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MINIMIZING COSTS INVOLVING
ECONOMIES OF SCALE:
A GLOBAL OPTIMIZATION PROBLEM

by
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ABSTRACT

Cost minimization with an objective function which reflects the effects of economies of scale can lead to nonlinear programming problems which possess more than one local optimum. A computational procedure is presented for such cost minimization subject to linear inequality constraints.

I. Introduction:

The unit cost of purchased items frequently decreases with increased order quantities. Figure 1 shows how the total cost might depend on quantity ordered, when the sellers pricing policy is:

- C_1 per unit for the first P_1 units,
- C_2 per unit for the next P_2 units,
- C_3 per unit for the next P_3 units,
- C_4 per unit for the rest of the order.

In a similar fashion unit production costs tend to decrease with amount produced. Figure 2 shows how total cost might depend on quantity produced in a situation where the employees' familiarity with the job brings about greater productivity rates for larger production quantities.

Linear programming is concerned with finding the combination of order (or production) quantities, which minimize a linear cost function subject to linear constraints. A linear cost function, $\sum_{i=1}^n c_i x_i$, represents situations where unit costs are independent of order quantity. An objective function which reflects economies of scale must be nonlinear.

In this paper, we are concerned with minimizing, subject to linear inequalities, a cost function of the form $f(x) = \sum_{i=1}^n f_i(x_i)$, where each f_i reflects economics of scale, such as occur in figures 1 and 2. Such an objective function is concave. $f(x)$

Figure 1
Order Costs

Figure 2
Production Costs

restricted to a convex set may possess many local optima. Not only is linear programming unsuitable, but also nonlinear programming techniques, such as proposed by Griffith and Stewart [1], Rosen [5], Goldfarb [2], etc., are unsuitable because they merely seek out a local optimum.

Some work has been done on minimizing a concave function subject to linear inequality constraints. Tui[6] presents a couple of approaches to this problem. Ritter [3], [4], deals with the case where the f_i 's are any quadratic functions. It has been shown in [7] that the first method presented by Tui, and the procedure of Ritter's are not computationally sound.

The approach presented in this paper, is much like Tui's second suggestion, but takes particular advantage of special properties of a concave cost function which reflects economies of scale. The problem is defined in section II. Then special aspects of the problem are discussed in section III. The algorithm is presented in section IV. It is not known whether or not the procedure is finite. So a method for obtaining a lower bound on the global minimum is described in the last section.

II. The Problem:

Our problem is:

$$\text{Maximize } f(x) = \sum_{i=1}^n f_i(x_i)$$

$$\begin{aligned} \text{Subject to: } Ax &\leq b \\ x &\geq 0 \end{aligned}$$

where x is an n -component column vector, b is a given m -component column vector, A is a given $m \times n$ matrix, and S^l and S^u are given n -component column vectors, and the f_i are nonnegative continuous piecewise differentiable functions whose derivatives are monotonically non-increasing and satisfy $0 \leq S^l \leq \frac{df}{dx} \leq S^u$.

The requirements on the f_i are consistent with our desire that they represent total cost functions which reflect economies of scale. Concavity follows from the fact that unit costs ($\frac{df}{dx}$) do not increase as quantity increases. The requirement that $0 \leq S^l \leq \frac{df}{dx} \leq S^u$, merely says that each item's cost function f_i , possesses some maximum unit cost S_i^u (which is $\left. \frac{df_i}{dx_i} \right|_{x_i=0}$) and some minimum unit cost S_i^l (which is $\lim_{x_i \rightarrow \infty} \frac{df_i}{dx_i}$).

III. Some Special Considerations:

A few ideas related to our particular problem should be presented prior to a description of the algorithm.

a. Extension of the Objective Function.

The f_i 's, being cost functions, are not defined for negative x_i 's, which are excluded by the requirement $x \geq 0$ in (1). The f_i 's can be extended to include negative x_i 's by using:

$$\tilde{f}_i(x_i) = \begin{cases} f_i(0) + S_i^u x_i, & x_i < 0 \\ f_i(x_i) & , x_i \geq 0 \end{cases}.$$

Claim: $\tilde{f}_i$ is concave.

Proof: $\frac{d\tilde{f}_i}{dx_i}$ is a monotonically nonincreasing function of x_i .

Therefore, $\tilde{f}_i$ is concave.

b. An Additional Constraint.

Let v represent an objective function value. Set

$$I(v) = \{i \mid \exists y_i(v) > 0 \text{ s.t. } f_i(y_i(v)) = v\}$$

Note: If $S_j^l = 0$, then it is possible that $j \notin I(v)$.

Claim: $x \geq 0, \sum_{i \in I(v)} (x_i / y_i(v)) > 1$ implies $f(x) > v$.

Proof: Suppose x satisfies $\sum_{i \in I(v)} (x_i / y_i(v)) = \alpha > 1$.

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Proof: Suppose x satisfies $\sum_{i \in I(v)} (x_i / y_i(v)) = \alpha > 1$.

Then, set $z_i = \begin{cases} 0, & \text{if } i \notin I(v) \\ x_i, & \text{if } i \in I(v) \end{cases}$

We have $\sum_{i \in I(v)} (z_i / \alpha y_i(v)) = 1$ and

$z = \sum_{i \in I(v)} (z_i / \alpha y_i(v)) \alpha y_i(v) e^i$, where e^i are the coordinate vectors,

i.e. $e_j^i = \delta_{ij}$.

Thus, $f(z) \geq \sum_{i \in I(v)} (z_i / \alpha y_i(v)) f_i(\alpha y_i(v))$
 $> \sum_{i \in I(v)} (z_i / \alpha y_i(v)) v$
 $\geq v$.

And, $f(x) = \sum_{i=1}^n f_i(x_i) \geq \sum_{i \in I(v)} f_i(x_i) = f(z) > v$. $a(v)$ will refer to the vector a , with $a_i = \begin{cases} 0, & \text{if } i \notin I(v) \\ 1/y_i(v), & \text{if } i \in I(v) \end{cases}$

c. One-Dimensional Search Result Recognition.

Suppose we wish to find $\min_{t \geq 0} \tilde{f}(y+td)$, where y and d are fixed, and $\frac{d\tilde{f}}{dt}(y+td)|_{t=0} > 0$. Set $S(d) = \sum_{i=1}^n \min(s_i^l d_i, s_i^u d_i)$.

Claim: $\min_{t \geq 0} \tilde{f}(y+td) = \tilde{f}(y)$ iff $S(d) \geq 0$.

Proof: $\tilde{f}(y+td)$ is a concave function of t .

$$\tilde{f}(y) < \tilde{f}(y+td) \text{ iff } \frac{d\tilde{f}}{dt}(y+td) > 0, t > 0.$$

$$\text{That is, iff } \lim_{t \rightarrow \infty} \frac{d\tilde{f}}{dt}(y+td) > 0.$$

$$\begin{aligned} \text{Now } \lim_{t \rightarrow \infty} \frac{d\tilde{f}}{dt}(y+td) &= \sum_{i=1}^n \lim_{t \rightarrow \infty} \frac{d\tilde{f}_i}{dt}(y_i+td_i) \\ &= \sum_{i=1}^n \min(s_i^l d_i, s_i^u d_i). \end{aligned}$$

d. Constraint Addition After One-Dimensional Searches.

Suppose $z, \theta^1, \dots, \theta^n$ have been determined such that the $\theta^j, j = 1, \dots, n$ are linearly independent and

$$\begin{aligned} \tilde{f}(z) &\geq v, \\ \tilde{f}(z+t_j\theta^j) &= v, t_j > 0, j \in J \\ \tilde{f}(z+t\theta^i) &\geq v, \text{ for all } t \geq 0, i \notin J. \end{aligned}$$

Let a be the uniquely determined n vector which satisfies

$$\begin{aligned} a \cdot t_j \theta^j &= 1, j \in J \\ a \cdot \theta^i &= 0, i \notin J. \end{aligned}$$

Claim: If $x = z + \sum_{j=1}^n \alpha_j \theta^j$, $\alpha_j \geq 0$, satisfies $a \cdot x < 1 + a \cdot z$.
then $\tilde{f}(x) \geq v$.

$$\text{Proof: } a \cdot x = a \cdot z + \sum_{j=1}^n (\alpha_j/t_j) a \cdot t_j \theta^j$$

$$= a \cdot z + \sum_{j \in J} (\alpha_j/t_j)$$

$$a \cdot x < 1 + a \cdot z \quad \sum_{j \in J} (\alpha_j/t_j) < 1.$$

$$\text{Now, } x = z + \sum_{j=1}^n \alpha_j \theta^j$$

$$= z (1 - \sum_{j \in J} (\alpha_j/t_j)) + \sum_{j \in J} (\alpha_j/t_j) (z + t_j \theta^j) + \sum_{j \notin J} \alpha_j \theta^j$$

$$= \delta z + \sum_{j \in J} (\alpha_j/t_j) (z + t_j \theta^j) + \sum_{j \notin J} \delta (z + (\alpha_j/\delta) \theta^j),$$

$$\text{where } \delta = (1 - \sum_{j \in J} (\alpha_j/t_j)) / (1 + n - \text{no. of subscripts in } J).$$

And, since $\tilde{f}$ is concave,

$$\tilde{f}(x) \geq \delta \tilde{f}(z) + \sum_{j \in J} (\alpha_j/t_j) \tilde{f}(z + t_j \theta^j) + \sum_{j \notin J} \delta \tilde{f}(z + (\alpha_j/\delta) \theta^j)$$

$$\geq (\delta + \sum_{j \in J} (\alpha_j/t_j) + \sum_{j \notin J} \delta) v$$

$$= v.$$

The uniquely determined a will be referred to as $a(z, \theta)$.

e. Finding a Local Minimum Extreme Point.

Consider the following procedure:

- i) Let z be an extreme point for $Ax \leq b^*, x \geq 0$, which does not minimize $\nabla f(z) \cdot x$.

Find $\hat{z}$, an extreme point of $Ax \leq b^*, x \geq 0$, which minimizes $\nabla f(z) \cdot x$.

- ii) If $\hat{z}$ also minimizes $\nabla f(\hat{z}) \cdot x$, then $\hat{z}$ is a local minimum extreme point. If not, then use $\hat{z}$ as a new z and go to i).

Claim: The above procedure yields a local minimum extreme point in a finite number of steps.

Proof: Since $\nabla f(z) \cdot (\hat{z} - z) < 0$, and f is concave, we must have $f(\hat{z}) < f(z)$. The procedure yields a sequence of distinct (different values of f) extreme points. Since there are only a finite number of extreme points, the procedure must terminate.

IV. The Algorithm:

We start with $A^*=A$, $b^*=b$. And we assume that the original constraint set possesses a feasible solution.

1. Find an extreme point, z , which is a local minimum for f subject to $Ax \leq b^*$. If there are no feasible solutions, then f takes on its global minimum at x^* . Otherwise, go to step 2.a.

- 2.a. From those constraints which hold as equalities at z , pick n constraints, say $a_{ij} \cdot x \leq b_{ij}$, $j = 1, \dots, n$, which

are linearly independent, such that z is a local minimum of f over $\{x | a_{ij} \cdot x \leq b_{ij}, j = 1, \dots, n\}$. Go to step 2.b.

b. Determine n vectors $\theta^1, \dots, \theta^n$ as solutions to $a_{ij} \cdot \theta^j = -\delta_{ij}$ (Kronecker delta) $j = 1, \dots, n$. Go to step 2.c.

c. Set $J = \{j | \exists t_j > 0, \text{ such that } \tilde{f}(z + t_j \theta^j) = f(x^*)\}$. (See section III b.). Go to step 2.d.

d. If, for some $j \in J, \exists t > 0$, such that $z + t\theta^j$ satisfies $A^*x \leq b^*$, and $f(z + t\theta^j) < f(x^*)$ then go to step 2e. Otherwise go to 3.

e. Determine $\tilde{t}_j, j \in J$, as

$$\tilde{t}_j = \max \{t | z + t\theta^j \text{ satisfies } A^*x \leq b^*\}.$$

Set $x^* = z + \tilde{t}_{\hat{j}}\theta^{\hat{j}}$ where $\hat{j}$ is determined by $f(z + \tilde{t}_{\hat{j}}\theta^{\hat{j}}) = \max_{j \in J} \{f(z + \tilde{t}_j\theta^j)\}$.
Go to step 2c.

3. Add $-a(z, \theta) \cdot x \leq -a(z, \theta) \cdot z - 1$ (see section III d.) to the constraint set $A^*x \leq b^*$. Go to step 4.

4. Add $a(f(x^*)) \cdot x \leq 1$ to the constraint set $A^*x \leq b^*$ (see section III b.). Go to step 1. (Notice that if this has been done before, then the new constraint can replace the previously added constraint.)

To see that termination in step 1 implies that x^* gives the global minimum of f , we must show that the constraints added

in 3 and 4 eliminate only feasible points y such that $f(y) > \text{global minimum of } f$.

In section III b., it is shown that all points y eliminated by $a(f(x^*)) \cdot x < 1$ must satisfy $f(y) > f(x^*)$. And, since $f(x^*) \geq \text{global minimum of } f$, step 3 is o.k.

In section III d., it is shown that all points of $y = \sum_{i=1}^n \alpha_i \theta^i$, $\alpha_i \geq 0$, which satisfy $a(z, \theta) \cdot y < 1 + a(z, \theta) \cdot z$ must satisfy $\tilde{f}(y) \geq f(x^*)$. Now any point which satisfies $a_{ij} \cdot x \leq b_{ij}$, $j=1, \dots, n$ can be expressed as $\sum_{i=1}^n \alpha_i \theta^i$, $\alpha_i \geq 0$, where the θ^i are determined by step 2b. Thus, in step 4, any feasible point y , eliminated by the new constraint, must have $\tilde{f}(y) > f(x^*) \geq \text{global minimum of } f$.

It is an open question as to whether or not the procedure is finite. However, it should be noted that in step 1., z could be chosen as any non-degenerate extreme point. In fact, if a particular local minimum, yields little reduction of the feasible region, it may be better to choose some other extreme point.

The next section describes how to obtain a bound on the global minimum of f , if it is necessary to discontinue calculations before termination by step 1.

V. Bound on Minimum:

Suppose z is a point of $Ax \leq b$, $x \geq 0$, which is the minimum for some linear function $c \cdot x$, where $c > 0$.

Claim: Minimum of f subject to $Ax \leq b$ is

$$\geq \min_j \{f_j(c \cdot z / c_j)\}.$$

Proof: For any feasible x ,

let y^k , $k = 1, \dots, n$ represent the n vectors with components

$$y_i^k = \delta_{ik} c \cdot x / c_i, \quad k, i = 1, \dots, n.$$

Then $x_i = (c_i x_i / c \cdot x) y_i^1$, and

$$x = \sum_{k=1}^n (c_k x_k / c \cdot x) y^k.$$

$$\text{Also, } \sum_{k=1}^n (c_k x_k / c \cdot x) = 1.$$

$$\text{So, } f(x) \geq \sum_{k=1}^n (c_k x_k / c \cdot x) f(y^k).$$

$$\text{Now, } f(y^k) = f_k(c \cdot x / c_k) \geq f_k(c \cdot z / c_k).$$

$$\begin{aligned} \text{So, } f(x) &\geq \sum_{k=1}^n (c_k x_k / c \cdot x) f_k(c \cdot z / c_k) \\ &\geq \sum_{k=1}^n (c_k x_k / c \cdot x) \min_j \{f_j(c \cdot z / c_j)\} \end{aligned}$$

$$= \min_j \{f_j(c \cdot z / c_j)\}.$$

This result is particularly handy because, if z is a local minimum for f then z is a minimum for $\nabla f(z) \cdot x$. Moreover, if it is known that $x \geq d \geq 0$, then a translation to $\bar{x} = x - d$ shows that $f_j(c \cdot z / c_j)$ can be replaced with $f(d) + f_j(c \cdot z / c_j - d_j) - f_j(d_j)$.

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