DETAILS OF FORMALISM OF $a + b \rightarrow 1 + 2 + 3$ INCLUDING
SUBENERGY UNITARITY*

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ABSTRACT

A formalism to include subenergy unitarity in $a + b \rightarrow 1 + 2 + 3$ was presented earlier by Goradia and Lasinski.¹ In this paper we discuss the details of this formalism and its application to $\pi N \rightarrow \pi\pi N$.

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I. INTRODUCTION

In a previous paper,¹ we have presented a K-matrix formalism to deal with the subenergy branch cuts in the reaction $a + b \rightarrow \alpha + \beta + \gamma$. The set of unitarity equations developed in it made use of the two- and three-particle canonical states, recoupling coefficients and other kinematical details which were introduced without derivation. The purpose of these notes is to discuss these and other important aspects of the formalism. In particular, we will construct the canonical states, derive an expression for the recoupling coefficient, study the requirements of Boson symmetry, do the partial-wave analysis and work out the cross section formula, discuss the kinematics and give explicit formulas for the rotation matrices up to $J = 7/2$. Reference 1, to be henceforth referred to as Gl., is recommended as a companion paper.

Recently, a generalized isobar model formalism was published by Herndon, Söding and Cashmore.⁴ These authors use helicity, instead of the canonical, states to do the calculations. The isobar model, however, quite naturally requires the use of orbital angular momenta and parity which are not a part of the helicity representation. One is thus compelled to make a unitarity transformation to introduce these quantum numbers. For this reason we have chosen to work directly in terms of states which are eigenstates of orbital momenta and parity (canonical states). It is hoped that the detailed construction of these states presented here will be helpful in understanding the formalism.

II. CONSTRUCTION OF CANONICAL STATES

A. One-Particle States

Consider a particle at rest in frame F. Let its spin projection σ_z be ν . We have the state

$$|A_1\rangle = |\vec{p}=0; \sigma, \nu\rangle \quad (2.1)$$

If we now apply a pure Lorentz transformation $L(\vec{p})$ in an arbitrary direction, i.e., go over to a reference frame F' in which the particle has a momentum $\vec{p}$, we will in general have a mixed spin state for the same particle in this new frame. A basic ket in F' can be written as

$$|b_1\rangle = |\vec{p}; \sigma, \mu\rangle \quad (2.2)$$

The mixed state $|B_1\rangle$ can now be expressed as a linear combination of our basic states,

$$\begin{aligned} |B_1\rangle &\equiv L(\vec{p}) |A_1\rangle \\ &= L(\vec{p}) |\vec{p}=0; \sigma, \nu\rangle \\ &= \sum_{\mu} D_{\mu\nu}^{\sigma}(\xi) |\vec{p}; \sigma, \mu\rangle \end{aligned} \quad (2.3)$$

where ξ is the Lorentz spin rotation. For the special case in which the two frames F and F' are moving with respect to each other along the direction of quantization — the z-axis — ξ will of course be zero. A general expression for ξ is derived in Appendix C.

B. Two-Particle States

Two-particle states can be built up from one-particle states. Let F be the c.m. frame in which the particles have momenta $\vec{q}_1$ and $\vec{q}_2$ along an arbitrary direction $\omega = (\theta, \phi)$ and spin projections ν_1 and ν_2 along the z-axis, so that $\vec{q}_1 + \vec{q}_2 = 0$ holds. We have the state

$$|A_2\rangle = |\vec{p}_{12}=0; \vec{q}; \sigma_1 \nu_1, \sigma_2 \nu_2\rangle \quad (2.4)$$

where $\vec{q}_1 = -\vec{q}_2 = \vec{q}$. Again, in a general frame F' moving with respect to F with velocity $\vec{v}$ in an arbitrary direction, the two particles will be in a mixed spin

state, say $|B_2\rangle$. As before, we can take our basic state to be

$$|b_2\rangle = |\vec{p}_{12}; \vec{p}_1^{\mu_1}, \vec{p}_2^{\mu_2}\rangle \quad (2.5)$$

with $\vec{p}_{12} = \vec{p}_1 + \vec{p}_2$. $|B_2\rangle$ can be written in terms of these basic kets.

$$\begin{aligned} |B_2\rangle &= L(\vec{p}_{12}) |A_2\rangle \\ &= \sum_{\mu_1 \mu_2} D_{\mu_1 \nu_1}^{\sigma_1}(\xi_1) D_{\mu_2 \nu_2}^{\sigma_2}(\xi_2) |\vec{p}_{12}; \vec{p}_1^{\mu_1}, \vec{p}_2^{\mu_2}\rangle \end{aligned} \quad (2.6)$$

Besides these plane-wave states, we shall also be interested in the angular momentum states. For the plane-wave states it is clear that one needs eight quantum numbers, six momentum variables and two spin projections, to completely specify the state. Since this number must remain the same for any other representation, we must select eight commuting variables to construct our angular momentum states. In F and F', we shall write these states as

$$|\phi_2\rangle = |\vec{p}_{12}=0; W; \sigma \ell J M\rangle \quad (2.7)$$

$$|\psi_2\rangle = L(\vec{p}_{12}) |\phi_2\rangle = |\vec{p}_{12}; W; \sigma \ell J M\rangle \quad (2.8)$$

respectively. Here σ is the total spin, $\vec{\sigma} = \vec{\sigma}_1 + \vec{\sigma}_2$; $\vec{\ell}$ is the orbital angular momentum; $\vec{J}$ is the total angular momentum, $\vec{J} = \vec{\ell} + \vec{\sigma}$, and M its z-projection.

An important problem in any partial-wave analysis is to express the plane-wave states in terms of the angular momentum states and vice versa. Let us consider the two representations in F. We can write

$$|\phi_2\rangle = \int \sum |A_2\rangle dA_2 \langle A_2 | \phi_2 \rangle \quad (2.9)$$

where the volume element dA_2 includes the normalization factor also. The transformation function $\langle A_2 | \phi_2 \rangle$ can be worked out as follows.

$$\begin{aligned} \langle A_2 | \phi_2 \rangle &= \langle \vec{p}_{12}=0; \vec{q}'(W'); \sigma_1 \nu_1, \sigma_2 \nu_2 | \vec{p}_{12}=0; W; \sigma \ell J M \rangle \\ &\equiv \langle \vec{p}_{12}=0; \vec{q}'(W') \theta \phi; \sigma_1 \nu_1 \sigma_2 \nu_2 | \vec{p}_{12}=0; W; \sigma_1 \sigma_2 \ell J M \rangle \\ &= \langle \vec{p}_{12}=0; \vec{q}'(W') | \vec{p}_{12}=0; W \rangle \langle \theta \phi; \sigma_1 \nu_1 \sigma_2 \nu_2 | \sigma_1 \sigma_2 \ell J M \rangle \end{aligned}$$

where we have separated off the kinematic part. First consider the spin-angular part.

$$\begin{aligned}
 \langle \sigma_1 v_1 \sigma_2 v_2 | &= \sum_{\sigma' v'} \langle \sigma_1 v_1 \sigma_2 v_2 | \sigma_1 \sigma_2 \sigma' v' \rangle \langle \sigma_1 \sigma_2 \sigma' v' | \\
 &= \sum_{\sigma' v'} C(\sigma_1 \sigma_2 \sigma'; v_1 v_2 v') \langle \sigma_1 \sigma_2 \sigma' v' | \\
 \langle \theta \phi | &= \sum_{\ell' m'} \langle \theta \phi | \ell' m' \rangle \langle \ell' m' | \\
 &= \sum_{\ell' m'} Y_{\ell' m'}(\theta \phi) \langle \ell' m' |
 \end{aligned}$$

Let us now couple the spin and the orbital parts.

$$\begin{aligned}
 &\langle \sigma_1 v_1; \sigma_2 v_2 | \otimes \langle \theta \phi | \\
 &= \sum_{\sigma' v' \ell' m'} C(\sigma_1 \sigma_2 \sigma'; v_1 v_2 v') Y_{\ell' m'}(\theta \phi) \langle \sigma_1 \sigma_2 \sigma' v' | \otimes \langle \ell' m' | \\
 &= \sum_{\sigma' v' \ell' m'} C(\sigma_1 \sigma_2 \sigma'; v_1 v_2 v') Y_{\ell' m'}(\theta \phi) \sum_{J' M'} \langle \sigma_1 \sigma_2 \sigma' v' \ell' m' | \sigma_1 \sigma_2 \sigma' \ell' J' M' \rangle \langle \sigma_1 \sigma_2 \sigma' \ell' J' M' | \\
 &= \sum_{\sigma' v' \ell' m' J' M'} C(\sigma_1 \sigma_2 \sigma'; v_1 v_2 v') C(\ell' \sigma' J'; m' v' M') Y_{\ell' m'}(\theta \phi) \langle \sigma_1 \sigma_2 \sigma' \ell' J' M' |
 \end{aligned}$$

Thus the spin-angular part of the transformation function becomes

$$\begin{aligned}
 &\langle \theta \phi; \sigma_1 v_1, \sigma_2 v_2 | \sigma_1 \sigma_2 \ell J M \rangle \\
 &= \sum_{\sigma' v' \ell' m' J' M'} C(\sigma_1 \sigma_2 \sigma'; v_1 v_2 v') C(\ell' \sigma' J'; m' v' M') Y_{\ell' m'}(\theta \phi) \underbrace{\langle \sigma_1 \sigma_2 \sigma' \ell' J' M' | \sigma_1 \sigma_2 \ell J M \rangle}_{\delta_{\sigma' \sigma} \delta_{\ell' \ell} \delta_{J' J} \delta_{M' M}} \\
 &= \sum_{\nu m} C(\sigma_1 \sigma_2 \sigma; v_1 v_2 v) C(\ell \sigma J; m \nu M) Y_{\ell m}(\theta \phi)
 \end{aligned}$$

If we normalize the states as

$$\langle A_2' | A_2 \rangle = \frac{4W}{q} \delta(W' - W) \delta(\cos\theta' - \cos\theta) \delta(\phi' - \phi) \delta_{v_1' v_1} \delta_{v_2' v_2} \quad (2.10)$$

$$\langle \phi_2' | \phi_2 \rangle = \frac{4W}{q} \delta(W' - W) \delta_{\sigma' \sigma} \delta_{\ell' \ell} \delta_{J' J} \delta_{M' M} \quad (2.11)$$

we get for the kinematic part

$$\langle \vec{p}_{12} = 0; q'(W') | \vec{p}_{12} = 0; W \rangle = \frac{4W}{q} \delta(W' - W)$$

which gives us the desired transformation.

$$\langle A_2' | \phi_2 \rangle = \frac{4W}{q} \delta(W' - W) \sum_{\nu m} C(\sigma_1 \sigma_2 \sigma; v_1 v_2 \nu) C(\ell \sigma J; m \nu M) Y_{\ell m}(\theta \phi) \quad (2.12)$$

Using this result in (2.9) one gets

$$\begin{aligned} |\phi_2\rangle &= \int \sum_{v_1 v_2} |\vec{p}_{12}=0; q' \theta \phi; \sigma_1 v_1 \sigma_2 v_2\rangle \frac{q'}{4W'} dW' d(\cos\theta) d\phi \\ &\quad \langle \vec{p}_{12}=0; q' \theta \phi; \sigma_1 v_1 \sigma_2 v_2 | \vec{p}_{12}=0; W; \sigma_1 \sigma_2 \ell J M \rangle \\ &= \int \sum_{v_1 v_2} \frac{q'}{4W'} dW' d(\cos\theta) d\phi \cdot \frac{4W}{q} \delta(W' - W) \sum_{\nu m} C(\sigma_1 \sigma_2 \sigma; v_1 v_2 \nu) \\ &\quad \times C(\ell \sigma J; \pi \nu M) Y_{\ell m}(\theta \phi) |\vec{p}_{12}=0; q' \theta \phi; \sigma_1 v_1 \sigma_2 v_2\rangle \\ &= \sum_{v_1 v_2} \int d\omega C(\sigma_1 \sigma_2 \sigma; v_1 v_2 \nu) C(\ell \sigma J; m \nu M) Y_{\ell m}(\theta \phi) |\vec{p}_{12}=0; q \theta \phi; \sigma_1 v_1 \sigma_2 v_2\rangle \end{aligned} \quad (2.13)$$

whose inverse is

$$\begin{aligned} |\vec{p}_{12}=0; q \theta \phi; \sigma_1 v_1 \sigma_2 v_2\rangle &= \sum_{\substack{\sigma \nu \\ \ell m \\ J M}} C(\sigma_1 \sigma_2 \sigma; v_1 v_2 \nu) C(\ell \sigma J; m \nu M) Y_{\ell m}^*(\theta \phi) \times \\ &\times |\vec{p}_{12}=0; W; \sigma_1 \sigma_2 \ell J M\rangle \end{aligned} \quad (2.13')$$

This is the relationship between the plane-wave and the angular momentum representations in the c.m. frame F. To derive it for the general frame F' we only have to give a Lorentz boost. Substituting (2.13) in the r.h.s. of (2.8) we have

$$\begin{aligned}
 |\psi_2\rangle &= L(\vec{p}_{12})|\phi_2\rangle \\
 &= \sum_{\nu_1 \nu_2 \nu m} d\omega C(\sigma_1 \sigma_2 \sigma; \nu_1 \nu_2 \nu) C(\ell \sigma J; m \nu M) Y_{\ell m}(\theta\phi) L(\vec{p}_{12})|A_2\rangle \\
 &= \sum_{\substack{\nu_1 \nu_2 \nu m \\ \mu_1 \mu_2}} d\omega C(\sigma_1 \sigma_2 \sigma; \nu_1 \nu_2 \nu) C(\ell \sigma J; m \nu M) Y_{\ell m}(\theta\phi) \\
 &\quad \times D_{\nu_1 \mu_2}^{\sigma_1}(\xi_1) D_{\mu_2 \nu_2}^{\sigma_2}(\xi_2) |\vec{p}_{12}; \vec{p}_1 \mu_1; \vec{p}_2 \mu_2\rangle
 \end{aligned} \tag{2.14}$$

where we have made use of (2.6). We can also write down the inverse of this expansion.

$$\begin{aligned}
 |\vec{p}_{12}; \vec{p}_1 \mu_1, \vec{p}_2 \mu_2\rangle &= \sum_{\substack{\nu_1 \nu_2 \nu m \\ \sigma \ell J M}} C(\sigma_1 \sigma_2 \sigma; \nu_1 \nu_2 \nu) C(\ell \sigma J; m \nu M) \\
 &\quad \times Y_{\ell m}^*(\theta\phi) D_{\mu_1 \nu_1}^{\sigma_1*}(\xi_1) D_{\mu_2 \nu_2}^{\sigma_2*}(\xi_2) |\vec{p}_{12}; W; \sigma \ell J M\rangle
 \end{aligned} \tag{2.15}$$

C. Three-Particle States

Three-particle states can be built up from the two-particle states. In the overall center-of-mass frame F, let the three momenta be $\vec{Q}_1$, $\vec{Q}_2$, $\vec{Q}_3$ and the spin projections along the z-axis be ν_1 , ν_2 and ν_3 . We have the plane-wave state

$$|A_3\rangle = |\vec{p}_{123}=0; \vec{Q}_1 \nu_1, \vec{Q}_2 \nu_2, \vec{Q}_3 \nu_3\rangle, \quad \vec{Q}_1 + \vec{Q}_2 + \vec{Q}_3 = 0 \tag{2.16}$$

For the angular momentum states, we note that there are three equivalent representations depending upon which two particles are coupled first. Let us choose this pair to be particles 1 and 2 and denote this representation by a superscript 3. Our basic ket then is

$$|\phi_3\rangle = |\vec{p}_{123}=0; W s^3; \sigma^3 \ell^3 j^3 \Sigma^3 L^3 J M\rangle \quad (2.17)$$

where W is the total energy; s is the square of c.m. energy for particles 1 and 2; $\vec{\sigma} = \vec{\sigma}_1 + \vec{\sigma}_2$; ℓ and j are the orbital and total angular momenta ($\vec{j} = \vec{\ell} + \vec{s}$) for the subsystem 1-2 in their c.m.; Σ is the total spin of the subsystem and particle 3, $\vec{\Sigma} = \vec{j} + \vec{\sigma}_3$; L and J are the orbital and total angular momenta for the subsystem and particle 3 ($\vec{J} = \vec{L} + \vec{\Sigma}$) and $M = J_z$. Unless required for clarity, we shall omit the superscript 3 in what follows.

As before, we can express $|\phi_3\rangle$ in terms of $|A_3\rangle$.

$$|\phi_3\rangle = \int \sum |A_3\rangle dA_3 \langle A_3 | \phi_3 \rangle \quad (2.18)$$

where dA_3 includes the normalization factor. Although one could proceed in this manner, it is much easier to rewrite $|A_3\rangle$ making use of what we have learned about the two-particle states.

$$\begin{aligned} |A_3\rangle &= |\vec{Q}_1 v_1; \vec{Q}_2 v_2\rangle \otimes |\vec{Q}_3 v_3\rangle \\ &= \sum_{\substack{\mu_1 \mu_2 \mu m \\ \sigma \ell j n}} C(\sigma_1 \sigma_2 \sigma; \mu_1 \mu_2 \mu) C(\ell \sigma j; m \mu n) Y_{\ell m}^*(\theta \phi) \\ &\quad \times D_{\nu_1 \mu_1}^{\sigma_1*}(\xi_1) D_{\nu_2 \mu_2}^{\sigma_2*}(\xi_2) |\vec{Q}_{12}; s; \sigma \ell j n\rangle \otimes |\vec{Q}_3 v_3\rangle \end{aligned}$$

where we used (2.15) for the two-particle part. Once again, the state on the r.h.s. describes a "particle 1-2" and the uncoupled particle 3 with spins j and σ_3 and their z-projections n and ν_3 respectively in their center of mass system which is the o.c.m., i.e. this can also be regarded as a two-particle state in its c.m. frame. Using (2.13'), we thus get

$$\begin{aligned} |\vec{Q}_{12}; s; \sigma \ell j n\rangle \otimes |\vec{Q}_3 v_3\rangle &= |\vec{p}_{123}=0; Q \Theta \Phi; j n; \sigma_3 v_3(s, \sigma, \ell)\rangle \\ &= \sum_{\substack{\Sigma \nu \\ L L_z \\ J M}} C(j \sigma_3 \Sigma; n \nu_3 \nu) C(L \Sigma J; L_z \nu M) Y_{L L_z}^*(\Theta \Phi) |\vec{p}_{123}=0; W; j \sigma_3 \Sigma L J M(s \sigma \ell)\rangle \end{aligned}$$

so that $|A_3\rangle$ becomes

$$\begin{aligned}
 |A_3\rangle = & \sum_{\substack{\mu_1 \mu_2 \mu m \\ \sigma \ell j n}} \sum_{\substack{\Sigma \nu \\ LL_2 JM}} C(\sigma_1 \sigma_2 \sigma; \mu_1 \mu_2 \mu) C(\ell \sigma j; m \mu n) C(j \sigma_3 \Sigma; n \nu_3 \nu) \\
 & \times C(L \Sigma J; L_2 \nu M) Y_{\ell m}^*(\theta \phi) Y_{LL_2}^*(\Theta \Phi) D_{\nu_1 \mu_1}^{\sigma_1^*}(\xi_1) \\
 & \times D_{\nu_2 \mu_2}^{\sigma_2^*}(\xi_2) |\vec{p}_{123}=0; Ws; \sigma \ell j \Sigma L J M\rangle
 \end{aligned} \quad (2.19)$$

which is the desired relation in the o.c.m.. Its inverse is given by

$$\begin{aligned}
 |\vec{p}_{123}=0; Ws; \sigma \ell j \Sigma L J M\rangle = & \sum_{\substack{\mu_1 \mu_2 \mu m n \\ \nu_1 \nu_2 \nu_3 \nu L_2}} \int d\omega d\Omega C(\sigma_1 \sigma_2 \sigma; \mu_1 \mu_2 \mu) C(\ell \sigma j; m \mu n) \\
 & \times C(j \sigma_3 \Sigma; n \nu_3 \nu) C(L \Sigma J; L_2 \nu M) Y_{\ell m}(\theta \phi) Y_{LL_2}(\Theta \Phi) D_{\nu_1 \mu_1}^{\sigma_1}(\xi_1) \\
 & \times D_{\nu_2 \mu_2}^{\sigma_2}(\xi_2) |\vec{p}_{123}=0; \vec{Q}_1 \nu_1, \vec{Q}_2 \nu_2, \vec{Q}_3 \nu_3\rangle
 \end{aligned} \quad (2.20)$$

The transformation function $\langle A_3 | \phi_3 \rangle$ appearing in (2.18) can now be extracted from (2.20). The differential element dA_3 is equal to $d^3 p_1 d^3 p_2 d^3 p_3 / 2E_1 \cdot 2E_2 \cdot 2E_3$, and the sum in (2.18) extends over ν_1, ν_2, ν_3 . To bring this out from (2.20), we note that

$$\int d\omega d\Omega = \frac{16W\sqrt{s^3}}{Q^3 q^3} \int \frac{d^3 p_1 d^3 p_2 d^3 p_3}{2E_1 \cdot 2E_2 \cdot 2E_3} \delta^3(\vec{p}_1 + \vec{p}_2 + \vec{p}_3) \delta(E_1 + E_2 + E_3 - W) \delta((p_1 + p_2)^2 - s^3) \quad (2.21)$$

Using this in (2.20) and comparing the result with (2.18), we easily get

$$\begin{aligned}
 \langle \vec{p}_\alpha \vec{p}_\beta \vec{p}_\gamma; \nu_\alpha \nu_\beta \nu_\gamma | \vec{p}_{123}=0, Ws^\alpha; \sigma^\alpha \ell^\alpha j^\alpha \Sigma^\alpha L^\alpha J^\alpha M^\alpha \rangle = & \frac{16W\sqrt{s^\alpha}}{Q^\alpha q^\alpha} \delta^3(\vec{p}_\alpha + \vec{p}_\beta + \vec{p}_\gamma) \delta(W - (E_\alpha + E_\beta + E_\gamma)) \\
 & \times \delta((p_\beta + p_\gamma)^2 - s^\alpha) \sum_{\substack{\mu_\beta \mu_\gamma \mu m n \nu L_2^\alpha}} C(\sigma_\beta \sigma_\gamma \sigma^\alpha; \mu_\beta \mu_\gamma \mu) C(\ell^\alpha \sigma^\alpha j^\alpha; m \mu n) C(j^\alpha \sigma_\alpha \Sigma^\alpha; n \nu_\alpha \nu) \\
 & \times C(L^\alpha \Sigma^\alpha J^\alpha; L_2^\alpha \nu M^\alpha) Y_{\ell^\alpha m^\alpha}(\theta^\alpha \phi^\alpha) Y_{L^\alpha L_2^\alpha}(\Theta^\alpha \Phi^\alpha) D_{\nu_\beta \mu_\beta}^{\sigma_\beta}(\xi_\beta^\alpha) D_{\nu_\gamma \mu_\gamma}^{\sigma_\gamma}(\xi_\gamma^\alpha)
 \end{aligned} \quad (2.22)$$

III. RECOUPLING COEFFICIENT

It was mentioned in Section II that for a three-particle system there are three linearly independent representations in the angular momentum space. These however are equivalent in the sense that they are connected by unitary transformations. In this section we derive an explicit expression for this so-called recoupling coefficient for any two of the three representations. Let us make the following points at the outset:

- i) We shall consider the states in the o.c.m. only; generalization to an arbitrary frame F' will only introduce a minor change.
- ii) We use cyclic notation and coupling schemes throughout. This means we order the three particles as 123, 231, 312 and the three representations will be denoted for brevity as $|1\rangle$, $|2\rangle$, $|3\rangle$, in which the particles 23, 31, and 12 are coupled first respectively and in that order.
- iii) We use Rose's convention³ for rotation operators. In particular, let a general Euler rotation $(\alpha\beta\gamma)$ be required to go from a frame F to $\tilde{F}$. Then, if a physical system is in a definite angular momentum state $|JM\rangle$ in frame F , it will, in general, be in a mixed state $R(\alpha\beta\gamma)|JM\rangle$ in frame $\tilde{F}$. This mixed state, however, can be expressed as a linear combination of the complete set $|JN\rangle$.

$$\begin{aligned}
 R|JM\rangle &= \sum_N |JN\rangle \langle JN|R|JM\rangle \\
 &= \sum_N D_{NM}^J(\alpha\beta\gamma) |JN\rangle
 \end{aligned} \tag{3.1}$$

The rotation operator is given by

$$R(\alpha\beta\gamma) = e^{-i\alpha J_z} e^{-i\beta J_y} e^{-i\gamma J_z} \tag{3.2}$$

A rotation, say α in the x-y plane, is positive (negative) if it advances a right-handed screw along the positive (negative) z-axis.

With these preliminary remarks, we are now equipped to consider a recoupling coefficient, say $\langle 1|3\rangle$. We shall omit the total momentum $\vec{p}_{123} = 0$ and the total energy W , and again, for easy notation, indicate the quantum numbers of $\langle 1|$ with

a prime and those of $|3\rangle$ without a prime. We wish to calculate

$$\begin{aligned} \langle 1|3\rangle &\equiv \langle J^1 M^1; \sigma^1 \ell^1 j^1 \Sigma^1 L^1; s^1 | J^3 M^3; \sigma^3 \ell^3 j^3 \Sigma^3 L^3; s^3 \rangle \\ &\equiv \langle J' M'; \sigma' \ell' j' \Sigma' L'; s' | J M; \sigma \ell j \Sigma L; s \rangle \end{aligned} \quad (3.3)$$

The essential steps are as follows.

Step 1

For the r.h.s. of (3.3) we make use of the expansion (2.20) thus relating the two different angular momentum representations to a single plane-wave representation. Before doing so, however, we note that although the l.h.s. of (2.20) is a $|JM\rangle$ state, and hence must transform as in (3.1), it is not obvious from its r.h.s. that it actually does so. To make this explicit, let us manipulate (3.1). Applying R^{-1} from left,

$$|JM\rangle = \sum_N D_{NM}^J(\alpha\beta\gamma) R^{-1}(\alpha\beta\gamma) |JN\rangle \quad \text{for a ket, and} \quad (3.4)$$

$$\langle J'M'| = \sum_{N'} D_{N'M'}^{J'*}(\alpha'\beta'\gamma') \langle J'N'| R'(\alpha'\beta'\gamma') \quad \text{for a bra.} \quad (3.5)$$

Substitution of these into (3.3) gives

$$\langle 1|3\rangle = \sum_{NN'} D_{N'M'}^{J'*}(R') D_{NM}^J(R) \langle J'N' \dots | R' R^{-1} | JN \dots \rangle \quad (3.6)$$

where we have suppressed some indices. Now the J-dependence is clear through the rotation matrices. We also abbreviate (2.20) to save space and write it as

$$|JM \dots\rangle = \int G(\vec{Q}_1 v_1 \vec{Q}_2 v_2 \vec{Q}_3 v_3) \quad (3.7)$$

and a similar expression holds for the primed state. Let us put this in the r.h.s. of (3.6).

$$\begin{aligned} \langle 1|3\rangle &= \sum_{NN'} \sum' \int d\omega d\Omega \int d\omega' d\Omega' D_{N'M'}^{J'*}(R') D_{NM}^J(R) G G'^* \\ &\times \langle \vec{Q}_1' v_1' \vec{Q}_2' v_2' \vec{Q}_3' v_3' | R' R^{-1} | \vec{Q}_1 v_1 \vec{Q}_2 v_2 \vec{Q}_3 v_3 \rangle \end{aligned} \quad (3.8)$$

Step 2

So far nothing has been said about the choice of coordinate axes in the o.c.m. so that the results derived up to the present hold true in any frame. Indeed, as is evident from (3.3), the function $\langle 1|3 \rangle$ can only depend upon s, s', L, L' , etc. whose values are independent of how the coordinate axes are selected for calculations. Since the three particles form a plane, it is natural to have two axes, say z and x , lie in this plane. Thus, let A be an arbitrary set of axes with the z -axis in the three-particle plane in an arbitrary direction, the y -axis perpendicular to, and out of, the plane and the x -axis such that the $Oxyz$ forms a right-handed system (see Fig. 2a).

Let us now understand the expansion (3.7). Figure 2b shows the angle variables for a typical configuration of the three momenta. The integration in (3.7) has to be performed over all such configurations keeping the axes fixed.

At this stage we also note that the arguments in the rotations R and R' are yet completely unspecified. Let us choose $(\alpha\beta\gamma) = (0\ 0\ 0)$ and give meaning to the action of $R^{-1}(\alpha\beta\gamma) = R(0\ 0\ 0)$ on the ket in (3.8). For simplicity, first omit the spins and consider the momenta only. Clearly, $R(-\theta) |\vec{Q}_1 \vec{Q}_2 \vec{Q}_3\rangle$ is the state $|\vec{Q}_1' \vec{Q}_2' \vec{Q}_3'\rangle$ as viewed from a new frame B reached from A by rotation through $-\theta$. The angle variables of the three momenta in B are shown in Fig. 2c. We will from now on understand these to be the angles in the function G and also in $d\omega d\Omega$. As far as the spins are concerned, we know that $|\sigma_1 v_1 \sigma_2 v_2 \sigma_3 v_3\rangle$ transforms as

$$R|\sigma_1 v_1 \sigma_2 v_2 \sigma_3 v_3\rangle = \sum_{\alpha_1 \alpha_2 \alpha_3} D_{\alpha_1 v_1}^{\sigma_1}(R) D_{\alpha_2 v_2}^{\sigma_2}(R) D_{\alpha_3 v_3}^{\sigma_3}(R) |\sigma_1 \alpha_1 \sigma_2 \alpha_2 \sigma_3 \alpha_3\rangle \quad (3.9)$$

Similar considerations apply for the primed state for which $(\alpha'\beta'\gamma') = (0\ -\theta'\ 0)$ so that the action of $R'(0\ \theta'\ 0)$ on the bra gives the desired state in B' and the relevant diagrams are Fig. 2d and Fig. 2e.

The statement is often made that one uses two sets of axes to calculate the recoupling coefficient. We have gone into such great detail to show how this comes about in a natural way by starting with an arbitrary set of axes. Equation (3.8) now becomes

$$\begin{aligned}
 \langle 1|3 \rangle &= \sum_{NN'} \sum_{\alpha_1 \alpha_2 \alpha_3} \sum_{\alpha'_1 \alpha'_2 \alpha'_3} \int d\omega d\Omega \int d\omega' d\Omega' D_{N'M'}^{J'*, (R'^{-1})} D_{NM}^{J, (R^{-1})} \\
 &\times GG'^* D_{\alpha_1 \nu_1}^{\sigma_1, (R)} D_{\alpha_2 \nu_2}^{\sigma_2, (R)} D_{\alpha_3 \nu_3}^{\sigma_3, (R)} D_{\alpha'_1 \nu'_1}^{\sigma'_1*, (R')} D_{\alpha'_2 \nu'_2}^{\sigma'_2*, (R')} D_{\alpha'_3 \nu'_3}^{\sigma'_3*, (R')} \\
 &\times \langle \vec{Q}'_1 \alpha'_1 \vec{Q}'_2 \alpha'_2 \vec{Q}'_3 \alpha'_3 | \vec{Q}_1 \alpha_1 \vec{Q}_2 \alpha_2 \vec{Q}_3 \alpha_3 \rangle
 \end{aligned} \tag{3.10}$$

We emphasize again for clarity that $R = R(0 - \Theta 0)$ and $R' = R'(0 \Theta' 0)$ so that the arguments for the D^J and $D^{J'}$ must involve R^{-1} and R'^{-1} as indicated in (3.4) and (3.5).

Step 3

We mentioned that the recoupling coefficient does not depend upon the arbitrary choice of axes. In fact, intuitively speaking, it can only depend upon the relative orientation of B and B'. Indeed, there exists a relationship between the rotations R and R' which we now obtain.

Since R takes us from A to B, $R A = B$,
 and since R' takes us from A to B', $R' A = B'$,
 let $\bar{R}$ take us from B to B' so that $\bar{R} B = B'$.

Substituting for B and B' from the first two relations, the third relation gives

$$\bar{R} R = R' = R \bar{R} \tag{3.11}$$

where the second equality follows from the fact that R and $\bar{R}$ being in the x-z plane commute with each other. Furthermore, looking at Figs. 2c and 2e, we find that $\bar{R}$ involves a rotation $-\chi$ in the x-z plane, i.e.,

$$\bar{R} = e^{-i(-\chi)J_y} \tag{3.12}$$

Step 4

Let us now consider the integration

$$X \equiv \int d\omega' d\Omega' \langle \vec{Q}'_1 \alpha'_1 \vec{Q}'_2 \alpha'_2 \vec{Q}'_3 \alpha'_3 | \vec{Q}_1 \alpha_1 \vec{Q}_2 \alpha_2 \vec{Q}_3 \alpha_3 \rangle$$

again leaving out the spins from the states. Our three-particle states are normalized such that

$$\langle \vec{Q}_1' \vec{Q}_2' \vec{Q}_3' | \vec{Q}_1 \vec{Q}_2 \vec{Q}_3 \rangle = 2E_1' \cdot 2E_2' \cdot 2E_3' \cdot \delta^3(\vec{Q}_1' - \vec{Q}_1) \delta^3(\vec{Q}_2' - \vec{Q}_2) d^3(\vec{Q}_3' - \vec{Q}_3)$$

and the volume element in the momentum space is related to that in the space of angle variables by

$$\frac{d^3 p_1'}{2E_1'} \frac{d^3 p_2'}{2E_2'} \frac{d^3 p_3'}{2E_3'} = \frac{Q' q'}{16W' \sqrt{s'}} d^3 p' dW' ds' d\omega' d\Omega'$$

Or,

$$\int \frac{16W' \sqrt{s'}}{Q' q'} \frac{d^3 p_1'}{2E_1'} \frac{d^3 p_2'}{2E_2'} \frac{d^3 p_3'}{2E_3'} \delta^4(P') \delta((p_2' - p_3')^2 - s') = \int d\omega' d\Omega'$$

Using these to calculate X, we get

$$X = \int \delta((Q_2 + Q_3)^2 - s') \frac{16W \sqrt{s'}}{Q' q'}$$

where, to be consistent with our procedure, we have left out the total 4-momentum δ -function. This X will be further subject to integration over $d\omega$ and $d\Omega$. Set

$$Y = \int d\omega d\Omega X = \int \sin\theta d\theta d\phi \cdot \sin\theta d\theta d\phi X = \int d(-\cos\theta) dR X$$

with

$$dR \equiv \sin\theta d\theta d\phi d\phi$$

Since $\cos\theta$ is related to $(Q_2 + Q_3)^2$ by (the Q's here are, of course, four momenta)

$$\cos\theta = \frac{1}{4W\sqrt{s}} \frac{1}{qQ} \left[(W^2 - m_3^2)(m_2^2 - m_1^2) + s(W^2 + m_1^2 + m_2^2 + m_3^2 - s) - 2s(Q_2 + Q_3)^2 \right]$$

we can replace $d(\cos\theta)$ by $d(Q_2 + Q_3)^2$.

$$d(-\cos\theta) = -\frac{2s}{4W\sqrt{s} qQ} d(Q_2 + Q_3)^2$$

This yields for Y

$$Y = \frac{8\sqrt{s} \sqrt{s'}}{qQ q' Q'} \int dR$$

so that we are left with the integration over dR only. The normalization over the spin variables trivially gives

$$\langle \alpha'_1 \alpha'_2 \alpha'_3 | \alpha_1 \alpha_2 \alpha_3 \rangle = \delta_{\alpha'_1 \alpha_1} \delta_{\alpha'_2 \alpha_2} \delta_{\alpha'_3 \alpha_3}$$

so that $\langle 1|3 \rangle$ becomes, after summing over $\alpha'_1, \alpha'_2, \alpha'_3$ using the above δ -functions

$$\begin{aligned} \langle 1|3 \rangle &= \sum_{NN'} \sum \sum' \sum_{\alpha_1 \alpha_2 \alpha_3} \int dR \left(\frac{8\sqrt{s s'}}{qQ q'Q'} \right) D_{N'M'}^{J'^*}(R'^{-1}) D_{NM}^J(R^{-1}) G G'^* \\ &\times D_{\alpha_1 \nu_1}^{\sigma_1}(R) D_{\alpha_2 \nu_2}^{\sigma_2}(R) D_{\alpha_3 \nu_3}^{\sigma_3}(R) D_{\alpha_1 \nu_1}^{\sigma_1*}(R') D_{\alpha_2 \nu_2}^{\sigma_2*}(R') D_{\alpha_3 \nu_3}^{\sigma_3*}(R') \end{aligned} \quad (3.13)$$

Step 5

The next step is to consider the rotation matrices alone, leaving out the C-G coefficients. There are two matrices in G [see (2.20)], two in C^* and eight as shown in (3.13). Let us write C and C' for the Clebsch-Gordan parts of G and G' respectively as given in (2.20) and collect the d-functions. We get

$$\begin{aligned} \langle 1|3 \rangle &= \sum_{NN'} \sum_{\substack{\mu_1 \mu_2 \mu m n \\ \nu_1 \nu_2 \nu_3 \nu L_z}} \sum_{\substack{\mu'_1 \mu'_3 \mu' m' n' \\ \nu'_1 \nu'_2 \nu'_3 \nu' L'_z}} \sum_{\alpha_1 \alpha_2 \alpha_3} \int dR \left(\frac{8\sqrt{s s'}}{qQ q'Q'} \right) D_{N'M'}^{J'^*}(R'^{-1}) \\ &\times D_{NM}^J(R^{-1}) Y_{lm}(\theta) \left(\frac{2L+1}{4\pi} \right)^{\frac{1}{2}} Y_{l'm'}^*(\theta', 0) \left(\frac{2L'+1}{4\pi} \right)^{\frac{1}{2}} d_{\nu_1 \mu_1}^{\sigma_1}(\xi_1) \\ &\times d_{\nu_2 \mu_2}^{\sigma_2}(\xi_2) d_{\nu_2' \mu_2'}^{\sigma_2}(\xi_2') d_{\nu_3' \mu_3'}^{\sigma_3}(\xi_3') D_{\alpha_1 \nu_1}^{\sigma_1}(R) D_{\alpha_2 \nu_2}^{\sigma_2}(R) D_{\alpha_3 \nu_3}^{\sigma_3}(R) \\ &\times D_{\alpha_1 \nu_1}^{\sigma_1*}(R') D_{\alpha_2 \nu_2}^{\sigma_2*}(R') D_{\alpha_3 \nu_3}^{\sigma_3*}(R') \times C C' \end{aligned} \quad (3.14)$$

where

$$\left(\frac{2L+1}{4\pi} \right)^{\frac{1}{2}} = Y_{LL_z}(\theta=0, \phi=0), \quad \left(\frac{2L'+1}{4\pi} \right)^{\frac{1}{2}} = Y_{L'L'_z}(\theta'=0, \phi'=0).$$

We now perform the following calculations which are self-explanatory.

$$(i) \quad D_{\alpha_1 \nu_1'}^{\sigma_1}(R') = D_{\alpha_1 \nu_1'}^{\sigma_1}(R \bar{R}) = \sum_{\chi_1} D_{\alpha_1 \chi_1}^{\sigma_1}(R) D_{\chi_1 \nu_1'}^{\sigma_1}(\bar{R})$$

so that

$$\begin{aligned} \sum_{\alpha_1} D_{\alpha_1 \nu_1'}^{\sigma_1}(R) D_{\alpha_1 \nu_1'}^{\sigma_1*}(R') &= \sum_{\chi_1} D_{\chi_1 \nu_1'}^{\sigma_1*}(\bar{R}) \underbrace{\sum_{\alpha_1} D_{\alpha_1 \nu_1'}^{\sigma_1}(R) D_{\alpha_1 \chi_1}^{\sigma_1*}(R)}_{\delta_{\nu_1 \chi_1}} \\ &= D_{\nu_1 \nu_1'}^{\sigma_1*}(\bar{R}) \\ &= d_{\nu_1 \nu_1'}^{\sigma_1}(\bar{R}) \end{aligned}$$

and similarly for the two D^{σ_2} and two D^{σ_3} terms. This takes care of the sum over α_1, α_2 and α_3 . The last step in the above calculation is a result of the fact that $\bar{R}$ is a rotation about the y-axis only so that $D \rightarrow d$ which happens to be a real function.

(ii) These d-functions can now be combined with the four d-functions that involve Lorentz spin rotations.

$$\begin{aligned} \sum_{\nu_1'} d_{\nu_1' \nu_1'}^{\sigma_1}(\bar{R}) d_{\nu_1' \mu_1}^{\sigma_1}(\xi_1) &= \sum_{\nu_1'} d_{\nu_1' \nu_1'}^{\sigma_1}(\bar{R}^{-1}) d_{\nu_1' \mu_1}^{\sigma_1}(\xi_1) = d_{\nu_1' \mu_1}^{\sigma_1}(\bar{R}^{-1} + \xi_1) , \\ \sum_{\nu_2'} d_{\nu_2' \mu_2'}^{\sigma_2}(\xi_2') d_{\nu_2' \nu_2'}^{\sigma_2}(\bar{R}) d_{\nu_2' \mu_2}^{\sigma_2}(\xi_2) &= \sum_{\nu_2'} d_{\mu_2' \nu_2'}^{\sigma_2}(-\xi_2') d_{\nu_2' \nu_2'}^{\sigma_2}(\bar{R}^{-1}) d_{\nu_2' \mu_2}^{\sigma_2}(\xi_2) \\ &= d_{\mu_2' \mu_2}^{\sigma_2}(-\xi_2' + \bar{R}^{-1} + \xi_2) , \\ \sum_{\nu_3'} d_{\nu_3' \nu_3'}^{\sigma_3}(\bar{R}) d_{\nu_3' \mu_3'}^{\sigma_3}(\xi_3') &= d_{\nu_3' \mu_3'}^{\sigma_3}(\bar{R} + \xi_3') \end{aligned}$$

Note that the μ 's and the μ' 's indices could not be summed over in the above calculations since they are contained in the C-G coefficients also.

(iii) Next we look at the D^J and $D^{J'}$ functions.

$$D_{N'M'}^{J'}(R'^{-1}) = D_{N'M'}^{J'}(\bar{R}^{-1}R^{-1}) = \sum_{\Lambda'} D_{N'\Lambda'}^{J'}(\bar{R}^{-1}) D_{\Lambda'M'}^{J'}(R^{-1})$$

so that

$$\begin{aligned} \int D_{N'M'}^{J'^*}(R'^{-1}) D_{NM}^J(R^{-1}) dR &= \sum_{\Lambda'} D_{N'\Lambda'}^{J'^*}(\bar{R}^{-1}) \underbrace{\int D_{\Lambda'M'}^{J'^*}(R^{-1}) D_{NM}^J(R^{-1}) dR^{-1}}_{\frac{8\pi^2}{2J+1} \delta_{J'J} \delta_{\Lambda'N} \delta_{M'M}} \\ &= \frac{8\pi^2}{2J+1} \delta_{J'J} \delta_{M'M} D_{N'N}^{J*}(\bar{R}^{-1}) \\ &= \frac{8\pi^2}{2J+1} \delta_{J'J} \delta_{M'M} d_{N'N}^J(\bar{R}^{-1}) \end{aligned}$$

(iv) Let us put the results of (i), (ii) and (iii) into Eq. (3.14).

$$\begin{aligned} \langle 1|3 \rangle &= \left(\frac{8\sqrt{s s'}}{qQ q'Q'} \right) \sum_{NN'} \sum_{\substack{\mu_1 \mu_2 \mu m \\ \nu_3 \nu L_z}} \sum_{\substack{\mu'_1 \mu'_2 \mu' m' n' \\ \nu'_1 \nu' L'_z}} \left(\frac{2L+1}{4\pi} \right)^{1/2} \left(\frac{2L'+1}{4\pi} \right)^{1/2} Y_{2m}(\theta_0) \\ &\times Y_{\ell'm'}^*(\theta'0) d_{\nu'_1 \mu'_1}^{\sigma_1}(\bar{R}^{-1} + \xi_1) d_{\mu'_2 \mu'_2}^{\sigma_2}(-\xi'_2 + \bar{R}^{-1} + \xi_2) d_{\nu_3 \mu_3}^{\sigma_3}(\bar{R} + \xi'_3) \\ &\times d_{N'N}^J(\bar{R}^{-1}) C C' \times \left(\frac{8\pi^2}{2J+1} \right) \delta_{J'J} \delta_{M'M} \end{aligned}$$

This can be further simplified if we note that since the three particles were chosen to lie in the x-z plane, $L_z = L'_z = 0$ so that $N = \nu + L_z$ is no longer different from ν . Similarly N' is no longer an independent variable and can be replaced by ν' . We also replace the dummy indices ν_3 and ν'_1 by μ_3 and μ'_1 for easy notation, set $Y_{\ell m}(\theta_0)$ equal to $\left(\frac{2\ell+1}{4\pi} \right)^{1/2} d_{m0}^{\ell}(\theta)$ and likewise for $Y_{\ell'm'}(\theta')$, and use (3.12) to

write $\bar{R} = -\chi$ and $\bar{R}^{-1} = +\chi$. Then, using (2.20) to write down the Clebsch-Gordan parts C and C', we finally arrive at

$$\begin{aligned}
 \langle 1|3 \rangle &= \langle J^1 M^1; \sigma^1 \ell^1 j^1 \Sigma^1 L^1; s^1 | J^3 M^3; \sigma^3 \ell^3 j^3 \Sigma^3 L^3; s^3 \rangle \\
 &= \delta_{J^1 J^3} \delta_{M^1 M^3} \left(\frac{8\sqrt{s^1 s^3}}{q^1 q^3 Q^1 Q^3} \right) \left(\frac{8\pi^2}{2J^1 + 1} \right) \left[\frac{(2\ell^1 + 1)}{4\pi} \cdot \frac{(2\ell^3 + 1)}{4\pi} \cdot \frac{(2L^1 + 1)}{4\pi} \cdot \frac{(2L^3 + 1)}{4\pi} \right]^{\frac{1}{2}} \\
 &\times \sum_{\substack{\mu_1 \mu_2 \mu_3 \\ \mu \nu m n}} \sum_{\substack{\mu'_1 \mu'_2 \mu'_3 \\ \mu' \nu' m' n'}} C(\sigma_1 \sigma_2 \sigma^3; \mu_1 \mu_2 \mu) C(\ell^3 \sigma^3 j^3; m \mu n) C(j^3 \sigma_3 \Sigma^3; n \mu_3 \nu) C(L^3 \Sigma^3 J^1; 0 \nu \nu) \\
 &\times C(\sigma_2 \sigma_3 \sigma^1; \mu'_2 \mu'_3 \mu'_1) C(\ell^1 \sigma^1 j^1; m' \mu' n') C(j^1 \sigma_1 \Sigma^1; n' \mu'_1 \nu') C(L^1 \Sigma^1 J^1; 0 \nu' \nu') \\
 &\times d_{m0}^{\ell^3}(\theta^3) d_{m'0}^{\ell^1}(\theta^1) d_{\mu'_1 \mu_1}^{\sigma_1}(\chi + \xi_1^3) d_{\mu_2 \mu_2}^{\sigma_2}(\chi + \xi_2^3 - \xi_2^1) d_{\mu_3 \mu_3}^{\sigma_3}(\chi - \xi_3^1) d_{\nu' \nu}^{J^1}(\chi)
 \end{aligned} \tag{3.15}$$

where we used $d_{\mu_3 \mu_3}^{\sigma_3}(\bar{R} + \xi_3^1) = d_{\mu_3 \mu_3}^{\sigma_3}(-\bar{R} - \xi_3^1)$ and replaced the primed and unprimed variables by superscripts 1 and 3 respectively. This is the desired recoupling coefficient for $\langle 1(23)|3(12) \rangle$. If we make one permutation of the indices throughout in the above expression so that $1 \rightarrow 2$, $2 \rightarrow 3$, $3 \rightarrow 1$, we obtain $\langle 2(31)|1(23) \rangle$ and another such permutation will yield $\langle 3(12)|2(31) \rangle$. Furthermore, since this coefficient is real, we have $\langle 3(12)|1(23) \rangle \equiv \langle 1(23)|3(12) \rangle^* = \langle 1(23)|3(12) \rangle$ and similarly for others. There are 14 summation indices in Eq. (3.15). Not all of these are independent since the Clebsch-Gordan coefficients impose the following restrictions.

$$\begin{aligned}
 \mu &= \mu_1 + \mu_2, \\
 n &= \mu + m = \mu_1 + \mu_2 + m, \\
 \nu &= n + \mu_3 = \mu_1 + \mu_2 + \mu_3 + m,
 \end{aligned}$$

so that one actually has 8 independent variables $(\mu_1, \mu_2, \mu_3, m; \mu'_1, \mu'_2, \mu'_3, m')$ to sum over. The Lorentz rotations are derived in Appendix C and the angles θ and χ are worked out in Appendix A.

IV. ISOSPIN SPACE

What has been discussed so far for the configuration space has its counterpart in the isospin space. Thus one has the charge states (plane-wave states), the isospin states (angular momentum states) and the recoupling coefficient which is essentially the Racah coefficient. We first introduce the states and then work out the recoupling coefficient.

A. One-Particle States

Consider a single particle whose isospin is I and its z -component I_z . We have the state

$$|\bar{A}_1\rangle = |I I_z\rangle \quad (4.1)$$

B. Two-Particle States

Two-particle charge states (plane-wave) can be built up from two one-particle states.

$$|\bar{A}_2\rangle = |I_1 i_1; I_2 i_2\rangle \quad (4.2)$$

To construct states of total isospin I (angular momentum states), we will need four quantum numbers as in the case of charge states. These are I_1, I_2 , the total isospin I and its z -component i so that we have the state

$$|\bar{\phi}_2\rangle = |I_1 I_2 I i\rangle \quad (4.3)$$

The unitary transformation which connects (4.2) and (4.3) is simply the Clebsch-Gordan coefficient.

$$|\bar{A}_2\rangle = \sum_{I i} |\bar{\phi}_2\rangle \langle \bar{\phi}_2 | \bar{A}_2\rangle = \sum_{I i} C(I_1 I_2 I; i_1 i_2 i) |I_1 I_2 I i\rangle \quad (4.4)$$

$$|\bar{\phi}_2\rangle = \sum_{i_1 i_2} |\bar{A}_2\rangle \langle \bar{A}_2 | \bar{\phi}_2\rangle = \sum_{i_1 i_2} C(I_1 I_2 I; i_1 i_2 i) |I_1 i_1; I_2 i_2\rangle \quad (4.5)$$

$$\langle \bar{\phi}_2 | \bar{A}_2\rangle = \langle \bar{A}_2 | \bar{\phi}_2\rangle = C(I_1 I_2 I; i_1 i_2 i) \quad (4.6)$$

C. Three-Particle States

Three-particle charge states can be built up from three one-particle states.

$$|\bar{A}_3\rangle = |I_1 i_1; I_2 i_2; I_3 i_3\rangle \quad (4.7)$$

The states of total isospin will require six quantum numbers. As in the configuration space, there are three ways in which we can choose these six variables. Using the now familiar superscripts 1, 2 and 3 to denote these representations, we can write a basic ket in, say, representation 3 as

$$|\bar{\phi}_3^3\rangle = |I_1 I_2 I_3 I^3 I i\rangle \quad (4.8)$$

in which $\vec{I}^3 = \vec{I}_1 + \vec{I}_2$ is the intermediate isospin.

The transformation connecting (4.7) and (4.8) again involves only the C-G coefficients. One could proceed like

$$|\bar{A}_3\rangle = \sum_{I^3 I i} |\bar{\phi}_3^3\rangle \langle \bar{\phi}_3^3 | \bar{A}_3 \rangle = \sum_{I^3 I i} \langle I_1 I_2 I_3 I^3 I i | I_1 i_1; I_2 i_2; I_3 i_3 \rangle | I_1 I_2 I_3 I^3 I i \rangle$$

but it will be much easier to use (4.4) twice directly.

$$\begin{aligned} |\bar{A}_3\rangle &= |I_1 i_1; I_2 i_2\rangle \otimes |I_3 i_3\rangle \\ &= \sum_{I^3 I i} C(I_1 I_2 I^3; i_1 i_2 i^3) |(I_1 I_2) I^3 I i\rangle |I_3 i_3\rangle \\ &= \sum_{I^3 I i} C(I_1 I_2 I^3; i_1 i_2 i^3) \sum_{I i} C(I^3 I_3 I; i^3 i_3 i) |(I_1 I_2) I^3 I i\rangle |I_3 i_3\rangle \\ &= \sum_{I^3 I^3 I i} C(I_1 I_2 I^3; i_1 i_2 i^3) C(I^3 I_3 I; i^3 i_3 i) |I_1 I_2 I_3 I^3 I i\rangle \end{aligned} \quad (4.10)$$

The inverse of (4.10) can be similarly worked out.

$$|\bar{\phi}_3^3\rangle = \sum_{i_1 i_2 i_3 i^3} C(I_1 I_2 I^3; i_1 i_2 i^3) C(I^3 I_3 I; i^3 i_3 i) |I_1 i_1; I_2 i_2; I_3 i_3\rangle \quad (4.11)$$

The transformation coefficient, which is real, is thus given by

$$\langle \bar{A}_3 | \bar{\phi}_3^3 \rangle = \langle \bar{\phi}_3^3 | \bar{A}_3 \rangle = \sum_i C(I_1 I_2 I^3; i_1 i_2 i^3) C(I^3 I_3 I; i^3 i_3 i) \quad (4.12)$$

A look at (4.10) will reveal that i^3 and i must satisfy $i^3 = i_1 + i_2$ and $i = i_1 + i_2 + i_3$ so that one actually has only I and I^3 to sum over. Similarly, there are actually only two summation indices in (4.11) and none in (4.12).

D. Recoupling Coefficient

For our three-particle system, the three representations, $|\bar{\phi}_3^1\rangle$, $|\bar{\phi}_3^2\rangle$ and $|\bar{\phi}_3^3\rangle$ are connected to one another by unitary transformations. Now denoting these representations by $|1(23)\rangle$, $|2(31)\rangle$ and $|3(12)\rangle$ to be consistent with our established notation, let us consider

$$\langle 1(23) | 3(12) \rangle = \langle I_1 I_2 I_3 I^1 i' | I_1 I_2 I_3 I^3 i \rangle$$

Inserting the unit operator $|\bar{A}_3\rangle\langle\bar{A}_3|$ and utilizing (4.12) this becomes

$$\begin{aligned} \langle 1 | 3 \rangle &= \sum_{i_1 i_2 i_3} \langle I_1 I_2 I_3 I^1 i' | I_1 I_2 I_3 i_1 i_2 i_3 \rangle \langle I_1 I_2 I_3 i_1 i_2 i_3 | I_1 I_2 I_3 I^3 i \rangle \\ &= \sum_{i_1 i_2 i_3} \sum_{i'} \sum_{i^3} C_1(I_2 I_3 I^1; i_2 i_3 i^1) C_2(I^1 I_1 I'; i^1 i_1 i') \\ &\quad \times C_3(I_1 I_2 I^3; i_1 i_2 i^3) C_4(I^3 I_3 I; i^3 i_3 i) \\ &= \sum_{i_1 i_2 i_3} \sum_{i'} \sum_{i^3} \delta_{i^1, i_2+i_3} \delta_{i', i^1+i_1} \delta_{i^3, i_1+i_2} \delta_{i, i^3+i_3} C_1 C_2 C_3 C_4 \\ &= \sum_{i_1 i_2 i_3} \delta_{i', i_2+i_3+i_1} \delta_{i, i_1+i_2+i_3} C_1 C_2 C_3 C_4 \end{aligned}$$

Instead of i_1 , i_2 and i_3 we may choose i_1 , i_2 and $i^* = i_1 + i_2 + i_3$ as the three variables so that

$$\langle 1 | 3 \rangle = \sum_{i_1 i_2 i^*} \delta_{i', i^*} \delta_{i, i^*} C_1 C_2 C_3 C_4 =$$

$$= \sum_{i_1 i_2} \delta_{i, i'} C_1(I_2 I_3 I^1; i_2, i_3) C_2(I^1 I_1 I'; i_2 + i_3, i_1) \\ \times C_3(I_1 I_2 I^3; i_1 i_2) C_4(I^3 I_3 I; i_1 + i_2, i_3)$$

To go further, we note that the Clebsch-Gordan coefficients appearing in (4.12) which we have used twice require that

$$\begin{aligned} \vec{I}_1 + \vec{I}_2 = \vec{I}_3; \quad \vec{I}^3 + \vec{I}_3 = \vec{I}_1 + \vec{I}_2 + \vec{I}_3 = \vec{I} \quad \text{for} \quad |3(12)\rangle, \\ \text{and} \quad \vec{I}_2 + \vec{I}_3 = \vec{I}^1; \quad \vec{I}^1 + \vec{I}_1 = \vec{I}_2 + \vec{I}_3 + \vec{I}_1 = \vec{I}' \quad \text{for} \quad |1(23)\rangle, \end{aligned}$$

from which it follows that I must be equal to I' or else the recoupling coefficient would vanish. We now have

$$\langle 1|3\rangle = \delta_{II}, \delta_{ii}, (-)^{I^1 + I_1 - I} \sum_{i_1 i_2} C_3(I_1 I_2 I^3; i_1 i_2) C_4(I^3 I_3 I; i_1 + i_2, i_3) \\ \times C_2(I_1 I^1 I; i_1, i_2 + i_3) C_1(I_2 I_3 I^1; i_2, i_3)$$

where the phase factor is a result of reordering in C_2 . A careful comparison of this expression with Eq. (6.6a) and Eq. (6.3) of Rose³ shows that

$$\langle 1(23)|3(12)\rangle = \delta_{II}, \delta_{ii}, (-)^{I^1 + I_1 - I} [(2I^1 + 1)(2I^3 + 1)]^{\frac{1}{2}} W(I_1 I_2 I I_3; I^3 I^1) \quad (4.13)$$

where W is the Racah coefficient. Equation (4.13) is real. By cyclic permutations of the indices we can obtain other recoupling coefficients. For future reference, it will be useful to write down W in general notation.

$$\sum_{\mu_a \mu_b} C(a b e; \mu_a \mu_b) C(e d c; \mu_a + \mu_b, \mu_d) C(b d f; \mu_b \mu_d) C(a f c; \mu_a, \mu_b + \mu_d) \\ = [(2e + 1)(2f + 1)]^{\frac{1}{2}} W(a b c d; e f) \quad (4.14)$$

V. BOSE SYMMETRIZATION

Our fundamental unitarity equations were derived in GL and they are (IV-11) and (IV-12).

$$T^{\alpha}(W, S^{\alpha}) = J^{\alpha}(W, S^{\alpha}) + \frac{i\Delta^{\alpha}}{2} \sum_{\beta \neq \alpha} \int \langle \alpha | \beta \rangle M^{\beta}(s^{\beta}) T^{\beta}(W, s^{\beta}) \rho^{\beta} ds^{\beta} \quad (5.1)$$

Or,

$$T = J + \mathcal{K}T, \quad (5.2)$$

in a more compact form. In this section we want to see how these equations are affected by the presence of two or three identical bosons in the final state. The three equations implied by (5.2) are

$$\left. \begin{aligned} T^1 &= J^1 + \mathcal{K}^{12} T^2 + \mathcal{K}^{13} T^3 \\ T^2 &= J^2 + \mathcal{K}^{23} T^3 + \mathcal{K}^{21} T^1 \\ T^3 &= J^3 + \mathcal{K}^{31} T^1 + \mathcal{K}^{32} T^2 \end{aligned} \right\} \quad (5.3)$$

where

$$\mathcal{K}^{\alpha\beta}(s^{\alpha}, s^{\beta}) = \left(\frac{i\Delta^{\alpha}}{2} \int \langle \alpha | \beta \rangle M^{\beta}(s^{\beta}) \rho^{\beta} ds^{\beta} \right)_{\text{op}} \quad (5.4)$$

First let us study the behavior of states, amplitudes and the recoupling coefficients under the interchange of two particles.

A. States

We are interested in the three-particle angular momentum states of the type introduced in (2.17). A typical one, $|\phi_{12}^3\rangle$, was expanded in (2.20). Our convention for this state is that the X-axis is toward particle 1, the Z-axis opposite to particle 3 and the system is right-handed. This makes the positive sense of rotation in the X-Z plane ccw. This is shown in Fig. 4a. In Eq. (2.20) the angles are

$$\begin{aligned} \theta^3 &= \text{polar angle of vector } \hat{n} \\ \xi_1^3 &= \text{spin rotation of particle 1} = -\chi^2 + \beta_{12} + \theta_{21} \\ \xi_2^3 &= \text{spin rotation of particle 2} = \chi^1 - \beta_{21} - \theta_{12} \end{aligned}$$

When we interchange particles 1 and 2, the X-axis flips and we have the configuration shown in Fig. 4b in which the positive rotations are CW. Applying p_{12} to (2.20) we get

$$\begin{aligned} p_{12} |3(12) \rangle \equiv \phi_{12}^3 \rangle &= \int d\omega d\Omega \ C(\sigma_2 \sigma_1 \sigma; \mu_2 \mu_1 \mu) C(\ell \sigma j; m \mu n) C(j \sigma_3 \Sigma; n \nu_3 \nu) \\ &\times C(L \Sigma J; L_z \nu M) Y_{\ell m}(\theta^{3'}) Y_{LL_z}(0^3) D_{\nu_2 \mu_2}^{\sigma_2}(\xi_1^{3'}) \\ &\times D_{\nu_1 \mu_1}^{\sigma_1}(\xi_2^{3'}) |\vec{Q}_1 \nu_1 \vec{Q}_2 \nu_2 \vec{Q}_3 \nu_3 \rangle \end{aligned}$$

where the transformed angles are indicated by primes. From Fig. 4b we see at once that

$$\begin{aligned} \xi_1^{3'} &\equiv p_{12}(\xi_1^3) = (-)(-\chi^1 + \beta_{21} + \theta_{12}) = (\chi^1 - \beta_{21} - \theta_{12}) = \xi_2^3 \\ \xi_2^3 &\equiv p_{12}(\xi_2^3) = (-)(\chi^2 - \beta_{12} - \theta_{21}) = (-\chi^2 + \beta_{12} + \theta_{21}) = \xi_1^{3'} \\ \hat{n}' &\equiv p_{12}(\hat{n}) = (\pi - \theta^3) \end{aligned}$$

Furthermore,

$$C(\sigma_2 \sigma_1 \sigma; \mu_2 \mu_1 \mu) = (-)^{\sigma_1 + \sigma_2 - \sigma} C(\sigma_1 \sigma_2 \sigma; \mu_1 \mu_2 \mu)$$

Thus we have

$$p_{12} |3(12) \rangle \equiv |3(21) \rangle = (-)^{\sigma_1 + \sigma_2 - \sigma} (-)^{\ell} |3(12) \rangle$$

Or, in general,

$$p_{\beta\gamma} |\alpha(\beta\gamma) \rangle_{\text{config.}} \equiv |\alpha(\gamma\beta) \rangle = (-)^{\ell^\alpha + \sigma_\beta + \sigma_\gamma - \sigma^\alpha} |\alpha(\beta\gamma) \rangle \quad (5.5)$$

The corresponding property for the isospin state follows similarly.

$$p_{\beta\gamma} |\alpha(\beta\gamma) \rangle_{\text{isospin}} \equiv |\alpha(\gamma\beta) \rangle = (-)^{I_\beta + I_\gamma - I^\alpha} |\alpha(\beta\gamma) \rangle \quad (5.6)$$

In general we shall combine (5.5) and (5.6) and write

$$p_{\beta\gamma} |\alpha(\beta\gamma); n^\alpha, s^\alpha \rangle = \eta_\alpha |\alpha(\beta\gamma); n^\alpha, s^\alpha \rangle \quad (5.7)$$

$$p_{12}(1(23); n^1=n, s^1=s | T^1 | a) = \eta_n(2(31); n^2=n, s^2=s | T^2 | a) \quad (5.10)$$

likewise for T^3 we have

$$p_{12}(3(12); n^3, s^3 | T^3 | a) = \eta_{n_1}(3(12); n^3, s^3 | T^3 | a) \quad (5.11)$$

To properly symmetrize these partial wave amplitudes, we must thus form their linear combinations. Actually, T^3 will always satisfy the necessary symmetrization condition since the space-spin and the isospin parts are chosen in such a way that $\eta_{n_1} = +1$ for two identical bosons and -1 for two identical fermions. To deal with the other two amplitudes, we define

$$T_n^\pm(s) = \frac{1}{\sqrt{2}} (T_n^1(s) \pm \eta_n T_n^2(s)) \quad (5.12)$$

The $+$ sign ($-$ sign) yields a symmetric (antisymmetric) amplitude and hence should be used for two identical bosons (fermions).

C. Recoupling Coefficients

Again, let particles 1 and 2 be identical and consider the three recoupling coefficients $\langle 1(23); n_1 s_1 | 2(31); n_2 s_2 \rangle$, $\langle 2(31); n_2 s_2 | 3(12); n_3 s_3 \rangle$ and $\langle 3(12); n_3 s_3 | 1(23); n_1 s_1 \rangle$. Since $p_{12}^{-1} p_{12} = 1$, we can write these as

$$\begin{aligned} \langle 1(23); n_1 s_1 | 2(31); n_2 s_2 \rangle &= \langle 1(23); n_1 s_1 | p_{12}^{-1} p_{12} | 2(31); n_2 s_2 \rangle \\ &= \eta_{n_1} \eta_{n_2} \langle 2(31); n_1 s_1 | 1(23); n_2 s_2 \rangle \end{aligned} \quad (5.13a)$$

$$\begin{aligned} \langle 2(31); n_2 s_2 | 3(12); n_3 s_3 \rangle &= \langle 2(31); n_2 s_2 | p_{12}^{-1} p_{12} | 3(12); n_3 s_3 \rangle \\ &= \eta_{n_2} \eta_{n_3} \langle 1(23); n_2 s_2 | 3(12); n_3 s_3 \rangle \end{aligned} \quad (5.13b)$$

$$\begin{aligned} \langle 3(12); n_3 s_3 | 1(23); n_1 s_1 \rangle &= \langle 3(12); n_3 s_3 | p_{12}^{-1} p_{12} | 1(23); n_1 s_1 \rangle \\ &= \eta_{n_3} \eta_{n_1} \langle 2(31); n_3 s_3 | 2(31); n_1 s_1 \rangle \end{aligned} \quad (5.13c)$$

These relations are to be distinguished from those that result from the reality condition on the recoupling coefficients (see the discussion following Eq. (3.15)).

$$\langle 1(23); n_1 s_1 | 2(31); n_2 s_2 \rangle = \langle 2(31); n_2 s_2 | 1(23); n_1 s_1 \rangle \quad (5.14a)$$

$$\langle 2(31); n_2 s_2 | 3(12); n_3 s_3 \rangle = \langle 3(12); n_3 s_3 | 2(31); n_2 s_2 \rangle \quad (5.14b)$$

$$\langle 3(12); n_3 s_3 | 1(23); n_1 s_1 \rangle = \langle 1(23); n_1 s_1 | 3(12); n_3 s_3 \rangle \quad (5.14c)$$

0. Equations

We consider the case of two identical bosons carrying labels 1 and 2 for which (5.12) holds with the plus sign. Now we must rewrite equations (5.3) so that they only involve T^+ and T^3 . First we write them in full detail.

$$\begin{aligned} T_n^1(s) = & J_n^1(s) + \frac{i}{2} \Delta(s) \sum_{n'} \int \langle 1(23); ns | 2(31); n' s' \rangle M(s') \rho(s') T_{n'}^2(s') ds' \\ & + \frac{i}{2} \Delta(s) \sum_{n_3} \int \langle 1(23); ns | 3(12); n_3 s_3' \rangle M^3(s_3') \rho^3(s_3') T_{n_3}^3(s_3') ds_3' \end{aligned} \quad (5.15a)$$

$$\begin{aligned} T_n^2(s) = & J_n^2(s) + \frac{i}{2} \Delta(s) \sum_{n_3} \int \langle 2(31); ns | 3(12); n_3 s_3' \rangle M^3(s_3') \rho^3(s_3') T_{n_3}^3(s_3') ds_3' \\ & + \frac{i}{2} \Delta(s) \sum_{n'} \int \langle 2(31); ns | 1(23); n' s' \rangle M(s') \rho(s') T_{n'}^1(s') ds' \end{aligned} \quad (5.15b)$$

$$\begin{aligned} T_{n_3}^3(s_3) = & J_{n_3}^3(s_3) + \frac{i}{2} \Delta(s_3) \sum_{n'} \int \langle 3(12); n_3 s_3 | 1(23); n' s' \rangle M(s') \rho(s') T_{n'}^1(s') ds' \\ & + \frac{i}{2} \Delta(s_3) \sum_{n'} \int \langle 3(12); n_3 s_3 | 2(31); n' s' \rangle M(s') \rho(s') T_{n'}^2(s') ds' \end{aligned} \quad (5.15c)$$

Let us multiply (5.15b) by η_n and add it to (5.15a). We then use (5.13a) in the second term on the r.h.s. of (5.15a) so that the integral over ds' takes the form

$$\sum_{n'} \int \langle 2(31); ns | 1(23); n' s' \rangle M(s') \rho(s') \eta_n (T_{n'}^1(s') + \eta_n T_{n'}^2(s')) ds'$$

Similarly, we use (5.13b) in the second term on the r.h.s. of (5.15b) and get for the integral over ds'_3

$$\sum_{n'_3} \int \langle 1(23); ns | 3(12); n'_3 s'_3 \rangle M^3(s'_3) \rho^3(s'_3) T_{n'_3}^5(s'_3) (1 + \eta_{n'_3}) ds'_3$$

Finally, we use (5.13c) in the third term on the r.h.s. of (5.15c) which can then be combined with its second term on the r.h.s. to get

$$\sum_{n'} \int \langle 3(12); n_3 s_3 | 1(23); n' s' \rangle M(s') \rho(s') \left(T_{n'}^1(s') + \eta_{n_3} \eta_{n'} T_{n'}^2(s') \right) ds'$$

Noting that the phase factor for channel 3, η_{n_3} , is always +1, we obtain the desired set of equations.

$$\left. \begin{aligned} T_{n'}^+(s) &= J_{n'}^+(s) + \eta_n \frac{i\Delta(s)}{2} \sum_{n'} \int \langle 2(31); ns | 1(23); n' s' \rangle M(s') \rho(s') T_{n'}^+(s') ds' \\ &\quad + \sqrt{2} \frac{i\Delta(s)}{2} \sum_{n'_3} \int \langle 1(23); ns | 3(12); n'_3 s'_3 \rangle M^3(s'_3) \rho^3(s'_3) T_{n'_3}^3(s'_3) ds'_3 \\ T_{n_3}^3(s_3) &= J_{n_3}^3(s_3) + \sqrt{2} \frac{i\Delta(s_3)}{2} \sum_{n'} \int \langle 3(12); n_3 s_3 | 1(23); n' s' \rangle M(s') \rho(s') T_{n'}^+(s') ds' \end{aligned} \right\} \quad (5.16)$$

Thus the presence of additional boson symmetry reduces the number of equations to two. Along similar lines, we would only have one equation for three identical particles in the final state. In terms of the kernels defined in (5.4), we can rewrite (5.16) in a condensed form.

$$\begin{aligned} T^+ &= J^+ + \eta \kappa^{21} T^+ + \sqrt{2} \kappa^{13} T^3 \\ T^3 &= J^3 + \sqrt{2} \kappa^{31} T^+ \end{aligned} \quad (5.17)$$

It should be noted that these equations, in view of (5.14c), involve only two types of recoupling coefficients.

VI. PARTIAL-WAVE ANALYSIS

The essential results of the partial wave analysis were presented in Section III of GL. Here we fill in the details. An important feature of the isobar model for the amplitude is that the partial-wave expansion necessarily involves the use of three different angular momentum representations and hence three different sets of coordinate axes. We begin with the basic decomposition of the scattering operator.

$$T_{23} = \sum_{\beta=1}^5 \frac{M_{T_{23}}^{\beta}}{\Lambda^{\beta}} = \sum_{\beta=1}^3 T_{23}^{\beta} \quad (6.1)$$

Let us look at two equivalent ways of taking the matrix elements of (6.1).

A. Method 1

Using the initial and final state plane-wave states, we get

$$\langle \vec{p}_a \vec{p}_\beta \vec{p}_\gamma; \mu_a \mu_\beta \mu_\gamma | T_{23} | \vec{p}_a \vec{p}_b; \mu_a \mu_b \rangle = \sum_{\beta=1}^3 \langle \vec{p}_a \vec{p}_\beta \vec{p}_\gamma; \mu_a \mu_\beta \mu_\gamma | T_{23}^{\beta} | \vec{p}_a \vec{p}_b; \mu_a \mu_b \rangle \quad (6.2)$$

We now insert two unit operators in the r.h.s. and denote the initial angular momentum state by $|a\rangle$ (this state is the same as (2.7) whose normalization is given in (2.11)).

$$\langle f | T_{23} | i \rangle = \sum_{\beta=1}^3 \sum_{\beta, a} \langle f | \beta \rangle \rho^{\beta} \langle \beta | T_{23}^{\beta} | a \rangle \rho^{\beta} \langle a | i \rangle \quad (6.3)$$

In the above $|i\rangle$ and $\langle f|$ denote the initial and final plane-wave states respectively as specified in (6.2). It is clear that in this method, one makes use of three different representations $|\alpha\rangle$, $|\beta\rangle$ and $|\gamma\rangle$ to start with in doing the expansion.

B. Method 2

It is, however, possible to begin with inserting one particular angular momentum representation, say $|\alpha\rangle$, in (6.2).

$$\begin{aligned}
 \langle f|T_{23}|i\rangle &= \sum_{\beta=1}^3 \langle f|T_{23}^{\beta}|i\rangle = \sum_{\beta=1}^3 \sum_{\alpha,a} \langle f|\alpha\rangle \rho^{\alpha}(\alpha|T_{23}^{\beta}|a) \rho^a(a|i) \\
 &= \sum_{\alpha,a} \langle f|\alpha\rangle \rho^{\alpha} \rho^a(a|i) \left[\langle \alpha|T_{23}^{\alpha}|a\rangle + \sum_{\beta} \langle \alpha|\beta\rangle \rho^{\beta}(\beta|T_{23}^{\beta}|a) + \sum_{\gamma} \langle \alpha|\gamma\rangle \rho^{\gamma}(\gamma|T_{23}^{\gamma}|a) \right] \\
 &= \sum_a \left[\sum_{\alpha} \langle f|\alpha\rangle \rho^{\alpha}(\alpha|T_{23}^{\alpha}|a) + \sum_{\beta} \langle f|\beta\rangle \rho^{\beta}(\beta|T_{23}^{\beta}|a) + \sum_{\gamma} \langle f|\gamma\rangle \rho^{\gamma}(\gamma|T_{23}^{\gamma}|a) \right] \rho^a(a|i) \\
 &= \sum_{\beta=1}^3 \sum_{\beta,a} \langle f|\beta\rangle \rho^{\beta}(\beta|T_{23}^{\beta}|a) \rho^a(a|i)
 \end{aligned}$$

which is the same as (6.3). This shows that the two methods are equivalent although their starting points are different.

C. Further Development

Equation (6.3) appears in GL as (III-15). Next, using (2.12) for $\langle i|a\rangle$ and (2.22) for $\langle f|\alpha\rangle$ we wish to derive Eq. (III-18) of GL. A typical term in Eq. (6.3), say for $\beta = \alpha$, then becomes

$$\begin{aligned}
 &\sum_{\alpha,a} \langle f|\alpha\rangle \rho^{\alpha}(\alpha|T_{23}^{\alpha}|a) \rho^a(a|i) \\
 &\sum_{\alpha,\alpha',a} \int \langle f|\alpha\rangle \left(\frac{Qq}{16W\sqrt{s}} \right) \langle \alpha|M^{\alpha}|\alpha'\rangle \left(\frac{Q'q'}{16W'\sqrt{s'}} \right) \langle \alpha'|T^{\alpha}|a\rangle \left(\frac{q_i}{4W_i} \right) \langle a|i\rangle \cdot \frac{1}{\Delta^{\alpha}}
 \end{aligned}$$

The superscripts α in the kinematic variables for $|\alpha\rangle$ and $|\alpha'\rangle$ have been omitted for brevity. The two matrix elements can be written as

$$\langle \alpha|M^{\alpha}|\alpha'\rangle = \left(\delta(W' - W)\delta(s' - s)\delta_{J'J}\delta_{M'M}\delta_{L'L}\delta_{\Sigma'\Sigma}\delta_{j'j} \right) \frac{4W}{Q} M_{\ell\sigma j\ell'\sigma'I\alpha}^{\alpha}(s) \\
 \times \delta_{II'}\delta_{ii'}\delta_{I\alpha'I\alpha}$$

$$\langle \alpha'|T^{\alpha}|a\rangle = \delta(W' - W_i)\delta_{J'J_i}\delta_{M'M_i}\delta_{I'I_i}\delta_{i'i_i} T_{L'\Sigma'j'j_i\ell'\sigma'I_i\alpha'}^{\alpha}(W_i,s')$$

so that when (2.12) and (2.22) are included we get

$$\begin{aligned}
 & \sum_{\alpha, a} \langle f | \alpha \rangle \rho^{-1} \langle \alpha | T_{23}^{-1} | a \rangle \Delta^a \langle a | i \rangle \\
 &= \sum_{\substack{J M L \Sigma j j' \sigma \\ J' M' L' \Sigma' j' j' \sigma' \\ J_1 M_1 L_1 \Sigma_1 \sigma_1}} \sum_{\substack{I_1 I_2 \alpha_i \\ I_1' I_2' \alpha_i' \\ I_1 i_1}} \int dW' dW dW_i ds' ds \sum_{\substack{\nu_F \nu_Y; \mu m \nu L_z \\ i'}} \sum_{i_1 m_i} \\
 & \cdot \left(\frac{16W\sqrt{s}}{Qq} \right) \left(\frac{Qq}{16W\sqrt{s}} \right) \left(\frac{4W}{Q} \right) \left(\frac{Q'q'}{16W'\sqrt{s'}} \right) \left(\frac{q_j}{4W_i} \right) \left(\frac{4W_i}{q_i} \right) \left(\frac{4\sqrt{s'}}{q'} \right) \\
 & \times \delta(W - (E_\alpha + E_\beta + E_\gamma)) \delta(s - (p_\beta + p_\gamma)^2) \cdot \delta(W' - W) \delta(s' - s) \cdot \delta(W_i - W_i) \cdot \delta(W_i - (E_\beta + E_\gamma)) \\
 & \times \delta_{J'J} \delta_{M'M} \delta_{L'L} \delta_{\Sigma'\Sigma} \delta_{j'j} \cdot \delta_{J'J_i} \delta_{M'M_i} \cdot \delta_{I_1'I_1} \delta_{i_1'i_1} \delta_{I_1\alpha} I_1 \alpha \delta_{I_1'I_1} \delta_{i_1'i_1} \\
 & \times \left\{ C(\nu_\beta \sigma_\beta \sigma; \nu_\beta \nu_\gamma \mu) C(\ell \sigma j; m \mu n) C(j \sigma_\alpha \Sigma; n \mu_\alpha \nu) C(L \Sigma J; L_z \nu M) Y_{\ell m}(\theta \phi) Y_{L L_z}(\omega \phi) \right. \\
 & \cdot D_{\mu_\beta \nu_\beta}^{\beta \beta}(\xi_\beta^\alpha) D_{\mu_\gamma \nu_\gamma}^{\gamma \gamma}(\xi_\gamma^\alpha) \times C(\sigma_a \sigma_b \sigma_i; \mu_a \mu_b \mu_i) C(\ell_i \sigma_i J_i; m_i \mu_i M_i) Y_{\ell_i m_i}^*(\theta_i \phi_i) \\
 & \times \left. C(I_\beta I_\gamma I^\alpha; i_\alpha i_\beta i^\alpha) C(I^\alpha I_\alpha I; i^\alpha i_\alpha i) C(I_a I_b I_i; i_a i_b i_i) \times M^\alpha T^\alpha \right\}
 \end{aligned}$$

In the above we also included (4.6) and (4.12) for the isospin parts. Now, because of the energy and subenergy δ -functions, the entire kinematical factor becomes unity. Also, s and s' get replaced by $(p_\beta + p_\gamma)^2$ and the indices $(J_1 M_1 L_1 \Sigma_1 j_1 I_1^\alpha i_1)$ go to $(J M L \Sigma j j' I^\alpha i)$ respectively. Substituting the resulting expression in (6.3), we finally arrive at

$$\begin{aligned}
 & (\vec{p}_\alpha \vec{p}_\beta \vec{p}_\gamma; \mu_\alpha \mu_\beta \mu_\gamma; i_\alpha i_\beta i_\gamma | T_{23} | \vec{p}_a \vec{p}_b; \mu_a \mu_b; i_a i_b) \\
 &= (p_f - p_i) \sum_{\substack{\alpha, \beta, \gamma \\ \text{cyclic}}} \sum_{\substack{L \Sigma j j' \sigma'; I^\alpha I \\ \ell_i \sigma_i J}} \left[\sum_{\substack{\ell \sigma \nu_\beta \nu_\gamma \mu m \\ \nu L_z \mu_i m_i M \\ i^\alpha i}} C(\sigma_a \sigma_b \sigma_i; \mu_a \mu_b \mu_i) C(\ell_i \sigma_i J; m_i \mu_i M) \times \right.
 \end{aligned}$$

$$\begin{aligned}
 & \times C(\sigma_{\beta} \sigma_{\gamma} \sigma; \nu_{\beta} \nu_{\gamma} \mu) C(\ell \sigma j; m \mu n) C(j \sigma_{\alpha} \ell; n \mu_{\alpha} \nu) C(L \ell J; L_z \nu M) \\
 & \times C(I_{\beta} I_{\gamma} I^{\alpha}; i_{\alpha} i_{\beta} i^{\alpha}) C(I_{\alpha} I_{\alpha} I; i_{\alpha} i_{\alpha} i) C(I_{\alpha} I_{\beta} I; i_{\alpha} i_{\beta} i) Y_{\ell_1 m_1}^*(\theta_1 \phi_1) Y_{\ell m}(\theta^2 \phi^2) \\
 & \times Y_{LL_z}(\theta^2 \phi^2) D_{\mu_{\beta} \nu_{\beta}}^{\sigma_{\beta}}(\xi_{\beta}^{\alpha}) D_{\mu_{\gamma} \nu_{\gamma}}^{\sigma_{\gamma}}(\xi_{\gamma}^{\alpha}) M_{\ell \sigma j \ell' \sigma' j'}^{\alpha}((p_{\beta} + p_{\gamma})^2) \left. \right] T_{\ell_1 \sigma_1 \ell' \sigma' j' j}^{\alpha} I^{\alpha} \\
 & \qquad \qquad \qquad (6.4)
 \end{aligned}$$

This expression is quite general and is valid for any choice of the coordinate axes. Further simplifications may also result from the requirement of the conservation of parity as was pointed out in GL for the case of $\pi N \rightarrow \pi \pi N$. Equation (III-18) of GL agrees with the above equation if we recall that the former was written down for the special case of $\pi N \rightarrow \pi \pi N$ without the isospin part. For later use we write (6.4) in an abbreviated form.

$$\langle f | T_{23} | i \rangle = \sum_{\alpha, x} g_x^{\alpha}(W, s^{\alpha}) T_x^{\alpha}(W, s^{\alpha}) \quad (6.5)$$

where g_x^{α} stands for the square bracket and x denotes the subscripts on T in Eq. (6.4). The overall δ -function is understood to be present.

D. Identical Particles

If we have two identical bosons in the final state, say particles 1 and 2, the final state $\langle f |$ in (6.5) must be symmetric under the interchange of these two bosons. Let $\vec{p}$, $\vec{q}$, $\vec{r}$ denote the three final momenta in the o.c.m. Then we can construct such a symmetric state as

$$\langle \hat{f} | = \frac{1}{\sqrt{2}} (\langle \vec{p}^1 \vec{q}^2 \vec{r}^3 | + \langle \vec{p}^2 \vec{q}^1 \vec{r}^3 |) = \frac{1}{\sqrt{2}} (\langle \vec{p}^1 \vec{q}^2 \vec{r}^3 | + \langle \vec{p}^1 \vec{q}^2 \vec{r}^3 |_{P_{12}^{-1}}) \quad (6.6)$$

Here the isospin labels are suppressed. We must now consider the amplitude $\langle \hat{f} | T_{23} | i \rangle$. To do this, we first rewrite (6.3).

$$\langle f | T_{23} | i \rangle = \sum_1 \langle f | 1 \rangle T^1 X^1 + \sum_2 \langle f | 2 \rangle T^2 X^2 + \sum_3 \langle f | 3 \rangle T^3 X^3 \quad (6.7)$$

where the X 's contain the density of states and the initial transformation function

which are not explicitly required for our purpose. The symmetrized amplitude now takes the form

$$\begin{aligned}\sqrt{2} \langle \hat{f} | T_{23} | i \rangle &= \langle f | T_{23} | i \rangle + \langle f | p_{12}^{-1} T_{23} | i \rangle \\ &= \sum_1 \langle f | 1(23) \rangle T^1 X^1 + \sum_2 \langle f | 2(31) \rangle T^2 X^2 + \sum_3 \langle f | 3(12) \rangle T^3 X^3 \\ &+ \sum_1 \langle f | p_{12}^{-1} | 1(23) \rangle T^1 X^1 + \sum_2 \langle f | p_{12}^{-1} | 2(31) \rangle T^2 X^2 \\ &+ \sum_3 \langle f | p_{12}^{-1} | 3(12) \rangle T^3 X^3\end{aligned}$$

We apply $p_{12}^{-1} = p_{12}$ on the kets and use Eqs. (5.9). Since particles 1 and 2 are identical, the summation indices for them will be identical too. We thus get

$$\begin{aligned}\sqrt{2} \langle \hat{f} | T_{23} | i \rangle &= \sum \langle f | 1(23) \rangle X(T^1 + \eta T^2) + \eta \langle f | 2(31) \rangle X(T^1 + \eta T^2) \\ &+ 2 \sum_3 \langle f | 3(12) \rangle T^3 X^3\end{aligned}$$

where we set $X = X^1 = X^2$ since they are identical and also $\eta_{n_3} = +1$ as we remarked in connection with (5.12). In terms of the amplitudes T^+ introduced in (5.12) we have

$$\langle \hat{f} | T_{23} | i \rangle = \sum \langle f | 1(23) \rangle + \eta \langle f | 2(31) \rangle X T^+ + \sqrt{2} \sum_3 \langle f | 3(12) \rangle T^3 X^3 \quad (6.8)$$

which also may be written using the g-functions as given in (6.5). This is our symmetrized amplitude in the isobar model for two identical bosons in the final state.

E. Elastic Amplitude

Equation (6.4) contains the elastic amplitude M. This matrix element is defined between the states normalized as in (2.11). In the nonrelativistic normalization, the kinematic factor $4W/q$ is usually absent and one sets the S-matrix equal to $e^{2i\delta}$. Thus,

$$\langle |s \rangle_{n.r.} = e^{2i\delta}$$

Comparing the normalizations,

$$\sqrt{\frac{q}{4W}} | \rangle_{\text{rel.}} = | \rangle_{\text{n.r.}}$$

so that

$$\langle |s| \rangle_{\text{rel.}} = \frac{4W}{q} e^{2i\delta}$$

For the T-matrix (M is the matrix element of T) defined through

$$S = 1 + iNT, \quad N = \text{a constant}$$

the above element becomes

$$M \equiv \langle |T| \rangle_{\text{rel.}} = \frac{4W}{q} \frac{(e^{2i\delta} - 1)}{iN}$$

or,

$$M = \frac{1}{N} \frac{8W}{q} e^{i\delta} \sin\delta \quad (6.9)$$

In the formalism developed in GL, N was set equal to one.

VII. CROSS SECTION

Formulas for cross section can be found in any good book on quantum mechanics; what is generally not found, however, is explicit definition of states, their normalizations, the relationship of the R and T matrices with the S-matrix and the effect of various arbitrary constants and kinematical factors on the final expression for the cross section. The purpose of this Section is to elucidate on these points in deriving an expression for cross section.

(i) The R matrix is usually defined by

$$S = 1 + R \quad (7.1)$$

The probability that a transition from $|i\rangle$ to $|f\rangle$ will occur is given by the square of the amplitude $\langle f|S-1|i\rangle = \langle f|R|i\rangle$.

$$P_{fi} = |\langle f|R|i\rangle|^2 \quad (7.2)$$

(ii) The probability that $|i\rangle$ will go into *any* $|f\rangle$ is obtained by integrating the above P_{fi} over all possible final states for a given final set of particles.

$$W = \int |\langle \vec{p}_1 \dots \vec{p}_n | R | \vec{p}_a \vec{p}_b \rangle|^2 \rho_f d^3p_1 d^3p_2 \dots d^3p_n \quad (7.3)$$

where ρ_f is the density of final states or the weight factor in the momentum space, i.e.,

$$\int |\vec{p}_1 \dots \vec{p}_n\rangle \rho_f d^3p_1 \dots d^3p_n \langle \vec{p}_1 \dots \vec{p}_n| = 1 \quad (7.4)$$

(iii) Next we introduce new states in terms of the total energy and momentum variables $P = (W, \vec{P})$. If we let ξ denote the remaining $3n-4$ variables, we can relate the volume elements by

$$d^3p_1 \dots d^3p_n = \mathcal{G}_n d^3P dW d\xi \quad (7.5)$$

For example, for a two-particle state we have

$$d^3p_1 d^3p_2 = (2E_1 \cdot 2E_2) \frac{q}{4W} d^3P dW d\xi, \quad \mathcal{G}_2 = \frac{E_1 E_2 q}{W} \quad (7.6)$$

and for a three-particle state

$$d^3p_1 d^3p_2 d^3p_3 = (2E_1 \cdot 2E_2 \cdot 2E_3) \frac{q_i Q_i}{16W \sqrt{s_i}} d^3p dW d\xi_i, \quad g_{3i} = \frac{E_1 E_2 E_3 q_i Q_i}{2W \sqrt{s_i}} \quad (7.7)$$

Let the new state $|\vec{p}, W, \xi\rangle$ have the density $\bar{\rho}$ with the completeness relation

$$|\vec{p}, W, \xi\rangle \bar{\rho} d^3p dW d\xi \langle \vec{p}, W, \xi| = 1 \quad (7.8)$$

Then, if we take

$$\bar{\rho} = \rho g \quad (7.9)$$

it follows from (7.4), (7.5), (7.8) and (7.9) that

$$|\vec{p}, W, \xi\rangle = |\vec{p}_1 \dots \vec{p}_n\rangle \quad (7.10)$$

(iv) We now remove the total energy-momentum δ -function from the matrix element.

$$\langle \vec{p}_1 \dots \vec{p}_n | R | \vec{p}_a \vec{p}_b \rangle = \langle \vec{p}_f W_f \xi_f | R | \vec{p}_i W_i \xi_i \rangle = \delta^4(p_f - p_i) \langle \xi_f | R | \xi_i \rangle \quad (7.11)$$

where we used (7.10). When this expression is squared, we will have to deal with $[\delta^4(Q)]^2$. To do this, one uses the δ -function trick and writes

$$[\delta^4(Q)]^2 \rightarrow \delta^4(Q) \frac{1}{(2\pi)^4} \int_{Vt} e^{iQx} dx = \delta^4(Q) \frac{Vt}{(2\pi)^4} \quad (7.12)$$

where V is the normalization volume and t is the interaction time.

(v) Integrating out p_f we get

$$W = \frac{Vt}{(2\pi)^4} \int |\langle \xi_f | R | \xi_i \rangle|^2 \bar{\rho}_f d\xi_f$$

Or,

$$\frac{\delta W}{\delta V \delta t} = \frac{1}{(2\pi)^4} \int |\langle \xi_f | R | \xi_i \rangle|^2 \bar{\rho}_f d\xi_f \quad (7.13)$$

which gives the total transition probability per unit volume per unit time.

(vi) For the initial system $a+b$, the incident flux is given by

$$\begin{aligned}
 F &= \text{relative velocity}/V_a V_b \\
 &= \left(\frac{pE}{E_a E_b} \right) \frac{1}{V_a V_b}
 \end{aligned} \tag{7.14}$$

(vii) The cross section then becomes

$$\begin{aligned}
 \sigma &= \frac{\left(\frac{\delta W}{\delta V \delta t} \right)}{\text{incident flux}} \\
 &= V_a V_b \left(\frac{E_a E_b}{pE} \right) \frac{1}{(2\pi)^4} \int |\langle \xi_f | R | \xi_i \rangle|^2 \bar{\rho}_f d\xi_f
 \end{aligned} \tag{7.15}$$

(viii) Now let us talk about the various arbitrary factors people introduce for their convenience. The cross sections are usually given in terms of T-matrix which is generally related to S by

$$S = 1 + A \cdot T \tag{7.16}$$

i.e., $R = AT$. Here A is an arbitrary factor of the type i, 2i, $i2\pi$, etc. Sometimes $\delta^4(P_f - P_i)$ is also included in A but we will ignore it since we have already pulled it out in (7.11).

The other arbitrary factor enters through the normalization of momentum states. Let us consider a single particle. We know from elementary quantum mechanics that there is one state in h^3 volume of phase space (momentum x configuration space). This amounts to

$$\begin{aligned}
 \frac{\text{no. of states per unit volume of phase space}}{} &= 1/h^3 \\
 &= \frac{1}{(2\pi)^3 (\hbar)^3} \\
 &= \frac{1}{(2\pi)^3}
 \end{aligned}$$

setting $\hbar = 1$. On the other hand, with the density of states ρ in the momentum space, there are ρ number of states per unit volume in momentum space. If the particle is confined to volume V in the configuration space, we then have ρ/V

number of states in unit volume of phase space. Comparing this with the above number, we have the relationship

$$V = (2\pi)^3 \rho \quad (7.17)$$

Let us substitute this result for V_a, V_b in (7.15) and write $R=AT$ to get

$$\sigma = \left(\frac{(2\pi)^2}{pW} \right) (\rho_a \rho_b E_a E_b) |A|^2 \int |\langle \xi_f | T | \xi_i \rangle|^2 \bar{\rho}_f d\xi_f \quad (7.18)$$

(ix) We have chosen for $2 \rightarrow 3$ transition,

$$\begin{aligned} \rho_a &= \frac{1}{2E_a} ; & \rho_b &= \frac{1}{2E_b} ; \\ \bar{\rho}_{3,i} &= \rho_3 g_3 = \frac{1}{2E_1 \cdot 2E_2 \cdot 2E_3} \times \frac{E_1 E_2 E_3 q_i Q_i}{2W \sqrt{s_i}} = \frac{q_i Q_i}{16W \sqrt{s_i}} \end{aligned}$$

$$A = 1$$

so that (7.18) becomes

$$\sigma = \frac{\pi^2}{pW} \int |\langle \xi_{3,i} | T_{2 \rightarrow 3} | \xi_i \rangle|^2 \bar{\rho}_{3,i} d\xi_{3,i} \quad (7.19)$$

The partial wave expansion carried out in Section VI can now be inserted in (7.19) to obtain the cross section in the isobar model. If no polarization is observed, one should take the usual sum over the final states and average over the initial polarization states. We will not enter into these trivial details here.

VIII. APPLICATION TO $\pi N \rightarrow \pi \pi N$

We wish to apply the unitarity equations to the process $\pi N \rightarrow \pi \pi N$. The Boson symmetry requirement is that it should make no difference which pion is labelled 1, i.e., one must consider only the symmetrized amplitudes and Eqs. (5.17) for this process. We solve these integral equations by matrix method, dividing the continuous range of integration into discrete number of equal parts. The calculations proceed in the following way.

A. Choice of Partial Waves

For a given IJP state of the initial π -N system there are, in principle, an infinite number of partial waves for the final state. However, we limit ourselves to the isobars ρ ($\ell = 1$ for π - π system), ϵ ($\ell = 0$ for π - π system), and Δ ($\ell = 1$ for π -N system). Furthermore, we restrict the maximum value of L to ≤ 3 . To conform to the recent Berkeley-SLAC analysis,⁵ we chose the 28 partial waves given in Table I, ref. 2.

B. The Phase η

Let us calculate the phase η . We label the nucleon by 3. From (5.8) we have

$$\begin{aligned} \eta &\equiv \eta_{\Delta} = \eta_1 = \eta_2 \approx \text{phase for } \Delta\text{-isobar in either channel 1 or 2} \\ &= (-)^{\ell^{\Delta} + \sigma_N + \sigma_{\pi} - \sigma^{\Delta}} (-)^{I_N + I_{\pi} - I^{\Delta}} \\ &= (-)^{1 + \frac{1}{2} + 0 - \frac{1}{2}} (-)^{\frac{1}{2} + 1 - \frac{3}{2}} \\ &= (-) \quad (\text{This is the phase that appears in (5.17)}). \\ \eta_3^{(\rho)} &= \text{phase for } \rho\text{-isobar in channel 3} \\ &= (-)^{\ell^{\rho} + \sigma_{\pi} + \sigma_{\pi} - \sigma^{\rho}} (-)^{I_{\pi} + I_{\pi} - I^{\rho}} \\ &= (-)^{1 + 0 + 0 - 0} (-)^{1 + 1 - 1} \\ &= (+), \quad \text{as we asserted before.} \end{aligned}$$

$$\begin{aligned}
 \eta_3^{(\epsilon)} &= \text{phase for } \epsilon\text{-isobar in channel 3} \\
 &= (-)^{l^\epsilon + \sigma_\pi + \sigma_\pi - \sigma^\epsilon} (-)^{I_\pi + I_\pi - I^\epsilon} \\
 &= (-)^{0+0+0-0} (-)^{1+1-0} \\
 &= (+), \quad \text{as we asserted before.}
 \end{aligned}$$

C. The Recoupling Coefficients

We need $\langle 2(31) | 1(23) \rangle$ and $\langle 1(23) | 3(12) \rangle$ in (5.17). Remember that $\langle 1(23) | 3(12) \rangle = \langle 3(12) | 1(23) \rangle$ even though the same is not true for their kernels. Also keep in mind that we have to include the isospin part in the coupling.

(i) The isospin part

From (4.13), the required coefficients can be obtained by suitable permutations.

$$\begin{aligned}
 \langle 2(31) | 1(23) \rangle_{\text{isospin}} &= (-)^{I^2 + I_2 - I} [(2I^2+1)(2I^1+1)]^{\frac{1}{2}} W(I_2 I_3 I I_1; I^1 I^2) \\
 &= (-)^{\frac{3}{2} + 1 - I} [(2 \times \frac{3}{2} + 1)(2 \times \frac{3}{2} + 1)]^{\frac{1}{2}} W(1 \frac{1}{2} 1 1; \frac{3}{2} \frac{3}{2}) \\
 &= (-)^{\frac{5}{2} - I} 4W(1 \frac{1}{2} 1 1; \frac{3}{2} \frac{3}{2})
 \end{aligned}$$

Note that this is $\langle \Delta | \Delta \rangle$ coupling. We can calculate W from (4.14) or use the Appendix in Rose's book.³ The results are given in Table I.

Again, from (4.13),

$$\begin{aligned}
 \langle 1(23) | 3(12) \rangle &= (-)^{I^1 + I_1 - I} [(2I^1+1)(2I^3+1)]^{\frac{1}{2}} W(I_1 I_2 I I_3; I^3 I^1) \\
 &= (-)^{\frac{3}{2} + 1 - I} [(2 \times \frac{3}{2} + 1)(2I^3+1)]^{\frac{1}{2}} W(1 1 1 \frac{1}{2}; I^3 \frac{3}{2}) \\
 &= (-)^{\frac{5}{2} - I} 2(2I^3+1)^{\frac{1}{2}} W(1 1 1 \frac{1}{2}; I^3 \frac{3}{2})
 \end{aligned}$$

Note that this is either $\langle \Delta | \rho \rangle$ or $\langle \Delta | \epsilon \rangle$ coupling. See Table I.

(ii) The configuration part

Here the basic expression is (3.15) and its permutation to get $\langle 2(31) | (1(23)) \rangle$.

$$\langle 2(31) | 1(23) \rangle_{\text{config.}} = \frac{64\pi^2}{2J+1} \frac{\sqrt{s^2 s^1}}{q^2 q^1 Q^2 Q^1} \frac{(9(2L^1+1)(2L^2+1))^{\frac{1}{2}}}{16\pi^2} \sum^{21} \quad (8.1)$$

$$\langle 1(23) | 3(12) \rangle_{\text{config.}} = \frac{64\pi^2}{2J+1} \frac{\sqrt{s^1 s^3}}{q^1 q^3 Q^1 Q^3} \frac{(3(2L^1+1)(2L^3+1)(2L^2+1))^{\frac{1}{2}}}{16\pi^2} \sum^{13} \quad (8.2)$$

where the summation is to be done as shown in (3.15).

D. The Kernels

Equation (5.4) defines the kernels. We rewrite it for convenience as

$$\begin{aligned} \mathcal{K}^{\alpha\beta} &= \int \frac{i\Delta^\alpha}{2} \langle \alpha | \beta \rangle_{\text{config}} \rho^\beta \cdot 8(BW)^\beta \langle \alpha | \beta \rangle_{\text{isospin}} ds^\beta \\ &\equiv \int A^{\alpha\beta} \langle \alpha | \beta \rangle_{\text{isospin}} (BW)^\beta ds^\beta \end{aligned} \quad (8.3)$$

with

$$M^\beta \equiv 8(BW)^\beta \quad (8.4)$$

$$A^{\alpha\beta} \equiv i4\Delta^\alpha \langle \alpha | \beta \rangle_{\text{config}} \rho^\beta \quad (8.5)$$

Furthermore, using GL (III-3) and (III-9), we have

$$\rho^\beta = \frac{Q^\beta q^\beta}{16W\sqrt{s^\beta}} \quad (8.6)$$

$$\Delta^\alpha = \frac{q^\alpha}{4\sqrt{s^\alpha}} \quad (8.7)$$

Substituting (8.1), (8.2) and the above two equations into (8.5), $A^{\alpha\beta}$ becomes

$$\begin{aligned} A^{\alpha\beta} &= \frac{i}{WQ^\alpha} \frac{[(2L^\alpha+1)(2L^\beta+1)(2L^\alpha+1)(2L^\beta+1)]^{\frac{1}{2}}}{4(2J+1)} \sum^{\alpha\beta} \\ &\equiv \frac{i}{WQ^\alpha} (\text{CPL})^{\alpha\beta} \end{aligned} \quad (8.8)$$

Putting (8.8) into (8.3) we get

$$\mathcal{K}^{\alpha\beta} = \frac{i}{WQ^\alpha} \langle \alpha | \beta \rangle_{\text{isospin}} \int (\text{CPL})^{\alpha\beta} (\text{BW})^\beta ds^\beta \quad (8.9)$$

This expression is programmed in OVRLAP.

E. The Equations

With the phase $\eta = -1$ and kernels as defined in (8.9), equations (5.17) become, for a given W, IJP ,

$$\begin{aligned} T^{\Delta}(s^{\Delta}) &= J^{\Delta}(s^{\Delta}) - \sum_{\Delta'} \int \mathcal{K}_{(s^{\Delta}, s^{\Delta'})}^{\Delta\Delta'} T_{(s^{\Delta'})}^{\Delta'} ds^{\Delta'} + \sqrt{2} \sum_R \int \mathcal{K}_{(s^{\Delta}, s^R)}^{\Delta, R} T_{(s^R)}^R ds^R \\ T^R(s^R) &= J^R(s^R) + \sqrt{2} \sum_{\Delta'} \int \mathcal{K}_{(s^R, s^{\Delta'})}^{R\Delta'} T_{(s^{\Delta'})}^{\Delta'} ds^{\Delta'} \end{aligned} \quad (8.10)$$

where R indicates a typical resonance in the π - π system. These equations were solved by matrix methods and the T amplitudes were expressed in terms of the J amplitudes through the mixing matrix H as outlined in GL. The functions $(\text{CPL})^{\alpha\beta}$ were further manipulated to simplify them but these calculations are rather long and tedious and the resulting expressions are still quite complicated; we shall not present them here. For numerical results see ref. 2.

TABLE I. Isospin Recoupling Coefficients

I	$\langle 2(31) 1(23) \rangle$	$\langle 1(23) 3(12) \rangle$	
	$\langle \Delta \Delta \rangle$	$\langle \Delta \rho \rangle$	$\langle \Delta \epsilon \rangle$
1/2	1/3	$1/\sqrt{3}$	$\sqrt{2/3}$
3/2	-2/3	$-\sqrt{5/6}$	0*
* I = 3/2 state is not possible for ϵ -N system			

APPENDIX A

Kinematics

In this Appendix we derive and list some important kinematical quantities for our problem. The angle variables are shown in Fig. 3 and the notation is as follows:

- q_i = magnitude of momentum in c.m. $_{jk}$
 s_i = square of total energy in c.m. $_{jk}$
 Q_i = momentum of the i^{th} particle in the o.c.m.
 W = total energy of the three-particle system in the o.c.m.
 $S = W^2$

A. q_i and Q_i

These are given by

$$q_i = \left[\frac{s_i^2 + m_j^4 + m_k^4 - 2s_i m_j^2 - 2s_i m_k^2 - 2m_j^2 m_k^2}{4s_i} \right]^{1/2} \quad (\text{A-1})$$

$$Q_i = \left[\frac{s^2 + s_i^2 + m_i^4 - 2ss_i - 2sm_i^2 - 2s_i m_i^2}{4s} \right]^{1/2} \quad (\text{A-2})$$

Another useful relation is

$$\sum_i s_i = S + \sum_i m_i^2 \quad (\text{A-3})$$

B. The Angle χ

Let us calculate χ_2 which is the angle between $\vec{Q}_1$ and $\vec{Q}_3$ in the o.c.m. Since $\vec{Q}_1 + \vec{Q}_2 + \vec{Q}_3 = 0$,

$$(\vec{Q}_1 + \vec{Q}_3)^2 = (\vec{Q}_2)^2,$$

or,

$$\cos \chi_2 = \frac{Q_2^2 - (Q_1^2 + Q_3^2)}{2Q_1 Q_3}.$$

Using (A-2) to substitute for Q_2 and retaining only s_1, s_3, Q_1 and Q_3 , we have

$$\cos \chi_2 = \frac{1}{4SQ_1Q_3} \left[s_1 s_3 + (S - m_3^2) s_1 + (S - m_1^2) s_3 - (S + 2m_2^2 - m_1^2 - m_3^2) S + m_1^2 m_3^2 \right]$$

Or, in general,

$$\cos \chi_\alpha = \frac{1}{4SQ_\alpha Q_\beta} \left[s_\beta s_\gamma + (S - m_\gamma^2) s_\beta + (S - m_\beta^2) s_\gamma - (S + 2m_\alpha^2 - m_\beta^2 - m_\gamma^2) S + m_\beta^2 m_\gamma^2 \right] \quad (A-4)$$

C. The Angle $\beta_{\alpha\beta}$

$\beta_{\alpha\beta}$ is the angle between the directions $\alpha \rightarrow \beta$ and $\alpha \rightarrow \text{o.c.m.}$ in the rest frame of the α particle. It will be convenient to consider c.m. $_{\alpha\beta}$ and the o.c.m. as "particles" $\alpha\beta$ and $\alpha\gamma$ respectively with the 4-momenta q and Q respectively. We calculate the scalar product of these two 4-vectors, which is an invariant, first in the rest frame of α and then in the o.c.m.. Refer to Fig. 3 for further insight.

(i) In the rest frame of α ,

$$q \cdot Q = e \cdot E - \vec{q} \cdot \vec{Q}$$

Now,

$$\begin{aligned} \vec{q} &= \text{momentum of } \alpha\beta \text{ in } \alpha\text{-rest frame} \\ &= -\vec{q}_\gamma = \frac{\sqrt{s_\gamma}}{m_\alpha} \end{aligned}$$

Similarly,

$$\vec{Q} = -\vec{Q}_\alpha \frac{W}{m_\alpha}$$

so that

$$q \cdot Q = eE - \left(\frac{q_\gamma \sqrt{s_\gamma}}{m_\alpha} \right) \left(\frac{Q_\alpha W}{m_\alpha} \right) \cos \beta_{\alpha\beta}$$

(ii) In the o.c.m.,

$$\begin{aligned} q \cdot Q &= e' E' - \vec{q}' \cdot \vec{Q}' \\ &= e' E' \\ &= \sqrt{s_\gamma + Q_\gamma^2} \cdot W \end{aligned}$$

(iii) Equating the two expressions for $q \cdot Q$,

$$\sqrt{s_Y + Q_Y^2} = \frac{eE}{W} - q_Y Q_\alpha \left(\frac{\sqrt{s_Y}}{m_\alpha^2} \right) \cos \beta_{\alpha\beta}$$

Now,

$$\begin{aligned} \frac{eE}{W} &= \frac{1}{W} \sqrt{\frac{q_Y^2 s_Y}{m_\alpha^2} + s_Y} \sqrt{\frac{Q_\alpha^2 W^2}{m_\alpha^2} + W^2} \\ &= \left(\frac{\sqrt{s_Y}}{m_\alpha^2} \right) \sqrt{q_Y^2 + m_\alpha^2} \sqrt{Q_\alpha^2 + m_\alpha^2} \end{aligned}$$

Using this and solving for $\cos \beta_{\alpha\beta}$, we have

$$\cos \beta_{\alpha\beta} = \frac{1}{q_Y Q_\alpha} \left[\sqrt{q_Y^2 + m_\alpha^2} \sqrt{Q_\alpha^2 + m_\alpha^2} - \frac{m_\alpha^2}{\sqrt{s_Y}} \sqrt{Q_Y^2 + s_Y} \right]$$

To proceed further, we use (A-1) and (A-2) to substitute for q_Y^2 , Q_α^2 and Q_Y^2 . Simple algebra leads to

$$\sqrt{q_Y^2 + m_\alpha^2} = \frac{1}{2\sqrt{s_Y}} (s_Y + m_\alpha^2 - m_\beta^2)$$

$$\sqrt{Q_\alpha^2 + m_\alpha^2} = \frac{1}{2W} (W^2 + m_\alpha^2 - s_\alpha)$$

$$\sqrt{Q_Y^2 + s_Y} = \frac{1}{2W} (W^2 + s_Y - m_Y^2)$$

so that $\cos \beta_{\alpha\beta}$ becomes

$$\cos \beta_{\alpha\beta} = \frac{1}{4WQ_\alpha q_Y \sqrt{s_Y}} \left[(s_Y + m_\alpha^2 - m_\beta^2)(W^2 + m_\alpha^2 - s_\alpha) - 2m_\alpha^2(W^2 + s_Y - m_Y^2) \right]$$

which can be written in an alternate form as

$$\cos \beta_{\alpha\beta} = \frac{1}{4WQ_\alpha q_Y \sqrt{s_Y}} \left[s_Y(s - m_\alpha^2) + s_\alpha(m_\beta^2 - m_\alpha^2) - s_\alpha s_Y - s(m_\alpha^2 + m_\beta^2) + m_\alpha^2(m_\alpha^2 - m_\beta^2 + 2m_Y^2) \right] \quad (A-5)$$

There are six such angles all of which can be obtained by interchange of two subscripts $\alpha\beta$ or cyclic permutation of all three indices $\alpha\beta\gamma$.

D. The Angle θ^α

Let us calculate $\theta^\alpha \equiv \theta_{\beta\gamma}^\alpha$ which is the angle between the direction of particle α and the z-axis in the c.m. $_{\beta\gamma}$. Again, we consider particle β and the o.c.m. as two "particles" with 4-momenta q and Q respectively and find the invariant $q \cdot Q$ first in the c.m. $_{\beta\gamma}$ and then in the o.c.m.

(i) In the c.m. $_{\beta\gamma}$

$$\begin{aligned} q \cdot Q &= eE - \vec{q} \cdot \vec{Q} = eE + \vec{q} \cdot (-\vec{Q}) \\ &= eE + qQ \cos\theta^\alpha \end{aligned}$$

But $|q| = |q_\alpha|$ and $|Q| = \left| Q_\alpha \frac{W}{\sqrt{s_\alpha}} \right|$ so that

$$q \cdot Q = \frac{W}{\sqrt{s_\alpha}} \left[\sqrt{q_\alpha^2 + m_\beta^2} \sqrt{Q_\alpha^2 + s_\alpha} + q_\alpha Q_\alpha \cos\theta^\alpha \right]$$

(ii) In the o.c.m.,

$$q \cdot Q = e'E' - \vec{q}' \cdot \vec{Q}' = e'E' = \sqrt{Q_\beta^2 + m_\beta^2} \cdot W$$

(iii) Equating the two expressions for $q \cdot Q$ and solving for $\cos\theta^\alpha$, we have

$$\cos\theta^\alpha = \frac{\sqrt{s_\alpha} \sqrt{Q_\beta^2 + m_\beta^2} - \sqrt{q_\alpha^2 + m_\beta^2} \sqrt{Q_\alpha^2 + s_\alpha}}{q_\alpha Q_\alpha}$$

Proceeding exactly in the same manner as we did for the case of $\cos\theta_{\alpha\beta}$, we get the final expression.

$$\cos\theta^\alpha \equiv \cos\theta_{\beta\gamma}^\alpha = \frac{1}{4WQ_\alpha q_\alpha \sqrt{s_\alpha}} \left[(W^2 - m_\alpha^2)(m_Y^2 - m_\beta^2) + s_\alpha(s_Y - s_\beta) \right] \quad (\text{A-6})$$

There are six such angles all of which can be obtained from the above by interchange or permutation of suitable indices. In particular, interchange of β and

γ brings about a negative sign in (A-6) which is also evident from Fig. 3 since $\theta_{\gamma\beta}^{\alpha} = \pi - \theta_{\beta\gamma}^{\alpha}$. We should like to point out that the θ angles appearing in (3.15) actually carry subscripts, i.e., $\theta^3 \equiv \theta_{12}^3$, $\theta^1 \equiv \theta_{23}^1$. The use of subscripts will be required in discussing identical particles.

APPENDIX B
 $d_{m,n}^J(\beta)$ Functions

Notation: $c \equiv \cos(\beta/2)$; $s \equiv \sin(\beta/2)$

$J = 1/2$	$d_{1/2,1/2} = d_{-1/2,-1/2}$	$= c$
	$d_{1/2,-1/2} = -d_{-1/2,1/2}$	$= -s$
$J = 1$	$d_{1,1} = d_{-1,-1}$	$= c^2$
	$d_{1,0} = -d_{0,1} = d_{0,-1} = -d_{-1,0}$	$= -\sqrt{2} cs$
	$d_{1,-1} = d_{-1,1}$	$= s^2$
	d_{00}	$= c^2 - s^2$
$J = 3/2$	$d_{3/2,3/2} = d_{-3/2,-3/2}$	$= c^3$
	$d_{3/2,1/2} = -d_{1/2,3/2} = d_{-1/2,-3/2} = -d_{-3/2,-1/2}$	$= -\sqrt{3} c^2 s$
	$d_{3/2,-1/2} = d_{1/2,-3/2} = d_{-1/2,3/2} = d_{-3/2,1/2}$	$= \sqrt{3} cs^2$
	$d_{3/2,-3/2} = -d_{-3/2,3/2}$	$= -s^3$
	$d_{1/2,1/2} = d_{-1/2,-1/2}$	$= c^3 - 2cs^2$
	$d_{1/2,-1/2} = -d_{-1/2,1/2}$	$= -2c^2 s + s^3$
$J = 2$	$d_{2,2} = d_{-2,-2}$	$= c^4$
	$d_{2,1} = -d_{1,2} = -d_{-2,-1} = d_{-1,-2}$	$= -2c^3 s$
	$d_{2,0} = d_{0,2} = d_{-2,0} = d_{0,-2}$	$= \sqrt{6} c^2 s^2$
	$d_{2,-1} = -d_{-1,2} = -d_{-2,1} = d_{1,-2}$	$= -2cs^3$
	$d_{2,-2} = d_{-2,2}$	$= s^4$
	$d_{1,1} = d_{-1,-1}$	$= c^4 - 3c^2 s^2$
	$d_{1,0} = -d_{0,1} = d_{0,-1} = -d_{-1,0}$	$= \sqrt{6} (-c^3 s + cs^3)$
	$d_{1,-1} = d_{-1,1}$	$= 3c^2 s^2 - s^4$
	$d_{0,0}$	$= c^4 - 4c^2 s^2 + s^4$

$$\begin{aligned}
 J = 5/2 \quad d_{5/2,5/2} &= d_{-5/2,-5/2} &= c^5 \\
 d_{5/2,3/2} &= -d_{3/2,5/2} = d_{-3/2,-5/2} = -d_{-5/2,-3/2} &= -\sqrt{5} c^4 s \\
 d_{5/2,1/2} &= d_{1/2,5/2} = d_{-1/2,-5/2} = d_{-5/2,-1/2} &= \sqrt{10} c^3 s^2 \\
 d_{5/2,-1/2} &= d_{1/2,-5/2} = -d_{-1/2,5/2} = -d_{-5/2,1/2} &= -\sqrt{10} c^3 s^3 \\
 d_{5/2,-3/2} &= d_{3/2,-5/2} = d_{-3/2,5/2} = d_{-5/2,3/2} &= \sqrt{5} c s^4 \\
 d_{5/2,-5/2} &= -d_{-5/2,5/2} &= -s^5 \\
 d_{3/2,3/2} &= d_{-3/2,-3/2} &= c^5 - 4c^3 s^2 \\
 d_{3/2,1/2} &= -d_{1/2,3/2} = d_{-1/2,-3/2} = -d_{-3/2,-1/2} &= -\sqrt{8} c^4 s + \sqrt{18} c^2 s^3 \\
 d_{3/2,-1/2} &= d_{1/2,-3/2} = d_{-1/2,3/2} = d_{-3/2,1/2} &= \sqrt{18} c^3 s^2 - \sqrt{8} c s^4 \\
 d_{3/2,-3/2} &= -d_{-3/2,3/2} &= -4c^2 s^3 + s^5 \\
 d_{1/2,1/2} &= d_{-1/2,-1/2} &= c^5 - 6c^3 s^2 + 3c s^4 \\
 d_{1/2,-1/2} &= -d_{-1/2,1/2} &= -3c^4 s + 6c^2 s^3 - s^5
 \end{aligned}$$

$$\begin{aligned}
 J = 3 \quad d_{3,3} &= d_{-3,-3} &= c^6 \\
 d_{3,2} &= -d_{2,3} = -d_{-3,-2} = d_{-2,-3} &= -\sqrt{6} c^5 s \\
 d_{3,1} &= d_{1,3} = d_{-3,-1} = d_{-1,-3} &= \sqrt{15} c^4 s^2 \\
 d_{3,0} &= -d_{0,3} = -d_{-3,0} = d_{0,-3} &= -\sqrt{20} c^3 s^3 \\
 d_{3,-1} &= d_{-1,3} = d_{-3,1} = d_{1,-3} &= \sqrt{15} c^2 s^4 \\
 d_{3,-2} &= -d_{-2,3} = -d_{-3,2} = d_{2,-3} &= -\sqrt{6} c s^5 \\
 d_{3,-3} &= d_{-3,3} &= s^6 \\
 d_{2,2} &= d_{-2,-2} &= c^6 - 5c^4 s^2 \\
 d_{2,1} &= -d_{1,2} = -d_{-2,-1} = d_{-1,-2} &= -\sqrt{10} c^5 s + \sqrt{40} c^3 s^3 \\
 d_{2,0} &= d_{0,2} = d_{-2,0} = d_{0,-2} &= \sqrt{30} (c^4 s^2 - c^2 s^4) \\
 d_{2,-1} &= -d_{-1,2} = -d_{-2,1} = d_{1,-2} &= -\sqrt{40} c^3 s^3 + \sqrt{10} c s^5 \\
 d_{2,-2} &= d_{-2,2} &= 5c^2 s^4 - s^6 \\
 d_{1,1} &= d_{-1,-1} &= c^6 - 8c^4 s^2 + 6c^2 s^4 \\
 d_{1,0} &= -d_{0,1} = d_{0,-1} = -d_{-1,0} &= -\sqrt{12} c^5 s + \sqrt{108} c^3 s^3 - \sqrt{12} c s^5 \\
 d_{1,-1} &= d_{-1,1} &= 6c^4 s^2 - 8c^2 s^4 + s^6 \\
 d_{0,0} & &= c^6 - 9c^4 s^2 + 9c^2 s^4 - s^6
 \end{aligned}$$

$$\begin{aligned}
 J = 7/2 \quad d_{7/2,7/2} &= d_{-7/2,-7/2} &= c^7 \\
 d_{7/2,5/2} &= -d_{5/2,7/2} = -d_{-7/2,-5/2} = d_{-5/2,-7/2} &= -\sqrt{7} c^6 s \\
 d_{7/2,3/2} &= d_{3/2,7/2} = d_{-7/2,-3/2} = d_{-3/2,-7/2} &= \sqrt{21} c^5 s^2 \\
 d_{7/2,1/2} &= -d_{1/2,7/2} = -d_{-7/2,-1/2} = d_{-1/2,-7/2} &= -\sqrt{35} c^4 s^3 \\
 d_{7/2,-1/2} &= d_{-1/2,7/2} = d_{-7/2,1/2} = d_{1/2,-7/2} &= \sqrt{35} c^3 s^4 \\
 d_{7/2,-3/2} &= -d_{-3/2,7/2} = -d_{-7/2,3/2} = d_{3/2,-7/2} &= -\sqrt{21} c^2 s^5 \\
 d_{7/2,-5/2} &= d_{-5/2,7/2} = d_{-7/2,5/2} = d_{5/2,-7/2} &= \sqrt{7} c s^6 \\
 d_{7/2,-7/2} &= -d_{-7/2,7/2} &= -s^7 \\
 d_{5/2,5/2} &= d_{-5/2,-5/2} &= c^7 - 6c^5 s^2 \\
 d_{5/2,3/2} &= -d_{3/2,5/2} = d_{-3/2,-5/2} = -d_{-5/2,-3/2} &= -\sqrt{12} c^6 s + \sqrt{75} c^4 s^3 \\
 d_{5/2,1/2} &= d_{1/2,5/2} = d_{-1/2,-5/2} = d_{-5/2,-1/2} &= \sqrt{45} c^5 s^2 - \sqrt{80} c^3 s^4 \\
 d_{5/2,-1/2} &= d_{1/2,-5/2} = -d_{-1/2,5/2} = -d_{-5/2,1/2} &= -\sqrt{80} c^4 s^3 + \sqrt{45} c^2 s^5 \\
 d_{5/2,-3/2} &= d_{3/2,-5/2} = d_{-3/2,5/2} = d_{-5/2,3/2} &= \sqrt{75} c^3 s^4 - \sqrt{12} c s^6 \\
 d_{5/2,-5/2} &= -d_{-5/2,5/2} &= -6c^2 s^5 + s^7 \\
 d_{3/2,3/2} &= d_{-3/2,-3/2} &= c^7 - 10c^5 s^2 + 10c^3 s^4 \\
 d_{3/2,1/2} &= -d_{1/2,3/2} = d_{-1/2,-3/2} = -d_{-3/2,-1/2} &= -\sqrt{15} c^6 s + \sqrt{240} c^4 s^3 \\
 & & - \sqrt{60} c^2 s^5 \\
 d_{3/2,-1/2} &= d_{1/2,-3/2} = d_{-1/2,3/2} = d_{-3/2,1/2} &= \sqrt{60} c^5 s^2 - \sqrt{240} c^3 s^4 \\
 & & + \sqrt{15} c s^6 \\
 d_{3/2,-3/2} &= -d_{-3/2,3/2} &= -10c^4 s^3 + 10c^2 s^5 - s^7 \\
 d_{1/2,1/2} &= d_{-1/2,-1/2} &= c^7 - 12c^5 s^2 + 18c^3 s^4 \\
 & & - 4c s^6 \\
 d_{1/2,-1/2} &= -d_{-1/2,1/2} &= -4c^6 s + 18c^4 s^3 - 12c^2 s^5 \\
 & & + s^7
 \end{aligned}$$

APPENDIX C
Lorentz Spin Rotation

The six generators of the homogeneous Lorentz group obey the commutation relations

$$[J_{\mu\nu}, J_{\lambda\sigma}] = i[J_{\nu\sigma}\delta_{\mu\lambda} - J_{\nu\lambda}\delta_{\mu\sigma} + J_{\mu\lambda}\delta_{\nu\sigma} - J_{\mu\sigma}\delta_{\nu\lambda}] \quad (C-1)$$

In particular, $J_3 = J_{12}$ and $K_3 = i J_{34}$ commute. Therefore, an eigenstate of J_3 will also be an eigenstate of K_3 .

Consider a frame F in which a particle with spin σ has a momentum $\vec{q}_0$ along the z -axis and its spin projection σ_z equal to μ (see Fig. 5). In another frame F' moving with respect to F along the z -axis, the spin component will still be $\sigma_z = \mu$ because $[J_3, K_3] = 0$ but the momentum will be, say, $\vec{q}'_0$ along the z -axis. We thus have the states

$$|\vec{q}_0, \mu\rangle \quad \text{in } F,$$

$$|\vec{q}'_0, \mu\rangle \quad \text{in } F',$$

and the connection between the two is given by

$$|\vec{q}'_0, \mu\rangle = L(\vec{p})|\vec{q}_0, \mu\rangle \quad (C-2)$$

If we now rotate the state in F by an angle θ about the y -axis [this corresponds to the rotation of the frame F by $-\theta$ and, according to our convention, the operator is $R(0, -\theta, 0)$] we will get the state

$$|\vec{q}, \mu\rangle = R^L(-\theta)|\vec{q}_0, \mu\rangle \quad \text{in } F \quad (C-3)$$

Note that since $\vec{J} = \vec{L} + \vec{\sigma}$, R^L only affects the orbital or angular part and leaves the spin unchanged. A Lorentz boost by the amount $L(\vec{p})$ to this state will yield a new state with momentum $\vec{q}'$ in F' and mixed spin. If we take a basic ket in F' to be $|\vec{q}', \nu\rangle$, we can write this new state as

$$L(\vec{p})|\vec{q}, \mu\rangle = \sum_{\nu} D^{\sigma}_{\nu\mu}(\alpha) |\vec{q}', \nu\rangle \quad \text{in } F' \quad (C-4)$$

where α is the rotation we wish to find. Of course, the state $|\vec{q}'\nu\rangle$ itself can be constructed from $|\vec{q}_0\nu\rangle$ by first giving it the boost $L(\vec{p})$ and then subjecting it to a rotation ψ (again note the minus sign in the operator below).

$$|\vec{q}'\nu\rangle = R^L(-\psi) L(\vec{p}) |\vec{q}_0\nu\rangle \quad (C-5)$$

Let us now consider (C-4) and substitute (C-3) in its l.h.s. and (C-5) in its r.h.s. thus getting

$$L(\vec{p}) R^L(-\theta) |\vec{q}_0\mu\rangle = \sum_{\nu} D_{\nu\mu}^{\sigma}(\alpha) \cdot R^L(-\psi) L(\vec{p}) |\vec{q}_0\nu\rangle$$

Or,

$$|\vec{q}_0\mu\rangle = \sum_{\nu} D_{\nu\mu}^{\sigma}(\alpha) \left(R^{-1}L(-\theta)L^{-1}(\vec{p})R^L(-\psi)L(\vec{p}) \right) |\vec{q}_0\nu\rangle \quad (C-6)$$

Now we invoke the familiar argument that the parenthesis in the above equation can only contain a spin rotation operator in the x-z plane since the relation is a simple transformation from a $|\nu\rangle$ state to a $|\mu\rangle$ state, the momentum remaining unchanged. Let this operator be $R^{\sigma}(-\alpha')$. (This means we will rotate the state by $+\alpha'$.) Then, multiplying the above equation from the l.h.s. by $\langle\vec{q}_0\nu'|$,

$$\langle\vec{q}_0\nu'|\vec{q}_0\mu\rangle = \sum_{\nu} D_{\nu\mu}^{\sigma}(\alpha) \langle\vec{q}_0\nu'|R^{\sigma}(-\alpha')|\vec{q}_0\nu\rangle$$

Or,

$$\begin{aligned} \delta_{\nu'\mu} &= \sum_{\nu} D_{\nu\mu}^{\sigma}(\alpha) D_{\nu'\nu}^{\sigma}(-\alpha') \\ &= \sum_{\nu} D_{\mu\nu}^{\sigma*}(-\alpha) D_{\nu'\nu}^{\sigma}(-\alpha') \end{aligned}$$

In order for this to be true, it follows from the orthogonality of the rotation matrices that we must have

$$-\alpha = -\alpha' \quad \text{or} \quad \alpha = \alpha' \quad (C-7)$$

Now α' can be worked out from the geometry of Fig. 5. It is equal to the net rotation after we carry out the four operations contained in the parenthesis of (C-6).

- i) Apply $L(\vec{p})$. This simply takes us from F to F' .
- ii) Next, rotate the *state* by ψ . This gives us the momentum $\vec{q}'$ in F' .
- iii) Now apply $L^{-1}(\vec{p})$ to $\vec{q}'$. This will give us $\vec{q}$ in F but will involve a rotation β .
- iv) Finally, the *state* has to be rotated by θ by the operator R^{-1} , i.e., by $-\theta$.

These four steps are indicated in Fig. 5 along with the appropriate rotations. We get

$$\alpha = \alpha' = \psi + \beta - \theta \quad (C-8)$$

Comparing Fig. 5 with the upper left hand part of Fig. 3, we see that the Lorentz spin rotation just worked out is for particle α when we go from c.m. _{$\alpha\beta$} to the o.c.m. Labeling the angles and writing $\psi = \pi - \chi^\beta$,

$$\xi_\alpha^\gamma = \pi - \chi^\beta + \beta_{\alpha\beta} - \theta_{\alpha\beta} = -\chi^\beta + \beta_{\alpha\beta} + \theta_{\beta\alpha} \quad (C-9)$$

Such spin rotations can be worked out for each particle simply by looking at the geometry and carrying out the required transformations as outlined above. One must be careful, however, about the sense of rotation angles and the signs. For example, for particle β , the spin rotation between F and F' of Fig. 5 is

$$\xi_\beta^\gamma = -(\pi - \chi^\alpha) - \beta_{\beta\alpha} + \theta_{\beta\alpha} = \chi^\alpha - \beta_{\beta\alpha} - \theta_{\alpha\beta} \quad (C-10)$$

Other rotations can be obtained from (C-9) and (C-10) by cyclic permutation of the three indices α , β and γ .

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FIGURE CAPTIONS

- Fig. 1. Production of the α -isobar in the intermediate state and its subsequent decay into the β and γ particles in the final state.
- Fig. 2. Diagrams used in calculating the recoupling coefficient.
- (a) Arbitrary orientation of z- and x-axes in the 3-particle plane. The Y-axis is out of the paper.
 - (b) Angle variables of representation $|3(12) \rangle$ in the arbitrary set of axes A.
 - (c) Angle variables of representation $|3(12) \rangle$ in the special set of axes B.
 - (d) Angle variables of representation $|1(23) \rangle$ in the arbitrary set of axes A.
 - (e) Angle variables of representation $|1(23) \rangle$ in the special set of axes B'.
- Fig. 3. Three-particle state in the overall center-of-mass frame with the three different sets of coordinate axes to correspond to three different angular momentum representations.
- Fig. 4. (a) The definition of positive angles and positive axes *before* particles 1 and 2 are interchanged.
- (b) The definition of positive angles and positive axes *after* particles 1 and 2 are interchanged. The transformed angles for particle 1 are indicated by single line, those for particle 2 by double lines.
- Fig. 5. Momenta and angles describing Lorentz spin rotation from the inertial frame F to F'.

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Fig. 1

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Fig. 2

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Fig. 3

(b)

(a)

Fig. 4

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Fig. 5