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CYLINDRICAL ARRANGEMENTS OF UNIFORM SPHERES

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CYLINDRICAL ARRANGEMENTS OF UNIFORM SPHERES

Introduction

Cylindrical arrangements of uniform spheres have been discussed on a general basis where the ratio of cylinder diameter to sphere diameter is large [1], [2], [9]. These papers have been directed at the study of the filling of cylindrical containers to achieve maximum density without regard for the systematic geometric arrangements which might occur. This paper deals primarily with the systematic arrangement of uniform spheres such that the envelope of the set of uniform spheres is an infinite cylinder. The particular arrangements which fill the cylinder to maximum density are specified in terms of the ratio of cylinder diameter to sphere diameter and have been evaluated for values of the ratio between one and two. Additional arrangements possessing what the authors believe to be the greatest density have been determined for values of the ratio from two to the quantity $(1 + 2/\sqrt{3})$. These results indicate a method of covering cylinders with systematic arrangements of uniform spheres similar to the covering of a plane with equal circles therefore yielding the greatest density of uniform spheres in cylindrical shells.

Dense Arrangements With Continuously Variable Parameters

We may consider the arrangements of unit spheres as representative of arrangements of equal spheres having arbitrary diameter and discuss only the special case of arrangements of unit spheres with their centers on cylinders of arbitrary diameter. The elementary case exists when this cylinder containing sphere centers has diameter zero. Here, the spheres are placed one on top of the other. Initially, let the cylinder expand and require adjacent spheres to remain in contact. Choosing any sphere center as reference, adjacent sphere centers lie on a sphere about the reference sphere. Now fix the distance in the direction of the cylinder axis between adjacent sphere centers at some value ζ where

$\frac{1}{2} < \zeta \leq 1$ and contract the cylinder diameter to its minimum value for the arbitrarily fixed value of ζ . Let d represent this minimum value of cylinder diameter. Then $d = \sqrt{1 - \zeta^2}$ for $\frac{1}{2} < \zeta \leq 1$. If the spheres are numbered in sequence according to their axial distance from the reference sphere, then the lower bound on ζ assures that even-numbered spheres have no points in common and odd-numbered spheres have no points in common. The greatest average density of spheres in this arrangement is obtained when the odd-numbered sphere centers lie on a line parallel to the line passing through the centers of even numbered spheres and parallel to the cylinder axis. The density of a segment of a cylindrical arrangement depends upon the location of the segment if its axial length differs from an integer multiple of ζ . Consequently we define the average density of the arrangement as the limit of the density of a sector of arbitrary length Z as $Z \rightarrow \infty$.

Theorem:

In the arrangement of unit spheres, let the spheres be numbered according to the magnitude of the axial distance of their centers from some reference sphere. Let ζ be the axial distance from the center of the reference sphere to the center of the sphere tangent to the reference sphere and having the largest index N relative to the reference sphere. Let d represent the minimum of all possible cylinder diameters which pass through the centers of every sphere in the arrangement. Let z be the arbitrary axial length of a section of the arrangement. Let $D(N, d, \zeta)$ be the average density of the arrangement.

Then

$$I) \quad D(N, d, \zeta) = \left\{ \begin{array}{l} \lim_{z \rightarrow \infty} \left\{ \frac{N \frac{\pi}{6} [\frac{z}{\zeta}]}{\frac{\pi}{4} (d+1)^2 z} \right\}, \quad 0 \leq d < 1 \\ \lim_{z \rightarrow \infty} \left\{ \frac{N \frac{\pi}{6} [\frac{z}{\zeta}]}{\frac{\pi}{4} \{ (d+1)^2 - (d-1)^2 \} z} \right\}, \quad 1 \leq d \leq \frac{2}{\sqrt{3}} \end{array} \right\}, \quad \frac{1}{2} < \zeta \leq 1.$$

Here, the symbol $[q]$ means the largest integer less than q . Evaluating the limits,

$$\text{II)} \quad D(N, d, \zeta) = \begin{cases} \frac{2}{3} \frac{N}{\zeta(d+1)^2}, & 0 \leq d < 1 \\ \frac{1}{6} \frac{N}{\zeta d}, & 1 \leq d \leq \frac{2}{\sqrt{3}} \end{cases}, \frac{1}{2} < \zeta \leq 1.$$

A proper choice of ζ is one for which N is an integer equal to the number of spheres in the sector of axial length ζ and independent of the placement of the sector relative to the sphere centers.

The values of N and d are defined as follows:

1. (a) If $0 \leq d < \frac{\sqrt{3}}{2}$, $\frac{1}{2} < \zeta \leq 1$ then $d = \sqrt{1 - \zeta^2}$, $N = 1$,
and every sphere touches two spheres.
- (b) If $\frac{\sqrt{3}}{2} \leq d \leq \frac{4\sqrt{3}}{7}$, $\frac{2}{\sqrt{7}} \leq \zeta \leq 1$, then $d = \frac{2}{\sqrt{3}} (1 - \frac{\zeta^2}{4})$, $N = 2$,
and every sphere touches four spheres.
- (c) If $\frac{4\sqrt{3}}{7} \leq d < 1$, $\frac{1}{\sqrt{2}} < \zeta \leq \frac{2}{\sqrt{7}}$, then $d = 2 \zeta \sqrt{1 - \zeta^2}$, $N = 2$,
and every sphere touches four spheres.
- (d) If $1 \leq d < \frac{3}{5} \sqrt{3}$, $\frac{2}{\sqrt{5}} < \zeta \leq \frac{1}{\sqrt{2}}$, then $d = \sqrt{1 + \zeta^2 (1 - 2\zeta^2)}$, $N = 2$,
and every sphere touches five spheres.
- (e) If $\frac{3\sqrt{3}}{5} < d < \frac{2}{\sqrt{3}}$, $\frac{2}{\sqrt{3}} < \zeta < \frac{3}{\sqrt{10}}$, then $d = \frac{\sqrt{3}}{4} [\zeta^2 + \zeta \sqrt{16 - 15\zeta^2}]$, $N = 3$.
Here, $\frac{1}{3}$ of the spheres touch six spheres, the others touch five spheres.

Proof:

- 1(a) The reference sphere and the first sphere have centers on a cylinder and are tangent. The axial distance separating their centers is defined to be ζ . Consequently the distance separating their centers projected to the plane perpendicular to the cylinder axis is $\sqrt{1 - \zeta^2}$ and is a chord of the circular cross-section of the cylinder in the

projection plane. The cylinder diameter is minimum when it equals the chord length $\sqrt{1 - \zeta^2}$. The sector of length ζ always contains one sphere for any position relative to the reference sphere. The spherical cap removed from the sector by one plane boundary is included in the sector by the other plane boundary so that the density of the sector is equivalent to the average density of the infinitely long cylinder for any placement of the sector relative to the sphere centers. This fulfills the requirements that ζ be a proper choice.

- 1(b) Here, every sphere is in contact with four spheres. The reference sphere and the second sphere are separated by an axial length of ζ . The first sphere is in contact with both the reference sphere and the second sphere; consequently the center of the first sphere lies on the locus of points equidistant from the centers of the reference sphere and the second sphere. The three spheres also have centers on a cylinder of diameter d . We seek the smallest d for which this arrangement of spheres is possible. Erect a rectangular cartesian coordinate system (figure 1) at the point of contact of the reference sphere and the second sphere so that the cylinder axis is parallel to the z axis and the x axis is the perpendicular bisector of the line segment connecting the centers of the reference sphere and second sphere. The centers of the reference sphere and the second sphere lie in the y - z plane and obey

$$(i) \quad y^2 + z^2 = \frac{1}{4}.$$

The axial distance between these spheres is defined as ζ . Hence the coordinates of the reference are $P_0 (0, -\frac{1}{2} \sqrt{1 - \zeta^2}, -\frac{\zeta}{2})$ and the coordinates of the second sphere are $P_2 (0, \frac{1}{2} \sqrt{1 - \zeta^2}, \frac{\zeta}{2})$. The line connecting P_0 to P_2 obeys the relation

$$(ii) \quad z = \frac{\zeta}{\sqrt{1 - \zeta^2}} y.$$

The first sphere center lies in the plane perpendicular to the plane ii and therefore obeys the relation

$$(iii) \quad z = -\frac{\sqrt{1 - \zeta^2}}{\zeta} y.$$

The first sphere is on the altitude of an equilateral triangle with unit edge, the altitude passing through the origin of the coordinate system. Therefore, it also has a center on a sphere of radius $\sqrt{3}/2$.

$$(iv) \quad x^2 + y^2 + z^2 = \frac{3}{4}.$$

The trace of the center of the first sphere in the x-y plane is yielded by using iii to eliminate z from iv.

$$(v) \quad \frac{x^2}{(\frac{3}{4})} + \frac{y^2}{\frac{3}{4} \zeta^2} = 1.$$

All centers lie on a cylinder of diameter d; hence the projections of these sphere centers lie on a circle of diameter d in the x-y plane.

The projections of P_0 and P_2 on to the x-y plane become

$$P_0 (0, -\frac{\sqrt{1 - \zeta^2}}{2}), \quad P_2 (0, +\frac{\sqrt{1 - \zeta^2}}{2}).$$

Since the circle of

diameter d must pass through P_0 and P_2 and since P_0 and P_2 are symmetric about the origin and have zero x coordinate, the center of the circle must lie on the x axis at x_0 and have the form

$$(vi) \quad (x - x_0)^2 + y^2 = \frac{d^2}{4},$$

where

$$(vii) \quad x_0^2 = \frac{1}{4} (d^2 - 1 + \zeta^2).$$

The smallest choice for d occurs when the circle vi and the ellipse v have one point in common. This occurs only if $y = 0$ for which v yields $x = \frac{\sqrt{3}}{2}$. Using these values for x and y in vi and eliminating x_0 with vii , the desired relationship between d and ζ is obtained.

$$(viii) \quad d = \frac{1}{2\sqrt{3}} (4 - \zeta^2).$$

As ζ decreases, the minor axis of the ellipse v decreases until a critical value of ζ for which the circle and ellipse have common curvature at the point of contact. Any further decrease of ζ must result in four, three, two or no points of contact depending on the value of d . Taking derivatives of v and vi with respect to y , since y goes to zero and x' goes to zero,

$$(ix) \quad xx'' + \frac{1}{\zeta^2} = 0.$$

The curvature is given by $x'' = -\frac{2}{d}$ since $x' = 0$ and the coordinate system is reversed. Using the fact that $x = \frac{\sqrt{3}}{2}$, and eliminating d from $viii$ yields the lower bound on ζ and the upper bound on d .

$$(x) \quad \zeta \geq \frac{2}{\sqrt{7}}, \quad d \leq \frac{4\sqrt{3}}{7}.$$

1(c) A further decrease of ζ from that given by inequality x , requires that the circle and ellipse have at least two points in common.

Either point, but not both points may be used as a position for the first sphere center. Arbitrarily selecting the point with minimum axial distance from the reference sphere center as the position for the first sphere center forces all odd numbered spheres to occupy the

same relative positions with respect to their adjacent numbered spheres. Consequently all even numbered spheres occupy the other position relative to their adjacent even numbered spheres; hence the arrangement uses both positions. A derivation of $l(c)$ follows using sphere (0) as the reference sphere and choosing the position with minimum axial distance from the center of sphere (0) for the center of sphere (1).

Equating first derivatives with respect to y of v and v_i ,

$$(xi) \quad x = \frac{x_0}{1 - \zeta^2}.$$

Eliminating y^2 from v and v_i ,

$$(xii) \quad \zeta^2 \left(\frac{3}{4} - x^2 \right) = \frac{d^2}{4} - (x - x_0)^2.$$

Eliminating x from xi and xii and using vii to eliminate x_0 ,

$$(xiii) \quad d = 25 \sqrt{1 - \zeta^2}$$

This equation holds until the axial distance between sphere (0) and sphere (1) is zero; i.e., sphere (0) has z coordinate identical with

$P_0 (0, -\frac{1}{2} \sqrt{1 - \zeta^2}, -\frac{\zeta}{2})$. Therefore from iii , the y coordinate of sphere

(1) is $\frac{1}{2} \frac{\zeta^2}{\sqrt{1 - \zeta^2}}$ and from v , the x coordinate of sphere (1)

is $\frac{1}{2} \sqrt{\frac{3 - 4\zeta^2}{1 - \zeta^2}}$. Evaluating x_0 from xi and from vii ,

$x_0 = \sqrt{(1 - \zeta^2) \left(\frac{3}{4} - \zeta^2 \right)} = \sqrt{(1 - \zeta^2) \left(\zeta^2 - \frac{1}{4} \right)}$. Here, ζ has an upper bound of $\frac{2}{\sqrt{7}}$; hence the factor $(1 - \zeta^2) \neq 0$, yielding the lower bound for ζ and the upper bound for d .

$$(xiv) \quad \zeta > \frac{1}{\sqrt{2}}, d < 1.$$

In the above discussions for $l(b)$ and $l(c)$, the choice for ζ is sufficient to insure that half of sphere (0) and half of sphere (2) are in the sector. Sphere (3) has the same axial distance from sphere (2) as

sphere (1) has from sphere (0). If the sector boundary at $z = -\frac{\zeta}{2}$ excludes a segment from sphere (1), the sector boundary at $z = \frac{\zeta}{2}$ includes an identical segment from sphere (3). Consequently the sector of axial length ζ contains two spheres regardless of its location and constitutes a proper choice of ζ . Here $N = 2$ for both 1(b) and 1(c).

1(d) When $\zeta = \frac{1}{\sqrt{2}}$ and $d = 1$, then the line segments connecting the centers of spheres (0), (1), (2), and (3) form the edges of a tetrahedron. The edge (0-1) is perpendicular to the cylinder axis, and the edge (2-3) is perpendicular to the cylinder axis, at a height of $\zeta = 1/\sqrt{2}$ above the edge (0-1). The same is true of the centers of spheres (4-5) relative to the centers of sphere (2-3). The arrangement of line segments connecting sphere centers consists of a column of tetrahedrons circumscribed in a cylinder, the tetrahedrons having an edge in common. Letting the cylinder expand, we seek the way to achieve maximum density for the arrangement of spheres.

Suppose that the tetrahedrons remain tetrahedrons with edges in contact and that a pair of faces of two tetrahedrons having a common edge move so that the angle between faces is decreased. Then the adjacent faces having the same common edge must move apart by the same amount. Also, the edges opposite the common edge must make an inclination with respect to the plane perpendicular to the cylinder axis equal to half of the change in the angle between the original two faces. Since the choice of the common edge is arbitrary, let each edge take an inclination such that the axial distance separating the center of sphere (0) from the center of sphere (1) is α . Then the axial distance separating the center of sphere (2) from the center of sphere (3) is also α . Let the axial distance separating the center

of sphere (2) from the center of sphere (0) be ζ . Finally, the sphere centers lie on a cylinder of diameter d . With these requirements we deduce a relationship between d and ζ .

Erect a right handed (x, y, z) coordinate system (Fig. 2) with its origin at the center of the line connecting the center of the reference sphere (0) to the center of sphere (3) with the line in the y - z plane and the z axis parallel to the cylinder axis. Let the points P_0, P_1, P_2, P_3 , represent the coordinates of the centers of spheres 0, 1, 2, and 3 respectively.

Let

$$(i) \quad P_i = (a_i, b_i, c_i), \quad i = 0, 1, 2, 3,$$

where a_i, b_i, c_i , are to be evaluated from the given information.

Consider the c_i only.

$$(ii) \quad \left\{ \begin{array}{l} \alpha = c_1 - c_0 \\ \alpha = c_3 - c_2 \\ \zeta = c_2 - c_0 \\ 0 = c_3 + c_0 \end{array} \right\}.$$

The first three equations were assigned; the fourth was obtained by choice of the coordinate system.

Solving (ii) for the c_i in terms of α and ζ ,

$$(ii) \quad \begin{pmatrix} c_0 = -\frac{1}{2}(\zeta + \alpha) \\ c_1 = -\frac{1}{2}(\zeta - \alpha) \\ c_2 = +\frac{1}{2}(\zeta - \alpha) \\ c_3 = +\frac{1}{2}(\zeta + \alpha) \end{pmatrix} .$$

Orient the tetrahedron in the coordinate system so that the edge connecting the center of sphere (1) to the center of sphere (2) lies in a plane parallel to the y-z plane. The mid-point of the (1-2) edge must be on the x axis at a distance of $1/\sqrt{2}$ from the origin since the minimum distance between opposite edges of a tetrahedron with unit edges is $1/\sqrt{2}$. Choose this midpoint to be on the positive x axis. This choice of coordinate system defines the a_i ,

$$(iii) \quad a_0 = a_3 = 0, \quad a_1 = a_2 = 1/\sqrt{2} .$$

With ii and iii, i becomes

$$(i) \quad P_i = \begin{cases} [0, b_0, -\frac{1}{2}(\zeta + \alpha)], i = 0 \\ [\frac{1}{\sqrt{2}}, b_1, -\frac{1}{2}(\zeta - \alpha)], i = 2 \\ [\frac{1}{\sqrt{2}}, b_2, +\frac{1}{2}(\zeta - \alpha)], i = 2 \\ [0, b_3, +\frac{1}{2}(\zeta + \alpha)], i = 3 \end{cases} \dots$$

The choice of coordinate system requires

$$(iv) \quad \begin{cases} b_0 + b_3 = 0 \\ b_1 + b_2 = 0 \end{cases} .$$

Since the edges are of unit length,

$$(v) \quad |\bar{P}_i - \bar{P}_j| = 1, i \neq j, i, j = 0, 1, 2, 3.$$

Using iv and v in i yields

$$(vi) \quad \begin{cases} b_3 = -b_0 = \frac{1}{2} \sqrt{1 - (\zeta + \alpha)^2} \\ b_1 = -b_2 = \frac{1}{2} \sqrt{1 - (\zeta - \alpha)^2} \end{cases} .$$

Finally, i becomes

$$(i) P_i \left\{ \begin{array}{l} [0, -\frac{1}{2}\sqrt{1-(\zeta+\alpha)^2}, -\frac{1}{2}(\zeta+\alpha)], i=0 \\ [\frac{1}{\sqrt{2}}, \frac{1}{2}\sqrt{1-(\zeta-\alpha)^2}, -\frac{1}{2}(\zeta-\alpha)], i=1 \\ [\frac{1}{\sqrt{2}}, -\frac{1}{2}\sqrt{1-(\zeta-\alpha)^2}, \frac{1}{2}(\zeta-\alpha)], i=2 \\ [0, \frac{1}{2}\sqrt{1-(\zeta+\alpha)^2}, \frac{1}{2}(\zeta+\alpha)], i=3 \end{array} \right\} .$$

Using the above result i in v for the unit lengths $|\vec{P}_1 - \vec{P}_0|, |\vec{P}_2 - \vec{P}_0|, |\vec{P}_3 - \vec{P}_1|$ and $|\vec{P}_3 - \vec{P}_2|$ yields a relation between ζ and α .

$$(vii) \quad \zeta^2 + \alpha^2 = \frac{1}{2}$$

With vii, the coordinates of the points are functions of the single parameter ζ . Hence the diameter of the cylinder passing through the four points is a function of ζ and the axial orientation of the cylinder. Since the cylinder axis has been chosen parallel to the z axis, the relation between d and ζ is unique and may be determined by passing a circle of radius d through the projected points P_i onto the x - y plane.

The symmetry of the points dictates that the circle have the form

$$(viii) \quad (x - x_0)^2 + y^2 = \frac{d^2}{4} .$$

Substituting from i for x and y in viii,

$$P_0) \quad x_0^2 + \frac{1}{4}[1 - (\zeta+\alpha)^2] = \frac{d^2}{4} \rightarrow x_0 = \frac{1}{2}\sqrt{d^2 - 1 + (\zeta+\alpha)^2} .$$

$$P_1) \quad (\frac{1}{\sqrt{2}} - x_0)^2 + \frac{1}{4}[1 - (\zeta - \alpha)^2] = \frac{d^2}{4} \rightarrow x_0 = \frac{1}{\sqrt{2}} - \frac{1}{2}\sqrt{d^2 - 1 + (\zeta - \alpha)^2} .$$

The other points repeat P_0 and P_1 . Eliminating x_0 ,

$$(ix) \quad d = \sqrt{1 + \zeta^2(1 - 2\zeta^2)} .$$

The lower bound on ζ occurs when sphere (4) comes in contact with sphere (1). The axial distance between the centers of spheres (0) and (1) is α , and between the centers of spheres (2) and (3) is α . So, the distance between the centers of spheres (4) and (5) is also α . The axial distance between the centers of spheres (0) and (2) is ζ . From ii, we

compute the axial distance between the centers of spheres (1) and (3) to be ζ and require the axial distance between the centers of spheres (2) and (4) to be ζ .

$$(x) \quad C_4 - C_2 = \zeta$$

With x and ii, we find that

$$(xi) \quad C = \frac{3}{2}\zeta - \frac{1}{2}\alpha .$$

$$(xii) \quad C_4 - C_1 = 2\zeta - \alpha .$$

The minimum value of ζ is achieved when

$$(xiii) \quad C_4 - C_1 = C_3 - C_0 .$$

Using ii,

$$(xiv) \quad C_3 - C_0 = \zeta + \alpha .$$

Consequently from xii and xiii,

$$(xv) \quad \alpha = \frac{\zeta}{2}$$

From vii, xv and ix we deduce the lower bound on ζ and the upper bound on d.

$$(xvi) \quad \zeta > \sqrt{\frac{2}{5}} , \quad d < \frac{3}{5}\sqrt{3} .$$

The choice for ζ from the center of sphere (0) to the center of sphere (2) is seen to be proper since the cap of sphere (2) excluded from the sector by the lower sector boundary has the same volume as the cap of sphere (3) included in the sector by the upper sector boundary and the sector density remains constant as the sector is moved. Here, as in 1(b) and 1(c), $N = 2$.

1(e) At the limit of 1(d), sphere (4) is tangent to sphere (1). In this arrangement, the spheres have their centers on the vertices of tetrahedrons with common faces, and each vertex is on a cylinder of diameter d. If the cylinder diameter is increased, some sphere must lose contact with an adjacent sphere. There are many ways to choose the pairs of spheres which are no longer in contact. However, the following method provides

a way for the diameter to change as a continuous function of ζ . The tetrahedral arrangement of sphere centers in which the tetrahedrons have common faces changes into an octahedral arrangement of sphere centers in which the octahedrons have common faces. The octahedral arrangement is more readily recognized as a column of spheres with three tangent spheres in each plane perpendicular to the cylinder axis. The sphere centers in alternate layers are rotated 60 degrees about the cylinder axis.

Suppose in the tetrahedral arrangement of sphere centers described above the second sphere breaks contact with the third sphere, the fifth sphere breaks contact with the sixth sphere and in general the sphere $3K-1$ breaks contact with the sphere $3K$. Originally, every sphere was in contact with six adjacent spheres; now, sphere (3) is in contact with spheres (0), (1), (4), (5), (6); sphere (4) with (1), (2), (3), (5), (6), (7); sphere (5) with (2), (3), (4), (7), (8). Since spheres (0), (1), (2) are each in contact with the others, the lines connecting their centers are the sides of an equilateral triangle. Let the edge (0-1) have an axial projection of α ; also, let the edge (1-2) have an axial projection of α . The centers of spheres (3), (4), and (5) also form an equilateral triangle. Let the edge (3-4) have an axial projection of α and the edge (4-5) have an axial projection of α . These assignments of pitch to two edges of the equilateral triangles fix their orientation relative to the cylinder axis. Let the axial projection between spheres (0-3), (1-4), and (2-5) be ζ . Except for the angular rotation of the center of triangle (3-4-5) about the center of triangle (0-1-2), the arrangement is specified by these requirements.

Erect a right-handed x, y, z coordinate system (Fig. 3) at the mid-point of the unit segment from the center of sphere (1) to the center of sphere (4) so that the z axis is parallel to the cylinder axis. Let the $+y$ axis point in the direction of the projection of the directed segment (1-4) onto the $x-y$ plane. Then the y coordinate of point 4 is positive, the y coordinate of point 1 is negative. Vectors from the origin to points 1 and 4 have equal magnitude and zero x coordinate.

Let the coordinates of the six points be

(i) $P_i = (a_i, b_i, c_i), \quad i = 0, 1, 2, 3, 4, 5.$

Consider the z components.

$$(ii) \left\{ \begin{array}{l} c_1 - c_0 = \alpha \\ c_2 - c_1 = \alpha \\ c_4 - c_3 = \alpha \\ c_5 - c_4 = \alpha \\ c_3 - c_0 = \zeta \\ c_4 - c_1 = \zeta \\ c_5 - c_2 = \zeta \\ c_1 + c_4 = 0 \\ c_2 + c_3 = 0 \\ c_0 + c_5 = 0 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} c_0 = -\alpha - \frac{\zeta}{2} \\ c_1 = -\frac{\zeta}{2} \\ c_2 = -\frac{\zeta}{2} + \alpha \\ c_3 = \frac{\zeta}{2} - \alpha \\ c_4 = \frac{\zeta}{2} \\ c_5 = \frac{\zeta}{2} + \alpha \end{array} \right\}$$

The altitude of triangle (0-1-2) from vertex (1) lies in a plane parallel to the $x-y$ plane at $z = -\frac{\zeta}{2}$. The altitude of triangle (3-4-5) from vertex (4) lies in a plane parallel to the $x-y$ plane at $z = \frac{\zeta}{2}$. These facts will be needed later.

From the symmetry and location of the points,

$$(iii) \left\{ \begin{array}{l} a_1 = a_4 = 0, b_1 + b_4 = 0 \\ a_2 = a_3, b_2 + b_3 = 0 \\ a_0 = a_5, b_0 + b_5 = 0 \end{array} \right\} .$$

The distance between tangent spheres is unity. Consequently the following formulas provide useful relationships.

$$(iv) |\vec{P}_i - \vec{P}_j| = 1 \text{ for } (i,j) = (1,0), (2,0), (3,0), (2,1), (3,1), (4,1).$$

The equations iv are listed for convenience. Here, the values given by ii and iii have been substituted and the parameters $a_i, b_i, i = 3,4,5$ and $c_i, i = 0, 5$ have been eliminated.

$$(iv) \left\{ \begin{array}{l} (1,0), a_0^2 + (b_1 - b_0)^2 + \alpha^2 = 1 \\ (2,0), (a_2 - a_0)^2 + (b_2 - b_0)^2 + 4\alpha^2 = 1 \\ (3,0), (a_2 - a_0)^2 + (b_2 + b_0)^2 + \zeta^2 = 1 \\ (2,1), a_2^2 + (b_2 - b_1)^2 + \alpha^2 = 1 \\ (3,1), a_2^2 + (b_2 + b_1)^2 + (\zeta - \alpha)^2 = 1 \\ (4,1), 4b_1^2 + \zeta^2 = 1 \end{array} \right\} .$$

From (4,1) of iv,

$$(v) b_1 = -\frac{1}{2} \sqrt{1 - \zeta^2} \quad (\text{the choice of sign is dictated by the coordinate system}).$$

Eliminating a_2^2 from (2,1) and (3,1) and substituting v,

$$(vi) b_2 = \frac{1}{2} \frac{\zeta(\zeta - 2\alpha)}{\sqrt{1 - \zeta^2}} .$$

Eliminating $(a_2 - a_0)$ from (2,0), (3,0) and substituting vi,

$$(vii) b_0 = -\frac{1}{2} \frac{(\zeta + 2\alpha) \sqrt{1 - \zeta^2}}{\zeta} .$$

Computing the sums and differences for future reference,

$$(viii) \quad b_1 - b_0 = \frac{\alpha}{\zeta} \sqrt{1 - \zeta^2}$$

$$b_2 - b_0 = \frac{1}{2} \frac{\zeta + 2\alpha - 4\alpha\zeta^2}{\zeta \sqrt{1 - \zeta^2}}, \quad b_2 + b_0 = \frac{1}{2} \frac{2\zeta^3 - \zeta - 2\alpha}{\zeta \sqrt{1 - \zeta^2}}$$

$$b_2 - b_1 = \frac{1}{2} \frac{(1 - 2\alpha\zeta)}{\sqrt{1 - \zeta^2}}, \quad b_2 + b_1 = \frac{1}{2} \frac{2\zeta^2 - 2\alpha\zeta - 1}{\sqrt{1 - \zeta^2}}$$

Using the results of viii in iv,

$$(ix) \quad a_0 = \frac{\sqrt{\zeta^2 - \alpha^2}}{\zeta}, \quad a_2 = \frac{1}{2} \sqrt{\frac{3 - 4(\zeta^2 - \alpha\zeta + \alpha^2)}{1 - \zeta^2}}$$

The coordinates of the six points are now determined in terms of α and ζ .

Equation (3,0) of iv yields a relationship between α and ζ .

$$(x) \quad 3\zeta^3 - 9\alpha\zeta^2 + (8\alpha^2 - 2)\zeta - 4\alpha^3 + 4\alpha = 0$$

Equation x factors as follows:

$$(x) \quad \left(\alpha - \frac{\zeta}{2}\right) \left(\alpha - \frac{3\zeta}{4} - \sqrt{1 - \frac{15}{16}} \zeta^2\right) \left(\alpha - \frac{3\zeta}{4} + \sqrt{1 - \frac{15}{16}} \zeta^2\right) = 0$$

The initial point of the interval l(e) must satisfy one of these factors.

The values of α and ζ may be found from the analysis for l(d). There,

the axial distance from sphere (0) to sphere (1) is α . The axial distance from sphere (0) to sphere (2) at the lower bound is $\sqrt{\frac{2}{5}}$.

From xv of l(d), $\alpha = \sqrt{1/10}$. From ii of l(d) the axial distance from

sphere (0) to sphere (3) may be evaluated and is equivalent to the

upper limit on the current variable ζ . Therefore, the present

value of ζ is given by

$$(xi) \quad \zeta = (\zeta^* + \alpha^*) = \sqrt{\frac{2}{5}} + \sqrt{\frac{1}{10}} = 3 \sqrt{1/10}$$

$$\alpha = \alpha^* = \sqrt{1/10}$$

Here, α^* and ζ^* are equivalent to the variable α and ζ of l(d) and are

evaluated at the lower limit of ζ in l(d). By substituting the values

of xi and x, the third factor is the desired relationship between α and ζ .

$$(xii) \quad \alpha = \frac{3\zeta}{4} - \sqrt{1 - \frac{15\zeta^2}{16}}, \zeta \leq 3\sqrt{1/10}.$$

With xii all points of i are known in terms of ζ . The additional requirement is that these points satisfy the equation of a cylinder with its axis parallel to the z axis. By symmetry of the coordinate system, its center intersects the positive x axis at x_0 . The equation of the cylinder projected onto the x - y plane is

$$(xiii) \quad (x - x_0)^2 + y^2 = \frac{d^2}{4}.$$

Choosing $P_1 (a_1, b_1, c_1)$ for simplicity, we may evaluate x_0 :

$$(xiv) \quad x_0 = \frac{1}{2} \sqrt{d^2 - 1 + \zeta^2}.$$

Now we use the lengths of the projections of lines (1,0) and (2,1) onto the x - y plane. Note from iv that

$$a_0^2 + (b_1 - b_0)^2 = 1 - \alpha^2 = \alpha_2^2 + (b_2 - b_1)^2.$$

Therefore, the projection of triangle (0-1-2) onto the x - y plane is an isosceles triangle with sides of $\sqrt{1 - \zeta^2}$ and third side of $\sqrt{1 - 4\alpha^2}$. We may verify that the altitude from the third side is $\sqrt{3}/2$. Heron's formula gives the radius of the circle which circumscribes a triangle as the product of the sides of the triangle divided by four times the area of the triangle.

Therefore,

$$(xv) \quad d = \frac{2}{\sqrt{3}} (1 - \alpha^2).$$

Using xii in xv yields the desired relation between d and ζ .

$$(xvi) \quad d = \frac{\sqrt{3}}{4} \zeta (\zeta + \sqrt{16 - 15\zeta^2}).$$

This policy terminates when each sphere is again in contact with six adjacent spheres. Sphere (5) originally was in contact with spheres (2), (3), (4), (7), (8), and eventually must come in contact with

sphere (0). Sphere (3) was originally in contact with spheres (0), (1), (4), (5), (6), and eventually must come in contact with sphere (8).

The assumption, in the terminal arrangement, that the axial distance between the centers of spheres (0) and (5) is identical with the axial distance between the centers of spheres (1) and (4) requires by ii

that

$$(xvii) \quad \alpha = 0.$$

From xii we evaluate the lower limit on ζ , namely $\zeta \geq \sqrt{2/3}$. The bounds on ζ from xii and xvii when substituted in xvi yield the bounds on d for which xvi holds.

$$(xviii) \quad \frac{3\sqrt{3}}{5} \leq d \leq \frac{2}{\sqrt{3}}.$$

To show that ζ is a proper choice, note that the altitude of triangle (3-4-5) from vertex (4) lies in a plane parallel to the x-y plane at $z = \frac{\zeta}{2}$, and that the altitude of the triangle (0-1-2) lies in a plane parallel to the x-y plane at $z = -\frac{\zeta}{2}$. Furthermore, the altitudes pass through the tangent points of the spheres in their respective triangles. Hence the upper bound on ζ excludes as much volume from sphere (3) as it includes from sphere (5) and the lower bound on ζ excludes as much volume from sphere (2) as it includes from sphere (0) regardless of the placement of the upper sector boundary in the axial interval between the centers of spheres whose axial separation is ζ . Therefore ζ is a proper choice since its density remains constant regardless of placement in the cylinder of arranged spheres. Since half of sphere (1), half of sphere (4) and a volume equivalent to all of spheres (2) and (3) are in the sector, $N = 3$.

A graph of the relations between d and ζ given in 1(a) through 1(e) appears on the following page. The scales are in terms of d^2 vs ζ^2 to illustrate the interdependence of the relations. With a specified value for d^2 , a unique value of ζ^2 may be obtained from the portions of the curves which are shown in solid colors with the exception of a few discontinuities indicated by a change in color. At these discontinuities, two different values of ζ^2 specify the same geometrical arrangement. Equation II giving the average density for d in the range 0 to $2/\sqrt{3}$ is shown as a solid line on the graph following the next section on arrangements with discrete parameters. A cusp on the curve appears for the arrangement common to 1(d) and 1(e). These two graphs help illustrate the complexity of the relation between density and cylinder diameter.

FIGURE 1

FIGURE 2

FIGURE 3

Dense Arrangements With Discrete Parameters

The arrangements of spheres in $l(e)$ for which ζ takes on the boundary points of its interval are suggestive in both cases of the densest packing of circles in a plane. In the plane, the densest packing of circles has each circle tangent to six circles. The arrangements of spheres in both cases have six spheres tangent to each sphere. The arrangement of circles covers the plane whereas the arrangement of spheres covers the cylinder. We find the centers of circles lying on straight lines in the plane; then we should expect the centers of spheres to lie on helices in the cylinder since helices are geodesics of a cylinder. Suppose there exists an arrangement of spheres with centers on a cylinder of radius r and that each sphere is tangent to six spheres. Then there exist three helices passing through the center of each sphere. Choosing a reference sphere center, each of the three helices passing through the center of the reference sphere also passes through the centers of the spheres on opposite sides of the reference sphere.

Let a right-handed (x, y, z) coordinate system be established with the z axis co-linear with the axis of the cylinder and with the center of the reference sphere on the x axis. Let the three helices passing through the center of the reference sphere be represented by the coordinates:

$$(i) \quad P(\theta) = (r \cos \theta, r \sin \theta, \xi_i \theta), \quad -\infty < \theta, \xi_i < \infty, \quad i = 1, 2, 3.$$

Unit tangent vectors at any point on the helix have the coordinates:

$$(ii) \quad \frac{dP(\theta)}{ds_i} = (-r \sin \theta, r \cos \theta, \xi_i) \frac{d\theta}{ds_i}, \quad i = 1, 2, 3; \quad s_i \text{ being arclength.}$$

Since the tangent vectors have unit length,

$$(iii) \quad \frac{ds_i}{d\theta} = \sqrt{r^2 + \xi_i^2}, \quad i = 1, 2, 3.$$

Suppose the projected angle on the x - y plane from the first sphere center encountered on the i^{th} helix is $\theta = \gamma_i, i = 1, 2, 3$. Then iii may be integrated

with respect to θ to give the arclength between consecutive sphere centers on the i^{th} helix.

$$(iv) \quad s_i = \gamma_i \sqrt{r^2 + \xi_i^2}, \quad \gamma_i \geq 0, \quad i = 1, 2, 3.$$

Let $\vec{t}_i$ be tangent vectors to the helices at $\theta = 0$ with magnitude equal to the arclength between adjacent sphere centers on their respective helices. Then,

$$(v) \quad \vec{t}_i = \frac{d\vec{P}(0)}{ds} s_i = (0, r \gamma_i, \xi_i \gamma_i), \quad i = 1, 2, 3.$$

Since the $\vec{t}_i$ have no x component they are linearly dependent. Consequently a pair of them may be used to form a basis which defines every point in the rectification plane of the helices. In this rectification plane all helices which are on the cylinder become straight lines. The $\vec{t}_i$ are indistinguishable, therefore we may arbitrarily assign the parameters ξ_i and γ_i , $i = 1, 2, 3$.

Let $\vec{t}_1$ and $\vec{t}_2$ be the reference vectors where

$$(vi) \quad \left\{ \begin{array}{l} \xi_1 < 0, h_1 = |\xi_1| \\ \xi_2 \geq 0, h_2 = \xi_2 \\ \xi_3 > 0, h_3 = \xi_3 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} \vec{t}_1 = (0, r \gamma_1, -h_1 \gamma_1) \\ \vec{t}_2 = (0, r \gamma_2, h_2 \gamma_2) \\ \vec{t}_3 = (0, r \gamma_3, h_3 \gamma_3) \end{array} \right\}.$$

Choose the ξ_i and γ_i such that

$$(vii) \quad \vec{t}_2 = \vec{t}_1 + \vec{t}_3.$$

The combination of vi and vii imply

$$(viii) \quad \left\{ \begin{array}{l} \gamma_3 = \gamma_1 + \gamma_2 \\ h_3 \gamma_3 = h_1 \gamma_1 + h_2 \gamma_2 \end{array} \right\}.$$

The above choices orient the triangle in the rectification plane so that the reference sphere center, the first sphere center on helix one and the first sphere center on helix two form a triangle. This triangle lies in the rectification plane which is parallel to the y-z plane. The y coordinates of the first sphere centers on helices one and two are positive. The z coordinates of these same centers have opposite signs. This choice is dictated by the additional

constraint of regularity in the arrangement of unit sphere, namely that every sphere be in contact with exactly six adjacent spheres. This implies a path around the cylinder which begins and ends on the reference sphere. For example this path may be chosen on helix one from the reference sphere center to sphere center p, then on helix two from sphere center p to sphere center q. The point (p,q) in the rectification plane is removed a distance equal to $2\pi r$ units from the point (0,0) and falls on the +y axis. This new restriction is an additional linear relation on the vectors $\vec{t}_1$ and $\vec{t}_2$.

$$(ix) \quad p \vec{t}_1 + q \vec{t}_2 = (0, 2\pi r, 0).$$

The component equations of ix require

$$(x) \quad \left\{ \begin{array}{l} p \gamma_1 + q \gamma_2 = 2\pi \\ p \gamma_1 h_1 = q \gamma_2 h_2 \end{array} \right\}.$$

It seems natural to assign integer coordinates to points in the rectification plane. These integer coordinates locate the centers of spheres relative to the reference sphere center. However, it also seems natural to assign numbers to each sphere in the original arrangement of spheres according to the magnitude of their z component relative to the reference sphere. Then sphere (l) in the arrangement of spheres has coordinates (i,j) in the rectification plane. We seek a relation between the integers (i,j) and (l). The index i runs along a helix which has an intercept with each helix having tangent vector $\vec{t}_2$. There are (p-1) such parallel helices in the arrangement. The spherical caps of spheres with centers on these helices cover the cylinder. Sphere (l) must then lie on one of these helices at coordinates which depend on (i) and (j). The path from the reference sphere to sphere (l) must consist of (i) steps on helix one and (j) steps on helix two. Then sphere center (l) must have coordinates

$$(xi) \quad \vec{P}_l (i \gamma_1 + j \gamma_2) = \{r \cos (i \gamma_1 + j \gamma_2), r \sin (i \gamma_1 + j \gamma_2), j h_2 \gamma_2 - i h_1 \gamma_1\}.$$

The index (ℓ) may be generated by ordering the z components of P_ℓ . Let (a_ℓ, b_ℓ, c_ℓ) be the components of the position of sphere (ℓ). The equations of the straight lines in the rectification plane representing the helices with tangent vectors parallel to $\vec{t}_2$ must satisfy the points of the rectification plane $T_0(Y, z) = v\vec{t}_1$ and $T_1(Y, z) = v\vec{t}_1 + \vec{t}_2$, $v = 0, 1, 2, \dots, p-1$. In terms of components,

$$(xii) \quad \left\{ \begin{array}{l} T_0(Y, z) = (vr \gamma_1, -vh_1\gamma_1) \\ T_1(Y, z) = (vr \gamma_1 + r\gamma_2, h_2\gamma_2 - vh_1\gamma_1) \end{array} \right\}, \quad v = 0, 1, 2, \dots, p-1.$$

Here, the Y component measures arclength around the cylinder and is related to θ in the following way:

$$(xiii) \quad Y = r \theta.$$

The variable y in the (x, y, z) coordinate system has been previously specified in i. Hence the relation between (x, y, z) and (Y, z) is

$$(xiv) \quad \left\{ \begin{array}{l} x = r \cos \frac{Y}{r} \\ y = r \sin \frac{Y}{r} \\ z = z \end{array} \right\}.$$

The straight lines in (Y, z) corresponding to helices in (x, y, z) are

$$(xv) \quad Y = \frac{r}{h_2} z + r \gamma_1 \left(\frac{h_1}{h_2} + 1 \right), \quad v = 0, 1, 2, \dots, p-1.$$

Consequently,

$$(xvi) \quad \theta = \frac{z}{h_2} + v\gamma_1 \left(\frac{h_1}{h_2} + 1 \right), \quad v = 0, 1, 2, \dots, p-1.$$

Setting $z = 0$ in (xvi) yields the θ intercepts of the helices.

$$(xvii) \quad \theta_v = v \gamma_1 \left(\frac{h_1}{h_2} + 1 \right).$$

In view of x , xvii also may be represented by

$$(xvii) \quad \theta_v = \frac{2\pi v}{p}, \quad v = 0, 1, 2, \dots, p-1.$$

Using xvii in xvi and xv,

$$(xvi) \quad \theta = \frac{z}{h_2} + \frac{2\pi v}{p}, \quad v = 0, 1, 2, \dots, p-1.$$

$$(xv) \quad Y = r \left(\frac{z}{h_2} + \frac{2\pi v}{p} \right), \quad v = 0, 1, 2, \dots, p-1.$$

This mapping of points on the cylinder to points on the rectification plane results in many points (Y,z) for a single point (x,y,z) because of the multiple valued character of the periodic functions. If we choose to retain continuous helices as a function of position, then xvi yields the correct choice for the parameter θ . Consequently, corresponding points are uniquely defined in terms of z only. Then if we have a point $T(Y_{ij}, z_{ij}) = i \vec{t}_1 + j \vec{t}_2$, the corresponding point is $P(x_{ij}, y_{ij}, z_{ij})$, where $i = 0, 1, 2, \dots, p-1$; $j = 0, 1, 2, \dots$. Writing out the components,

$$(xviii) \quad T[r(i \gamma_1 + j \gamma_2), j h_2 \gamma_2 - i h_1 \gamma_1] \neq P[r \cos(i \gamma_1 + j \gamma_2), r \sin(i \gamma_1 + j \gamma_2), j \gamma_2 h_2 - i \gamma_1 h_1],$$

$$i = 0, 1, 2, \dots, p-1, j = 0, 1, 2, \dots, 0 \leq h_1, h_2 < \infty, 0 < \gamma_1, \gamma_2.$$

One set of restrictions on the parameters $r, p, q, \gamma_i, h_i, i = 1, 2, 3$ are needed. These are the stipulations that each unit sphere be tangent to six surrounding unit spheres. Consequently the distance separating their centers is unity in the (x,y,z) coordinate system. Let the coordinates of the central sphere be the integers i and j . And let the coordinates of the surrounding spheres be μ and v . Then

$$(xix) \quad |\vec{P}(i,j) - \vec{P}(\mu,v)| = 1, \mu - i = \pm 1 \text{ and } v = j, \mu = i \text{ and } v - j = \pm 1,$$

$$\mu - i = 1 \text{ and } v - j = 1, \mu - i = -1, v - j = -1.$$

Substituting the components of $P(i,j)$ given in xviii yields three independent equations from xix:

$$(xx) \quad \left\{ \begin{array}{l} 2r^2 \{1 - \cos \gamma_i\} + (h_i \gamma_i)^2 = 1, \\ \text{or, equivalently,} \\ d^2 \sin^2 \frac{\gamma_i}{2} + (h_i \gamma_i)^2 = 1, \end{array} \right\} \quad , i = 1, 2, 3.$$

Some fundamental inequalities are needed to eliminate extraneous solutions. In iv, arclength was computed using the parameter ξ_i^2 which is equivalent to

$$h_i^2, i = 1, 2, 3.$$

$$(iv) \quad s_i = \gamma_i \sqrt{r^2 + h_i^2}, \quad i = 1, 2, 3.$$

Let us restrict iv to obey some fundamental geometric concepts. If two unit spheres are on a helix with infinite pitch then the arclength between their centers is unity (since we have maintained tangency). If three unit spheres are tangent with centers on a cylinder of radius r , then the arclengths between their centers forms a triangle in the rectification plane.

Consequently, if two tangent unit spheres have their centers on a cylinder of radius r , then the minimum arclength(s) between sphere centers is bounded:

$$(xxi) \quad 1 \leq s_i \leq \gamma_i r, \quad i = 1, 2, 3.$$

Let three unit spheres be tangent with centers on a cylinder of radius r and choose the arclengths s_i to obey the inequalities

$$(xxii) \quad 1 \leq s_3 \leq s_1 \leq s_2 \leq 2\pi r/q, \quad r \geq \frac{3\sqrt{3}}{10}.$$

The equations which specify the geometry of the cylindrical arrangement of spheres are:

$$(vii) \quad \left\{ \begin{array}{l} \gamma_1 - \gamma_2 + \gamma_3 = 0 \\ h_1 \gamma_1 + h_2 \gamma_2 - h_3 \gamma_3 = 0 \end{array} \right\},$$

$$(x) \quad \left\{ \begin{array}{l} p \gamma_1 + q \gamma_2 = 2\pi \\ ph_1 \gamma_1 - q h_2 \gamma_2 = 0 \end{array} \right\},$$

and

$$(xx) \quad d^2 \sin^2 \frac{\gamma_i}{2} + (h_i \gamma_i)^2 = 1, \quad i = 1, 2, 3.$$

Equations vii and x are used to develop the following relations,

$$(xxiii) \left\{ \begin{array}{l} \gamma_1 = \frac{2\pi - q \gamma_2}{p} = \frac{2\pi - q \gamma_3}{p+q} \\ \gamma_2 = \frac{2\pi - p \gamma_1}{q} = \frac{2\pi + p \gamma_3}{p+q} \\ \gamma_3 = \frac{2\pi - (p+q)\gamma_1}{q} = \frac{(p+q)\gamma_2 - 2\pi}{p} \\ h_1 \gamma_1 = \frac{q}{p} h_2 \gamma_2 = \frac{q}{p+q} h_3 \gamma_3 \\ h_2 \gamma_2 = \frac{p}{q} h_1 \gamma_1 = \frac{p}{p+q} h_3 \gamma_3 \\ h_3 \gamma_3 = \frac{p+q}{q} h_1 \gamma_1 = \frac{p+q}{p} h_2 \gamma_2 \end{array} \right.$$

These relations xxiii are used to put the equations xx in terms of γ_i and h_i , $i = 1, 2, 3$. Choosing to evaluate γ_3 , we substitute for γ_i , $h_i \gamma_i$, $i = 1, 2$, and $d = 2r$ in xx to obtain

$$(xx) \left\{ \begin{array}{l} 2r^2 \left\{ 1 - \cos \left(\frac{2\pi - q\gamma_3}{p+q} \right) \right\} + \left(\frac{q}{p+q} h_3 \gamma_3 \right)^2 = 1 \\ 2r^2 \left\{ 1 - \cos \left(\frac{2\pi + p\gamma_3}{p+q} \right) \right\} + \left(\frac{p}{p+q} h_3 \gamma_3 \right)^2 = 1 \\ 2r^2 \{ 1 - \cos \gamma_3 \} + (h_3 \gamma_3)^2 = 1 \end{array} \right.$$

Eliminating $2r^2$ and $(h_3 \gamma_3)^2$ from xx,

$$(xxiv) \left| \begin{array}{ccc} \cos \left\{ \frac{2\pi - q\gamma_3}{p+q} \right\} & \left(\frac{q}{p+q} \right)^2 & 1 \\ \cos \left\{ \frac{2\pi + p\gamma_3}{p+q} \right\} & \left(\frac{p}{p+q} \right)^2 & 1 \\ \cos \gamma_3 & 1 & 1 \end{array} \right| = 0.$$

Obtaining the value of γ_3 from xxiv for specified integers p and q will, in general, require the use of numerical techniques. One geometrically significant value of γ_3 satisfying xxiv should fall in the interval $0 \leq \gamma_3 \leq \frac{\pi}{q}$.

If we have evaluated r , γ_2 , and h_i , $i = 1, 2, 3$, for some specified whole numbers p and q , then we may evaluate the density of the arrangement in terms of these known parameters. In the correspondences given by xviii, we replace r , i , and j by the variables ρ , U , and V respectively

$$\text{xxv} \quad T [\rho (\gamma_1 U + \gamma_2 V), h_1 \gamma_1 V - h_1 \gamma_1 U] \quad \ddagger$$

$$P [\rho \cos (\gamma_1 U + \gamma_2 V), \rho \sin (\gamma_1 U + \gamma_2 V), h_2 \gamma_2 V - h_1 \gamma_1 U],$$

$$r - \frac{1}{2} \leq \rho \leq r + \frac{1}{2}, \quad 0 \leq U < p, \quad \text{for all } V, \text{ and}$$

for arbitrarily fixed p , q , r , γ_i , and h_i , $i = 1, 2, 3$.

The left side xxv gives the x , y , z components of the locus of points on helicoids in the x , y , z coordinate system. These helicoids correspond to lines in the rectification plane given by the right side of xxv. Setting U and ρ constant yields lines with tangent vector $\vec{t}_2$ in the rectification plane and corresponding helices in x , y , z coordinates. Setting V and ρ constant yields lines with tangent vector $\vec{t}_1$ in the rectification plane and corresponding helices in x , y , z coordinates. The parallelopiped with vertices at the intersection points of surfaces $\rho = r - \frac{1}{2}$, $r + \frac{1}{2}$; $U, V = 0, 1$ contains one sphere. The density of this parallelopiped is equivalent to the average density of spheres in the cylindrical shell specified by the above values of ρ . Consequently we need only compute the volume of the parallelopiped in the x , y , z coordinate system and divide the result into the volume of a sphere to obtain the average density for any given arrangement having every sphere tangent to six spheres.

$$(xxvi) \quad V = \int_{\rho=r-\frac{1}{2}}^{r+\frac{1}{2}} \int_{u=0}^1 \int_{v=0}^1 J\left(\frac{x,y,z}{\rho,u,v}\right) d\rho du dv$$

where V is the volume of the parallelopiped and $J\left(\frac{x,y,z}{\rho,u,v}\right) = \gamma_1 \gamma_2 (h_1 + h_2) \rho$ is the Jacobian evaluated using the x, y, z components of the left side of XXV. Evaluating V in terms of the parameters,

$$(xxvi) \quad V = \gamma_1 \gamma_2 (h_1 + h_2) r$$

The density is found to be

$$(xxvii) \quad D = \frac{\Pi}{6 r (h_1 + h_2) \gamma_1 \gamma_2}$$

Using relations xxiii in xxvii to obtain density in terms of sphere center diameter d , and axial height $h_i \gamma_i$, $i = 1, 2, 3$, we compare the result of this computation with density given by II for $d \geq 1$. Finally xxvii becomes,

$$(xxvii) \quad D = \frac{q}{6 h_1 \gamma_1 d} = \frac{p}{6 h_2 \gamma_2 d} = \frac{p+q}{6 h_3 \gamma_3 d} = \frac{N_i}{6 \zeta_i d}$$

From the comparison, we find the proper choice of axial height $\zeta_i = h_i \gamma_i$, $i = 1, 2, 3$, and the corresponding number of spheres in the sector to be $N_1 = q$, $N_2 = p$, and $N_3 = p + q$ respectively.

Particular classes of solutions to the set of equations vii, x, and xx are immediately available. One class of arrangements occurs when the slope of a helix is zero, resulting in sphere centers lying in planes perpendicular to the cylindrical axis. The other class of solutions occurs when the slope of a helix is infinite.

Arrangements having helices with zero pitch occur when the arclength S_2 takes its maximum value according to inequality xxii and $h_2 = 0$. Using the geometrically obvious fact that h_1 and γ_1 differ from zero when $h_2 = 0$ for sufficiently large q , equations x dictate $p = 0$, and $\gamma_2 = 2\pi/q$. Using these results in vii, $\gamma_1 + \gamma_3 = \frac{2\pi}{q}$ and $h_1 \gamma_1 = h_3 \gamma_3$. Comparing equations i = 1, and i = 3 of xx, we find that $\cos \gamma_1 = \cos \gamma_3$. If we restrict $\gamma_1, \gamma_3 < \pi$, then $\gamma_1 = \gamma_3$ since the angles have been designated positive. Also, this restriction requires $q \geq 2$.

Tabulating the values of the parameters for this class of arrangements for

$$q = 2, 3, 4, \dots, p = 0,$$

$$(xxviii) \quad \gamma_1 = \gamma_3 = \frac{\pi}{q}, \quad \gamma_2 = 2\pi/q,$$

$$(xxix) \quad h_1 = h_3 = \frac{q}{\pi} \sqrt{1 - \frac{\sin^2 \frac{\pi}{2q}}{\sin^2 \frac{\pi}{q}}}, \quad h_2 = 0,$$

$$(xxx) \quad s_1 = s_3 = \sqrt{\left(\frac{\frac{\pi}{2q}}{\sin \frac{\pi}{q}}\right)^2 + 1 - \left(\frac{\sin \frac{\pi}{2q}}{\sin \frac{\pi}{q}}\right)^2}, \quad s_2 = \frac{\frac{\pi}{q}}{\sin \frac{\pi}{q}}.$$

From equation i = 2 of xx,

$$(xxxi) \quad d = \frac{1}{\sin \frac{\pi}{q}}.$$

The proper choice for the axial thickness ζ of a segment with density equivalent to the average density is

$$(xxxii) \quad \zeta = h_3 \gamma_3 = \sqrt{1 - \frac{\sin^2 \frac{\pi}{2q}}{\sin^2 \frac{\pi}{q}}}.$$

The number of spheres in the sector is

$$N = q.$$

The average density for $d \geq 1$ is

$$(xxxiii) \quad D = \frac{N}{6\zeta d} = \frac{1}{6} \frac{q \sin \frac{\pi}{q}}{1 - \frac{\sin^2 \frac{\pi}{2q}}{\sin^2 \frac{\pi}{q}}}$$

It is reassuring to note that

$$\lim_{q \rightarrow \infty} D = \frac{\pi}{3\sqrt{3}}$$

equivalent to the density of a sphere inscribed in a hexagonal prism.

Arrangements having helices with infinite pitch occur when the arclength

$s_3 = 1$ for which $h_3 = \infty$. These values substituted into iv for finite r yield

$$1 = \lim_{h_3 \rightarrow \infty} s_3 = h_3 \gamma_3, \text{ implying } \gamma_3 = 0.$$

Evaluating vii, $\gamma_1 = \gamma_2$ and $h_1 \gamma_1 + h_2 \gamma_2 = 1$.

Comparing equations $i = 1, 2$ of xx, $h_1 = h_2$.

Tabulating the values of the parameters for this class of arrangements for $p = q = 1, 2, 3, \dots$

$$(xxxiv) \quad \gamma_1 = \gamma_2 = \frac{\pi}{p}, \gamma_3 = 0.$$

$$(xxxv) \quad h_1 = h_2 = \frac{p}{2\pi}, h_3 = \infty.$$

$$(xxxvi) \quad s_1 = s_2 = \frac{1}{2} \sqrt{3 \left(\frac{\frac{\pi}{2p}}{\sin \frac{\pi}{2p}} \right)^2 + 1}, \quad s_3 = 1.$$

From equation $i = 1$ of xx,

$$(xxxvii) \quad d = \frac{\sqrt{3}}{2} \frac{1}{\sin \frac{\pi}{2p}}.$$

The proper choice for the axial thickness ζ of a segment with density equivalent to the average density is

$$\zeta = h_3 \gamma_3 = 1 .$$

The number of spheres in the sector is

$$N = 2p .$$

The average density for $d = \frac{\sqrt{3}}{2}$, $p = 1$ is

$$(xxxviii) \quad D = \frac{2}{3} \frac{N}{\zeta(1+d)^2} = \frac{16}{3} (7-4\sqrt{3}) \approx 0.3829 .$$

The average density for $d > 1$ is

$$(xxxix) \quad D = \frac{N}{6\zeta d} = \frac{2p \sin \frac{\pi}{2p}}{3\sqrt{3}} .$$

It is more reassuring to note that

$$\lim_{p \rightarrow \infty} D = \frac{\pi}{3\sqrt{3}} ,$$

equivalent to the result obtained for the class of arrangements containing helices with zero pitch.

The class of arrangements containing helices with zero pitch will be used in the following section to evaluate models consisting of one such arrangement nested inside another arrangement from the same class. Other nested arrangements of uniform spheres may be explored by following an analytical procedure similar to the one given in the next section.

Values of the average shell density obtained using equation II in the section on arrangements with variable parameters appear as points in the graph on the next page. Points indicated with circles represent arrangements having helices with zero pitch. Points indicated with plus signs represent arrangements having helices with infinite pitch. Finally, points indicated with X's represent arrangements with helices with finite non-zero pitch.

Models For The Dense Packing of Particles in Cylindrical Shells

The packing of particles in cylindrical shells has produced fuel elements with acceptable densities when the component screen sizes and relative weight percentages of each size were calculated from models of uniform spheres packed regularly in a cylindrical shell. The heuristic argument in favor of such a model is based on the premise that no particles are as difficult to pack to a high density as are uniform spheres in a cylindrical container. If a spherical model can be formulated with a calculated density in the range of interest, then replacing each sphere with a convex particle and vibrationally compacting the aggregate should occasionally produce a fuel element with density in excess of the model density. The sequel presents an algorithm for evaluating sizes and corresponding weight-percentages of particles to be used as initial mixes in the fabrication of vibrationally compacted fuel elements. These initial mixes should be refined by experimentation to take advantage of other probably more sensitive factors in the process of fuel element fabrication.

Spherical Model

The most dense arrangement of uniform spheres of arbitrary diameter in a given cylindrical shell is not known analytically. However, we have found the most dense cylindrical layer of uniform spheres in the preceding section. If we use a single layer of spheres, the tolerances on sphere diameter and cylinder diameter become prohibitive. If we use many layers, then the particles must be small. Consequently the second and third sizes must be correspondingly small to fit within the void structures of the largest size. As the diameter of the particles is decreased, screening and compacting become increasingly difficult. These conditions dictate the choice of a model having as its first size a set of uniform spheres with maximum diameter that will form two cylindrical (not necessarily concentric) shells of spheres within the given hollow cylindrical container.

The analysis is accomplished using unit spheres in cylinders of arbitrary diameter as in the preceeding section. Conversion of the results to given cylindrical dimensions is performed on the analytical results.

Suppose we are given an arrangement of equal spheres specified by the integers $p = 0, q = n$. We seek the arrangement $p = 0, q = n-k$ which for minimum k will occupy the interior of the given arrangement with periodic interference between the two shells. We use the equations xxviii through xxxiii when applicable in the analysis and extend the equations to include the arrangement $p = 0, q = 1$ with the following definitions:

Diameter of cylinder containing sphere centers,

$$i) \quad d(q) = \left\{ \begin{array}{l} 0, q = 1, \\ \csc(\frac{\pi}{q}), q = 2, 3, 4, \dots \end{array} \right\}.$$

Axial height between centers of adjacent layers of spheres.

$$ii) \quad z(q) = \left\{ \begin{array}{l} 1, q = 1, \\ \frac{1}{\sqrt{\sin(\frac{\pi}{2q}) \sin(\frac{3\pi}{2q})} \csc(\frac{\pi}{q}), q = 2, 3, 4, \dots \end{array} \right\}.$$

Density of spheres in annular region between tangent cylinders.

$$iii) \quad D(q) = \left\{ \begin{array}{l} 2/3, q = 1, \\ \frac{q \sin^2(\frac{\pi}{q})}{6 \sqrt{\sin(\frac{\pi}{2q}) \sin(\frac{3\pi}{2q})}}, q = 2, 3, 4, \dots \end{array} \right\}.$$

The difference between the diameters of the two cylinders $d(n) - d(n-k)$ is bounded by the following conditions:

- (a) No interference of spheres in one shell with spheres in another shell occurs when a sphere in one shell and a sphere in the other shell are tangent and their centers lie on a radius of the cylinder.

- (b) Maximum interference of spheres in one shell with spheres in another shell occurs when the lines connecting the centers of three mutually tangent spheres in the outer shell form a regular tetrahedron with the center of a sphere in the inner shell.

These conditions are fulfilled algebraically by

$$(iv) \quad \frac{2}{3} \tan \frac{\pi}{2n} + \frac{2\sqrt{2}}{3} \sqrt{4 - \sec^2 \frac{\pi}{2n}} \leq d(n) - d(n-k) \leq 2.$$

As $n \rightarrow \infty$, the left side of iv $\rightarrow 2\sqrt{2}/3$. In the limit, the difference is bounded implying an estimate of k for sufficiently large n . Using the absolutely and uniformly converging series

$$(v) \quad \csc x = \frac{1}{x} + \frac{x}{3!} + \frac{7x}{3 \cdot 5!} + \frac{31x}{3 \cdot 5!} + \dots, \quad -\pi < x < \pi,$$

we are able to obtain the estimate of k for large n .

$$(vi) \quad 2\sqrt{\frac{2}{3}} \leq \csc \frac{\pi}{n} - \csc \frac{\pi}{n-k} \leq 2$$

$$\csc \frac{\pi}{n} - \csc \frac{\pi}{n-k} = \frac{n}{\pi} - \frac{n-k}{\pi} + o(o) = \frac{k}{\pi} + o(o).$$

Finally,

$$(vii) \quad 2\sqrt{\frac{2}{3}} \pi \leq k \leq 2\pi.$$

Since k is an integer, the only choice is $k = 6$ for large n . To determine when n is sufficiently large, we solve inequality iv for k and evaluate the bounds on k for assigned values of n .

$$(viii) \quad n - \frac{\pi}{\sin^{-1} \left[\frac{1}{d(n) - f(n)} \right]} \leq k \leq n - \frac{\pi}{\sin^{-1} \left[\frac{1}{d(n) - 2} \right]},$$

where $f(n)$ is equal to the left side of iv. In this way, we find $k = 6$ for $n \geq 9$.

there exist particular values of the difference $d_{(n)} - d_{(n-k)}$ which are important. These are:

- (a) $n = 6, k = 1$ for which iv assumes its upper bound,
- (b) $n = 12, k = 6$ for which the difference is $\sqrt{3}$.

In these two arrangements, there exists no interference between spheres of the outer shell with spheres of the inner shell. In all other cases where $n \geq 9, k = 6$, layers in the inner shell of spheres are separated by layers of the outer shell of spheres as a result of the interference between spheres of the outer shell with spheres of the inner shell. Noting that $\zeta(q+1) > \zeta(q)$ for $q = 2, 3, 4, \dots$, we see that each successive layer of an outer shell could support a corresponding layer of an inner shell. The axial separation between planes passing through the centers of adjacent layers of spheres in the outer supporting shell; i.e., the axial separation $\zeta(q)$ tends to propagate radially inward from outer shells to inner shells. The density of m nested cylindrical arrangements of equal spheres is bounded from below by assuming $\zeta(n)$ for every shell and from above by assuming each of the m shells has a $\zeta(n-6k), k = 0, 1, 2, \dots, m-1$, given by equation ii. The bounds are computed from the weighted average of the individual shell densities. The weights for the upper bound are the shell areas ($\Pi d(n-6k), k = 0, 1, 2, \dots, m-1$). The weights for the lower bound are the shell areas augmented by the ratio of layer heights ($\Pi d(n-6k) \zeta(n-6k)/\zeta(n), k=0,1,2,\dots,m-1$), thus accounting for the loss of density caused by the inward propagation of $\zeta(n)$. Dividing the sum of weighted densities by the total area yields formulas for the bounds on the composite density $D(m,n)$ of m shells of equal spheres.

$$(ix) \quad D(m,n) \left\{ \begin{array}{l} \geq \frac{\sum_{k=0}^{m-1} D(n-6k)d(n-6k)\zeta(n-6k)/\zeta(n)}{[d(n)+d(n-6m+6)][2+d(n)-d(n-6m+6)]} \\ \\ \leq \frac{\sum_{k=0}^{m-1} D(n-6k)d(n-6k)}{[d(n) + d(n-6m+6)][2+d(n) - d(n) - d(n-6m+6)]} \end{array} \right.$$

$$n - m \geq 8, n > 6m, m = 1, 2, 3, \dots$$

The upper and lower bound on $D(m,n)$ approach the same number as n becomes large.

$$(x) \quad D(m,\infty) = \frac{\pi^2}{9\sqrt{3}} \left(\frac{m}{m + \frac{\pi}{3} - 1} \right)$$

As m becomes large, $D(m,\infty)$ approaches an asymptote representing the density of nested cylindrical arrangements of spheres in space.

$$(xi) \quad D(\infty,\infty) = \pi^2/9 \sqrt{3} \approx .6331$$

Comparing $D(\infty,\infty)$ with the empirical value $D(\infty,\infty) \approx .635 \pm .01$ obtained experimentally [9] we may conclude that in the process of filling cylindrical containers with equal spheres, the cylindrical boundary influence is manifest in the arrangement of all spheres in the container causing the spheres to arrange themselves in nested cylindrical shells rather than the more dense rhombohedral pattern.

Model Parameter Evaluation

The analysis was conducted using unit spheres in cylinders of arbitrary diameter. The physical problem of filling a given cylindrical container demands cylinders of given diameter and sphere diameters to be determined from the geometry of a dense model having two shells of spheres in the annular region between two concentric cylinders. Let a be the outside diameter, b be the inside

diameter of the annulus to be filled, δ be the diameter of the spheres where n is the smallest whole number of spheres in a layer of the outer shell such that

$$(xii) \quad \left\{ \begin{array}{l} \delta[d(n)+1] = a, \\ \delta[d(n-6)-1] > b \end{array} \right\}.$$

Taking the ratio $r = b/a$ eliminates δ and permits evaluation of n by substitution for $d(n)$ and $d(n-6)$ from i.

$$(xiii) \quad \frac{-1 + \csc \frac{\pi}{n-6}}{1 + \csc \frac{\pi}{n}} \geq r, \quad n \text{ minimum}.$$

By choosing n in this way, we match the outer boundary and allow for a gap at the inner boundary. If $n(r)$ satisfies xiii, then

$$(xiv) \quad \delta = \frac{a}{1 + \csc \frac{\pi}{n(r)}}.$$

Table 1 has been evaluated by assigning whole numbers of $n(r)$ and evaluating r , δ/a , and the smaller sizes per unit of a .

The determination of the diameters of the spheres which fill the interstitial voids of the rhombohedral lattice has been studied by Dereziwicz [2] and Hudson [10]. These studies indicate the diameters of the spheres which best fill the square and triangular interstices to be $0.1782 \delta_n$ and $0.1716 \delta_n$ respectively. In consideration of the non-spherical nature of the particles and the difficulties of obtaining uniform particle sizes using screening methods, the dependence of density on the second and third size fractions is obscured. The vibratory techniques used to compact the mix could be expected to grind the second and third sizes to sizes compatible with the interstitial void structure.

Therefore, we have chosen the second and third particle diameters to be $0.1782 \delta/a$ and $(0.1782)^2 \delta/a$ respectively with the thought that designed statistical experiments may indicate better choices. Equation ix yields densities of the order 61.5% for the first size if $n \geq 12$. Table 5 of [11] gives a value of 37.14 for the percentage of void space filled by the second size which agrees with the experimental results of [9]. Empirical results indicate a value of 60 for the percentage of remaining void space filled by the third size. These factors combine to give an overall density of 90% for the mix.

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The numerical analysis of Mr. C. A. Oster in preparing Table I disclosed the existence of inequality viii page 40 resulting in the development of the section entitled "Models for the Dense Packing of Particles in Cylindrical Shells." His help was invaluable.

TABLE I

PARTICLE SIZE DISTRIBUTIONS FOR CYLINDRICAL COMPACTS

<u>ID/OD of Fuel Tube</u>		<u>Particle Size/OD of Fuel Tube</u>		
<u>Min.</u>	<u>Max.</u>	<u>Course</u> <u>(68 wt%)</u>	<u>Medium</u> <u>(15 wt%)</u>	<u>Fine</u> <u>(17 wt%)</u>
.1701	.2056	.2056	.0366	.0065 *
.2215	.2520	.1931	.0344	.0061
.2674	.2936	.1820	.0324	.0058
.3097	.3311	.1721	.0307	.0055
.3475	.3650	.1632	.0291	.0052
.3814	.3957	.1552	.0277	.0049
.4120	.4237	.1480	.0264	.0047 *
.4398	.4492	.1413	.0252	.0045
.4651	.4726	.1353	.0241	.0043
.4882	.4942	.1297	.0231	.0041
.5094	.5140	.1246	.0222	.0040
.5290	.5324	.1198	.0214	.0038
.5470	.5494	.1155	.0206	.0037 *
.5638	.5653	.1114	.0198	.0035
.5793	.5801	.1076	.0192	.0034
.5938	.5939	.1040	.0185	.0033
0		.3333	.0594	.0106 **

* Rods or annular compacts

** Rods only

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