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ON THE DEFINITION AND OBSERVATION OF A SIGNAL VELOCITY
FOR PULSED WAVE PROPAGATION IN A MEDIUM WITH A RESONANCE
EXHIBITING COLLISION AND DOPPLER BROADENING

by

Dennis Benedict Trizna

Ph. D. Thesis, February 1970

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Dennis Benedict Trizna

— An Abstract of —
A Dissertation Submitted to the
Graduate Faculty in Partial Fulfillment of
The Requirements for the Degree of
DOCTOR OF PHILOSOPHY

Approved

In Charge of Major Work

Head of Major Department

Dean of Graduate College

Iowa State University
of Science and Technology
Ames, Iowa

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Dennis Benedict Trizna

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A theoretical and experimental study is made of the definition of the signal velocity for pulse propagation in a dilute, resonant medium in the region of anomalous dispersion.

First, previous work on the subject is reviewed which deals with the case of a Lorentz profile and a strongly absorbed signal, $\sigma x \gg 1$, where σ is the absorption coefficient for the resonance and x is the path length traversed by the signal in the medium (1,2,3). Then a theoretical study is made for the case of weak absorption for the same line profile; various signal velocity definitions are made and compared with those for strong absorption. These signal velocity definitions make sense only from a mathematical point of view, since it will be shown later that one cannot discriminate experimentally between the signal and precursor to define a signal arrival.

The techniques of the early works are used to define a signal velocity for the case of a resonance exhibiting a combination of Lorentz and Doppler line broadening for a strongly attenuated signal. Results are presented for the transition from a pure Lorentz profile to a pure Doppler profile as a function of the ratio of the half-

widths of the two shapes.

A computer calculation is performed using the exact theory derived earlier to predict the display of a broad band detection system for the case of a Lorentz profile. A pulse ringing effect is demonstrated graphically for this case. The results of an interferometer experiment are also predicted and presented in graphical form for the purpose of exhibiting the dispersion of the leading edge of the pulse which has traveled through the medium.

A similar analysis is made for the case of a narrow band detection system, in which an attempt is made to discriminate against the precursor by means of a filter with a Lorentz shaped frequency selectivity curve. It is found that one cannot effectively discriminate against the precursor, so that the concept of a signal velocity in the region of anomalous dispersion appears to make sense only from a mathematical point of view. The predictions of various signal velocity definitions are tested against the pulse shapes given by the computer analysis. Finally, the results of a previous experiment (4) in which a delayed pulse was observed are interpreted in terms of this work.

Experimental results are reported in which an attempt was made to measure the signal velocity in the region of anomalous dispersion of a Lorentz profile for weak absorption, using a microwave resonance of ammonia. Various techniques were used, including interference and filtering of the signal, but a delayed pulse was not observed. A ringing effect is displayed, along with the results of filtering the signal; estimates of the absorption and linewidth are made from the first display.

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4. Shiren, N. S. Measurement of a Signal Velocity in a Region of Resonance Absorption by Ultrasonic Paramagnetic Resonance. Phys. Rev. 128: 2103-2112. 1962.

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INTRODUCTION

A recent feasibility study has indicated the possibility of detecting molecular constituents of planetary atmospheres and the interstellar medium by measuring time delays of pulsed radio signals (1). Such time delays are due to dispersion by radio frequency resonances of the constituents and may be measurable under conditions for which resonance absorption is too weak to detect. In order to be able to use the method as a tool for determining number densities, the experimental techniques employed should be tested and calibrated in the laboratory under predetermined conditions for which the physical parameters are known. This dissertation presents a theoretical study of time delays for the dispersion rules describing the above mentioned media, followed by an experimental investigation of the same phenomenon.

There are two types of line broadening encountered in microwave spectroscopy: collision broadening and Doppler broadening, each with a characteristic absorption coefficient frequency profile. Collision broadening is the result of a perturbation in the energy level of the resonance due to collisions with other molecules and produces a Lorentzian profile, the halfwidth proportional to pressure. Doppler broadening is caused by a Maxwellian distribution of the line-of-sight velocities of the molecules which causes a Doppler shift in the resonance frequency of each molecule; the frequency profile is Gaussian, with a halfwidth proportional to the square root of the temperature. For typical atmospheric temperatures, collision

broadening dominates for high pressure, Doppler broadening for extremely low pressure. When the pressure is such that the collision width is of the order of the Doppler width, both effects must be taken into account simultaneously.

Previous theoretical analyses of the definition of a signal velocity for the case of anomalous dispersion have been concerned with the case of a Lorentz profile, for the asymptotic condition, $\sigma x \gg 1$, where σ is the absorption coefficient for the resonance and x is the path length (2,3,4). Signal velocities were mathematically defined in these treatments by considering the integral representing the electric field vector of the pulse in the medium. Evaluation of this integral results in expressions for the signal and the precursor; a signal velocity is defined by comparing these expressions as a function of time. In this dissertation the same integral is evaluated exactly. Signal velocities are defined for the condition of weak absorption, $\sigma x \ll 1$, using techniques similar to the early works. In addition, the case of Doppler broadening is treated in the asymptotic limit and signal velocities are defined.

When an experiment was performed to measure the signal velocity for weak absorption, no positive results could be reported. This led to a reconsideration as to whether the mathematical definitions had any physical meaning. That is, although one can distinguish between the precursor and signal mathematically, can one do so experimentally? Because the exact solution for the dispersed pulse was at hand, various experiments could be performed in a computer calculation where the predicted pulse shape could be displayed. One could follow numeri-

cally the time dependent contributions of the precursor and signal as a function of time. Then by observing the predicted shape of the directly detected pulse, the result of interference with an undispersed pulse, or the result of filtering the pulse before detection, one could determine whether a signal velocity is indeed observable. These calculations were performed and the results are presented herein. It appears that the experimental detection of the signal velocity is not possible in the region of anomalous dispersion.

Because the details of the pulse shape depend upon the physical parameters of the medium, these parameters can be determined by fitting experimental data to the predicted pulse shapes. This was done as a function of intensity and as a function of pressure. For the case of the interference experiment, no signal was detected, as predicted by the calculation. The filtered signal also behaved as predicted by the calculation, and no signal velocity was observed. All of these results are presented herein, along with some comments about an experiment reported by Shiren (6). He has observed a pulse delay and his results are explained in light of the computer calculation.

A summary and conclusions are presented at the end of the dissertation along with some comments about possible additional work on the problem. The remaining sections present the literature cited, acknowledgements, and appendices.

NON-ASYMPTOTIC THEORY

The problem of defining a signal velocity for pulsed electromagnetic waves propagating in a medium of resonant absorbers has been discussed in some detail previously (2). The results of that and previous work will be outlined here insofar as they relate to the problem to be dealt with in this section, defining the signal velocity for the non-asymptotic condition, $\sigma x < 1$, where σ is the absorption coefficient for the resonance of the medium, and x the path length traversed by the signal. The notation will be the same as that of reference 2.

For a sinusoidal step input signal of the form

$$\begin{aligned} E(0,t) &= \sin \omega' t & t &\geq 0 \\ &= 0 & t &< 0 \end{aligned} \quad (1)$$

the solution for the wave equation for a plane wave in a medium of resonant absorbers is given by

$$E(x,t) = \lim_{\epsilon \rightarrow 0} (-2\pi)^{-1} \operatorname{Re} \int_{-\infty}^{\infty} \frac{e^{-i\omega' t + i k x}}{\omega - \omega' + i\epsilon} d\omega \quad (2)$$

For a medium of N absorbers per unit volume, with resonant frequency ω_0 , and line halfwidth ρ , the dispersion rule for natural broadening is

$$k(\omega) = \frac{\omega}{c} [1 - 4a^2 / (\omega^2 - \omega_0^2 + 2i\rho\omega)]^{\frac{1}{2}} \quad (3)$$

where

$$a^2 = \pi N e^2 / m \quad (4-a)$$

$$e, m = \text{electron charge, mass.} \quad (4-b)$$

The group velocity derived from Equation 3 has no physical meaning near resonance, where it gives the non-causal result that the signal travels faster than the speed of light in a vacuum. Using the saddle point method of integration, Brillouin analyzed the problem of defining a signal velocity for the case of signal frequencies outside the region of anomalous dispersion (3). His evaluation of Equation 2 results in two terms: one represents the steady state signal, with sinusoidal frequency dependence the same as the impressed signal; the second term contains a wide spectrum of frequencies, is sinusoidally independent of ω' , and represents the precursor. Brillouin defined the arrival of the signal as that value of x/ct when the steady state contribution reached one half its final value; mathematically, this corresponds to the pole crossing the path of steepest descent. He then extended this definition to the case of signal frequencies near resonance, but without a detailed analysis of the integral for this case.

Baerwald later used a more precise generalized saddle point method of integration and defined a signal velocity for all frequencies, in the asymptotic limit defined by $\sigma x \gg 1$ (4). He defined the arrival of the signal as that time when the amplitude of the signal term equals that of the precursor. This signal velocity, as a function of frequency, has a minimum at resonance in contrast to Brillouin's definition which equals c there. Neither of the above results are expressible in closed form.

We have recently applied a method of asymptotic expansion of Fourier type integrals (5) to the problem, approximating the dispersion rule for the case of a dilute medium and narrow line profile, describable near resonance by

$$ck(\omega) = \omega - a^2/(\omega - \omega_0 + i\rho) . \quad (5)$$

Both methods of defining the arrival of the signal, that of Brillouin and Baerwald, were used. The corresponding results each have essentially the same shape as a function of frequency as their values. In the asymptotic limit, $\sigma x \gg 1$, the signal velocity is (2)

$$v/c = [1 + c \sigma(\omega)/\rho]^{-1} . \quad (6)$$

Shiren has measured time delays of pulsed ultrasonic signals propagating in a medium with a paramagnetic resonance. The resultant time delays are an order of magnitude, and more, larger than those predicted by the above theories. He therefore postulates very narrow local resonances, each of which delay the signal by an amount somewhere between the values given by the theories of Brillouin and Baerwald. Their theories are then used in an averaging process in analyzing his data, the Brillouin theory as a lower limit.

There are arguments as to why Brillouin's result is not a valid lower limit for the signal velocity, which shall be presented later. Because of this, and because conditions for which $\sigma x \ll 1$ are of interest in atmospheric measurements (1), it was felt that a solution to the problem for weak absorption would be of value, with a signal velocity defined if possible. This work now follows.

Evaluation of the Integral

In this section we shall evaluate the integral exactly, expressing the result in series form. From an approximation of this expression for $ox \ll 1$, signal velocities will be defined in the spirit of the previous definitions.

The integral of Equation 2 is transformed to a Fourier integral by the transformation equation

$$z = (1-\xi)(\eta + A^2/\eta) \quad (7)$$

where

$$\xi = x/ct \quad (8)$$

$$A^2 = a^2 \xi / (1-\xi) \quad (9)$$

$$\eta = \omega - \omega_0 + i\rho \quad (10)$$

$$|\eta'|^2 = D \quad (11)$$

The inverse transformation is

$$\eta = [z + (z-z_+)^{\frac{1}{2}}(z-z_-)^{\frac{1}{2}}]/2(1-\xi) \quad (12)$$

where

$$z_{\pm} = \pm 2A(1-\xi) \quad (13)$$

Equation 2 is now written, using $\tau = t-x/c$, as

$$E(x,t) = -\frac{1}{2\pi} \lim_{\epsilon \rightarrow 0} \operatorname{Re} e^{-i\omega_0 \tau - \rho \tau} (I_1 + I_2) \quad (14)$$

with

$$I_1 = \frac{1}{2} \oint e^{-izt} (z-z_p)^{-1} dz \quad (15)$$

$$I_2 = \frac{1}{2} \oint e^{-izt} \frac{(z-z_1)}{(z-z_p)(z-z_+)^{\frac{1}{2}}(z-z_-)^{\frac{1}{2}}} dz \quad (16)$$

where

$$z_1 = 2 A^2 (1-\xi)/\eta' \quad (17)$$

$$z_p = z(\eta'). \quad (18)$$

The contour and the singularities of the integrand move about in the transformed plane as a function of ξ , as shown in Figure 1. The branch points lie on the real axis: both are at the origin at $\xi = 1$, move to $\pm \sqrt{2}a$ at $\xi = \frac{1}{2}$, and back to the origin for $\xi = 0$. The pole lies in the lower half plane for $\xi = 1$, crosses the real axis between the branch points for ξ_c given by

$$A^2(\xi_c) = D(\omega') \quad (19)$$

and remains above it for all $\xi < \xi_c$. It lies to the left (right) of the imaginary axis for $\omega' - \omega_0 > 0$ (< 0), and on it for $\omega' = \omega_0$, for all ξ . The exact position of the contour, Γ , the transformed real ω -axis, is irrelevant since it will be deformed in the z -plane to evaluate the integrals. Suffice it to note that all the singularities lie below the contour in the z -plane for all ξ .

This transformation is the same that was used in our asymptotic theory (2). The branch cuts in Figure 1 are just the paths of steepest descent of the ω -plane. To evaluate the integral by the asymptotic method of expansion, one distorts the contour to lie along these cuts, as in Figures 2a and 2b. For very asymptotic conditions, the singularity which lies highest in this transformed plane gives the dominant contribution to the integral. Under these conditions one can define the signal arrival as the time when the pole in the figure, representing the steady state oscillations, climbs above the

Figure 1 .

Figure 1. Transformed z-plane with contour, Γ , branch points, $z_{\pm}$, and pole, z_p .

Figure 2 a. Transformed z -plane for incident frequency far from resonance.

Figure 2 b. Transformed z -plane for incident frequency near resonance.

real axis and dominates the branch points.

Alternatively one could define the arrival of the signal, for not so strong asymptotic conditions, as the time when the pole passes through the branch cut and its residue first begins to contribute. This is the analogue to the definition of Brillouin. Such a definition appears straightforward in a first order calculation. However a more exact method of asymptotic expansion, in which two singularities are treated jointly (5), should be used for the case of weakly asymptotic conditions, cases for which $\alpha x \sim 1$, for example. For frequencies far off resonance the pole is near one of the branch points as it passes under the branch cut, as is shown in Figure 2a. These two singularities are treated jointly in the expansion and it can be seen how the residue term reaches one half its final value as the pole passes through the branch cut. However for frequencies in the region of anomalous dispersion, as the pole passes through the branch cut, the two branch points lie near one another and are to be treated jointly. It is not apparent how the pole contributes now so that an exact method of integration to show this behavior is necessary.

To evaluate the integral, one first writes Equation 16 as

$$I_2 = I_{21} + I_{22} \quad (20)$$

$$I_{21} = \frac{1}{2} \oint e^{-izt} (z-z_+)^{-\frac{1}{2}} (z-z_-)^{\frac{1}{2}} dz \quad (21)$$

$$I_{22} = \frac{1}{2}(z_p - z_1) \oint e^{-izt} (z-z_p)^{-1} \cdot (z-z_+)^{-\frac{1}{2}} (z-z_-)^{-\frac{1}{2}} dz \quad (22)$$

The branch cuts are drawn up to lie along the positive z -axis, cancelling one another to the right of z_+ , as indicated in Figure 3. The contour is distorted downward to lie around the branch cut and pole, pinched off when necessary, and down to the infinite semi-circle along which the integrand damps exponentially and contributes nothing. The integral about the cut is simply proportional to the Bessel function $J_0(z_{\pm}t)$, as is shown in Appendix A. The integral I_{22} is evaluated using the integral representation of one of the degenerate hypergeometric functions of two variables (7):

$$\beta(a,b) \Phi_1(a,c,a+b;Y,Z) = \int_0^1 \zeta^{a-1} (1-\zeta)^{b-1} (1-Y\zeta)^{-c} e^{Z\zeta} d\zeta \quad (23)$$

with $\text{Re } a, b > 0$, $|\arg(1-Y)| < \pi$. Translating z_- (z_+) to the origin for $\omega' > (<) \omega_0$, one gets from the branch cut integration

$$I_{22} = \frac{1}{2}(z_p - z_{\pm}) 2\pi i \{ e^{-iz_{\pm}t} (z_p - z_{\pm})^{-1} \cdot \Phi_1\left(\frac{1}{2}, 1, 1; 2Y/(Y-1), Z\right) - \frac{e^{-iz_p t}}{(z_p^2 - z_{\pm}^2)^{\frac{1}{2}}} \} \quad (24)$$

where

$$Y = z_{\pm}/z_p \quad (25)$$

$$Z = 2iz_{\pm}t \quad (26)$$

for $\omega' \leq \omega_0$. Therefore the entire integral can now be written,

for $\omega' > \omega_0$,

$$I_1 + I_2 = -i\pi e^{-iz_p t} (1 \pm 1) - i\pi J_0(z_{\pm}t) - i\pi e^{-iz_{\pm}t} \cdot \left(\frac{A - \eta'}{A + \eta'} \right) \Phi_1\left(\frac{1}{2}, 1, 1; 2Y/(Y-1), Z\right) \quad (27)$$

Figure 3. Transformed plane, with branch cuts drawn to $+\infty$, with contour wrapped about the resultant cut between $z_{\pm}$.

where the $\pm$ holds for $A^2 \lesseqgtr D$. Hence, for the pole in the lower half plane the residue does not contribute; however the Φ_1 function will begin to oscillate like the pole term before it does contribute so that a measurable signal will be present. The expression given by Equation 27 is exact but is difficult to interpret as it stands. In the following it will be cast into more familiar expressions for various positions of the singularities as a function of A^2 .

$$\infty > A^2 > D, \quad |2Y/(Y-1)| < 1$$

For this case the pole lies in the lower half plane and outside the circle centered at z_- with radius $|2z_-|$, as is shown in Figure

3. The Φ_1 function can be written in its series representation

$$\Phi_1\left(\frac{1}{2}, 1, 1; \frac{2Y}{(Y-1)}, z\right) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\frac{1}{2}, k+m)(1, k)}{(1, k+m)} \frac{\left(\frac{2Y}{Y-1}\right)^k}{k!} \frac{z^m}{m!} \quad (28)$$

which converges for $|2Y/(Y-1)| < 1$. (The terms of the form (a, m) are Appell symbols defined by

$$(a, m) = a(a+1)(a+2) \cdots (a+m-1) \quad (29)$$

Note that the representation given by Gradshteyn and Ryzhik is in error. They have a $(1, m)$ instead of $(1, k)$, which is demonstrated as correct in Appendix B. Equation 28 is now written in terms of a series of hypergeometric functions,

$$\Phi_1 = \sum_{m=0}^{\infty} \frac{(\frac{1}{2}, m)}{(1, m)} \frac{z^m}{m!} {}_2F_1\left(m+\frac{1}{2}, 1; m+1; 2Y/(Y-1)\right). \quad (30)$$

The hypergeometric functions are convergent in the unit circle

of the argument as required by the condition presently under consideration. The time variable is associated with $z_- = 2A(1-\xi)$, from which the precursor frequency arises. The argument of the hypergeometric function is a function of relative position of the singularities only. Hence at this point the Φ_1 function is not yet oscillating with the steady state frequency. Since both arguments of the Φ_1 function are of the same order of magnitude no further expansion of Equation 30 will be attempted.

$$\infty > A^2 > D, \quad |2Y/(Y-1)| > 1$$

For this case the pole still lies in the lower half plane and the residue is not yet contributing, but it lies within the circle with center at z_- . The hypergeometric functions in the expansion of Equation 30 are no longer convergent; however one can analytically continue them by a linear transformation. (See for example, Equation 9.5.9 of Lebedev (8).) It follows that

$$\Phi_1 = \left(\frac{1-Y}{1+Y}\right) e^{Z(Y-1)/2Y} - \frac{Z(Y-1)}{2Y} \sum_{n=0}^{\infty} \frac{(\frac{1}{2}, n)}{(2, n)} \frac{Z^n}{n!} \quad (31)$$

$${}_2F_1(-n, 1; -n+\frac{1}{2}; (Y-1)/2Y)$$

$$= " - \frac{Z(Y-1)}{2Y} \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{(\frac{1}{2}, n)(-n, k)(1, k)}{(2, n)(-n+\frac{1}{2}, k)} \frac{Z^n}{n!} \frac{(\frac{Y-1}{2Y})^k}{k!} \quad (32)$$

which now converges for $|2Y/(Y-1)| < 1$. Note that

$$Z(Y-1)/2Y = -i(z_p - z_-)t \quad (33)$$

so that now oscillations begin to appear containing the pole due to the $k = n$ term of the double sum, in addition to the exponential term of Equation 32. For $A^2 > A_1^2$, where A_1^2 is that value for which $|2Y/(Y-1)| = 1$, the exponential term is being cancelled by the double sum, but ever less so as A^2 further decreases. Hence for some value A_2^2 , $A_1^2 < A_2^2 < D$, one half of the steady state term will result from the effects of this semi-cancellation.

To find A_2 , one associates the time variable with the pole in the z -plane, as in Equation 33 for example, and examines the resulting expression. This is accomplished by a shift of indices of the double sum of Equation 32. Letting $n = k+m$,

$$\sum_{n=0}^{\infty} \sum_{k=0}^n \rightarrow \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \quad (34)$$

and making use of the properties of the Appell symbols,

$$(a, m+n) = (a, m) (a+m, n), \quad (35)$$

$$(a, -m) = (-1)^m / (1-a, m), \quad (36)$$

one has from Equation 32:

$$\begin{aligned} \Phi_1 = & \left(\frac{1-Y}{1+Y} \right)^{\frac{1}{2}} e^{Z(Y-1)/2Y} + \frac{Z(Y-1)}{2Y} \sum_{m=0}^{\infty} \frac{(\frac{1}{2}, m)}{(2, m)} \frac{Z^m}{m!} \cdot \\ & \cdot \sum_{k=0}^{\infty} \frac{(1, k)}{(2+m, k)} \frac{[Z(Y-1)/2Y]^k}{k!} \end{aligned} \quad (37)$$

Before approximating this expression, consider the magnitude of the arguments of the double sum as a function of A . Using

$A^2 = \sigma_0 x \rho / \tau$ again, one has

$$|Z(Y-1)/2Y| = |(z_p - z_-)t| = \sigma_0 x \left| \frac{\rho}{\eta'} (1 + \frac{\eta'}{A})^2 \right| \quad (38)$$

$$|Z| = |2 z_- t| = 4 \sigma_0 x (\rho/A) \quad (39)$$

so that the small variable is Z for the range of A^2 under consideration. Hence if one expands Equation 37 in terms of this small variable, one has to the $m=1$ term,

$$\begin{aligned} \Phi_1 = & \pm \left(\frac{\eta' + A}{\eta' - A} \right) e^{-i(z_p - z_-)t} + e^{-i(z_p - z_-)t} - 1 + \\ & + (\sigma_0 x)^2 \frac{\rho^2}{A \eta'} [1 - (\eta'/A)^2] {}_1F_1(1, 3; -i(z_p - z_-)t) \end{aligned} \quad (40)$$

and the expression for the wave is therefore

$$\begin{aligned} E(x, t) = & \frac{1}{2} \operatorname{Re} i \left\{ \frac{2\eta'}{\eta' + A} e^{-i\omega' t + i k_r x - \sigma x} + J_0(2A\tau) e^{-i\omega_0 \tau - \rho \tau} \right. \\ & \left. + \frac{A - \eta'}{A + \eta'} e^{-i(\omega_0 - 2A)\tau - \rho \tau} + O[(\sigma x)^2] \right\}. \end{aligned} \quad (41)$$

Signal Velocity Definitions

From the results attained thus far one could define a signal velocity three different ways using analogies to Brillouin's criterion. He defined the arrival of the signal as that value of x/ct for which the residue first begins to contribute. At this time the signal begins to oscillate with its steady state frequency and the amplitude of the residue is one half its steady state value.

If one uses the results of the above analysis and defines the arrival of the signal as that time at which the signal first begins to oscillate with its steady state frequency, the signal velocity is

$$v/c = \xi = (1 + a^2/A^2)^{-1} \quad (42)$$

with A_1^2 , the solution to $|2Y/(Y-1)| = 1$, to be used:

$$A_1^2 = D \left\{ 2 - \frac{|\omega' - \omega_0|}{\sqrt{D}} - \left[3 + \frac{(\omega' - \omega_0)}{D} - 4 \frac{|\omega' - \omega_0|}{\sqrt{D}} \right]^{\frac{1}{2}} \right\}^{-2} \quad (43)$$

At this time the pole lies on the circle about z_- with radius $|2z_-|$, Equation 37 begins to hold, and the exponential term in it is just being cancelled by some of the terms of the double sum.

One can also define the signal arrival as that time when the steady state contribution reaches one half its final value. This can be considered in two different ways. Writing

$$\eta' / (\eta' + A) = F_r + iF_i = F \quad (44)$$

one has from the first term of Equation 41,

$$\begin{aligned} \operatorname{Re} i(F_r + iF_i) e^{-i\omega'\tau - i\phi - \sigma x} = \\ e^{-\sigma x} [F_r \sin(\omega'\tau + \phi) - F_i \cos(\omega'\tau + \phi)] \end{aligned} \quad (45)$$

As τ goes to infinity, F_r goes to one and F_i to zero. Hence one half the steady state is reached at $F_r = \frac{1}{2}$, which is satisfied at $A^2 = D$. Note that it is at this time that the pole passes through the cut in Figure 3 into the upper half plane and the residue of the pole first begins to contribute. This is also the arrival time of the signal as predicted by the asymptotic theory. This definition thus satisfies two of the three criteria used by Brillouin and produces a result which agrees with that of the asymptotic theory.

Although the above result is mathematically pleasing, it is not a good working definition experimentally using the solution of Equation 41. Because the cosine term contributes oscillations with the frequency of the incident signal, it would contribute to the arrival of the signal as measured by a detector. Hence it must be properly accounted for in defining a signal velocity. To do so one must first consider the function F more closely as a function of time. F_r rises monotonically from zero at $\tau = 0$, to one half at $A^2 = D$, and to one as $\tau \rightarrow \infty$. F_i rises from zero at $\tau = 0$, to a maximum at $A^2 = D$, then to zero again as $\tau \rightarrow \infty$. This maximum is one half for $\omega' = \omega_0$, and ever less for ω' further off resonance. Hence the cosine oscillation is yielding its greatest contribution at the time defined as that of the signal arrival in the last section. The energy density delivered by this contribution is also a maximum at this time and is not accounted for by this definition. To take both the sine and cosine oscillations into account equally one can write Equation 45 as equal to

$$|F|^2 e^{-\sigma x} \sin(\omega' \tau + \varphi - \Theta) \quad (46)$$

where

$$\tan \Theta = F_i / F_r = \rho A / (D + A\omega) \quad (47)$$

At $\tau = 0$, $\Theta = \pi/2$ and the oscillation is a cosine. At $A^2 = D$, $\Theta = \pi/4$ for $\omega' = \omega_0$ and is $< \pi/4$ for ω' off resonance; that is, one has an equal mixture of sine and cosine at this time for ω' on resonance, with the sine term dominating ever more so for ω' further off resonance. At $\tau \rightarrow \infty$, $\Theta \rightarrow 0$, and the signal oscillation approaches a

pure sine as it should. Hence one could alternatively define the signal arrival as that time when a given fraction of the steady state energy density with main frequency component ω' arrived.

For this fraction equal to a quarter, in analogy with Brillouin, one would set

$$|\eta' / (\eta' + A)| = |F| = \frac{1}{2}. \quad (48)$$

The value of A^2 satisfying this equation, to be used in Equation 42, is

$$A_2^2 = \bar{v} \left\{ \frac{\omega' - \omega_0}{3\sqrt{D}} + \left[1/3 + \frac{(\omega' - \omega_0)^2}{9D} \right]^{\frac{1}{2}} \right\}^{-2} \quad (49)$$

Therefore the three criteria for defining the signal arrival which gave a single result in Brillouin's analysis give three different expressions for the signal velocity now, depending upon what one interprets as the correct signal expression. All of these three are larger lower limits than Brillouin's value so that the arguments made earlier against using his result as a lower limit for the case $\sigma x \approx 1$ appear to be borne out. That is, the signal velocity for the mildly asymptotic condition $\sigma x \approx 1$ should not be less than that for the case of a near vacuum, $\sigma x \ll 1$.

Time delays derived from these signal velocities, defined by

$$\Delta t = x/v - x/c \quad (50)$$

are shown in Figure 4, along with that from the asymptotic theory, Δt_A , and the Brillouin analogue, Δt_B . Units for delay are $\sigma_0 x / \rho$ and for frequency, ρ . These results suggest that the arrival of the steady state for the case $\sigma x \ll 1$ does not take place instantaneously.

Figure 4. Time delays as a function of frequency. Δt_a is the time delay from the asymptotic theory, Δt_b is the Brillouin analogue, and Δt_1 and Δt_2 are defined from the non-asymptotic theory.

neously as the Brillouin definition would imply if used under this condition, but is spread over a reasonable length of time. Hence the measurement of the signal arrival may depend quite heavily upon the detection system used. This factor will be considered further in a later discussion within this text.

To complete the analysis of this section, an expression for the wave will now be derived for $0 < A^2 < D$, i.e. for the pole in the upper half plane. For this case Equation 27 is convenient with the plus sign in the factor multiplying $e^{-iz_p t}$. The Φ_1 function is now written in terms of confluent hypergeometric functions with argument Z . The only time z_p appears associated with the time variable is in the residue term:

$$(I_1 + I_2)/\pi i = -J_0(2A\tau) - 2e^{-iz_p t} + \left(\frac{1+A/\eta'}{1-A/\eta'} \right) e^{i2A\tau} \cdot \sum_{k=0}^{\infty} \left(\frac{1}{2}, k \right) \frac{[2Y/(Y-1)]^k}{k!} {}_1F_1\left(k+\frac{1}{2}, k+1; Z\right) \quad (51)$$

As it stands the confluent hypergeometric functions converge very slowly because of the summation index in both Appell symbols are nearly equal for large k . Making use of

$${}_1F_1(a, b; z) = e^z {}_1F_1(b-a, b; -z) \quad (52)$$

(See for example, Lebedev (8), Equation 9.11.2, corrected from his previous equation), the first term of the sum is a Bessel function of zeroth order which adds to the one outside the sum so that

$$I_1 + I_2 = -2\pi i e^{-iz_p t} + \frac{A/\eta'}{1-A/\eta'} J_0(2A\tau) + \frac{2A/\eta'}{[1-(A/\eta')^2]} e^{-i2A\tau}. \quad (53)$$

$$\cdot {}_1F_1\left(\frac{1}{2}, 2; 4A\tau\right) + O[(A/\eta')^2]$$

The resulting expression the the wave is then

$$E(x, t) = e^{-\sigma x} \sin(\omega' \tau - k_r x) - \frac{1}{2} e^{-\rho \tau} \operatorname{Re} \frac{(A/\eta')}{[1-(A/\eta')^2]} e^{-i(\omega_0 - 2A)\tau}. \quad (54)$$

$$\cdot {}_1F_1\left(\frac{1}{2}, 2; 4A\tau\right) - \frac{1}{2} e^{-\rho \tau} \operatorname{Re} \frac{A/\eta'}{1-A/\eta'} i e^{-i\omega_0 \tau} J_0(2A\tau),$$

to first order in A. As A goes to zero, the precursor term goes to zero and the steady state remains.

BESSEL FUNCTION SERIES AND SOMMERFELD'S RESULT

The integral of Equation 2 can be solved exactly in the η -plane using the dispersion rule of Equation 5. The results can be compared with previous work by Sommerfeld and the asymptotic calculation and a signal velocity can be defined in some sense which agrees with the result of the asymptotic calculation. Using the same notation as before, and writing $I_1 + I_2$ as I_s ,

$$I_s = \int_{-\infty + i\rho}^{\infty + i\rho} e^{-i(\eta + A^2/\eta)\tau} \frac{d\eta}{\eta - \eta'} \quad (55)$$

The contour can be completed in the lower half plane along the infinite semi-circle, along which the exponent of the integrand is damping so that this contour contributes nothing. Transforming the variable to

$$z = -i\eta/A$$

Equation 55 becomes

$$I_s = \oint e^{\frac{1}{2}(2A\tau)(z - 1/z)} \frac{dz}{z - z'} \quad (56)$$

For $|z'/z| < 1$, the pole can be expanded so that

$$I_s = \sum_{k=0}^{\infty} (z')^k \oint e^{\frac{1}{2}(2A\tau)(z - 1/z)} z^{-k-1} dz \quad (57)$$

where the order of summation and integration have been interchanged.

This is allowed because the expansion made is absolutely convergent for the contour $|z_c| > |z'|$. Making use of the integral representation of the Bessel function (8)

$$J_k(z) = (2\pi i)^{-1} \oint e^{\frac{1}{2}z(t-1/t)} t^{-k-1} dt \quad (58)$$

Equation 57 becomes

$$I_s = -2\pi i \sum_{k=0}^{\infty} (z')^k J_k(2A\tau). \quad (57a)$$

One could also evaluate Equation 56 by first distorting the contour around the origin, such that $|z_c| < |z'_p|$. In the process one picks up the residue of the pole. Now the denominator is expanded for $|z'/z| > 1$:

$$I_s = -2\pi i e^{A\tau(z' - 1/z')} - (1/z') \sum_{k=0}^{\infty} (z')^{-k} \oint e^{\frac{1}{2}(2A\tau)(z-1/z)} z^k dz \quad (59)$$

$$I_s = -2\pi i e^{A\tau(z' - 1/z')} + 2\pi i \sum_{k=0}^{\infty} (z')^{-k-1} J_{-k-1}(2A\tau). \quad (59a)$$

Using

$$J_{-k}(z) = (-1)^k J_k(z)$$

and letting $n=k+1$, one finally has

$$I_s = -2\pi i e^{A\tau(z' - 1/z')} + 2\pi i \sum_{n=1}^{\infty} (-z')^{-n} J_n(2A\tau). \quad (59b)$$

Using the generating function for the Bessel functions (See for example Abramowitz and Stegun (9), Equation 9.1.41)

$$e^{\frac{1}{2}t(z-1/z)} = \sum_{k=-\infty}^{\infty} (z)^k J_k(t) \quad (60)$$

one can show that Equations 57a and 59b are identical. Both solutions are also absolutely convergent for z' finite and non-zero, as can be demonstrated by using the ratio test for large k on the terms in the expansion of either equation. Therefore there is no strict mathematical reason for using one solution instead of the other for arbitrary values of z' and At . However, if one wanted the best approximation to the exact solution for the least number of terms taken in the series, $|z'| = 1$ is the natural dividing line for choosing one expression over the other. If one then identifies the precursor with the series and the steady state with the exponential, using this criterion the steady state begins to contribute at $|z'| = 1$, or at $A^2 = D$. The signal velocity so defined is then the same as that given by the asymptotic theory. However such a definition is naive since the series of Bessel functions of Equation 57a is oscillating like the steady state before $A^2 = D$, the same way the Φ_1 function did in the last section before the pole passed through the cut and the residue began to contribute. A detailed examination of the series solution of the present section will not be carried out here as was done for the Φ_1 function.

The first few terms of the solution of Equation 57a give a good

approximation to the integral for $\tau \sim 0$ and can be compared with earlier work for this case. Writing the expression for the wave using the first few terms of the series of Equation 57a, one has

$$E(x,t) = e^{-\rho\tau} J_0(2A\tau) \sin \omega_0 \tau + e^{-\rho\tau} \operatorname{Re} \frac{\eta'}{A} e^{-i\omega_0 \tau} J_1(2A\tau). \quad (61)$$

The first order term is exactly the same as the result of the asymptotic theory when two branch points are treated jointly for ξ close to one. The second term is similar to the result of Sommerfeld, Equation 2-33 of Brillouin (3):

$$E(x,t) = \frac{\omega'}{A} J_1(2A\tau). \quad (62)$$

However in his derivation, Sommerfeld approximated the exact dispersion rule, Equation 3, for ξ close to one such that the first term of his expansion is independent of ω_0 . He also assumed ρ to be zero in his calculation. If these two terms are set equal to zero in Equation 61, the first order term disappears and Equation 62 results. Hence it appears that by assuming ρ to be zero and ω_0 small, Sommerfeld may have not accounted for the first term properly. However, the dispersion rule in the present analysis was approximated for very specific physical conditions and it is difficult to compare the two results rigorously. Hence no other comparison shall be made other than to note their similarity.

The expression, using Equation 59b, and $\tau = t - x/c$ again,

$$E(x,t) = e^{-\sigma x} \sin(\omega' t - k_r x) - \operatorname{Re} i \sum_{n=1}^{\infty} \left(-\frac{iA}{\eta'} \right)^n \cdot J_n(2A\tau) e^{-i\omega_0 \tau - \rho \tau} \quad (63)$$

is analogous to the solution to the differential equation for a driven, damped harmonic oscillator. The first term can be considered as the analogue to the steady state response, the solution to the inhomogeneous oscillator equation. The second term is then the analogue to the transient response of the oscillator, the solution to the homogeneous equation. For the case of the harmonic oscillator initially at rest, the transient and steady state responses cancel at time equal to zero, when the force is applied, as required by the initial conditions. The growth of the oscillator amplitude to the final steady state results from the decay in cancellation of these two responses.

The analogy of Equation 63 to this oscillator behavior is the motivation for the definition of the signal velocity. At $\tau = 0$, $E(x,t) = 0$, and the two terms in Equation 63 cancel. Just after $\tau = 0$, this cancellation between the two responses is not quite complete; the oscillations which result are called the precursor. For longer times the signal arrives, with main frequency components equal to that of the final steady state signal. At this time the precursor still has a large amplitude and the amplitude behavior of the signal depends upon the particular mathematical solution employed, as does the signal velocity definition. From the mathematical results it is not as yet certain as to whether one can discriminate between the precursor and signal experimentally to measure a signal velocity.

COMBINED DOPPLER AND COLLISION BROADENING

Dispersion Rule

As indicated earlier there are two types of line broadening encountered in microwave spectroscopy: collision broadening, exhibiting a Lorentz profile; and Doppler broadening, exhibiting a Gaussian profile. When pressures are such that the collision frequency is of the order of the Doppler halfwidth, both effects must be considered simultaneously and the resulting resonance profile has a shape which differs from either of the two. In the treatments dealing with this effect for absorption measurements, an average absorption coefficient generally is derived by taking an appropriate average of the collision broadened absorption coefficient over a Maxwellian distribution of resonant frequencies (10). The frequency dependence of this absorption coefficient is given by an expression related to the error function.

It is dangerous to assume that a similar averaging of the time delay for the Lorentz profile over a Maxwellian distribution of resonant frequencies will give a correct result for the Doppler-collision time delay. The time delay effect is related to the dispersive properties of the medium. Hence any averaging must be done in frequency components before performing the integration for $E(x,t)$ to properly account for coherence effects between scatterings from the molecules of the gas. Therefore in the following treatment the dispersion rule is derived for combined collision and

Doppler broadening.

A gas of molecules at temperature, T , has a Maxwellian distribution of velocities along a given direction, and hence a similar distribution of shifts in resonance frequencies. The fraction of molecules with their resonant frequency shifted by an amount Ω in a band of frequencies $d\Omega$ wide is given by

$$\frac{dN(\Omega)}{N} = e^{-(\Omega/\omega_D)^2} d\Omega / \pi \omega_D \quad (64)$$

where N is the number per unit volume, and ω_D is the Doppler halfwidth defined by

$$\omega_D = \omega_0 (2kT/Mc^2)^{\frac{1}{2}} \quad (65)$$

with k Boltzmann's constant, M the molecular weight, and c the velocity of light. In the derivation of a dispersion rule using a gas of classical oscillators, one Fourier analyzes the electric field, polarization, and oscillator displacement (2). Then for a given Fourier component, the polarization

$$P_\omega = [\epsilon(\omega) - 4\pi] E_\omega = N e y_\omega \quad (66)$$

defines the dielectric constant, $\epsilon(\omega)$, where

$$y_\omega = - \frac{eE_\omega}{m} (\omega^2 - \omega_0^2 + 2i\rho\omega)^{-1} \quad (67)$$

$$\approx - \frac{eE_\omega}{m} [\omega(\omega - \omega_0 + i\rho)]^{-1} \quad (68)$$

is the Fourier transform of the displacement of the oscillator, in the same approximation made in the earlier treatment on $k(\omega)$ for a narrow linewidth. For the case of a gas with a non-zero temperature the polarization is averaged over a unit volume:

$$P_{\omega} = e \int \gamma_{\omega}(\omega_0) dN(\Omega) \quad (69)$$

Using the Maxwellian distribution of Equation 64 and the fact that an arbitrary oscillator has its frequency shifted to $\omega_0 + \Omega$, one obtains

$$P_{\omega} = \frac{e^2 E_{\omega}}{m} i \frac{1}{\pi^2} \frac{N}{\omega' \omega_D} w(\eta) \quad (70)$$

where now

$$\eta = (\omega - \omega_0 + i\rho)/\omega_D \quad (71)$$

and

$$w(\eta) = \frac{1}{i\pi} \int_{-\infty}^{\infty} \frac{e^{-t^2}}{t-\eta} dt \quad (72)$$

is the plasma dispersion function, related to the error function.

(See for example, Abramowitz and Stegun (9), page 297.)

Solving for $\epsilon(\omega)$ from Equations 66, 68 and 70, using the relation

$$ck(\omega) = \omega [\epsilon(\omega)]^{\frac{1}{2}} \quad (73)$$

and expanding the square root of Equation 73 under the condition of low density of oscillators, as was done for the case of a Lorentz profile, one has the dispersion rule

$$ck(\omega) = \omega \left[1 + \frac{a^2}{\omega \omega_D} i \pi^{\frac{1}{2}} w(\eta) \right] \quad (74)$$

where a^2 has the same meaning as before. In the limit as $\omega_D \rightarrow 0$, $\eta \rightarrow \infty$, and one can use the asymptotic representation of $w(\eta)$ to show that the dispersion rule goes to that of the previous section.

Equation 74 is also causal: it is an entire function, so that it has no branch points or poles in the finite plane; its poles at infinity lie within the quarter section in the lower half plane $5\pi/4 < \arg \eta < 7\pi/4$. Hence when distorting the contour into the upper half plane for $t > x/c$, there are no singularities for it to wrap around; the contour along the semi-circle at infinity contributes nothing since Equation 74 goes to the dispersion rule of the last section for this case. Therefore $E(x,t)$ is zero for $t > x/c$ and Equation 74 is causal.

Definition of the Signal Velocity

In the following analysis the actual integration leading to the solution for $E(x,t)$ will not be performed. The reason for this is that, although the transformation to the z -plane can be made, in analogy to Equation 7,

$$z = (1-\xi) \left[\eta - i \pi^{\frac{1}{2}} A^2 w(\eta) \right], \quad (75)$$

where now

$$A^2 = \frac{a^2}{\omega_D^2} \frac{\xi}{(1-\xi)}; \quad (76)$$

the inverse transformation, $\eta = \eta(z)$, cannot be carried out.

In the asymptotic method of expansion this is convenient for the transformed integral to be expanded and the solution obtained. However the signal velocity for the asymptotic condition can be defined without actually evaluating the integral, but by merely considering the positions of the singularities of the integrand in the transformed plane. Because the non-asymptotic signal velocities appeared not to differ greatly compared to the asymptotic definition, for the case of a Lorentz profile, the definition for the case of Doppler-collision broadening using the asymptotic criterion would appear to be useful for either condition, $\sigma x \gtrsim 1$.

It can be shown for the case of a Lorentz profile that the saddle points in the η -plane, $\eta_{sp} = \pm A$, transform to the branch points in the z -plane, $z_{\pm} = \pm 2a(1-\xi)$, and the pole in the η -plane transforms into the pole in the z -plane. The same behavior holds for the case of Doppler broadening included in the dispersion rule. If one were able to perform the transformation of the integrand for this case to a function of z only and evaluate the integral, the result would consist of contributions from integrating about the pole and along the branch cuts. These contributions would represent the steady state signal and precursors, respectively. Therefore the signal velocity for the asymptotic condition can again be defined by that value of x/ct for which the pole and branch points have equal imaginary parts. The result then is the analogue to Δt_A of the first section and should be the upper limit to the time delay for combined Doppler and collision broadening.

The saddle points for the transformation of Equation 75 are just

Figure 5. Saddle points in the η -plane for the dispersion for Doppler broadening.

the roots of

$$\frac{dz}{d\eta} = (1-\xi) \left[1 - i\pi^{\frac{1}{2}} w'(\eta_s) \right] = 0 \quad (77)$$

Using the recursion relation for $w'(\eta)$, (78)

$$w'(\eta) = -2\eta w(\eta) + 2i\pi^{-\frac{1}{2}}, \quad (79)$$

the saddle points are the roots of

$$\eta_s w(\eta_s) = i\pi^{-\frac{1}{2}} (1 + (2A^2)^{-1}) \quad (80)$$

The roots of this equation lie in pairs in the lower half η -plane, symmetric about the imaginary η -axis. For $A^2 = \infty$, ($\xi = 1$), these are located about the zeros of $w(\eta)$; the way they vary in position with decreasing A is plotted in Figure 5 for the two most important ones.

Now these saddle points transform into branch points in the z -plane located at

$$z_{bj} = (1-\xi) [\eta_{sj} + (A^2 + \frac{1}{2})/\eta_{sj}] \quad (81)$$

with the j a labeling index. Defining

$$\eta_{sj} = x_j - iy_j \quad (82)$$

Equation 81 becomes

$$z_{bj} = (1-\xi) \left[x_j \left(1 + \frac{A^2 + \frac{1}{2}}{|\eta_{sj}|^2} \right) - iy_j \left(1 - \frac{A^2 + \frac{1}{2}}{|\eta_{sj}|^2} \right) \right]. \quad (83)$$

To define the signal velocity under the asymptotic condition, one must know the dominant branch point as a function of A^2 in order to compare its imaginary part with that of the pole. For the Lorentz profile there were just a pair of branch points to consider, with equal imaginary parts. For this case one need consider the transformed position of all the saddle points of Figure 5 in order to know which dominates for a given value of A . For the present discussion the branch points will be labeled as follows: that which comes in along the real axis from infinity with decreasing A is z_{b0} ; that which lies near the zero of $w'(\eta) = 0$ closest to the origin at $\xi = 1$, z_{b1} , and so on for z_{bj} which lies near the j th closest zero to the origin for $\xi = 1$.

Before determining which branch point dominates as a function of A , one must separate those branch points which lie below the contour from those which lie above it, since those above will not contribute to the integral for $\xi < 1$, and hence need not be considered further. This analysis is most easily carried out for ξ close to, but less than, one; then the j th branch point, $j \geq 1$, is located approximately at

$$z_{bj} = (1-\xi)A^2(x_j + iy_j)/|x_j^2 + y_j^2| \quad (84)$$

To compare the contour and branch point, one first sets

$$\text{Re } z_c = \text{Re } z_{bj} \quad (85)$$

where z_c is a point on the contour in the transformed plane. Solving this for a value of η , one substitutes this value into $\text{Im } z_c$ to compare with the imaginary part of the branch point to determine whether the branch point lies above or below the contour. Using Equation 84, Equation 85 becomes

$$W_c + \frac{1}{\pi^2} A^2 \text{Im } w(W_c + \rho) = A^2 x_j / |\eta_{sj}|^2 \quad (86)$$

where $W_c = w_c - w_0$. This equation is impossible to solve for W_c in closed form so that some assumptions are necessary. If one assumes first that

$$(i) \quad |W_c| \ll 1$$

using (9)

$$w(\eta) = e^{-\eta^2} + 2i\eta\pi^{\frac{1}{2}} {}_1F_1(1, 3/2, -\eta^2) \quad (87)$$

Equation 86 becomes

$$W_c + A^2 [2(W_c \phi_r - \rho \phi_i) - \pi^{\frac{1}{2}} e^{\rho^2 - W_c^2} \sin 2\rho W_c] = A^2 x_j / |\eta_{sj}|^2 \quad (88)$$

where ϕ_r (ϕ_i) is the real (imaginary) part of the ${}_1F_1$ function. This equation cannot be satisfied for small W_c . A second limiting possibility is

$$(ii) \quad |W_c| \gg 1$$

for which case, using the asymptotic representation of the w -function,

(See for example, Abramowitz and Stegun (9))

$$w(z) \sim \pi^{-\frac{1}{2}} (-iz)^{-1} [1 - \frac{1}{2}(-iz)^{-2} + \dots] \quad (89)$$

Equation 86 becomes

$$w_c [1 + \frac{A^2}{w_c^2} (1 + \frac{1}{2} w_c)] = A^2 x_j / |\eta_{sj}|^2 \quad (90)$$

The value

$$w_c = A^2 x_j / |\eta_{sj}|^2 \quad (91)$$

does satisfy this equation and the assumption made above. Substituting this value into $\text{Im } z_c \geq \text{Im } z_b$, using the asymptotic representation for $w(\eta)$ again, one has

$$\rho(1 - \frac{|\eta_{sj}|^4}{A^2 x_j^2}) < A^2 y_j / |\eta_{sj}|^2. \quad (92)$$

Hence it follows that the branch points z_{bj} , $j \geq 1$, lie above the contour for $\xi \lesssim 1$, and therefore for all $\xi < 1$. Therefore these branch points need not be considered any further in the comparison with the pole.

For $j = 0$ the comparison with the contour is again most easily carried out for $A \sim \infty$. For this case, using the asymptotic representation for $w(\eta)$ in Equation 80, one has

$$|\eta_{s0}|^2 \approx A^2 + 3/2 = x_0^2 \quad (93)$$

and Equation 85 for this case becomes

$$W_c + \pi^{\frac{1}{2}} A^2 \operatorname{Im} w(W_c + i\rho) = 2A \quad (94)$$

One must again make assumptions as to the size of W to solve this equation:

(i) $|W_c| > 1$, as satisfied the previous case. Using the asymptotic representation for the w -function again, Equation 94 is

$$W_c (1 + A^2/W_c) = 2A \quad (95)$$

which is satisfied by $W_c = 2A$. Substituting this into $\operatorname{Im} z_c \geq \operatorname{Im} z_{b0}$:

$$\rho \left[1 - \frac{A^2}{4A^2} + O(A^{-2}) \right] > \gamma_0 \left(\frac{A^2 + \frac{1}{2}}{A^2 - \frac{1}{2}} - 1 \right) \approx -\gamma_0/A^2 \quad (96)$$

so that the contour lies above z_{b0} for $\xi < 1$. Therefore it is the only branch point that will contribute to the integral and the only singularity that need be compared with the pole with respect to determining a signal velocity.

One need not consider the positions of the branch points, z_{bj} , $j \geq 1$, for $\xi > 1$ to prove causality since the dispersion rule was shown to be causal by the analysis in the w -plane earlier. Because of this analysis all branch points must lie under the contour in the z -plane or on another sheet for $\xi > 1$.

The signal velocity is now defined in the asymptotic limit, $\sigma x \gg 1$. The criteria for the arrival of the steady state is that value of $A(\xi)$ for which the amplitude of the residue of the pole equals that of the expression representing the precursor. Mathema-

tically, this is expressed, for $\sigma x \gg 1$, by equating the imaginary parts of the branch point and the pole. One solves

$$\text{Im } z_p = \text{Im } z_b \quad (97)$$

i.e.,

$$\rho - \pi^{\frac{1}{2}} A^2 \text{Re } w(W' + i\rho) = \gamma_0 \left(\frac{A^2 + \frac{1}{2}}{|\eta_{s0}|^2} - 1 \right) \quad (98)$$

for a value of A^2 to be used in Equation 42 to define the signal velocity.

A computer program was written to solve this equation numerically and executed on the Iowa State I.B.M. 360-65 machine. The w -function was represented by Equation 87 with a series representation for the confluent hypergeometric function, ${}_1F_1$, which is convergent over the entire complex η -plane. A subroutine was written for Equation 80 which gave the position of the saddle point for a given A^2 and which was then used in Equation 98. This equation was solved for $\omega' = \omega_0$ by a hand calculation, the solution of which was used as the starting point in the machine calculation.

For a fixed ρ , Equation 98 was solved as a function of frequency out to three halfwidths off resonance. Values of ρ ranged from $\rho = 10 \omega_D$ to $10^{-3} \omega_D$, so that the transition from a pure Lorentzian profile to a pure Doppler profile was covered. One can write the asymptotic time delay as

$$\Delta t_A = \alpha(\omega) \sigma(\omega) \times / [\rho + \beta(\rho/\omega_D) \omega_D] = \tau_A \quad (99)$$

Then $\alpha(\omega)$ describes the difference of the time delay frequency profile from that of absorption. Its effect is presented for various values of ρ in Figures 6 to 13, where the normalized time delay and absorption profiles are plotted against frequency. Each is symmetric about resonance so that they are plotted only on one side of resonance. It can be seen that $\alpha(\omega)$ becomes different from one at $\rho \sim 1.5 \omega_D$, slowly increasing off resonance with decreasing ρ . Far from resonance, where the Lorentz tails dominate the profile, $\alpha(\omega)$ again approaches one for all values of ρ . The parameter β is a kind of normalization constant for a given ρ , and is a function of ρ/ω_D . It is equal to zero for pure collision broadening and one half for Doppler broadening and its variation in the transition is shown in Figure 14. It begins to have an effect on the time delay a bit before the parameter α does.

The asymptotic signal velocity can then be written as

$$v_A = c(1 + c\tau_A/x)^{-1} \leq c. \quad (100)$$

Presumably one could also calculate the analogue to the Brillouin signal velocity and v_1 presented earlier. For the Brillouin definition the residue begins to contribute at $\text{Re } z_p = \text{Re } z_b$, which can be solved numerically in the same manner as Equation 97. To calculate v_1 one must assume that the integral first begins to oscillate with the steady state frequency at the value of A^2 for which the pole lay on the circle about z_{b0-} with radius $|z_{b0-} - z_{b0+}|$, as earlier. Each of these definitions depends only on the relative positions of the branch points and pole. The signal velocity v_2 depended on the solution to the integral for $E(x,t)$ and cannot be derived in this case. These

definitions will not be calculated here because only a qualitative estimate was desired for the present. The result is that there is a qualitative change in the time delay in the transition to Doppler broadening when compared to the absorption coefficient.

One of the more important results of this treatment is that the time delay for pure Doppler broadening is proportional to the total absorption over the Doppler width, rather than over the collision width. This second case seemed intuitively attractive since it was felt that perhaps the time delay might be proportional to the lifetime of the excited state of the transition described by the dispersion rule. For Doppler broadening dominating collisions in determining the line shape, the lifetime of the excited state is still the reciprocal of the collision frequency, while the Doppler width is simply an effect due to the motion of the molecules. Were the time delay proportional to ρ^{-1} , then β would have remained zero in the transition from collision broadening to Doppler broadening. Hence, proper derivation of the dispersion rule for this case shows that the time delay for Doppler broadening is not simply the delay for collision broadening, averaged over a Maxwellian distribution of resonance frequencies, which would be the result were β zero.

The preceding analysis is also important for future work on the problem. One can get an expansion of the solution for the integral expression for the wave in the medium for Doppler broadening by expanding the integrand in the transformed plane and integrating term by term. One must know the position of the important singularities in the transformed plane before expanding the integrand, however. The

preceeding analysis has traced these singularities as a function of time and appears to be a good starting point for continuation of this problem.

Figure 6. Absorption and time delay for collisions dominating Doppler broadening.

Figure 7. Absorption and time delay for Doppler broadening effect beginning to appear.

Figure 8. Absorption and time delay for collisions and Doppler broadening of equal strengths.

Figure 9. Absorption and time delay.

Figure 10. Absorption and time delay.

Figure 11. Absorption and time delay.

Figure 12. Absorption and time delay.

Figure 13. Absorption and time delay for Doppler broadening dominating collisions.

Figure 14. The normalization parameter, β , as a function of halfwidth ratio.

NUMERICAL ANALYSIS

Broad Band Detecting

The theory presented thus far has been concerned with defining the arrival time of the steady state signal. This can be quite arbitrary from an experimental point of view because to do so involves differentiating between the steady state and the precursor. The amplitude of the precursor may be of the same order as that of the steady state when its frequency is far from the input signal's frequency, ω' , and at a time long before the arrival time of the steady state as predicted by theory. The time at which the instantaneous frequency of the precursor is A frequency units off resonance, using $\tau = t - x/c$, is

$$\tau = \sigma x \rho / A^2 = \tau_A (\rho / A)^2 \quad (101)$$

where τ_A is the arrival time of the steady state as predicted by the asymptotic theory. Hence for a broad band detector with a bandwidth λ , a response will be measured at, and for all times after,

$$\tau = \tau_A (\rho / \lambda)^2. \quad (102)$$

Unless the amplitude of the precursor behaves quite differently than that of the steady state before τ_A , if $\lambda \gg \rho$ one will not be able to define the arrival time of a steady state by such a detection system.

For simple crystal detection of microwave signals with no filtering, the bandwidth of the system is determined by the cutoff of the waveguide at low frequencies and by the crystal response at

high frequencies. The resulting bandwidth may therefore be as large as one half the resonance frequency used. Using these typical quantities, for a 30 MHz linewidth and a 25 GHz resonance frequency for the gas, the precursor produces a response at the detector at

$$\tau \approx 10^{-6} \tau_A,$$

long before the onset of the steady state oscillations.

Using the exact series solutions calculated earlier, those in terms of the Bessel function and the Φ function, a computer experiment was performed to investigate the behavior of the precursor to see if it could be discriminated against in some manner. The first case considered was that of a broad band d.c. detection system with the properties as described earlier. In the calculation it was assumed that the non-linear response of the crystal detector can be well approximated by a square law behavior so that the detected signal is simply E^2 . For the solution given by the Bessel function series, using $S = S_r + iS_i$ for the sum of Equation 63, one has

$$E(x, \tau) = e^{-\sigma x} \sin(\omega' \tau + \phi) - e^{-\rho \tau} [S_r \sin \omega_0 \tau - S_i \cos \omega_0 \tau] \quad (103)$$

where

$$\phi = a^2 (\omega' - \omega_0) x / cD. \quad (104)$$

For the solution employing the Φ function, using Equation 40 for Φ , Equations 26 and 44, and defining

$$G = G_r + iG_i = -(A - \eta') / (A + \eta') \quad (105)$$

$$\chi = \chi_r + i\chi_i = -i\sigma x \frac{\rho}{\eta'} (1 + \eta'/A)^2 \quad (106)$$

$$\zeta = 1 - \chi \sum_{k=1}^{\infty} \frac{(\frac{1}{2}, k)}{(2, k)} \cdot \frac{z^k}{k!} {}_1F_1(1, 2+k, \chi) \quad (107)$$

one has

$$\begin{aligned} E(x, \tau) = e^{-\sigma x} [F_r \sin(\omega' \tau + \varphi) - F_i \sin(\omega' \tau + \varphi)] \\ + \frac{1}{2} e^{-\rho \tau} [J_0(2A\tau) \sin \omega_0 \tau - G_r \zeta_r \sin(\omega_0 - 2A)\tau \\ + G_i \zeta_i \cos(\omega_0 - 2A)\tau] \end{aligned} \quad (108)$$

which is always valid, though slowly convergent for large τ . This solution gives more of an intuitive feeling for the problem because it shows the sinusoidal frequency dependence of the precursor on $A\tau$ and the time dependent amplitude of the signal.

Squaring the above expressions, one obtains products of sines and cosines of $\omega' \tau$ and $\omega_0 \tau$ which can then be expressed as sinusoidal functions of the sum and difference of these phases. For example, using Equation 103,

$$\begin{aligned} E^2(x, \tau) = e^{-2\sigma x} \frac{1}{2} [1 - \cos 2(\omega' \tau + \varphi)] + e^{-2\rho \tau} \frac{1}{2} \cdot \\ \cdot [S_r^2(1 + \cos 2\omega_0 \tau) + S_i^2(1 - \cos 2\omega_0 \tau) + \\ + 2 S_r S_i \sin(2\omega_0 \tau)] - e^{-\rho \tau - \sigma x} \{S_r [\cos(W\tau + \varphi) - \\ - \cos(\omega_0 \tau + \omega' \tau + \varphi)] - S_i [\sin(\omega_0 \tau + \omega' \tau + \varphi) + \\ + \sin(W\tau + \varphi)]\} \end{aligned} \quad (109)$$

Where $W = \omega' - \omega_0$, and S_r and S_i are slowly varying functions of time compared to the sinusoidal oscillations. A similar result is had

when Equation 108 is squared. For a resonance near 25 GHz, the terms containing the sums of frequencies in their argument oscillate with a frequency of the order of 50 GHz. For the experimental apparatus which was of interest, coaxial line which carries the signal from the detector to the oscilloscope acts as a low pass filter to cut off these high frequencies, allowing only the d.c. signal and the difference frequencies to pass. Therefore the terms of Equation 109 which would contribute to the displayed signal are the d. c. terms and the slowly varying Bessel function series:

$$E^2(x, \tau)_{d.c.} = \frac{1}{2} e^{-2\sigma x} + \frac{1}{2} e^{-2\rho\tau} |S| - e^{-\sigma x - \rho\tau} \cdot [S_r \cos(W\tau + \varphi) - S_i \sin(W\tau + \varphi)] \quad (110)$$

The equivalent expression obtained from Equation 108 is

$$\begin{aligned} E^2(x, \tau)_{d.c.} = & \frac{1}{2} e^{-2\sigma x} |F|^2 + 1/8 e^{-2\rho\tau} (J_0^2 + |\zeta \cdot \chi|^2) \\ & - \frac{1}{2} e^{-\rho\tau - \sigma x} \{ F_r \zeta_r \chi_r + F_i \zeta_i \chi_i \} \cos(W\tau + 2A\tau + \varphi) \\ & + F_i \zeta_r \chi_r - F_r \zeta_i \chi_i \sin(W\tau + 2A\tau + \varphi) \} \quad (111) \\ & + \frac{1}{2} e^{-\rho\tau - \sigma x} J_0 [F_r \cos(W\tau + \varphi) + F_i \sin(W\tau + \varphi)] \\ & - 1/4 e^{-2\rho\tau} J_0 [G_r \zeta_r G_r \cos(2A\tau) - \zeta_i G_i \sin(2A\tau)] \end{aligned}$$

Each of these expressions were tested and were found to give identical numerical results for all values of τ used. Because the Bessel function series was close to an order of magnitude faster than Equation 111, it was used in running the various "experiments"

which were tried. The results of the calculations are presented in graphical form and represent the expected oscilloscope display for various types of measurements as plotted by the IBM simplotter.

Direct Detection of Pulse

First consider the properties of the d.c. detected signals. These are given for values of αx up to .75 in Figures 15 to 20 by the set of plots with their initial points at .5, and for stronger αx in Figures 21 through 24. Each of these representations are for a fixed value of αx as a function of time for different values of $W = \omega' - \omega_0$. The results are symmetric about resonance. The signals at resonance behave as predicted by Baerwald: a slow decay from the unattenuated precursor to the final steady state. There are no strange oscillations of the precursor for times before τ_A ($= \alpha x$ in the units presented) nor a translation of the pulse which would allow one to mark τ_A as a significant time. The transition is smooth and the precursor still gives a noticeable contribution to the signal for times up to $10 \tau_A$.

As the signal is shifted off resonance and interesting effect occurs. Pulse ringing begins to appear due to the interference between the precursor and the steady state terms in $E^2(x,t)$. Even for the weakest absorption chosen, $\alpha x = .01$, the ringing is present with a decaying envelope, as is shown in the magnified pulses of Figures 15a to d. For this small a value of absorption the sum S in Equation 110 rapidly converges and shows little variation with time. Hence, the ringing is due simply to the sinusoidal oscillations with

Figure 15 a. Pulse shape for weak absorption, the figures with initial points at .50. The interference signal is not visible; the normalized interference signal rises from zero initially to one at large times.

Figure 15 b. Expanded view of ringing phenomenon for previous figure.

Figure 15 c. Expanded view of ringing phenomenon for weak absorption.

Figure 15 d. Expanded view of ringing phenomenon for weak absorption.

Figure 15 e. Expanded view of ringing phenomenon for weak absorption.

Figure 16. Pulse shapes with initial values of .5, interference signal at bottom of figure, and normalized interference signal rising to one for large times.

Figure 17. Pulse shapes with initial values of .5, interference signal at bottom of figure, and normalized interference signal rising to one for large time.

Figure 18. Pulse shapes with initial values of .5, interference signal at bottom of figure, and normalized interference signal rising to one for large time.

Figure 19. Pulse shapes with initial values of .5; interference signal at bottom of figure, and normalized interference signal rising to one for large time.

Figure 20. Pulse shapes with initial values of .5, interference signal at bottom of figure, and normalized interference signal rising to one for large time.

Figure 21. Pulse shapes as a function of time for changing frequency.

Figure 22. Pulse shapes for changing frequency. The dashed lines are hypothetical steady state extensions; the squiggly lines are the range of arrival times predicted by the Brillouin definition to that of the asymptotic theory.

Figure 23. Pulse shapes for changing frequency. The dashed lines are hypothetical steady state extensions; the squiggly lines are the range of arrival times predicted by the Brillouin definition to that of the asymptotic theory.

Figure 24. Pulse shapes for strong absorption.

multiplying S with the exponentially decaying envelope.

This effect presents an interesting possibility for measuring parameters of a weakly absorbing medium. Absorption is usually measured by a null technique and may be difficult if the probe signal is extremely weak in the first place. However, amplification of a d.c. detected pulsed signal will produce the ringing effects of Figures 15 a - d, from which one can measure ρ , the decay time of the envelope, and ϕ , the phase shift of the dispersed signal compared to that originally sent out. From these quantities one can then get an estimate of the pressure and the number density of the constituent producing the dispersion.

The magnitude of the ringing is larger than the absorption at the same frequency as can be seen by considering Figure 15 a for $W = 5$. The relative magnitude of the first few oscillations is of the order of 3 parts in 500 or .6% whereas the absorption at the same frequency, as detected by a null measurement, is

$$1 - e^{-\sigma x} \approx 1 - (1 - .01/26) \approx .04\%, \quad (112)$$

a factor of 15 weaker. At $W = 10$ the ratio is even larger: .4%/.01%, or a factor of 40. Even at resonance the null absorption is just .01, the same order of magnitude of the ringing at $W = 5$. Because of this wide band feature one would be able to work over a much more extended range of frequencies than allowed by absorption techniques. This would be advantageous for simply detecting the presence of weak, narrow lines before a more detailed analysis.

As one goes to stronger absorption the ringing effect increases.

in magnitude and begins to change in character in the figures near $\sigma x = 1.5$. Because the argument of the Bessel functions increases with increasing σx , more terms must be taken in the series S so that the oscillatory nature of this factor becomes evident. The first oscillation of the pulse ringing now begins to take on the shape of the front of a wave packet, ever more so farther off resonance. The jagged lines drawn on the figures mark the range of arrival times of the pulses as predicted by τ_B and τ_A . Although the packets are delayed in the proper sense as a function of frequency, these predicted arrival times do not appear fall on any given region of the leading edge of the packets consistently. In fact far off resonance they begin to climb up the falling edge of what one might interpret as the precursor. If the leading edge of the packets were due to the steady state contribution one would expect τ_A to fall in the first minimum in each of the plots, since it appears that the edge begins to dominate the "precursor" here. It was according to this criterion that the signal velocity leading to τ_A was defined.

For the case of resonance, for σx large, a hump occurs in the signal. This hump must be due to the precursor and the cross term because the steady state is essentially zero in this case. As one follows the signal off resonance for each case one sees the hump grow to the leading edge of the packet spoken of above. Hence by arguments of continuity, a reasonably large contribution to this leading edge must come from the precursor terms. Hence, as long as one detects non-linearly without filtering or using interference techniques, it does not appear possible to differentiate between the

precursor and the steady state signal.

In a few cases a dotted line has been drawn from the origin to the leading edge of the packet or one of the humps in the waveform, interpreting these as the arrival of the steady state. The jagged lines have then been projected downward upon them to see what fraction of the steady state amplitude is reached at this time. From the definitions made in the non-asymptotic analysis, one would expect these times to appear near one quarter the final amplitude. However in all cases they appear much lower, although there is more consistency between their range of values. Hence the same conclusions about the precursor representing partially the leading edge of the wave packet can be drawn.

Interference Measurement

Up until now the discussion has centered about the direct d.c. detection of the pulse shape. In this case it does not appear possible to differentiate the steady state contribution from the precursor. One possible attempt at doing so would be by interfering the dispersed pulse with a comparison signal to suppress one of the terms in the expression for $E(x,t)$ before detection. The most straightforward technique would be to suppress the final steady state of the dispersed signal, in which case the comparison signal would be

$$E(x,t)_c = e^{-\sigma x} \sin(\omega' t + \varphi + \pi). \quad (113)$$

If this signal is added to $E(x,t)$ before detection, the resultant

detected signal will have zero amplitude as the final steady state is reached. The expression for the displayed signal for this case is

$$\begin{aligned}
 E^2(x,t)_{int} = & \frac{1}{2} e^{-2\sigma x} [(F_r - 1)^2 + F_i^2] + 1/8 e^{-2\rho\tau} (J_0^2 + |G \cdot \zeta|^2) \\
 & - \frac{1}{2} e^{-\rho\tau - \sigma x} \{ [F_r G_r \zeta_r + F_i G_i \zeta_i] \cos(W\tau + \varphi + 2A\tau) \\
 & + [F_i G_r \zeta_r - F_r G_i \zeta_i] \sin(W\tau + \varphi + 2A\tau) \} \\
 & + \frac{1}{2} e^{-\rho\tau - \sigma x} J_0 [F_r \cos(W\tau + \varphi) + F_i \sin(W\tau + \varphi)] \\
 & - \frac{1}{2} e^{-2\rho\tau} J_0 [G_r \zeta_r \cos(2A\tau) - G_i \zeta_i \sin(2A\tau)] \\
 & - \frac{1}{2} e^{-\rho\tau - \sigma x} [J_0 \cos(W\tau + \varphi) - G_r \zeta_r \cos(W\tau + \varphi + 2A\tau) \\
 & + G_i \zeta_i \sin(W\tau + \varphi + 2A\tau)] \quad (114)
 \end{aligned}$$

where the Φ solution has been used for $E(x,t)$. It does not appear evident that better steady state - precursor discrimination can be made by this method over direct d.c. detection. The interference term which now appears may be larger than before for $\tau \leq \tau_A$ because of the difference $(1 - F_r)^2$. This factor may be large for early times for large σx .

The above expression was calculated for each of the cases treated earlier in the direct pulse display. The results are shown in Figures 15 to 20 as the set of graphs with smallest amplitude

at $\tau = 0$ and in 25 to 28 as those beginning with non-zero amplitude. In each case they decay to zero as expected. This signal was also normalized by setting

$$E^2(x,t)_{\text{int norm}} = 1 - E^2(x,t)_{\text{int}}/E^2(x,0) \quad (115)$$

to study the interference as a function of frequency more closely. This set of graphs starts from zero at $\tau = 0$ and rises to one for large times on the same figures.

The results of this calculation are interesting. For small αx the interference signal behaves as expected. It is essentially zero for $\alpha x = .01$ and grows slowly with increasing absorption. For small absorption, $F_r \approx 1$, and one is getting essentially complete cancellation plus a small precursor contribution. For larger absorption $F - 1$ becomes different from zero near $\tau = 0$, so that the semi-cancelled steady state oscillation is adding to the precursor to give a growing interference signal as expected.

The first point of interest to notice is the frequency independence of the interference signal for small absorption. All of the normalized plots are superimposed for αx up to .6 indicating that there are no relative differences between the interference signals as a function of frequency other than amplitude. This can be seen mathematically from Equation 103. For these cases the Bessel function series, S , represents the interference signal well:

$$e^{-\rho\tau} S \approx e^{-\rho\tau} \frac{iA}{\eta^2} J_1(2A\tau) \quad (116)$$

whence the frequency dependence cancels in the normalization above.

Figure 25. Interference and normalized interference signals for changing frequency.

Figure 26. Interference and normalized interference signals for changing W . Note the largest interference signal now occurs off resonance.

Figure 27. Interference and normalized interference signals for changing frequency. Note the largest interference signal now occurs off resonance.

Figure 28. Interference and normalized interference signals for changing frequency. The largest interference signal now occurs off resonance and is larger than the input signal.

In fact for vanishing absorption the normalized interference signal rises as

$$1 - \exp(-2\rho\tau). \quad (117)$$

This can be seen by expanding the Bessel function of Equation 116 for small argument:

$$e^{-\rho\tau} S \approx e^{-\rho\tau} \frac{iA}{\eta^1} (2A\tau) = e^{-\rho\tau} \frac{2i}{\eta^1} \sigma(\omega_0) \times \rho \quad (118)$$

so that the only time dependence is in the exponential. Equation 117 then results after normalization of this equation.

The next trend of interest is the behavior of the interference with increasing absorption. It becomes more prominent for larger absorption, indicating greater dispersion, yet remains frequency independent until $\alpha x \sim .75$. Now a slight separation can be seen in the normalized signal, the signal with the faster rising slope being that at resonance. The normalized signal now rises faster than $(1 - \exp(-2\rho\tau))$, indicating that more terms are taken in the Bessel function series than the first. It was felt that perhaps one could define a signal velocity from a study of the precursor as a function of frequency. One would expect the precursor at resonance to be more prominent than off resonance, lasting for a longer time, so that the normalized signal would be delayed compared to off resonance. A similar behavior might be expected for the signal at resonance with growing absorption. In each case the opposite effect is observed.

One of the most interesting results of the interference experiments occurs for fairly strong absorption. For these cases,

interference off resonance begins to grow with increasing αx faster than on resonance, so that at $\alpha x \approx 1.6$, the initial value of the interference signal at $\tau = 0$ is the same from resonance out to $W = 2\rho$. For larger absorption the maximum interference signal at $\tau = 0$ is given at some frequency off resonance, the value of W growing with increasing absorption. (For example, for $\alpha x = 6$, the maximum is about .77 at a frequency setting about 2.5 halfwidths off resonance.) The interference signal at resonance meanwhile approaches the limit .5 with increasing absorption.

The growth of the interference signal to a value greater than one half off resonance at $\tau = 0$ raises the question as to what type of solution best describes the physical situation: one which consists of an ever present steady state response plus a transient response, as is given by the Bessel function solution of Equation 103, or one in which the cancellation of these two terms is taken account of to give a precursor and a slowly growing steady state contribution, as is given by the Φ function, Equation 108, and the asymptotic theory. If the first solution does, it makes no sense to talk of a signal velocity since the final steady state contribution is present at $\tau = 0$ for all frequencies. The interference signal behavior for strong absorption will now be interpreted from both points of view to try to answer this question.

According to the solution given by the steady state and transient responses, the interference signal is simply the transient response itself. For the cases spoken of above then the transient

at $\tau = 0$ is of greater amplitude than the original signal sent into the medium for frequencies off resonance. It also follows that the transient off resonance is larger than that on resonance; because the transient is a result of the signal interacting with the resonance one would expect it to be weaker off resonance where the interaction is weaker.

In terms of the picture presented by the solutions which give the precursor and the steady state oscillations which arrive at a time later than $\tau = 0$, the behavior of the large interference signal off resonance is more easily interpreted. For this case the precursor value initially is less than the original signal sent into the medium for all frequencies. It adds to the small steady state oscillations, (whose phase at this time need not be the same as that of the final steady state signal) to produce the regular pulse shape with initial value one half. When the comparison signal is interfered with the pulse it is $\pi/2$ radians out of phase with the steady state oscillations, for the solution given by the Φ function. Hence it does not produce destructive interference of the signal initially but can add to give a value greater than one half for the interference signal off resonance. With this interpretation, the precursor can decrease as one goes off resonance, in contrast to the behavior of the transient for the same conditions. The large interference signal is mainly due to a phase difference between the comparison signal and the steady state oscillations for early times.

The interference experiment appears to show that the concept of a precursor and steady state oscillations with time-dependent amplitude gives a better physical description of the signal than does the steady state - transient response description. Hence the concept of a signal velocity does make sense. However the technique does not allow one to separate the precursor and steady state oscillations to measure this signal velocity.

Narrow Band Filtering

An alternative to interference techniques for discriminating against the precursor is the use of a narrow band filter before detection. To incorporate it into the theory, a detector function with a Lorentzian frequency selectivity profile was chosen because it is easily treated in the theory as a simple pole. It fit the actual specifications of the detector used in the experiment quite well: fitting the half power point on the response curve to the halfwidth of a Lorentz profile, the 10 db point on the response curve is about 2.1 halfwidths off resonance. Hence it is somewhat narrower than Lorentzian, closer to a Lorentzian squared. This profile was also used in a few runs, represented by a pole squared, to look at some pertinent features which did not change in character from those of the simple pole treatment.

For a sinusoidal detector response, the Fourier frequency spectrum is represented by a pair of poles located at

$$\omega = \pm \omega_D - i\lambda$$

in the complex ω - plane, where ω_D is the central frequency of

the detector and λ is its frequency response halfwidth. To include these properly in the integral formulation one must go back in the formalism to the point where the integrand has poles symmetrically located about the real axis.(2) Including these in the integral one has

$$E(x,t) = -\frac{\alpha}{2\pi} \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} e^{-i\omega t + ikx} \left[\frac{1}{\omega - \omega' + i\epsilon} - \frac{1}{\omega + \omega' + i\epsilon} \right] \cdot \left[\frac{1}{\omega - \omega_D + i\lambda} - \frac{1}{\omega + \omega_D + i\lambda} \right] d\omega \quad (119)$$

where $k(\omega)$ is given by Equation 3, and α is a normalization constant. Taking $k = \omega/c$ and $\omega' = \omega_D$, the response for the filter is

$$E(x,t) = (1 - e^{-\lambda\tau}) \sin \omega_D \tau, \quad (120)$$

if $\alpha = i\lambda$ is chosen as the normalization constant. Using the same assumptions made in arriving at Equations 2 and 5, narrow linewidths and low density of oscillators, one has for arbitrary ω_D :

$$E(x,t)_{\text{det}} = -\frac{1}{2\pi} \lim_{\epsilon \rightarrow 0} \text{Re } i\lambda \int_{-\infty}^{\infty} e^{-i\omega t + ikx} \frac{d\omega}{((\omega - \omega' + i\epsilon)(\omega - \omega_D + i\lambda))} \quad (121)$$

with the dispersion rule of Equation 5 used for $k(\omega)$. Using partial fractions to separate the poles one finally has

$$E(x,t)_{\text{det}} = -\frac{1}{2\pi} \lim_{\epsilon \rightarrow 0} \text{Re } \frac{i\lambda}{\omega' - \omega_D + i\lambda} \int_{-\infty}^{\infty} e^{-i\omega t + ikx} \cdot \left(\frac{1}{\omega - \omega' + i\epsilon} - \frac{1}{\omega - \omega_D + i\lambda} \right) d\omega \quad (122)$$

For practical use one detects at the frequency propagated, i.e., $\omega_D = \omega'$.

The solution for each term of Equation 122 is exactly that of the last section with ρ replaced by $\rho - \lambda$ for the second term. Both the Bessel function series and the Φ function solution were used in the preliminary calculations and each gave identical answers, the Bessel function representation converging more rapidly again. For $\rho \sim \lambda$, the first of the Bessel function solutions was used, Equation 57a, because of poor cancellation of the exponential and series of Equation 59a. The cancellation could be obtained, however, by setting the tolerance higher at the expense of computing time. The results of this calculation are presented in Figures 29 to 33, representing the oscilloscope display of the filtered signal. The time variable now is $T = \lambda \tau$ Where λ is the bandwidth of the filter and, hence, one over the rise time of its response. In each case the signals have been normalized to their steady state amplitudes:

$$E_N^2 = E_{det}^2(x, t) / \frac{1}{2} e^{-2\sigma x} \quad (123)$$

Figures 29 to 30 represent the signal as a function of W for various values of ρ and $\sigma x = .3$. Values of ρ and σx were chosen compatible with a real experiment which is discussed later in the text. As can be seen from the graphs there is a change in the shape of the leading edge of the pulse, which one could interpret as a time delay, but it is in the opposite sense from what was

Figure 29. Filtered signal for changing frequency. The signal at resonance is that rising most quickly, and appears to be time advanced.

Figure 30. Filtered signal for the linewidth larger than the filter width. The signal at resonance is even more advanced than that of the last figure.

Figure 31. Filtered signal for the linewidth just larger than the filter width. The large precursor is now less effectively filtered than either of the previous figures, giving the large leading edge.

expected. That is, the signal with the sharpest leading edge is that for propagation on resonance, giving a sort of time advance as a function of frequency. These results were first obtained with the Bessel function expansion and it was felt that perhaps the important frequency information of the signal was given by higher order terms in the expansion which were not showing up on the computer. An increase in the tolerance which demanded more terms in the expansion did not change the result. Hence the solution in terms of the Φ function was tested in which the amplitude term multiplying the steady state residue was known to grow from zero in an explicit manner. It was felt that perhaps this manipulation of the important residue term would produce results which would reflect the signal velocity definitions made from this expansion in Equations 43 to 49. However, results identical to the Bessel function expansion were obtained.

This apparent time advance of the signal at resonance compared to off resonance is best described by a consideration of the unfiltered signals of the last section. The precursor always gives its greatest contribution for early times. Therefore, it is all the more enhanced when compared to the attenuated steady state at these times, hence most strongly enhanced for propagation at resonance. Even though its instantaneous frequency lies outside the filter bandwidth for times before τ_A , for $\lambda = \rho$, because it still has a relatively strong effect on the signal for times long after τ_A , the precursor will cause the filtered signal to show

a sharper leading edge at resonance. If one chooses a bandwidth much less than the resonance, $\lambda \ll \rho$, one discriminates against the precursor for times longer than τ_A . This effect is shown by interpreting Figures 29 through 31 as for fixed ρ and decreasing λ . Hence, the effect of the precursor at resonance is simply muted. It still exists though, because at resonance the precursor is always enhanced compared to off resonance.

The next figure, 32, shows the filtered signal, at resonance, for $\tau_x = .3$ for various ratios of ρ/λ which were experimentally feasible. According to the theory, the time delay is linearly proportional to τ_x and hence the signal should show the sharpest leading edge for the smallest τ_x . The opposite is observed however, again in agreement with the interpretation given in the last paragraph. The stronger the attenuation, the more the precursor is enhanced with respect to it.

Figure 33 shows the filtered signal at resonance for $\sigma_x = .3$ for halfwidths less than those of the previous figure. As was pointed out earlier in Equation 102, it appeared necessary that $\lambda > \rho$ for proper discrimination of the precursor. It is seen that for $\lambda \sim \rho$ that the signal begins to get a peak upon its leading edge, indicating the precursor is no longer being filtered. In the limit of $\lambda \rightarrow \infty$, the signal would revert to the unfiltered, with the sharp nose of the precursor. These results are again in agreement with the earlier interpretation.

From the above analysis it does not appear that it is possible

Figure 32. Filtered signal with changing pressure, and hence, changing linewidth.

Figure 33. Resonance signal for linewidths both larger and smaller than filter width. For linewidth smaller than filter width, the signal reaches a value larger than the steady state.

to discriminate against the precursor using filtering techniques.

If one increases the absorption to a very large value to give a larger time delay, the precursor becomes even more enhanced compared to the steady state and the same behavior results. Hence filtering of the signal before detection has been no greater an aid than interference techniques in attempting to measure the arrival of the steady state.

Check of Various Signal Velocity Definitions

Because the contribution of each term in the expression for $E^2(x,t)$ was known as a function of time, in the computer calculation, one could make a check as to the accuracy of all the signal velocity definitions made. Consider first the definition made in the asymptotic theory which gave the largest time delay, τ_A . Because the cross term in the expression for $E^2(x,t)$ oscillates for $W \neq 0$, a comparison for this case is difficult off resonance. Hence, only the case of resonance is considered here. Even though the signal velocity defined in the asymptotic theory is applicable only to cases for which $\sigma x \gg 1$, one can consider the results of applying it to cases for $\sigma x \lesssim 1$. The signal arrival was defined as that time at which the amplitude of the precursor decreased to equal that of the steady state. Since there is no phase difference between the steady state and precursor for weak absorption, at τ_A

$$|E(x,t)|^2 = |E_{s.s.} + E_{prec.}|^2 = 4 E_{s.s.}^2 \quad (124)$$

The times at which this equation holds, as determined from the computer

output, will be referred to as τ_d . This equation begins to make sense only for $\alpha x \geq .7$, since for absorption weaker than this the steady state value of E_{ss}^2 is greater than four times the signal sent into the medium. Some of the values read off the output and compared to τ_A are:

$$\begin{array}{lll} \alpha x = .75, & \rho\tau_A = .75, & \rho\tau_d = .06 \\ & = .90, & = .24 \\ & = 1.5, & = .63. \end{array} \quad (125)$$

Although there is a trend in the correct direction for increasing absorption, the predictions are not good. Hence it appears that the criterion used for extreme asymptotic cases is not useful here.

For $\alpha x \gtrsim 1.5$, the precursor can change sign before τ_A so that at $|E_{ss}| = |E_{prec.}|$, $E(x,t)$ can be zero instead of $2 E_{ss}$. Hence for very strong absorption the criterion of Equation 124 makes little sense and different considerations must be made. Consider the extremely asymptotic conditions, $\alpha x = 6, 7.5$, and 9 . These cases are shown expanded in Figures 34, 35, and 36, respectively. The solid line is the entire signal, steady state response plus transient response, squared. The dashed line which always remains positive is the squared transient; that which oscillates to negative values is the cross term between the steady state and transient responses. The times marked by the dashed vertical lines mark the time τ_A . In terms of the precursor and steady state signal it appears that the final steady state does rise to a reasonably constant value at a time near τ_A on the average. If one defined the signal arrival

Figure 34. Test of the asymptotic signal velocity.
Solid line is the signal. Vertical line is the arrival
time predicted by the asymptotic theory.

Figure 35. Test of the asymptotic signal velocity.

Solid line is the signal. Vertical line is the arrival time predicted by the asymptotic theory.

Figure 36. Test of asymptotic signal velocity. Solid line is the signal. Vertical line is the arrival time predicted by the asymptotic theory.

as the time marked by the maximum of the final hump, the times given by τ_A are off by 17, 22, and 15%, respectively. However in each case the definitions do make physical sense in that the signal has reached a steady state and no longer oscillates for times after τ_A .

Consider next the signal velocity leading to the time delay, τ_2 . The plots of the signal, or "steady state oscillations", defined by

$$S_0 = .5|F|^2 e^{-2\sigma x} \quad (126)$$

have been superimposed upon the directly detected pulse shapes in Figures 37 and 38 for two typical cases of strong and weak absorption: $\sigma x = 6$, and $.45$. Although the packet-like leading edges were shown to consist partially of precursor contributions, dotted lines have been extended up to meet this leading edge for purpose of comparison. The plot of this function rises quickly for early times, but continues to do so rather slowly after τ_A , so that the final steady state amplitude is not reached on these graphs. The reason for this is that probably more terms should be subtracted from the sum in ζ and added to F for times long after τ_A , since they become more important then. However for early times these plots should give a good representation of the squared amplitudes of the steady state oscillations. The problem is that one cannot detect the rise of this amplitude by interference or filtering techniques to prove their validity. Numerically they are one half at the times predicted, but because they were defined numerically, which is somewhat redundant.

The analogue of the definition made by Brillouin, which gives τ_B , cannot be estimated numerically. However its behavior as a func-

Figure 37. Wave forms with the function $|F|^2$ superimposed.

Figure 38. Wave forms with the function $|F|^2$ superimposed.

tion of frequency does appear similar to the anomalous behavior of the interference results for strong absorption. That is, the large interference signal off resonance can be attributed to the interference of the precursor and the comparison signal since the steady state contribution has not yet arrived according to Brillouin. However one would then expect a maximum in this behavior at a frequency between one and two halfwidths off resonance, according to Figure 4. Instead this maximum continues to move further off resonance with increasing absorption.

Brillouin's calculations were done for the case ω' far from resonance. The parameters he chose were for optical frequencies, in which case $\omega_0 \approx a \approx .1 \rho$, conditions for which the approximations made on $k(\omega)$ in this work are not valid. Nevertheless his calculation is a first order asymptotic expansion, which produces expressions similar to the asymptotic theory (Sec, for example, page 73, Equation 20, of reference 3, and page 23, Equation 54, of reference 2.). Hence there must be some limit on the absorption for which his calculation is valid. If one argues that it is good in the region, $\sigma x \approx 1.7$ to 2, one can say his results give as good a picture as the others for this case since the interference and filtering techniques cannot discriminate between one mathematical solution and another.

Finally consider the time delays predicted by the group velocity. It appears that the presence of the precursor prohibits any possible detection of the time delays predicted by the group velocity. Figures 39 and 40 represent filtered and unfiltered

Figure 39. Filtered signal for very strong absorption and filter width much larger than linewidth.

Figure 40. Unfiltered signal for same parameters as last figure. The points of inflection of the curves and maxima and minima occur at different times than for the filtered case.

(with $\lambda = 10 \rho$) pulse shapes for a strong and very narrow resonance: $\sigma x = 20$ and $\rho = .1 a^2 x/c$. Values of W used are 2, 4,, 30, and the time scales are the same for both graphs. For the filtered figures the rise time of the signals have been decreased due to the response time of the filter. In addition the precursors have not been effectively filtered as can be seen by the oscillations of the signals near resonance, for which the steady state is zero. If λ is decreased to better discriminate against the precursors, the group velocity estimate of the resultant signal will be all the more dependent on the response time, λ^{-1} , of the filter. Hence to make group velocity estimates, the unfiltered signal was chosen, but for a range of frequencies far enough off resonance that perhaps the precursor has little an effect on the leading edge.

Figure 41 shows the normalized unfiltered signal for the same conditions as for Figure 40, on an expanded time scale, and for a range of frequencies from 12 to 40 halfwidths off resonance. One quarter of the final steady state is represented by .25 on the horizontal axis. The jagged lines again represent the range of arrival times between τ_B and τ_A . For the large values of W chosen τ_A is the same as the group velocity. In a few cases the leading edge has been extended down to the origin, as if it were completely due to the final steady state. For $W < 18$, the group velocity predicts too late an arrival time of one quarter the final amplitude. For these cases it can be argued that the steady state oscillations have reached one quarter their final amplitude earlier, but are destruc-

Figure 41. Normalized pulse shapes for same parameters as last figure for frequencies far off resonance. Dotted lines are extensions of the leading edge, and the squiggly lines are ranges of arrival times between the Brillouin delay and the asymptotic delay.

tively interfered with by the precursor to produce an earlier value of one quarter in the figure. For $W > 18$, the group time delay gives too early an arrival time. For these cases one can say that the leading edges are still affected by the precursor and that the extended leading edge is not valid. The steady state oscillations are really rising more slowly. In either case the precursor plays a non-negligible role and prevents the detection of the steady state oscillations.

The Experiment of Shiren

The results of the preceding analysis indicate that it is impossible to filter out the precursor for times near or before τ_A . Hence it appears that the definitions of the signal velocity in the region of anomalous dispersion made earlier are not verifiable by these techniques. However Shiren reported a time delay for wave propagation in a resonant medium in which he claims the precursors were filtered properly. His results show a distortion and translation of the leading edge of the pulse. In the following we shall attempt to explain his results in terms of the theory presented thus far.

In the measurements reported by Shiren the filter bandwidth he used was of the order of, or less than, the width of the three absorption lines measured: $\lambda = 10$ MHz, $\rho_1 = 8$ MHz, $\rho_2 = 47$ MHz, and $\rho_3 = 715$ MHz. The reported time delays are at least an order of magnitude larger than τ_A predicted by the asymptotic theory, using these measured

absorption widths in the expression for τ_A . To explain the time delays reported Shiren invokes the concept of a resonance with a very narrow line width, much narrower than that of the absorption reported, which is shifted in varying amounts by local inhomogeneities in the medium to produce the measured absorption profile.

If the dimension of the inhomogeneity is much less than a wavelength the dispersion rule of the Doppler theory would apply, with ω_D equal to the observed absorption width. The resultant τ_A 's calculated are too small by 1, 2, and 3 orders of magnitude, respectively.

However if the dimension over which the resonance is unshifted is much larger than a wavelength, a large scale inhomogeneity, the pulse locally sees a medium of very narrow resonances, each delaying the signal by an amount τ_A , but with the extremely narrow width used for ρ in τ_A . Shiren then averages this expression over the profile given by absorption. The result is

$$\tau_{avg} = \sigma x / \rho_{loc} \quad (127)$$

where σx has the frequency dependence and strength of the measured absorption, and ρ_{loc} is the narrow halfwidth of the local resonance being averaged. The result is a time delay much larger than the regular τ_A :

$$\tau_{avg} = \tau_A (\rho_{meas} / \rho_{loc}) \gg \tau_A \quad (128)$$

Shiren then used this expression to estimate ρ_{loc} , with a measure of

τ_{avg} .

Shiren's explanation appears to be in agreement with the results of our analysis, except for the statement that all the precursors were properly discriminated against. For each very narrow local resonance the filter width is now much larger than ρ . (For the values of ρ_{loc} derived by Shiren, λ/ρ is between 4.5 and 47.5.) If the narrow resonances that Shiren postulates over local regions were not inhomogeneously broadened, the resultant absorption profile would be an order of magnitude larger at $W = 0$, resulting in an extremely narrow, strong line. Such a resonance can be approximated by Figure 39, used previously, for which $\lambda/\rho = 10$ presents a value in the region derived by Shiren. As applied to Shiren's experiment, one can think of the figure as for fixed ω' and a varying ω_0 , to produce the values of W listed.

One would then explain Shiren's time delay by saying that, on the average, the pulse frequency is on the wings of most of the local shifted resonances with which it interacts. Contributions to the pulse deformation due to the few lines with which it interacts for which $W \approx 0$ will simply produce an overall absorption contribution as for $W = 2$ and less, but in less magnitude. For ω' on the observed center of the broadened resonance, for example, the pulse will interact with the greatest number of $W = 0$ resonances, each giving an absorption contribution. However it will also interact with the wings of many more $W \neq 0$ resonances to form some distorted leading edge, some average of those pictured in Figure 40. As one tunes off the center of the observed resonance, the pulse is interacting with fewer

$W = 0$ local resonances, and its frequency is further on the wings of the non-zero resonances, so that the averaged pulse shape is rising faster to produce less a time delay. Finally, far off resonance, the filtered signal will rise with the rise time of the filter, for which case W is larger than those shown in the figure, and there is no pulse distortion or time delay due to the resonance.

From the above discussion it appears that Shiren's physical interpretation of the time delays he saw is correct. However, his estimates of ρ_{loc} as derived from the theoretical time delay expressions are probably incorrect. According to the computer analysis, one quarter the final amplitude of the pulse occurs at a time later than any of the theoretical definitions, and it is a function of the ratio of the detector width over the line width. One could make an estimate of ρ_{loc} from the pulse shapes of Figure 39. To do so, one would define the arrival time of one quarter the final amplitude of the pulse numerically from the figure by an equation of the form, for example,

$$\tau_{fig} = \gamma(\omega, \lambda_D / \rho_{loc}) \sigma(\omega) \times / \rho_{loc} \quad (129)$$

where $\sigma(\omega)$ has the shape of the postulated local resonance. The parameter, γ , takes into account the additional frequency dependence that the time delays have, similar to the case of Doppler broadening treated earlier. One would take this expression and average it over the observed absorption profile in a manner analogous to Shiren's. He did this for $\gamma = 1$, using the asymptotic time delay, τ_A , and for a γ which describes Brillouin's delay.

Although such an averaging process appears useful in a first approximation, it is not valid mathematically. In such a process one is assuming that one has a square pulse impinging on each of the local resonances, which is not true. After traveling through one local region the pulse is dispersed, and it is this signal that impinges upon the next local region. One would have to Fourier analyze the signal at the interface of each local region, which results in a sum on $k(\omega)$ in the integrand of Equation 2. Because each local region is much larger than a wavelength, the sum cannot be converted to an integral, as could have been done in the Doppler theory if such a method were used there in deriving $k(\omega)$. This summation is not straightforward and will not be delved further into in this treatment.

EXPERIMENTAL BACKGROUND

Line Broadening Mechanism

The sources of spectral line broadening which arise in microwave spectroscopy are (11):

1. natural line breadth,
2. saturation broadening,
3. collisions with walls of containing vessel,
4. collisions with other molecules, and
5. Doppler broadening.

The natural line breadth is due to radiation damping from a classical point of view, or, quantum mechanically, due to the zero point vibration of electromagnetic fields which are always present in free space. (11). It is of the order of 10^{-7} hertz for resonances near 25 GHz and hence too small to be of interest presently. Saturation broadening arises due to the use of a high intensity signal in spectroscopic work, altering the population levels of the atom from the thermodynamic equilibrium condition. It is an effect which must be avoided in the study of time delays experimentally because the present theory does not account for it. Broadening due to collisions with the walls of the gas cell appear when the pressure becomes so low that the mean free path of the molecules becomes larger than the smallest dimension of the cell. However Doppler broadening appears before this for 25 GHz waveguide and is about an order of magnitude larger than this effect for room temperature. For atmospheric physics collision and Doppler broadening are the only two effects which must be considered and these have been described in sufficient detail

earlier.

For pressures down to the order of 10^{-5} atmosphere, the principle line broadening mechanism for microwave resonances is due to collisions between molecules of the gas. The absorption coefficient for this case has a Lorentz profile and is given by the imaginary part of Equation 5

$$\sigma(\omega) = \frac{\pi N_a e^2 f}{m c} \frac{\rho}{(\omega - \omega_0)^2 + \rho^2} \quad (130)$$

where N_a is the number density difference between the lower and upper states involved in the transition; f is the oscillator strength for the transition, and the rest of the symbols are as given earlier.

For this case of collision broadening the halfwidth is proportional to pressure (or number density):

$$\rho = \sum_i \Gamma_i P_i = \sum_i \Gamma_i N_i k T \quad (131)$$

where Γ_i is the line broadening parameter and P_i the partial pressure for the i th constituent of the gas, k is Boltzman's constant and T is the temperature. For a gas consisting of a single constituent, or one with a constant relative number density, the absorption coefficient at resonance is independent of changes in the total number density:

$$\sigma(\omega_0) = \frac{\pi e^2 f}{m c k T} \frac{\alpha_i}{\sum_i \Gamma_i \alpha_i} \quad (132)$$

where the relative density of the i th constituent is

$$\alpha_i = N_i / N_{\text{total}} \quad (133)$$

Making use of Equation 99 for the time delay for a Lorentz profile, the time delay at resonance is inversely proportional to number density and therefore increases with decreasing pressure until Doppler broadening effects begin to appear.

When the pressure is reduced so that the collision width becomes of the order of the Doppler width, the pressure dependence of both the absorption coefficient and the time delay can be numerically calculated from the theory presented earlier. For further reduction in pressure Doppler broadening dominates and the absorption coefficient takes on a Gaussian profile and is now linearly dependent upon the number density, decreasing proportionally with pressure:

$$\sigma(\omega) = \frac{\pi N_a e^2 f}{m c} \frac{e^{-(\omega-\omega_0)^2/\omega_D^2}}{\omega_D/\sqrt{\pi}} \quad (134)$$

From Equation 99 the time delay is also proportional to number density in the limit of pure Doppler broadening, now decreasing with pressure. Hence, for a given path length, the maximum time delay occurs at pressures at which the line broadening mechanism undergoes the transition from collision broadening to Doppler broadening.

For collision broadening, the absorption coefficient and time delay exhibit different temperature dependence. Making use of the inverse temperature dependence of N_a (due to expanding the Boltzmann factor, $\exp(-h\omega_0/kT)$, which appears in taking the difference in number densities between the two levels involved in the transition) and the linear temperature dependence of ρ , one has

$$\sigma(\omega_0) \propto T^{-2}. \quad (135)$$

Because of the ρ^{-2} dependence of the time delays

$$\Delta t(\omega_0) \propto T^{-3} \quad (136)$$

Hence, making simultaneous measurements of absorption and time delay, one can extract the temperature dependence unambiguously, whereas absorption measurements alone require a knowledge of all other physical parameters of the gas before a value of the temperature can be obtained.

In the case of pure Doppler broadening, making use of the linear ω_0 dependence of f_{jk} , the inverse temperature dependence of N_a due to the Boltzmann factor, and the expression for the Doppler width, Equation (65), one has

$$\sigma(\omega') \propto \omega_0 M^{1/2} T^{-3/2} \quad (137)$$

and

$$\Delta t(\omega') \propto M T^{-2} \quad (138)$$

The same statements can be made about the temperature dependence as in the case above. A second interesting difference for this case is the dependence on the resonance frequency of the two expressions.

The linear dependence of σ on ω_0 is one reason for the lack of absorption lines below microwave frequencies for gases in the earth's atmosphere. However, the time delay for Doppler broadening is independent of ω_0 , other than through the line shape, so that equal number densities of two gaseous species may produce equal time delays, yet have entirely different absorption line strengths. Noting the range of frequencies spanned by the radio window of the earth's atmosphere (50 MHz to 23,000 MHz), it is apparent that time delays may be measurable for conditions for which absorption is not.

Experimental Apparatus

Microwave Resonance Used

The microwave spectrum of ammonia has a series of very strong absorption lines throughout the K-band range of frequencies (18 to 26 GHz). The $J = 6, K = 6$ line (where J is the total angular momentum of the molecule and K is the projection of the total angular momentum onto the symmetry axis of the molecule) was chosen for two reasons. It is the second strongest line of the spectrum, and it has no neighboring resonances as does the strongest line, $J = 3, K = 3$. The resonance frequency is 25.05604 GHz, its intensity is $\sigma(\omega_0) = 6.9 \times 10^{-4} \text{ cm}^{-1}$, and its collision width parameter is $\Gamma = 28 \text{ MHz/mm Hg}$ (11). For room temperature, the Doppler width is 37.6 kHz. Hence the transition from collision broadening to Doppler broadening occurs for pressures of the order of $1.3 \times 10^{-3} \text{ mm Hg}$. Using the time delay predicted by the asymptotic theory for collision broadening, for the (6,6) ammonia line,

$$\Delta t = .392 \times (\text{cm}) / P(\text{mm Hg}) \quad \text{n-sec} \quad (139)$$

At the transition to Doppler broadening

$$\Delta t = 300 \times (\text{cm}) \quad \text{n-sec} \quad (140)$$

Pressures down to 0.010 mm Hg are easily attainable so that path lengths of the order of three meters appeared sufficient to produce measurable time delays initially.

The basic experimental system employed was a K-band microwave gaseous spectrometer, modified appropriately to measure pulsed signals. Five different arrangements of the system were used and the detailed differences between each will be described in separate sections devoted to each method. However common to all methods were some basic pieces of microwave equipment which shall be described presently.

Gas Cell

The gas cell was constructed from sections of K-band wave guide with waveguide windows at each end of the cell. Holes of diameter $3/64$ inch ($\approx 10 \lambda$) were drilled into the waveguide to allow gas to enter the cell, pumping from the cell, and pressure gauge connections. Because of the fact that the brass waveguide outgasses various unknown gases other than that of interest below pressures of ~ 50 microns of mercury, the test gas was allowed in at one end of the cell through a needle valve and pumped out the other end. Therefore outgassing products were never allowed to become a large fraction of the total gas content using this technique. The gas cell was pumped on with a 1402 Welch pump and pressure measured from .500 mm Hg to .005 mm Hg with a Stokes McLeod gauge, model 276 BB. For pressures below this, an ion gauge was to be used to measure the pressure.

Klystron and Power Supply

The signal source was an OKI-24V10A, K-band klystron with power available from 18 to 26 GHz. The klystron was mounted in an oil bath and water cooled with soft copper tubing wrapped about it.

This was necessary to reduce the noise bandwidth of the klystron to a value much narrower than the line widths of the resonance which was to be probed. The 3 db width was determined to be less than 25 kHz with the water cooled oil bath. The power supply was a PRD Electronics 812 model which could power two klystrons with similar operating characteristics simultaneously, and which was used in such a manner with the heterodyne technique using identical klystrons.

Pulse System

The signal was pulsed by means of a Philco switching diode, L4774, which is inserted directly into the waveguide line in a diode mount. A negative bias applied to the switch produces a large impedance mismatch with the result that just a fraction of energy continues past the switch down the waveguide, the fraction depending on the magnitude of the bias voltage. The output of a pulse generator was then applied as the bias voltage and this signal then impressed upon the r.f. energy. Two pulse generators were available: a Rutherford B14, with a maximum voltage of ± 10 volts, a 10 n-sec rise time, and pulse lengths up to the order of milliseconds; and an EH-120D, with a negative pulse of -25 volts maximum, a 1.2 n-sec rise time, and a maximum pulse length of 160 n-sec.

It was desirable to use pulse lengths much longer than that attainable with the EH pulse generator for two reasons. The first is because the maximum time delays expected were of the order of the longest pulse length attainable with this piece of equipment. Assuming that the front edge of the pulse up to τ_A consists of the precursor

with its frequency different from ω' , the signal would just begin to reach its steady state oscillation about the time the pulse ended. Hence for even longer time delays the entire pulse might consist of just a precursor, which would make the definition of a signal velocity impossible.

The second reason for the long pulse lengths is so that the width of the Fourier spectrum of the pulsed signal is much less than the width of the resonance line. Then the main Fourier components of the signal are contained in a frequency range over which the resonance varies slowly, as is the case in the theory. Repeating a given wave form at a repetition rate $1/T$ yields a Fourier spectrum consisting of a series of lines separated by $1/T$ frequency units with an envelope given by the Fourier transform of the waveform being repeated. The Fourier transform of a single pulse of frequency ω' and pulse length τ is the function (12)

$$f(\omega) = [\tau/4\pi i] \left\{ \frac{\sin(\omega+\omega')\tau/2}{(\omega+\omega')\tau/2} - \frac{\sin(\omega-\omega')\tau/2}{(\omega-\omega')\tau/2} \right\} \quad (141)$$

which is illustrated in Figure (42) for the positive half frequency plane. The function has zeros at frequencies $\pm\omega' + 2n\pi/\tau$; hence as long as $2\pi/\tau \ll \rho$, the shape of the resonance will appear slowly varying to the main frequency components of the pulse. The EH pulse generator was not able to meet this requirement with its .16 μ -sec pulse length maximum for line widths of interest (< 3 MHz), because $1/(.16 \mu\text{-sec}) = 6.25$ MHz. Hence the Rutherford model was used in conjunction with a Hewlett-Packard 460BR broad band amplifier to

Figure 42. Fourier spectrum for a square pulse of sinusoidal oscillations with main frequency component

Figure 43. Response curve of detecting crystal.

produce a reasonable bias with which to run the diode switch efficiently. The response curve of the switching diode is shown in Figure (43) as a function of bias voltage. With the above arrangement, the transmission loss was approximately 12 db.

Detection

The signal was detected by means of a 1N26 crystal rectifier and displayed on either a Hewlett-Packard 175-B or 185 sampling oscilloscope, the 175-B having a rise time of 10 n-sec. The 1N26 is a non-linear element with a current-voltage response similar to that shown in Figure (44). The signal was detected both by d.c. and the heterodyne technique. In the case of d.c. detection one merely detects the rectified output of the crystal. For a 25 volt pulse applied across the diode switch one obtains ~12 db transmission loss so that less than 10% of the energy passes the switch as a c.w. signal, the remainder responding to the signal of the pulse generator. The small c.w. signal acts as a bias, shifting the operating point of the crystal off the axis slightly, and the pulsed signal is rectified as is indicated in Figure (44). Because the coaxial line which carries the signal from the detector acts as a low pass filter as indicated earlier, the resulting display is simply the d.c. signal and low frequency oscillations. For extremely narrow linewidths, the intensity of the r.f. signal had to be reduced to avoid saturating the resonance, and the resulting signal was too weak to be effectively displayed by this method.

Figure 44. Response curve for a typical detector with rectification, or non-linear detection, exhibited.

Figure 45. Time dependence of the precursor frequency exhibited.

A convenient method to measure small signal amplitudes is the heterodyne technique, in which the non-linear characteristic of the crystal is taken advantage of in a second way. The small amplitude signal, δe , is mixed with a second, stronger c.w. signal, e , at a different frequency from the first, which acts as a bias on the crystal. Using (13)

$$\delta e = \delta E(t) \sin \omega' t \quad (142)$$

$$e = E \sin (\omega' + \Delta) t \quad (143)$$

one can make a Taylor series expansion, about the origin if no d.c. bias is applied to the crystal:

$$i = f(e + \delta e) = f(0) + f'(0) \cdot (e + \delta e) + \frac{1}{2} f''(0) \cdot (e + \delta e)^2 + \dots \quad (144)$$

The quadratic term will contribute an expression due to the cross product proportional to

$$E \cdot \delta E(t) \cos \Delta t. \quad (145)$$

The quartic term will also produce terms proportional to $\cos \Delta t$, but mixing crystals are chosen such that $f^{iv}(0)$ is small enough so that it may be neglected. All harmonics of ω' and Δ will also appear with some small amplitude due to higher order terms in the series. Making use of the term given by Equation (145), one has the time dependence of $\delta E(t)$, the pulse shape in this case, reproduced at an inter-

mediate carrier frequency and magnified in amplitude. This is because the signal is now proportional to $E \cdot \delta E(t)$ rather than $\delta E(t)^2$, as was the case for square law detection. This method will be considered later in more detail.

EXPERIMENTAL TECHNIQUES

Because a precise frequency meter was not available, the frequency for a fixed pressure was measured by plotting out the measured absorption profile. Knowing the pressure to a reasonable degree of accuracy, one could then establish the half power point of the Lorentz profile and from this the frequency of the rest of the data points. This method was used in the first three techniques which follow.

Interferometer

The equipment was originally designed to operate as an interferometer, as diagrammed in the schematic of Figure 46. The c.w. signal from the klystron passes through the isolator, a one way transmission device which protects the klystron from reflected signals. The variable attenuator is adjusted such that the resonance line is not saturated by the signal. After the c.w. signal is pulsed at the diode switch, it passes into the hybrid tee where it divides and propagates into the gas cell and comparison arm. The hybrid tee is a four-port device which is the microwave analogue to the optical half silvered mirror. A signal entering port 1 divides equally and passes out through ports 2 and 3. In the case of Figure 46 they are both then reflected back into the ports after passing through the gas cell and comparison arm respectively. If these reflected signals are in phase they will pass out through port 1 again, if 180° out of phase, through port 4. A tunable short was used at the end of the

Figure 46. Block diagram for interferometer experiment.

comparison cell by which the relative phase of the two reflected signals could be continuously adjusted. The resulting signal out of port 4 was d.c.-detected at the crystal and displayed on the oscilloscope, which was synchronized to the pulse generator.

The comparison arm was the same length as the gas cell to within a wavelength, and hence to within 0.03 n-sec travel time. A variable attenuator was included in the arm so that the absorption due to the gaseous resonance could be balanced. If the interferometer is tuned to a null in the steady state, a time delay of the signal due to the interaction with the gaseous resonance would be measured as a pulse, the length of which would equal the time delay. The leading edge of this pulse would be the leading edge of the signal which has passed through the comparison arm of the interferometer at speed c ; the trailing edge of the pulse, some time Δt later, would be the inverted leading edge of the signal which has interacted with the gaseous resonance and been delayed by an amount Δt . For a six foot length absorption cell, the path length is 365.8cm. Using Equation (139), for pressures ≤ 0.100 mm Hg, the time delay should be ≥ 14.3 n-sec. No pulse due to a time delay was observed; The rise time of the pulse generator prevented the observation of the interference of Figure 17.

When the interferometer was tuned for constructive interference, or when the attenuator in the comparison arm was set to full attenuation so that only the signal passing through the gas cell was detected, the pulse shape exhibited a "ringing" phenomenon which was associated with the resonance. That is, for frequencies far off resonance, a

periodic oscillation with an amplitude of the order of ten percent or less of the pulse height and a frequency of a few megahertz became discernible, superimposed upon the pulse front. After a few oscillations it decayed away to the original pulse shape. As the signal frequency was tuned toward resonance, the period of the oscillation increased, until at resonance an oscillation could no longer be observed, the effective period being infinite. This behavior was symmetric about the resonance frequency.

A ringing phenomena is observed in propagation through waveguides over extremely long path lengths due to the dispersive properties of the guide and results in pulse distortion. However, in that case it does not exhibit a resonance-like behavior with frequency since the dispersion rule for a waveguide is just that of a plasma, i.e. Equation (3), with $\omega_0=0$. This phenomenon has been tabulated for various types of waveguide, and a calculation (Appendix C) shows that such a phenomena should not be detectable for the lengths of waveguide which were used in this experiment.

That the phenomenon was associated with the ammonia resonance was verified by being able to identify four other resonances by associating this ringing effect with them: $\nu(2,2) = 23.722$ GHz, $\nu(3,3) = 23,870$ GHz, $\nu(4,4) = 24.139$ GHz, and $\nu(5,5) = 24,532$ GHz.

D. C. with Heterodyne Detection

Pleshko and Palocz (14) have performed an experiment in which they have observed Sommerfeld precursors associated with

dispersion in a waveguide filled with a ferromagnetic material. They were able to observe directly the oscillations of the signal, and hence the continuously changing frequency of the precursor. The frequency of the signal during the first oscillation, for example, was roughly one fifth the final steady state frequency. Using this method one could roughly define the arrival of the steady state as that time when the frequency of the signal no longer changed as a function of time. However such a method is limited by the relative accuracy with which one can measure time differentials with respect to the period of the oscillations. Hence they made no quantitative estimates of a time delay other than stating the velocity of propagation measured was less than that predicted by the group velocity. As stated earlier, the time at which the instantaneous frequency of the precursor is A frequency units off resonance is

$$\tau(A) = \tau_A (\rho/A)^2$$

where τ_A is the arrival time of the steady state for $\omega' = \omega_0$.

As an example, for a line width of 0.3 MHz and a time delay of 160 n-sec for the 12 foot path length, the time at which the frequency of the precursor is 10 halfwidths from resonance is

$$\tau(3 \text{ MHz}) = 1.6 \text{ n-sec.} \quad (146)$$

Similarly, $\tau(2 \text{ MHz}) = 3.6 \text{ n-sec}$, $\tau(1 \text{ MHz}) = 14.4 \text{ n-sec}$, $\tau(.5 \text{ MHz}) = 48 \text{ n-sec}$, etc.

Because the carrier frequency in the present case is so high (for 25 GHz, $\omega^{-1} = 0.04$ n-sec), the oscillations of the steady state signal for this case cannot be observed on an oscilloscope as they were by Pleshko and Palocz. Even if this were possible one would have to detect differences of 3 parts in 25,000, for example, if one wished to discriminate between the precursor 3 MHz off resonance and the steady state. This would involve counting one extra quarter oscillation in 2,000 which would take 83 n-sec to count, a time over which the instantaneous frequency of the precursor will have changed drastically.

The above problem could be circumvented in the following way. Making use of the heterodyne technique one could reproduce the pulse shape of the signal at an intermediate frequency, the oscillations of which one could observe directly on an oscilloscope display. If the local oscillator is set at ω_{10} , the intermediate frequency is $\Omega = \omega - \omega_{10}$, and the instantaneous frequency of the precursor is now $\Omega_0 \pm A$. From the discussion in the above paragraph it would appear that one would like to have the intermediate frequency as low as possible to detect relative changes more easily. For example, for an intermediate frequency of 30 MHz, to detect the precursor 3 MHz off resonance, one need only detect changes of one part in ten, and therefore one half oscillation out of five. This alone is misleading however because the time over which five oscillations at 30 MHz occurs is still large (16 n-sec) compared to the time over which the instantaneous frequency is changing rapidly (e.g., a few n-sec for $A = 3-2$ MHz).

If the intermediate frequency is set to zero, i.e. $\omega_{10} = \omega_0$, one could presumably measure A directly, observing an oscillation of continuously changing period. The number of oscillations such a signal undergoes before the arrival of the steady state is just the phase as the steady state begins, over 2π :

$$n = \phi_{ss}/2\pi = 2 A\tau_A/2\pi = \sigma_0 x/\pi. \quad (147)$$

For the case $\sigma_0 x = .3$, $n \sim .1$, i.e. the sine wave has not yet undergone a quarter oscillation, which would appear to be the minimum necessary to perform such an analysis. In addition, because the two klystrons had noise bandwidths of ~ 25 kHz, A would not be measurable to closer than 50 kHz at best; this is sufficient as long as $\rho/2\pi \gg 50$ kHz, since the steady state dominates for $A < \rho$.

Because the heterodyne system was in operation, all the above methods were tried, in case the precursor frequency was other than that predicted by the theory, allowing it to be observed. None of them gave positive results however.

Narrow Band Filtering

Another experimental technique attempted was narrow band filtering of the signal before rectification. There are no tunable narrow band filters commercially available as waveguide components at K-band frequencies. However an alternative method of filtering was possible, similar to the one used by Shiren. A narrow band 30 MHz i.f. amplifier was employed in conjunction with the heterodyne

detection system used, which is diagrammed in block form in Figure 47. The signal which is available at the output of the hybrid tee is the 25 GHz signal, which is rectified, plus that given by Equation 145. Here E is the c.w. signal of the local oscillator and $\delta E(t)$ the electric field amplitude of the wave which has traveled through the dispersive medium. The 30 MHz signal is linear in δE so that rectification of this signal has not yet taken place. A picture of this signal is shown in Figure 48.

The i.f. frequency was set to 30 MHz, the central frequency of the General Radio, 1216-A, narrow band amplifier which was used. The filtering of the signal takes place within the amplification system and the bandwidth is a function of the number and type of amplifier tubes used. The rectification (non-linear detection) of this signal takes place after filtering, at a double diode in the amplifier. The signal out of the amplifier is that of Equation 145, but rectified. Therefore the theory used in the numerical analysis is still valid for this experiment.

The bandwidth at half maximum of this amplifier is .700 MHz ($\lambda = .350$ MHz) with a rise time of the order of $1.3 \mu\text{-sec}$. For pressures greater than $10 \mu\text{ Hg}$, $\rho \geq .300$ MHz so that $\lambda \leq \rho$ for pressures greater than $12 \mu\text{ Hg}$. The graphs of Figure 32 are the resonance predictions for the filter output for a filter with a Lorentz profile for pressures down to this value.

Figure 49a is a picture of the output of the amplifier for the signal as taken on an oscilloscope camera, for various pressures. The amplifier was set at automatic volume control so that the signals

Figure 47. Block diagram of the narrow band filtering system with heterodyne detection.

Figure 48. Heterodyne detection of pulse for both signal oscillator and local oscillator set to the same intensity. Both d. c. and heterodyned signals are visible. The upper trace is for an external sweep, the lower on internal sweep.

Figure 49a. Top photograph represents filtered signal, far off resonance for all values of ρ , and on resonance for large ρ . Middle and bottom are for on resonance, ρ decreasing ($\rho \approx 1.15$ and 0.30 MHz, respectively).

are normalized. The top picture is that for a pressure of about 100 μ Hg. As the signal was tuned through resonance no change in the output could be seen at this pressure, so that the signal also represents the off resonance output. The middle picture is the amplifier output for a pressure of 25 μ Hg, which exhibits a faster rise in the pulse shape, as predicted by the computer analysis. The off resonance signal was the same as that of the top picture. The bottom picture is the amplifier output for a pressure of 10 μ Hg, which shows even a greater enhancement of the precursor. The ratio of the detector width to the line width has increased to a point, according to the linewidth determined by the pressure measurement, that the signal should rise above the final steady state.

The pictures of Figure 49a appear to verify the predictions of the computer calculation. Because the Lorentz filter does not give an exact reproduction of the shape of the filter used, no numerical comparison was made to the theory. There would be no point in doing so because no information about the medium can be obtained from such an estimate. The pictures are presented because they are a successful observation which confirms the somewhat unexpected prediction of the theory.

Broad Band Detecting

The final experimental technique presented is that of d. c. broad band detection. For this case the predicted signal from the computer calculation is given in Figure 17, in which case the ringing effect is exhibited. For this weak an absorption the ringing can be

described by a sinusoidal oscillation with a decaying envelope. The envelope is a combination of a decaying exponential and a slowly varying Bessel function of first order. For the parameters of interest, the Bessel function was dominated by the exponential in determining the envelope's shape, as shall be shown later. Because the ringing amplitude is a function of the parameters of the medium, presumably, one should be able to fit data to the theory to determine these parameters for given experimental conditions. In the following treatments, a fit of the data from two different experiments will be performed.

Because the Bessel function is a slowly varying function of time for the parameters of interest, the frequency of oscillation of the ringing phenomenon is just that of the sinusoidal oscillations of the precursor-signal interference term: $\omega' - \omega_0$. Although no direct, accurate method of measuring frequency was available, one could use the heterodyne principle to prove this fact. With the local oscillator tuned to the gas resonance, ω_0 , the signal klystron's frequency, ω' , could be tuned through the resonance and the beat frequency compared with the ringing. This was accomplished in the following manner. First, the signal klystron was tuned to the resonance of the gas as determined by the ringing effect. Then the method of detection was switched to heterodyning with the internal sweep on the oscilloscope, so that the difference frequency of the two klystrons could be observed. The local oscillator was then tuned to the resonance frequency of the gas. This was done by observing the heterodyne frequency on the oscilloscope until it exhibited an infi-

nite wavelength by tuning. Now one could tune the signal klystron off resonance and read the frequency difference, W , from the resultant heterodyne frequency. Pictures of the ringing effect were taken, using d. c. broad band detection. The heterodyne frequency was then displayed and photographed on the same plate so that a frequency comparison could be made. Two of such pictures are shown in Figure 49b, both on a time scale of $.2 \mu\text{-sec/cm}$. Hence the higher frequency oscillation is about 20 MHz, the lower frequency 6.35 MHz. More important the heterodyne frequency is seen to coincide with the ringing frequency, so that it is indeed the difference, W .

Data were taken for the ringing effect as a function of changing intensity and as a function of pressure in two separate operations. The signal was passed through a variable attenuator, which was used to vary the intensity of the signal in the first experiment performed, then through the gas cell, the pressure of which could be varied for the second measurement made at constant intensity. The signal was then detected at the end of the cell. The crystal current and voltage were measured by a Hewlett Packard 425A d. c. voltmeter-ammeter, from which the radiation intensity could be calculated. The voltage reading of the oscilloscope was also checked by this meter and was found to be properly calibrated.

The ringing effect was photographed by a Polaroid oscilloscope camera, five times for each variable setting. The photographs were projected onto graph paper hung on a wall, producing roughly ten-to-one magnification of the trace. Greater magnification produced poor

Figure 49b. Examples of the pulse ringing effect along with a heterodyned signal of frequency equal to the difference between the incident frequency minus the resonant frequency. Both are for an absorption of .25 and a pressure of approximately 15μ Hg.

Figure 49c. Examples of the pulse ringing effect: The top picture presents the effect as a function of incident frequency, the frequency of the ringing being just $\omega' - \omega_0$; the bottom picture is a presentation of a series of the data used and shows the exponential decay of the ringing amplitude.

black-to-white definition of the trace. The graticules of the oscilloscope screen were calibrated to the graph paper divisions. The resultant trace was about an inch wide and each one produced two curves, i. e., the black-to-white definition of the top and bottom of each trace. Because the greatest amplitude of the ringing was three to four divisions on the graph paper, only the maxima and minima of the curves were considered. Two maxima and one minimum were analyzed for each trace, and averaged over ten values. Hence, three averaged data points are available for each set of traces.

Ringling Effect versus Saturation Intensity

One of the problems encountered in measuring very narrow, pressure broadened line widths in microwave spectroscopy is that the intensity of the radiation begins to interfere with the measurement (11). This occurs in two ways: first, it affects the linewidth; second, it changes the population levels of the two states of the absorbing transition, affecting the absorption coefficient. According to Townes (11), when collision and saturation are the only important broadening mechanisms, one can write

$$\rho^2 = \rho_c^2 + \rho_c B I \quad (148)$$

where ρ is the resultant linewidth, ρ_c is the linewidth due to collisions without saturation, I is the radiation intensity, and B is a coefficient with proper units. The absorption coefficient for saturation effects present is

$$\sigma = \sigma_0(1 + \sigma_0 B' I)^{-1} \quad (149)$$

where σ_0 is the absorption coefficient for no saturation, B' is a coefficient, and I is the intensity again. The saturation effect occurs because stimulated emission begins to become important compared to collisions in de-exciting the upper state of the absorbing transition. Because σ is a function of the intensity, the dispersion rule is non-linear in the electric field, $E(x,t)$, and the linear dispersion theory may not apply to the propagation problem under consideration. Because the intensity was a variable in the experimental setup, it was decided to take saturation measurements to see if the linear theory would describe the observations.

According to the computer analysis, the relative amplitude of the ringing compared to the pulse height is a function of the total absorption of the medium. The envelope of the ringing amplitude depends upon the halfwidth of the resonance. The data were fit to the theory by varying these two parameters until a theoretical curve was generated which best fit the three data points for a given intensity. These curves are presented in Figures 50 through 57. From each fitting a value of σx and ϕ is obtained. These two parameters are then plotted as a function of intensity and are presented in Figures 58 and 59.

Consider first the absorption versus intensity curve, Figure 58. The curve passing through the data points was fit to them by the computer and is not a function of the form of Equation 149. The other two curves, C_1 and C_2 , are described by this functions for values of B' equal to

Figure 50. Data fitting for intensity of $13.1 \mu \text{ watt/cm}^2$

Figure 51. Data fitting for an intensity of $3.7 \mu \text{ watt/cm}^2$.

Figure 52. Data fitting for an intensity of $27.8 \mu \text{ watt/cm}^2$.

Figure 53. Data fitting for an intensity of $51.1 \mu \text{ watt/cm}$.

Figure 54. Data fitting for an intensity of $82.9 \mu \text{ watt/cm}^2$.

Figure 55. Data fitting for an intensity of $139 \mu \text{ watt/cm}^2$.

Figure 56. Data fitting for an intensity of $271 \mu \text{ watt/cm}^2$.

Figure 57. Data fitting for an intensity of $458 \mu \text{ watt/cm}^2$.

Figure 59. Halfwidth squared versus intensity graph.

Figure 58. Relative absorption versus intensity graph.

2.5 and 18.9 cm/ μ watt, respectively. From the curves it appears that either the absorption does not behave according to Equation 149, or that possibly B' was varying during the experiment. Because the intensity at which saturation occurs is a function of pressure, proportional to pressure squared (11), it follows that B' is also a function of pressure, going like P^{-2} . It varies, for example, from .515 cm/ μ watt at 50 μ Hg, to 51.8 cm/ μ watt at 5 μ Hg, so that it appears to be a sensitive function of pressure. The two values used to generate the curves were derived by fitting the first and last data points taken to Equation 149, and correspond to pressures of 21 and 8.2 μ Hg, respectively. Fitting the rest of the data points to the same equation and solving for B' gives a smooth transition of the parameter between these two values. Hence it appears that a slow pressure variation over this range could produce the data points of Figure 58, and Equation 149 does indeed describe the variation of σ as determined from a measurement of the ringing effect.

Data were taken for a pressure which appeared to stabilize at 10 μ Hg. The variation apparent in the data could be described in the following way. Because the inlet valve at one end of the gas cell was shut off, there was probably a pressure gradient along the cell, the lowest pressure at the end which was pumped on, the highest at the end with the inlet valve. Pressure was measured at the middle of the gas cell. This gradient probably builds up due to the high rate of outgassing at the low pressure end of the cell. The value of the pressure at the center of the cell could therefore appear to remain constant for such conditions. If the pressure actually reached 5 μ Hg near the end of

taking the data, it is possible that it was not measured by the McLeod gauge because of the pin size hole used to connect the gauge to the waveguide. A reasonable pressure gradient may exist across this hole at this low a pressure, making an accurate measurement difficult.

Typical microwave absorption measurements are made at pressures ranging from $\sim 100 \mu \text{ Hg}$ to $\sim 30 \mu \text{ Hg}$, low enough to separate neighboring lines. Information about the molecular parameters are a function of collision line widths and Doppler broadening acts as a lower limit to measuring these parameters. Hence it is considered a nuisance to such measurements and low pressures which were attained here are not used normally. In conclusion, it appears that the absorption coefficient, as determined from the ringing effect probably does fit Equation 149, and the discrepancy in these measurements can be explained by pressure variations due to outgassing by the metal waveguide at the low pressures which were attained.

Consider now the linewidth-squared versus intensity curve, Figure 59. These data do not fit a straight line as well as the absorption data fit Equation 149. There are two possible reasons for this. First, whereas the absorption data were a function of ringing amplitude to pulse amplitude, which could be measured with reasonable accuracy, the linewidth is measured by comparing relative amplitudes of the ringing maxima and minimum. In most cases the second minimum was of the order of the error bars in the measurement. Secondly, if the pressure was indeed changing, or if there was a pressure gradient along the cell, ρ_c is not constant for the measurements as was

assumed, so that Equation 148 will not be fit by the data. Nevertheless, a straight line was fit to the data and is presented in Figure 59. The intercept give some average value of the collision width of the gas during the measurement. From this value one can determine an average pressure, which varied either as a function of time if it were changing so, or as a function of distance along the gas cell if there were a pressure gradient present. In either case it appears that outgassing of the waveguide at this low pressure is the reason for the deviation of both absorption and halfwidth from their predicted values. From the graph, the average linewidth is .78 MHz, and the average pressure is $24.3 \mu \text{ Hg}$, large because of the poor fit of data to the line.

Ringng Effect versus Pressure

The second experiment performed was measurement of the ringing effect as a function of pressure. In the same manner as the previous section, the theory was fit to the data and values of ρ and αx were determined for each pressure setting. The data fitting is exhibited in Figures 60 through 66, and ρ and αx are plotted as a function of pressure in Figure 67.

The absorption coefficient appears to be a constant with pressure, indicating no saturation effects. If saturation were present, αx would show a decrease at low pressures where the effect would begin to appear. The intensity used for this experiment was $\sim 7 \mu \text{ watts}$, which would begin to saturate the resonance at a pressure of $< 5 \mu \text{ Hg}$. The average value of αx from the data is 0.111, whereas the value given

Figure 60. Data fitting for pressure equal 55 μ Hg.

Figure 61. Data fitting for pressure equal $36 \mu \text{ Hg}$.

Figure 62. Data fitting for pressure equal 29 μ Hg.

Figure 63. Data fitting for pressure equal 23 μ Hg.

Figure 64. Data fitting for pressure equal $20 \mu \text{ Hg}$.

Figure 65. Data fitting for 16 μ Hg.

Figure 66. Data fitting for pressure equal 8μ Hg.

Figure 67. Absorption and linewidth plotted as a function of pressure.

by Townes is .1271 for this path length. The systematic error in σ , due to the uncertainty in reading the pressure, is roughly 18 %, larger than the comparison of this quantity with Townes' value.

The value of the halfwidth determined by these measurements is roughly as accurate as the absorption. The value of Γ determined from the graph, using $\rho = \Gamma P$, is 23.97 MHz/mm Hg, with an error of 18% again. Bleaney and Penrose's value (15), as determined from absorption profile measurements, is 28 MHz/mm Hg, with an error of 5%. It appears that the value of Γ measured would be better if the true pressure is lower than that measured. This may be the case because of the pinhole connection of the McLeod gauge connection spoken of earlier. Although equilibrium appeared to be reached before photographs were taken, a pressure differential could have existed across the pinhole connection. The error would have been a subjective one, in establishing when equilibrium was reached, and would have propagated through the experiment. Hence, the reasonably good fit of the data to a straight line. The percentage error here is

In addition to the sources of error discussed thus far, there are those associated with the data reduction. Assuming the voltage at the crystal detector is accurate, errors are introduced, first, in transferring the signal to the oscilloscope screen. The amplitude of the ringing oscillations may not be reproduced as accurately as the d. c. pulse height. It was observed that when heterodyne detecting, the heterodyne oscillations disappear as their frequency approaches the

upper end of the bandwidth of the oscilloscope amplifier, leaving just the d. c. signal. The Hewlett Packard oscilloscope has a 25 MHz bandwidth while the ringing frequency was about 5 MHz for the two experiments which were performed (5.1 and 5.21, respectively).

In addition to the oscilloscope, errors are introduced in the photographing, projection, and tracing of the oscilloscope displays onto graph paper. In the ringing versus intensity measurements, the noise generated in the crystal is a function of intensity, which produces some distortion.

All of the above errors could be drastically reduced by a different system of taking data. First, rather than using d. c. detection, one could use a good heterodyne detection system. This would involve building a good stabilization circuit such that the local oscillator is tied in frequency to the signal klystron very accurately. For the experiments being done, this would require a tunable system, which would add more sophistication. Rather than displaying the ringing on an oscilloscope screen, one could store the pulse shape, for example, in a multi-channel analyzer. This storage could then be used to drive a plotter, which would reproduce the pulse shape for measurement. Such a method involves no subjective reduction by the experimenter.

The problems involved with pressure at the low values reached are inherent in a metal system and are not as easily solved. For the problem caused by outgassing, perhaps one could pump on the system at more than one point along the gas cell so that the pressure gradient is not encountered at low pressures. For the ringing versus pressure

measurement, one must allow very long periods of time for true equilibrium to be reached. Neither of these two problems appear insurmountable.

The results of the preceeding measurements indicate that indeed parameters of the medium can be measured by the ringing effect. The resulting data have low enough scatter in most cases so that theoretical curves can be fit to them quite well. The curves are determined well enough so that the variation from curves fit by other experimental methods can be explained by pressure problems within the system. These were known to exist and are apparently showing up in the data. Conversely, the variation may be real and perhaps the real part of $k(\omega)$ varies differently with intensity than the absorption coefficient as given in Equation 149. Such a variation is not described by the linear dispersion theory used here.

The accuracy of the ringing measurements compared to absorption techniques have been discussed elsewhere in the text. The beauty of the ringing method is that a measurement at a single frequency gives both a value for the absorption and halfwidth for a given pressure or intensity setting. If absorption techniques were used, one would have to scan the line in measuring these quantities. In concluding, it appears that the ringing effect is properly described by the theory, and this effect can be used to measure parameters of the medium.

SUMMARY AND CONCLUSIONS

The exact solution has been derived for the problem of electromagnetic pulse propagation in a medium exhibiting a Lorentz profile with a dispersion rule approximated for low density of absorbers. Using an approximation of this solution for weak absorption, three different criteria were used to define mathematically signal velocities of the steady state oscillations for signal frequencies in the region of anomalous dispersion of the resonance. These were found to agree with results for the asymptotic case of strong absorption better than with the definition of Brillouin. Because of this agreement the first step in the solution for the case of a mixture of Lorentz and Doppler broadening was taken: defining the signal velocity for the asymptotic condition of strong absorption. This could be done without solving for an expression for the wave in the medium.

Two different expressions for the exact solution mentioned above were used in a computer analysis of wave propagation in the medium. This analysis predicts the waveforms for various experiments which could be performed using square law detection. The main result of the analysis is that, for weak absorption, the concept of a signal velocity in the region of anomalous dispersion appears to make sense only from a mathematical point of view. One cannot discriminate between the steady state oscillations to define the signal velocity experimentally, neither by direct detection of the pulse, interferometry, nor narrow band filtering of the signal.

For the case of weak absorption the computer analysis predicts a so-called ringing phenomenon of the pulse, in which case the pulse top oscillates with an amplitude small compared to the pulse height. This oscillation amplitude is a function of the parameters of the medium, disappearing in the limit of a vacuum. The envelope of these oscillations is a decaying exponential for extremely weak absorption, the decay time being the halfwidth of the resonance. The frequency of the oscillations is just the difference between the signal's main frequency component and the resonance frequency of the medium.

For increasing absorption the oscillatory nature of the ringing changes, according to the computer analysis. Originally a decaying sinusoidal, the oscillations change to a much more complex form which is a mixture of Bessel function and sinusoidal oscillation, according to the exact solution for the wave. The most important feature is that the first oscillation begins to form the leading edge of a wave packet, which exhibits a delay as a function of frequency in the correct sense as predicted by most of the signal velocities. When the signal velocities, including the group velocity, are tested for predicting the arrival times of these packets, none give a correct answer which is consistent for all frequencies. The apparent explanation is that, because the squared amplitude is being detected, the cross term between the precursor and steady state oscillations produces interference effects. These prohibit labeling the wave packet's leading edge as due completely to the steady state oscillations, in terms of which the signal velocities are defined.

For the case of the signal on resonance and very strong absorption, the delay produced from an asymptotic theory appears to give a somewhat a consistent a definition of the steady state arrival.

The main conclusion of the computer analysis is that the signal velocity has little meaning from the experimental view taken here. However, because one has the solution for the waveforms one can glean some information about parameters of the medium from an experimental measurement of the features of the dispersed pulse's shape.

The experimental results reported verify the predictions of the computer analysis in a qualitative manner. The rising of the leading edge of the narrow band filtered signal where one would expect a decay is verified. It can be explained as due to an amplitude modulation effect of the precursor which has not been effectively filtered. It appears that such filtering is impossible for reasons explained in the text. The ringing effect predicted by the theory is also verified experimentally. It is shown to have the correct frequency dependence, and the decaying amplitude is estimated to give a measure of the halfwidth of the resonance.

It appears that further theoretical work on the case of combined Lorentz and Doppler broadening is necessary if information about the medium is sought from pulse shape measurements. The signal velocities defined for this case have been shown not to have much physical significance except for the case of very strong absorption. As was pointed out earlier, the problem cannot be solved exactly for this case. However one can get an asymptotic expansion of the solution

for this problem for the case of strong absorption. To get an estimate for how weak an absorption this solution is still valid, a comparison between the asymptotic theory and exact theory can be made for the case of collision broadening. This would involve applying the asymptotic expansion for the case of two singularities treated jointly to a computer analysis and comparing it for changing parameters to that performed herein. Perhaps with this information and the experience gained in doing so, a non-asymptotic expansion could then be performed to cover the whole range of parameters possible.

With respect to further experimental work, it appears that a good superheterodyne receiving system would allow a better look at the pulse ringing effect for extremely weak signals. In addition to the two klystrons which were available for this experiment, one would need stabilization circuits by which one klystron would be locked to the second in frequency. This involves very high Q cavities or a very accurate crystal oscillator to maintain a constant difference frequency between the two signals. Furthermore, the use of a multi-channel analyzer and plotter to reproduce the pulse shape would remove subjective errors introduced in the method of data reduction used in this experiment.

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APPENDIX A

The integral I_{21} around the cut between z_+ and z_- is first written as the integral below the cut, from z_+ to z_- , plus the integral above the cut, from z_- to z_+ , with the phases on the branch cuts chosen to be zero to the right of the branch points:

$$\begin{aligned}
 I_{21} &= \frac{1}{2} \int_{z_+}^{z_-} e^{-izt} \frac{dz}{|(z-z_+)^{\frac{1}{2}}| e^{i\pi} |(z-z_-)^{\frac{1}{2}}| e^{i\pi}} \\
 &\quad \text{(below)} \\
 &\quad + \frac{1}{2} \int_{z_-}^{z_+} e^{-izt} \frac{dz}{|(z-z_+)^{\frac{1}{2}}| e^{i0} |(z-z_-)^{\frac{1}{2}}| e^{i\pi}} \\
 &\quad \text{(above)}
 \end{aligned} \tag{A-1}$$

$$= \int_{z_-}^{z_+} e^{-izt} \frac{dz}{(z-z_+)^{\frac{1}{2}} (z-z_-)^{\frac{1}{2}}} \tag{A-2}$$

Then, since $z_- = -z_+$ and letting $\zeta = z/z_+$, one has

$$I_{21} = \int_{-1}^1 e^{-i\zeta T} \frac{d\zeta}{(\zeta^2 - 1)^{\frac{1}{2}}} \tag{A-3}$$

where $T = z_+ t$. Because $\sin \zeta z_+ t$ is an odd function, it contributes nothing to the integral between the symmetric end points of the contour so that

$$I_{21} = \int_{-1}^1 \frac{\cos \zeta T}{(\zeta^2 - 1)^{\frac{1}{2}}} d\zeta \tag{A-4}$$

Using an integral representation for the Bessel function (See for example, Lebedev (8), Equation 5.10.3.),

$$J_0(z) = \pi^{-1} \int_{-1}^1 \frac{\cos zt}{(1-t^2)^{\frac{1}{2}}} dt \quad (\text{A-5})$$

with $|\arg z| < \pi$, it follows that

$$I_{21} = -i\pi J_0(z_+ t) \quad (\text{A-6})$$

The same expression results for the choice of z_- in the argument because J_0 is an even function of its argument.

APPENDIX B

The first generalized confluent hypergeometric function of two variables has the integral representation:

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \Phi_1(a, c, a+b; Y, Z) = \int_0^1 \zeta^{a-1} (1-\zeta)^{b-1} (1-Y\zeta)^{-c} e^{Z\zeta} d\zeta \quad (B-1)$$

To get the correct series representation for $|Y| < 1$, one expands the factor

$$(1-Y\zeta)^{-c} = \sum_{k=0}^{\infty} (c, k) \frac{Y^k \zeta^k}{k!} = {}_1F_0(c, -; Y\zeta) \quad (B-2)$$

which converges for $|Y| < 1$. Then

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \Phi_1 = \sum_{k=0}^{\infty} (c, k) \frac{Y^k}{k!} \int_0^1 \zeta^{k+a-1} (1-\zeta)^{b-1} e^{Z\zeta} d\zeta \quad (B-3)$$

Now one uses an integral representation for the confluent hypergeometric function:

$$\frac{\Gamma(B-A)\Gamma(A)}{\Gamma(B)} \varphi(A, B, Z) = \int_0^1 e^{Z\zeta} \zeta^{A-1} (1-\zeta)^{B-A-1} d\zeta \quad (B-4)$$

so that, using

$$A = a + k \quad (B-5)$$

$$B = A + b = a + b + k \quad (B-6)$$

one has for the generalized function,

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \Phi_1 = \sum_{k=0}^{\infty} (c,k) \frac{y^k}{k!} \frac{\Gamma(b)\Gamma(a+k)}{\Gamma(a+b+k)} \varphi(a+k, a+b+k; z) \quad (B-7)$$

Now using the Appell symbol in terms of the gamma functions

$$(a,k) = \frac{\Gamma(a+k)}{\Gamma(a)} \quad (B-8)$$

so that

$$\Gamma(a+k) = (a,k) \Gamma(a) \quad (B-9)$$

$$\Gamma(a+b+k) = (a+b,k) \Gamma(a+b) \quad (B-10)$$

and using the series representation for the confluent hypergeometric function:

$$\varphi(a+k, a+b+k, z) = \sum_{m=0}^{\infty} \frac{(a+k,m)}{(a+b+k,m)} \frac{z^m}{m!} \quad (B-11)$$

one has that

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \Phi_1 = \sum_{k=0}^{\infty} (c,k) \frac{y^k}{k!} \frac{\Gamma(b)\Gamma(a)}{\Gamma(a+b)} \frac{(a,k)}{(a+b,k)} \sum_{m=0}^{\infty} \frac{(a+k,m)}{(a+b+k,m)} \frac{z^m}{m!} \quad (B-12)$$

Cancelling the gamma functions and using

$$(a,k)(a+k,m) = (a,k+m) \quad (B-13)$$

one finally has the correct series representation for the generalized confluent hypergeometric function of two variables

$$\Phi_1(a, c, a+b, y, z) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(a, k+m)(c, k)}{(a+b, k+m)} \frac{y^k}{k!} \frac{z^m}{m!} \quad (B-14)$$

APPENDIX C

In this section the pulse distortion effects expected due to waveguide dispersion are calculated. Elliott has derived a parameter which when evaluated and compared with calculated waveforms can give an estimate of the pulse distortion to expect when propagating signals in a waveguide (16). The parameter is

$$a = 4 T^{-1} (BL)^{\frac{1}{2}} \quad (C 1)$$

where T^{-1} is the repetition rate of the signal; L is the path length of waveguide used; and B is given by

$$B = \frac{1}{2} \omega_c^2 v^{-1} (\omega^2 - \omega_c^2)^{-3/2} \quad (C 2)$$

ω_c is the cutoff frequency of the waveguide (~15 GHz for K band waveguide) and v is the velocity of the signal in the medium in the waveguide, c in this case. Pulse shapes for various values of a are shown in Figure C1. Using values of $T^{-1} = 200$ kHz, $L = 367$ cm, $\omega_c = 15$ GHz, and $v = 3 \times 10^{10}$ cm/sec, one obtains

$$a = 0.0015 \quad (C 3)$$

From the graphs it is seen that an effect akin to pulse ringing occurs at $a = 0.10$, which is about two orders of magnitude larger than the value for the present experiment. Therefore it is apparent that there should be no pulse ringing effects due to the waveguide based upon Elliott's calculation as large as those which were observed in the experiment.

Figure C1. Examples of dispersion in a waveguide as calculated by Elliott (16).