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INVERSION OF THE LEVEL MATRIX IN R-MATRIX
THEORY BY THE METHOD OF RANK ANNIHILATION

T. Watanabe

IDAHO NUCLEAR CORPORATION
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ABSTRACT

The level matrix approach to the problem of matrix inversion in R-matrix theory still requires inverting the level matrix if the collision matrix is to be used for calculating cross sections. It is shown that the method of rank annihilation inverts the level matrix in analytical form for the general case. The general results are then specialized to the cases of two and three channels, and the expressions for the scattering and fission cross sections are obtained. Although the results of the specialized treatment are applied to the two level situation, the relevant expressions are in a form that allows a generalization to any number of levels (channels are still restricted to 2 and 3 in number). The approximations, which can be made to simplify the reduced widths and which can eliminate off-diagonal channels as well as the application of these modifications, will be discussed.

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INVERSION OF THE LEVEL MATRIX IN R-MATRIX THEORY BY THE METHOD OF RANK ANNIHILATION

I. INTRODUCTION

In the resonance theory of Wigner and Eisenbud^[1], the connection between the collision matrix and nuclear parameters is expressed through a quantity called the R-matrix. The relationship between the collision matrix and the R-matrix is a matrix equation in which it is necessary to invert a matrix, called the channel matrix, whose rows and columns refer to channels. To avoid this particular matrix inversion problem, Thomas^[2] has shown that it is possible to transform the matrix equation relating the collision matrix of Wigner-Eisenbud and the R-matrix into a form which no longer contains the R-matrix explicitly. However, the problem of matrix inversion is still not avoided. In the latter form of the collision matrix, a matrix termed "the level matrix" must be inverted. The purpose of this report is to describe the application of a matrix-inversion technique termed "method of rank annihilation" to invert the level matrix and to point out some approximations which can be made with this method of inverting the level matrix.

II. THE COLLISION MATRIX

Expressed with the use of the Lane and Thomas^[3] notation, the relationship between the derivative matrix (R) and the collision matrix (U) has the following form:

$$U^J = \Omega W^J \Omega \quad (1)$$

with

$$W^J = I + \frac{P^{1/2} R^J P^{1/2} \omega}{[I - R^J L^O]} \quad (2)$$

I is the unit matrix and the quantities Ω , $P^{1/2}$, L^O , and ω are "surface" diagonal matrices whose elements contain the properties of the external regions in the various channels. The R, or derivative, matrix contains the specifically nuclear terms, and has the following form:

$$R = \sum_{\lambda} \frac{\gamma_{\lambda} \times \gamma_{\lambda}}{E_{\lambda} - E} \quad (3)$$

where $\gamma_{\lambda} \times \gamma_{\lambda}$ is the direct product of the vectors γ_{λ} ^[4]. The cross sections are related to elements U_{ij} of the collision matrix by the relations

$$\sigma_{ij} = \pi \lambda_{ig}^2 \left| \delta_{ij} - U_{ij} \right|^2 \quad (4)$$

[It must be noted that only resonance levels having the same total angular momentum (J) and parity contribute to the cross section.] Accordingly, in order to relate the R-matrix to the cross section, it is necessary to invert the channel matrix $(I - RL^O)$ contained in W. Rather than invert this matrix expression, Lane and Thomas perform a transformation which takes $(I - RL^O)$ to a matrix whose rows and columns are labeled according to levels.

According to Lane and Thomas^[3], the transformation is effected by assuming that $(I - RL^O)^{-1}$ may be expressed as

$$(I - RL^O)^{-1} = I + \sum_{\mu} \sum_{\nu} \gamma_{\mu} \times \beta_{\nu} A_{\mu\nu} \quad (5)$$

The quantities γ and β are vectors and $A_{\mu\nu}$ are called "level" quantities which must be determined. Multiplying both sides of Equation (5) by $(I - RL^O)$ and rearranging terms, Lane and Thomas obtain for the level equations:

$$(e_\lambda - E) A_{\lambda\nu} - \sum_\mu \xi_{\lambda\mu} A_{\mu\nu} = \delta_{\lambda\mu} \quad (6)$$

By introducing level matrix A with components $A_{\mu\nu}$, the symmetric matrix ξ with component $\xi_{\lambda\mu} = (\beta_\lambda \alpha_\mu)$, the real diagonal matrix e with components e_λ , and the unit matrix I with components $\delta_{\lambda\mu}$, Equation (6) may be expressed as

$$A = (e - E - \xi)^{-1} \quad (7)$$

where the matrix E is scalar E times the unit matrix. As an example, the matrix A can take on the following form for two levels:

$$A = \begin{bmatrix} e_1 - E - 1/2 i \sum_c \Gamma_1^c & - 1/2 i \sum_c (\Gamma_1^c \Gamma_2^c)^{1/2} \\ - 1/2 i \sum_c (\Gamma_2^c \Gamma_1^c)^{1/2} & e_2 - E - 1/2 i \sum_c \Gamma_2^c \end{bmatrix}^{-1} \quad (8)$$

It is to be noted that in this example, certain simplifying approximations have been made.

With the inverse of $(I - RL^0)$ from Equation (5), the matrix expression $R/I - RL^0$ becomes

$$\frac{R}{I - RL^0} = \sum_\lambda \sum_\mu \gamma_\lambda \times \gamma_\mu A_{\lambda\mu} \quad (9)$$

and

$$W = I + P^{1/2} \left(\sum_\lambda \sum_\mu \gamma_\lambda \times \gamma_\mu A_{\lambda\mu} \right) P^{1/2} \omega \quad (10)$$

III. LEVEL MATRIX INVERSION

The level matrix

$$A = (e - E - \xi)^{-1} \quad (11)$$

may be expressed in the following form

$$A = \left(e - E - \sum_c \gamma^c \times \beta^c \right)^{-1} \quad (12)$$

with column vectors

$$\gamma^c = \begin{bmatrix} \gamma_1^c \\ \gamma_2^c \\ \vdots \\ \gamma_n^c \end{bmatrix} \quad (13)$$

and

$$\beta^c = \begin{bmatrix} \beta_1^c \\ \beta_2^c \\ \vdots \\ \beta_n^c \end{bmatrix} \quad (14)$$

The vectors refer to some particular channel c and the components of these vectors refer to levels. The matrix A^{-1} may be rewritten as follows:

$$A^{-1} = \left(e - E - \sum \gamma^c \times \beta^c \right) \quad (15)$$

$$= (e - E) \left(I - \sum \alpha^c \times \beta^c \right) \quad (16)$$

where

$$\alpha^c = (e - E)^{-1} \gamma^c \quad (17)$$

To solve for $A_{\mu\lambda}$, it is then necessary to obtain the following inverse:

$$A = \left(I - \sum_c \alpha^c \times \beta^c \right)^{-1} (e - E)^{-1} . \quad (18)$$

The inverse $(e - E)^{-1}$ is easily obtained, for this is a diagonal matrix. The inversion of $\left(I - \sum_c \alpha^c \times \beta^c \right)$ is carried out by the method of rank of annihilation [5]. For this purpose, one introduces the matrix C_0 to be the inverse of the unit matrix, or

$$C_0^{-1} = (I) . \quad (19)$$

Next, one lets the matrix C_1 be equal to the partial sum

$$C_1 = \left(I - \alpha^1 \times \beta^1 \right)^{-1} = \left(C_0^{-1} - \alpha^1 \times \beta^1 \right)^{-1} . \quad (20)$$

To obtain the indicated inverse by the method of rank annihilation [5], one applies the formula

$$(B + u \times v)^{-1} = \left[B^{-1} - \frac{(B^{-1} u) \times (v B^{-1})}{1 + (v B^{-1} u)} \right] \quad (21)$$

where B is a matrix whose inverse is known and u, v are column vectors. According to Equation (21), one finds

$$C_1 = \left(C_0 + \frac{C_0 \alpha^1 \times \beta^1 C_0}{1 - (\beta^1 C_0 \alpha^1)} \right) . \quad (22)$$

Clearly from the definition of the matrix C_1 , one can define the following partial sum termed C_2 as

$$C_2 = \left(I - \alpha^1 \times \beta^1 - \alpha^2 \times \beta^2 \right)^{-1} \quad (23)$$

and since

$$C_1^{-1} = (I - \alpha^1 \times \beta^1)$$

C_2 may be written

$$C_2 = \left(C_1^{-1} - \alpha^2 \times \beta^2 \right)^{-1} . \quad (24)$$

According to the method of rank annihilation^[5], one obtains for Equation (24)

$$C_2 = \left[C_1 + \frac{C_1 \alpha^2 \times \beta^2 C_1}{1 - (\beta^2 C_1 \alpha^2)} \right] \quad (25)$$

By proceeding in this manner, one finds

$$C_n = \left(C_{n-1}^{-1} - \alpha^n \times \beta^n \right)^{-1} = \left(I - \sum_c \alpha^c \times \beta^c \right)^{-1} \quad (26)$$

$$C_n = \left[C_{n-1} + \frac{C_{n-1} \alpha^n \times \beta^n C_{n-1}}{1 - (\beta^n C_{n-1} \alpha^n)} \right] \quad (27)$$

Thus in principle, the desired matrix inverse of $\left(I - \sum \alpha^c \times \beta^c \right)$ has been obtained (see Appendix A). It is to be noted that in each inversion that is carried out, the condition that B be a matrix having a known inverse is satisfied.

IV. APPLICATIONS AND APPROXIMATIONS

In Part 1 of this section, the inverse of the level matrix is obtained for a two-channel and two-level case. In Part 2, an approximate three-channel inversion is made. In Part 3, the approximations which are made for the $\ell = 0$ case are discussed.

PART 1

The expression for A^{-1} for a two-channel and two-level situation becomes

$$A^{-1} = \left[\begin{pmatrix} e_1 - E & 0 \\ 0 & e_2 - E \end{pmatrix} - \gamma^1 \times \beta^1 - \gamma^2 \times \beta^2 \right] \quad (28)$$

where $\gamma \times \beta$'s are 2×2 matrices. With the diagonal matrix taken outside, A^{-1} becomes

$$A^{-1} = \begin{pmatrix} e_1 - E & 0 \\ 0 & e_2 - E \end{pmatrix} (I - \alpha^1 \times \beta^1 - \alpha^2 \times \beta^2) \quad (29)$$

where

$$\alpha^c = (e - E)^{-1} \gamma^c, \text{ and } c = 1, 2 \quad (30)$$

and

$$(e - E)^{-1} = \begin{pmatrix} e_1 - E & 0 \\ 0 & e_2 - E \end{pmatrix}^{-1} \quad (31)$$

Inverting A^{-1} , one obtains

$$A = \left(I - \alpha^1 \times \beta^1 - \alpha^2 \times \beta^2 \right)^{-1} (e - E)^{-1} \quad (32)$$

At this point, it will be convenient to let

$$C = \left(I - \alpha^1 \times \beta^1 - \alpha^2 \times \beta^2 \right)^{-1} \quad (33)$$

where C is a 2×2 matrix. By the matrix inversion method of rank annihilation, matrix C is determined as follows: According to Equation (19),

$$C_0 = (I)^{-1} \quad (34)$$

From Equation (22),

$$\begin{aligned} C_1 &= \left(C_0^{-1} - \alpha^1 \times \beta^1 \right)^{-1} \\ &= \left(C_0 + \frac{\alpha^1 \times \beta^1}{1 - (\beta^1 \alpha^1)} \right) \end{aligned} \quad (35)$$

Finally, from Equation (25),

$$C = C_2 = \left(C_0 + \frac{\alpha^1 \times \beta^1}{1 - (\beta^1 \alpha^1)} + \frac{C_1 \alpha^2 \times \beta^2 C_1}{1 - (\beta^2 C_1 \alpha^2)} \right) \quad (36)$$

On expansion, C becomes

$$\begin{aligned} C = & \left(I + \frac{(1 - \beta^2 \alpha^2)}{D} \alpha^1 \times \beta^1 + \frac{(1 - \beta^1 \alpha^1)}{D} \alpha^2 \times \beta^2 + \frac{(\beta^1 \alpha^2)}{D} \alpha^1 \times \beta^2 \right. \\ & \left. + \frac{(\beta^2 \alpha^1)}{D} \alpha^2 \times \beta^1 \right) \end{aligned} \quad (37)$$

where

$$D = (1 - \beta^2 \alpha^2) (1 - \beta^1 \alpha^1) - (\beta^2 \alpha^1) (\beta^1 \alpha^2) \quad (38)$$

Substituting the matrix C in Equation (32), one obtains for matrix A,

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \begin{pmatrix} \frac{1}{e_1 - E} & 0 \\ 0 & \frac{1}{e_2 - E} \end{pmatrix} \quad (39)$$

and the resulting components of matrix A are

$$\begin{aligned} A_{11} = & \frac{1}{(e_1 - E)} + \frac{(1 - \beta^2 \alpha^2)}{(e_1 - E) D} \alpha_1^1 \beta_1^1 + \frac{(1 - \beta^1 \alpha^1)}{(e_1 - E) D} \alpha_1^2 \beta_1^2 \\ & + \frac{(\beta^1 \alpha^2)}{(e_1 - E) D} \alpha_1^1 \beta_1^2 + \frac{(\beta^2 \alpha^1)}{(e_1 - E) D} \alpha_1^2 \beta_1^1 \end{aligned} \quad (40a)$$

$$\begin{aligned}
A_{12} = & \frac{(1 - \beta^2 \alpha^2)}{(e_2 - E) D} \alpha_1^1 \beta_2^1 + \frac{(1 - \beta^1 \alpha^1)}{(e_2 - E) D} \alpha_1^2 \beta_2^2 + \frac{(\beta^1 \alpha^2)}{(e_2 - E) D} \alpha_1^1 \beta_2^2 \\
& + \frac{(\beta^2 \alpha^1)}{(e_2 - E) D} \alpha_1^2 \beta_2^1
\end{aligned} \tag{40b}$$

$$\begin{aligned}
A_{21} = & \frac{(1 - \beta^2 \alpha^2)}{(e_1 - E) D} \alpha_2^1 \beta_1^1 + \frac{(1 - \beta^1 \alpha^1)}{(e_1 - E) D} \alpha_2^2 \beta_1^2 + \frac{(\beta^1 \alpha^2)}{(e_1 - E) D} \alpha_2^1 \beta_1^2 \\
& + \frac{(\beta^2 \alpha^1)}{(e_1 - E) D} \alpha_2^2 \beta_1^1
\end{aligned} \tag{40c}$$

$$\begin{aligned}
A_{22} = & \frac{1}{(e_2 - E)} + \frac{(1 - \beta^2 \alpha^2)}{(e_2 - E) D} \alpha_2^1 \beta_2^1 + \frac{(1 - \beta^1 \alpha^1)}{(e_2 - E) D} \alpha_2^2 \beta_2^2 \\
& + \frac{(\beta^1 \alpha^2)}{(e_2 - E) D} \alpha_2^1 \beta_2^2 + \frac{(\beta^2 \alpha^1)}{(e_2 - E) D} \alpha_2^2 \beta_2^1
\end{aligned} \tag{40d}$$

where

$$D = (1 - \beta^2 \alpha^2) (1 - \beta^1 \alpha^1) - (\beta^2 \alpha^1) (\beta^1 \alpha^2) \quad . \tag{40e}$$

On expanding and collecting terms, one obtains

$$A_{11} = \frac{1}{d} [(e_2 - E) - \beta_2^1 \alpha_2^1 - \beta_2^2 \alpha_2^2] \tag{41a}$$

$$A_{12} = \frac{1}{d} [\alpha_1^1 \beta_2^1 + \alpha_1^2 \beta_2^2] \tag{41b}$$

$$A_{21} = \frac{1}{d} [\alpha_2^1 \beta_1^1 + \alpha_2^2 \beta_1^2] \tag{41c}$$

$$A_{22} = \frac{1}{d} [(e_1 - E) - \beta_1^1 \alpha_1^1 - \beta_1^2 \alpha_1^2] \tag{41d}$$

where

$$\begin{aligned}
d = & \left\{ (e_1 - E) (e_2 - E) - [(e_2 - E) \beta_1^1 \alpha_1^1 + (e_1 - E) \beta_2^1 \alpha_2^1] \right. \\
& - [(e_2 - E) \beta_1^2 \alpha_1^2 + (e_1 - E) \beta_2^2 \alpha_2^2] + \beta_1^1 \beta_2^2 \alpha_1^1 \alpha_2^2 - \beta_1^2 \beta_2^1 \alpha_1^1 \alpha_2^2 \\
& \left. + \beta_1^2 \beta_2^1 \alpha_2^1 \alpha_1^2 - \beta_1^1 \beta_2^2 \alpha_2^1 \alpha_1^2 \right\} \quad .
\end{aligned} \tag{41e}$$

PART 2

In this section, the level matrix inversion for an approximate three-channel and two-level expression will be considered. The general two-level expression for the matrix A^{-1} is

$$A^{-1} = \left[\begin{pmatrix} e_1 - E & 0 \\ 0 & e_2 - E \end{pmatrix} - \begin{pmatrix} \gamma^1 \times \beta^1 & \gamma^2 \times \beta^2 & \dots & \gamma^n \times \beta^n \end{pmatrix} \right] \quad (42)$$

where $\alpha \times \beta$ are 2×2 matrix. Suppose the approximation is made that

$$\gamma^3 \times \beta^3 + \gamma^4 \times \beta^4 + \dots + \gamma^n \times \beta^n = \begin{pmatrix} \sum_{c=3}^n \gamma_1^c \beta_1^c & 0 \\ 0 & \sum_{c=3}^n \gamma_2^c \beta_2^c \end{pmatrix} = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} \quad (43)$$

then Equation (42) becomes

$$A^{-1} = \left[\begin{pmatrix} e_1 - E - g_1 & 0 \\ 0 & e_2 - E - g_2 \end{pmatrix} - \begin{pmatrix} \gamma^1 \times \beta^1 & \gamma^2 \times \beta^2 \end{pmatrix} \right]. \quad (44)$$

If the diagonal matrix is taken outside, one obtains

$$A^{-1} = \epsilon \left(I - \gamma^1 \times \beta^1 - \gamma^2 \times \beta^2 \right) \quad (45)$$

where

$$\alpha^c = \epsilon^{-1} \gamma^c \quad (46)$$

and

$$\epsilon = \begin{pmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{pmatrix} = \begin{pmatrix} e_1 - E - g_1 & 0 \\ 0 & e_2 - E - g_2 \end{pmatrix}. \quad (47)$$

The matrix A may be written

$$A = (I - \alpha^1 \times \beta^1 - \alpha^2 \times \beta^2)^{-1} \epsilon^{-1} = C \epsilon^{-1} \quad (48)$$

where the matrix C is introduced for convenience and represents

$$(I - \alpha^1 \times \beta^1 - \alpha^2 \times \beta^2)^{-1}. \quad (49)$$

From Part 1, one obtains for $(I - \alpha^1 \times \beta^1 - \alpha^2 \times \beta^2)^{-1}$ the equation

$$C = \left[I + \frac{(1 - \beta^2 \alpha^2)}{D} \alpha^1 \times \beta^1 + \frac{(1 - \beta^1 \alpha^1)}{D} \alpha^2 \times \beta^2 + \frac{(\beta^1 \alpha^2)}{D} \alpha^1 \times \beta^2 + \frac{(\beta^2 \alpha^1)}{D} \alpha^2 \times \beta^1 \right] \quad (50a)$$

and

$$D = (1 - \beta^2 \alpha^2) (1 - \beta^1 \alpha^1) - (\beta^2 \alpha^1) (\beta^1 \alpha^2) \quad (50b)$$

Substituting C in Equation 48, one obtains the matrix A. The components of A will be similar to Equation (40a), (40b), (40c), (40d) of Part 1 except where $(e_i - E)$ occurs, it is replaced by ϵ_i . [See Appendix B for proof that Equation (50) is the required inverse.]

Since $\epsilon_i = e_i - E - g_i$ and since g_i is complex, ϵ_i cannot vanish ($\neq 0$) for any real value of E if $\text{Im } g_i \neq 0$. This is advantageous since the terms in A_{ij} need not be expanded and multiplied out in order to remove the singularity as was the case in Part 1. What is meant by removal of singularity in Part 1 is the following: From Equation (40a) Part 1, consider A_{11} as an example,

$$A_{11} = \frac{1}{e_1 - E} + \frac{(1 - \beta^2 \alpha^2)}{(e_1 - E) D} \alpha_1^1 \beta_1^1 + \frac{(1 - \beta^1 \alpha^1)}{(e_1 - E) D} \alpha_1^2 \beta_1^2 + \frac{(\beta^1 \alpha^2)}{(e_1 - E) D} \alpha_1^1 \beta_1^2 + \frac{(\beta^2 \alpha^1)}{(e_1 - E) D} \alpha_1^2 \beta_1^1 \quad (40a)$$

Since $e_1 - E = 0$ for some real value of E, the term singularity is used. However, if all the terms are expanded and multiplied as indicated, all the singular terms drop out, eg,

$$A_{11} = \frac{1}{d} [(e_2 - E) - \beta_2^1 \gamma_2^1 - \beta_2^2 \gamma_2^2], \quad d \neq 0 \quad (41a)$$

PART 3

In this section, results of Part 2 will be simplified by making certain assumptions. It will also become apparent that with these approximations the result can be extended to the many level case using parameters which are easily associated with the single resonance B-W formula.

Channel vector β^C is a product of a scalar L_C^0 and reduced width amplitude vector γ^C .

$$\beta^c = L_c^O \gamma^c = L_c^O \begin{pmatrix} \gamma_1^c \\ \gamma_2^c \end{pmatrix} \quad (51)$$

with

$$L_c^O = S_c^O - B_c + i P_c \quad (52)$$

Consider the case: $\ell = 0$ for all channels, only positive energy channels exist, and neutrons are the incident particles. For this case, $S_c^O - B_c = 0$ in the neutron channel. For the remaining channels, it will be assumed that $S_c^O - B_c = 0$ by a proper choice for B_c [6,7]. This is possible since B_c is an unspecified constant in the Lane and Thomas derivation [3] of R-matrix theory. Similar arguments [1,8] are made to simplify the shift matrix found in Wigner's [1] development of R-matrix theory. With this approximation, the channel vector β^c becomes

$$\beta^c = i P_c \gamma^c = i P_c \begin{pmatrix} \gamma_1^c \\ \gamma_2^c \end{pmatrix} = \begin{pmatrix} i P_c \gamma_1^c \\ i P_c \gamma_2^c \end{pmatrix} \quad (53)$$

For the $\ell = 0$ case, it will be further assumed that penetration factor Γ_c is a constant for all the channels except the neutron channel. In the neutron channel for the $\ell = 0$ case, $P_c = k_c r_c$ (r_c = channel radius). Under these conditions, the following definitions can be made:

$$\Gamma_\lambda^1 = 2 P_1 \gamma_\lambda^1 \gamma_\lambda^1 \quad (\text{channel 1 and level } \lambda) \quad (54a)$$

$$\Gamma_\lambda^2 = 2 P_2 \gamma_\lambda^2 \gamma_\lambda^2 \quad (\text{channel 2 and level } \lambda) \quad (54b)$$

$$\Gamma_\lambda^3 = 2 P_3 \gamma_\lambda^3 \gamma_\lambda^3 \quad (\text{channel 3 and level } \lambda) \quad (54c)$$

$$\Gamma_\lambda^n = 2 P_n \gamma_\lambda^n \gamma_\lambda^n \quad (\text{channel } n \text{ and level } \lambda) \quad (54d)$$

By designating channel 1 as neutron, channel 2 as fission channel, and channel 3 as a gamma channel, the reduced width amplitude can be defined in terms of single-level partial widths:

$$\gamma_\lambda^1 = \gamma_\lambda^n = \left(\frac{\Gamma_\lambda^n}{2 P_1} \right)^{1/2} = \left(\frac{\Gamma_{n\lambda}}{2 P_1} \right)^{1/2} \quad (55a)$$

$$\gamma_\lambda^2 = \gamma_\lambda^f = \left(\frac{\Gamma_\lambda^f}{2 P_2} \right)^{1/2} = \left(\frac{\Gamma_{f\lambda}}{2 P_2} \right)^{1/2} \quad (55b)$$

$$\gamma_{\lambda}^3 = \gamma_{\lambda}^{\gamma} = \left(\frac{\Gamma_{\lambda}^{\gamma}}{2 P_3} \right)^{1/2} = \left(\frac{\Gamma_{\gamma\lambda}}{2 P_3} \right)^{1/2} \quad (55c)$$

If all channels from 3 to n are considered as partial channels of some common exit channel (eg, radiation channel), then one can define $g_j = \sum_3^n \gamma_j^c \beta_j^c$, Equation (43), as

$$g_j = \frac{i \Gamma_{3j}}{2} \quad (55d)$$

The approximation given by Equation (43) is based on a statistical argument. For the off-diagonal terms, it is assumed that reduced widths γ_{λ}^c have random signs and exhibit random size variation for the various channels, $c^{[9]}$, such that these terms vanish when taken over a large number of channels.

From the discussion in Part 1 of this section, the components α_{λ}^c and β_{λ}^c in the expressions for A_{ij} may be expressed in terms of Γ 's. Thus in the expressions

$$\begin{aligned} A_{11} = & \frac{1}{\epsilon_1} + \frac{(1 - \beta^2 \alpha^2)}{\epsilon_1 D} \alpha_1^1 \beta_1^1 + \frac{(1 - \beta^1 \alpha^1)}{\epsilon_1 D} \alpha_1^2 \beta_1^2 + \frac{(\beta^1 \alpha^2)}{\epsilon_1 D} \alpha_1^1 \beta_1^2 \\ & + \frac{(\beta^2 \alpha^1)}{\epsilon_1 D} \alpha_1^2 \beta_1^1 \end{aligned} \quad (56a)$$

$$\begin{aligned} A_{12} = & \frac{(1 - \beta^2 \alpha^2)}{\epsilon_2 D} \alpha_1^1 \beta_2^1 + \frac{(1 - \beta^1 \alpha^1)}{\epsilon_2 D} \alpha_1^2 \beta_2^2 + \frac{(\beta^1 \alpha^2)}{\epsilon_2 D} \alpha_1^1 \beta_2^2 \\ & + \frac{(\beta^2 \alpha^1)}{\epsilon_2 D} \alpha_1^2 \beta_2^1 \end{aligned} \quad (56b)$$

$$\begin{aligned} A_{21} = & \frac{(1 - \beta^2 \alpha^2)}{\epsilon_1 D} \alpha_2^1 \beta_1^1 + \frac{(1 - \beta^1 \alpha^1)}{\epsilon_1 D} \alpha_2^2 \beta_1^2 + \frac{(\beta^1 \alpha^2)}{\epsilon_1 D} \alpha_2^1 \beta_1^2 \\ & + \frac{(\beta^2 \alpha^1)}{\epsilon_1 D} \alpha_2^2 \beta_1^1 \end{aligned} \quad (56c)$$

$$\begin{aligned}
A_{22} = & \frac{1}{\epsilon_2} + \frac{(1 - \beta^2 \alpha^2)}{\epsilon_2 D} \alpha_2^1 \beta_2^1 + \frac{(1 - \beta^1 \alpha^1)}{\epsilon_2 D} \alpha_2^2 \beta_2^2 + \frac{(\beta^1 \alpha^2)}{\epsilon_2 D} \alpha_2^1 \beta_2^2 \\
& + \frac{(\beta^2 \alpha^1)}{\epsilon_2 D} \alpha_2^2 \beta_2^1
\end{aligned} \tag{56d}$$

one can make the following replacements

$$g_\lambda = i \frac{\Gamma_{3\lambda}}{2} \tag{56e}$$

$$\alpha_\lambda^c = \frac{\Gamma_{c\lambda}^{1/2}}{\left(\epsilon_\lambda - E - i \frac{\Gamma_{3\lambda}}{2} \right)} \tag{56f}$$

and

$$\beta_\lambda^c = i \Gamma_{c\lambda}^{1/2} \tag{56g}$$

where c refers to channels and λ to levels. It is to be noted that these replacements, Equations (56f) and (56g), are allowed because the following situations occur:

For

$$\begin{aligned}
\alpha_2^1 \beta_2^1 &= \frac{i P_1 \gamma^{1/2} \gamma^{1/2}}{\left(\epsilon_2 - E - i \frac{\Gamma_{32}}{2} \right)} = \frac{P_1}{\left(\epsilon_2 - E - i \frac{\Gamma_{32}}{2} \right)} i \frac{(\Gamma_{12})^{1/2}}{(2P_1)^{1/2}} \frac{(\Gamma_{12})^{1/2}}{(2P_1)^{1/2}} \\
&= i \frac{(\Gamma_{12})^{1/2} (\Gamma_{12})^{1/2}}{2 \left(\epsilon_2 - E - i \frac{\Gamma_{32}}{2} \right)} .
\end{aligned} \tag{57}$$

For

$$\begin{aligned}
(\beta^1 \alpha^2) (\alpha_2^1 \beta_2^2) &= (\beta_1^1 \alpha_1^2 + \beta_2^1 \alpha_2^2) \alpha_2^1 \beta_2^2 \\
&= \beta_1^1 \alpha_1^2 \alpha_2^1 \beta_2^2 + \beta_2^1 \alpha_2^2 \alpha_2^1 \beta_2^2
\end{aligned}$$

$$\begin{aligned}
&= i^2 \frac{P_1 \gamma_1^1 \gamma_2^1}{e_2} \frac{P_2 \gamma_2^2 \gamma_1^2}{e_1} + i^2 \frac{P_1 \gamma_2^1 \gamma_2^1}{e_2} \frac{P_2 \gamma_2^1 \gamma_2^2}{e_2} \\
&= \frac{i^2}{e_2 e_1} P_1 P_2 \frac{\Gamma_{11}^{1/2}}{2 P_1^{1/2}} \frac{\Gamma_{12}^{1/2}}{P_1^{1/2}} \frac{\Gamma_{22}^{1/2}}{2 P_2^{1/2}} \frac{\Gamma_{21}^{1/2}}{P_2^{1/2}} \\
&\quad + \frac{i^2}{e_2 e_2} P_1 P_2 \frac{\Gamma_{12}^{1/2}}{2 P_1^{1/2}} \frac{\Gamma_{12}^{1/2}}{P_1^{1/2}} \frac{\Gamma_{22}^{1/2}}{2 P_2^{1/2}} \frac{\Gamma_{22}^{1/2}}{P_2^{1/2}} \\
&= i^2 \frac{\Gamma_{11}^{1/2} \Gamma_{12}^{1/2} \Gamma_{22}^{1/2} \Gamma_{21}^{1/2}}{4 \left(e_1 - E - i \frac{\Gamma_{31}}{2} \right) \left(e_2 - E - i \frac{\Gamma_{32}}{2} \right)} \\
&\quad + i^2 \frac{\Gamma_{12}^{1/2} \Gamma_{12}^{1/2} \Gamma_{22}^{1/2} \Gamma_{22}^{1/2}}{4 \left(e_2 - E - i \frac{\Gamma_{32}}{2} \right) \left(e_1 - E - i \frac{\Gamma_{31}}{2} \right)} . \tag{58}
\end{aligned}$$

These two examples illustrate the condition in which the reduced amplitude, γ , can be replaced by the $\Gamma_{c\lambda}^{1/2}$'s instead of $\left(\frac{\Gamma_{c\lambda}}{2 P_c} \right)^{1/2}$. Extension to more levels increases both the dimension of matrix C and the number of terms in the scalar product $(\alpha^C_\beta C)$.

V. DISCUSSION

It is quite evident that matrix inverse by rank annihilation will easily invert an $n \times n$ level matrix A for the case of few channels. The examples given in the text were for two levels, but extension to more levels can be made with little difficulty. Extension to more channels, while quite straightforward, requires working with more terms. If the approximation that beyond the first two or three channels only diagonal elements are to be considered, extension to more channels is quite simple in the present method of obtaining the inverse of the level matrix.

By considering the result described in a report^[10] on matrix inversion of $(I - RL^0)$ and the present result, it is evident that one may approach the matrix inversion problem in R -matrix theory by two different paths: (a) The inversion of the channel matrix $(I - RL^0)$ by the method of rank annihilation is most amenable to a situation where there are many channels but few levels. (b) For the reverse situation of many levels and few channels, inversion of the level matrix by this method is the most logical way to go.

In the section on application, a gain is made when an additional imaginary term is included in the diagonal element. This immediately eliminates the question of singularity which would arise had this term not been present. In actuality, however, the singular terms do cancel out, but it is necessary to make a complete expansion of all the terms. With the imaginary term, the expansion does not necessarily have to be made.

It has been commented that useful general expression for the inverse of $(I - RL^0)$ cannot be given explicitly^[11]. It is agreed that this statement is true for the case of many channels and many levels; however, if one is allowed to relax either one of the requirements, the method of rank annihilation is capable of giving a useful expression for the inverse. It may be noted that this method described applies because of the special construction of either the channel matrix $(I - RL^0)$ or the level matrix, A^{-1} .

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APPENDIX A

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APPENDIX A

To show that Equation (27) is the inverse of $\left[I - \sum_c \alpha^c x \beta^c \right]$, an inductive proof will be used.

From Equations (20) and (22), one obtains C_1 and C_1^{-1}

$$C_1^{-1} = \begin{pmatrix} C_0^{-1} & -\alpha^1 x \beta^1 \end{pmatrix} \quad (20)$$

$$C_1 = \begin{bmatrix} C_0 + \frac{C_0 \alpha^1 x \beta^1 C_0}{1 - (\beta^1 C_0 \alpha^1)} \end{bmatrix} \quad (22)$$

Multiplying C_1^{-1} and C_1 , one finds

$$\begin{aligned} \begin{pmatrix} C_1 & C_1^{-1} \end{pmatrix} &= \begin{bmatrix} C_0 & C_0^{-1} + \frac{C_0 \alpha^1 x \beta^1}{1 - \beta^1 C_0 \alpha^1} - C_0 \alpha^1 x \beta^1 - \frac{C_0 \alpha^1 x \beta^1 (\beta^1 C_0 \alpha^1)}{1 - (\beta^1 C_0 \alpha^1)} \end{bmatrix} \\ &= \begin{pmatrix} C_0 & C_0^{-1} \end{pmatrix} \\ &= \begin{pmatrix} I \end{pmatrix} . \end{aligned}$$

Assuming that $C_n C_n^{-1} = (I)$, it will be demonstrated that $C_{n+1} C_{n+1}^{-1} = (I)$. Consider Equations (26) and (27),

$$C_{n+1}^{-1} = \begin{pmatrix} C_n^{-1} & -\alpha^{n+1} x \beta^{n+1} \end{pmatrix} \quad (26)$$

$$C_{n+1} = \begin{bmatrix} C_n + \frac{C_n \alpha^{n+1} x \beta^{n+1} C_n}{(1 - \beta^{n+1} C_n \alpha^{n+1})} \end{bmatrix} . \quad (27)$$

Multiplying C_{n+1}^{-1} and C_{n+1} , one finds

$$\begin{aligned} C_{n+1} C_{n+1}^{-1} &= \begin{bmatrix} C_n C_n^{-1} + \frac{C_n \alpha^{n+1} x \beta^{n+1}}{1 - (\beta^{n+1} C_n \alpha^{n+1})} - C_n \alpha^{n+1} x \beta^{n+1} \\ - \frac{C_n \alpha^{n+1} x \beta^{n+1}}{1 - (\beta^{n+1} C_n \alpha^{n+1})} (\beta^{n+1} C_n \alpha^{n+1}) \end{bmatrix} \\ &= \begin{pmatrix} C_n & C_n^{-1} \end{pmatrix} \\ &= \begin{pmatrix} I \end{pmatrix} \end{aligned}$$

which is the desired result.

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APPENDIX B

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APPENDIX B

To show that Equation (50) from Section IV, Part 2, is the required inverse, one need only to demonstrate that

$$CC^{-1} = (I) .$$

From Equation (50)

$$C = \left[I + \frac{(1 - \beta^2 \alpha^2)}{D} \alpha^1 \times \beta^1 + \frac{(1 - \beta^1 \alpha^1)}{D} \alpha^2 \times \beta^2 + \frac{(\beta^1 \alpha^2)}{D} \alpha^1 \times \beta^2 + \frac{(\beta^2 \alpha^1)}{D} \alpha^2 \times \beta^1 \right]$$

and from Equation (49)

$$C^{-1} = (I - \alpha^1 \times \beta^1 - \alpha^2 \times \beta^2) .$$

Multiplying the matrix C by the matrix C^{-1} , and collecting terms, one finds that

$$CC^{-1} = C (I - \alpha^1 \times \beta^1 - \alpha^2 \times \beta^2)$$

$$= (C - C \alpha^1 \times \beta^1 - C \alpha^2 \times \beta^2)$$

$$= \left[C - \frac{(1 - \beta^2 \alpha^2)}{D} \alpha^1 \times \beta^1 - \frac{(1 - \beta^1 \alpha^1)}{D} \alpha^2 \times \beta^2 - \frac{(\beta^2 \alpha^1)}{D} \alpha^2 \times \beta^1 - \frac{(\beta^1 \alpha^2)}{D} \alpha^1 \times \beta^2 \right]$$

$$= (I)$$

which is the desired result.

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APPENDIX C

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APPENDIX C

As an illustration, the collision amplitude, U_{12} and U_{11} for the two level case will be obtained from the results of Section IV, Part 2, and the cross section, σ_{12} and σ_{11} , will be calculated from given sets of parameters. For sake of utility and simplicity, channel 1 will be assumed to be a neutron channel. In addition, the approximations described in Section IV, Part 3, will be assumed to hold. Accordingly, matrix W, Equation (2), becomes

$$W = 1 + P^{1/2} \left[\sum_{\lambda} \sum_{\mu} \gamma_{\lambda} \times \gamma_{\mu} A_{\lambda\mu} \right] P^{1/2} \omega$$

$$= 1 + 2i \sum_{\lambda} \sum_{\mu} a_{\lambda} \times a_{\mu} A_{\lambda\mu}$$

where

$$a_{\lambda} = P_c^{1/2} \gamma_{\lambda} \quad \text{and} \quad \omega = 2i$$

PART A Reaction Cross Section, σ_{12} .

The element of matrix W referring to channel 1 and channel 2 is

$$W_{12} = 2i [A_{11} a_1^1 a_1^2 + A_{12} a_1^1 a_2^2 + A_{21} a_2^1 a_1^2 + A_{22} a_2^1 a_2^2] \quad (C.1)$$

Elements A_{11} , A_{12} , A_{21} , and A_{22} are obtained from Equations (41a), (41b), (41c), (41d), and (41e). Since the approximate 3 channel case is desired, the expressions $(e_i - E)$ are replaced by ϵ_i which is defined

$$c_i - c_i - E - g_i \quad i = 1, 2 \quad (C.2)$$

By applying the approximations mentioned previously in Section IV, Part 3, we obtain the following quantities

$$\Gamma_{c\lambda} = 2 P_c \gamma_{\lambda}^c \gamma_{\lambda}^c \quad (C.3a)$$

$$\beta_{\lambda}^c = i P_c \gamma_{\lambda}^c \quad (C.3b)$$

$$g_{\lambda} = \frac{i \Gamma_{3\lambda}}{2} \quad (C.3c)$$

Substituting these into W_{12} , the following expression is obtained

$$\begin{aligned}
W_{12} = \frac{1}{d} & \left[\left(\frac{\Gamma_{12} + \Gamma_{22} + \Gamma_{32}}{2} \right) \Gamma_{11}^{1/2} \Gamma_{21}^{1/2} \right. \\
& - \frac{1}{2} \left(\Gamma_{12}^{1/2} \Gamma_{11}^{1/2} + \Gamma_{22}^{1/2} \Gamma_{21}^{1/2} \right) \left(\Gamma_{11}^{1/2} \Gamma_{22}^{1/2} + \Gamma_{12}^{1/2} \Gamma_{21}^{1/2} \right) \\
& + \frac{(\Gamma_{11} + \Gamma_{21} + \Gamma_{31})}{2} \Gamma_{12}^{1/2} \Gamma_{22}^{1/2} + i(e_2 - E) \Gamma_{11}^{1/2} \Gamma_{21}^{1/2} \\
& \left. + i(e_1 - E) \Gamma_{12}^{1/2} \Gamma_{22}^{1/2} \right] \quad (C.4)
\end{aligned}$$

where d is a complex expression whose real part is

$$\begin{aligned}
\text{Re } d = (e_1 - E)(e_2 - E) - \frac{1}{4} & (\Gamma_{32} \Gamma_{11} + \Gamma_{31} \Gamma_{12} + \Gamma_{32} \Gamma_{21} + \Gamma_{31} \Gamma_{22} \\
& + \Gamma_{11} \Gamma_{22} + \Gamma_{21} \Gamma_{12} + \Gamma_{31} \Gamma_{32}) \\
& + 1/4 (\Gamma_{22}^{1/2} \Gamma_{12}^{1/2} \Gamma_{11}^{1/2} \Gamma_{21}^{1/2}) + 1/4 (\Gamma_{11}^{1/2} \Gamma_{21}^{1/2} \Gamma_{12}^{1/2} \Gamma_{22}^{1/2}) \quad (C.5)
\end{aligned}$$

and whose imaginary part is

$$\text{Im } d = - (e_2 - E) \frac{(\Gamma_{11} + \Gamma_{21} + \Gamma_{31})}{2} - (e_1 - E) \frac{(\Gamma_{12} + \Gamma_{22} + \Gamma_{32})}{2} \quad (C.7)$$

Cross section σ_{12} is given by

$$\sigma_{12} = \pi \lambda_1^2 \sum_J g_J |U_{12}^J|^2 \quad (C.8)$$

Since

$$U_{12} = \Omega_1 W_{12} \Omega_2 \quad (C.9)$$

$$|U_{12}|^2 = U_{12} U_{12}^* = W_{12} W_{12}^* \quad (C.10)$$

therefore, for a given J

$$\sigma_{12} = g \pi \lambda_1^2 W_{12} W_{12}^* \quad (C.11)$$

A plot of σ_{12} versus E is given in Figure 1. The points were calculated using the value of parameters as indicated. For comparison, the results, obtained by the Reich-Moore [12] formula for the same set of parameters, are indicated. In the top plot, the parameters used are those of Lynn [13] and therefore the curves exhibit the interference effect first described by Lynn [14]. Disagreement over the peak in this plot is most likely to round off error in the computer program of Reich-Moore formula.

FIG. 1 σ_{12} CALCULATED BY EQUATION (C-11), APPENDIX C, AND REICH-MOORE FORMULA FOR INDICATED SETS OF RESONANCE PARAMETERS.

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APPENDIX C

PART B Scattering Cross Section, σ_{11} .

The element of matrix W referring to an incoming channel 1 and an outgoing channel 1 is

$$W_{11} = 1 + 2i [A_{11} a_1^1 a_1^1 + A_{12} a_1^1 a_2^1 + A_{21} a_2^1 a_1^1 + A_{22} a_2^1 a_2^1],$$

(C.12)

and from Equation (1)

$$U_{11} = \Omega_1 W_{11} \Omega_1 = e^{-i\phi_1} W_{11} e^{-i\phi_1}$$

(C.13)

where

$$k_1 r_1 = \phi_1.$$

(C.14)

At this point, it will be assumed that only two channels, neutron and radiation, are involved. It will be assumed further that the approximation described by Equation (43) in Section IV, Part 2, applies to the radiation channel; ie, only diagonal terms contribute. Based on these assumptions, the matrix A^{-1} may be written

$$A^{-1} = \left[\begin{pmatrix} e_1 - E - g_1 & 0 \\ 0 & e_2 - E - g_2 \end{pmatrix} - \gamma^1 x \beta^1 \right].$$

(C.15)

Taking the diagonal matrix outside, one obtains

$$A^{-1} = \epsilon (I - \alpha^1 x \beta^1)$$

(C.16)

where

$$\alpha^1 = \epsilon^{-1} \gamma^1$$

(C.17)

and

$$\epsilon = \begin{pmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{pmatrix} = \begin{pmatrix} e_1 - E - g_1 & 0 \\ 0 & e_2 - E - g_2 \end{pmatrix}.$$

(C.18)

The matrix A then becomes

$$A = \left(I - \alpha^1 x \beta^1 \right)^{-1} \epsilon^{-1} \quad (C.19)$$

The inverse $\left(I - \alpha^1 x \beta^1 \right)^{-1}$ using Equation (35) from Section IV, Part 1 is

$$\left(I - \alpha^1 x \beta^1 \right)^{-1} = \left(I + \frac{\alpha^1 x \beta^1}{1 - \beta^1 \alpha^1} \right) \quad (C.20)$$

From the results of Section IV, Part 1, Equations (39) to (41), the components of the level matrix A are

$$A_{11} = \frac{1}{d} \left(\epsilon_2 - \beta_2^1 \gamma_2^1 \right) \quad (C.21a)$$

$$A_{12} = \frac{1}{d} \gamma_1^1 \beta_2^1 \quad (C.21b)$$

$$A_{21} = \frac{1}{d} \gamma_2^1 \beta_1^1 \quad (C.21c)$$

$$A_{22} = \frac{1}{d} \left(\epsilon_1 - \beta_1^1 \gamma_1^1 \right) \quad (C.21d)$$

where

$$d = \left(\epsilon_1 \epsilon_2 - \epsilon_2 \beta_1^1 \gamma_1^1 - \epsilon_1 \beta_2^1 \gamma_2^1 \right) \quad (C.21e)$$

Using the approximations mentioned in Section IV, Part 3, and equations (54), (55a), (55c) and (55d), one obtains the following:

$$\Gamma_{c\lambda} = 2 P_c \gamma_\lambda^c \gamma_\lambda^c \quad (C.22a)$$

$$\beta_\lambda^c = i P_c \gamma_\lambda^c \quad (C.22b)$$

$$g_\lambda = i \frac{\Gamma_{2\lambda}}{2} \quad (C.22c)$$

$$\gamma_\lambda^c = \left(\frac{\Gamma_{c\lambda}}{2 P_c} \right)^{1/2} \quad (C.22d)$$

Substituting these quantities in the expression W_{11} , one obtains

$$A_{11} a_1^1 a_1^1 = \frac{1}{d} \left[(e_2 - E) - i \frac{(\Gamma_2^1 + \Gamma_2^2)}{2} \right] \frac{\Gamma_1^1}{2} \quad (C.23a)$$

$$A_{12} a_1^1 a_2^2 = \frac{i}{d} \frac{\left[\Gamma_1^1 \Gamma_2^1 \right]}{4} \quad (C.23b)$$

$$A_{21} a_2^1 a_1^1 = \frac{i}{d} \frac{\left[\Gamma_1^1 \Gamma_2^1 \right]}{4} \quad (C.23c)$$

$$A_{22} a_2^1 a_2^1 = \frac{1}{d} \left[(e_1 - E) - i \frac{(\Gamma_1^1 + \Gamma_1^2)}{2} \right] \frac{\Gamma_2^1}{2} \quad (C.23d)$$

and

$$W_{11} = 1 + \frac{i}{d} \left\{ \frac{\Gamma_2^2 \Gamma_1^1}{2} + \frac{\Gamma_1^2 \Gamma_2^1}{2} + i \left[(e_1 - E) \Gamma_2^1 + (e_2 - E) \Gamma_1^1 \right] \right\} \quad (C.24)$$

The complex quantity d is composed of

$$\text{Re } d = (e_1 - E) (e_2 - E) - \frac{\Gamma_1^2 \Gamma_2^2}{4} - \frac{\Gamma_2^2 \Gamma_1^1}{4} - \frac{\Gamma_1^2 \Gamma_2^1}{4} \quad (C.25a)$$

and

$$\text{Im } d = - \left[(e_2 - E) \left(\frac{\Gamma_1^1 + \Gamma_1^2}{2} \right) + (e_1 - E) \left(\frac{\Gamma_2^1 + \Gamma_2^2}{2} \right) \right] \quad (C.25b)$$

From the quantity U_{11} given by

$$U_{11} = e^{-i\phi_1} W_{11} e^{-i\phi_1} \quad (C.26)$$

one forms the expression for the cross section σ_{11} as

$$\sigma_{11} = g\pi \lambda_1^2 \left| 1 - U_{11} \right|^2 \quad (C.27)$$

where

$$\left| 1 - U_{11} \right|^2 = 4 \sin^2 \phi + 4 \frac{X}{dd^*} \sin^2 \phi - 2 \frac{Y}{dd^*} \sin 2\phi + \frac{C^2 + D^2}{dd^*} \quad (C.28a)$$

and

$$C = \frac{\Gamma_{12}^{21,1}}{2} + \frac{\Gamma_{21}^2 \Gamma_1^1}{2} \quad (C.28b)$$

$$D = (e_1 - E) \Gamma_2^1 + (e_2 - E) \Gamma_1^1 \quad (C.28c)$$

$$a = \text{Re } d \quad (C.28d)$$

$$b = \text{Im } d \quad (C.28e)$$

$$(C.28f)$$

$$X = Ca + Db$$

$$Y = Da - Cb \quad (C.28g)$$

A plot of σ_{11} versus E is given in Figure 2. The calculations were made using the parameters as indicated. For comparison, the same cross section obtained by the Combco program ⁽¹⁵⁾ for the same set of parameters are indicated. It is to be noted that the calculation was made for S-Wave Neutrons and a target with Spin $I = 0$. This simplifies the calculation since there is only one compound state ($J = 1/2$) which contributes to the cross section. It is observed that the cross section σ_{11} obtained from the Combo program ⁽¹⁵⁾ goes slightly negative in one region. This is the result of using the single level Breit Wigner formula which has been pointed out by Otter ⁽¹⁶⁾.

FIG. 2 σ_{11} CALCULATED BY EQUATION (C-27), APPENDIX C, AND COMBO FOR INDICATED RESONANCE PARAMETERS. $\sigma_{POT} = 10.2$ BARNs.