W A S H I N G T O N   U N I V E R S I T Y

Department of Applied Mathematics and Computer Science

THE TRANSPORTATION-LOCATION PROBLEM

by

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1. Introduction to the Problem

In a previous paper [1], the present author formulated a problem which was a generalization of both the Hitchcock "transportation problem" (see Hadley [2]) and a problem that is sometimes referred to as the "location-allocation" problem with unlimited capacities. (See Cooper [3], [4].) We should like to consider here both that problem and an even more general formulation in much greater detail.

We are given n fixed locations, to be called destinations, whose positions in Euclidean space are known. Their co-ordinates will be given by (x_{Dj}, y_{Dj}), $j = 1,\ldots,n$. We are also given a set of known requirements, r_j, $j = 1,\ldots,n$ for some commodity or product at each of the n destinations. We wish to locate m sources, where m is a given number, from which the product is to be shipped. These sources are supposed to have certain limitations on their capacity to ship the product. These numbers are known and will be designated c_i, $i = 1,\ldots,m$. Finally, there may be "weights" relating to destination requirements, e.g., multiplicity of trips in a time period, or other possible weights. These will be designated β_j, $j = 1,\ldots,n$. We shall designate by (x_i, y_i) $i = 1,\ldots,m$, the locations which are to be determined, for the sources and by w_{ij}, the amounts to be "shipped" from source i to destination j, which are also to be determined. Finally we will define a set of "cost functions" which depend on the relative locations of the sources with respect to the destinations. These will be designated:

$\Psi(x_{Dj}, y_{Dj}; x_i, y_i) \equiv$ cost of supplying the $j^{\underline{th}}$ destination from the $i^{\underline{th}}$ source

We assume that the Ψ functions are continuous. Our object is to minimize total cost subject to capacity and requirement constraints. This problem may be formulated as follows:

$$\text{Min } z = \sum_{i=1}^{m} \sum_{j=1}^{n} \beta_j w_{ij} \, \Psi(x_{Dj}, y_{Dj}; x_i, y_i)$$

$$\sum_{j=1}^{n} w_{ij} \leq c_i, \quad i = 1, \ldots, m \qquad (1)$$

$$\sum_{i=1}^{m} w_{ij} = r_j, \quad j = 1, \ldots, n$$

$$x_i \geq 0, \; y_i \geq 0, \; w_{ij} \geq 0, \quad \text{all } i,j$$

The relationship of the problem given in (1) to the transportation problem and the location-allocation problem is readily seen. If the $\Psi(x_{Dj}, y_{Dj}; x_i, y_i) \equiv a_{ij}$, a set of constant costs, and if $a_{ij}\beta_j$ is designated as γ_{ij}, then (1) can be written as:

$$\text{Min } z_1 = \sum_{i=1}^{m} \sum_{j=1}^{n} \gamma_{ij} w_{ij}$$

$$\begin{aligned} &\sum_{j=1}^{n} w_{ij} \leq c_i, \quad i = 1,\ldots,m \\ &\sum_{i=1}^{m} w_{ij} = r_j, \quad j = 1,\ldots,n \\ &w_{ij} \geq 0, \quad \text{all } i,j \end{aligned} \qquad (2)$$

which is the usual form of the transportation problem. Hence, for a set of fixed costs, the problem (1) becomes a transportation problem.

Returning to equations (1), we may also note that 1) if the requirements are stated in terms of the β_j and 2) the capacities associated with each source are unlimited, then the set of m+n constraints of (1) are no longer explicitly required and (1) now becomes:

$$\text{Min } z_2 = \sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_{ij}\beta_j \, \Psi(x_{Dj}, y_{Dj}; x_i, y_i) \qquad (3)$$

$$\alpha_{ij} = 0,1$$

This is the location-allocation problem discussed in [3], [4].

The particular form of the functions Ψ, that we shall be most interested in, throughout this paper is:

$$\Psi(x_{Dj}, y_{Dj}; x_i, y_i) = [(x_{Dj}-x_i)^2 + (y_{Dj}-y_i)^2]^{\frac{1}{2}} \qquad (4)$$

which is the usual Euclidean distance in a two-dimensional space. However, we shall give some consideration to the more general form given in (1).

2. Characteristics of the General Transportation-Location Problem

We shall refer to the problem stated in equations (1) as the "general transportation-location problem". There are a number of important observations that can be made concerning this problem. We will state these results in the form of theorems for convenience.

Theorem 1: A necessary and sufficient condition for (1) to have a feasible set of w_{ij} is that $\sum_{j=1}^{n} r_j \leq \sum_{i=1}^{m} c_i$.

Proof: This is easily shown since if we sum:

$$\sum_{i=1}^{m} w_{ij} = r_j, \quad j = 1,\ldots,n$$

over j we have:

$$\sum_{j=1}^{n} \sum_{i=1}^{m} w_{ij} = \sum_{j=1}^{n} r_j \tag{5}$$

and if we sum:

$$\sum_{j=1}^{n} w_{ij} \leq c_i, \quad i = 1,\ldots,m$$

over i we have:

$$\sum_{i=1}^{m} \sum_{j=1}^{n} w_{ij} \leq \sum_{i=1}^{m} c_i \tag{6}$$

From (5) and (6) we see that:

$$\sum_{j=1}^{n} r_j \leq \sum_{i=1}^{m} c_i \tag{7}$$

Therefore if (7) is true then there is a feasible set of w_{ij}.

In what follows and throughout this paper, we shall assume that the functions $\Psi(x_{Dj},y_{Dj};x_i,y_i)$ are such that (1) has a finite minimum. The next theorem is an important result.

Theorem 2: For the problem:

$$\begin{aligned} \text{Min } z &= \sum_{i=1}^{m}\sum_{j=1}^{n} \beta_j w_{ij}\, \Psi(x_{Dj},y_{Dj};x_i,y_i) \\ &\sum_{j=1}^{n} w_{ij} \leq c_i, \quad i = 1,\ldots,m \\ &\sum_{i=1}^{m} w_{ij} = r_j, \quad j = 1,\ldots,n \\ &x_i \geq 0,\ y_i \geq 0,\ w_{ij} \geq 0, \ \text{all } i,j \end{aligned} \tag{1}$$

an optimal solution will occur at an extreme point of the convex set of feasible solutions to (1).

Proof: Let $W = \{w_{ij} | A\bar{w} \leq \bar{b},\ \bar{w} \geq \bar{0},\ \bar{x} \geq \bar{0},\ \bar{y} \geq \bar{0}\}$

where

$$A = \begin{bmatrix} \bar{I}_n & \bar{0} & \bar{0} & \cdots & \bar{0} \\ \bar{0} & \bar{I}_n & \bar{0} & \cdots & \bar{0} \\ \vdots & \vdots & \vdots & & \vdots \\ \bar{0} & \bar{0} & \bar{0} & \cdots & \bar{I}_n \\ I_n & I_n & I_n & \cdots & I_n \end{bmatrix} \begin{array}{l} \left.\begin{array}{l} \\ \\ \\ \\ \end{array}\right\} \text{m rows} \\ \left.\begin{array}{l} \\ \end{array}\right\} \text{n rows} \end{array}$$

mn columns

$$\bar{b} = [c_1, c_2, \ldots, c_m, r_1, r_2, \ldots, r_n]$$

$$\bar{w} = [w_{11}, w_{12}, \ldots, w_{1n} w_{21}, \ldots, w_{2n}, \ldots, w_{mn}]$$

We shall prove the theorem by contradiction. Assume that $\{\bar{x}^*, \bar{y}^*, \bar{w}^*\}$ is an optimal solution to (1) where $\bar{x} = (x_1, x_2, \ldots, x_m)$, $\bar{y} = (y_1, y_2, \ldots, y_m)$. Further, we assume that the $\bar{w}^*$ vector has $k > m+n$ positive values of w_{ij}. Hence $\bar{w}^*$ is not an extreme point of W.

Let us now consider the following problem:

$$\text{Min } z = \sum_{i=1}^{m} \sum_{j=1}^{n} \beta_j w_{ij} \; \Psi(x_{Dj}, y_{Dj}; x_i^*, y_i^*)$$

$$A \bar{w} \leq \bar{b} \tag{8}$$

$$\bar{w} \geq \bar{0}, \; \bar{x} \geq \bar{0}, \; \bar{y} \geq \bar{0}$$

The functions $\Psi(x_{Dj}, y_{Dj}; x_i^*, y_i^*)$ are simply constants. Let us designate them ψ_{ij}^*. If we define $d_{ij}^* = \beta_j \psi_{ij}^*$, and $\bar{d}^* = (d_{11}^*, d_{12}^*, \ldots, d_{mn}^*)$ then we can write (8) as:

$$\begin{aligned} \text{Min } z &= \bar{d}^*\bar{w} \\ A\,\bar{w} &\leq \bar{b} \\ \bar{w} \geq \bar{0},\ \bar{x} &\geq \bar{0},\ \bar{y} \geq \bar{0} \end{aligned} \tag{9}$$

Consider the optimal solution to (9). Problem (9) is a transportation problem and hence its optimal solution will be at an extreme point of W, say, $\bar{w}_E$, with no more than m+n positive w_{ij}. $\bar{w}_E$ is also a feasible solution to (1) since the set of feasible solutions W is the same for the problems of (1) and of (9).

Now consider the solution $\bar{w}^*$ which is also a feasible solution to (9). For this solution $\bar{d}^*\bar{w}^* \geq \bar{d}^*\bar{w}_E$, since $\bar{w}_E$ was the minimal solution. However, $z = \bar{d}^*\bar{w}^*$ was assumed optimal for (1). Yet, we have found another solution $\{\bar{x}^*, \bar{y}^*, \bar{w}_E\}$ which is feasible and yields a lower or equal value of z than $\{\bar{x}^*, \bar{y}^*, \bar{w}^*\}$. Hence, we have obtained a contradiction and established the result of the theorem.

3. The Transportation-Location Problem

Let us now consider the problem of equations (1) for a particular form of the function, Ψ:

$$\Psi(x_{Dj}, y_{Dj}; x_i, y_i) = [(x_{Dj}-x_i)^2 + (y_{Dj}-y_i)^2]^{\frac{1}{2}} \tag{4}$$

Hence, what we shall now call the "transportation-location problem" is the following:

$$\text{Min } z = \sum_{i=1}^{m} \sum_{j=1}^{n} \beta_j w_{ij} \left[(x_{Dj}-x_i)^2 + (y_{Dj}-y_i)^2 \right]^{\frac{1}{2}}$$

$$\begin{aligned} &\sum_{j=1}^{n} w_{ij} \le c_i, \quad i = 1,\ldots,m \\ &\sum_{i=1}^{m} w_{ij} = r_j, \quad j = 1,\ldots,n \\ &x_i \ge 0,\ y_i \ge 0,\ w_{ij} \ge 0, \quad \text{all } i,j \end{aligned} \tag{10}$$

We shall now characterize the transportation-location problem. The constraint set is clearly a convex set. What of the objective function? Theorem 3 has some less than comforting information.

<u>Theorem 3</u>: The function $z = \sum_{i=1}^{m} \sum_{j=1}^{n} \beta_j w_{ij} [(x_{Dj}-x_i)^2+(y_{Dj}-y_i)^2]^{\frac{1}{2}}$ is neither a convex function nor a concave function of the vector $(\bar{x},\bar{y},\bar{w})$.

<u>Proof</u>: To establish this, we make use of the well known result, [5], that if a function $f(u_1,u_2,\ldots,u_p)$ is a twice continuously differentiable function on an open set, then it is convex, concave or neither on this set, depending on whether the quadratic form:

$$Q(\bar{u},\bar{h}) = \sum_{k=1}^{p} \sum_{\ell=1}^{p} \frac{\partial^2 f}{\partial u_k \partial u_\ell} h_k\, h_\ell \tag{11}$$

is positive semi-definite, negative semi-definite or indefinite. The variables in $\bar{u}$ are the x_i, y_i and w_{ij}. We now compute the necessary second partial derivatives to use in (11). If we let $D_{ij} \equiv [(x_{Dj}-x_i)^2 + (y_{Dj}-y_i)^2]^{\frac{1}{2}}$, then they are as follows:

$$\frac{\partial^2 z}{\partial x_i^2} = \sum_{j=1}^{n} \frac{\beta_j w_{ij}}{D_{ij}} \left[1 - \left(\frac{x_{Dj}-x_i}{D_{ij}} \right)^2 \right], \quad i = 1,\ldots,m \tag{12}$$

$$\frac{\partial^2 z}{\partial y_i^2} = \sum_{j=1}^{n} \frac{\beta_j w_{ij}}{D_{ij}} \left[1 - \left(\frac{y_{Dj}-y_i}{D_{ij}} \right)^2 \right], \quad i = 1,\ldots,m \tag{13}$$

$$\frac{\partial^2 z}{\partial w_{ij}^2} = 0 \quad , \quad \begin{array}{l} i = 1,\ldots,m \\ j = 1,\ldots,n \end{array} \tag{14}$$

$$\frac{\partial^2 z}{\partial x_i \partial y_i} = - \sum_{j=1}^{n} \frac{\beta_j w_{ij} (x_{Dj}-x_i)(y_{Dj}-y_i)}{D_{ij}^3} \quad , \quad i = 1,\ldots,m \tag{15}$$

$$\frac{\partial^2 z}{\partial x_i \partial w_{ij}} = - \frac{\beta_j (x_{Dj}-x_i)}{D_{ij}} \quad , \quad \begin{array}{l} i = 1,\ldots,m \\ j = 1,\ldots,n \end{array} \tag{16}$$

$$\frac{\partial^2 z}{\partial y_i \partial w_{ij}} = - \frac{\beta_j (y_{Dj}-y_i)}{D_{ij}} \quad , \quad \begin{array}{l} i = 1,\ldots,m \\ j = 1,\ldots,n \end{array} \tag{17}$$

$$\frac{\partial^2 z}{\partial x_i \partial x_k} = 0 \;, \quad i \neq k \tag{18}$$

$$\frac{\partial^2 z}{\partial y_i \partial y_k} = 0 \;, \quad i \neq k \tag{19}$$

$$\frac{\partial^2 z}{\partial x_i \partial w_{kj}} = 0 \;, \quad i \neq k \tag{20}$$

$$\frac{\partial^2 z}{\partial y_i \partial w_{kj}} = 0 \quad , \qquad i \neq k \tag{21}$$

$$\frac{\partial^2 z}{\partial w_{ij} \partial w_{kj}} = 0 \quad , \qquad i \neq k \tag{22}$$

$$\frac{\partial^2 z}{\partial x_i \partial y_k} = 0 \quad , \qquad i \neq k \tag{23}$$

In order to associate the arbitrary scalars of the vector $\bar{h}$ with the appropriate expression, we shall call them α_i (for the derivatives with respect to x_i), λ_i (for those with respect to y_i) and γ_{ij} (for those with respect to w_{ij}). If we do this, and express Q in terms of the non-vanishing second partial derivatives, we have:

$$Q = \sum_{i=1}^{m} \frac{\partial^2 z}{\partial x_i^2} \alpha_i^2 + \sum_{i=1}^{m} \frac{\partial^2 z}{\partial y_i^2} \lambda_i^2 + 2 \sum_{i=1}^{m} \frac{\partial^2 z}{\partial x_i \partial y_i} \alpha_i \lambda_i$$
$$+ 2 \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{\partial^2 z}{\partial x_i \partial w_{ij}} \alpha_i \gamma_{ij} + 2 \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{\partial^2 z}{\partial y_i \partial w_{ij}} \lambda_i \gamma_{ij} \tag{24}$$

Substituting from equations (12) - (23) into (24) we have:

$$Q = \sum_{i=1}^{m} \sum_{j=1}^{n} \left\{ \frac{\beta_j w_{ij}}{D_{ij}} \left[1 - \left(\frac{x_{Dj}-x_i}{D_{ij}} \right)^2 \right] \alpha_i^2 + \sum_{i=1}^{m} \sum_{j=1}^{n} \left\{ \frac{\beta_j w_{ij}}{D_{ij}} \left[1 - \left(\frac{y_{Dj}-y_i}{D_{ij}} \right)^2 \right] \right\} \lambda_i^2 \right.$$

$$- 2 \sum_{i=1}^{m} \sum_{j=1}^{n} \left[\frac{\beta_j w_{ij} (x_{Dj}-x_i)(y_{Dj}-y_i)}{D_{ij}^3} \right] \alpha_i \lambda_i - 2 \sum_{i=1}^{m} \sum_{j=1}^{n} \left[\frac{\beta_j (x_{Dj}-x_i)}{D_{ij}} \right] \alpha_i \gamma_{ij}$$

$$- 2 \sum_{i=1}^{m} \sum_{j=1}^{n} \left[\frac{\beta_j (y_{Dj}-y_i)}{D_{ij}} \right] \lambda_i \gamma_{ij} \qquad (25)$$

Let us rewrite (25) as:

$$Q = \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{\beta_j w_{ij}}{D_{ij}^3} \left\{ \alpha_i^2 [D_{ij}^2 - (x_{Dj}-x_i)^2] + \lambda_i^2 [D_{ij}^2 - (y_{Dj}-y_i)^2] \right.$$

$$\left. - 2\alpha_i \lambda_i (x_{Dj}-x_i)(y_{Dj}-y_i) \right\} - 2 \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{\beta_j \gamma_{ij}}{D_{ij}} [\alpha_i (x_{Dj}-x_i) + \lambda_i (y_{Dj}-y_i)] \qquad (26)$$

Since $D_{ij}^2 = (x_{Dj}-x_i)^2 + (y_{Dj}-y_i)^2$ we can simplify (26) to yield:

$$Q = \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{\beta_j w_{ij}}{D_{ij}^3} [\alpha_i^2 (y_{Dj}-y_i)^2 + \lambda_i^2 (x_{Dj}-x_i)^2 - 2\alpha_i \lambda_i (x_{Dj}-x_i)(y_{Dj}-y_i)]$$

$$- 2 \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{\beta_j \gamma_{ij}}{D_{ij}} [\alpha_i ({}_{Dj}-x_i) + \lambda_i (y_{Dj}-y_i)] \qquad (27)$$

which yields in turn:

$$Q = \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{\beta_j w_{ij}}{D_{ij}^3} \left[\alpha_i (y_{Dj}-y_i) - \lambda_i \; (x_{Dj}-x_i)\right]^2$$

$$- 2 \sum_{i=1} \sum_{j=1}^{n} \frac{\beta_i \gamma_{ij}}{D_{ij}} \quad \left[\alpha_i (x_{Dj}-x_i)+\lambda_i (y_{Dj}-y_i)\right] \tag{28}$$

The first term in (28) is a non-negative real scalar. Let us designate this term $M(x_i, y_i, w_{ij}, \alpha_i, \lambda_i) \geq 0$. For some specified value of M, we have also determined, automatically, a value of:

$$\alpha_i (x_{Dj}-x_i) + \lambda_i (y_{Dj}-y_i) \equiv N_{ij}$$

Therefore we can write (28) as:

$$Q = M - 2 \sum_{i=1}^{m} \sum_{j=1}^{n} \gamma_{ij} N_{ij} \tag{29}$$

It is clear that with suitably chosen γ_{ij}, which are completely arbitrary, we can make $Q > 0$ or $Q < 0$ or $Q = 0$. Hence z is neither a concave nor a convex function, which is what we wished to show.

A consequence of theorem 3 is that the transportation-location problem, which is given by equations (10), is a non-convex non-linear programming problem in which we are minimizing a non-convex objective function over a convex set. However, because of Theorem 2, we can state the equivalent result for the transportation-location problem.

Theorem 4: For the problem:

$$\text{Min } z = \sum_{i=1}^{m} \sum_{j=1}^{n} \beta_j w_{ij} \left[(x_{Dj}-x_i)^2 + (y_{Dj}-y_i)^2\right]^{\frac{1}{2}}$$

$$\sum_{j=1}^{n} w_{ij} \le c_i, \quad i = 1,\ldots, m$$

$$\sum_{i=1}^{m} w_{ij} = r_j, \quad i = 1,\ldots, n \tag{10}$$

$$x_i \ge 0, \; y_i \ge 0, \; w_{ij} \ge 0, \text{ all } i,j$$

An optimal solution will occur at an extreme point of the convex set of feasible solutions to (10).

Proof: The problem above is a special case of the general formulation proven in theorem 2.

The importance of this result is that it makes it unnecessary to consider any but basic feasible solutions to the constraints of (10). This will be important when we consider computational approaches in subsequent sections of this report.

Let us consider a result which is somewhat related to the previous result. Suppose, as is often the case in practice, there is more capacity than is required, i.e., $\sum_{i=1}^{m} c_i > \sum_{j=1}^{n} r_j$. A consequence of this is that, under certain conditions, no destination will be served by more than one source. First, let us rewrite the transportation-location problem with equality constraints by adding slack variables. If we do, equations (10) become:

$$
\begin{aligned}
&\text{Min } z = \sum_{i=1}^{m} \sum_{j=1}^{n} \beta_j w_{ij} \left[(x_{Dj}-x_i)^2+(y_{Dj}-y_i)^2\right]^{1/2} \\
&\sum_{j=1}^{n} w_{ij} + w_{is} = c_i, \quad i = 1,\ldots,m \\
&\sum_{i=1}^{m} w_{ij} = r_j \qquad , \quad j = 1,\ldots,n
\end{aligned}
\tag{30}
$$

$$x_i \geq 0, \; y_i \geq 0, \; w_{ij} \geq 0, \; w_{is} \geq 0, \quad \text{all } i,j$$

The result we seek is in the following theorem.

<u>Theorem 5</u>: A sufficient condition for the transportation location problem to have no more than n positive w_{ij} is that

$$c_i \geq \sum_{j=1}^{n} r_j, \; i = 1,\ldots, m \text{ or equivalently, } m \leq \frac{\sum_{i=1}^{m} c_i}{\sum_{j=1}^{n} r_j}$$

<u>Proof</u>: First, we note that if there are m sources (or any number ≥ 2) and n destinations, and if each destination were to be supplied by one and only one source, then there would indeed be only n positive w_{ij}. We will now show that it is sufficient for this to be true that, $c_i \geq \sum_{j=1}^{n} r_j$, $i = 1,\ldots,m$ or $m \leq \frac{\sum_{i=1}^{m} c_i}{\sum_{j=1}^{n} r_j}$. The proof is by contradiction.

Let us assume that the optimal solution to (10) or equivalently, to (30), z^*, is such that there are $n+k$ positive w_{ij} where $k \neq 0$. Let S_1 be the set of destinations supplied by one source and let S_2 be the set of destinations supplied by more than one source. If $D_{ij} = (x_{Dj}-x_i)^2+(y_{Dj}-y_i)^2{}^{\frac{1}{2}}$, then we note that:

$$z^* = \sum_{i=1}^{m} \sum_{j\varepsilon S_1} w_{ij}\ D_{ij} + \sum_{i=1}^{m} \sum_{j\varepsilon S_2} w_{ij}\ D_{ij} \tag{31}$$

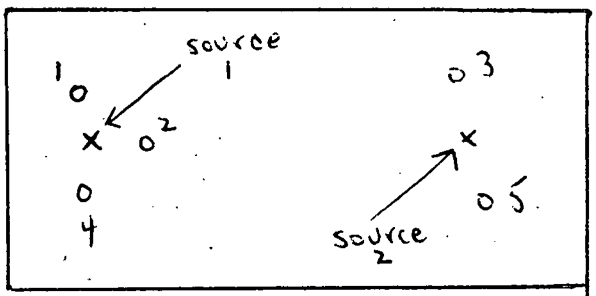
Consider some destination $d_e \varepsilon S_2$. One of its contributions to the second term in (31) is:

$w_{p\ell}\left[(x_{D\ell}-x_q)^2 + (y_{D\ell}-y_q)^2\right]^{\frac{1}{2}}$. These can be written as $w_{p\ell}D_{p\ell}$ and $w_{q\ell}D_{q\ell}$, respectively. Since the sources designated by the subscripts p and q are distinct, the distances $D_{p\ell}$ and $D_{q\ell}$ are unequal. Let us say that $D_{p\ell} < D_{q\ell}$. Therefore, we could reduce $w_{q\ell}$ to zero and supply destination d_ℓ by source p, which we can do since by hypothesis $c_p \geq \sum_{j=1}^{n} r_j$ and hence, there is sufficient capacity. If we do this, we reduce the sum z^* by an amount $w_{q\ell} - w_{q\ell}D_{p\ell}$. Therefore,

$$z' = z^* - (w_{q\ell}\ D_{q\ell} - w_{p\ell}D_{p\ell})$$

Since $(w_{q\ell}D_{q\ell} - w_{p\ell}D_{p\ell}) > 0$, we have that $z' < z^*$ which is a contradiction. Therefore, in the optimal solution, each destination is supplied by a single source. It should be noted that under these conditions, it is obvious that the slack variables w_{is} in in equations (30) are always positive.

That theorem 5 gives only a sufficient condition and is not a necessary condition is easily shown by a simple example. Consider a 2 source-5 destination problem $c_1 = 160$, $c_2 = 100$, $r_1 = 20$, $r_2 = 30$, $r_3 = 40$, $r_4 = 50$, $r_5 = 60$. It can be seen that $\sum_{j=1}^{5} r_j = 200$ and that $c_1 < \sum_{j=1}^{5} r_j$ and $c_2 < \sum_{j=1}^{5} r_j$, thus violating the hypothesis of theorem 5. Nevertheless, suppose the destinations were arranged as follows:



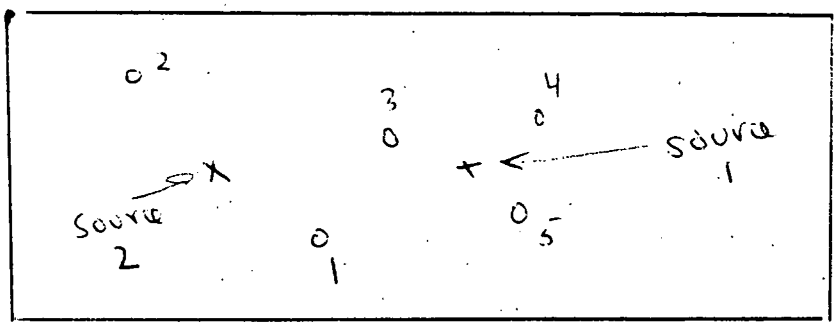
We could optimally supply them with the w_{ij} as shown in the following tableau:

Destinations

Sources	1	2	3	4	5	c_i
1	0	0	40	0	60	160
2	20	30	0	50	0	100
r_j	20	30	40	50	60	200 / 260

It can be seen that there are only n = 5 positive w_{ij} so that the condition $c_i > \sum_{j=1}^{n} r_j$ is not necessary.

It can be seen from the previous example that although the condition of theorem 5 is not met, it is true that $\sum_{i=1}^{m} c_i > \sum_{j=1}^{n} r_j$. However, while this is necessary and sufficient for the existence of a feasible solution to the transportation-location problem (Theorem 1), it says nothing about the existence of a solution with exactly n positive w_{ij}. For example, with the following arrangement of destinations:



and with $c_1 = 260$, $c_2 = 40$, $r_1 = 25$, $r_2 = 30$, $r_3 = 70$, $r_4 = 80$, $r_5 = 90$ we have:

Sources						
1	20	0	70	80	90	260
2	5	30	0	0	0	40
r_j	25	30	70	80	90	300 / 295

It can be seen that $\sum_{i=1}^{m} c_i > \sum_{j=1}^{n} r_j$, the sufficient condition is not satisfied, and we require n+1 positive w_{ij}. It does not seem simple to state a necessary and sufficient condition or even a necessary condition, that there exist a solution with exactly n positive w_{ij}.

4. The First Exact Algorithm For The Transportation-Location Problem

The first algorithm we shall consider is exact and relatively simple in concept. However, its use will be limited, as will be evident, to relatively small problems.

We may note, according to theorems 2 and 4, that an optimal solution to the transportation-location problem will occur at an extreme point of the convex set of solutions, $W=\{\bar{w}_{ij} | A\bar{w} \leq \bar{b}, \bar{w} \geq \bar{0}\}$ where A, $\bar{w}$ are as defined in Theorem 2. We know that the extreme points of W correspond to basic feasible solutions of $A\bar{w} \leq b$, $\bar{w} \geq \bar{0}$. If the number of basic feasible solutions is designated as N_{BFS}, for an m source, n destination problem, $N_{BFS} \leq \binom{mn}{m+n-1}$. This follows from the basic theory of the transportation problem [2]. Actually, $\binom{mn}{m+n-1}$ is the number of basic solutions, most of which are infeasible. Hence, it is usually the case that $N_{BFS} << \binom{mn}{m+n-1}$. For example, in the simple example, we shall present, with m = 2, n = 4, $\binom{mn}{m+n-1} = \binom{8}{5} = \frac{8!}{5!3!} = 56$. However, there are only 9 different non-degenerate basic feasible solutions.

According to results of Demuth [6] and Doig [7], the minimum number of basic feasible solutions, for $m \leq n$, to a transportation problem of order $m \times n$ is $m!/(n-m+1)!$ For m = 2, n = 4, we have $4!/3! = 4$. Hence, the number of bases we have had to examine, 9 is much closer to 4, the minimum number, than the unrealistic upper bound, 56. This is somewhat encouraging.

In any case, N_{BFS} is a finite number. Suppose we generate all basic feasible solutions. Let us designate any such solution $\hat{\bar{w}} = \{\hat{w}_{ij}\}$. (We consider, subsequently, how to do this). For each such solution, we can then solve the problem:

$$\text{Min } \hat{z} = \sum_{i=1}^{m} \sum_{j=1}^{n} \hat{w}_{ij} \left[(x_{Dj}-x_i)^2+(y_{Dj}-y_i)^2\right]^{\frac{1}{2}} \qquad (32)$$

by considering this problem as a set of problems of the form:

$$\text{Min } \hat{z}_i = \sum_{j=1}^{n} \hat{w}_{ij} \left[(x_{Dj}-x_i)^2+(y_{Dj}-y_i)^2\right]^{\frac{1}{2}}, \ i=1,\ldots,m \qquad (33)$$

We can do this since the $\{\hat{w}_{ij}\}$ are simply known non-negative constants or weights. An iterative technique for solving problems of the form given by equations (33) was given by the author in [3], [4] and is repeated here for convenience. The initial estimates of (x_i,y_i) are given by:

$$x_i^0 = \frac{\sum_{j=1}^{n} \hat{w}_{ij} x_{Dj}}{\sum_{j=1}^{n} \hat{w}_{ij}} \ ; \ y_i^0 = \frac{\sum_{j=1}^{n} \hat{w}_{ij} y_{Dj}}{\sum_{j=1}^{n} \hat{w}_{ij}}, \ i = 1,\ldots,m \qquad (34)$$

and the general iteration equations are:

$$x_i^{k+1} = \frac{\sum_{j=1}^{n} \frac{\hat{w}_{ij} x_{Dj}}{D_{ij}^k}}{\sum_{j=1}^{n} \frac{\hat{w}_{ij}}{D_{ij}^k}} \ ; \ y_i^{k+1} = \frac{\sum_{j=1}^{n} \frac{\hat{w}_{ij} y_{Dj}}{D_{ij}^k}}{\sum_{j=1}^{n} \frac{\hat{w}_{ij}}{D_{ij}^k}}, \ i=1,\ldots,m \qquad (35)$$

where the superscript on the x_i and y_i is the iteration parameter and

$$D_{ij}^{k} = \left[(x_{Dj}-x_i^k)^2+(y_{Dj}-y_i^k)^2\right]^{\frac{1}{2}} \tag{36}$$

If we now designate the minimum value of $\hat{z}$, for the $\ell^{\underline{th}}$ basic feasible solution as z_ℓ^*, then it is obvious that the optimal value of the objective function, z^*, for the transportation-location problem will be:

$$z^* = \underset{\forall \ell}{\text{Min}} \; z_\ell^* \tag{37}$$

If the minimum is taken on at $\ell = s$, then the optimal values of the variables are (x_{is}, y_{is}), $i = 1,\ldots,m$ and $\hat{w}_{ijs}$, $i = 1,\ldots,m$; $j = 1,\ldots,n$ where the designation (x_{is}, y_{is}) indicates (x_i, y_i) for the $s^{\underline{th}}$ basic feasible solution and $\hat{w}_{ijs}$ are the set of $\hat{w}_{ij}$ for this solution.

Let us now return to a point we glossed over, earlier, viz., generating all the basic feasible solutions to the constraints of the transportation-location problem, i.e., the constraints:

$$\begin{aligned} \sum_{j=1}^{n} w_{ij} &\le c_i, \quad i = 1,\ldots,m \\ \sum_{i=1}^{m} w_{ij} &= r_j, \quad j = 1,\ldots,n \\ w_{ij} &\ge 0 \;, \quad j = 1,\ldots,n \end{aligned} \tag{38}$$

As has already been mentioned, these are simply the constraints of the standard transportation problem. We can make use of the transportation problem tableau [2] to advantage in order to generate only basic feasible solutions. The alternative would be to find all basic solutions and discard the infeasible solutions. This may require orders of magnitude more work. The method described below is more efficient.

As an example, consider a problem with m = 2, n = 3. There will be at most 2+3-1 = 4 non-zero values of w_{ij} in a basic feasible solution. Suppose $c_1 = 80$, $c_2 = 120$, $r_1 = 70$, $r_2 = 90$, $r_3 = 40$. An initial tableau might be:

	1	2	3	c_i
1	70	10		80
2		80	40	120
r_j	70	90	40	

(1)

The blank squares have zero values of w_{ij}, i.e., $w_{13} = 0$, $w_{21} = 0$. It is a simple matter to find all basic feasible solutions from this initial tableau (1). We can use the standard "loop" method for allowing a zero variable to become positive and still remain feasible. (See [2]). This is merely the application of the simplex method to the special case of the transportation problem. From the tableau (1), we can generate two new tableaux. These would be:

	1	2	3	c_i
1		80		80
2	70	10	40	120
r_j	70	90	40	

2

	1	2	3	c_i
1	70		10	80
2		90	30	120
r_j	70	90	40	

3

From 2 we can generate one new tableau (the other one would be 1). It is as follows:

	1	2	3	c_i
1		40	40	80
2	70	50		120
r_j	70	90	40	

4

From 3 we can generate:

	1	2	3	c_i
1	40		40	80
2	30	90		120
r_j	70	90	40	

5

From 4 we can generate:

	1	2	3	c_i
1	40		40	80
2	30	90		120
r_j	70	90	40	

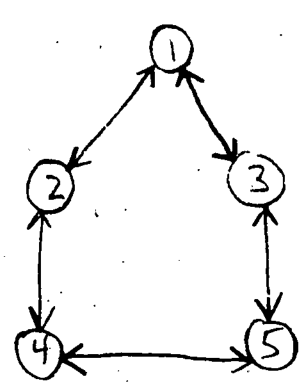
5

From 5 we can generate:

	1	2	3	c_i
1		40	40	80
2	70	50		120
r_j	70	90	40	

4

We have now generated all the basic feasible solutions. The relationships between them for this example can be represented as a directed graph:



It can be seen that there are 5 basic feasible solutions. If all basic solutions had been found, we would have had to solve 4 simultaneous equations in 4 variables, $\binom{6}{4} = 15$ times. The above procedure is simpler and very much less work.

We now state the algorithm we have been discussing for solving the transportation-location problem and then present a numerical example.

Enumeration Algorithm for Transportation-Location Problem

1. Using the transportation problem tableau, starting with any basic feasible solution, generate the connected graph of all basic feasible solutions. Designate each such solution, $\{\hat{w}_{ij\ell}\}$, $\ell = 1,\ldots,T$, where there are T basic feasible solutions.

2. For each such solution, solve the set of location problems:

$$\text{Min } \hat{z}_{i\ell} = \sum_{j=1}^{n} \hat{w}_{ij\ell} \left[(x_{Dj}-x_{i\ell})^2+(y_{Dj}-y_{i\ell})^2\right]^{\frac{1}{2}}, \quad i=1,\ldots,m$$

and $$z_\ell^* = \sum_{i=1}^{m} \hat{z}_{i\ell}$$

3. The optimal solution is found by:

$$z^* = \min_{\ell=1,\ldots,T} z_\ell^*$$

with $W_{ij\ell}^*$ and $(x_{i\ell}^*, y_{i\ell}^*)$ being the corresponding values of the variables.

Enumeration Algorithm - Sample problem:

Let $(x_{D1}, y_{D1}) = (0,0)$; $(x_{D2}, y_{D2}) = (0,1)$; $(x_{D3}, y_{D3}) = (1,1)$; $(x_{D4}, y_{D4}) = (1,0)$; $m = 2$, $n = 4$; $c_1 = 50$, $c_2 = 100$; $r_1 = 20$, $r_2 = 40$, $r_3 = 60$, $r_4 = 30$. A basic feasible solution is given in the tableau:

	1	2	3	4	c_i
1	20	30			50
2		10	60	30	100
r_j	20	40	60	30	

(1)

From (1) we can generate:

	1	2	3	4	c_i
1	10	40			50
2	10		60	30	100
r_j	20	40	60	30	

(2)

	1	2	3	4	c_i
1	20		30		50
2		40	30	30	100
r_j	20	40	60	30	

(3)

	1	2	3	4	c_i
1	20	0		30	50
2		40	60		100
r_j	20	40	60	30	

(4)

From (2) we can generate:

	1	2	3	4	c_i
1		40	10		50
2	20		50	30	100
r_j	20	40	60	30	

(5)

	1	2	3	4	c_i
1		40		10	50
2	20		60	20	100
r_j	20	40	60	30	

(6)

From 3 we can generate:

	1	2	3	4	c_i
1			50		50
2	20	40	10	30	100
r_j	20	40	60	30	

7

	1	2	3	4	c_i
1	20		0	30	50
2		40	60		100
r_j	20	40	60	30	

4

From 4 we can generate:

	1	2	3	4	c_i
1		20		30	50
2	20	20	60		100
r_j	20	40	60	30	

8

	1	2	3	4	c_i
1	20		0	30	50
2		40	60		100
r_j	20	40	60	30	

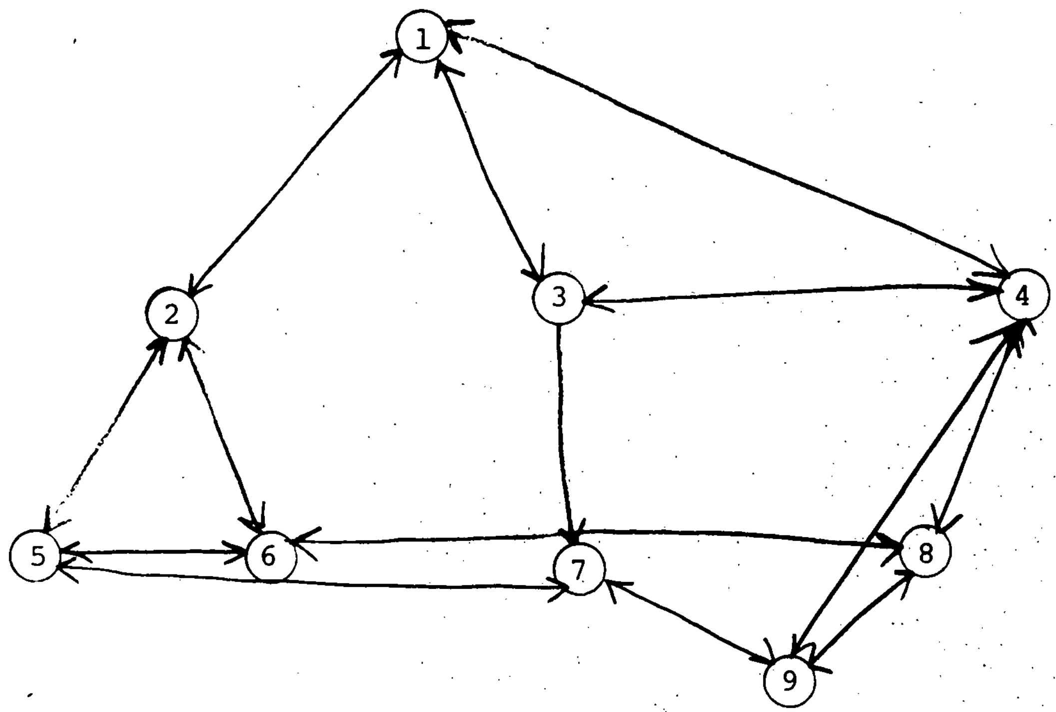
4

From 5 we can generate 6 and 7 (which are not given again to conserve space). From 6 we can generate 5 and 8 . From 7 we generate 5 and a new tableau:

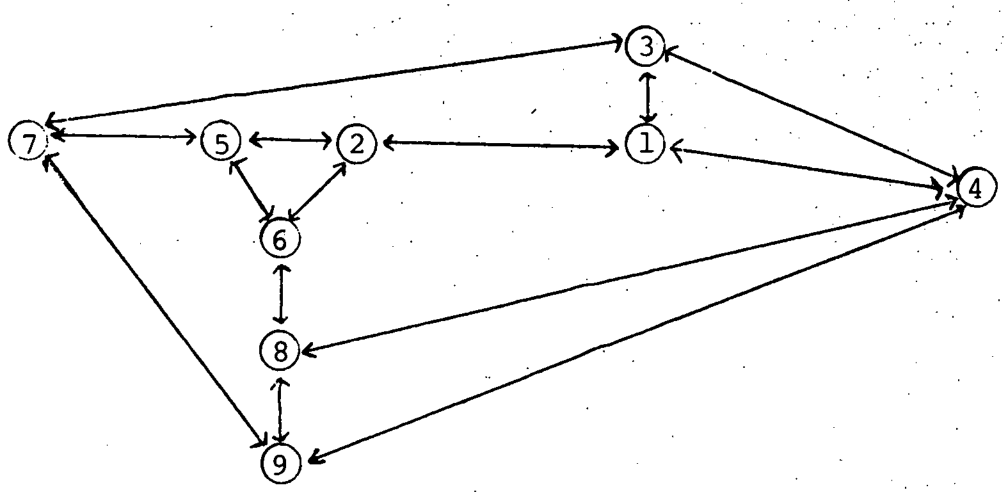
		20	30	50
20	40	40		100
20	40	60	30	

9

From 8 we generate 6 and 9. From 9 we generate 4 and 8. Hence we are done. The graph of the nine basic feasible solutions is as follows:



A more simple representation is as follows:



For each of these nine basic feasible solutions to the constraints of the transportation-location problem, two location problems were solved with the following results:

BASIC FEASIBLE SOLUTION	(x_i, y_i)	$z_{i\ell}$	z^*_ℓ
1	(0,1) (1,1)	20 40	60
2	(0,1) (1,1)	10 44.143	54.143
3	(1,1) (0.641,0.792)	28.286 65.490	93.776
4	(1,0) (1,1)	20 40	60
5	(0,1) (1,1)	10 58.284	68.284
6	(0,1) (1,1)	14.142 48.286	62.428
7	(1,1) (0.287,0.596)	0 69.126	69.126
8	(1,0) (1,1)	28.286 48.286	76.572
9	(1,0) (0.173,0.924)	20 59.579	79.579

It can be seen that the minimum occurs for solution 2 . Hence the solution to our problem is: $w_{11} = 10$, $w_{12} = 40$, $w_{13} = 0$, $w_{14} = 0$; $w_{21} = 10$, $w_{22} = 0$, $w_{23} = 60$, $w_{24} = 30$; $(x_1, y_1) = (0,1)$, $(x_2, y_2) = (1,1)$; $z^* = 54.143$.

It is not difficult to see that the number of basic feasible solutions grows combinatorially. In addition, large amounts of storage are required for larger problems. Hence, this method is restricted to problems of relatively small size.

5. The Second Exact Algorithm for the Transportation-Location Problem

It is possible to formulate the transportation-location problem, given by equations (10) in the form of a mixed integer-continuous variable linear programming problem. The importance of being able to do this is that since algorithms exist for solving such problems, one can obtain the global minimum to what was originally a non-convex nonlinear programming problem. Unfortunately, the size of the problem that results precludes the use of presently existing mixed integer-continuous variable linear programming algorithms. We now derive this formulation.

We first seek to formulate equations (10) in spearable form*. As a first step, we may rewrite our problem as follows:

$$\text{Min } z = \sum_{i=1}^{m} \sum_{j=1}^{n} \beta_j w_{ij} D_{ij}$$

$$\sum_{j=1}^{n} w_{ij} \leq c_j, \quad i = 1,\ldots,m \tag{39}$$

$$\sum_{i=1}^{m} w_{ij} = r_j, \quad j = 1,\ldots,n$$

$$(x_{Dj}-x_i)^2 + (y_{Dj}-y_i)^2 = D_{ij}^2, \quad \begin{array}{l} i = 1,\ldots,m \\ j = 1,\ldots,n \end{array}$$

$$x_i, y_i \geq 0, \quad i = 1,\ldots,m$$

$$w_{ij} \geq 0, \; D_{ij} \geq 0, \quad \begin{array}{l} i = 1,\ldots,m \\ j = 1,\ldots,n \end{array}$$

---

*By separable, we mean that all functions in the constraints or objective function can be expressed as $f(\bar{x}) = \sum_{r=1}^{N} f_r(x_r)$.

Equations (39) are still not in separable form since there are cross-products, $w_{ij}D_{ij}$ of variables in the objective function. These can be separated as follows:

$$w_{ij}D_{ij} = u_{ij}^2 - v_{ij}^2 \tag{40}$$

where

$$u_{ij} = \frac{1}{2}(w_{ij}+D_{ij}); \quad v_{ij} = \frac{1}{2}(w_{ij}-D_{ij}) \tag{41}$$

Using (40) and (41) we can now rewrite (39) as follows:

$$\text{Min } z = \sum_{i=1}^{m} \sum_{j=1}^{n} \beta_j (u_{ij}^2 - v_{ij}^2)$$

$$\begin{aligned}
&\sum_{j=1}^{n} w_{ij} \leq c_i, \quad i = 1,\ldots,m \\
&\sum_{i=1}^{m} w_{ij} = r_j, \quad j = 1,\ldots,n \\
&(x_{Dj}-x_i)^2 + (y_{Dj}-y_i)^2 = D_{ij}^2 && i = 1,\ldots,m;\ j = 1,\ldots,n \\
&u_{ij} = \frac{1}{2}(w_{ij}+D_{ij}) && i = 1,\ldots,m;\ j = 1,\ldots,n \\
&v_{ij} = \frac{1}{2}(w_{ij}-D_{ij}) && i = 1,\ldots,m;\ j = 1,\ldots,n \\
&x_i, y_i \geq 0 && i = 1,\ldots,m \\
&w_{ij} \geq 0,\ D_{ij} \geq 0,\ u_{ij} \geq 0 && i = 1,\ldots,m;\ j = 1,\ldots,n \\
&v_{ij} \text{ unrestricted}
\end{aligned} \tag{42}$$

Equations (42) are equivalent to the original transportation-location problem given by equations (10). They are in separable form. There are 3mn+m+n constraints and 4mn+2m variables. All the constraints are linear except for mn of them, viz, $(x_{Dj}-x_i)^2 + (y_{Dj}+y_i)^2 = D_{ij}^2$ for all i,j. In order to use separable programming (see [2]), we shall make polygonal approximations to the nonlinear variables in these constraints. Similarly, the variables in the objective function are non-linear and so they also will be approximated in the same way. The variables which require the polygonal approximations are: u_{ij}^2, v_{ij}^2, x_i^2, y_i^2, D_{ij}^2.

Let us briefly recapitulate the "δ-form" of the approximation problem [2]. Let the original problem be:

$$\text{Min } Z = \sum_{t=1}^{q} f_t(x_t)$$

$$\sum_{t=1}^{q} g_{st}(x_t) \leq b_s, \quad s = 1,\ldots,p \tag{43}$$

$$x_t \geq 0, \quad t = 1,\ldots,q$$

If we have or impose upper bounds on the x_t so that $0 \leq x_t \leq \alpha_t$, we subdivide each x_t into R_t intervals so that we have R_t+1

points x_{kt}, $x_{0t} = 0$, $x_{1t}, x_{2t}, \ldots, x_{R_t,t}$ We then define, for each x_t:

$$f_{kt} = f_t(x_{kt}), \quad t = 1,\ldots,q$$

$$g_{kst} = g_{st}(x_{kt}), \quad \begin{array}{l} s = 1,\ldots,p \\ t = 1,\ldots,q \end{array} \tag{44}$$

With these conventions we now define:

$$\begin{aligned} \Delta f_{kt} &= f_{kt} - f_{k-1,t} \\ \Delta g_{kst} &= g_{kst} - g_{k-1,st} \qquad k = 1,\ldots,R_t \\ \Delta x_{kt} &= x_{kt} - x_{k-1,t} \end{aligned} \tag{45}$$

For $x_{k-1,t} \leq x_t \leq x_{kt}$ we also define:

$$\delta_{kt} = \frac{x_t - x_{k-1,t}}{\Delta x_{kt}} \tag{46}$$

Using (44), (45) and (46), we can approximate the original form of the problem, given by (43) with the following:

$$\text{Min } \hat{Z} = \sum_{t=1}^{q} \sum_{k=1}^{R_t} (\Delta f_{kt})\delta_{kt}$$

$$\sum_{t=1}^{q} \sum_{k=1}^{R_t} (\Delta g_{kst})\delta_{kt} \leq b_s - \sum_{t=1}^{q} g_{0st}, \quad s = 1,\ldots,p \tag{47}$$

$$0 \leq \delta_{kt} \leq 1, \quad \begin{array}{l} k = 1,\ldots,R_t \\ t = 1,\ldots,q \end{array}$$

where we also require that if $\delta_{kt} > 0$, $\delta_{\ell t} = 1$, $\ell = 1,\ldots,k-1$.

Using the above theory on the δ-form of the approximation problem, let us now cast the problem given by equation (42) in that form. In order to do so, we require the following definitions:

$$
\begin{aligned}
f_{ijk} &= \beta_j u_{ijk}^2 \\
g_{ijk} &= \beta_j v_{ijk}^2 \\
F_{ik} &= x_{ik}^2 \\
G_{ik} &= y_{ik}^2 \\
H_{ijk} &= D_{ijk}^2
\end{aligned}
\qquad
\begin{aligned}
i &= 1,\ldots,m \\
j &= 1,\ldots,n \\
k &= 1,\ldots,R_{ij}
\end{aligned}
\tag{48}
$$

Using the definitions given in (45) and (46) we have quantities corresponding to the definitions given in (48): Δf_{ij}, Δg_{ijk}, ΔF_{ik}, ΔG_{ik}, ΔH_{ijk}. Using these, we can now phrase our problem given by equations (42) as:

$$\text{Min } \hat{z} = \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{R_{ij}} (\Delta f_{ijk} - \Delta g_{ijk})\delta_{ijk}$$

$$\sum_{j=1}^{n} w_{ij} \leq c_i, \quad i = 1,\ldots,m$$

$$\sum_{i=1}^{m} w_{ij} = r_j, \quad j = 1,\ldots,n$$

$$\sum_{k=1}^{R_{ij}} (F_{ik}+G_{ik}-H_{ijk})\delta_{ijk} - 2x_{Dj}x_i - 2y_{Dj}y_i = H_{ij0} - F_{i0} - G_{i0} - x_{Dj}^2 - y_{Dj}^2, \quad \begin{array}{l} i = 1,\ldots,m \\ j = 1,\ldots,n \end{array} \qquad (49)$$

$$2u_{ij} - w_{ij} - D_{ij} = 0 \quad \begin{array}{l} i = 1,\ldots,m \\ j = 1,\ldots,n \end{array}$$

$$2v_{ij} - w_{ij} + D_{ij} = 0 \quad \begin{array}{l} i = 1,\ldots,m \\ j = 1,\ldots,n \end{array}$$

$$x_i y_i \geq 0 \quad i = 1,\ldots,m$$

$$w_{ij} \geq 0,\ D_{ij} \geq 0,\ u_{ij} \geq 0, \quad \begin{array}{l} i = 1,\ldots,m \\ j = 1,\ldots,n \end{array}$$

$$v_{ij} \text{ unrestricted}$$

and if $\delta_{ijk} > 0$, $\delta_{ij\ell} = 1$, $\ell = 1,\ldots,k-1$.

We now have $3mn+m+n$ constraints((as before) but the number of variables has become $4mn+mn \sum_{i,j} R_{ij}+m+n$.

The last step is to convert the separable programming problem (49) into a mixed integer-continuous variable linear programming problem so that an approximate global minimum can be found. Following Hadley [2], the problem in (49) can be represented as:

$$\text{Min } \hat{Z} = \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{R_{ij}} (\Delta f_{ijk} - \Delta g_{ijk}) \delta_{ijk}$$

$$\sum_{j=1}^{n} w_{ij} \leq c_i, \qquad i = 1,\ldots,m$$

$$\sum_{i=1}^{m} w_{ij} = r_j, \qquad j = 1,\ldots,n$$

$$\sum_{k=1}^{R_{ij}} (F_{ik}+G_{ik}-H_{ijk})\delta_{ijk} - 2x_{Dj}x_i - 2y_{Dj}y_i = H_{ij0} - F_{i0} - G_{i0} - x_{Dj}^2 - y_{Dj}^2, \qquad \begin{array}{l} i = 1,\ldots,m \\ j = 1,\ldots,n \end{array} \tag{50}$$

$$2u_{ij} - w_{ij} - D_{ij} = 0 \qquad \begin{array}{l} i = 1,\ldots,m \\ j = 1,\ldots,n \end{array}$$

$$2v_{ij} - w_{ij} + D_{ij} = 0 \qquad \begin{array}{l} i = 1,\ldots,m \\ j = 1,\ldots,n \end{array}$$

$$\delta_{ijk} - \psi_{ijk} \geq 0 \qquad \begin{array}{l} i = 1,\ldots,m \\ j = 1,\ldots,n \\ k = 1,\ldots,R_{ij} \end{array}$$

$$\delta_{ij,k+1} - \psi_{ijk} \leq 0 \qquad \begin{array}{l} i = 1,\ldots,m \\ j = 1,\ldots,n \\ k = 1,\ldots,R_{ij} \end{array}$$

$$0 \leq \delta_{ijk} \leq 1 \qquad \begin{array}{l} i = 1,\ldots,m \\ j = 1,\ldots,n \\ k = 1,\ldots,R_{ij} \end{array}$$

$$x_i y_i \geq 0 \qquad i = 1,\ldots,m$$

$$w_{ij} \geq 0,\ D_{ij} \geq 0,\ u_{ij} \geq 0 \qquad \begin{array}{l} i = 1,\ldots,m \\ j = 1,\ldots,n \end{array}$$

$$\psi_{ijk} \geq 0,\ \psi_{ijk} \text{ integers} \qquad \begin{array}{l} i = 1,\ldots,m \\ j = 1,\ldots,n \\ k = 1,\ldots,R_{ij} \end{array}$$

$$v_{ij} \text{ unrestricted}$$

As can be seen from equations (50), the original problem has grown to be quite large. The formulation given has $3mn\left(\sum_{i=1}^{m} \sum_{j=1}^{n} R_{ij}+1\right)+m+n$ constraints (including upper bounds). This is an exceedingly large problem. Nevertheless, if mixed integer-continuous variable linear programming codes become available so that large problems can be solved efficiently, the solution obtained would be an approximate global minimum to the original problem given by equations (10).

6. Heuristic Algorithm No. 1 for the Transportation Location Problem

The heuristic algorithm described in this section was a first attempt to devise a rapid suboptimal method for solving transportation-location problems. It was suggested by a heuristic method previously developed for the pure location-allocation problem and described in [4]. This method, called the "alternate location-allocation method", is as follows:

1) Select some subset, m of the n destinations which are given and consider these as source locations.
2) Allocate each of the remaining n-m destinations to the closest of the m sources selected in Step 1.
3) Within each of the m sets of destinations determined in Step 2, use the iteration method given in [3], [4] (also used in equations (34) and (35) of this paper) to find the exact location of the optimal source location.
4) Determine for each destination whether or not it is closer to another of the sources located in Step 3 than the one to which it is allocated. This defines a new grouping of m subsets of destinations.
5) Repeat Steps 3 and 4 until no further changes are possible.

It is shown in [4] that this algorithm is a moderately successful one, but by no means the best of the several heuristics tested for the pure location-allocation problem. A modification of this method for the transportation-location problem can be made as follows.

Alternate Transportation-Location Heuristic

1) Arbitrarilv select m of the (x_{Dj}, y_{Dj}) and let these be the m initial source locations. This then yields a set of distances between each of the destinations and the assumed sources.

2) Using these distances as cost coefficients, γ_{ij} (as in equations (2)) we can solve an ordinary transportation problem to find a set of $\{w_{ij}\}$.

3) Using the $\{w_{ij}\}$ from Step 2 we can solve a location problem using equations (34) and (35) and find a new set of source locations.

4) We now iterate Steps 2 and 3 until no further changes are obtained in two successive cycles.

It can be seen that the general notion behind this approximate method is to alternately locate sources given a pattern of allocation given a set of source locations. The location-allocation problem methodology and the usual transportation problem methodology are alternately applied to perform the calculations. It can readily be seen that this iteration method yields a convergent monotone non-increasing sequence of values for z. However, there is no guarantee that it will converge to the global maximum we seek. However, what experience we have with this and similar algorithms indicates that the result, when not optimal, lies within ~10% and usually within 2-3% of the optimal solution. Table I indicates the results with this heuristic method for the first seven problems given in [3]. These are location-allocation

TABLE I

Alternate Transportation-Location Heuristic Results

PROBLEM NO.	ALLOCATIONS OBTAINED	OPTIMAL ALLOCATIONS	Z OBTAINED	OPTIMAL Z
1	(1,2,3,4,5) (6,7)	(1,2,4,5) (3,6,7)	52.118	50.450
2	(1,3,4,6,7) (2,5)	(1,3,4,6) (2,5,7)	81.764	72.000
3	(1,2,3,4) (5,6,7)	(1,2,3,4) (5,6,7)	38.323	38.323
4	(1,2,3,7) (4,5,6)	(1,2,3,7) (4,5,6)	48.850	48.850
5	(1,2,3,4,5) (6,7)	(1,2,3,5) (4,6,7)	38.560	38.033
6	(1,2,3) (4,5,6,7)	(1,2,3,4) (5,6,7)	44.564	36.175
7	(1,5,6,7) (2,3,4)	(1,3,4,5,6,7) (2)	61.935	59.716

problems with $m = 2$, $n = 7$. They were solved as transportation-location problems by using a set of capacities and requirements that satisfied the sufficient condition given in Theorem 5. The starting points in each of these problems was completely random. For problems 3 and 4 the optimal solutions were obtained. In the others, results of varying degrees of closeness to optimality are obtained.

In order to study how the results obtained by the heuristic are related to the starting value, 17 different (randomly selected) starting values were chosen and the heuristic then applied for Problem 1 of Table I. Table II indicates the results obtained. It will be noted from Table I, that for problem number 1, the optimal allocations are (1,2,4,5) (3,6,7) and the optimal value of z is 50.450.

From the results of Table II we can see that in 3 of the 17 trials we obtained the optimal solution. In roughly 60% of the trials we obtained a solution no worse then about 3% of optimal and usually better than this. The maximum error was about 15%. However, it can be seen that repeated use of the heuristic method, with varying starting values, is apt to give a reasonably good good approximation to the optimal solution, if not the optimal solution itself.

TABLE II

Summary of Local Minima Found for Problem I

Allocation Obtained	z Obtained	No. of Times (out of 17) This Solution Occurs
(1,2,3,4,5) (6,7)	52.118	3
(1,2,4,5) (3,6,7)	50.450	3 (optimal solution)
(1,2,3,4) (5,6,7)	52.001	4
(1,3,4,5,6,7) (2)	59.705	3
(1,2,3,7) (4,5,6)	60.128	2
(1,2,4,5,6) (3,7)	57.672	2

7. Heuristic Algorithm No. 2 for the Transportation-Location Problem

This algorithm is based on earlier work reported in [1]. It seems to be extremely efficient as will be seen. The heuristic method that has been developed is as follows.

Transportation-Location Heuristic Method

1. Use of one of the heuristic methods reported in [4] (the "Alternate Location-Allocation" method referred to in the previous section is one of these) to find a solution with unspecified destination requirements and unlimited source capacities. Let matrix $A = ||\alpha_{ij}||$ be the allocation matrix for the solution obtained, i.e., $\alpha_{ij} = 1$ if source i supplies destination j and $\alpha_{ij} = 0$ otherwise. A is a matrix of zeros and ones such that each column contains exactly one "1". There are no restrictions on the rows since one source may supply more than one destination. The allocation matrix A is merely a convenient way of indicating the subdivision of n destinations into m subsets, i.e., m subsets served by each of the m sources.

2. Replace the non-zero elements of A by their respective r_j thus forming a new matrix $W_1 = ||w'_{ij}||$ where

$$w'_{ij} = \begin{cases} r_j, & a_{ij} = 1 \\ 0, & a_{ij} = 0 \end{cases}$$

3. Using the original matrix A of allocations we now derive two new matrices, analogous to W_1 of Step 2 above as follows. Find the pair of points being served by the same source such that distance between them is a maximum. Let the sources be s_u, $u = 1,\ldots,m$ and Q_u be the subsets of destinations. Then we wish to find a pair of points p_s, p_t as follows:

$$D_u = \underset{k,\ell \varepsilon Q_u}{\text{Max}} [(x_{Dk}-x_{D\ell})^2 + (y_{Dk}-y_D)^2]^{\frac{1}{2}}$$

(8)

$$[(x_{Ds}-x_{Dt})^2 + (y_{Ds}-y_{Dt})^2]^{\frac{1}{2}} = \underset{u=1,\ldots,m}{\text{Max}} D_u$$

Let the subset in which this occurs be Q_h with source s_h. We then eliminate source s_h and replace it with points p_s, p_t. We now have a set of m+1 sources $(s_u,\ u=1,\ldots,m;\ u\neq h),\ p_s, p_t$. Therefore we have one more source than is desired. Let $R = \{(s_u,\ u=1,\ldots,m;\ u\neq h),\ p_s, p_t\}$. The source coordinates are (x_u, y_u) for $u\neq h$, (x_{Ds}, y_{Ds}), (x_{Dt}, y_{Dt}). For notational simplicity we rename the sources as s_v. Therefore $R = \{s_v | v = 1,\ldots,m+1\}$, where the first m-1 sources are $s_u (u\neq h)$ and $s_m = p_s$, $s_{m+1} = p_t$. We now wish to find the pair of sources s_a and s_b that are closest together. Let us designate the set of indices v, corresponding to $s_v \varepsilon R$, as V, i.e., $V = \{v | s_v \varepsilon R\}$. The pair of sources s_a and s_b are now determined by:

$$[(x_a-x_b)^2 + (y_a-y_b)^2]^{\frac{1}{2}} = \underset{\substack{k,\ell \varepsilon V \\ k\neq \ell}}{\text{Min}} [(x_k-x_\ell)^2 + (y_k-y_\ell)^2]^{\frac{1}{2}}$$

Having found the pair of sources, s_a, s_b that are closest together, first we eliminate s_a and apply the "alternate" method of successive location and allocation until convergence as referred to under step 1 above. This will determine a second solution. Next we restore source s_a and eliminate s_b, apply the "Alternate" procedure and obtain a third solution. We calculate two new matrices W_2 and W_3 corresponding to these solutions, as we did in step 2.

4. Sum the requirements for each subset of destinations served by one source, i.e., calculate $\sum_{j \varepsilon s_i} r_j$ for each subset S_i, $i = 1,\ldots,m$. We now calculate the differences, $c_i - \sum_{j \varepsilon s_i} r_j$, $i = 1,\ldots,m$. A negative difference implies a capacity deficit and a positive difference indicates a capacity surplus.

5. The iterative calculation begins with this step. Let I_- be any set with a capacity deficit and let I_+ be any set with a capacity surplus. Choose a destination point, p_k, in the set I_- such that the difference in the distances from p_k to the source of I_- and from p_k to the source of I_+ is a minimum. Symbolically, if p_k has co-ordinates (x_{pk}, y_{pk}) and the source location of I_- is (x_-, y_-) and of I_+ is (x_+, y_+), then we choose p_k as the index j from I_- such that

$$\theta = \operatorname*{Min}_{j \varepsilon I_-} \left\{ [(x_{pj}-x_-)^2 + (y_{Dj}-y_-)^2]^{1/2} - [(x_{Dj}-x_+)^2 + (y_{Dj}-y_+)^2]^{1/2} \right\}$$

6. We now reallocate part or all (if possible) of the requirement at p_k from the source in I_- to the source in I_+. The amount reallocated depends on the size of the deficit at I_- and the surplus at I_+. The four cases are

a) If the (requirement at p_k) $\geq$ (deficit for I_-) and the (surplus for I_+) $\geq$ (deficit for I_-), then we reallocate the deficit from I_- by supplying that amount to p_k from I_+ instead of I_-.

b) If the (requirement at p_k) $\geq$ (deficit for I_-) and the (surplus for I_+) < (deficit for I_-), then we reallocate the amount of the surplus at I_+ by supplying that amount to p_k from I_+ instead of I_-.

c) If the (requirement at p_k) < (deficit for I_-) and the (surplus at I_+) $\geq$ (requirement at p_k), then we reallocate the entire requirement at p_k from I_- to I_+.

d) If the (requirement at p_k) < (deficit for I_-) and the (surplus at I_+) < (requirement at p_k), then we reallocate the amount of the surplus at I_+ by supplying that amount to p_k from I_+ instead of I_-.

7. With the new allocations, new source locations are computed for each subset of destinations, S_i by the use of equations (34) and (35), using the allocations as weights.

8. Each subset of destination points, S_i, is now examined for those points which might be closer to a different source with an excess capacity. If possible, we reallocate part or all of its requirement, depending on the amount of the surplus at that source.

9. We now repeat steps 7 and 8 (exact source location and subsequent reallocation to satisfy requirements), until no further change in the allocation matrix occurs.

10. The entire reallocation process (steps 5-9) is now repeated until all capacity deficits are removed. The value of z for this allocation matrix, W_1 is computed.

11. The procedure of steps 4-10 is repeated for allocation matrices W_2 and W_3.

12. The minimum of the three values of z obtained is chosen as the solution, together with its source locations and destination allocations.

This heuristic method was tested on the eight 2 source - 7 destination problems listed in [3]. The requirements were generated randomly. The sum of the capacities were chosen as five per cent higher than the sum of the requirements and were all equal. In order to have exact solutions to compare the heuristic against, the exact extremal equations (34) and (35) were used to solve for all possible allocations. For these eight problems, the heuristic method produced the optimal solution.

In [1] this basic method was also applied to 100 randomly generated problems with apparently good results, although the correct solutions were not known in advance.

8. Recommendations for Further Work

In section 4 of this paper an exact enumerative algorithm is presented for solving the transportation=location problem. However, as was indicated, it will not be computationally attractive for any but small problems. However, the possibility exists that a truncated enumeration method of the "branch and bound" variety might be constructed to drastically reduce the number of basic feasible solutions examined in order to find the optimal basic feasible solution. In this connection references [8], [9] may be consulted.

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