we describe the effective coupling coefficient and the damping coefficients of interest.

In the third section the solution of the dispersion relation including the effects of a spatially nonuniform ($k_0 \neq 0$) pump field is examined. We present computer solutions of the complete equation. Analytical results are obtained for the important case in

which the pump is an electrostatic plasma wave. For this case significant threshold suppression is shown to occur for the unstable low frequency modes at wave numbers k [and simultaneously for electron modes at $\pm \left(\frac{k-k_{pm}}{4} \right)$] due to pump waves k_0 which satisfy

$$1/2 \left(k - k_{pm} \right) < k_0 \sqrt{\left(\frac{k - k_{pm}}{4} \right)^2 + k^2} - \left(\frac{k - k_{pm}}{4} \right) \quad (3.2)$$

where $k_{pm} \lambda_{DE} \equiv 2/3 \left(\frac{M_e}{M_i} \right)^{1/2} \approx$ perfect phase matching (satisfying eq. (3.1) for $k_0=0$ subject to the conditions $k > k_{pm}$ and the pump wave frequency $\omega_0 = \omega_{ek}$, the Bohm Gross frequency. In the case with a finite wavelength dispersionless pump with frequency $\omega_0 = \text{constant} > \omega_{pe}$, the minimum thresholds and maximum growth rates shift in wave number space with k_0 essentially as would be expected from the shifting of the wave number of perfect resonance. Or similarly, the shifting of the intersections of the uncoupled Brillouin diagram can be used. These were described in Section B.

We obtain unstable electron waves with wave numbers larger than $k_{co} \equiv \left\{ k \left| \omega_0 = \omega_{ek} \right. \right\}$ the wave number above which (in the $k_0=0$ case) this is stabilized. Thus the decay can proceed to electron waves at wave numbers is greater than the cut off wave number for the $k_0=0$ case. This type of pump wave is however, non physical for it is a dispersionless electrostatic wave in a plasma.

B. The Dispersion Equation

In this section the coupled mode equations derived in Chapter II are used to obtain a dispersion relation for the parametric excitation of the coupled modes. An intense high frequency wave is assumed present. The dispersion relation obtained will be examined and the known results for $k_0=0$ summarized. Finally the specific linear dispersion curves, damping coefficients, and effective coupling coefficient relevant to laser-plasma fusion are reviewed.

1. Formulation of the Problem

The coupled mode equations eqs. (2.6) and (2.13) describing the time evolution of high frequency electrostatic plasma oscillations and low frequency ion acoustic waves

result from the two species plasma hydrodynamic equations. These were derived in Chapter II upon applying the following approximations. First, unequal temperatures ($\theta_e \gg \theta_i$) as well as a large mass ratio ($M_i/M_e \gg 1$) are assumed yielding the normal modes eqs. (1.1) and (1.2) from the linearized fluid equations. These oscillate at quite different frequencies such that a two time scale expansion is valid. (The slow time $\equiv (\omega \text{ ion acoustic})^{-1}$ fast time $\equiv (\omega_{pe})^{-1}$). Finally an expansion in the fluctuating field amplitudes is made keeping terms up to second order. The small parameter is $\epsilon \equiv n_{\text{pump}}^2 k_{\text{DE}}^2$ where $n_{\text{pump}}^2 \equiv \frac{E^2}{4\pi n_0}$.

In terms of the spatial Fourier amplitudes of the high frequency electric fields, E_k^{ℓ} , and the low frequency ion density perturbations, $n_{ik} \equiv n_i(k)^{\dagger}$, the coupled mode equations eqs. (2.6') and (2.13) are:

$$\left(\frac{\partial^2}{\partial t^2} + \nu_{ek} \frac{\partial}{\partial t} + \omega_{ek}^2 \right) E_k^{\ell} = -\omega_{pe}^2 \sum_{k'} \frac{n_i(k-k')}{n_0} E_{k'}^{\ell} \quad (3.3)$$

$$\left(\frac{\partial^2}{\partial t^2} + \nu_{ik} \frac{\partial}{\partial t} + \Omega_k^2 \right) \frac{n_i(k)}{n_0} = \frac{-k}{4\pi M_i n_0} \sum_{k'} k' E_{k-k'}^{\ell} E_{k'}^{\ell}$$

where $\omega_{ek}^2 = \omega_{pe}^2 \left(1 + 3k^2 \lambda_{\text{DE}}^2 \right)$, $\Omega_k^2 = k^2 \lambda_{\text{DE}}^2 \omega_{pe}^2 \left(M_e/M_i \right)$

as before and where n_{σ} , ν_{σ} , M_{σ} are the species density, damping and mass for $\sigma = \text{ion or electron}$. A strong high frequency wave is seen to bring about a coupling of these modes.

We assume as in laser plasma interactions that the pump wave has frequency $\omega_0 \sim \omega_{pe}$ and wave number k_0 . Eqs. (3.3) are transformed in time. Then, following Nishikawa's¹³ technique we assume ω is a low frequency $\sim \Omega_k \equiv \text{ion acoustic frequency}$. We include on the left hand sides of equations (3.3) only the resonant or near resonant terms (i.e. $|\omega| \approx \Omega_k$ and $|\omega \pm \omega_0| \sim \omega_{pe}$ are resonant; $|\omega \pm n\omega_0| \sim n\omega_{pe}$ are not for $n \geq 2$). The equations can then be written as follows:

[†] For notational simplicity, I hope.

$$\left\{ \omega^2 + i\omega \gamma_k - \Omega_k^2 \right\} \frac{n_i(\omega, k)}{n_0} = \frac{k}{4\pi M_i n_0} \sum_{k', \omega'} E_{\omega-\omega'}^{\mathcal{L}} \times E_{\omega', k'}^{\text{pump}} \quad (3.4)$$

$$\left\{ \left(\omega \pm \omega_0 \right)^2 + i \left(\omega \pm \omega_0 \right) \nu_{e, k \pm k_0} - \omega_e^2 \right\} E_{\omega \pm \omega_0, k \pm k_0} = \frac{\omega_{pe}^2}{n_0} \sum_{k', \omega'} \left[E_{\omega', k'}^{\text{pump}} \times \left[n_i \left(\omega \pm \omega_0 - \omega', k \pm k_0 - k' \right) \right] \right]$$

Applying the resonance approximation to each summation on the right hand sides and linearizing them with respect to the fluctuating quantities assuming the pump is zero order, we find $k' = \pm k_0$ are the only relevant terms in the summation. This greatly satisfies equations (3.4). (In his parametric review paper, Nishikawa²¹ also adds to the LHS (left hand side) equations $\frac{\nu_i k^2}{4} \frac{n_i(\omega, k)}{n_0}$ and $\frac{\nu_e^2 (k+k_0)}{4} E_{\omega \pm \omega_0, k \pm k_0}$, respectively in order to complete the square. This shall be included here also to allow easy comparison with his papers. On the other hand, it could be ignored because our applications involve weak damping.)

$$\text{Define } \Gamma_0 \equiv \frac{\nu_0}{2} \quad \text{and } D_0(\omega, k) \equiv -\omega^2 - 2i\omega \Gamma_0 + \Gamma_0^2 + \omega_0^2 \quad (3.5)$$

Notice $D_0(\omega, k) = 0$ is the dispersion relation of the uncoupled mode. The following dispersion relation

$$\begin{vmatrix} D_i(\omega, k) & \frac{k}{4\pi M_i n_0} (k+k_0) E_{-\omega_0, -k_0} & \frac{k}{4\pi M_i n_0} (k-k_0) E_{\omega_0, k_0} \\ \omega_{pe}^2 E_{\omega_0, k_0} & D_e(\omega + \omega_0, k + k_0) & 0 \\ \omega_{pe}^2 E_{-\omega_0, -k_0} & 0 & D_e(\omega - \omega_0, k - k_0) \end{vmatrix} = 0 \quad (3.6)$$

is then obtained from the following eq. (3.7). This is the matrix form of eqs. (3.4) in which the summations have been cut off as described.

$$\underline{D} \times \begin{bmatrix} n_i(k, \omega) / n_0 \\ E_{\omega + \omega_0, k + k_0} \\ E_{\omega - \omega_0, k - k_0} \end{bmatrix} = 0 \quad (3.7)$$

Eqs. (3.6) can be written

$$1 = \frac{\omega_0^2 e^2}{4\pi M_i n_0} \frac{|E_{k_0, \omega_0}|^2}{D_i(\omega, k)} \left\{ \frac{k(k+k_0)}{D_e(\omega+\omega_0, k+k_0)} + \frac{k(k-k_0)}{D_e(\omega-\omega_0, k-k_0)} \right\} \quad (3.8)$$

Note the " $k+k_0$ " and " $k-k_0$ " factors are different from Nishikawa's finite k_0 dispersion relation.²¹ Nishikawa ignored the k_0 with respect to k assuming $k_0/k \ll 1$. We do not make that restriction on the dispersion relation.

Next define:

$$\alpha = \frac{\omega_e(k+k_0) - \omega_e(k-k_0)}{2} = 3\omega_{pe} k k_0 \lambda_{De}^2 \quad (3.9)$$

$$\text{and} \quad \delta' = \omega_0 - \omega_{ek} - 3\omega_{pe} k_0^2 \lambda_{De}^2 / 2$$

both subject to $(k\lambda_{De})^2 \ll 1$.

The resonance approximation can be applied to $D_e(\omega \pm \omega_0, k \pm k_0)$ to obtain:

$$D_e(\omega \pm \omega_0, k \pm k_0) \approx \pm 2\omega_0 (\omega - \alpha \pm \delta' + i\Gamma_e) \quad (3.10)$$

However, $D_i(\omega, k)$ cannot be simplified in this way because the frequency shift is not necessarily small compared with the frequency of the ion wave, ω . Converting to dimensionless units where $k\lambda_{De} = \omega_{pe}/\omega_{pe}$, and $\eta_{k_0} = E_{k_0}^2 / (4\pi n_0 e)$; the dispersion relation becomes

$$\omega^2 + 2i\omega\Gamma_i - (\Gamma_i^2 + \Omega_k^2) = \left(\eta_{k_0} \frac{M_e}{M_i} \frac{k}{2\omega_0} \right) \left[\frac{k+k_0}{\omega - \alpha + \delta' + i\Gamma_e} + \frac{k-k_0}{\omega - \alpha - \delta' + i\Gamma_e} \right] \quad (3.11)$$

This equation can be written as two real polynomials analogous to equations (43), (44), (45) in Nishikawa's 1968 paper¹³ assuming $\omega \approx x + iy$

$$x^2 - (y + \Gamma_i)^2 - \Omega_k^2 = \frac{k^2 \left\{ \delta' \left[\delta'^2 - (x - \alpha)^2 + (\Gamma_e + y)^2 \right] + \frac{k_0}{k} (x - \alpha) \left[(\Gamma_e + y)^2 + (x - \alpha)^2 - \delta'^2 \right] \right\}}{F} \quad (3.12)$$

$$2x(y + \Gamma_i) = k^2 \left\{ 2\delta' (x - \alpha) (\Gamma_e + y) - \frac{k_0}{k} (\Gamma_e + y) \left[(x - \alpha)^2 + \delta'^2 + (\Gamma_e + y)^2 \right] \right\} / F \quad (3.13)$$

where

$$F = \left(\frac{\omega_0}{2} \frac{M_e}{\eta_{k_0} M_i} \right) \left\{ (x - \alpha + \delta')^2 + (\Gamma_e + y)^2 \right\} \left\{ (x - \alpha - \delta')^2 + (\Gamma_e + y)^2 \right\} \quad (3.14)$$

where the difference between $\Gamma_e(k, k_0)$ and $\Gamma_e(k-k_0)$ has been neglected. For $k_0 = 0$, these reduce exactly to Nishikawa's equations for a spatially inhomogeneous pump¹² if the terms multiplied by (k_0/k) on the right hand side are ignored. Thus in his analysis Nishikawa ignored the k_0 dependence of the coupling terms.

A finite k_0 introduces added complexity to equations (3.12), (3.13), and (3.14). We found it profitable to deal directly with equation (3.11) which can be rewritten in the following form:

$$\left\{ \frac{\omega + i\Gamma_e(k-k_0)}{2} \frac{M_e}{M_i} \frac{k_0^2}{\omega_0^2} \left[\frac{\omega + i\Gamma_e(k-k_0)}{2} \frac{M_e}{M_i} \frac{k_0^2}{\omega_0^2} \left[\frac{\omega + i\Gamma_e(k-k_0)}{2} \frac{M_e}{M_i} \frac{k_0^2}{\omega_0^2} \right] \right] \right\} \quad (3.15)$$

where $\Gamma_e \equiv \Gamma_e(k \pm k_0)$

and $\Delta_+ \equiv \omega_0 - \omega_{ck+k_0} = \omega_0 - \omega_0 + k_0 v_{Te}$

$\Delta_- \equiv \omega_0 - \omega_{ck-k_0} = \omega_0 - \omega_0 - k_0 v_{Te}$

In this form one clearly sees that the dispersion relation describes the coupling of the plasma's linear modes expressed in normal mode form by the left hand side (LHS) of equation (3.15) driven by the effective coupling coefficient which is the right hand side (RHS). We shall study the dispersion relation as a combination of the normal mode terms (LHS) and effective coupling coefficient (RHS) in greater detail later. First, we review the known solution for the $k_0 = 0$ case to gain more insight into the behavior of our dispersion relation, equation (3.15).

2. Review of Dispersion Equation Solution for $k_0 = 0$

Nishikawa's two papers entitled "Parametric Excitation of Coupled Waves, General Formulation and Parametric Plasmon-Photon Interaction"¹³ provide a very thorough analysis of the unstable roots of the coupled mode dispersion relation for an infinite wavelength pump. In the general formulation, Part I he analyzes the interaction of three modes at (ω_1, k_1) , (ω_2, k_2) and (ω_0, k_0) corresponding to two normal modes of the system plus a pump with $k_0 = 0$. He obtains the threshold pump power necessary for on-

set of instability as well as the growth rates and frequencies of the unstable modes in various limiting cases. Also Nishikawa derives the minimum threshold power and maximum growth rate as a function of the frequency difference, $\delta = \omega_0 - \omega_2$, between the pump wave at ω_0 and the high frequency wave at ω_2 (or analogously as a function of the wave number of the unstable mode since ω_0 constant and $\omega_2 = \omega_2(k)$: $\delta = \delta(k)$ in various approximations, e.g., strong and weak damping or pump power near threshold and well above threshold. Nishikawa¹³ found that the case of parametric coupling of the linear modes ω_1 and ω_2 due to a pump at ω_0 for which $\omega_1 \ll \omega_2 \approx \omega_0$ exhibits two types of unstable roots depending upon the sign of δ or alternatively upon the sign of the effective coupling coefficient, the RHS of equation (3.15).

When the effective coupling coefficient is negative or $\delta > 0$ the parametric decay (PD) type of solution is obtained. This is the well-known three wave resonant interactions which occurs for coupled modes when the phase matching conditions eq. (3.1) i.e. $\omega_0 = \omega_1 + \omega_2$ and $k_0 = k_1 + k_2$ are satisfied or nearly satisfied. (See references: 9, 10, 11, 12, 13, 14, 15, 16, or 17) The threshold pump power is determined by the frequency mismatch, $\omega_0 - \omega_2 = \omega_1 \equiv \Delta$ and by the damping of coefficients. For negligible damping the fastest growing mode occurs for modes with perfect frequency matching, $\Delta = 0$, as does the minimum threshold. In fact, the threshold becomes infinitesimal as the damping rates go to zero. For higher values of the damping coefficients the most unstable real frequency solution for the low frequency mode is off resonant by an amount proportional to $\sqrt{\Gamma_1 \Gamma_2}$.

On the other hand when the effective coupling coefficient, i.e. RHS of equation (3.15) in $k_0 = 0$ limit is positive ($\delta > 0$), Nishikawa¹³ obtained a non-resonant purely growing (i.e. $\text{Re}\omega = 0$) type of instability called the Oscillating Two Stream Instability (OTSI). For this instability the threshold pump power is no longer dependent on the damping of the low frequency mode, Γ_1 , since the low frequency mode is nonoscillatory. Also the topology of the unstable roots is different as we shall describe later. (See for example Figures 3-7a and 3-8.) This type of unstable solution can be thought of as a four-wave interaction¹⁷ and can occur only for the

parametric coupling satisfying $\left| \frac{\omega_1}{\omega_0} \right| < \left| \frac{\omega_2}{\omega_0} \right|$. This inequality allows both the sum and difference frequencies $\omega_{\pm 0}$ and $\omega_{\mp 0}$ where ω is low frequency solution ω_1 to be near resonance. The threshold power never becomes infinitesimal. It is limited by $\left| \frac{\omega_1}{\omega_0} \right|$ which can never be zero for this non-resonant process.

The analysis of this dispersion relation eq. (3.15) in terms of its linear Brillouin diagrams, that is, in terms of the ω vs. k plots of the uncoupled linear modes will clarify the distinction between the non-resonant instability (OTSI) and the resonant instability (PD). The Brillouin diagrams will also be used to understand the effects of the spatial inhomogeneities of the pump. A final point to remember in reference to the $k_0=0$ solutions is that the nonlinear mode coupling due to this instability leads to a shifting of the power spectrum in wave number space. This can be an important consideration because in plasmas collisionless absorption occurs when the phase velocity of the wave is near the particle thermal velocity. Thus the direction of energy transfer in terms of the wave's phase velocity is crucial. For electron plasma waves $v_{\text{phase}} = \omega_{\text{ek}}/k \approx \omega_{\text{pe}}/k$. The parametric decay instability (PD) occurs for $\delta > 0$ spreading energy to higher phase velocities. Whereas the oscillating two stream instability (OTSI) occurring for $\delta < 0$ feeds energy to higher wavenumbers and thus lower phase velocities leading to strong wave-particle coupling.

Based on this knowledge of the unstable roots for the $k_0=0$ case in the next section we shall re-examine the full dispersion equation. Also in the next section, the Brillouin diagrams for the uncoupled normal modes and the effective coupling coefficient [i.e. RHS of equation (3.15)] will be described as well as the facts they reveal about the instability. The damping coefficients relevant to the laser-plasma regime will be given.

3. Finite k_0 Dispersion Equation

a. Uncoupled Brillouin Diagrams

The dispersion equation for coupled plasmon-phonon excitation,

$$\left(\omega + i\Gamma_i - \Omega_k \right) \left(\omega + i\Gamma_i + \Omega_k \right) \left(\omega + i\Gamma_e + \Delta^+ \right) \left(\omega + i\Gamma_e - \Delta^- \right) = -\eta^2 \left[\delta^+ + R(\omega - \alpha - i\Gamma_e) \right] \quad (3.15)$$

ignoring the difference between Γ_e^+ and Γ_e^-

$$\text{where } \eta_2 = \frac{\eta_k^2 \omega_k^2}{\omega_0} \quad \text{and} \quad R = \frac{k_0}{k}$$

is written in normal mode form.¹⁴ When the pump power ($\eta_{k_0}^2$) goes to zero the solution to this equation is trivial. The oscillations are described by the uncoupled normal modes, (the equation factors).

The Brillouin diagram for the zero modulation case evolves continuously into the ω, k diagram of unstable waves for values of the pump power above threshold. The intersections in the uncoupled Brillouin diagram of one normal mode with the other shifted by the pump frequency indicate the region of interaction between those modes when the coupling is not zero. The intersections also yield the approximate frequency of the interaction and allow the identification of backward and traveling wave processes depending upon the signs of the intersecting slopes. Opposite signs occur for a backward wave process and like signs for a traveling wave process. The lack of intersections implies stability due to an inability to satisfy the frequency and wave number conservation conditions. In other words this represents a forbidden selection rule.

The Brillouin diagram corresponding to zero coupling in eq. (3.15) is shown in Figure 3-2. Equivalently these are the linear dispersion curves for ion acoustic waves and for the frequency down shifted Langmuir (electron plasma) waves. The intersections are labeled 1, 2, and 3. One corresponds to parametric decay type, two to OTSI. Three is stable.¹⁷ This can be seen from the following form of the dispersion equation given in equation (3.8a).

$$1 = \frac{\eta_{k_0}^2 k_{DE}^2 \left(\frac{M_e}{M_i} \right) \omega_{pe}^4}{D_i(\omega, k)} \left\{ \frac{1 + \frac{k_0}{k}}{D_e(\omega + \omega_0, k + k_0)} + \frac{1 - \frac{k_0}{k}}{D_e(\omega - \omega_0, k - k_0)} \right\} \quad (3.8a)$$

For this equation $\frac{2}{\eta_{k_0}} k \lambda_{DE} \ll 1$ was assumed.

Equation (3.8a) therefore can be satisfied either when

$$D_i(\omega, k) = 0 \quad \text{or} \quad D_e(\omega + \omega_0, k + k_0) = 0 \quad \text{and} \quad D_e(\omega - \omega_0, k - k_0) = 0$$

Two of these must be zero to obtain growing solutions. Thus we have:

$$1) D_i(\omega, k) = 0 \text{ and } D_e(\omega + \omega_0, k + k_0) = 0 \text{ or } D_e(\omega - \omega_0, k - k_0) = 0$$

corresponding to PD or intersections #1. This is a resonant point. Or we have:

$$2) D_e(\omega + \omega_0, k + k_0) = 0 \text{ and } D_e(\omega - \omega_0, k - k_0) = 0$$

corresponding to OTSI or intersections #2. This point is non-resonant in that $\text{Re } \omega = 0 \neq \omega_k$ when $k \neq 0$. For a non-zero pump wave number, $k_0 \neq 0$, the uncoupled dispersion diagram for the shifted langmuir wave translate in the k direction also according to equation (3.15) as shown in Figure 3-3. Thus the intersections of the uncoupled modes shift due to the finite pump wave number. The shifting of the linear modes will be used to understand the unstable roots obtained in solving the complete dispersion equation in the next section.

b. Effective Coupling Coefficient

The right hand side (RHS) of equation (3.15) is the effective coupling coefficient for this parametric coupling process. This term couples the linear modes and drives them unstable at and around particular intersections of the linear modes as discussed above.

For Nishikawa's $k_0 = 0$ solution, the sign of the coupling term was crucial in distinguishing between the resonant (PD) and nonresonant (OTSI) instabilities. Similarly we expect

$$N_{\text{eff}}^2 \equiv - \frac{n_{k_0}^2 k^2}{\omega_0} \left(\frac{M_e}{M_i} \right) \left\{ \delta' - \frac{k_0}{k} (\omega - \omega_{Te}) \right\} \quad (3.16)$$

to act in much the same way. However, for $k_0 \neq 0$, the coupling term is complex and frequency dependent. Its sign though is still important determining the type of coupling possible even though the linear modes are now asymmetric in k -space. At the zeros of N_{eff}^2 the uncoupled solutions hold. The nature of the instability switches from one type to the other at or near this point. We shall see this in the next section describing the unstable roots obtained for the $k_0 \neq 0$ case, especially in the graphs of the

the computer solutions to the dispersion equation for a given k_{DE} and $n_{k_0}^2$ as a function of $k_0^1_{De}$.

c. Damping Coefficients

The damping coefficients as a function of k_{DE} used in this analysis illustrated in Figure 3-4 are given in the following equations:

$$\nu_e = \nu_{e01} + \left(\frac{\pi}{8}\right)^{1/2} \frac{10_{De}}{ik_{\lambda De}^3} \exp\left(-\frac{1}{2k_{\lambda De}^2} + \frac{3}{2}\right) \quad (3.17)$$

$$\nu_i = \nu_{i01} + \frac{i \operatorname{Re} \omega (\pi/8)^{1/2}}{(1 + k_{\lambda De}^2)^{3/2}} \left(\frac{T_e}{T_i}\right)^{3/2} \exp\left(-\frac{T_e/T_i}{2(1 + k_{\lambda De}^2)}\right) \quad (3.18)$$

$$+ \left(\frac{M_e}{M_i}\right)^{1/2}$$

assuming the equilibrium distributions are Maxwellian and

$$\left(\frac{\kappa T_i}{M_i}\right)^{1/2} < \frac{\omega}{k} \left(\frac{\kappa T_e}{M_e}\right)^{1/2}$$

where T_σ = species temperature, κ = Boltzmann constant

These correspond to collisional damping ν_{c01} plus the wave number dependent Landau damping formulae for electrons and ions,²² the dominant damping in the laser plasma fusion regime.

C. The Solution to the Dispersion Relation for a Finite Wavelength Pump Field Driving the Coupled Plasmon-Phonon Interaction

1. Introduction

The coupled mode dispersion relation, equation (3.15), describes the normal modes in a one-dimensional, unequal temperature, homogeneous, unbounded plasma model and their nonlinear interaction due to a strong finite wave length pump field subject to

to the condition that $\eta_{k_0}^2 k_{\lambda_{DE}} \ll 1$. The solutions to this equation were obtained for the unstable modes driven both by an intense electron plasma wave of finite wave number, i.e. (ω_0, k_0) and by an intense dispersionless electrostatic wave i.e. (ω_0, k_0) . The dispersion relation was solved for the complex frequencies assuming real wave numbers, as Nishikawa has done¹³. (The complete mapping from the complex wave number space to the corresponding complex frequencies yields information as to the absolute or convection nature of the unstable roots. Cassedy¹⁷ has done this for the $k_0 = 0$ equation finding the threshold pump power for absolute instability.) The solutions were obtained by computer applying a polynomial root finder to eq. (3.15). The complex frequency solutions were obtained as a function of the wave number of the low frequency wave $k_{\lambda_{DE}}$, the wave number of the pump wave, k_0 , and the pump power normalized to the plasma thermal energy $\eta_{k_0}^2 \equiv \frac{E^2 k_0}{4\pi n_0^2 e}$, for various pump frequencies, ω_0 , mass ratios M_e/M_i , and damping coefficients, ν_e and ν_i . These solutions are presented.

The appendix briefly describes the computer program DISREL used to solve the dispersion relation, eq. (3.15). In the section following, solutions obtained will be presented and explained. The $k_0 = 0$ case will be used as an example and check. These solutions reveal the variation of the growth rate, frequency, and threshold of the instability as well as the topology of its roots. That is, the unstable roots are seen to evolve from the linear modes coupled by the nonlinearities. We see that the linear modes still have meaning in the weak pump field case. The threshold reduction obtained for an electron plasma wave as pump is then derived. Finally the effect of a finite wave length pump on the parametric decay instability is shown.

2. The Computer Solutions

The dispersion relation, eq. (3.15), was solved using a polynomial root finder. ω , the complex frequency of the low frequency mode, was found in dimensionless units for various choices of the real parameters, (ω_0, k_0) pump frequency and wave number, $k_{\lambda_{DE}}$ low frequency mode's wave number, $M_e/M_i \equiv$ mass ratio, $\eta_{k_0}^2 \equiv$ normalized pump power. The solutions obtained were graphed as a function of $k_{\lambda_{DE}}, k_0, \omega_{DE}$, or $\eta_{k_0}^2$ hold-

ing all else constant.

The solutions, $\text{Re } \omega$ and $\text{Im } \omega$ versus $k \lambda_{DE}$ for the $k_0=0$ case are quite well known (for example, see Nishikawa¹³ or Refs. 24,25). The first frames of Figures 3-5a and 3-5b show two examples of these for $\frac{\omega}{\omega_{pe}} = 1.0$ and $\omega_{0\lambda_{pe}} = 1.04$ respectively. Here we see exactly the behavior as predicted by Nishikawa¹³. The frequencies of the unstable modes are seen to exhibit the usual complex conjugate behavior. At the onset of the instability two real roots becoming complex conjugates with $\frac{\partial}{\partial k} (\text{Re } \omega) = 0$.¹⁷ The computer generated plots of ω vs. $k \lambda_{DE}$ become the usual uncoupled linear mode Brillouin diagrams (Figures 3-2 and 3-3) for $\eta_{k_0}^2 = 0$. A comparison with them is instructive.

The continuous variation of ω from the $\eta_{k_0}^2 = 0$ case is seen clearly in the solutions of the dispersion equation obtained as a function of $\eta_{k_0}^2$ for particular wave numbers k and k_0 . The value of $\eta_{k_0}^2$ is varied from a value below the threshold pump power to $\eta_{k_0}^2 = 1$. These graphically reveal the manner in which the linear modes are driven together, couple, and go unstable. The three dimensional graph, Figure 3-6, illustrates how the ω vs. $\eta_{k_0}^2$ plots evolve from the uncoupled diagram. Figures 3-7a and 3-8 are the ω vs. $\eta_{k_0}^2$ for an oscillating two stream (OTSI) and parametric decay (PD) type of interaction respectively. The onset of instability is easily seen here agreeing with Nishikawa's prediction.¹³ The topological differences between the two types of unstable roots are also evident. The OTSI instability is seen to occur when the two branches of the ion acoustic waves $\pm \Omega_k$ (or the two frequency down shifted electron plasma waves $\pm \omega_{pe}$) couple due to the pump wave forming a 4-wave¹⁷ type interaction. In this situation perfect phase matching can never occur and the OTSI is referred to as non-resonant. Parametric decay (PD) Figure 3-8 is obtained when an ion wave and a frequency down shifted electron wave couple due to the pump wave. This is a resonant instability because perfect phase matching can occur for this situation at specific k 's and k_0 's. Figures 3-7 and 3-8 will be referred to again in following section which discusses the threshold reduction when the OTSI driven by an electron plasma wave becomes of PD type as k_0 increases from zero. The progression of graphs of ω vs. k and ω vs. $\eta_{k_0}^2$ for increasing pump wave number can be examined to obtain an impression of the effect of a finite k_0 .

Finally for a particular low frequency wave number k and pump power the complex frequency solutions as a function of k_0 , the pump wave number, are shown in Figures 3-9 and 3-10. The uncoupled ($n_{kn}^2=0$) case is shown in Figures 3-9a and 3-10a. The linear modes which couple to bring about the instability can be seen to change as the pump wave number increases. The wave number at which this occurs can be predicted by the points at which $N_{eff}^2=0$. In addition a most important result is also evident. That is that the growth rate varies only slightly due to the finite pump wavelength. Thus the expression for the growth rates obtained in the $k_0=0$ case yield good approximations to the values for the $k_0 \neq 0$ case. In the PD instability Figure 3-10 the unstable frequency is no longer "locked" due to the more complex coupling coefficient. It oscillates about ω_k as k_0 varies. At the points when $\text{Re} \omega$ equals the ion acoustic frequency, i. e. at perfect resonance, the growth rate has a maximum as is seen in Figures 3-9, 3-10b, and 3-10c. In the case with an electron plasma wave as pump the growth rate does go to zero at the pump wave number where the effective coupling coefficient goes through zero and changes sign. In general, though, the growth rate remains constant over a range of $k_0 \lambda_{DE}$ beyond the onset of strong Landau damping. Remember that the damping of the pump wave was not included in the calculation.

3. Threshold Reduction for the Oscillating Two Stream Instability (OTSI) in the Weak Damping Domain

The dispersion relation (for the parametric plasmon-phonon interaction driven by an electrostatic wave at (ω_0, k_0) assuming $\Gamma_{e_{k+k_0}} \approx \Gamma_{e_{k-k_0}} \approx \Gamma_e$

$$\left(\omega + i\Gamma_e - \omega_k \right) \left(\omega + i\Gamma_e + \omega_k \right) \left(\omega + i\Gamma_e + \Delta_+ \right) \left(\omega + i\Gamma_e - \Delta_- \right) = \frac{-\eta k_0^2 \left(\frac{M_e}{M_i} \right) k^2}{\omega_0} \left(\delta \frac{k_0}{k} (\omega - \alpha + i\Gamma_e) \right) \quad (3.15)$$

where $\Delta_{\pm} \equiv \omega_0 - \omega_{ek \pm k_0} \equiv \delta_{\pm} \mp \alpha$; $\alpha = 3kk_0$, $\delta_{\pm} = \omega_0 - \omega_{ek} - 3k_0^2/2$ for $k\lambda_{DE}$ and $k_0\lambda_{DE} \ll 1$

yields growing roots in complex conjugate form when the nonlinear coupling term i.e. RHS of equation (3.15), or of equations (3.1) and (3.2), "drives" two of the linear

modes such that their real frequencies are equal, or nearly so and a positive imaginary root occurs. (Strong damping can allow the unstable solutions ω_A and ω_B not to be complex conjugates with $|\text{Re}\omega_A - \text{Re}\omega_B| \lesssim \Gamma_e$. They still act in general like complex conjugate pairs. The unstable roots depend upon the interaction of two roots. In the same way eq. (3.15) can be solved by neglecting the ω dependence of two of the normal mode terms reducing the equation from quartic to quadratic.²³ The approximation is valid whenever the unstable $\omega \ll$ two of the four resonant frequencies, Δ_+ , Δ_- , Ω_k or $-\Omega_k$. This condition holds for the OTSI case in which the ion frequency is much less than the other characteristic frequencies. Transformation to coordinates centered at the intersections of the linear dispersion curves allows this type of solution in the more general case.¹⁷

This technique yields results analogous to Nishikawa's.¹³ For unstable wave numbers k such that $k\lambda_{DE} \frac{2}{3} \left(\frac{M_e}{M_i} \right)^{1/2}$ assuming

$$\omega_{-A_-} \sim \Delta_-, \omega_{+A_+} \sim \Delta_+ \text{ and } \Omega_k - \Delta_- < 0 \text{ i.e. } k_0 < \frac{k}{2} - \frac{1}{3} \left(\frac{M_e}{M_i} \right)^{1/2}$$

$$\text{we obtain } (\omega + i\Gamma_i)^2 \approx \Omega_k^2 \left(1 - \frac{\eta_{k0}^2}{\eta^2 \frac{k_0}{TR}} \right) \quad (3.19)$$

$$\text{where } \eta_{TR}^2 \equiv \frac{-\omega_0^2 \delta'^2 + \Gamma_e^2}{\delta'^2 + R\alpha} - \alpha^2 \quad (3.19')$$

$$\text{and } k_0 < \frac{k}{2} - \frac{1}{3} \left(\frac{M_e}{M_i} \right)^{1/2} \quad R \equiv k_0/k, \quad \alpha \equiv 3kk_0 \text{ for } k\lambda_{DE}, k_0\lambda_{DE} \ll 1 \quad (3.19'')$$

This expression yields a threshold suppression for finite k_0 provided $3/2 k^2 > \Gamma_e^2$, satisfied in the laser plasma regime. Figure 3-7 illustrates this.

Expressions for the real frequency near threshold and the growth rate near threshold follow from equation (3.19):

$$\text{Re } \omega_{TR} \approx \frac{\Omega_k \omega_k}{1\delta'^2 + R\alpha} \quad (3.20)$$

$$\gamma_{TR}^{\text{near}} \approx \Omega_k \left(\frac{\eta_{k0}^2}{\eta^2 \frac{k_0}{TR}} - 1 \right)^{1/2} \quad (3.21)$$

Solving by iteration for the growth rate we have

$$\gamma_{\text{near TR}} \approx \Omega_k \sqrt{\frac{\eta_{k0}^2}{\eta_{\text{TR}}^2} - 1} + \Omega_k \left(\frac{\eta_{k0}^2}{\eta_{\text{TR}}^2 + \Omega_k^2} \left(\frac{\eta_{k0}^2}{1\delta^2 + R\alpha} \left(\frac{\eta_{k0}^2}{\eta_{\text{TR}}^2} + \frac{\Omega_k^2}{1\delta^2 + R\alpha} \right) \dots \right) - 1 \right)^{1/2} \quad (3.22)$$

These values agree well with the computer solutions for the threshold pump power and γ_{near} near the threshold as shown by Tables 3-I and 3-II. For the smaller wave number modes, i.e., those for which $k\lambda_{\text{DE}} < \frac{2}{3} \left(\frac{M_e}{M_i} \right)^{1/2}$ the same sort of approximation is made ignoring ω cf. $\pm \Omega_k$. In this way, the dispersion relation can be solved to obtain for $k_0 \cdot k/2$

$$\omega \approx \alpha - i\Gamma_e + \delta + \left(\frac{\eta_{k0}^2}{-\omega_0^2 \delta^2 / (\delta^2 + R\alpha)} - 1 \right)^{1/2} \quad (3.23)$$

Here again the minimum threshold pump power for onset of the unstable roots is given by an expression reducing to Nishikawa's for $k_0 = 0$.

$$\eta_{\text{TH}}^2 = - \frac{(\delta^2 + R\alpha) \omega_0^2}{\sigma^2 + R\alpha} \quad (3.24)$$

This threshold value does not show a reduction for k_0 nonzero. Thus the important threshold suppression occurs only for the higher ion wave numbers such that $k_0 \lambda_{\text{DE}} > \frac{2}{3} \left(\frac{M_e}{M_i} \right)^{1/2}$ provided $k_0 \lambda_{\text{DE}}$ is small enough to satisfy $\Delta_- \ll k$.

When the pump wave number increases further such that

$$\Omega_k + \Delta_- > 0 \quad (3.25)$$

which reduces to

$$k_0 > \frac{k}{2} - \frac{1}{3} \left(\frac{M_e}{M_i} \right)^{1/2} \quad \text{for } \omega_0 = \omega_{ek_0} \quad (3.26)$$

a resonant type of unstable solution may be obtained. Again using the same technique to approximate the solution, we find

$$\omega = \frac{\Omega_k + \Delta_-}{2} - i \left(\frac{\Gamma_i + \Gamma_e}{2} \right) - \left(\frac{\eta_{\text{R}}^2}{2B_1} \right) \quad (\text{continued on next page})$$

$$+ \left\{ \left(\frac{n_R^2}{B} \right)^2 + \frac{4n^2}{B^2} (k_0 - k_{omin})(k_0 - k_{omax}) + \left[(\Lambda_- - \Omega_k) + i(\Gamma_e - \Gamma_i) \right]^2 \right\}^{1/2}$$

where

$$\rho \equiv \frac{M_e}{M_i}, \quad B = 2\Omega_k \left(-(\Lambda_+ + \Omega_k) - \Gamma_i \Gamma_e + i \Gamma_i \left[(\Lambda_+ + \Omega_k) + 2\Gamma_e \Omega_k \right] \right) \quad (3.27)$$

$$R \equiv \frac{k_0}{k}, \quad n^2 = n_{ko}^2 \left(\frac{M_e}{M_i} \right) k^2 / \omega_0^2$$

$k_{omin/max}$ are the solutions to the equation

$$\delta^2 - R \left(\frac{\Omega_k + \Lambda_-}{2} \right) + R\alpha = 0 \quad (3.27a)$$

for an electron plasma wave as pump the solution to (3.27a) is

$$k_{omin}^{max} = \frac{\left(\frac{k}{2} - \frac{\rho}{3} \right)^{1/2}}{2} \mp \frac{1}{2} \left(\left(\frac{k}{2} - \frac{\rho}{3} \right)^2 + 4k^2 \right)^{1/2} \quad (3.27b)$$

This solution eq. (3.27) appears to be in good agreement with the complete solution to eq. (3.15) plotted in Figure 3-11. At the threshold of the instability the real solution is the average of the two uncoupled solutions. It decreases with n_{ko}^2 increasing.

Equation (3.27) yields a condition for the range of pump wave numbers which lead to instabilities of this form as well as a condition on the pump power for instability. k_0 must satisfy the following inequality for instability.

$$\delta^2 - R \left(\frac{\Omega_k + \Lambda_-}{2} \right) + R\alpha < 0 \quad (3.28)$$

where $\frac{\Omega_k + \Lambda_-}{2}$ is an approximation to ω . For an electron plasma wave pump and $k\lambda_{oe} < 1$, $k_0\lambda_{oe} < 1$ this reduces to the condition that k_0 must satisfy the following inequality.

$$k_{omin} < k_0 < k_{omax} \quad (3.29)$$

The symbols are defined in eq. (3.27b). Thus this type of solution can occur and yield growing roots for pump wave numbers satisfying the following inequality assuming an electron plasma wave pump (ω_{ek_0}, k_0)

$$k - \frac{\rho}{3} \frac{1/2}{\omega_0} < k_0 < k_{\text{omax}} \quad (3.30)$$

Equation (3.28) is cubic in k_0 for a dispersionless ($\omega_0 = \text{constant}$) finite k_0 pump. The solution is not as simple but has the same effect of restricting the possible unstable k_0 's. It could be solved in various approximations by reducing (3.28) to quadratic.

Also the pump power must satisfy the following inequality to obtain these growing solutions:

$$n_-^2 < n_+^2 < n_+^2 \quad \text{where} \quad n_+^2 = \frac{\eta_{k_0}^2 \rho k^2}{\omega_0} \quad (3.31)$$

$$\text{and} \quad n_{\pm}^2 = 2 \left(k_0 - k_{\text{omin}} \right) \left(k_{\text{omax}} - k_0 \right) / R^2 \quad (3.32)$$

$$\pm \frac{1}{R^2} \left[4 \left(k_0 - k_{\text{omin}} \right)^2 \left(k_{\text{omax}} - k_0 \right)^2 - R^2 \left(\Delta_- - \Omega_k \right)^2 \right]^{1/2}$$

n_-^2 and n_+^2 are the roots of the equation

$$n^4 \left(\frac{k_0}{k} \right)^2 + 4n^2 \left(\delta + \frac{k_0}{k} \left(\alpha - \frac{\Omega_k + 1}{2} \right) \right) + \left(\Omega_k - 1 \right)^2 = 0 \quad (3.32a)$$

The threshold power becomes, when k_0 is not near k_{omax}

$$n_{\text{th}}^2 = \frac{B \left[\left(\Gamma_i + \Gamma_e \right)^2 + \left(\Delta_- - \Omega_k + i \left(\Gamma_i - \Gamma_e \right) \right)^2 \right]}{4 \left(k_0 - k_{\text{omin}} \right) \left(k_{\text{omax}} - k_0 \right)} \quad (3.33)$$

This expression was used in Table 3-I obtaining agreement with the computer solution. Also equation (3.21) for the growth rates was used in Table 3-II. They agreed well with the computer solution of the full equation.

n_{TR}^2 eq. (3.33) reduces to the following simple expression for negligible damping and k_0 well below k_{omax}

$$n_{\text{TR}}^2 = \frac{\left(\Omega_k - 1 \right)^2}{-4\delta^2} \quad (3.34)$$

or including damping we obtain

$$n_{\text{TR}}^2 = \left[\left(\Gamma_i + \Gamma_e \right)^2 + \left(\Omega_k - 1 \right)^2 \right] / \left(4\delta^2 \right) \quad (3.34a)$$

Here the resonance nature of the instability is shown by the fact that $\eta_{IR}^2 \rightarrow 0$ for perfect matching and negligible damping. Thus the dispersion of the pump wave and of the normal modes of the system allow this resonant instability to occur in the given ranges of k_0 and $\eta_{k_0}^2$ space specified by inequalities (3.30) and (3.31).

When $k_0 \sim k_{omax}$ this mode is stabilized as $\eta_{k_0}^2$ increases. This is predicted in eqs. (3.31) and (3.32) and clearly seen in Figure 3-12. The dependence of the growth rate upon pump power is also shown. The unstable region with the lower pump power (PD like) has a growth rate γ which increases only as η_{k_0} . The OTSI region is obtained above η_+^2 and has $\gamma \propto \eta_{k_0}^{2/3}$. The same type of analysis can be done for a dispersionless pump although it is somewhat non-physical, obtaining similar results. Finally the wave numbers of the unstable electron modes in the region of reduced threshold are seen to be $k-k_0$ since the coupled unstable modes are ω_k and the down shifted frequency $\Delta_- = \omega_0 - \omega_{ek-k_0}$. As usual for the PD branch longer wavelength (Shorter k) modes are driven. Thus we have shown that finite values of k_0 can alter the behavior of the OTSI branch. A resonant type decay process is made possible due to the dispersion of the ion acoustic and electron plasma waves because the relevant phase matching condition is now shifted in k space. This reduces significantly the pump power necessary for threshold of the instability in the small damping case because of its resonant nature. This reduction in threshold was derived analytically by solving the dispersion relation in the case $Re\omega \neq 0$ as for OTSI, by reducing it to a quadratic equation. These solutions compared favorably with the solutions to the full dispersion relation. Finally Figure (3-13) shows the threshold pump power vs. $k_0 \lambda_{DE}$ for several wave numbers $k \lambda_{DE}$ of the unstable low frequency growing mode. This illustrates the large threshold reduction obtained with the instability is resonant satisfying the perfect phase matching conditions eq. (3.1). This behavior could be applied to the nonlinear mode conversion properties of a spectrum of electron plasma waves as in Figure 3-1. We have shown a threshold reduction occurs for the exponential growth of modes driven by the finite k_0 electron plasma waves just as those in this spectrum. This threshold reduction takes place due to a change in

the nature of the instability from non resonant to resonant. This in turn changes the wave numbers of the growing modes. Our analysis would therefore predict the following sequence of events. A pump wave at $\omega_0 = \omega_{pe}$ which can be thought of as electrostatic (cf Figure 1-1) derives primarily the most unstable wave numbers of the electron plasma waves. These finite k_0 electron plasma waves experience the threshold reduction we have discussed above. They therefore quickly begin to drive unstable modes themselves. The latter unstable modes are driven by the finite k_0 pumps in a PD type process. This feeds the energy to lower wave numbers causing the spectrum to flatten out for low wave numbers.

For wave numbers above the original fastest growing modes the spectrum of E_k^2 vs. k takes on the $1/k^2$ shape as shown in the Figure 3-1. These higher k modes are driven both by the original incident pump wave and the finite k unstable electron plasma waves with powers above the threshold value n_+^2 from eqs. (3.31) and (3.32).

3. Asymmetry of Unstable Wave Numbers for the Parametric Decay Branch

The general dispersion relation equation (3.15) has also been solved assuming a dispersionless electrostatic pump with $\omega_0 > \omega_{pe}$ i.e. the underdense case. This is somewhat nonphysical because an electrostatic wave in a plasma would satisfy the normal Bohm Gross dispersion relation, eq. (1.1). This situation is considered to illustrate the effect of the finite pump wave number on the parametric decay branch. Referring to Figures 3-2b and 3-3b the asymmetry brought about as k_0 increases from zero is easily seen. The intersection of the frequency of the $k-k_0$ electron plasma wave with Ω_k corresponding to the PD interaction (i.e. intersection #1, a perfect phase matching point where eqs. (3.1) are satisfied), approaches intersection #2 as k_0 increases. The region in k space with unstable roots therefore shrinks and eventually goes to zero when the two intersections coalesce. On the other hand the $k + k_0$ Langmuir wave's dispersion curve moves to the left. The region in k space dominated by the PD intersection expands. We obtain therefore an increase in the width of the unstable spectrum in k space. (See the progression of ω vs. k for increasing k_0 ; Figure 3-5b 1), 2), 3). Also ω vs. $k_0 \lambda_{DE}$ for PD Figure 3-10 [especially

part C)] exhibits a variation in the value of the growth rate with $k_0 \lambda_{DE}$. The maximum values occur near the perfect phase matching points where $\text{Re} \omega = \omega_k$. The wave number of the fastest growing low frequency mode does decrease in magnitude as the Brillouin diagrams predict. Again see Figure 3-5b. However, it does not decrease as fast as $|k_0|$ increases. This causes the fastest growing electron mode which is at $k_{\text{fastest growing ion mode}} + k_0$ in turn to increase and leads to the growth of higher wave number, slower phase velocity electron waves. For example with $k_0 \lambda_{DE} = 0.1$, $k_{\text{fastest growing ion mode}} = 0.09$ and thus $k_{\text{electron}} = 0.19$. This could also be an important point in considering the nonlinear mode conversion in the parametric interactions driven by finite wavelength pump fields. As discussed earlier the growth of electron waves with high wave numbers ($k \lambda_{DE} \geq 0.2$) leads to strong wave-particle interactions. The situation therefore of a finite wavelength, "underdense", (i.e. $\omega_0 > \omega_{pe}$) dispersionless pump, if it arose, could bring about strong anomalous heating thru mode conversion to higher k modes and then Landau damping of the waves by the particles.

D. Summary

In this chapter we have studied the parametric coupling which occurs among the electrostatic modes of an unequal temperature ($\theta_e \gg \theta_i$) plasma due to high frequency (ω_{pe}) electromagnetic (reduced to one dimension (See Figure 1-1)) or electrostatic pump wave. A finite wavelength pump wave was allowed.

The problem of parametric coupling in the limit of an infinite wavelength pump field has been previously treated as described in the introduction to this chapter. The results are well known (eq. Ref. 13) and have often been applied to the laser fusion plasma regime (eq. Ref. 8, 24, 25, 26). As stated in the introduction the effect of a finite wave length pump field has not been thoroughly investigated. This was the purpose of Chapter III.

In this Chapter first, the dispersion relation of eq. (3.15) governing this interaction was derived using the two fluid model of a plasma and assuming $n_{k0}^2 k \times \lambda_{DE} < 1$. Secondly, the dispersion relation was examined, solved by computer, and

analytically in various approximations. The main results different from the $k_0=0$ case were emphasized. Finally we used these results to predict their effect on the evolution of the plasma wave spectrum in a plasma irradiated by an intense electromagnetic wave.

A spatially inhomogeneous pump wave can significantly reduce the threshold power for onset of the instability from the $k_0=0$ case. This is the most important result of a finite wave number pump. Nishikawa¹³ obtained a threshold reduction for the strong damping PD regime. That is, in the case for which

$$n_k^2 (= \omega_1^2 \text{ in Nishikawa's notation}) \ll \Gamma_e \tau_i$$

the parametric decay (PD) branch is known to oscillate far from its resonant frequency at $\omega_k \approx \omega_e \tau_i^{-1/2}$. The finite wavelength pump field introduces an extra parameter, k_0 , upon which ω depends. This parameter allows the frequencies to become resonant for some k_0 greatly reducing the threshold power necessary for onset of the instability. The damping coefficients of interest for the laser-plasma interaction satisfy the opposite inequality. In this range also we found that an important threshold suppression occurs due to a change in the nature of the instability from non-resonant (OTS1) to resonant (PD). This takes place because of the shifting of the resonant frequencies due to dispersion. The analytic expression for the finite k_0 , reduced threshold pump power are given by eq. (3.19') for small k_0 satisfying (3.19'') and by eq. (3.33) or (3.34) for k_0 satisfying inequality (3.2) (or equivalently, (3.30)). This behavior of the parametric instability for a finite wave length pump wave should cause the unstable plasma wave spectrum to flatten out for modes with wave numbers less than that of the most unstable mode in the $k_0=0$ case.

For the (somewhat non-physical) case of an intense dispersionless electrostatic wave in an underdense plasma (i.e. $\omega_0 = \text{const.} > \omega_{pe}$), $k_0 \neq 0$ the parametric decay (PD) instability is affected by the pump's wave number, k_0 . The perfect phase matching conditions shift in k -space. An asymmetry in the spectrum of growing modes occurs if only one sign of k_0 is present. Electron modes with wave numbers higher

than that of the pump are also predicted. This fact could lead to stronger "collisionless" plasma interactions than those predicted by the $k_0=0$ results. This is, of course, an important result for the anomalous heating of plasmas. Finally it was shown that the growth rate of the instability is not significantly effected by the spatial inhomogeneities of the pump wave. This fact allows one to use the $k_0=0$ solutions for the growth rates in cases of $k_0 \neq 0$. We have used this useful result in the analysis of Chapter IV: Mode Coupling of Electron Plasma Waves.

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E. Appendix: THE COMPUTER PROGRAM, DISREL

DISREL, a program to solve equation (3.15), used a polynomial root finder allowing complex coefficients, CPOLY, written by Dr. C. Sherman.* The damping coefficients described by equations (3.17) and (3.18) were used. The four solutions for complex ω were obtained as a function of $k\lambda_{DE}$, the unstable low frequency wave number, $k_0\lambda_{DE}$ the pump wave number, and η_{ko}^2 the normalized pump power. The program was checked by solving the dispersion relation for $\eta_{ko}^2=0$ to obtain the uncoupled Brillouin diagrams. In addition the roots were substituted back into the dispersion equation to insure that they were indeed roots. The approximation to the Bohn-Gross frequency for small $k\lambda_{DE}$, $\omega_{ek} \approx \omega_{pe} (1 + 3/2 k\lambda_{DE})^2$ in one-dimension is assumed in DISREL, thus restricting the validity of the solution to $k\lambda_{DE} \lesssim 0.6$. This is well above the onset of strong damping which occurs at $k\lambda_{DE} \approx 0.25$. Thus the restriction to $k\lambda_{DE} \lesssim 0.6$ is of little importance. The input parameters are the mass ratio $M_e/M_i = \rho$, the temperature ratio T_e/T_i which is used only in calculating the ion damping but it must be $\gg 1$ for this model to be valid, the pump wave number, the ion wave number to be considered and the pump power. Two of the last three are specified and the third is varied to obtain the solution as a function of this parameter. The computer graphs and computer solutions referred to in this chapter were all obtained using DISREL.

*The algorithm used is that of Jenkins and Traub CACM 15, 1972, Algorithm 419.

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Figure 3-1

The turbulent field spectrum from a computer solution of the coupled mode equations, eqs.(3.1) and (3.2) for $n_{\text{pump}}^2 = 0.13$ and $T=1250$.

Figure 3-2

Brillouin diagrams for uncoupled ion acoustic wave and electron plasma wave shifted by a pump wave with frequency ω_0 , wave number $k_0=0$, $m_e/M_i=10^{-2}$.

- a) $\omega_0 = \omega_{pe}$, $\delta = \omega_0 - \omega_{ek} < 0$ for all k 's.
- b) $\omega_0 = 1.04 \omega_{pe}$, sign of δ indicated on drawing.

Intersection points indicated correspond to:

1. Backward wave, PD type; $\delta > 0$
2. Backward wave, OTSI type; $\delta < 0$.
3. Travelling wave, stable.

Brillouin diagrams for uncoupled ion acoustic waves and electron plasma wave shifted by pump at ω_0 and $k_0 \lambda_{De} = 0.1$. $\omega_i = 10^2$. Intersections shift compared with Figure 3-2.

- a) $\omega_0 = \omega_{ek0}$; pump wave is an electron plasma wave. Intersection at origin remains. New intersections, type #1 and #3 arise as indicated.
- b) $\omega_0 = 1.04 \omega_{pe}$. Corresponding intersections are shifted from $k_0 = 0$ positions and labeled with primes.

Figure 3-4

High and low frequency damping coefficients ν_e and ν_i versus $k\lambda_{De}$ corresponding to eqs. (3.17) and (3.18) are plotted with

$$\nu_{e\text{col}} / \omega_{pe} = 1.6 \cdot 10^{-4}$$

$$\nu_{i\text{col}} / \omega_{pe} = 1.6 \cdot 10^{-5}$$

$$T_e/T_i = 30.0$$

Figure 3-5

This figure consists of two sets of Brillouin diagrams (ω vs. k plots) for a system of ion acoustic waves and electron plasma waves coupled by a finite wavelength pump wave with wave number k_0 . In each set $k_0 \lambda_{De}$ was varied from zero as indicated. $m_e/M_i = 10^{-2}$, $n_{k0}^2 = 0.1$. Compare these with Figures 3-2 and 3-3 showing the uncoupled Brillouin diagrams. Only the first and third quadrants are given here because of the logarithmic axes. The results are symmetric. The complete solution all four roots to the dispersion equation, eq. (3.15) are plotted with the real part as "dotted" lines and the imaginary values of ω as "solid" lines.

- a) For this first set of solutions the pump wave frequency followed the electron plasma wave dispersion relation, $\omega_0 = \omega_{ek_0}$ except for part 5 where $\omega_0 = \omega_{pe}$. The pump wave numbers are

- 1) $k_0 \lambda_{De} = 0.0$
- 2) $k_0 \lambda_{De} = 0.005$
- 3) $k_0 \lambda_{De} = 0.05$
- 4) $k_0 \lambda_{De} = 0.1$
- 5) $k_0 \lambda_{De} = 0.1$ and $\omega_0 = \omega_{pe}$
- 6) $k_0 \lambda_{De} = 0.17$

Note that the maximum growth rate remains nearly constant. The exact functional dependence of $\text{Im } \omega$ vs. k does vary with k_0 . The unstable $\text{Re } \omega$ solution is no longer zero for $k_0 = 0$. Also different unstable branches form as k_0 increases due to the fact that ω_{ek_0} increases. Graph 5) is included for completeness. Compare it with 4). These graphs should be compared with the uncoupled plots of Figure 3-2 and 3-3.

Figure 3-5 (Continued)

b) Here the pump wave is dispersionless with a frequency $\omega_0 = 1.04 \omega_{pe}$ and k_0 as listed:

- a) $k_0 \lambda_{De} = 0.0$
- b) $k_0 \lambda_{De} = 10^{-2}$
- c) $k_0 \lambda_{De} = 0.1$
- d) $k_0 \lambda_{De} = 0.17$

Notice the two unstable branches which divide at the point where $\delta=0$. This equation defines a "cut off" wave number, k_{co} . As k_0 increases one of these, the PD branch, itself splits due to asymmetry introduced by $k_0 \neq 0$. OTSI branch behaves as in Part 2.

Figure 3-6

The three dimensional graph shown illustrates the ω vs. k dependence of the ion acoustic and down shifted electron plasma wave as well as the dependence of the frequency, ω , on the normalized pump power, η_{ko}^2 . In the $\eta_{ko}^2=0$ plane the uncoupled Brillouin diagrams are obtained just as in Figure 3-2. In a plane of constant $k\lambda_{De}$ the merging of two uncoupled solution and the onset of instability is illustrated.

Figure 3-7

Real $[\text{Re}\omega]$ and imaginary $[\text{Im}\omega]$ frequencies satisfying eq. (3.15) as a function of normalized pump power, $n_{k_0}^2$. As the pump power increases an unstable root (i.e. $\text{Im}\omega > 0$) occurs corresponding to the OTSI branch. Low frequency wave number, $k\lambda_{De} = 0.15$. The pump wave is an electron plasma wave, $\omega_0 = \omega_{ek_0}$ and has a wave number k_0 as listed. Δ_- and Δ_+ indicate the shifted electron plasma wave frequencies. The threshold pump power for onset of the instability is given by equation (3.19).

a) $k_0\lambda_{De} = 0.0$

b) $k_0\lambda_{De} = 0.04$

Figure 3-8

ω satisfying eq. (3.15) is plotted versus n_{ko}^2 . In this figure the two unstable roots correspond to the PD branch. The pump wave has $\omega_o = 1.04 \omega_{pe}$ and $k_o \lambda_{De} = 0.0$. Also $m_e/M_i = 10^{-2}$. The low frequency wave number, $k \lambda_{De} = 0.12$. Please note well: In this figure the code to denote the real and imaginary parts of the frequency is the reverse of that used in the other figures.

Figure 3-9

Complex frequency solutions, ω , versus the wave number of the pump wave, k_0 , λ_{De} given an electron plasma wave as pump, $\omega_0 = \omega_{ek_0}$ [except part d) where $\omega_0 = \omega_{pe}$]. The pump power, $\eta_{k_0}^2$, and the low frequency wave number, $k\lambda_{De}$ are held constant for each graph.

- In a) $\eta_{k_0}^2 = 0.0$, $k\lambda_{De} = 0.16$, $\omega_0 = \omega_{ek_0}$
 b) $\eta_{k_0}^2 = 0.1$, $k\lambda_{De} = 0.16$, $\omega_0 = \omega_{ek_0}$
 c) $\eta_{k_0}^2 = 0.1$, $k\lambda_{De} = 0.1$, $\omega_0 = \omega_{ek_0}$
 d) $\eta_{k_0}^2 = 0.1$, $k\lambda_{De} = 0.1$, $\omega_0 = \omega_{pe}$

Part a) is the uncoupled stable solution and part b) the corresponding unstable roots. The uncoupled real frequencies are labeled by $\pm \Omega_k$, ion acoustic frequency or $\Delta_{\pm} \equiv$ the frequency up or down shifted plasma waves $\equiv \omega_{ek_0} - \omega_{ek \pm k_0}$

The coupled unstable solutions are labeled PD and OTSI. Compare c) and d) showing the different roots for a pump with dispersion c) $\omega_0 = \omega_{ek_0}$ and without d) $\omega_0 = \omega_{pe}$.

Figure 3-10

Complex frequency solution to eq. (3.15) vs. $k_0 \lambda_{De}$ for constant n_{ko}^2 , $k \lambda_{De}$, ω_0 as given below.

- a) $n_{ko}^2=0.0$, $k \lambda_{De}=0.16$, $\omega_0=1.04\omega_{pe}$
- b) $n_{ko}^2=0.1$, $k \lambda_{De}=0.16$, $\omega_0=1.04\omega_{pe}$
- c) $n_{ko}^2=0.1$, $k \lambda_{De}=0.1$, $\omega_0=1.04\omega_{pe}$

The width of the unstable region in $k_0 \lambda_{De}$ space is a function of $k \lambda_{De}$ as seen by comparing parts b) and c), and of pump power compare a) and b). In all cases the $k_0 \lambda_{De}=0.0$ solution corresponds to the parametric decay (PD) type of instability in which the ion acoustic and shifted electron plasma branches couple.

Figure 3-11

Real and imaginary frequencies satisfying eq. (3.15) as a function of normalized pump power are shown. As the pump power increases an unstable root appears with a threshold pump power predicted by eq. (3.34). The wave numbers satisfy the inequality eq. (3.30) as required. In this particular graph $\omega_0 = \omega_{ek_0}$, $k_0 \lambda_{De} = 0.113$, $k \lambda_{De} = 0.15$, $m_e/M_i = 10^{-2}$. Notice the 4th real solution is far off scale and therefore not included in this graph. It would have a value $\Delta_+ = 7.99.10$ at $n_{k0}^2 = 0$.

Figure 3-12

Real and imaginary frequencies satisfying eq. (3.15) as a function of normalized pump power, η_{k0}^2 are shown. Here $\omega_0 = \omega_{ek_0}$, $k_0 \lambda_{De} = 0.12$, $k \lambda_{De} = 0.15$, and $m_e/M_i = 10^{-2}$. $k_0 \lambda_{De} \sim k_0 \lambda_{De}^{\max}$ as defined in eq. (3.27b) = 0.13. For the unstable region specified by $\gamma_{an_{k0}}$ the pump power satisfies inequality (3.31). Again the Δ_+ solution is off scale.

Figure 3-13

Threshold pump power, $\frac{n_{TR}}{2}$ versus $k_0 \lambda_{De}$ for unstable wave numbers, $k \lambda_{De} = 0.06667$, 0.1, 0.15, are plotted. A large reduction in threshold pump power occurs for the value of pump wave number k_0 which satisfies the perfect phase matching condition, equations (3.1), just as should occur for resonant type instabilities.

THRESHOLD PUMP POWER SUPPRESSION DUE TO
FINITE k_0 ELECTRON PLASMA WAVE PUMP

Pump Wave Number $k_0 \lambda_{De}$	Unstable Ion Wave Number $k \lambda_{De}$	η_{TR}^2 Calculated Using Eq (3.19)	η_{TR}^2 From Solution To Eq (3.15)
0.0	0.15	$3.38 \cdot 10^{-2}$	$3.4 \cdot 10^{-2}$
0.04	0.15	$2.74 \cdot 10^{-2}$	$2.7 \cdot 10^{-2}$
0.042	0.15	$2.75 \cdot 10^{-2}$	$2.7 \cdot 10^{-2}$
0.13	0.15	0.138	0.14
0.20	0.15	$8.07 \cdot 10^{-2}$	0.08
		Using Eq (3.33)	
0.113	0.15	$0.206 \cdot 10^{-2}$	$0.22 \cdot 10^{-2}$
	where $k_0 \max$	= 0.113	
0.06	0.06667	$3.0 \cdot 10^{-4}$	$2.4 \cdot 10^{-4}$
	where $k_0 \max$	= 0.06667	

Table 3-I

This table gives a comparison of the threshold pump power, n_{TK}^2 , obtained using the approximate analytical solution eq. (3.19) or (3.33) and the computer solution of eq. (3.15), the full dispersion equation. Here $\Delta_0 = 1.585 \cdot 10^{-2}$. The two solutions vary by at most 6% and 20% for eq. (3.19) and (3.33) respectively.

Unstable Mode's Wave Number $k_{\lambda De}$	Pump's Wave Number $k_o \lambda De$	Pump Power $\eta_{ko}^2 = \frac{E_k^2}{8\pi n k T_e}$	Calculated* Growth Rate γ_{CAL}/ω_{pe}	Computer Solution of Eq (3.15) $\gamma_{DISREL}/\omega_{pe}$
0.15	0.0	0.1	0.021	0.018
0.15	0.0	0.2	0.033	0.024
0.15	0.0	0.42	0.051 to 0.025	0.035
0.15	0.042	0.1	0.024	0.017
0.15	0.042	0.2	0.037	0.025
0.15	0.042	0.4	0.056 to 0.021	0.032
0.15	0.042	0.55	0.065	0.040

*Using eq (3.21) and (3.22) when there are two values .

Table 3-II

A comparison of the approximate analytical results using eqs. (3.21) and (3.22) and the computer solution of the full dispersion relation, eq. (3.15), are given. The results are always within a factor of 1.75 or less of each other.

CHAPTER IV

Mode Coupling of Finite Amplitude Electron

Plasma Waves

A. INTRODUCTION

One of the important topics of concern for laser fusion is the interaction of electromagnetic radiation with a plasma, specifically the process by which the laser energy can be fed into the plasma and can cause heating of the plasma particles. This involves the transfer of energy from electromagnetic energy to plasma thermal energy. For collisionless plasmas the energy flow proceeds as follows. First by means of the parametric instabilities¹ the laser drives long wavelength standing electron plasma waves and ion waves. The electron waves have phase velocities much larger than the particles' thermal velocity. For very large field strengths ($n^2 \geq \frac{E^2}{4\pi n_0 k T_e} > 1$) electrons are trapped by the unstable waves and heated strongly. It has been shown that the trapping mechanism explains the unstable wave saturation and the anomalous particle heating for $n > 1$.^{2,3} For small field strengths, $n \lesssim 0.5$ such that no trapping occurs² Katz, et. al.³ showed that the anomalous heating mechanism is diffusion in velocity space due to the broad spectrum of plasma waves caused by mode coupling. In addition short wavelength (low phase velocity) electron plasma oscillations, excited by mode coupling, heat the plasma particles by Landau damping. The detailed mechanism of this mode coupling is not understood. Here we study a simple case to clarify the mode coupling processes of such long wavelength, weak ($n^2 \ll 1$) electron plasma waves.

In this investigation, a one dimensional, cloud-in-cell plasma⁴ simulation code was used. Initially, electrostatic plasma waves were excited at wave numbers corresponding to those which result from the parametric decay of laser light. Instead of the expected exponential growth, t^2 growth of specific ion waves and t^3 growth of electron modes near the initially perturbed pump waves in k-space was observed. This growth occurred faster than that predicted for parametric excitation. These results led us to the direct driving model for finite wave number mode coupling which is explained and justified in Section B. In Section C. the computer calculations are

presented. They are shown to agree with the theoretical predictions obtained in part B. In Section D. we summarize the results and argue that these processes occurring for a system with only a few resonant modes can be expanded to the case in which many modes are present. Therefore, these results imply that this sort of two step direct driving is the dominant mode coupling mechanism leading to the turbulent spectrum obtained in simulation codes as well as in numerical solutions to the model equations.^{5,8}

8. COUPLED MODE EQUATIONS AND DIRECT DRIVING

The time evolution of the resonant modes in an unequal temperature, two species plasma in one dimension can be described by the coupled mode equations. These are derived from the two species fluid equations by separating them into low and high frequency contributions (where low frequency implies frequencies $\sim$ the ion acoustic frequencies and high frequencies are those $\sim$ the electron plasma frequency) and keeping terms up to second order in all the field components, so that any of these could act as pump.⁵ They are the following, ignoring damping:

$$\frac{1}{n_0} \frac{\partial^2 n_i(k)}{\partial t^2} + \Omega_k^2 \frac{n_i(k)}{n_0} = - \frac{k}{4\pi M_i n_0} \sum_{K'} k' E_{k'}^{\omega} E_{k-k'}^{\omega}, \quad (4.1)$$

$$\frac{\partial^2}{\partial t^2} E_k^{\omega} + \omega_{ek}^2 E_k^{\omega} = - \frac{\omega_{pe}}{n_0} \sum_{k'} E_{k-k'}^{\omega} \propto n_i(k')$$

where $n_i(k) \equiv k^{\text{th}}$ component of ion density fluctuation

$n_0 \equiv$ the background density

$E_k^{\omega} \equiv k^{\text{th}}$ component of the high frequency fields

$M_i \equiv$ ion mass

$\Omega_k \equiv$ ion acoustic frequency for k^{th} mode

$\omega_{ek} \equiv$ electron plasma wave (Bohm-Gross) frequency for k^{th} mode, eq.(1.1)

Notice the right hand side of equation (4.1) is the force term due to the high frequency electric field pressure gradient $\frac{\nabla(E_k^{\omega})^2}{8\pi}$ ^{6,7}. The right hand of equation (4.2) is a high frequency force term due to the charge fluctuations caused by a high frequency pump wave oscillating in the presence of a low frequency density inhomogeneity, setup

by an ion wave.⁷

These equations are quite unmanageable algebraically without making some approximations. They have been solved by computer for a finite set of wavenumbers.^{5,8} The results were found to reproduce the simulation results for both mode growth and mode spectrum. However, the physical mechanism of the mode coupling was not clarified by these solutions.

A resonance approximation can be applied to equations (4.1) and (4.2). One then obtains a dispersion relation for these waves which can be solved to obtain the unstable modes (i.e. those with $\text{Im} > 0$). This was done by K. Nishikawa¹ and others (see Introduction, Chapter III) for an infinite wavelength pump and by the author allowing for a finite pump wavelength.¹⁰ In this approximation the ion and electron modes necessarily exponentiate simultaneously with growth rate γ given by the imaginary part of the frequency solution. Nishikawa's¹ results are well-known. He finds for pump wave number $k_0=0$ two unstable branches depending upon the relative magnitude of the pump frequency compared with the resonant frequency of the high frequency mode.

Here we are primarily interested in the so called oscillating two stream branch where $\omega_{\text{pump}} < \omega_{ek}$. For electron plasma waves it is this branch which excites modes with wave numbers higher than the pump's. Thus the pump's energy is fed toward lower phase velocity waves which can be Landau damped heating the particles. Nishikawa derives the following expression for the growth rate of the fastest growing mode, well above threshold assuming the pump wavenumber $k_0=0$ as is done throughout this paper:

$$\frac{\gamma_{\text{max}}}{\omega_{pe}} = \left(\frac{M_e}{M_i} \frac{k^2 \lambda_{De}^2 n^2}{2} \right)^{1/3} \quad (4.3)$$

$$\text{where } \lambda_{De} = \frac{\kappa T_e}{4\pi n e^2} \quad 1/2$$

κ = Boltzmann Constant

T_e = Electron temperature

M_e = electron Mass

n = plasma density

$n^2 = (E^2)/(8\pi \kappa T_e)$

A computer solution of the general dispersion relation, including the possibility of spatially inhomogeneous pump waves, reveals that the growth rates for the oscillating two stream branch depend only weakly on the pump wavenumber, k_0 . For a typical pump power ($\eta_{\text{pump}}^2 = 0.1$), Figure 4-1 shows growth rate of a particular electron plasma wave $k\lambda_{pe} = .08$ versus $k_0\lambda_{De}$. The growth rate remains nearly constant for pump wavenumbers of interest (not strongly Landau damped). Thus Nishikawa's expression for $\gamma_{\text{max}}/\omega_{pe}$ of (4.3) can be used to obtain a characteristic time for mode growth due to finite wave number pumping fields. This will be compared with the characteristic time for a similar growth due to the other possible mode coupling process proposed here, namely two step direct driving.

This direct driving mode coupling process can be understood by considering the coupled mode equations one at a time as harmonic oscillator equations directly driven near their resonant frequencies. We were led to this type of analysis by the behavior of the growing modes in our simulation calculations, as shall be described in detail in Section C. Analogous to initial conditions of the simulation calculations, assume a simple system in which only a few finite amplitude electron plasma oscillations are initially excited. The growth of the ion and electron modes is thought of as occurring one after the other instead of simultaneously.

Analytically one has an initially excited, standing electron plasma wave at (k_0, ω_{ek_0}) called the pump.

$$E^{\text{pump}} = 2E_{k_0} \sin(k_0 x) \cos(\omega_{ek_0} t) \quad (4.4)$$

All the other modes in the system are small in comparison to the pump and will be ignored. Equation (4.1) then yields a driven harmonic oscillator equation in which the driver term has a wavenumber equal to $\pm(2k_0)$ and frequencies $\pm\omega_{ek_0}$ and 0. The high frequency terms are ignored because the ions cannot react on that fast time scale. The zero frequency contribution is near the resonant frequency of the ion waves. It can, therefore, strongly drive ion waves at $\pm 2k_0$. If more than one pump mode in k -space were initially excited, ion waves at all possible sums and differences of the pump wavenumbers

would be directly driven in this way. Solving equation (4.1) using equations (4.4) and the initial conditions that at $t=0$ $n_i(k)=0$ and $\frac{\partial n_i(k)}{\partial t}=0$. One obtains

$$\frac{n_i(2k_0)}{n_0} = \frac{n_{k_0}^2}{2} (1 - \cos n_{2k_0} t) \quad (4.5)$$

The directly driven solution is weakly dependent on the initial conditions as contrasted with the resonant instability analysis where $n_i = n_i(t=0) \times \exp(\gamma t)$. The solution eq. (4.5) is valid for times short compared with the damping time of the high frequency pump and with the growth time of the other high frequency modes. This condition is easily satisfied as the simulations will demonstrate. Equation (4.5) can be expanded for $n_{2k_0} t \ll 1$, i.e., $\omega_{pe} t \ll \left(\frac{M_i}{M_e}\right)^{1/2} \left(\frac{1}{2k_0 \lambda_{De}}\right) \approx 50$ for our simulations examples to yield

$$n_i(2k_0)/n_0 = n_{k_0}^2 \omega_{pe}^2 t^2 / 4 \quad (4.6)$$

Next in the environment of the originally excited electron plasma oscillations and the driven ion oscillations, there is a nonzero, high frequency driving term on the right hand side of equation (4.2). With one standing electron plasma wave (k_0 , ω_{ek_0}) and the driven ion wave at $\pm 2k_0$, the high frequency equation is driven only at $k=\pm k_0$. This gives no mode coupling of the high frequency modes but just coupling back to the original pump. If more than one high frequency mode were initially pumped, mode coupling among the high frequency modes is obtained. (More than one pump wave models the situation of interest since the parametric instability excites wave in a region of wavenumber space broader than the simulation mode spacing, $\Delta k \lambda_{De} = 0.0123$ or less). This behavior is observed in the simulations. Assuming two or more different wave number pump waves, the right hand side of equation (4.2) is non-zero at some wave numbers not equal to those initially present.

We take, for example, the simplest case in which just two high frequency standing electron plasma modes at $(\pm k_0, \omega_{ek_0})$ and $(\pm(k_0 + \delta), \omega_{ek_0 + \delta})$ are the pump waves. These non resonantly drive ion acoustic waves at wavenumbers $\pm \delta$, $\pm 2k_0$, $\pm (2k_0 + \delta)$, and $\pm (2k_0 + 2\delta)$. The forcing term in equation (4.2) can be non-zero for $k=\pm(k_0 + \delta)$ and $\pm(k_0 + 2\delta)$ as well as for the uninteresting cases of $k=\pm k_0$ and $\pm(k_0 + \delta)$. These non-zero

terms are just those corresponding to the possible wave number matching conditions. (Also modes at $\pm 3k_0$, etc. are driven. These can be important if the pump waves have small wavenumbers so that $3k_0\lambda_{De}$ is in the range of wavenumber space not strongly Landau damped.) Assuming only the above mentioned modes are present because others have not yet grown to significant levels, the "early-in-time" growth of these high frequency modes can be approximated. At early times the right hand side of equation (4.2) is proportional to $t^2 \cos(\omega_{ek_0} t)$ because $n_i(2k_0) \propto t^2$. Separating E_k^2 into two parts, one varying as $\exp(i\omega_{ek} t)$ and the other varying on a much slower time scale

$$E_k^2(t) = \epsilon_k(t) \left[\exp(i\omega_{ek} t) + \text{c.c.} \right] \quad (4.7)$$

$$\text{where } \dot{\epsilon}_k \equiv \frac{\partial \epsilon_k}{\partial t} \ll \omega_{ek} \epsilon_k$$

and substituting into equation (4.2), one obtains to lowest order in $\frac{\epsilon_k}{\epsilon_k}$ that

$$\frac{\epsilon_{k_0-\delta}(t)}{\left(\frac{4\pi n_0 k T_e}{\epsilon_0} \right)^{1/2}} = \frac{1}{24} \eta(k_0+\delta) \eta_{k_0}^2 \left(\frac{k_\lambda}{D_0} \right)^2 \left(\frac{m_e}{M_i} \right) \left(\omega_{pe} t \right)^3 \quad (4.8)$$

Likewise for $k = -(k_0 - \delta)$.

In Section C it will be shown that the simulation results verify the dominance of this two step, non-resonant in frequency, direct driving model for the mode coupling of finite amplitude electron plasma waves. An analysis of the time scales involved also demonstrates this point. It is well known that for infinitesimal pump powers the resonant solutions, based on expansions in $\eta_{\text{pump}} \equiv \frac{E_{\text{pump}}}{\sqrt{4\pi n_0 k T_e}}$, must dominate. However, for the unstable waves resulting from laser plasma interactions, the pump power is not infinitesimal but finite with $\eta_{\text{pump}}^2 \approx 0.1$. Comparing the time necessary for the growing modes to become significant for the different processes, inequalities, which specify whether non-resonant direct driving or resonant exponentiation dominates, are derived. These involve the ratio between the initial amplitudes of the pump and growing modes

as well as the pump wave number.

Define τ_{pe} as the time such that $n_k(\tau) = \frac{n_{\text{pump}}(t=0)}{3}$ for the fastest growing mode. Let $N_0 \equiv n_k^{(0)}$. Assume the two pump waves initially have equal amplitudes $n_{\text{pump}_1}^{(0)} = n_{\text{pump}_2}^{(0)} = n_{\text{pump}}$ and wavenumbers $k_0 \lambda_{De}$. We then obtain a characteristic time for resonant mode growth

$$\omega_{pe} \tau_{\text{Res}} = \frac{1}{\gamma} \ln \left(\frac{n_{\text{pump}}}{3N_0} \right) \quad (4.9)$$

where γ is the growth rate for parametric instabilities, and

$$\omega_{pe} \tau_{\text{DD}} = \left[\frac{8 \left(M_i / m_e \right)}{n_{\text{pump}_1} n_{\text{pump}_2} \left(k_0 \lambda_{De} \right)^2} \right]^{1/3} \quad (4.10)$$

which is the characteristic time for the direct driving process. From eq. (4.9) we see that for resonant growth τ_{Res} is a function of the initial level of the mode N_0 (even though a rather weak function). For direct driving creates the initial level which was assumed to be zero, has essentially no effect on the mode growth because of direct driving creates as well as amplifies the waves. Eq. (4.10) yields the fact that τ_{DD} depends on the presence of two pump waves as well as on the finite wave number of the pumps. As the pump wave number approaches zero, $k_0 \rightarrow 0, \tau_{\text{DD}} \rightarrow \infty$. That is, this direct driving process cannot occur for spatially homogeneous pump waves. The ion waves would not be directly driven at certain wavenumbers by a pump with infinite wavelength. Without these ion waves, the rapid coupling of the high frequency modes⁹ could not occur. In all of the above analysis, pump depletion is ignored and the solutions describe only the early in time evolution of the system.

The condition for the non-resonant direct-driving processes to dominate over resonant processes is just

$$\tau_{\text{Res}} > \tau_{\text{DD}}$$

or using equations (4.9) and (4.10)

$$\gamma \ln \left(\frac{n_{\text{pump}}}{3N_0} \right) \left[\frac{1}{8} \left(\frac{m_e}{M_i} \right) \left(n_{\text{pump}} k_0 \lambda_{De} \right)^2 \right]^{1/3} \quad (4.11)$$

Using equation (4.3) for γ , the maximum growth rate well above threshold for the oscillating two stream instability, and using $k \lambda_{De} = 0.15$ which is a maximum estimate for the wavenumber at which the maximum γ occurs, equation (4.11) results in

$$k_0 \lambda_{De} > 0.3 \left/ \left[\ln \left(\frac{\eta_{\text{pump}}}{3N_0} \right) \right] \right|^{3/2} \quad (4.12)$$

or in general

$$k_0 \lambda_{De} > \left[\frac{2 k \lambda_{De}}{\ln \left(\frac{\eta_{\text{pump}}}{3N_0} \right)} \right]^{3/2} \quad (4.13)$$

For the case in which initially the pump waves are about two orders of magnitude greater than the background fields i.e., $\frac{\eta_{\text{pump}}}{3N_0} \approx 10^2$ we find

$$k_0 \lambda_{De} > 0.2 k \lambda_{De} \quad (4.14)$$

Or taking $k_0 \lambda_{De}$ as given one obtains that when $k_0 \lambda_{De} > 0.1$, non resonant processes lead to faster mode growth than resonant ones unless the initial level of the background fields, $N_0 > \frac{\eta_{\text{pump}}}{30}$. Thus, in typical parametric decay cases, non-resonant processes will dominate the mode coupling of the electron plasma waves once the fastest growing modes have been driven by incident electromagnetic radiation. This is a very interesting result in that the dominance of a non-resonant process over a resonant one is only a very weak function of the level of the fields, depending specifically on the logarithm of the relative field amplitudes. The more important parameter is found to be the wavenumber of the pump, k_0 . However, at least two pump waves are required for this non-resonant process to occur.

C. SIMULATION CALCULATIONS OF THE MODE COUPLING

Computer simulation calculations confirm the results predicted above concerning the physical mechanism of the mode coupling of finite amplitude electron plasma waves. A one-dimensional, electrostatic, cloud-in-cell⁴ simulation code in which a large number of electrons and ions are allowed to evolve in time in their self-consistent fields was used. Typically 16,000 to 32,000 electrons and ions were followed in a system with a spatial periodicity of 512 electron Debye lengths (λ_{De}). The temperature ratio was normally set to $\frac{T_e}{T_i} = 30$ and the mass ratio to $\frac{m_e}{M_i} = \frac{1}{100}$. Our technique was to excite one or more standing long wavelength electron plasma modes such as are produced by the

parametric decay of an electromagnetic wave. This excitation was achieved by simply imposing a sinusoidal perturbation on the initial electron positions (and velocities if travelling waves are desired) at specific wavelengths.

The necessity of more than one high frequency pump wave to obtain electron plasma wave mode coupling was immediately obvious from the simulation calculations as Figures 4-2 and 4-3 demonstrate. These plot relative mode amplitude $\left(\begin{smallmatrix} 2 & 2 \\ n_k & n_{-k} \end{smallmatrix} \right)^{1/2}$ vs. $k\lambda_{De}$ at different points in time. For only one pump wave (Figure 4-2) essentially no mode coupling was seen. In Figure 4-2 the harmonics of the pump are immediately excited because they are part of the nonlinearities of the pump. However, the first harmonic at $\left(\pm 2k_0, 2\omega_{ek_0} \right)$ is far off resonant and does not grow. The third harmonic does grow due to direct driving by the pump and the driven ion wave as predicted above. Figure 4-3 covers the same time interval as Figure 4-2 for a simulation in which two standing electron plasma waves with different wavelengths act as the pumps. Here the mode spectrum changes significantly evolving to a generally turbulent state by $\omega_{pe}t \approx (28)\times(2\pi)$.

We use this typical simulation to illustrate other aspects of the computer calculations which confirm the direct driving picture. Here the 9th and 11th modes of the system (corresponding to $k_0\lambda_{De} = \frac{2\pi}{512} \times (\text{mode \#}) = 0.1107$ and 0.135 since the periodicity length is $512\lambda_{De}$) are initially excited to a level $n_{k_0} \approx 0.3$ for which no particle trapping in the wave occurs. In times very short ($t_{pe} \approx 30$) we have seen that these modes undergo significant mode coupling. According to the non-resonant, direct driving picture ion wave at mode numbers 18, 20, 22, and 2 should be strongly driven and grow initially as t^2 . This is observed. The time evolution of the 17th, 18th, 19th, and 20th ion modes given in Figure 4-4 clearly show t^2 growth on the log-log graph. The direct driving analysis predicts that next high frequency electron plasma waves should be driven at wavenumbers $k_0 - \delta$ and $k_0 + 2\delta$ for pumps at k_0 and $k_0 + \delta$. These modes correspond to numbers 7 and 13. They should grow in time as t^3 . This is clearly shown in Figure 4-5 which is the time history of the relative amplitude of the 7th and 13th modes. Thus the non-resonant direct driving model of

the mode coupling agrees with the simulation results. In addition no exponential mode growth as predicted by the resonant parametric instability analysis was observed, substantiating again the direct driving model for the mode coupling occurring here.

Figures 4-6 and 4-7 are another set of $|n_k|$ vs $v_{pe} t$ and $n_i(k)$ vs $v_{pe} t$ time histories for a similar simulation as Figure 4-3 with mass ratio $\frac{m_e}{M_i} = \frac{1}{100}$. The higher mass ratio slows down the mode coupling $\propto \left(\frac{m_e}{M_i}\right)^{1/3}$. For that reason the t^3 and t^2 growth are even more striking. Log-linear graphs are also included for completeness. They illustrate the obvious lack of exponential growth which would indicate mode coupling due to instabilities. Finally another set of $|n_k|$ time histories is given in Figure 4-8. These correspond to a simulation in which electron plasma waves oscillating at the 3rd and 4th modes of a system with spatial periodicity $1024 \lambda_{De}$ acted as the pump waves. Thus $k_0 \lambda_{De, \#3, \#4} \approx 0.019, 0.025$ and $n_{\#3, \#4} = 1.41$. {These waves have very high phase velocities so that trapping effects do not dominate even for this high value of n_{k_0} } According to our simple theory, for pump wavenumbers in this range direct driving and resonant exponentiation should occur on similar time scales. This is substantiated by the simulation results in which t^3 growth of the directly driven modes is easily seen. In addition exponential growth occurs for those modes which are not directly driven. See Figure 4-8. The growth rate agrees with the predicted values for parametric decay (oscillating two stream branch). Both of these mode coupling processes take place on comparable time scales.

D. SUMMARY

In conclusion, we have investigated the mode coupling mechanisms occurring among a small number of long wavelength electron plasma oscillations in an unequal temperature plasma. The coupled mode equations, eqs. (4.1) and (4.2) derived in Chapter II, result in two types of behavior, 1) the well-known linear instability, the parametric excitation of coupled waves¹ and 2) a two-step, off resonant direct driving process⁹.

We have shown using theory and simulation calculations that the latter mechanism dominates the mode coupling process for finite amplitude standing electron plasma waves such as result from parametric instabilities and with powers such that no particle trapping in the wave occurs. This mechanism proceeds as follows. First, standing

finite amplitude electron plasma modes directly drive standing ion acoustic waves at half each electron mode wavelength. Then the electron modes couple with the ion waves to drive other electron plasma waves directly. This process very rapidly results in a turbulent spectrum of electron modes as seen in Figure 4-3.

A comparison of the characteristic times for mode growth due to each of these mechanisms was also made to substantiate the dominance of the direct driving process for energy transfer among modes. This analysis yielded the fact that direct driving should dominate the resonant processes such as the parametric instabilities for finite amplitude electron plasma waves with wave numbers $k_{0\lambda_{De}} > 0.03$ (i.e. eq. (4.15)) assuming that the waves amplitude is two orders of magnitude (or more) above background levels and the fastest growing mode has wave number $k_{\lambda_{De}} \leq 0.15$. Since electron modes with wave numbers typically $k_{\lambda_{De}} \approx 0.1$ are driven unstable in the parametric decay of electromagnetic waves, these results suggest that this two-step process is the dominate mode coupling process. These results were obtained for the simple case of only a few pump modes, however, they are not limited to this case. Carrying thru the analysis for a large number of pump modes is impractical because of the huge number of terms involved, nevertheless, coupling would occur. Indeed we feel that it is this process which dominates the mode coupling as soon as finite wavelength standing electron plasma waves have grown to sufficient levels. Thus we have shown that although the parametric instabilities drive the unstable electron plasma waves to large amplitude, the subsequent formation of the broad band turbulence seen in simulation and experiments can be attributed to direct driving at least equally as well as to the initial instability. We wish to emphasize that in such a system with finite (not infinitely small) amplitude waves interacting the possibility of processes other than linear instabilities should be carefully considered.

ACKNOWLEDGEMENTS

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Figure 4-1

Plotted here are the real and imaginary frequency solution of the general dispersion equation, eq. (3.15), versus the pump wave number $k_0 \lambda_{De}$ for a particular electron plasma wave number, k_{epw} . Using the notation of Chapter III $k_{epw} = k + k_0$. Here the growth rates for both k_{epw} are shown on a logarithmic axes by changing the sign of the wave number.

a) $k_{epw} \lambda_{De} = 0.08$

b) $k_{epw} \lambda_{De} = -0.08$

The other parameters, held constant throughout are $n_{\text{pump}}^2 = 0.1$, $T_e/T_i = 30.0$, and $m_e/M_i = 10^{-2}$. The pump wave obeys the dispersion relation $\omega_0 = \omega_{ek_0}$ of an electron plasma wave given in eq. (1.1). For simplicity only the unstable solutions are plotted here. There are, of course, four complete sets of solutions to eq. (3.15). [This figure can be compared with Figure 3-9C) which plots ω/ω_{pe} vs $k \lambda_{pe}$ for a given unstable low frequency wave at $k \lambda_{De} = 0.1$. In that figure $k_{epw} = k + k_0$ and therefore k_{epw} shifts with k_0 .] The value of the growth rate γ_{max} predicted by eq. (4.3) is indicated.

Figure 4-2

Relative mode amplitudes, $\left(\eta_k^2 + \eta_k^2 \right)^{1/2} / \eta_0$, versus $k\lambda_{De}$ at different times are shown. In these graphs mode #9 is initially excited to a level such that no particles are trapped in the wave. $k\lambda_{De} = \frac{2\pi}{512} \times$ (mode number). $T_e/T_i = 30$, $m_i/m_e = 10^{-2}$.

a) $\nu_{pe} t = 4$

b) $\nu_{pe} t = 28$

where $\nu_{pe} = 2\pi \omega_{pe}$

All modes in the system are plotted. In 28 plasma periods the pump wave is still more than an order of magnitude above all the other modes.

Figure 4-3

Relative mode amplitude versus $k\lambda_{De}$ at different times. The parameters of this run were the same as in Figure 4-2 except that both modes #9 and #11 were initially perturbed.

a) $v_{pe}t=4$

b) $v_{pe}t=16$

c) $v_{pe}t=28$

Strong mode coupling is seen to have flattened the spectrum in 28 plasma periods.

Figure 4-4

The ion density fluctuations for specific mode numbers versus time from the simulation of Figure 4-3. Initially electron modes number 9 and 11 of the system were excited. Ion modes #18 and #20 are seen to grow in time as t^2 .

Figure 4-5

The normalized high frequency mode amplitude $= \left\{ n_k^2 + n_{-k}^2 \right\}^{1/2}$ versus time shows the mode growth of modes a) #7 and b) #13 for the simulation of Figure 4-3. The initial amplitude of the pump wave is indicated for reference. Both modes are seen to grow two orders of magnitude as t^3 .

Figure 4-6

n wave time plots for a simulation calculation with modes #9 and #11 initially excited and $T_e/T_i = 30$, $M_i/m_e = 1000$, $k\lambda_{De} = \frac{2\pi}{512}$ (mode #) Graphs a) and b) show the t^3 growth of the fastest growing mode #7. Part b) has a linear scale in time to show the lack of exponential growth. Part C) shows the n wave time plot of the pump wave at mode #9 illustrating the validity of neglecting pump depletion.

Figure 4-7

ion density fluctuation histories showing t^2 growth of certain modes (i.e. #2, #18, #20, #22) for a simulation with parameters as in Figure 4-6.

- a) Ion Modes #1, 2, 3, 4 are shown.
- b) Ion Modes #17, 18, 19, 20 are shown.
- c) Same as b) except linear axis in time.
- d) Ion modes #21, 22, 23, 24 are shown.

The growing modes are exactly as predicted.

Figure 4-8

These time histories were generated by our 1-D simulation code with parameters $M_i/m_e=100$, $T_e/T_i=30$, $k\lambda_{De}=(\frac{2\pi}{1024})$ (mode #). Modes #3 and 4 were excited at $t=0$.

- a) The "pump" wave amplitude at mode #3 $|n_{\{3\}}|$ versus $v_{pe}t$ is plotted.
- b) Log-log plot of $|n_{\{5\}}|$ vs. $v_{pe}t$ shows t^3 growth.
- c) This is the same as b) $|n_{\{5\}}|$ vs. $v_{pe}t$ on linear-log axes. No exponential growth is seen.
- d) $|n_{\{6\}}|$ vs. $v_{pe}t$ shows exponential growth with $\gamma=0.17 v_{pe}$.
For this case eq. (4.3) predicts $\gamma=0.19 v_{pe}$. The finite k_o solution gives $\gamma = 0.13 v_{pe}$.
- e) $|n_{\{11\}}|$ vs. $v_{pe}t$ illustrates that as predicted by theory mode #11 grows as t^3 due to a pump at #3 beating with the driven ion wave at #8.
- f) Linear-log version of e).

Appendix

Plasma Simulation Using Digital Computers;

Treating Specifically a One-Dimensional, Electrostatic Code CYCLOPS

A. Introduction

Plasma simulation using digital computers has become a widely used tool for plasma physicists confronted with the necessity of going beyond the linearized solutions of plasma problems. The advancements both in speed and capacity of available computers is, of course, an important factor leading to the rapid growth of plasma simulation. In general the plasma simulation code follows a large number of charged "super" particles in their own self fields as well as any applied fields. The system evolves in time and space according to the known physical laws (i.e. the equation of motion, Maxwell's equations). This is achieved by merely stepping through the time dependent equations with a small time step retaining the self-consistency at each cycle.

Plasma simulations are very useful because of the simplicity of their general organization (just stepping through the force and field equations). Also because at each step all the information describing the system is known, unlimited diagnostics can be performed, leading hopefully to better physical insight. In this appendix the general technique of plasma simulation will be illustrated by briefly describing the workings of the simplest type of simulation code, namely, a one-dimensional electrostatic code. The code used for the calculations reported in Chapter IV will be used as our example. In addition quiet start loading¹ of phase space and its drawbacks which were encountered in this study will be described. Finally some advantages and disadvantages of this important method of plasma research will be given.

B. General Operation

As has been discussed in this thesis the one-dimensional, electrostatic plasma model representing a highly simplified system does apply to some important problems of plasma physics. One-dimensional, electrostatic plasma simulation codes have been used to study these and many others. Simulation codes follow a large number of charged particles representing electrons and ions in time and space as they are acted upon and respond to applied external fields as well as their own self fields.

As an example we consider the code used for this work, named CYCLOPS. It consists of a one-dimensional system of some specific length. Periodic boundary conditions are applied modelling one period of an infinitely repeating system. For computational simplicity, this system is divided into cells of length Δx , which must be shorter than the shortest half wave length of interest, i.e. $\Delta x < \lambda/2$ or $k\Delta x < \pi$. A large number of particles representing electrons and ions are initialized in the system at specific positions and velocities according to some sort of loading scheme. The self fields due to this configuration of charged particles is calculated. From this point the main time loop proceeds.

1. Main Loop

In the main loop of the program the system is evolved forward in time self-consistently in small steps (shorter than the shortest time scale of interest) in the following manner. See Figure A-1. First the routine "OUTPUT" is called. This routine takes care of the diagnostics requested. The other routines also perform calculations for the diagnostics. Next the particle velocities and positions are incremented forward in time in routine MOVE. The fields at each particle position to be used for the acceleration are evaluated from the fields at the grid points with a linear weighting scheme, C. I. C.² Then these fields at the particles are employed in a finite difference approximation to the equation of motion to calculate the new velocities and positions. In CYCLOPS the "leap frog" scheme (eq. (A.1)) is used. Finally again with a linear weighting scheme the new particle positions are used to obtain a charge density, ρ , at each grid point. In

routine EFLD a finite difference approximation to the equation

$$\nabla \cdot E = 4\pi\rho \quad (A.2)$$

calculates the "new" plasma self field. From here the process begins again. This is illustrated in Figure A-1. The loop is repeated for as many cycles as are desired. Complex Foriner transformations of the electric field are performed in EFLD for diagnostic purposes and for purposes of smoothing in wave number space. The smoothing function used is a step function which zeros the $k=0$ mode as well as all modes above NMODE, an input parameter. All other modes are unchanged. The specific equations used for each step in CYCLOPS are given in Figure A-1. The weighting scheme which is used has been shown to be useful in cutting down the non-physical fluctuations due to the discrete system.^{2,4}

In addition to the main time loop, a useful simulation code must also consist of two other elements, a method of initializing the particle positions and velocities in a useful configuration and diagnostic routines which evaluate and "output" the simulation results in an understandable and useable format.

2. Phase Space Initialization

The loading scheme used to obtain initially a uniform distribution of particles in space and a Maxwellian velocity distribution were the quiet start¹ method and the random loading method. For the quiet start each particle is loaded uniformly in space with a velocity chosen from a finite set of "beam" velocities in such a way to yield a particular distribution function, $f(v)$. This is achieved using the cumulative distribution function, $F(v)$,

$$F(v) \equiv \int_{-\infty}^v f(v)dv \bigg/ \int_{-\infty}^{\infty} f(v) dv \quad (A.4)$$

Each "beam" velocity is obtained upon dividing the unit interval (0,1) into N_b equal parts and evaluating the inverse of $F(v)$ for a value of $F(v)$ chosen within each of the N_b intervals. The distribution of the velocities is then an approximation to $f(v)$ made up of N_b "beams" of particles with different locations in space but like velocities.

Using this method of loading the mode amplitudes up to but not including the N^{th} mode where N is given by $N = \frac{\text{\# of particles of 1 species}}{\text{\# of different beams}}$ are non-existent except for round off noise. The N^{th} mode is very strongly excited. Therefore when using a quiet start it is necessary to disallow the charge density of the N^{th} mode from producing a potential or acceleration since it is a non-physical effect of the loading. This was done by zeroing the modes $\geq N$ after the Fourier transformation. However, it was found that the mode coupling with this non-physical charge density at mode number N was very strong and dominated over physical nonlinearities. This is illustrated in figure A-2 which is a graph of the relative mode amplitudes vs. $k\lambda_{De}$ at $v_{pe}t=4$ in a simulation calculation. In this example mode #7 was initially excited. In addition a quiet start loading scheme was used. 8,000 electrons and 8,000 ions were loaded into 200 beams in velocity space. Thus the 40th mode is strongly excited. Only 39 modes were allowed in the Fourier transformation solutions. However, the strong mode coupling of the zero order charge density at #40 with the excited mode #7 is strikingly obvious by the high level of mode #33. It dominates over any "physically real" nonlinearities of interest. To avoid this pitfall the number N can be chosen to be greater than twice the # of modes allowed. Then the waves resulting from the coupling of this non-physical mode with the modes of the simulation plasma will fall in part of the mode spectrum which has been disallowed, avoiding the non-physics. The problem of the "Non Physical Effects of Finite Numbers of Particles in Quiet Start Computer Simulations" has been treated in greater detail by explicitly deriving the appropriate charge density. See Reference 5. Eventually the quiet start plasma undergoes a beaming instability bringing all the mode amplitudes up to the random fluctuation level which would have occurred if a random loader were used. The statistics in the tail of the distribution function were improved in CYCLOPS by using the same number of particles in each velocity. For "random loading" each particle is assigned a velocity randomly from a Maxwellian distribution of velocities. In this scheme the modes are all equally excited to the random thermal level.

3. Diagnostics

Finally it is the simulation diagnostics which render the computer results useful. The most important asset of a simulation calculation is that at each time step all the information about the system under consideration is known. Therefore snapshots at a given time of the particle densities, the potential, the field, the distribution functions, the phase space, the mode amplitudes, etc. can be obtained. In addition particular quantities can be saved at each time step to be plotted later as a function of time. For example the total energy, particle energy, field energy, mode amplitude, and temperatures as a function of time are useful diagnostics. These are limited only by computer size and running time requirements.

C. General Evaluation

Thus plasma simulation by high speed digital computers is a popular and important tool for plasma physicists. It has the advantage of being very simple and straightforward in its general organization. Even so it can treat very complex systems. Finally it allows one to perform any number of diagnostics since at each time step all the information about the system is known. One is limited only by the computer time available.

Simulation calculations do, make great demands on computer time and space following the huge number of particles necessary to "realistically" model a system. This especially becomes a problem when 2 and 3-dimensional simulations are undertaken. Also it must be remembered that simulation calculations are far from real plasma systems containing on the order of 10^4 - 10^5 particles cf. $\sim 10^{13}$ or more. In addition, the spatial grid as well as the finite size and numbers of particles and finite time steps do affect the simulation plasma. All of these limitations must be taken into account when evaluating computer simulation results.

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Figure A-1

A schematic flow chart of the 1-D plasma simulation code CYCLOPS, shows the overall operation of the code. The equations used in the leap frog mover are

$$v_{new} = v_{old} + E \Delta t \quad x_{new} = x_{old} + v_{new} \Delta t \quad (A.1)$$

The normalization of the positions, velocities, and accelerations to computer units is assumed in eq. (A.1). The calculation of the self field is made using the finite difference approximations to $\nabla \cdot E = 4\pi\rho$ (A.2)

which is

$$\frac{E(k+1) - E(k)}{\Delta x} = \sum_{\text{species}} Q(k) + Q(k+1) \quad (A.3)$$

where $E(K)$ = electrical field at the k^{th} grid, $e/M = 1.0$

$Q(k)$ = charge density at k^{th} grid

Δx = grid space

Again normalization of this equation is assumed.

Figure A-2

The relative mode amplitudes vs. $k\lambda_{De}$ show the rapid non physical growth of mode #33 due to a coupling of the pump, mode #7, with the charge density inhomogeneity at mode #40 due to the quiet start loading. This mode plot corresponds to a time $t_{pe}=4$. Notice mode #33 is more than two orders magnitude higher than the "physically real" driven 1st harmonic at mode #14.