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**STEADY-STATE TEMPERATURE SOLUTION
FOR A HEAT-GENERATING CIRCULAR CYLINDER
COOLED BY A RING OF EQUAL HOLES**

by

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ABSTRACT

An analytical solution is presented to the steady-state heat-conduction equation for a heat-generating circular region which is cooled by a ring of equal holes spaced uniformly on a circle concentric with the boundary of the cylinder. The solution is based upon a class of harmonic functions previously defined by Howland.

Numerical results are presented in the form of dimensionless curves for the particular examples of (a) uniform heat generation within the region with cooling at the holes and insulated outer boundary, and (b) the shape factor or conductance for this geometry.

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INTRODUCTION

The thermal design of reactor components often dictates that the designer consider the necessity of the removal of generated heat from the interior of a long circular control rod or fuel element in a situation where the cylindrical surface is not available as a heat transfer surface. One possible solution is to provide a ring of axial cooling holes as indicated in Fig. 1. It was with this problem in mind that the present analysis of the temperature distribution in such a region was undertaken.

The harmonic functions utilized in the present solution were originally defined by R. C. J. Howland¹ and subsequently have been used in several investigations.²⁻⁴ The latter two references are more closely related to the present work, being solutions of torsion problems for circular cylinders pierced by a ring of axial holes³ or inset with a ring of "reinforcing" rods.⁴ However, a search of the literature did not reveal an application of these functions to the present heat-conduction situation.

Fig. 1 Cross section of cylinder with geometry indicated

Statement of the Problem

The heat-conduction system is treated as mathematically linear, steady state, and axial conduction is neglected. Constant physical properties and uniform heat generation throughout the region will be assumed. Thus, the mathematical problem reduces to that of solving the heat-conduction equation

$$\nabla^2 T(r, \rho) = - \frac{q}{k} \quad (1)$$

Two applications of the general solution are considered which subject equation (1) to the boundary conditions,

$$\frac{\partial T}{\partial r} = 0 \quad (r = r_o) \quad (2a)$$

$$k \frac{\partial T}{\partial \rho} = bh(T - T_b) \quad (\rho = \lambda) \quad (2b)$$

for convection cooling at the holes and the outer surface insulated. Also, with $q = 0$ in equation (1), and

$$T = T_o \quad (r = r_o) \quad (3a)$$

$$T = 0 \quad (\rho = \lambda) \quad (3b)$$

the conductance or "shape factor" for the region is obtained. These two boundary value problems are solved and the results presented in the form of dimensionless curves suitable for design purposes.

Nomenclature

Any consistent system of units may be employed for the following quantities.

- a = Radius of cooling holes
- $A' = \pi r_o^2 (1 - m \lambda^2 / \beta^2)$ = net area of cross section of the cylinder
- b = Radius of hole-circle of the coolant holes
- r = Radial coordinate in polar coordinates referred to the axis of the cylinder as origin
- r_o = Outer radius of cylinder
- T = Temperature
- $\Delta T_{ref} = qb^2 / 4k$
- $\Delta T = qr_o^2 / 4k$
- h = Coefficient of convective heat transfer
- k = Thermal conductivity
- A_n, B_n, C_n = Arbitrary coefficients in series solution
- q = Uniform heat generation rate per unit volume

T_b	= Bulk temperature of coolant in cooling holes
ψ	= $(T - T_b)/\Delta T_{ref}$ = Dimensionless temperature function
X	= $(T - T_b)/\Delta T$ = Dimensionless temperature function
λ	= a/b = Dimensionless cooling hole radius
β	= r_o/b = Dimensionless outer radius
N_B	= ha/k = Biot modulus, dimensionless
ρ	= Dimensionless radial coordinate referred to the center of the coolant hole at $(r = b, \varnothing = 0)$; see Fig. 1
$\varnothing$	= Angular coordinate referred to the axis of the cylinder
θ	= Angular coordinate referred to the center of the coolant hole at $(r = b, \varnothing = 0)$
m	= Number of cooling holes, an integer greater than 1
U_n	= $\text{Re}(w_n)$ = Real part of Howland's analytic functions
V_n	= $\text{Im}(w_n)$ = Imaginary part of Howland's analytic functions
$\alpha_{n,s}^m$	= Coefficients in expansions of Howland's functions around $(r = b, \varnothing = 0)$
n, s, p	= Summation indicies, integers

$w_n(z)$ = $U_n + iV_n$ = Howland's analytic functions
 z = $x + iy = re^{i\theta}$ = Complex variable referred
to the axis of the cylinder as origin
 i = $\sqrt{-1}$
 j = Iteration index
 ζ = $b\rho e^{i\theta}$ = Complex variable referred to the
coolant hole at ($r = b$, $\theta = 0$) as origin
 $\overline{\quad}$ = An average value (superimposed bar)

Subscripts:

o = quantities evaluated on the outer radius
 a = quantities evaluated on the surface of
the holes
 c = quantities evaluated at the center of
the section, $r = 0$
 max = a maximum value
 min = a minimum value
 m = a quantity related to the m holes

ANALYSIS

A brief description of the analytical solution will now be presented with a more detailed consideration given in the Appendix. Under study is a circular region of radius r_o , pierced by a ring of m uniformly-spaced holes of radius a . The angular spacing of the holes is then

$$\phi_m = \frac{2\pi}{m} \quad (4)$$

The geometry is referred to the reference dimension b , the radius of the ring of hole-centers such that the non-dimensional quantities

$$\begin{aligned} \lambda &= \frac{a}{b} \\ \beta &= \frac{r_o}{b} \end{aligned} \quad (5)$$

can be conveniently defined.

The following inequalities for λ and B will hold,

$$\lambda < \sin \frac{\pi}{m} \quad (6)$$

$$1 + \lambda < B \quad (7)$$

if the cooling holes are not to overlap or run out of the outer boundary.

The solution then requires satisfaction of equation (1) and can be taken as the sum of four terms,

$$T = -\frac{qr^2}{4k} + T_2 + T_3 + T_b \quad (8)$$

where $-qr^2/4k$ is the particular solution for a uniform generation rate q , T_2 , and T_3 properly selected harmonic functions satisfying the homogeneous equation, and T_b the bulk temperature of the coolant. From the form of (5) and (8) and the periodicity and symmetry present, it is convenient to define a dimensionless temperature function ψ ,

$$\psi = \frac{(T - T_b)}{(qb^2)/4k} = -\frac{r^2}{b^2} + \sum_{n=0}^{\infty} B_n \left(\frac{r}{b}\right)^{nm} \cos(nm\theta) + \sum_{n=0}^{\infty} C_n U_n \quad (9)$$

and further the reference temperature difference

$$\Delta T_{\text{ref}} = \frac{qb^2}{4k} \quad (10)$$

is that which would be required to remove the uniform heat generation q from a cylinder of radius b of thermal conductivity k . The series

$$\psi_2 = \sum_{n=0}^{\infty} B_n \left(\frac{r}{b} \right)^{nm} \cos (nm\phi) \quad (11)$$

in equation (9) is the solution of (1) which is harmonic everywhere in the region having the proper periodicity. The functions U_n in the second summation of equation (9) are those defined by Howland which have the required periodicity and are singular at the centers of the m cooling holes located at the points

$$z_m = (x + iy)_m = be^{2\pi ip/m} \quad (p = 0, \dots, m - 1) \quad (12)$$

in the cross section plane. These functions must be differently expanded in two regions owing to the singularities on the circle $r = b$; and thus,

$$\psi_3 = \sum_{n=0}^{\infty} C_n U_n$$

is expanded as follows:

For $r < b$

$$U_0 = \sum_{s=1}^{\infty} \frac{1}{s} \left(\frac{r}{b}\right)^{sm} \cos(sm\theta)$$

$$U_n = m(-1)^n \sum_{s=1}^{\infty} \left\{ \begin{matrix} sm+n-1 \\ n-1 \end{matrix} \right\} \left(\frac{r}{b}\right)^{sm} \cos(sm\theta), \quad n = 1, 2, 3 \dots (13)$$

For $r > b$

$$U_0 = -m \log \left(\frac{r}{b}\right) + \sum_{s=1}^{\infty} \frac{1}{s} \left(\frac{b}{r}\right)^{sm} \cos(sm\theta)$$

$$U_n = m \left[(-1)^{n+1} + \sum_{s=1}^{\infty} \left\{ \begin{matrix} sm-1 \\ n-1 \end{matrix} \right\} \left(\frac{b}{r}\right)^{sm} \cos(sm\theta) \right], \quad n = 1, 2, 3 \dots (14)$$

with

$$\begin{aligned} \left\{ \begin{matrix} s \\ n \end{matrix} \right\} &= \frac{s!}{(s-n)!n!} & s > n \\ &= 1, & s = n, \quad n = 0 \\ &= 0, & s < n \\ &= s, & n = 1 \end{aligned} \tag{15}$$

denoting the familiar binominal coefficients.

In order to apply boundary conditions on the surface of the cooling holes, alternate expansions for (9), (13) and (14) are required. Use of the following identities,

$$r^2 = b^2 (1 + \rho^2 + 2\rho \cos \theta)$$

$$\begin{aligned} r^p \cos (p\theta) &= \operatorname{Re} (z^p) = \operatorname{Re} \left[b(1 + \rho e^{i\theta}) \right]^p \\ &= b^p \sum_{s=0}^p \left\{ \begin{matrix} p \\ s \end{matrix} \right\} \rho^s \cos (s\theta) \end{aligned} \quad (16)$$

where Re denotes the "real part of," together with the details outlined in the Appendix for the U_n functions, puts the solution in the form

$$\begin{aligned} \psi (\rho, \theta) &= - (1 + \rho^2 + 2\rho \cos \theta) \\ &+ \sum_{n=0}^{\infty} B_n \left[\sum_{s=0}^{mn} \left\{ \begin{matrix} mn \\ s \end{matrix} \right\} (\rho)^s \cos (s\theta) \right] \\ &+ \sum_{n=0}^{\infty} C_n U_n \end{aligned} \quad (17)$$

with

$$U_0 = - \left[\gamma_m + \log \rho + \sum_{s=1}^{\infty} (-1)^{s+1} a_{s,0}^m \rho^s \cos (s\theta) \right]$$

$$\begin{aligned}
U_n = & (-1)^{n+1} m + \frac{\cos(n\theta)}{\rho^n} \\
& + \sum_{s=0}^{\infty} (-1)^s \alpha_{s,n}^m \rho^s \cos(s\theta), \quad n \geq 1
\end{aligned} \tag{18}$$

where

$$\begin{aligned}
\alpha_{s,0}^m &= \frac{1}{s} \sum_{p=1}^{m-1} x_p^s \\
\alpha_{s,n}^m &= \left\{ \begin{matrix} s+n-1 \\ s \end{matrix} \right\} \sum_{p=1}^{m-1} (x_p-1)^n x_p^s \\
\gamma_m &= \frac{1}{2} \sum_{p=1}^{m-1} \log \left\{ 2 \left[1 - \cos\left(2\pi \frac{p}{m}\right) \right] \right\}
\end{aligned} \tag{19}$$

and the x_p are the $m-1$ roots of the equation

$$(x-1)^m - x^m = 0 \tag{20}$$

The quantities γ_m and $\alpha_{s,n}^m$ are tabulated in the Appendix for useful ranges of $m \geq 2$, s , and n .

The general solution for any specified pair of boundary conditions on the holes and the outer radius is now formally available through the use of equations (9), (13), (14), (17) and (18). In applications, satisfaction of any pair of

boundary conditions will result in an infinite system of simultaneous (linear) equations for the arbitrary set of coefficients B_n and C_n , which must be solved (approximately) to an acceptable degree of accuracy by a suitable numerical scheme. An iteration method is applied here with details given in the Appendix.

In the next section the results of the solutions of two problems of practical interest with boundary conditions (2) and (3) are presented.

RESULTS

Uniform Cooling

Boundary conditions (2), when put in dimensionless form, become

$$\frac{\partial \psi}{\partial r} = 0 \quad (r = bB) \quad (21a)$$

$$\lambda \frac{\partial \psi}{\partial \rho} = N_B \psi \quad (\rho = \lambda) \quad (21b)$$

where the outer surface of the cylinder has been taken as insulated, and uniform cooling by a fluid with a constant convection coefficient is assumed at each of the holes. Application of conditions (21) to the forms of the solutions given by (9), (14), (17) and (18) results in the following system of equations to evaluate the coefficients B_n and C_n :

$$C_0 = - \frac{2\beta^2}{m}$$

$$B_0 = - \frac{2\lambda^2}{N_B} + (1 + \lambda^2) + \frac{2\beta^2}{m} \left[\frac{1}{N_B} - (\gamma_m + \log \lambda) \right]$$

$$- \sum_{n=1}^{\infty} B_n - \sum_{n=1}^{\infty} C_n \left[m(-1)^{n+1} + \alpha_{0,n}^m \right]$$

$$C_1 = \lambda^2 \left[\frac{(N_B-1)}{(N_B+1)} \right] \left[2 - \frac{2\beta^2}{m} \alpha_{1,0}^m + \sum_{n=1}^{\infty} C_n \alpha_{1,n}^m \right.$$

$$\left. - m \sum_{n=1}^{\infty} n B_n \right]$$

$$B_s = \beta^{-2ms} \left[- \frac{2\beta^2}{ms} + m \sum_{n=1}^{sm} C_n \left\{ \begin{matrix} sm-1 \\ n-1 \end{matrix} \right\} \right], \quad s \geq 1$$

$$C_s = \lambda^{2s} \left[\frac{(N_B-s)}{(N_B+s)} \right] \left[\frac{2\beta^2}{m} (-1)^s \alpha_{s,0}^m - \sum_{n=1}^{\infty} B_n \left\{ \begin{matrix} mn \\ s \end{matrix} \right\} \right.$$

$$\left. - (-1)^s \sum_{n=1}^{\infty} C_n \alpha_{s,n}^m \right], \quad s \geq 2 \quad (22)$$

where the value of C_0 in the first of (22) has been used in the remaining equations. With the solution of system (22)

for the C_n and B_n , temperatures at any point of interest in the cross section can be obtained. In design applications temperatures at specific points are usually of major interest. The two local maxima at the center of the section and on the outer radius,

$$\psi_c = \psi(0,0) = B_0 \quad (23)$$

$$\begin{aligned} \psi_{o \max} &= \psi(r_o, \frac{\pi}{m}) \\ &= \psi_c - \beta^2 + \sum_{n=1}^{\infty} B_n \beta^{mn} (-1)^n \\ &\quad - \frac{2\beta^2}{m} \left[-m \log \beta + \sum_{s=1}^{\infty} (-1)^s \frac{\beta^{-sm}}{s} \right] \\ &\quad + m \sum_{n=1}^{\infty} C_n \left[(-1)^{n+1} + \sum_{s=1}^{\infty} \begin{Bmatrix} sm-1 \\ n-1 \end{Bmatrix} \beta^{-sm} (-1)^s \right] \end{aligned} \quad (24)$$

are of value in estimating the temperature peaking in the section. Also of interest is the value and location of the maximum temperature on the cooling hole surface,

$$\psi_{a \max} = \psi_{\max}(\rho = \lambda, \theta_a) \quad (25)$$

This value is not directly obtainable from the solution but can be evaluated by calculating the angles θ_a where the derivative of ψ vanishes on the surface of the holes, namely,

$$\frac{\partial \psi_a}{\partial \theta} = 0 \quad (26)$$

or from the form of the solution in terms of (ρ, θ) , equation (26) becomes

$$\begin{aligned} 2 \lambda \sin \theta_a = C_0 \sum_{s=1}^{\infty} s(-1)^s \alpha_{s,0}^m \lambda^s \sin(s\theta_a) \\ + \sum_{n=1}^{\infty} C_n \left[\frac{n \sin(n\theta_a)}{\lambda^n} + \sum_{s=1}^{\infty} s(-1)^s \alpha_{s,n}^m \lambda^s \sin(s\theta_a) \right] \\ + \sum_{n=1}^{\infty} B_n \left[\sum_{s=1}^{mn} \begin{Bmatrix} mn \\ s \end{Bmatrix} s \lambda^s \sin(s\theta_a) \right] \end{aligned} \quad (27)$$

Clearly, as expected,

$$\theta_a = 0, \pi \quad (28)$$

are roots of (27), each of which may be either a maximum or minimum depending on the geometry. Additionally one other

maximum or minimum may occur. This angle is denoted by

$$\theta_a = \theta^* \quad (29)$$

The solution for θ^* requires a trial and error solution of (27). A comparison of the temperatures at $\theta_a = 0, \pi$, and θ^* then reveals the desired maximum. This surface maximum is of interest if cladding/coating temperatures are of interest, or if surface reactions with the coolant are important.

The minimum temperature on the outer radius is also of interest and is

$$\begin{aligned} \psi_{o \min} = \psi(r_o, 0) = & -\beta^2 + \sum_{n=0}^{\infty} B_n \beta^{mn} \\ & + C_o \left[-m \log \beta + \sum_{s=1}^{\infty} \frac{1}{s} \beta^{-sm} \right] \\ & + m \sum_{n=1}^{\infty} C_n \left[(-1)^{n+1} + \sum_{s=1}^{\infty} \begin{Bmatrix} sm-1 \\ n-1 \end{Bmatrix} \beta^{-sm} \right] \end{aligned} \quad (30)$$

Three average temperatures are also useful. The hole surface average is given by

$$\begin{aligned}
\bar{\psi}_a &= \frac{1}{2 \pi a} \int_0^{2\pi} (\psi)_{\rho=\lambda} a d\theta \\
&= 2 \frac{\beta^2}{m} (\gamma_m + \log \lambda) - (1 + \lambda^2) + \sum_{n=0}^{\infty} \overline{B_n} \\
&\quad + m \sum_{n=1}^{\infty} C_n \left[(-1)^{n+1} + \frac{\alpha_{o,n}^m}{m} \right] \\
&= \frac{2\beta^2}{m N_B} \left(1 - \frac{m\lambda^2}{\beta^2} \right) \tag{31}
\end{aligned}$$

where the last, simpler alternate form in (31) results from a consideration of the energy removal by the coolant. The average on the outer radius is similarly calculated:

$$\begin{aligned}
\bar{\psi}_o &= \frac{1}{2 \pi r_o} \int_0^{2\pi} (\psi)_{r=r_o} r_o d\theta \\
&= B_o + 2 \beta^2 (\log \beta - \frac{1}{2}) \\
&\quad + m \sum_{n=1}^{\infty} (-1)^{n+1} C_n \tag{32}
\end{aligned}$$

The remaining average temperature calculated is that over the entire section,

$$\bar{\psi} = \frac{1}{A'} \iint_{A'} \psi \, dx \, dy \quad (33)$$

with A' the net area of the section,

$$A' = \pi r_o^2 - m\pi a^2 = \pi r_o^2 \left(1 - \frac{m\lambda^2}{\beta^2}\right) \quad (34)$$

Calculation of (33) was accomplished by elementary integrations for the portions of the solution which are regular on the region, but the integral for ψ_3 with singularities at the hole centers was accomplished by means of a contour integration method similar to that used by Ling.³ The result is

$$\begin{aligned} \bar{\psi} = \frac{\lambda^2}{2} + \frac{1}{1 - \frac{m\lambda^2}{\beta^2}} & \left[\bar{\psi}_o - \frac{m\lambda^2}{\beta^2} \bar{\psi}_a + 1 - \frac{1}{2} (\beta^2 - \lambda^2) \right. \\ & \left. + c_1 \left(\frac{2m}{\beta^2} \right) (\lambda - 1) \right] \quad (35) \end{aligned}$$

where the results of (31) and (32) were used to simplify the form of the final expression (35). Details of the calculation are to be found in the Appendix.

The results thus far presented are relatively complex; and, although experience has indicated that convergence is quite rapid (requiring at most 10 terms of the series for reasonable accuracy), to facilitate the production of design curves the problem has been coded for the IBM 704.

The function ψ , although suitable for the previous analysis, is now replaced by another dimensionless temperature function, X , defined by

$$X = \frac{T - T_b}{(q r_o^2)/4k} = \frac{1}{\beta^2} \psi \quad (36)$$

which is more convenient for use in design. This function is simply the local-to-coolant temperature difference which is the temperature difference required to remove the generated heat from the outer surface of a solid cylinder of radius r_o .

Before proceeding to the presentation of the design curves it is valuable to record the so-called void and solidity fraction for the present geometry, namely:

$$V_f = \frac{m\pi a^2}{\pi r_o^2} = \frac{m\lambda^2}{\beta^2} \quad (37)$$

$$S_f = 1 - V_f = 1 - \frac{m\lambda^2}{\beta^2}$$

These quantities are plotted in Fig. 2 as functions of $\lambda/\beta = a/r_o$ for $m = 2, 3, \dots, 8$.

It is noted that there is a physical limitation on the range of the parameter λ/β for each hole geometry since there is a maximum value where the hole size is such that the holes are tangent to each other and to the outer circumference [refer equations (A51) and (A52)]. Thus,

$$\left(\frac{\lambda}{\beta}\right)_{\max} \leq \frac{\sin(\pi/m)}{1 + \sin(\pi/m)} \quad (38)$$

and this limit is reflected in Figs. 2 and 3. Relation (38) is given in Table I, together with the largest void fraction possible as indicated by (37).

Fig. 2 Void fraction and solidity fraction

Fig. 3 Average surface temperature of coolant holes in dimensionless form

TABLE I
LIMITING VALUES OF λ/β

$\underline{m}$	$\underline{(\lambda/\beta)_{\max}}$	$\underline{(V_f)_{\max}}$
2	0.5000	0.500
3	0.4643	0.647
4	0.4141	0.686
5	0.3702	0.685
6	0.3333	0.667
7	0.3026	0.641
8	0.2768	0.613

The average value of the dimensionless temperature on the surface of the cooling holes, $\bar{T}_a$, is obtained from equation (31).

$$\bar{X}_a = \frac{\bar{T}_a - T_b}{qr_o^2 / 4k} = \frac{2}{m N_B} \left(1 - \frac{m\lambda^2}{\beta^2} \right) = \frac{2V_f}{m N_B} \quad (39)$$

and is seen to be simply related to the void fraction of the section and the Biot modulus N_B . This average surface temperature can be found from the curves in Fig. 3 where $\bar{X}_a N_B$ is plotted vs λ/β for $m = 2, 3, \dots, 8$.

General results of the temperatures of interest for useful ranges of the three parameters l/β , λ/β and N_B for

various numbers of holes would produce an impractical number of curves. The approach taken here was to select a limited but representative set of cooling hole patterns ($m = 2, 4, 6$) and for given values of the Biot modulus indicate how the various temperatures vary as the relative location of the circle of holes is varied from the center position outward toward the bounding circle r_o . Figures 4 through 21 show

X_c = center temperature

$X_{O \text{ max}}$ = maximum temperature on outer surface
at $(\phi = \frac{\pi}{m}, r = r_o)$

$X_{O \text{ min}}$ = minimum temperature on outer surface
at $(\phi = 0, r = r_o)$

and

$\bar{X}$ = average temperature over the cross section

$\bar{X}_o$ = average temperature on outer surface

$\bar{X}_a$ = average temperature on hole surface

for $N_B = 50, 5, 1, 0.5, 0.05, \text{ or } 0.005$ as functions of $1/B = b/r_o$ with limits

$$\frac{\lambda/B}{\sin(\pi/m)} \leq \frac{1}{B} \leq 1 - \frac{\lambda}{B}$$

(Text continues on page 69)

Fig. 4a Maximum and minimum outer radius temperatures and center section temperature in dimensionless form for $m = 2$, $N_B = 50$

Fig. 4b Average temperature on the hole surface, outer radius, and entire cross section in dimensionless form for $m = 2$, $N_B = 50$

Fig. 5a Maximum and minimum outer radius temperatures and center section temperature in dimensionless form for $m = 2$, $N_B = 5$

Fig. 5b Average temperatures on the hole surface, outer radius, and entire cross section in dimensionless form for $m = 2$, $N_B = 5$

Fig. 6a Maximum and minimum outer radius temperatures and center section temperature in dimensionless form for $m = 2$, $N_B = 1$

Fig. 6b Average temperatures on the hole surface, outer radius, and entire cross section in dimensionless form for $m = 2$, $N_B = 1$

Fig. 7a Maximum and minimum outer radius temperatures and center section temperature in dimensionless form for $m = 2$, $N_B = 0.5$

Fig. 7b Average temperatures on the hole surface, outer radius, and entire cross section in dimensionless form for $m = 2$, $N_B = 0.5$

Fig. 8a Maximum and minimum outer radius temperatures and center section temperature in dimensionless form for $m = 2$, $N_B = 0.05$

Fig. 8b Average temperatures on the hole surface, outer radius, and entire cross section in dimensionless form for $m = 2$, $N_B = 0.05$

Fig. 9a Maximum and minimum outer radius temperatures and center section temperature in dimensionless form for $m = 2$, $N_B = 0.005$

Fig. 9b Average temperatures on the hole surface, outer radius, and entire cross section in dimensionless form for $m = 2$, $N_B = 0.005$

Fig. 10a Maximum and minimum outer radius temperatures and center section temperature in dimensionless form for $m = 4$, $N_B = 50$

Fig. 10b Average temperatures on the hole surface, outer radius, and entire cross section in dimensionless form for $m = 4$, $N_B = 50$

Fig. 11a Maximum and minimum outer radius temperatures and center section temperature in dimensionless form for $m = 4$, $N_B = 5$

Fig. 11b Average temperatures on the hole surface, outer radius, and entire cross section in dimensionless form for $m = 4$, $N_B = 5$

Fig. 12a Maximum and minimum outer radius temperatures and center section temperature in dimensionless form for $m = 4$, $N_B = 1$

Fig. 12b Average temperatures on the hole surface, outer radius, and entire cross section in dimensionless form for $m = 4$, $N_F = 1$

Fig. 13a Maximum and minimum outer radius temperatures and center section temperature in dimensionless form for $m = 4$, $N_B = 0.5$

Fig. 13b Average temperatures on the hole surface, outer radius, and entire cross section in dimensionless form for $m = 4$, $N_B = 0.5$

Fig. 14a Maximum and minimum outer radius temperatures and center section temperature in dimensionless form for $m = 4$, $N_B = 0.05$

Fig. 14b Average temperatures on the hole surface, outer radius, and entire cross section in dimensionless form for $m = 4$, $N_B = 0.05$

Fig. 15a Maximum and minimum outer radius temperatures and center section temperature in dimensionless form for $m = 4$, $N_B = 0.005$

Fig. 15b Average temperatures on the hole surface, outer radius, and entire cross section in dimensionless form for $m = 4$, $N_B = 0.005$

Fig. 16a Maximum and minimum outer radius temperatures and center section temperature in dimensionless form for $m = 6$, $N_B = 50$

Fig. 16b Average temperatures on the hole surface, outer radius, and entire cross section in dimensionless form for $m = 6$, $N_B = 50$

Fig. 17a Maximum and minimum outer radius temperatures and center section temperature in dimensionless form for $m = 6$, $N_B = 5$

Fig. 17b Average temperatures on the hole surface, outer radius, and entire cross section in dimensionless form for $m = 6$, $N_p = 5$

Fig. 18a Maximum and minimum outer radius temperatures and center section temperature in dimensionless form for $m = 6$, $N_B = 1$

Fig. 18b Average temperatures on the hole surface, outer radius, and entire cross section in dimensionless form for $m = 6$, $N_B = 1$

Fig. 19a Maximum and minimum outer radius temperatures and center section temperature in dimensionless form for $m = 6$, $N_B = 0.5$

Fig. 19b Average temperatures on the hole surface, outer radius, and entire cross section in dimensionless form for $m = 6$, $N_B = 0.5$

Fig. 20a Maximum and minimum outer radius temperatures and center section temperature in dimensionless form for $m = 6$, $N_B = 0.05$

Fig. 20b Average temperatures on the hole surface, outer radius, and entire cross section in dimensionless form for $m = 6$, $N_B = 0.05$

Fig. 21a Maximum and minimum outer radius temperatures and center section temperature in dimensionless form for $m = 6$, $N_B = 0.005$

Fig. 21b Average temperatures on the hole surface, outer radius, and entire cross section in dimensionless form for $m = 6$, $N_B = 0.005$

and with $\lambda/\beta = a/r_0$ as parameter. The following conclusions can be drawn from these results:

- (i) $N_B = 50$ is essentially $N_B \rightarrow \infty$
- (ii) It is rather difficult to interpolate from curves of this type for intermediate values of either N_B or (λ/β) .
- (iii) An "optimum" design or location of the coolant holes is possible such that the maximum value of the temperature X_c or $X_{O \max}$ is minimized.

If an optimum design which is defined as the minimizing of the peak temperature is required, a more useful set of design curves can be obtained. By reference to Figs. 4-21 it is clear that this optimization occurs when either

$$X_{\max} = X_{O \max} = X_c, \quad m > 3 \quad (40)$$

or

$$X_{\max} = (X_{O \max})_{\min}, \quad m = 2, 3$$

If, for given values of (λ/β) and N_B , the value of $1/\beta$ where this optimization occurs is denoted by $1/\beta = (1/\beta)_{\text{opt}}$, a plot of the best radial location for the ring of coolant holes is possible. Hence in Fig. 22, $(1/\beta)_{\text{opt}}$ is plotted vs λ/β for $N_B = 0.005, 1, \text{ and } 50$.

Fig. 22a Optimum location of holes for $N_B = 0.005$

Fig. 22b Optimum location of holes for $N_B = 1.0$

Fig. 22c Optimum location of holes for $N_B = 50$

Figure 22 indicates that for a given number of holes m this optimum location is rather insensitive to the hole size a/r_o , and also, that the Biot modulus N_B affects this optimum only slightly. In addition, the limiting geometries for the coolant holes tangent to each other, tangent to the outer circumference, and simultaneously tangent to each other and the outer surface are indicated in Figure 22. If the geometry is then chosen to be optimum, the maximum temperature over the section is as indicated in Figures 23 through 29 vs N_B with λ/B as parameter and for $m = 2, 3, \dots 8$. It is noted that the curves in this form can be used for interpolation quite readily.

Thus for design purposes the optimum value of b/r_o is selected from Figure 22 and the corresponding temperature peaking read from Figures 23 through 29.

Shape Factor

The geometrical property of the region which relates the rate of heat flow or flux through the region when the outer surfaces and hole surfaces are maintained at different, constant temperatures has been variously termed the "characteristic length" by Jakob,⁵ "shape factor" by Langmuir,⁶ and "conductance" in the mathematical literature.⁷ The term shape factor and notation S_m is adopted here. S_m is defined

(Text continues on page 81)

Fig. 23 Maximum temperature over the cross section at the optimum value of b/r_0 for $m = 2$

Fig. 24 Maximum temperature over the cross section at the optimum value of b/r_0 for $m = 3$

Fig. 25 Maximum temperature over the cross section at the optimum value of b/r_0 for $m = 4$

Fig. 26 Maximum temperature over the cross section at the optimum value of b/r_0 for $m = 5$

Fig. 27 Maximum temperature over the cross section at the optimum value of b/r_0 for $m = 6$

Fig. 28 Maximum temperature over the cross section at the optimum value of b/r_o for $m = 7$

Fig. 29 Maximum temperature over the cross section at the optimum value of b/r_0 for $m = 8$

by

$$q'' = S_m k \Delta T \quad (41)$$

where q'' is the total flux per unit axial length through the region, k the thermal conductivity, and ΔT the difference in temperature between the hole surfaces and the exterior boundary.

The appropriate solution of the heat-conduction equation (1) is of the form

$$T = \sum_{n=0}^{\infty} C_n U_n + \sum_{n=0}^{\infty} B_n \left(\frac{r}{b} \right)^{mn} \cos(mn\theta) \quad (42)$$

Recall boundary conditions (3),

$$T = T_0 \quad (r = r_0)$$

$$T = 0 \quad (\rho = \lambda)$$

The heat flux then can be calculated from either

$$q'' = m \int_0^{2\pi} k \left[\frac{\partial T}{\partial \rho} \right]_{\rho=\lambda} \lambda d\theta$$

or

$$q'' = \int_0^{2\pi} k \left[\frac{\partial T}{\partial r} \right]_{r=r_0} r_0 d\theta$$

(43)

By using expansion (14) for the U_n when $r > b$ and expansions (16) and (18) around the holes, equations (43) give

$$q'' = -2\pi mk C_0$$

or a shape factor of

$$S_m = -2\pi m C_0 \quad (44)$$

where $\Delta T = 1$ for the present purpose. Satisfaction of the boundary conditions (3) results in the following system of equations to determine the C_n and B_n ,

$$C_0 = - \left[\frac{1 + \sum_{n=1}^{\infty} C_n \alpha_{0,n}^m + \sum_{n=1}^{\infty} B_n}{(m \log \beta - \log \lambda - \gamma_m)} \right]$$

$$B_0 = 1 + C_0 m \log \beta + m \sum_{n=1}^{\infty} (-1)^n C_n$$

$$C_s = -\lambda^{2s} \left[C_0 (-1)^s \alpha_{s,0}^m + \sum_{n=1}^{\infty} (-1)^s \alpha_{s,n}^m C_n \right. \quad (45)$$

$$\left. + \sum_{n=1}^{\infty} \left\{ \begin{matrix} mn \\ s \end{matrix} \right\} B_n \right], \quad s = 1, 2, 3 \dots$$

$$B_s = -\beta^{-2ms} \left[\frac{C_0}{s} + m \sum_{n=1}^{\infty} \left\{ \begin{matrix} ms-1 \\ n-1 \end{matrix} \right\} C_n \right], \quad s = 1, 2, 3 \dots$$

This system can be solved by iteration as suggested previously, and it is to be noted that only the value of C_0 is required for the evaluation of the shape factor.

Results of calculations of S_m for various numbers of holes ($m = 2, 3, \dots, 8$) as functions of $1/B = b/r_0$ with $\lambda/B = a/r_0$ as parameter are plotted in Figs. 30 through 36.

(Text continues on page 91)

Fig. 30 Shape factors for $m = 2$

Fig. 31 Shape factors for $m = 3$

Fig. 32 Shape factors for $m = 4$

Fig. 33 Shape factors for $m = 5$

Fig. 34 Shape factors for $m = 6$

Fig. 35 Shape factor for $m = 7$

Fig. 36 Shape factors for $m = 8$

CONCLUSIONS AND DISCUSSION

The potential functions of Howland have been used to provide a direct calculation of the steady-state temperature distribution in a long heat-generating circular cylinder cooled by a ring of holes. Design curves are provided for uniform heat removal at the coolant holes and for the shape factor for this geometry.

Several extensions and applications of this type of solution to other problems of interest are possible:

- (i) Other combinations of boundary conditions are possible, and nonlinear cases (e.g., thermal radiation on the outer surface) can be treated by iteration.
- (ii) An addition to the general solution in the form

$$T = A_0 \log \frac{r}{b} + \sum_{n=1}^{\infty} A_n \left(\frac{b}{r} \right)^{nm} \cos (nm\theta)$$

makes it possible to extend the solution for determination of temperatures in a cylinder with a central hole. This solution has also been carried out and coded for the IBM 704.

- (iii) The consideration of a thin liner or cladding on the surface of the coolant hole can be made if the thickness $t \ll a$, and only radial heat flux in the layer is considered. This is accomplished by replacing the heat transfer coefficient h by

$$h_s = \frac{h}{1 + h (t/k_s)}$$

and the bulk gas temperature T_b with

$$T_{bs} = T_b + \frac{q_s t^2}{2 k_s} + \frac{q_s t}{h}$$

where the subscript s refers to the surface layer. Then the results of the uniform cooling example are immediately available, with the interface between the film and material of the cylinder at $\rho = \lambda$.

(iv) The elastic thermal stress problem for this geometry is of considerable interest, and the extension of the present harmonic functions to the biharmonic functions appropriate for this problem is straight forward.² A rough estimate for the axial component of stress (assuming plane strain conditions) can be made using the present results by the relation

$$\sigma_{\text{axial}} \sim \alpha E [T(r, \theta) - \bar{T}] \quad (46)$$

where $\bar{T}$ is the average temperature over the cross section from equation (35).

A term containing the stress components in the cross section plane⁸

$$v(\sigma_x + \sigma_y) - \frac{1}{A} \iint v(\sigma_x + \sigma_y) \, dx dy$$

is omitted in (46). However, it is expected the maximum thermal stress component will be found to be in the plane of the section and probably on the hole surfaces, and thus a complete solution of the biharmonic equation will be required.

APPENDIX

The Harmonic Functions U_n

A function with a logarithmic singularity at the m points

$$z_m = b e^{2\pi p i / m}, \quad p = 0, 1, 2, \dots, m-1 \quad (A1)$$

uniformly spaced on the circle of radius $r = b$ is

$$\begin{aligned} w_0(z) &= m \log b - \log(z-b) - \sum_{p=1}^{m-1} \log \left[z - b e^{2\pi p i / m} \right] \\ &= m \log b - \log \left[\prod_{p=0}^{m-1} (z - b e^{2\pi p i / m}) \right] \\ &= - \log \left[\left(\frac{z}{b} \right)^m - 1 \right] \end{aligned} \quad (A2)$$

This is the function originally defined by Howland¹ except for the constant term, $m \log b$, introduced here to provide a consistent dimensionless form. The function (A2) is clearly

invariant with respect to rotations through angles in multiples of $2\pi/m$; and this same periodicity is shared by functions generated by differentiation of (A2) with respect to b . Thus, defining,

$$w_n(z) = \frac{b^n}{(n-1)!} \frac{d^n w_0}{db^n} \quad (A3)$$

the real and imaginary parts of the $w_n(z)$ can be separated,

$$w_n(z) = U_n + iV_n \quad (A4)$$

where the U_n and V_n will be harmonic in the region excluding the singularities at the points z_m and will also be invariant under angular rotations of $2\pi/m$. The U_n , or the real parts of the w_n ,

$$\text{Re } w_n(z) = U_n, \quad n = 0, 1, 2, 3, \dots$$

are suitable functions for the present application.

Useful expansions for the U_n in terms of polar coordinates (r, θ) and (ρ, θ) are required to apply boundary conditions on the holes and the outer radius. Such expansions have been obtained previously;^{2,3} however, due to the different normalization introduced in (A2), the results

have slightly different forms, and it is worthwhile to indicate formally the steps used to obtain them. Owing to the singularities on the circle $r = b$, the form of the expansion around the origin differs depending on whether (z) is greater or less than b . For $r < b$, from (A2)

$$\begin{aligned} w_o(z) &= -\log \left[- (b^m - z^m) \right] + m \log b \\ &= -i\pi - \log \left[1 - \left(\frac{z}{b} \right)^m \right] \end{aligned} \quad (A5)$$

and expanding the logarithm,

$$w_o(z) = -i\pi + \sum_{s=1}^{\infty} \frac{1}{s} \left(\frac{z}{b} \right)^{ms}, \quad \left| \frac{z}{b} \right| < 1 \quad (A6)$$

Thus, application of (A3) to (A6) readily gives

$$w_n(z) = m (-1)^n \sum_{s=1}^{\infty} \left\{ \begin{matrix} sm+n-1 \\ n-1 \end{matrix} \right\} \left(\frac{z}{b} \right)^{ms}, \quad n = 1, 2, 3, \dots \quad (A7)$$

and using the polar form,

$$z = r e^{i\theta}$$

The real parts of (A6) and (A7) yield the results given in (13). For $r > b$, take w_0 in the form

$$w_0(z) = -m \log \left(\frac{z}{b} \right) - \log \left[1 - \left(\frac{b}{z} \right)^m \right] \quad (\text{A8})$$

and expanding,

$$w_0(z) = -m \log \left(\frac{z}{b} \right) + \sum_{s=1}^{\infty} \frac{1}{s} \left(\frac{b}{z} \right)^{ms}, \quad \left| \frac{b}{z} \right| < 1 \quad (\text{A9})$$

Whence the derivatives with respect to b give

$$w_n(z) = m (-1)^{m+1} + m \sum_{s=1}^{\infty} \left\{ \begin{matrix} sm-1 \\ n-1 \end{matrix} \right\} \left(\frac{b}{z} \right)^{ms}, \quad n = 1, 2, 3 \dots \quad (\text{A10})$$

with the real parts of (A9) and (A10) as given in (14).

Expansion around the singularities at z_m requires consideration of only the one point at $(r = b, \varnothing = 0)$ due to the periodicity. Thus, introducing the translation

$$b \zeta = b \rho e^{i\theta} = z - b$$

and expressing $w_0(z)$ as

$$w_0(z) = -\log(z - b) - \sum_{p=1}^{m-1} \log(z - be^{2\pi pi/m}) + m \log b \quad (\text{A11})$$

substitution into (A11) yields

$$\begin{aligned}
 w_0(z) &= -\log b - \log \zeta + m \log b \\
 &\quad - \sum_{p=1}^{m-1} \log \left[b(\zeta + 1) - be^{2\pi pi/m} \right] \\
 &= - \sum_{p=1}^{m-1} \log \left(1 - e^{2\pi pi/m} \right) \\
 &\quad - \log \zeta - \sum_{p=1}^{m-1} \log \left(1 + u_p \zeta \right) \tag{A12}
 \end{aligned}$$

where

$$\begin{aligned}
 u_p &= \frac{1}{(1 - e^{2\pi pi/m})} \\
 1 - u_p^{-1} &= e^{2\pi pi/m} \tag{A13}
 \end{aligned}$$

The first term in (A12) is defined as

$$\gamma_m = \sum_{p=1}^{m-1} \log (u_p^{-1}) \tag{A14}$$

which can be shown to be real since

$$\log (1 - e^{i\beta}) = \frac{1}{2} \left[\log (2 - 2 \cos \beta) \right] \\ + i \left[\tan^{-1} \left(\frac{-\sin \beta}{1 - \cos \beta} \right) \right]$$

Thus

$$\sum_{p=1}^{m-1} \tan^{-1} \left(\frac{-\sin 2\pi p/m}{1 - \cos 2\pi p/m} \right) = - \sum_{p=1}^{m-1} \tan^{-1} \left(\frac{\cos \pi p/m}{\sin \pi p/m} \right) \\ = - \sum_{p=1}^{m-1} \tan^{-1} \left(\cot \frac{\pi p}{m} \right) = - \sum_{p=1}^{m-1} \left(\frac{\pi}{2} - \frac{\pi p}{m} \right) \equiv 0$$

and the γ_m become simply

$$\gamma_m = \frac{1}{2} \sum_{p=1}^{m-1} \log \left[2 \left(1 - \cos \frac{2\pi p}{m} \right) \right] \quad (A15)$$

Table A I records values of γ_m for the range $2 \leq m \leq 8$.

The last term in (A12) can be expanded as before, and

therefore

$$w_o(z) = -\gamma_m - \log \zeta + \sum_{s=1}^{\infty} (-1)^s \alpha_{s,o}^m \zeta^s \quad (A16)$$

where the coefficients

$$\alpha_{s,o}^m = \sum_{p=1}^{m-1} \frac{u_p^s}{s} \quad (A17)$$

TABLE A I

$$\gamma_m = \frac{1}{2} \sum_{p=1}^{m-1} \log \left[2 \left(1 - \cos 2\pi \frac{p}{m} \right) \right]$$

m	γ_m
2	0.69315
3	1.09861
4	1.38629
5	1.60944
6	1.79176
7	1.94591
8	2.07944

are found to be real. It is most convenient to apply (A3) to (A11) such that

$$w_n(z) = m(-1)^{n+1} + \frac{1}{\zeta^n} + \sum_{p=1}^{m-1} \frac{(u_p - 1)^n}{(1 + u_p \zeta)^n} \quad (A18)$$

but, expanding the denominator in the last term of (A18),

$$\frac{1}{(1 + u_p \zeta)^n} = \sum_{s=0}^{\infty} (-1)^s \begin{Bmatrix} n+s-1 \\ s \end{Bmatrix} (u_p \zeta)^s \quad (A19)$$

gives

$$w_n(z) = m(-1)^{n+1} + \zeta^{-n} + \sum_{s=0}^{\infty} (-1)^s \alpha_{s,n}^m \zeta^s, \quad n = 1, 2, 3, \dots \quad (A20)$$

where the general coefficients

$$\alpha_{s,n}^m = \sum_{p=1}^{m-1} (u_p - 1)^n \begin{Bmatrix} n+s-1 \\ s \end{Bmatrix} u_p^s, \quad n = 1, 2, 3, \dots \quad (A21)$$

together with those for $n = 0$ in (A17) are recorded in Tables A II-A VIII for practical ranges of m , s , and n . A rapid technique for calculation of the $\alpha_{s,n}^m$ is given in Ref. 1.

Calculation of the Average Value of ψ

The average value of the function ψ is defined by

$$\bar{\psi} = \frac{1}{A'} \int \int_{A'} \psi \, dA \quad (A22)$$

with the net area of the cross section A' given by

$$A' = \pi r_0^2 \left(1 - m \frac{\lambda^2}{\beta^2} \right) \quad (A23)$$

Consider the contributions to (A22) separately according to

$$\bar{\psi} = \bar{\psi}_1 + \bar{\psi}_2 + \bar{\psi}_3$$

TABLE A II

$a_{s,n}^2$									
s	0	1	2	3	4	5	6	7	8
n									
0		5.0000×10^{-1}	1.2500×10^{-1}	4.1667×10^{-2}	1.5625×10^{-2}	6.2500×10^{-3}	2.6042×10^{-3}	1.1161×10^{-3}	4.8828×10^{-4}
1	-3.0000×10^{-1}	-2.5000×10^{-1}	-1.2500×10^{-1}	-6.2500×10^{-2}	-3.1250×10^{-2}	-1.5625×10^{-2}	-7.8125×10^{-3}	-3.9062×10^{-3}	-1.9531×10^{-3}
2	2.8000×10^{-1}	2.5000×10^{-1}	1.8750×10^{-1}	1.2500×10^{-1}	7.8125×10^{-2}	4.6875×10^{-2}	2.7344×10^{-2}	1.5625×10^{-2}	8.7891×10^{-3}
3	-1.2500×10^{-1}	-1.8750×10^{-1}	-1.8750×10^{-1}	-1.5625×10^{-1}	-1.1719×10^{-1}	-8.2031×10^{-2}	-5.4687×10^{-2}	-3.5156×10^{-2}	-2.1973×10^{-2}
4	6.2500×10^{-2}	1.2500×10^{-1}	1.5625×10^{-1}	1.5625×10^{-1}	1.3672×10^{-1}	1.0937×10^{-1}	8.2031×10^{-2}	5.8594×10^{-2}	4.0283×10^{-2}
5	-3.1250×10^{-2}	-7.8125×10^{-2}	-1.1719×10^{-1}	-1.3672×10^{-1}	-1.3672×10^{-1}	-1.2305×10^{-1}	-1.0254×10^{-1}	-8.0566×10^{-2}	-6.0425×10^{-2}
6	1.5625×10^{-2}	4.6875×10^{-2}	8.2031×10^{-2}	1.0937×10^{-1}	1.2305×10^{-1}	1.2305×10^{-1}	1.1279×10^{-1}	9.6680×10^{-2}	7.8552×10^{-2}
7	-7.8125×10^{-3}	-2.7344×10^{-2}	-5.4687×10^{-2}	-8.2031×10^{-2}	-1.0254×10^{-1}	-1.1279×10^{-1}	-1.1279×10^{-1}	-1.0474×10^{-1}	-9.1644×10^{-2}
8	3.9063×10^{-3}	1.5625×10^{-2}	3.5156×10^{-2}	8.8594×10^{-2}	8.0566×10^{-2}	9.6680×10^{-2}	1.0474×10^{-1}	1.0474×10^{-1}	9.8190×10^{-2}
9	-1.9531×10^{-3}	-8.7891×10^{-3}	-2.1973×10^{-2}	-1.0283×10^{-2}	-6.0425×10^{-2}	-7.8552×10^{-2}	-9.1644×10^{-2}	-1.8190×10^{-2}	-9.8190×10^{-2}

TABLE A III

 $\alpha_{s,n}^3$

s =	0	1	2	3	4	5	6	7	8	9
n										
0		1.0000	1.6667×10^{-1}	1.3970×10^{-8}	-2.7778×10^{-2}	-2.2222×10^{-2}	-1.2346×10^{-2}	-5.2910×10^{-3}	-1.5432×10^{-3}	-5.4651×10^{-10}
1	-1.0000	-6.6667×10^{-1}	-3.3333×10^{-1}	-1.1111×10^{-1}	-1.3970×10^{-8}	3.7037×10^{-2}	3.7037×10^{-2}	2.4691×10^{-2}	1.2346×10^{-2}	4.1152×10^{-3}
2	3.3333×10^{-1}	6.6667×10^{-1}	6.6667×10^{-1}	4.4444×10^{-1}	1.8519×10^{-1}	2.9337×10^{-8}	-8.6420×10^{-2}	-9.8765×10^{-2}	-7.4074×10^{-2}	-4.1152×10^{-2}
3	-4.1910×10^{-8}	-3.3333×10^{-1}	-6.6667×10^{-1}	-7.4074×10^{-1}	-5.5556×10^{-1}	-2.5926×10^{-1}	-4.8126×10^{-8}	1.4815×10^{-1}	1.8519×10^{-1}	1.5089×10^{-1}
4	-1.1111×10^{-1}	5.5879×10^{-8}	3.7037×10^{-1}	7.4074×10^{-1}	8.6420×10^{-1}	6.9136×10^{-1}	3.4568×10^{-1}	6.9849×10^{-8}	-2.2634×10^{-1}	-3.0178×10^{-1}
5	1.1111×10^{-1}	1.8519×10^{-1}	-7.3342×10^{-8}	-4.3210×10^{-1}	-8.6420×10^{-1}	-1.0370	-8.6420×10^{-1}	-4.5267×10^{-1}	-9.1841×10^{-8}	3.2693×10^{-1}
6	-7.4074×10^{-2}	-2.2222×10^{-1}	-2.5926×10^{-1}	9.6253×10^{-8}	5.1852×10^{-1}	1.0370	1.2675	1.0864	5.8848×10^{-1}	1.2216×10^{-7}
7	3.7037×10^{-2}	1.7284×10^{-1}	3.4568×10^{-1}	3.4568×10^{-1}	-1.2224×10^{-7}	-6.3374×10^{-1}	-1.2675	-1.5693	-1.3731	-7.6284×10^{-1}
8	-1.2346×10^{-2}	-9.8765×10^{-2}	-2.9630×10^{-1}	-4.9383×10^{-1}	-4.5267×10^{-1}	1.4695×10^{-7}	7.8464×10^{-1}	1.5693	1.9616	1.7436
9	4.9185×10^{-9}	3.7037×10^{-2}	1.8519×10^{-1}	4.5267×10^{-1}	6.7901×10^{-1}	5.8848×10^{-1}	-1.8324×10^{-7}	-9.8080×10^{-1}	-1.9616	-2.4702

TABLE A IV

 $\alpha_{s,n}^i$

$s \setminus n$	0	1	2	3	4	5	6	7	8	9
0		1.5000	1.2500×10^{-1}	-1.2500×10^{-1}	-1.0937×10^{-1}	-4.3750×10^{-2}	2.6042×10^{-3}	1.8973×10^{-2}	1.6113×10^{-3}	7.1615×10^{-3}
1	-1.5000	-1.2500	-6.2500×10^{-1}	-6.3500×10^{-2}	2.1875×10^{-1}	2.3437×10^{-1}	1.1719×10^{-1}	-3.9062×10^{-3}	-6.4453×10^{-2}	-6.3477×10^{-2}
2	2.5000×10^{-1}	1.2500	1.6875	1.1250	7.8125×10^{-2}	-7.0312×10^{-1}	-8.4766×10^{-1}	-4.8437×10^{-1}	8.7888×10^{-3}	3.1738×10^{-1}
3	3.7500×10^{-1}	-1.8750×10^{-1}	-1.6875	-2.6562	-1.9922	-8.2032×10^{-2}	1.6953	2.2148	1.3843	-1.3427×10^{-2}
4	-4.3750×10^{-1}	-8.7500×10^{-1}	1.5625×10^{-1}	2.6562	4.5117	3.6094	8.2032×10^{-2}	-3.6914	-5.1160	-3.4106
5	3.1875×10^{-1}	1.1719	1.7578	-1.3672×10^{-1}	-4.5117	-7.9980	-6.6650	-8.0568×10^{-2}	7.6739	1.1128×10^1
6	1.5625×10^{-2}	-7.0312×10^{-1}	-2.5430	-3.3906	1.2305×10^{-1}	7.9980	1.4550×10^1	1.2472×10^1	7.8555×10^{-2}	-1.5580×10^1
7	-1.3281×10^{-1}	-2.7343×10^{-2}	1.6953	6.1680	6.4600	-1.1279×10^{-1}	-1.4550×10^1	-2.6917×10^1	-2.3553×10^1	-7.6375×10^{-2}
8	1.2891×10^{-1}	5.1562×10^{-1}	3.5155×10^{-2}	-3.6914	-1.0232×10^1	-1.2278×10^1	1.0474×10^{-1}	2.6917×10^1	5.0372×10^1	4.4775×10^1
9	-6.4453×10^{-2}	-5.7129×10^{-1}	-1.4282	-4.0281×10^{-2}	7.6739	2.0031×10^1	2.3369×10^1	-9.8197×10^{-2}	-8.0372×10^1	-9.5054×10^1

TABLE A ▽

 $\alpha_{s,n}^5$

s	0	1	2	3	4	5	6	7	8	9
n										
0		2.0000	1.5181×10^{-7}	-3.3333×10^{-1}	-2.0000×10^{-1}	-6.3283×10^{-8}	1.0000×10^{-1}	8.5714×10^{-2}	2.0000×10^{-2}	-3.1111×10^{-2}
1	-2.0000	-2.0000	-1.0000	2.0000×10^{-1}	8.0000×10^{-1}	6.0000×10^{-1}	2.3525×10^{-7}	-4.4000×10^{-1}	-4.4000×10^{-1}	-1.2000×10^{-1}
2	3.0361×10^{-7}	2.0000	3.6000	2.4000	-1.0000	-3.6000	-3.0800	-1.3859×10^{-6}	2.8800	3.2000
3	1.0000	6.0000×10^{-1}	-3.6000	-8.0000	-6.0000	3.3600	1.2320×10^1	1.1520×10^1	5.6673×10^{-6}	-1.2760×10^1
4	-8.0000×10^{-1}	-3.2000	-2.0000	8.0000	1.9600×10^1	1.5680×10^1	-1.0060×10^1	-3.8400×10^1	-3.8280×10^1	-1.9859×10^{-5}
5	3.1642×10^{-7}	3.0000	9.0000	5.6000	-1.9600×10^1	-5.0400×10^1	-4.2000×10^1	2.9040×10^1	1.1484×10^2	1.2012×10^2
6	6.0000×10^{-1}	-1.4115×10^{-6}	-9.2400	-2.4640×10^1	-1.5120×10^1	5.0400×10^1	1.3306×10^2	1.1405×10^2	-8.2368×10^1	-3.3634×10^2
7	-6.0000×10^{-1}	-3.0800	4.8507×10^{-6}	2.6880×10^1	6.7200×10^1	4.0656×10^1	-1.3306×10^2	-3.5693×10^2	-3.1231×10^2	2.3223×10^2
8	1.6000×10^{-1}	3.5200	1.1520×10^1	-1.5113×10^{-5}	-7.6560×10^1	-1.8374×10^2	-1.0982×10^2	3.5693×10^2	9.6782×10^2	8.6029×10^2
9	2.8000×10^{-1}	-1.0800	-1.4400×10^1	-3.8280×10^1	4.4682×10^{-5}	2.1622×10^2	5.0450×10^2	2.9858×10^2	-9.6782×10^2	-2.6449×10^3

TABLE A VI

 $a_{s,n}^6$

s	n	0	1	2	3	4	5	6	7	8	9
0			2.5000	-2.0833x10 ⁻¹	-6.2500x10 ⁻¹	-2.6215x10 ⁻¹	1.8493x10 ⁻¹	3.2359x10 ⁻¹	1.3868x10 ⁻¹	-1.2605x10 ⁻¹	-2.2200x10 ⁻¹
1	-2.5000	-2.9167	-1.4583	8.2639x10 ⁻¹	1.9687	1.9214	-9.7077x10 ⁻¹	-1.9792	-9.8961x10 ⁻¹	1.0031	
2	-4.1667x10 ⁻¹	2.9167	6.8542	4.5674	-4.7367	-1.1953x10 ¹	-7.0591	7.9168	1.7935x10 ¹	9.9637	
3	1.8750	2.4792	-6.8542	-2.0897x10 ¹	-1.5673x10 ¹	2.0659x10 ¹	5.5945x10 ¹	3.6113x10 ¹	-4.4837x10 ¹	-1.0986x10 ²	
4	-1.0486	-7.8750	-9.4734	3.0897x10 ¹	7.1001x10 ¹	5.6801x10 ¹	-8.3572x10 ¹	-2.3999x10 ²	-1.6519x10 ²	2.1972x10 ²	
5	-9.2014x10 ⁻¹	5.1071	2.9883x10 ¹	3.4431x10 ¹	-7.1001x10 ¹	-2.5315x10 ²	-2.1097x10 ²	3.2947x10 ²	9.8994x10 ²	7.1528x10 ³	
6	1.9415	5.8246	-3.1177x10 ¹	-1.1189x10 ²	-1.3536x10 ²	2.5316x10 ²	9.2538x10 ²	7.9318x10 ²	-1.2863x10 ³	-4.0039x10 ³	
7	-9.7078x10 ⁻¹	-1.3854x10 ¹	-2.7709x10 ¹	8.4264x10 ¹	4.1990x10 ²	1.6125x10 ²	-9.2538x10 ²	-3.4337x10 ³	-3.0045x10 ³	5.0041x10 ³	
8	-1.0084	7.9169	7.1739x10 ¹	1.1956x10 ²	-3.3037x10 ²	-1.5839x10 ³	-1.7151x10 ³	3.4337x10 ³	1.2872x10 ⁴	1.1442x10 ⁴	
9	1.9960	9.0382	-4.4837x10 ¹	-3.2959x10 ²	-1.9438x10 ²	1.2875x10 ³	6.0059x10 ³	6.4339x10 ³	-1.2872x10 ⁴	-4.8622x10 ⁴	

TABLE A VII

 $\alpha_{s,n}^7$

s =	0	1	2	3	4	5	6	7	8	9
n										
0		3.0000	-5.0000x10 ⁻¹	-1.0000	-2.5000x10 ⁻¹	6.0000x10 ⁻¹	6.9048x10 ⁻¹	3.7556x10 ⁻⁷	-6.9643x10 ⁻¹	-6.1905x10 ⁻¹
1	-3.0000	-4.0000	-2.0000	2.0000	4.0000	1.1429	-4.1429	-5.5714	-3.5069x10 ⁻⁶	7.4286
2	-1.0000	4.0000	1.2000x10 ¹	8.0000	-1.4286x10 ¹	-3.1714x10 ¹	-1.0000x10 ¹	4.4571x10 ¹	6.6857x10 ¹	4.6152x10 ⁻⁵
3	3.0000	6.0000	-1.2000x10 ¹	-4.8571x10 ¹	-3.6429x10 ¹	8.1000x10 ¹	1.9600x10 ²	6.6857x10 ¹	-3.3429x10 ²	-5.4326x10 ²
4	-1.0000	-1.6000x10 ¹	-2.8571x10 ¹	4.8571x10 ¹	2.2000x10 ²	1.7600x10 ²	-4.3200x10 ²	-1.1143x10 ³	-4.0408x10 ²	2.1731x10 ³
5	-3.0000	5.7143	7.9286x10 ¹	1.3500x10 ²	-2.2000x10 ²	-1.0440x10 ³	-8.7000x10 ²	2.2561x10 ³	6.1016x10 ³	2.3201x10 ³
6	4.1429	2.4857x10 ¹	-3.0000x10 ¹	-3.9200x10 ²	-6.4800x10 ²	1.0440x10 ³	5.0726x10 ³	4.3479x10 ³	-1.1688x10 ⁴	-3.2767x10 ⁴
7	-2.6289x10 ⁻⁶	-3.9000x10 ¹	-1.5600x10 ²	1.5600x10 ²	1.9500x10 ³	3.1586x10 ³	-5.0726x10 ³	-2.5005x10 ⁴	-2.1879x10 ⁴	6.0366x10 ⁴
8	-5.5714	2.8056x10 ⁻⁵	2.6743x10 ²	8.9143x10 ²	-8.0816x10 ²	-9.7626x10 ³	-1.5584x10 ⁴	2.5005x10 ⁴	1.2450x10 ⁵	1.1066x10 ⁵
9	5.5714	6.6857x10 ¹	-2.0769x10 ⁻⁴	-1.6298x10 ³	-4.8894x10 ³	4.1762x10 ³	4.9151x10 ⁴	7.7614x10 ⁴	-1.2450x10 ⁵	-6.2455x10 ⁵

TABLE A VIII

 α_n^B

n	0	1	2	3	4	5	6	7	8	9
0		3.5000	-3.7500×10^{-1}	-1.4583	-1.0938×10^{-1}	1.3562	1.1693	-6.9531×10^{-1}	-2.1083	-9.3728×10^{-1}
1	-3.5000	-5.2500	-2.6250	3.9375	7.2187	2.3438×10^{-1}	-1.1863×10^1	-1.2004×10^1	8.4355	2.8936×10^1
2	-1.7500	5.2500	1.9687×10^1	1.3125×10^1	-3.4922×10^1	-7.2703×10^1	-8.4769×10^{-1}	1.6352×10^2	1.8451×10^2	-1.4468×10^2
3	4.3750	1.1812×10^1	-1.9687×10^1	-1.0263×10^2	-7.6902×10^1	2.5192×10^2	5.7569×10^2	2.2152	-1.5736×10^3	-1.9250×10^3
4	-3.3750×10^{-1}	-2.6875×10^1	-6.3814×10^1	1.0268×10^2	5.0951×10^2	1.7961×10^2	-1.7219×10^3	-4.2037×10^3	-5.1186	1.3142×10^4
5	-6.7812	1.1719	7.8176×10^2	4.1986×10^2	-5.9951×10^2	-3.6520×10^3	-3.9517×10^3	1.1550×10^4	2.9594×10^4	1.1148×10^1
6	7.0156	7.3297×10^1	-2.5431	-1.1514×10^3	-2.5829×10^3	3.6520×10^3	2.2893×10^4	1.3614×10^4	-7.6898×10^4	-2.0422×10^5
7	4.8672	-8.4027×10^1	-5.7230×10^2	5.1688	7.3564×10^3	1.6170×10^4	-2.2883×10^4	-1.1503×10^5	-1.2690×10^5	5.1051×10^5
8	-1.6871×10^1	-6.7484×10^1	7.3893×10^4	1.1963×10^3	-1.0237×10^1	-1.7334×10^4	-1.0253×10^5	1.4503×10^5	9.2830×10^5	8.2515×10^5
9	8.4355	2.6043×10^2	6.5107×10^2	-5.7750×10^3	-2.9563×10^4	2.0066×10^1	3.0633×10^5	6.5637×10^5	-3.2830×10^5	-5.9864×10^5

$$\begin{aligned}
\bar{\psi}_1 &= \frac{1}{A'} \iint_{A'} \left(- \frac{r^2}{b^2} \right) dA \\
\bar{\psi}_2 &= \frac{1}{A'} \iint_{A'} \left[\sum_{n=0}^{\infty} B_n \left(\frac{r}{b} \right)^{mn} \cos (mn\phi) \right] dA \\
\bar{\psi}_3 &= \frac{1}{A'} \iint_{A'} \left[\sum_{n=0}^{\infty} C_n U_n \right] dA
\end{aligned} \tag{A24}$$

The first two integrals in (A24) are elementary and result in

$$\begin{aligned}
A' \bar{\psi}_1 &= \frac{\pi r_o^2}{2} \left[m \frac{\lambda^2}{\beta^2} (2 + \lambda^2) - \beta^2 \right] \\
A' \bar{\psi}_2 &= \pi r_o^2 \left[B_0 - m \frac{\lambda^2}{\beta^2} \sum_{n=0}^{\infty} B_n \right]
\end{aligned} \tag{A25}$$

The calculation of $\bar{\psi}_3$, however, requires a more sophisticated approach. C. B. Ling³ has indicated how this can be accomplished.* Consider an analytic function $F(z)$; then it is easily demonstrated by Green's theorem that the integral over an area S can be expressed in terms of the

*We note here that several errors occur in Ling's evaluation of some of his integrals of similar form.

line integral around its boundary C as

$$\iint_S \frac{dF}{dz} dx dy = \frac{i}{2} \oint_C F d\bar{z} \quad (A26)$$

If the w_n are broken into real and imaginary parts,

$$w_n(z) = U_n + iV_n \quad (A4)$$

and the indefinite integrals are defined

$$F_n(z) = \int^z w_n(z) dz \quad (A27)$$

then

$$\operatorname{Re} \left(\frac{dF}{dz} \right) = U_n$$

or from (A26)

$$\iint_S \operatorname{Re} \left(\frac{dF}{dz} \right) dx dy = \operatorname{Re} \left(\frac{i}{2} \oint_C F d\bar{z} \right) \quad (A28)$$

Thus

$$\begin{aligned} A' \bar{\psi}_3 &= 2m \sum_{n=0}^{\infty} C_n \iint_S U_n dx dy \\ &= m \sum_{n=0}^{\infty} \left\{ C_n \operatorname{Re} \left(i \oint_C \left[\int w_n(z) dz \right] d\bar{z} \right) \right\} \end{aligned} \quad (A29)$$

with the region S and boundary C as indicated in Fig. A1. Selection of $1/2m$ of the cross section avoids the possibility of multiple-valuedness in (A27); and, due to the periodicity of the U_n , consideration of integration over the depicted region is sufficient. The details of such integrations will be illustrated by demonstration of the integral for U_0 .

Consider the curved arcs first. Thus from (A29) the integral

$$\operatorname{Re} \left[i \int_{C_4} F_0(z) d\bar{z} \right] \quad (A30)$$

is required, and since

$$\begin{aligned} w_0(z) &= -m \log \left(\frac{z}{b} \right) + \sum_{s=1}^{\infty} \frac{1}{s} \left(\frac{b}{z} \right)^{ms} \\ F_0(z) &= \int^z w_0 dz = -m \left[z \log \left(\frac{z}{b} \right) - z \right] \\ &\quad - \sum_{s=1}^{\infty} \frac{(b)^{ms}}{s(sm-1) z^{ms-1}} \end{aligned} \quad (A31)$$

but, on the outer boundary,

$$z = r_0 e^{i\theta}, \quad d\bar{z} = -ir_0 e^{-i\theta} d\theta$$

Fig. A1 Region considered for calculation of $\bar{\psi}_3$

$$\begin{aligned}
F_0 \Big|_{r=r_0} &= -m \left[r_0 e^{i\theta} \log (\beta e^{i\theta}) - r_0 e^{i\theta} \right] \\
&- \sum_{s=1}^{\infty} \frac{r_0 e^{is\theta}}{s(ms-1)} \left(\frac{b}{r_0} \right)^{ms} \frac{1}{e^{ims\theta}} \quad (A32)
\end{aligned}$$

Therefore, from (A31) and (A32),

$$\begin{aligned}
\int_{C_4} F_0 d\bar{z} &= - \int_0^{\pi/m} F_0 \Big|_{r=r_0} i r_0 e^{-i\theta} d\theta \\
&= i r_0^2 \int_0^{\pi/m} \left[m (\log \beta + i\theta - 1) \right. \\
&\quad \left. + \sum_{s=1}^{\infty} \frac{e^{-ims\theta}}{s(ms-1) \beta^{sm}} \right] d\theta \quad (A33)
\end{aligned}$$

Only the imaginary part of (A33), which implies the real part of the integrand, is needed, whence

$$\operatorname{Re} \left(i \int_{C_4} F_0 d\bar{z} \right) = - r_0^2 \pi (\log \beta - 1) \quad (A34)$$

On the cooling hole, w_0 is given by

$$w_0(z) = -\gamma_m - \log(z-b) + \log b + \sum_{s=1}^{\infty} (-1)^s \alpha_{s,0}^m \frac{(z-b)^s}{b^s} \quad (A35)$$

and, thus,

$$F_0 = -\gamma_m z - \left[(z - b) \log (z - b) - z \right] \\ + z \log b + \sum_{s=1}^{\infty} \frac{(-1)^s \alpha_{s,0}^m (z - b)^{s+1}}{(s+1) b^s} \quad (A36)$$

On the path C_2

$$z - b = b \zeta = b \lambda e^{i\theta} \\ d\bar{z} = b d\bar{\zeta} = -i b \lambda e^{-i\theta} \quad (A37)$$

whence

$$\int_{C_2} F_0 d\bar{z} = i a^2 \int_0^\pi \left[-\gamma_m \left(\frac{e^{-i\theta}}{\lambda} + 1 \right) - \log a - i\theta \right. \\ \left. + \left(\frac{e^{-i\theta}}{\lambda} + 1 \right) + (\log b) \left(\frac{e^{-i\theta}}{\lambda} + 1 \right) \right. \\ \left. + \sum_{s=1}^{\infty} \frac{(-1)^s}{(s+1)} \alpha_{s,0}^m \lambda^s e^{is\theta} \right] d\theta (=)$$

$$\begin{aligned}
= ia^2 \left[-\gamma_m \left(\frac{2}{\lambda i} + \pi \right) - \pi \log a - \frac{i\pi^2}{2} \right. \\
+ \left(\frac{2}{\lambda i} + \pi \right) (1 + \log b) \\
\left. + 2 \sum_{s=1,3,5,\dots}^{\infty} \frac{(-1)^{s+1} \alpha_{s,0}^m}{i(s+1)^s} \lambda^s \right] \quad (A38)
\end{aligned}$$

and finally, taking the imaginary part of (A38),

$$\operatorname{Re} \left(i \int_{C_2} F_0 d\bar{z} \right) = \pi a^2 \left(\log \lambda + \gamma_m - 1 \right) \quad (A39)$$

The integrals over the "rays" C_1 , C_3 and C_5 are approached differently by considering the original definition of $w_0(z)$. Thus, from the second of (A2)

$$w_0(z) = m \log b - \sum_{p=0}^{m-1} \log \left(z - b e^{2\pi p i / m} \right) \quad (A40)$$

and

$$F_0(z) = z m \log b - \sum_{p=0}^{m-1} \left[\left(z - b e^{2\pi p i / m} \right) \log \left(z - b e^{2\pi p i / m} \right) - z \right] \quad (A41)$$

and thus,

$$\begin{aligned}
 i \int_{C_5} F_0 d\bar{z} &= i \int_{r_0}^0 F_0 \left| \frac{e^{-i\pi/m}}{z - r e^{i\pi/m}} \right| dr \\
 &= i \int_0^{r_0} \left\{ \sum_{p=0}^{m-1} \left[(r - b e^{\pi i(2p-1)/m}) \right. \right. \\
 &\quad \left. \left(\log(r - b e^{\pi i(2p-1)/m}) + \frac{i\pi}{m} \right) - r \right] \\
 &\quad \left. + r m \log b \right\} dr \quad (A42)
 \end{aligned}$$

Since only the real part of (A42) is required, consideration need only be given to the imaginary part of the integrand.

Let

$$z_p = b e^{\pi i(2p-1)/m} = b e^{\beta_p}$$

Then consider the imaginary part of the first product in the p summation

$$\begin{aligned}
 &\text{Im} \left[\sum_{p=0}^{m-1} (r - z_p) \log(r - z_p) \right] \\
 &= \sum_{p=0}^{m-1} \left[(r - \text{Re } z_p) \arg(r - z_p) + \text{Im } z_p \log |r - z_p| \right] \quad (=)
 \end{aligned}$$

$$\begin{aligned}
&= \sum_{p=0}^{m-1} r \tan^{-1} \left(\frac{b \sin \beta_p}{r - b \cos \beta_p} \right) \\
&- b \sum_{p=0}^{m-1} \cos \beta_p \tan^{-1} \left(\frac{b \sin \beta_p}{r - b \cos \beta_p} \right) \\
&+ \frac{b}{2} \sum_{p=0}^{m-1} \sin \beta_p \log \left(r^2 - 2br \cos \beta_p + b^2 \right) = 0 \quad (A43)
\end{aligned}$$

The individual vanishing of these summations can be easily demonstrated by referring to Fig. A2 and noting that $\beta_0 = -\beta_1$, $\beta_2 = -\beta_{m-1}$, etc. That is, the angles β_p occur in pairs of opposite signs. Thus the terms in (A43) will cancel. An exception is z_p which occurs at $\emptyset = \pi$ with m odd, but the terms in (A43) at $\beta_p = \pi$ identically vanish. The remaining terms in (A42) are

$$i \int_0^r \left[\sum_{p=0}^{m-1} (r - z_p) \left(\frac{i\pi}{m} \right) \right] dr \quad (A44)$$

However,

$$\sum_{p=0}^{m-1} z_p = 0 \quad (A45)$$

Fig. A2 The vectors z_p

as is clear from Fig. A2, and thus finally

$$\operatorname{Re} \left(i \int_{C_5} F_o d\bar{z} \right) = - \frac{\pi r_o^2}{2} \quad (\text{A46})$$

Consider next the integral along C_3 .

$$\begin{aligned} \operatorname{Re} \left(i \int_{C_3} F_o d\bar{z} \right) &= \operatorname{Re} \left(i \int_{b+a}^{r_o} F_o d\bar{z} \right) \\ &= \operatorname{Re} \left\{ i \int_{b+a}^{r_o} \left[- \sum_{p=0}^{m-1} (r - z_p) \log (r - z_p) - r \right. \right. \\ &\quad \left. \left. - r m \log b \right] dr \right\} \quad (\text{A47}) \end{aligned}$$

Since

$$\operatorname{Im} \left[\sum_{p=0}^{m-1} (r - z_p) \log (r - z_p) \right]$$

vanishes and the remainder of the integrand is real,

$$\operatorname{Re} \left[i \int_{C_3} F_o d\bar{z} \right] = 0 \quad (\text{A48})$$

If arguments similar to those in (A43) are applied, the imaginary part of the integrand reduces to

$$\operatorname{Re} \left(i \int_{C_1} F_0 d\bar{z} \right) = \operatorname{Re} \left(i \int_0^{(b-a)} -i\pi (r - b) dr \right)$$

or

$$\operatorname{Re} \left(i \int_{C_1} F_0 d\bar{z} \right) = -\frac{\pi}{2} (b^2 - a^2) \quad (\text{A49})$$

Then contribution of F_0 to the line integral in (A39) is the sum of (A34), (A39), (A46), (A48) and (A49) multiplied by m , or

$$\begin{aligned} m \operatorname{Re} \left(i \oint_C F_0 d\bar{z} \right) = m \left[-r_0^2 \pi (\log \beta - 1) \right. \\ \left. + \pi a^2 (\log \lambda + \gamma_m - 1) \right. \\ \left. - \frac{\pi r_0^2}{2} - \frac{\pi}{2} (b^2 - a^2) \right] \quad (\text{A50}) \end{aligned}$$

By continuing in a similar manner for $n = 1, 2, 3 \dots$, the complete form for $\bar{\psi}$ as given in (35) results.

Solution for the C_n and B_n

Solution of the infinite systems of simultaneous linear equations which result for the cooling and the shape factor problems is dependent upon the convergence of the infinite series forms of the solutions. To demonstrate analytically such convergence is rather difficult and has been discussed in references [1] and [9]. Such attempts generally indicate a rather restricted range of geometry, that is, values of λ for which convergence can be rigorously demonstrated. However, on physical grounds it would seem likely that convergence can be expected as long as the holes do not overlap,

$$a < b \sin \frac{\pi}{m} \quad , \quad \lambda < \sin \frac{\pi}{m} \quad (A51)$$

or touch the outer boundary

$$b + a < r_o, \quad \beta > 1 + \lambda \quad (A52)$$

Experience has indicated that for the numerical technique outlined subsequently, these inequalities do define the limits of convergence, and equality in (A51) and (A52) can be approached to a practical limit of about 2% without difficulty. A closer approach evidences difficulty through very slow convergence of the numerical scheme, and occasionally

oscillations without convergence will be experienced.

The numerical technique selected was an iteration scheme and will be illustrated for the shape factor calculations with the cooling problem approached in a similar fashion. The infinite system to be solved is equations (45).

$$C_o = - \left[\frac{1 + \sum_{n=1}^{\infty} C_n \alpha_{o,n}^m + \sum_{n=1}^{\infty} B_n}{m \log \beta - \log \lambda - \gamma_m} \right]$$

$$B_o = 1 + C_o m \log \beta + m \sum_{n=1}^{\infty} (-1)^n C_n$$

$$C_s = - \lambda^{2s} \left[(-1)^s C_o \alpha_{s,o}^n + \sum_{n=1}^{\infty} (-1)^s \alpha_{s,n}^m C_n \right], \quad s = 1, 2, \dots$$

$$B_s = - \beta^{-2ms} \left[\frac{C_o}{s} + m \sum_{n=1}^{\infty} \left\{ \begin{matrix} ms-1 \\ n-1 \end{matrix} \right\} C_n \right], \quad s = 1, 2, 3, \dots$$

Then define an iteration index $j = 0, 1, 2, \dots$ such that

$$C_o^{(j)} = - \left[\frac{1 + \sum_{n=1}^{\infty} C_n^{(j-1)} \alpha_{o,n}^m + \sum_{n=1}^{\infty} B_n^{(j-1)}}{m \log \beta - \log \lambda - \gamma_m} \right]$$

$$B_o^{(j)} = 1 + C_o^{(j)} m \log \beta + m \sum_{n=1}^{\infty} (-1)^n C_n^{(j-1)} \quad (A53)$$

$$C_s^{(j)} = - \lambda^{2s} \left[(-1)^s C_o^{(j)} \alpha_{s,o}^m + \sum_{n=1}^{\infty} (-1)^s \alpha_{s,n}^m C_n^{(j-1)} \right], \quad s = 1, 2, \dots$$

$$B_s^{(j)} = - \beta^{-2ms} \left[\frac{C_o^{(j)}}{s} + m \sum_{n=1}^{\infty} \left\{ \begin{matrix} ms-1 \\ n-1 \end{matrix} \right\} C_n^{(j)} \right], \quad s = 1, 2, 3, \dots$$

and take starting values of the coefficients as

$$C_o^{(o)} = - \frac{1}{m \log \beta - \log \lambda - \gamma_m}$$

$$B_o^{(o)} = 1 + C_o^{(o)} m \log \beta$$

(A54)

$$C_s^{(o)} = (-1)^{s+1} \lambda^{2s} \alpha_{s,o}^m C_o^{(o)}, \quad s = 1, 2, 3, \dots$$

$$B_s^{(o)} = - \frac{C_o^{(o)}}{s} \beta^{-2ms}, \quad s = 1, 2, 3, \dots$$

A truncation N is then decided upon with $n \leq N$, the infinite sums in (A53) are terminated at N , and s is taken to be $s = 1, 2, 3, \dots, N$. Then with $j = 1, 2, \dots$ the iteration is repeated according to (A53) until the values of the coefficients do not change significantly. In practice it has

proven sufficient to set this iteration termination to be a unit change or less in the 5th digit; that is,

$$\left(\frac{c_n^{(j)} - c_n^{(j-1)}}{c_n^{(j)}} \right) \leq 10^{-5} \quad (\text{A55})$$

It has also been determined that $N = 10$ is usually a practical limit on the number of coefficients required for reasonable accuracy of the solutions.

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