UCRL-9251

UC-34 Physics and Mathematics
TID-4500 (15th Ed.)

UNIVERSITY OF CALIFORNIA

Lawrence Radiation Laboratory
Berkeley, California

Contract No. W-7405-eng-48

LOW-ENERGY PHOTOPION PRODUCTION FROM PIONS AND NEUTRAL-PION DECAY

How-sen Wong

(Thesis)

June 2, 1960

Printed for the U. S. Atomic Energy Commission

Printed in USA. Price $1.00. Available from the
Office of Technical Services
U. S. Department of Commerce
Washington 25, D.C.

LOW-ENERGY PHOTOPION PRODUCTION FROM PIONS AND NEUTRAL-PION DECAY

Contents

LOW-ENERGY PHOTOPION PRODUCTION FROM PIONS AND NEUTRAL-PION DECAY

How-sen Wong

(Thesis)

Lawrence Radiation Laboratory
University of California
Berkeley, California

June 2, 1960

ABSTRACT

The Mandelstam representation is applied to the process $\gamma + \pi \rightarrow 2\pi$. It is shown that a homogeneous integral equation may be obtained for the p-wave amplitude whose solution allows one arbitrary real multiplicative constant, which at present must be determined from experiment. By the use of crossing symmetry, a simple and tractable approximate solution of the integral equation is obtained. Higher partial waves may be calculated in terms of the p wave. The order of magnitude of the new constant is estimated by considering the decay rate of the neutral pion, in which the amplitude for $\gamma + \pi \rightarrow 2\pi$ should play a prominent role.

I. INTRODUCTION

Recently, Chew and Mandelstam[1] have developed a new method for calculating the interactions of strongly interacting particles and have applied this method to the problem of the pion-pion interaction. Their procedure is based on the two-dimensional representation proposed by Mandelstam,[2] which prescribes a method of simultaneous analytic continuation of scattering amplitudes into the complex planes as a function of both the energy-and momentum-transfer variables. In particular, this representation gives the location and character of all singularities of a scattering amplitude and enables one to write partial-wave dispersion relations. Chew and Mandelstam have developed the theory further by adopting the philosophy that the functions are dominated by nearby singularities. Accepting this philosophy and applying the unitary condition, one obtains a system of integral equations in most scattering problems. Frazer and Fulco[3] have applied these ideas to the problem of $\pi + \pi \rightarrow N + \bar{N}$ and then to the nucleon electromagnetic-structure problem. Here we use the same approach to calculate the matrix element for low-energy photopion production from pions.*

If both the Mandelstam representation and the philosophy of the importance of nearby singularities are accepted, no further substantial theoretical work may be done in strong-coupling physics involving photons and other elementary particles until we understand something about the problem of photopion production from pions. In other words, this problem plays a role similar to that of π - π scattering in any phenomena involving at least one photon. For example, Ball[4] has recently investigated photopion production from

---

*A preliminary account of this work was given at the 1959 Thanksgiving meeting of the American Physical Society, November 27-28, 1959 [How-sen Wong, Bull. Am. Phys. Soc. 4, 407 (1959)].

nucleons and found that the process $\gamma + \pi \rightarrow 2\pi$ produces an additive correction to the CGLN formulas.[5] Further work on "photo" problems such as $\gamma + \pi \rightarrow \gamma + \pi$ and $\gamma + N \rightarrow \gamma + N$ will also require a knowledge of the $\gamma + \pi \rightarrow 2\pi$ reaction.

In the following section, kinematics, isotopic spin and partial-wave decompositions will be considered. It is shown by invariance requirements that the problem under consideration requires only a single invariant function. By assuming that this function has the Mandelstam representation, we are able to locate the singularities and hence to write dispersion relations for the partial-wave amplitudes. Using the unitary condition and the Omnès[6]-Frazer-Fulco[3] method, we find that the p-wave amplitude satisfies a homogeneous integral equation and depends on a single real parameter Λ. This constant, although not fundamental, cannot be related to fundamental constants at this stage of the theory and must be determined from experiment. Higher partial waves for our process are related directly to the p wave. The p-wave integral equation is solved by the Chew-Mandelstam technique of replacing unphysical singularities by a series of poles, whose positions and residues are determined by crossing symmetry.[7]

Finally in Section III, we discuss neutral-pion decay. It is shown that the decay rate is related to the unknown parameter Λ.

II. LOW-ENERGY PHOTOPION PRODUCTION FROM PIONS

A. Kinematics

Let the four-vector momenta of the pions be p_1, p_2, and p_3, and let K and ϵ be the four-vector photon momentum and polarization, respectively. Define the variables*

$$s_1 = -(K + p_1)^2,$$

$$s_2 = -(K - p_2)^2, \qquad (1)$$

and

$$s_3 = -(K - p_3)^2,$$

which are related by the condition

$$s_1 + s_2 + s_3 = 3. \qquad (2)$$

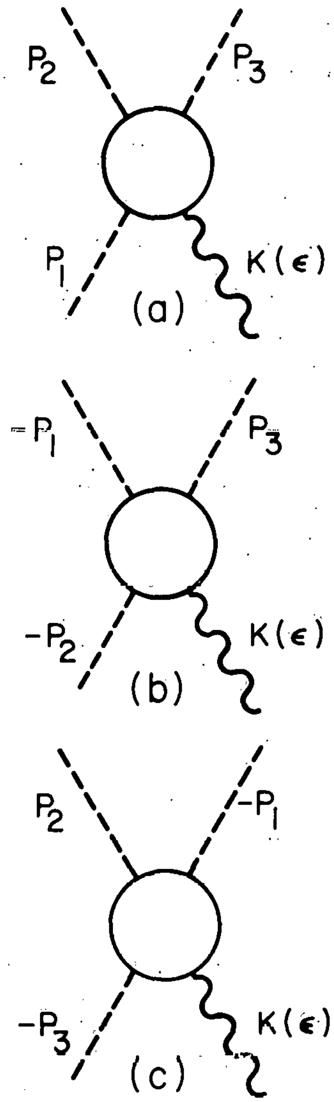
These three Lorentz-invariant variables are just the squares of energies in the barycentric systems of the corresponding processes a, b, and c in Fig. 1. We sue these variables because of the fundamental structure of the Mandelstam representation.

In the case when the photon K and meson p_1 are the incoming particles (we shall call this channel I), the variables s_1, s_2, and s_3 are related to the energy and momentum transfer in the following ways:

$$s_1 = -(p_1 + K)^2 = 4(1 + p^2)$$
$$s_2 = -(p_2 - K)^2 = 1 - 2kE + 2kp\cos\theta_1$$
$$s_3 = -(\mu_3 - K)^2 - 1 - 2kE - 2kp\cos\theta_1,$$

---

*We use the fundamental metric tensor such that $p_1 \cdot p_2 = \vec{p}_1 \cdot \vec{p}_2 - p_{10}p_{20}$. Units are used in which $\hbar = c = 1$ and $\mu = 1$, where μ is the mass of the pion.



MU-20472

Fig. 1. The three channels of the $\gamma + \pi \rightarrow 2\pi$ problem.

where p and E are the magnitudes of the outgoing pion momenta and energy respectively, k is the energy of the photon, and we have $\cos\theta_1 = \frac{\vec{p}\cdot\vec{k}}{pk}$, all in the barycentric system. Energy-momentum conservation leads to Eq. (2) and $k = (s_1 - 1)/2\sqrt{s_1}$.

The S matrix for photopion production from pions can be written as

$$S_{fi} = i\,(2\pi)^4 \delta^4 (K + p_1 - p_2 - p_3) \frac{T_{fi}}{(16\, P_{10} P_{20} P_{30} k)^{1/2}}, \qquad (4)$$

where P_{10}, P_{20}, and P_{30} are the energies of the mesons.

The decomposition of T will be considered as follows: First, we decompose T into the product or sum of the products of an isospin-dependent function and an isospin-independent function. Next, the isospin-independent function (functions) is (are) decomposed into the product or sum of products of a gauge-invariant, Lorentz-invariant function and a function of s_1, s_2, and s_3. We can perform all these decompositions by using the known conservation laws or invariance properties such as the conservation of parity, conservation of G-parity (G-conjugation is the combined operation of charge conjugation and 180-deg rotation about the y axis in isotopic spin space), gauge invariance etc. With these invariance properties and the pseudoscalar nature of pions, we find that only a single pseudoscalar quantity can be formed from the four independent kinematic four-vectors, and the final 2π state must have isotopic spin one. Thus the problem under consideration requires only one scalar transition amplitude. Therefore, we can define the scalar amplitude $M(s_1\ s_2\ s_3)$ by

$$T_{fi} = \Sigma\left(-\frac{i}{\sqrt{2}}\right) \epsilon_{\alpha\beta\gamma}\, \epsilon_{\lambda\delta\mu\nu}\, (p_1)_\lambda\, (p_2)_\delta\, k_\mu\, \epsilon_\nu\, M(s_1\ s_2\ s_3)\ , \qquad (5)$$

where α, β, and γ are the isotopic indices of the pions, and $\epsilon_{\alpha\beta\gamma}$ and $\epsilon_{\lambda\delta\mu\nu}$ are the conventional antisymmetric tensors of third and fourth rank, respectively. The fact that a single scalar transition amplitude is required for the problem is a great simplification; we believe this may be the unique situation in strong-coupling physics with such a nice property.

The consequences of crossing symmetry are very simple in this case. Interchanging the numbers of various pion pairs evidently amounts to interchange of the s variables. Thus we have

$$M(s_1\ s_2\ s_3) = M(s_2\ s_1\ s_3) = M(s_1\ s_3\ s_2) = M(s_3\ s_2\ s_1)\ , \tag{6}$$

since the product of the factors multiplying M in formula (5) are symmetric under any pion interchange.

B. The Mandelstam Representation

We assume that, except for subtractions that may be required, the invariant function M satisfies the two-dimensional representation proposed by Mandelstam:[2]

$$M = \frac{1}{\pi^2} \iint_4^\infty \frac{ds'_2 ds'_3 \rho_1(s'_2 s'_3)}{(s'_2 - s_2)(s'_3 - s_3)} + \frac{1}{\pi^2} \iint_4^\infty \frac{ds'_3 ds'_1 \rho_2(s'_3 s'_1)}{(s'_3 - s_3)(s'_1 - s_1)}$$

$$+ \frac{1}{\pi^2} \iint_4^\infty \frac{ds'_1 ds'_2 \rho_3(s'_1 s'_2)}{(s'_1 - s_1)(s'_2 - s_2)} \tag{7a}$$

Conservation laws preclude the presence of any poles.

Although the variables s_1, s_2, and s_3 are not independent, being related by (2), we shall often write them all out explicitly in order to see the symmetry of the representation. The assumption of the Mandelstam representation is the essential tool for setting up an integral equation for the p-wave amplitude. We use the representation not only to locate the singularities of the partial-wave amplitudes in the s_1 (energy square in Channel I) plane, but also to relate the unphysical and physical cuts in this plane. This is discussed in the following sections.

Using the crossing relations[Eq. (6)], we find that all spectral functions ρ in Eq. (7a) are equal and symmetric in the two variables. Therefore Eq. (7a) becomes

$$M = \frac{1}{\pi^2} \int_4^\infty \int_4^\infty \left[\frac{1}{(x-s_1)(y-s_2)} + \frac{1}{(x-s_2)(y-s_3)} + \frac{1}{(x-s_3)(y-s_1)} \right] \rho(x, y)\, dxdy. \tag{7b}$$

As shown by Mandelstam, one can easily derive a one-dimensional relation from Eq. (7a)with either s_1, s_2, or s_3 fixed. The spectral function ρ is non zero in regions whose boundary can be calculated from perturbation theory or whose formulas are given by Mandelstam. It is shown that ρ is bounded by the following two curves [see Fig. (2)]:

$$(s_1 - 4)s_2 = 8\,(2s_1 + 1) \tag{8a}$$

$$(s_2 - 4)s_1 = 8\,(2s_2 + 1). \tag{8b}$$

The curve for Eq. (8b) is obtained from Eq. (8a) by interchange of s_1 and s_2. From Eqs. (8a) and (8b) it is evident that the region in which ρ is not zero is asymptotically bounded by the limits of integration in Eq. (7a):

C. Analytic Properties and ~~Dispersion~~ Relations for Partial-Wave Amplitudes

Our approach to this problem requires the same sharp distinction between high and low angular-momentum states as in the problem of π-π scattering of Chew and Mandelstam. The discussion of their work in this connection may be repeated almost word by word. Here we separate out the p-wave part for special consideration, the higher part of the amplitude to be calculated later in terms of the p-wave part.

Using the method of Jacob and Wick,[8] we can write the partial-wave decomposition of our scalar amplitude M as

$$M(s_1, \cos\theta_1) = \sum_{\text{odd } \ell} M_\ell(s_1)\, P'_\ell(\cos\theta_1), \tag{9a}$$

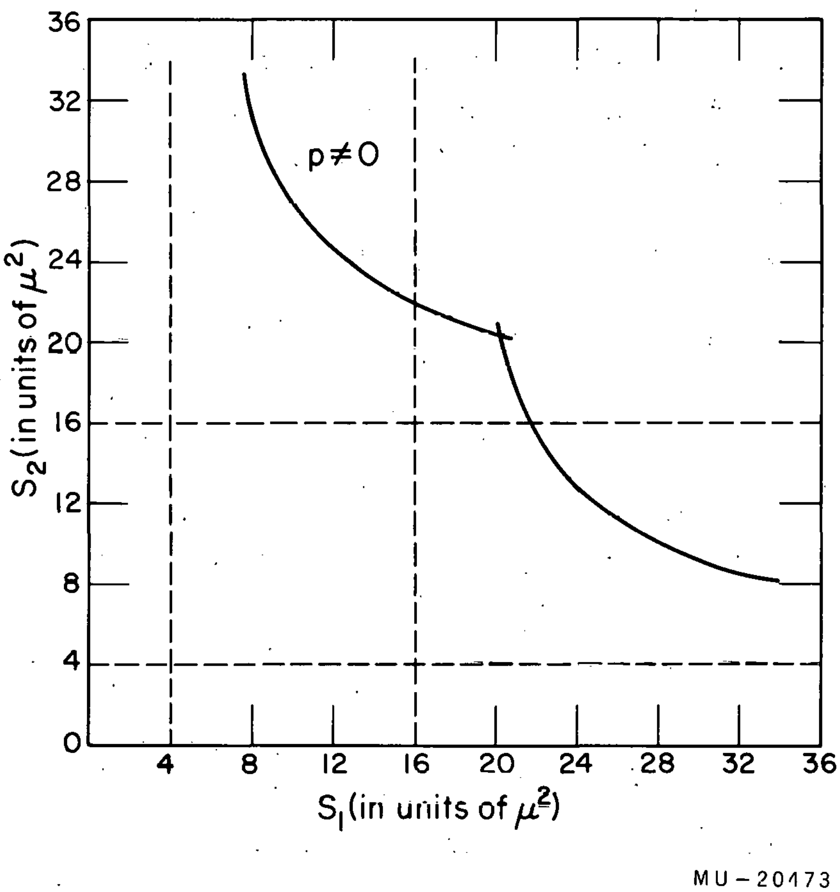


Fig. 2. Boundary curve of the spectral function $\rho\ (s_1, s_2)$.

where θ_1, the angle of emission, is given by

$$\cos\theta_1(s_1, s_2) = \frac{2s_2 + s_1 - 3}{s_1 - 1} \left(\frac{s_1}{s_1 - 4}\right)^{1/2}, \qquad (10)$$

and P'_ℓ is the derivative of Legendre polynomials of order ℓ.

In order to obtain the dispersion relations for partial-wave amplitudes, we write

$$M(s_1, \cos\theta_1) = \frac{1}{\pi} \int_4^\infty \rho'(s'_2, s_1) \left[\frac{1}{s'_2 - s_2(s_1, \cos\theta_1)} + \frac{1}{s'_2 - s_3, \cos\theta_2)}\right] ds'_2,$$

where (11)

$$\rho'(s'_2 s_1) = \frac{1}{\pi} \int_4^\infty \rho(x, s'_2) \left[\frac{1}{x - s_1} + \frac{1}{x + s_1 + s'_2 - 3}\right] dx. \qquad (12)$$

It is easy to see that $\rho'(s_2, s_1)$ is the imaginary part of M for $s_2 \geq 4$, and $s_1 \leq 0$ and the analytic continuation of this function is otherwise. In other words, the spectral function ρ' is the imaginary part of M when the photon and meson p_2 are the incoming particles (se shall call this channel II). Equation (11) is a one-dimensional representation of the amplitude M that shows explicitly the dependence on the momentum-transfer variables s_2 and s_3 and thus the angle θ_1, since

$$s_2 = \frac{1}{2}\left[3 - s_1 + (s_1 - 1)\left(\frac{s_1 - 4}{s_1}\right)^{1/2} \cos\theta_1\right]$$

$$s_3 = \frac{1}{2}\left[3 - s_1 - (s_1 - 1)\left(\frac{s_1 - 4}{s_1}\right)^{1/2} \cos\theta_1\right].$$

Using the formula

$$M_\ell(s_1) = \frac{2\ell + 1}{2\ell(\ell+1)} \int_{-1}^{1} (1 - z^2)\, P'_\ell(z)\, M(s_1, z)\, dz \qquad (13)$$

in our partial-wave decomposition, we find

$$M_\ell(s_1) = \frac{2(2\ell+1)}{\ell(\ell+1)\pi} \int_{-1}^{1} (1-z^2)P'_\ell(z)\, dz \int_4^\infty ds'_2 \frac{\operatorname{Im} M(s'_2, \cos\theta_2(s'_2, s_1))}{2s'_2+s_1-3+(s_1-1)\left(\frac{s_1-4}{s_1}\right)^{1/2} z} \tag{14}$$

where θ_2, the barycentric-system emission angle for channel II, can be obtained from formula (10) by interchanging s_1 and s_2.

Before discussing the analytic properties of partial-waves, let us make a subtraction by removing the p-wave part of Eq. (7b):

$$M(s_1, \cos\theta_1) = M_1(s_1) + \frac{1}{\pi}\int_4^\infty ds'_2 \operatorname{Im} M(s'_2, \cos\theta_2)$$
$$\times \left[\frac{1}{s'_2 - s_2(s_1, \cos\theta_1)} + \frac{1}{s'_2 - s_3(s_1, \cos\theta_1)} - K_1(s_1, s'_2)\right], \tag{15}$$

where

$$K_1(s_1, s'_2) = \frac{6s_1(2s'_2+s_1-3)}{(s_1-4)^2(s_1-4)} + \frac{3}{(s_1-1)}\left(\frac{s_1}{s_1-4}\right)^{1/2}\left[1 - \frac{s_1(2s'_2+s_1-3)^2}{(s_1-4)(s_1-1)^2}\right]$$
$$\times \ln\left[\frac{2s'_2+s_1-3+(s_1-1)\left(\frac{s_1-4}{s_1}\right)^{1/2}}{2s'_2+s_1-3-(s_1-1)\left(\frac{s_1-4}{s_1}\right)^{1/2}}\right]$$

In our approximation on below, we shall let $\operatorname{Im} M(s_2, \cos\theta_2) \cong \operatorname{Im} M_1(s_2)$, assuming that π-π phase shifts are small in all states for $\ell > 1$; so that Eqs. (13) and (15) give the formula for the higher-angular-momentum part of the amplitude in terms of the p wave.

It is not hard to locate the singularities of the M_ℓ's in the s_1 (energy-square) plane from Eqs. (11) and (12). The M_ℓ's are analytic in the whole complex s_1 plane except for left-hand and right-hand branch cuts on the real axis. The right-hand cut runs from 4, the physical threshold for two pions, to ∞. The vanishing of the denominators in formula (11) gives the left-hand cut from 0 to $-\infty$. The discontinuity across the left-hand cut is related to the absorptive part of the amplitude for channel II by crossing relations. The apparent singularity from the vanishing of the second denominator in formula (12) was introduced artificially through the separation into partial fractions of one of the terms in formula (7b). This singularity can be easily seen to vanish after the integration in Eq. (13) is performed.

In order to be able to write the dispersion relation for the partial-wave amplitudes, we must consider their asymptotic behavior. The unitary condition tells us that $M_\ell(s_1)$ goes to zero as s_1 approaches infinity at least as fast as does $s_1^{-3/2}$. Guided by this asymptotic behavior and using the analytic properties we have found, we can write the following dispersion relations without subtractions:

$$M_\ell(s_1) = \frac{1}{\pi}\int_{-\infty}^{0} \frac{\mathrm{Im}\, M_\ell(s'_1)\, ds'_1}{s'_1 - s_1} + \frac{1}{\pi}\int_{4}^{\infty} \frac{\mathrm{Im}\, M_\ell(s'_1)\, ds'_1}{s'_1 - s_1} . \qquad (16)$$

For consistency, it is necessary that $\mathrm{Im}\, M_\ell(s_1)$ vanish at $-\infty$ as well as $+\infty$. We shall consider this later.

Our next task is to evaluate $\mathrm{Im}\, M_\ell(s_1)$ on the unphysical cut $(-\infty < s_1 \leq 0)$. In this region, we find, from the crossing-relation formula (14),*

$$\mathrm{Im}\, M_\ell(s_1) = -\epsilon(s_1 + \tfrac{1}{2}) \int_4^{\nu(s_1)} h_\ell(s_2 s_1)\, \mathrm{Im}\, M[s_2, \cos\theta_2(s_2, s_1)]\, ds_2 . \qquad (17)$$

where

$$\nu(s_1) = \frac{1}{2}\left[3 - s_1 - (s_1 - 1)\left(\frac{s_1 - 4}{s_1}\right)^{1/2}\right],$$

and

---

*See Appendix.

$$h_\ell(s_2, s_1) = \frac{4(2\ell+1)\sqrt{s_1}}{(\ell+1)(s_1-1)(s_1-4)^{1/2}} \left[1 - s_1 \left(\frac{3-s_1-2\ s_2}{(s_1-1)(s_1-4)^{1/2}} \right)^2 \right]$$

$$\times P'_\ell \left[\frac{\sqrt{s_1}\ (3-s_1-2s_2)}{(s_1-1)(s_1-4)^{1/2}} \right].$$

Although in Eq. (17) the variable s_2 is the energy variable in the physical region for channel II, the upper limit $\nu(s_1)$ is such that we have $\cos\theta_2 < -1$. Therefore we must make an analytic continuation from the physical region. One method of continuation is to expand Im $M[s_2, \cos\theta_2(s_2, s_1)]$ in Legendre polynomials:

$$\text{Im } M(s_2, \cos\theta_2) = \sum_{\text{odd } \ell} \text{Im } M_\ell(s_2)\ P'_\ell(\cos\theta_2).$$

The region of convergence of our Legendre polynomial expansion can be determined from formula (12). Since a function of $\cos\theta_2$ can be expanded in Legendre polynomials within a singularity-free ellipse with foci at -1 and $+1$, we must find the position of the nearest singularity in $\cos\theta_2$. This singularity can be located from the vanishing of the denominator of formula (12) in the region where ρ is not zero. Using formulas (8a) and 8b) for the boundary curves of this region, we find that the expansion converges on the left-hand cut as long as we have $s_1 \geqslant -33.94$.

Beyond the region of convergence of the polynomial expansion, a more subtle method of analytic continuation will be necessary. However, since we know from general principles that Im $M_\ell(s_1)$ must vanish as s_1 approaches $-\infty$, it is reasonable to expect the contribution of Im $M_\ell(s_1)$ for $s_1 \geqslant -33.94$ to be small. If we keep only the p-wave amplitude in the partial-wave expansion (96), we see that Im $M_1(s_1)$ for $s_1 \leqslant 0$ oes to zero like $s_1^{-3/2}$, so that no cut-off parameter is needed. The situation is thus more favorable than in the problem of pion-pion scattering. The extra convergence here is a consequence of the gauge condition.

D. The Integral Equation

We shall proceed to transform the Cauchy integrals (16) into another form from which we hope to obtain solutions. Applying the unitary condition and adopting the assumption that the functions are determined by nearby singularities so that only the intermediate two-pion state need be considered, we write

$$M_\ell(s_1) = \frac{1}{\pi}\int_{-\infty}^{0} \frac{\text{Im } M_\ell(s'_1)\, ds'_1}{s'_1 - s_1} + \frac{1}{\pi}\int_4^{\infty} \frac{M_\ell(s'_1)\exp\left[i\delta_\ell(s'_1)\right]\sin\delta_\ell(s'_1) ds'_1}{s'_1 - s_1}, \tag{18}$$

where $\delta_\ell(s_1)$ is the pion-pion phase shift for the ℓ angular-momentum state.

Frazer and Fulco have extended the Omǹes investigations on the Chew-Low type Eq. (18) and find that $M_\ell(s_1)$'s satisfy

$$M_\ell(s_1) = \frac{1}{\pi D_\ell(s_1)} \int_{-\infty}^{0} \frac{D_\ell(s'_1)\text{ Im } M_\ell(s'_1)\, ds'_1}{s'_1 - s_1}, \tag{19}$$

where

$$D_\ell(s_1) = \exp\left[- \frac{s}{\pi} \int_4^{\infty} \frac{\delta_\ell(s'_1)\, ds'_1}{(s'_1 - s_1)\, s'_1} \right]. \tag{20}$$

In this problem, the imaginary part of $M_\ell(s_1)$ for $s_1 \leqslant 0$ is not known but is related to the p-wave amplitude through the crossing relation (17).

From now on we shall concentrate on the $\ell = 1$ solution and leave the calculation of higher waves to Eqs. (13) and (15). By substituting Eq. (17), where we approximate $\text{Im } M \cong \text{Im } M_1$ in the integrand, into the Omǹes-Frazer-Fulco solution (19), we obtain a homogeneous equation for the p-wave amplitude. The homogenity of this integral equation, whose solution is not unique at least with respect to a multiplicative factor, would be removed if we kept any contribution from inelastic processes. In the calculation below, we shall fix this multiplicative factor Λ as the value of $M_1(s_1)$ at $s_1 = 1$. At present, we do not know how to relate this unknown parameter Λ to fundamental constants, in particular to the electromagnetic coupling constant e, which certainly plays a fundamental role here. For the time being, Λ must be determined from experiment. Although we do not know the relationship between Λ and other constants, we have no reason to believe that it is itself fundamental. It should be possible to calculate Λ if and when techniques for handling high-mass singularities become available.

E. The Pole Approximation

To proceed further we need the denominator function D_1, for p-wave pion-pion scattering. The pion-pion calculations of Chew and Mandelstam have not yet reached a conclusive stage but these authors have given an approximate form for D_1 which corresponds to the replacement of the unphysical branch cut in the pion-pion amplitude by a finite number of poles. Further, they have shown that two poles lead to an accurate approximation in the physical region up to $s_1 \sim 40$. Once the parameters of the two-pole formula have been determined, it will be a straightforward problem to incorporate the information into the amplitude for $\gamma + \pi \rightarrow 2\pi$. The determination of the pion-pion parameters is still in progress,[7, 9] but we outline here, for further use, the form of the solution of our problem that corresponds to the pole approximation of Chew and Mandelstam. We shall illustrate the method with the one-pole, pion-pion amplitude, for which parameters have been given by Frazer and Fulco.

We thus propose to replace the left-hand cut of the amplitude M_1 by poles with appropriate positions and residues. This philosophy of replacing cuts by poles has been successful in many circumstances and can perhaps be best understood by looking into the connection with the effective-range formulas of low-energy nuclear physics. It is well-known that the effective-range formula gives a useful description of low-energy nucleon-nucleon scattering. If the formula is written in the form

$$f = \frac{1}{q} e^{i\delta} \sin \delta = (\frac{r}{2} q^2 - i q - \frac{1}{a})^{-1}, \qquad (21)$$

where q is the center-of-mass momentum, a is the scattering length, and r the effective range, we see that the s-wave effective-range formula implies that the amplitudes has two poles in the complex q plane at

$$q_1 = i \left[\frac{1}{r} + (\frac{1}{r^2} - \frac{2}{a\,r})^{1/2} \right]$$

and

$$q_2 = i \left[\frac{1}{r} - (\frac{1}{r^2} - \frac{2}{a\,r})^{1/2} \right] .$$

Usually, one of these poles q_1, is in the physical sheet ($-iq_1 > 0$) and the other one is either in the unphysical sheet ($-iq_2 < 0$) or becomes a bound state pole. The effective-range formula (22) can be now equivalently characterized by a single pole q_1, the interaction pole, and its residue

$$\Gamma_1 = \frac{1}{i} (\frac{q_1 + q_2}{q_1 - q_2}),$$

and may be called the "one-pole formula". This relation between poles and effective-range formulas has been known for some time, but was not emphasized and interpreted until recently by Chew and Wong after the Mandelstam representation was proposed.[10] In fact, the interaction pole lying in the negative real axis of the q^2 plane can be considered to be the replacement of left-hand singularities implied by the Mandelstam representation.

In general, we would obtain a "multiple-pole formula" by replacing the left-hand branch cuts by a series of poles. The "n-pole formula" thus obtained should contain $2n$ parameter--the residues and positions of the poles. For our $\gamma + \pi \rightarrow 2\pi$ problem, replacing the left cut by poles corresponds to approximating $M_1(s_1)$ for $s_1 \leqslant 0$ by a finite number of delta functions here and enables one to transform the integral Eq. (19) into an algebraic and trivially soluble equation. But the question now arises how to determine the residues and positions of the poles, which correspond to the strength and range of the various contributing interactions. Recall that a parameter Λ must enter into our final solution, so we might as well introduce it through the residues of one of the poles. The rest of the parameters may be determined from the crossing relation Eq. (6).

As in the π-π problem, we have a point of maximum symmetry. This occurs at the unphysical point $s_1 = s_2 = s_3 = 1$, or $s_1 = 1$ and $z_1 \equiv \cos\theta_1 = -\sqrt{3}\,i$, where M is real. By differentiating the general crossing relation (6) or the Mandelstam representation with respect to these variables, we can derive an infinite number of conditions on the derivatives of the amplitude M at the symmetry point (referred to hereafter as SP). Three of these conditions which may be useful for our further discussions, are

$$M(s_1\, z_1)\Big|_{s_1 = 1} = \text{constant}, \qquad (22a)$$

$$\left.\frac{\partial M(s_1\, z_1)}{\partial s_1}\right|_{SP} = 0, \qquad (22b)$$

and

$$\left.\frac{\partial^2 M(s_1\, z_1)}{\partial s_1\, \partial z_1}\right|_{SP} = 0. \qquad (22c)$$

In addition, from the Mandelstam representation we can also find the following relation:

$$\left.\frac{\partial^2 M(s_1\, z_1)}{\partial s_1^{\,2}}\right|_{s_1 = 1} = \text{constant } x(1 - z_1^2). \qquad (23)$$

Equation (24) together with the crossing conditions (22a) to 22c), imply

$$M(s_1\ s_2\ s_3)\Big|_{SP} = M_1\,(1) \equiv \Lambda$$

or $M_\ell\,(1) = 0$ (24a)

for $\ell \geq 3$, and

$$\frac{\partial M_\ell\,(s_1)}{\partial s_1}\Bigg|_{s_1=1} = 0 \tag{24b}$$

for all ℓ,

$$\frac{\partial^2 M_\ell\,(s_1)}{\partial s_1^{\,2}}\Bigg|_{s_1=1} = 0 \tag{24c}$$

for $\ell \geq 5$ and

$$\frac{\partial^2 M_3\,(s_1)}{\partial s_1^{\,2}}\Bigg|_{s_1=1} = -\frac{1}{6}\ \frac{\partial^2\, M_1\,(1)}{\partial s_1^{\,2}}\ . \tag{24d}$$

These relations are exact, but only Eqs. (24a), (24b), and (24d) are of direct use in determining the pole parameters for the p wave. The additional relations needed depend on Eq. (15) which is only approximate. To obtain these formulas, let us write Eq. (15) in the following form:[11]

$$M(s_1\ z_1) \cong \frac{1}{\pi}\int_4^\infty \frac{\mathrm{Im}\, M_1(x)\, dx}{x - s_1} + \frac{1}{\pi}\int_4^\infty \mathrm{Im}\, M_1\,(x)$$

$$\times \left[\frac{1}{x - s_2(s_1,\ z_1)} + \frac{1}{x - s_3(s_1,\ z_1)}\right] dx. \tag{25}$$

Evidently, the first integral is the contribution from the right-hand cut and the second integral that from the left. This equation, although not exact, satisfies all the exact crossing relations (22a) to (22c) and (23).

The importance of Eq. (25) lies in the fact that at the symmetry point the contribution from the left cuts is simply related to that from the right. Defining

$$M^R(s_1 z_1) = \frac{1}{\pi}\int_4^\infty \frac{\mathrm{Im}\, M_1(x)\, dx}{x - s_1}$$

and

$$M^L(s_1 z_1) = \frac{1}{\pi}\int_4^\infty \mathrm{Im}\, M_1(x)\left[\frac{1}{x - s_2(s_1, z_1)} + \frac{1}{x - s_3(s_1, z_1)}\right] dx,$$

with the obvious meaning for the notations, we see from Eq. (25) that at the symmetry point we have

$$M^L = 2M^R$$

and

$$\frac{\partial^2 M^L}{dS_1^2} = 5 \frac{\partial^2 M^R}{\partial S_1^2} .$$

Using these relations and Eqs. (24a) to (24d), and remembering

$$M^L = M_1^L(1),$$

$$M^R = M_1^R(1),$$

$$\frac{\partial^2 M^L}{\partial s_1^2} = \frac{\partial^2 M_1^L(1)}{\partial s_1^3} - 24 \frac{\partial^2 M_3(1)}{\partial s_1^2} ,$$

and

$$\frac{\partial^2 M^R}{\partial s_1^2} = \frac{\partial^2 M_1^R(1)}{\partial s_1^2} ,$$

we obtain

$$M_1^L(1) = 2 M_1^R(1), \tag{26a}$$

$$\frac{\partial M_1^L(1)}{\partial s_1} = - \frac{\partial M_1^R(1)}{\partial s_1}, \tag{26b}$$

and

$$\frac{\partial^2 M_1^L(1)}{\partial s_1^2} = \frac{1}{5} \frac{\partial^2 M_1^R(1)}{\partial s_1^2} . \tag{26c}$$

There are relations of this kind for all derivatives at $S_1 = 1$, but these three will serve our purpose if we consider only one- or two-pole formulas. In general, the more poles we put in to replace the left cut, the more accurate the p-wave solution we would obtain and the more derivative conditions we need.

We are now in a position to derive pole formulas for our problem. The "one-pole" case will be considered first. Let us write

$$M_1(s_1) = \frac{\Lambda'}{s_1+a} + \frac{1}{\pi} \int_4^\infty \frac{M_1(s'_1)\, e^{-i\delta_1(s'_1)} \sin\delta_1(s'_1)\, ds'_1}{s'_1 - s_1} ,$$

where Λ' is real, and a is real and positive. This assumption evidently corresponds to setting

$$\mathrm{Im}\, M_1(s_1) = -\Lambda' \pi\, \delta(s_1 + a)$$

for $s_1 \leqslant 0$, so that from Eq. (19) we have the p-wave solution

$$M_1(s_1) = \frac{\Lambda' D_1(-a)}{(s_1+a)\, D_1(s_1)} \tag{27}$$

where

$$\Lambda' = \frac{\Lambda(1+a)\, D_1(1)}{D_1(-a)} ,$$

since we have defined $\Lambda = M_1(1)$. If the Frazer-Fulco one-pole $D_1(s_1)$ function is used for a resonance at $s_1 = 10$ we find $a \cong 5.7$ from the crossing relation (26a). Since the calculation of the two-pole pion-pion parameters is still in progress, we are not able to give "good" two-pole $\gamma + \pi \rightarrow 2\pi$ results here, but we derive the two-pole formula for future reference. Writing

$$\operatorname{Im} M_1(s_1) = -\pi \Lambda' \left[\delta(s_1 + a) + \Lambda_1 \, \delta(s_1 + b) \right]$$

for $s_1 \leqslant 0$, where Λ' and Λ_1 are real, and a and b are real and positive, we find

$$M_1(s_1) = \Lambda' \left[\frac{D_1(-a)}{(s_1+a)\, D_1(s_1)} + \frac{\Lambda_1 D_1(-b)}{(s_1+b)\, D_1(s_1)} \right] , \qquad (28)$$

where

$$\Lambda' = \Lambda D_1(1) \left[\frac{D_1(-a)}{(1+a)} + \Lambda_1 \frac{D_1(-b)}{(1+b)} \right]^{-1} .$$

The parameters a, b, and Λ_1 may be determined from the crossing relations (26a) to (26c). It turns out that no a, b, and Λ can satisfy all three crossing conditions (26a) to (26c) if the Frazer-Fulco one-pole form is used. However, if we fix the position of one of the two poles between zero and -4.92 ($0 \leqslant a \leqslant 4.93$) and ignore the second-derivative condition, we find that the first two conditions of Eqs. (26) do have solutions for a, b and Λ_1. The results of this calculation show that the p-wave amplitude in the physical region is not sensitive to the positions of the poles, as can be seen from Table I. The table also lists the values of b, Λ_1, and Λ' for different values of a. The fact that a and b must be between 0 and -4.93 indicates that relatively small contributions to the p-wave amplitude come from the far-away left-hand singularities.

Note finally that in the physical region, the difference between our one-pole and two-pole solutions is not great. Thus we may be confident of the accuracy of our two-pole results once the parameters of the two-pole $\pi\pi$ scattering formula are known.

Table I

Table of parameters and p-wave amplitudes for various values of a.

a	b	$\frac{\Lambda'}{\Lambda}$	Λ_1	$M_1(s_1)/\Lambda$: $s_1=4$	$s_1=8$	$s_1=12$	$s_1=20$	$s_1\to\infty$
[a] 5.7		4.47		0.96	$\frac{0.453}{D_1(8)}$	$\frac{0.35}{D_1(12)}$	$\frac{0.241}{D_1(20)}$	$\frac{6.2}{s_1 D_1(s_1)}$
[b] 4.93	0	4.52	-0.021	0.98	$\frac{0.455}{D_1(8)}$	$\frac{0.345}{D_1(12)}$	$\frac{0.239}{D_1(20)}$	$\frac{5.82}{s_1 D_1(s_1)}$
[b] 4.0	1.13	5.2	-0.152	0.994	$\frac{0.46}{D_1(8)}$	$\frac{0.35}{D_1(12)}$	$\frac{0.236}{D_1(20)}$	$\frac{5.78}{s_1 D_1(s_1)}$
[b] 3.0	2.25	11.53	-0.169	0.995	$\frac{0.458}{D_1(8)}$	$\frac{0.35}{D_1(12)}$	$\frac{0.235}{D_1(20)}$	$\frac{5.73}{s_1 D_1(s_1)}$
[c] 2.62		5.85	-1.54	1.01	$\frac{0.465}{D_1(8)}$	$\frac{0.355}{D_1(12)}$	$\frac{0.241}{D_1(20)}$	$\frac{5.85}{s_1 D_1(s_1)}$

[a] One-pole solution see Eq. (27) .

[b] Two-pole solution see Eq. (28) .

[c] A single pole and a dipole placed at -a. In this case, $M_1(s_1)$ is given by

$$M_1(s_1 = \left[\frac{\Lambda'}{D_1(s_1)}\right] \left[\frac{1}{s_1+a} + \frac{\Lambda_1}{(s_1+a)^2}\right].$$

F. A Method of Determination of Λ

Recently, Chew and Low have proposed a general method for analyzing the scattering of particle **A** by particle B, leading to three or more final particles, in order to obtain the cross section for the interaction of A with a particle which is virtually contained in B.[12] This method is useful for unstable particles such as pions and neutrons from which free targets cannot be made, and hence can be applied to determine the unknown parameter Λ of photopion production from pions.

Let us consider the reactions $\gamma + p \rightarrow p + \pi^+ + \pi^-$ and $\gamma + p \rightarrow n + \pi^+ + \pi^0$. (Fig. 3). We conjecture the existence of a pole in the momentum-transfer variable $\Delta^2 \equiv (p_1{}^2 - p_2)^2$ at -1. The residue of the pole in Δ^2 is found from the general formula given by Chew and Low:

$$\lim_{\Delta^2 \rightarrow -1} \quad (\Delta^2+1)^2 \frac{\partial^2 \sigma(W^2, \Delta^2)}{\partial W^2 \partial \Delta^2} \rightarrow \left(\frac{1}{2}\right) \frac{f^2}{4\pi} \quad \frac{\Delta^2(W^2-1}{K_L{}^2} \quad \sigma^T(W), \tag{29}$$

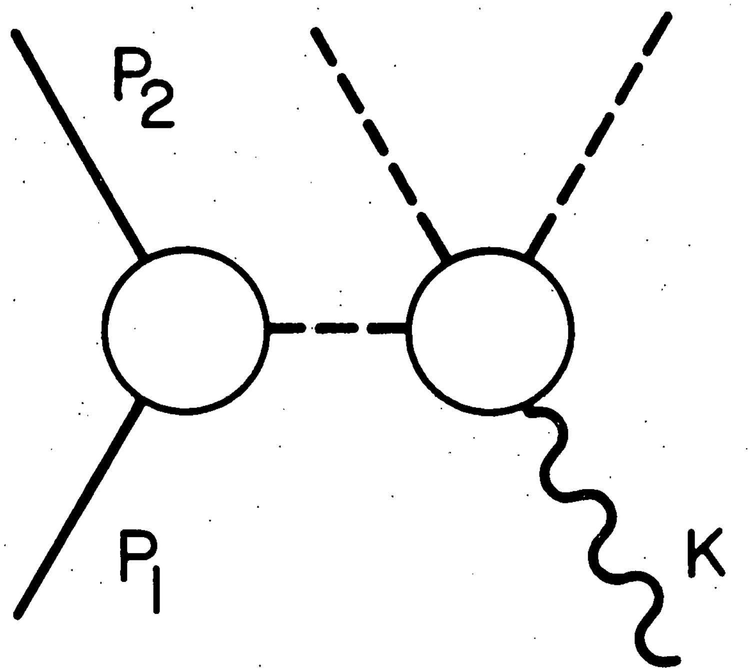
where $f^2 \cong 0.08$ and $\partial^2\sigma / \partial W^2 \partial \Delta^2$ is the differential cross section for $\gamma + p \rightarrow p + \pi^+ + \pi^-$ in the variables Δ^2 and the total energy of the two outgoing pions W in their barycentric system, K_L is the photon energy in the rest frame of the target proton, and $\sigma^T(W)$ is the total cross section for photopion production from pions which is given by*

$$\sigma^T(W) \cong \frac{1}{96\pi} \left(\frac{W^2-1}{2W}\right) \left(\frac{W^2}{4} - 1\right)^{3/2} \left| M_1 (W^2) \right|^2 , \tag{30}$$

provided we neglect all higher partial-wave contributions.

---

* The cross section for π^+, π^- and π^0 is the same.



MU-20474

Fig. 3. Diagram of $\gamma + p \rightarrow \begin{cases} \pi^+ + \pi^- + p. \\ \pi^+ + \pi^0 + n \end{cases}$

This figure shows the pole of interest.

III. NEUTRAL-PION DECAY

One application of the $\gamma + \pi \rightarrow \pi + \pi$ process is to calculate the decay rate of neutral pions. Goldberger and Treiman were the first to analyze π^0 decay by using dispersion relations, but they considered nucleon-antinucleon pairs as the most important intermediate states and neglected multi-pion states.[13] It seems to the author that this may not be a good approximation, since it involves only contributions from far-away singularities but not from near ones. Here we adopt a different approach and consider the contribution of the least massive state. This can be done if we extend a photon variable q^2 into the complex plane instead of the meson variable p^2 used by Goldberger and Treiman.

Following the standard method, one has (see Fig. 4)

$$\left\langle q\ (\mu)\ K(\nu) \left| T \right| p\ (3) \right\rangle = \frac{1}{\sqrt{2K_0}} \int d^4 x\ e^{ikx} \left\langle q(\mu) \left| J_\nu (x) \right| p(3) \right\rangle \epsilon'\nu\ . \tag{31}$$

Because of translation invariance, Eq. (31) becomes

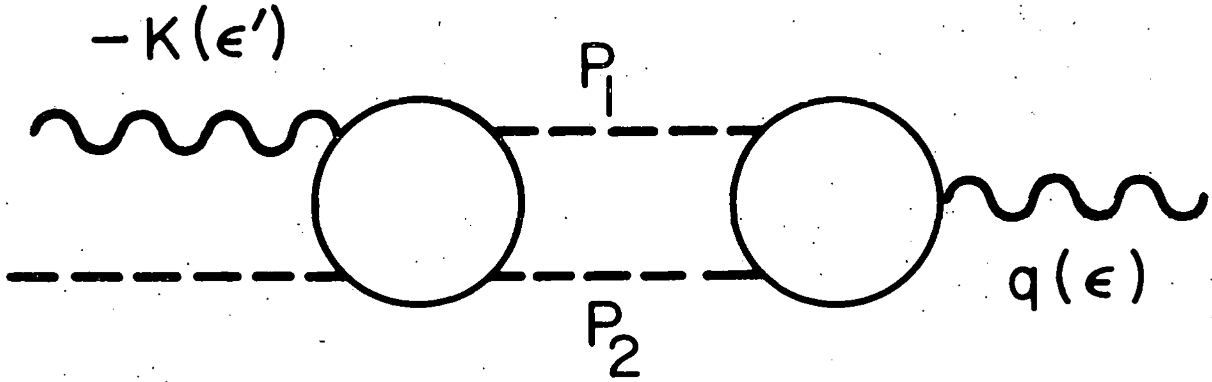
$$\left\langle q\ (\mu)\ k(\nu)\ \ T\ \ p\ (3) \right\rangle = \frac{i(2\pi)^4\ \delta^4\ (p-q-k)\ F_\nu\ (-q^2;\ -k^2;\ -p^2)\ \epsilon\nu'}{(8q_0k_0p_0)^{1/2}}\ ,$$

where we have

$$F_\nu = (4p_0q_0)^{1/2} \left\langle q(\mu) \left| J_\nu\ (0) \right| p(3) \right\rangle , \tag{32}$$

and p is the pion four-momentum. The indices μ and ν refer to the polarization state of the photons of momenta q and k, respectively. The number "3" inside the matrix element represents a neutral meson in the initial state; J is the source of the electromagnetic field and satisfies

$$\Box^2 A_\mu = J_\mu\ .$$



MU-20475

Fig. 4. Neutral pion decay, with only the 2π intermediate state considered. Wavy lines are photons; broken lines, pions.

From general invariance arguments, the F function can be written in the form:

$$F(-q^2; -k^2; -p^2) = \epsilon_{\alpha\beta\mu\nu} q_\alpha k_\beta \epsilon_\mu \epsilon_\nu^\pi f(-q^2; -k^2; -p^2). \tag{33}$$

We can write this form because F must satisfy three conditions: (a) $K \cdot \epsilon = 0$ (b) $q \cdot \epsilon' = 0$ and (c) $\epsilon \cdot \epsilon' = 0$. The gauge-invariant photons require the first two conditions. The last condition is due to the fact that the meson is a pseudoscalar spinless particle, and the polarizations of two photons decaying from it must be perpendicular to each other.

We assume that, with both p^2 and k^2 on the mass-shell, the scalar function $f(-q^2)$ is analytic in the whole complex q^2 plane except for a branch cut on the real axis from -4 to $-\infty$. Using these analytic properties, we can write the dispersion relation for $f(-q^2)$:

$$f(-q^2) = \frac{1}{\pi} \int_4^\infty \frac{\operatorname{Im} f(\sigma^2)\, d\sigma^2}{\sigma^2 - q^2}. \tag{34}$$

Using the unitary condition, we can express the absorptive part of F in formula (32) as

$$A_\nu(-q^2) = \pi\epsilon_\mu \sqrt{2p_0} \sum_n \delta^4(q-p_n) \left\langle 0 \middle| J_\mu(0) \middle| n, P_n \right\rangle \left\langle n, P_n \middle| J_\nu(0) \middle| p(3) \right\rangle .$$

Since our approach is to assume that the function is determined by nearby singularities, no intermediate states except the least massive state--the 2π state--will be considered here. Actually, we should not neglect the 3π contributions, especially if they produce a resonance or even form a bound state at roughly the same energy as the two-pion resonance.[14,*]

---

*The author wishes to thank Professor R. J. Eden of Cambridge University for indicating this point to him.

At present, however, we are not able to handle this part. By considering the 2π intermediate state only, we have

$$A_\nu(-q^2) = 2\pi\,\epsilon_\mu \sqrt{2p_0} \int \frac{d^3p_1\, d^2 l_2}{(2\pi)^3}\, \delta^4(q - p_1 - p_2)$$
$$\times \sum_{i\ \mathrm{spin}} \left\langle 0 \middle| J_\mu(0) \middle| P_1(i),\ P_2(j) \right\rangle \left\langle p_1(i),\ p_2(j) \middle| J_\nu(0) \middle| P(3) \right\rangle . \qquad (35)$$

The first factor of the integrand, i.e. the matrix element describing the disappearance of a pion pair with the creation of a photon, may be written as

$$\left\langle 0 \middle| J_\mu(0) \middle| p_1(i),\ p_2(j) \right\rangle = \frac{ie_r}{(4P_{10}P_{20})^{1/2}} (p_1 - p_2)_\mu \left(\frac{\delta_{1j}\,\delta_{2j} - \delta_{2i}\,\delta_{2j}}{\sqrt{2}} \right)$$
$$\times\ F_\pi^\dagger(-(p_1 + p_2)^2), \qquad (36)$$

where $F_\pi^\dagger(S)$ is the hermitian conjugate of $F_\pi(S)$, the pion form factor and is given by

$$F_\pi(S) = \frac{1}{D_1(S)},$$

where the D_1 function is given by formula (20). The second factor in Eq. (35) is just the matrix element for photopion production from pions:

$$\left\langle p_1(i),\ p_2(j) \middle| J_\nu(0) \middle| p(3) \right\rangle = \left(-\frac{i}{\sqrt{2}}\right) M \frac{\epsilon_{3ij}\ \epsilon_{\alpha\beta\delta\nu}(p_1)_\alpha (p_2)_\beta K_\delta}{(8 P_{10} P_{20} P_0)^{1/2}}$$

where

$$M = M\left[-(p_1 + p_2)^2,\ -(p_1 - p_2)^2 \right]. \qquad (37)$$

Substituting Eq. (36) and (37) into (35), letting $q' = p_1 + p_2$ and $Q = \frac{1}{2}(p_1 - p_2)$; carrying the isotopic-spin sum, and integrating over d^4q' and d^4Q, we have

$$A(-q^2) = \frac{e_r}{48\pi}\ \epsilon_{\alpha\beta\mu\nu}\, q_\alpha\, K_\beta\, \epsilon_\mu\, \epsilon'_\nu \frac{(-q^2 - 4)^{3/2}}{(-q^2)^{1/2}}\ F_\pi^\dagger(-q^2)\, M_1(-q^2).$$

(38)

Notice that the integration over d^4Q projects out the p wave of M only. This is also evident from the fact that the photon has spin one. Comparing Eqs. (33) and (38), we obtain

$$f(-q^2) = \frac{e_r}{48\pi^2} \int_4^{\infty} \frac{(\sigma^2 - 4)^{3/2}}{\sigma(\sigma^2 - q^2)} F_\pi^{\dagger}(\sigma^2) M_1(\sigma^2) d\sigma^2 ,$$

and the pion decay rate is given by

$$\omega = \frac{1}{64\pi} \left| f(0) \right|^2 .$$

The numerical evaluation of f(0) is carried out by using Simpson's rule in steps of 0.01 for t from 0 to 1, where $t = 4/\sigma^2$. We use Eq. (28b) for M_1 and find that the decay rate of the π^0 is given by

$$\tau \cong \frac{7.0 \times 10^{-16}}{\Lambda^2} \text{ sec},$$

where Λ is expressed in the unit of $e(e^2 = 1/137)$. For $\Lambda = \pm 1.3$,* τ is approximately 4×10^{-16} sec, the upper limit from π^0 lifetime experiments performed by Harris, Orear, and Taylor.[15]

*Dr. J. S. Ball, of Lawrence Radiation Laboratory, has applied the Mandelstam representation to the $\gamma + N \rightarrow \pi + N$ problem and finds that $|\Lambda|$ is less than 1.3 in order to make his calculated cross-section compatible with experimental data. The author wishes to thank Dr. Ball for information about his results before publication.

IV. CONCLUSION

From the assumption that the matrix element for photopion production from pions has the Mandelstam representation, we have been able to formulate a set of integral equations for the partial-wave amplitudes. In our approximation, this set of equations has been reduced to a single homogeneous integral equation for the p-wave amplitude, whose solution depends on an unknown parameter Λ, and to formulas for the higher partial waves in terms of the p wave.

Using the pole-approximation technique of Chew and Mandelstam, we have been able to solve the p-wave integral equation and have given the explicit form of the two-pole formula in terms of Λ and three other parameters. These parameters can be calculated in a straightforward manner from crossing relations, once the parameters of the two-pole π-π formula are known.

We have proposed a method to determine Λ by extrapolation of the cross section for $\gamma + p \rightarrow p + \pi^+ + \pi^-$ and $\gamma + p \rightarrow n + \pi^+ + \pi^0$. However, this experiment is very difficult and can only determine the parameter Λ up to its absolute value. We have estimated the order of magnitude of Λ here by considering the decay of neutral pions, assuming that the $\gamma+\pi\rightarrow 2\pi$ process should play a prominant role. Our calculation is based on the assumption that only the least-massive intermediate states contribute to the dispersion integral, but there is no good reason to expect the 3π contribution to be negligible. A better estimate of Λ may come from photopion production on nucleons, where Ball has shown that $\gamma+\pi\rightarrow 2\pi$ makes an important and characteristic contribution. Other problems in which Λ appears include the calculation of 3π contributions to the nucleon isotopic scalar form factors.[16] Up to now, however, no one has succeeded in treating the matrix element $\langle N\bar{N} \mid 3\pi \rangle$, which is also needed here.

ACKNOWLEDGMENTS

The author wishes to thank Professor G. F. Chew for suggesting the problem and for his guidance, constant advice, and encouragement throughout this work. He also thanks Drs. W. R. Frazer and D. Y. Wong for their helpful discussions.

This work was performed under the auspices of U. S. Atomic Energy Commission.

APPENDIX

The sign function $\epsilon(x)$ used in Eq. (17) is defined as

$$\epsilon(x) = \begin{Bmatrix} 1 \\ \\ -1 \end{Bmatrix}$$

$$\text{for} \quad x \begin{Bmatrix} >0 \\ \\ <0 \end{Bmatrix} .$$

The sign function arises because of the complicated s_1 dependence of the denominator in Eq. (17). Since we have

$$2i\,\mathrm{Im}\left[\frac{1}{2s_2+s_1-3+(s_1-1\left(\frac{s_1-4}{s_1}\right)^{1/2} z_1}\right]$$

$$= \frac{1}{2s_2+s_1+i\eta-3+(s+i\eta-1)\left(\frac{s_1+i\eta-4}{s_1+i\eta}\right)^{1/2} z_1}$$

$$- \frac{1}{2s_2+s_1-i\eta-3+(s_1-i\eta-1)\left(\frac{s_1-i\eta-4}{s_1-i\eta}\right)^{1/2} z_1}$$

$$= -2\pi i\,\epsilon\left[\frac{-6-2s_2(s_1^{\,2}-2s_1-2)}{s_1(s_1-1)(s_1-4)}\right] \times \delta\left(2s_2+s_1-3+(s_1-1)\left(\frac{s_1-4}{s_1}\right)^{1/2} z_1\right),$$

from the limits of s_2 $\left\{4 \text{ and } \frac{1}{2}\left[3-s_1-(s_1-1)\left(\frac{s_1-4}{s_1}\right)^{1/2} z_1\right] \text{ for } s_1 < 0\right\}$, one sees that the argument of the sign function always lies between the limits x_1 and x_2, where we have

$$x_1 = \frac{-2\,(2s_1 + 1)(2s_1 - 5)}{s_1(s_1-1)(s_1-4)}$$

and

$$x_2 = 1 - \frac{s_1^2 - 2s_1 - 2}{s_1\left[(s_1-4)\right]^{1/2}}$$

in the range $-\infty < s_1 \leqslant 0$. These two functions x_1 and x_2 vanish for the unique value $s_1 = -\frac{1}{2}$. They are both positive for $-\infty < s_1 < -\frac{1}{2}$ and both negative for $-\frac{1}{2} < s_1 \leqslant 0$ so that we may simply write $\epsilon\,(s_1+\frac{1}{2})$ in the left-hand cut discontinuity.

REFERENCES

1. G. F. Chew and S. Mandelstam, Theory of the Low-Energy Pion-Pion Interaction, UCRL-8728, April 15, 1959 and Phys. Rev. (to be published).
2. S. Mandelstam, Phys. Rev. 112, 1344 (1959); ibid. 115, 1741 and 1752 (1959).
3. W. R. Frazer and J. R. Fulco, Phys. Rev. Letters 2, 365 (1959); Phys. Rev. 117, 1603 and 1909 (1960).
4. J. S. Ball, The Application of the Mandelstam Representation to Photoproduction of Pions From Nucleons, UCRL-9172, April 11, 1960.
5. G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. 106, 1337 (1957).
6. R. Omnes, Nuovo cimento 8, 316 (1958).
7. G. F. Chew and S. Mandelstam, Theory of the Low-Energy Pion-Pion Interaction, Part II., UCRL-9126, March 24, 1960.
8. M. Jacob and G. C. Wick, Ann. Phys. 7, 404 (1959).
9. J. S. Ball and D. Y. Wong, Lawrence Radiation Laboratory, private communications.
10. Professor G. F. Chew and Dr. D. Y. Wong, Lawrence Radiation Laboratory, private communication.
11. M. Gourdin and A. Martin, Photoproduction of Pions on Pions, Preprint CERN 7687 TH/67 (no date). These authors have studied this problem and have used Eq. (25) for M instead of the double spectral representation. They have obtained essentially the same integral equation for the p wave, but their method of solution and numerical results are different from ours. In deriving Eq. (25), we use the fact that the contribution to the first term of Eq. (15) from the left-hand cut just cancels the integral over the K_1 function in the equation.
12. G. F. Chew and F. E. Low, Phys. Rev. 113, 1635 (1959).

13. M. L. Goldberger and S. Treiman, Nuovo cimento 9, 451 (1958).
14. G. F. Chew, Phys. Rev. Letters 4, 142 (1960).
15. G. Harris, J. Orear, and S. Taylor, Phys. Rev. 106, 327 (1957).
16. G. F. Chew, R. Karplus, S. Gasiorowicz, and F. Zachariasen, Phys. Rev. 110, 265 (1958).