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DEFLECTIONS AND STRESSES
IN HEMISPHERICAL LIGHTWEIGHT
CELLULAR SUPPORT STRUCTURE
WITH FLEXIBLE SUPPORTING SHELL

Livermore, California

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ABSTRACT

A spherical component is frequently supported by a hemispherical lightweight cellular core which is in turn supported by a flexible elastic support shell. This report gives design information on computing the normal and shear stresses in the core as well as circumferential and meridional stresses in the flexible support shell. A set of graphs is given showing the ratio of these stresses for the case of a flexible support shell to the corresponding maximum stress for the case of a rigid support shell, plotted against the meridional angular coordinate of the hemisphere. Two dimensionless geometrical and material parameters appear as variables in these graphs. A second set of graphs to be used for obtaining displacements of the flexible shell with respect to the supported sphere are also plotted in this manner. For computing the maximum stresses in the shell or core, a graph is given showing the maximum value of the stress ratio plotted against a dimensionless material-geometrical parameter.

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NOMENCLATURE

General

- d Radial thickness of core
- E Modulus of elasticity
- F Force applied by supported sphere on core
- G Shear modulus of elasticity
- p Normal stress of core at radius R
- R Mean radius of support shell
- t Thickness of support shell
- u Displacement at core-shell interface
- θ Meridional angular coordinate measured from pole
- σ In-plane stress of support shell
- τ Shear stress of core at radius R

Dimensionless Parameters

- C Stress parameter for core
- D Deflection parameter at core-shell interface
- G Modulus and geometrical ratio $E_c R^2 / E_s t d$
- M Modulus ratio G_c / E_c
- N Stress parameter giving support shell yield criterion
- S Stress parameter for support shell
- ν Poisson's ratio

Subscripts

c	Pertains to core
d	Hencky-von Mises yield criterion stress
f	Circumferential direction of support shell
m	Meridional direction of support shell
r	Pertains to rigid support shell
s	Pertains to support shell
x, y	Orthogonal coordinate direction, y along axis of symmetry

INTRODUCTION

A frequently occurring design problem is the supporting of rather heavy spherical components with a lightweight cellular material. A common configuration is shown in Fig. 1, where the spherical part rests on a core fabricated from the lightweight material, which is in turn supported by the remaining structure through an essentially rigid support shell.

With a rigid support shell, the allowable loads are usually governed by the maximum compressive stress of the core, which occurs at the pole ($\theta = 0$) of the support. The maximum shear stress developed in the core material (occurring at $\theta = \pi/2$) is generally considerably below the elastic limit.

By decreasing the stiffness of the support shell, some additional deformation of the core will occur, resulting in increased shear stress and decreased compressive stress.

It is desirable, then, to decrease the stiffness of the support shell for two reasons. First, a lighter, less bulky support shell results, and second, the full potential support capabilities of the core are more nearly realized, perhaps permitting the use of a lighter core material. In short, an optimum overall design can be achieved.

This report presents design data enabling the engineer to determine compressive and shear stresses in the core material as well as circumferential and meridional stresses in the flexible support shell for a wide range of shell and filler geometries and material properties.

BASIC THEORY

1. Rigid Support Shell Stresses

For the case of an absolutely rigid support shell it is easy to calculate the maximum compressive and shear stresses in the core (see Appendix). A simple sine and cosine distribution results, as shown in Eqs. (1) and (2) below.

$$p_r(\theta) = \frac{3}{2} \frac{F \cos \theta}{\pi R^2 [1 + 2 G_c/E_c]} \quad (1)$$

$$\tau_r(\theta) = \frac{3}{2} \frac{F \sin \theta}{\pi R^2 [2 + E_c/G_c]} \quad (2)$$

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Fig. 1. Basic configuration.

The maximum core stresses with a rigid shell are then

$$p_r(0) = p_r \max = \frac{3}{2} \frac{F}{\pi R^2 [1 + 2 G_c / E_c]}, \quad (3)$$

$$\tau_r\left(\frac{\pi}{2}\right) = \tau_r \max = p_r(0) \frac{G_c}{E_c}. \quad (4)$$

2. Flexible Support Shell Stresses

A. Definitions

For the case of a flexible support shell a sine and cosine series will be used to express the shear and normal stress, respectively, in the core. In order to have coefficients a and b in dimensionless form, the normal and shear stress distribution will be defined as

$$p(\theta) \equiv p_r \max \sum_{n=0, 1, 2, \dots} a_n \cos n\theta, \quad (5)$$

$$\tau(\theta) \equiv \tau_r \max \sum_{n=1, 2, 3, \dots} b_n \sin n\theta, \quad (6)$$

where $p_r \max$ and $\tau_r \max$ are as defined by Eqs. (3) and (4).

For a general case, the problem is to determine the value of the coefficients a_n and b_n . Since a_n has values from a_0 to a_n and b_n has values from b_1 to b_n , there are $2n + 1$ unknowns to find.

For solution of these unknowns, equations requiring compatibility of displacements and static equilibrium at the core-support shell interface have been written. A boundary condition requiring the meridional force at $\theta = \pi/2$ to be equal to the applied load F provides an additional equation. By solving these equations simultaneously in a least-squares sense at a large number of meridional angular positions θ , the required coefficients are found. It is seen for a rigid shell that only the coefficients a_1 and b_1 will be non-zero, and Eqs. (5) and (6) reduce to (1) and (2).

B. Statics

From the statics of the support shell, the meridional stress $\sigma_m(\theta)$ can be determined in terms of the core stress $\tau(\theta)$ and $p(\theta)$. Referring to Fig. 2, a force balance in the y direction ($\Sigma F_y = 0$) yields

$$\begin{aligned} \sigma_m(\theta) t 2\pi R \sin^2 \theta = 2\pi R^2 \int_0^\theta p(\phi) \cos \phi \sin \phi d\phi \\ + 2\pi R^2 \int_0^\theta \tau(\phi) \sin^2 \phi d\phi. \end{aligned} \quad (7)$$

Solving for $\sigma_m(\theta)$,

$$\sigma_m(\theta) = \frac{R}{t \sin^2 \theta} \left[\int_0^\theta p(\phi) \cos \phi \sin \phi d\phi + \int_0^\theta \tau(\phi) \sin^2 \phi d\phi \right].$$

Performing the indicated integrations results in

$$\sigma_m(\theta) = \frac{R}{t \sin^2 \theta} [I_1(\theta) + I_2(\theta)] \quad (7a)$$

where

$$\begin{aligned} I_1(\theta) = p_r \max \left[\frac{a_0}{2} \sin^2 \theta + \frac{a_1}{3} (1 - \cos^3 \theta) + \frac{a_2}{8} \sin^2 2\theta \right. \\ \left. - \sum_{n=3,4,5,\dots} a_n F_a(\theta, n) \right], \\ I_2(\theta) = p_r \max \frac{G_c}{E_c} \left[\frac{b_1}{3} (\cos^3 \theta + 2 - 3 \cos \theta) + \frac{b_2}{2} \sin^4 \theta \right. \\ \left. + \sum_{n=3,4,5,\dots} b_n F_b(\theta, n) \right], \end{aligned}$$

with

$$F_a(\theta, n) = \frac{1}{4} \frac{\cos(2-n)\theta}{(2-n)} + \frac{1}{4} \frac{\cos(2+n)\theta}{(2+n)} + \frac{1}{n^2 - 4},$$

$$F_b(\theta, n) = \frac{\cos(n-2)\theta}{4(n-2)} + \frac{\cos(n+2)\theta}{4(n+2)} - \frac{1}{2n} \cos n\theta + \frac{1}{2n} - \frac{1}{2} \left(\frac{n}{n^2 - 4} \right).$$

A. NOMENCLATURE

B. FORCE COMPONENTS

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Fig. 2. Statics of shell element.

Using membrane theory of shells, a relation between support shell stresses and normal stress can be derived.

$$\sigma_f(\theta) = \frac{p(\theta)R}{t} - \sigma_m(\theta). \quad (8)$$

C. Compatibility

A set of equations expressing compatibility of displacements at the core-support shell interface will be derived next, assuming a bonded interface. For the core material (see Fig. 3) the displacements in the x and y directions are given by

$$u_{xc}(\theta) = \frac{\tau(\theta)d \cos \theta}{G_c} - \frac{p(\theta)d \sin \theta}{E_c}, \quad (9)$$

$$u_{yc}(\theta) = \frac{\tau(\theta)d \sin \theta}{G_c} + \frac{p(\theta)d \cos \theta}{E_c}. \quad (10)$$

For the support shell (Fig. 4) the meridional and circumferential strains are known to be

$$\epsilon_m(\theta) = \frac{1}{E_s} [\sigma_m(\theta) - \nu \sigma_f(\theta)], \quad (11)$$

$$\epsilon_f(\theta) = \frac{1}{E_s} [\sigma_f(\theta) - \nu \sigma_m(\theta)]. \quad (12)$$

Then the displacements of the shell in the x and y directions are given by

$$u_{xs}(\theta) = \epsilon_f R \sin \theta \quad (13)$$

and

$$u_{ys}(\theta) = u_{ys}(0) + \int_0^\theta \epsilon_m R \sin \phi d\phi \quad (14)$$

where $u_{ys}(0)$ is the rigid body displacement of the support shell over a fixed center and is given by $p_r \max d/E_c$.

Substituting Eqs. (11) and (12) above in Eqs. (13) and (14) yields

$$u_{xs}(\theta) = \frac{R \sin \theta}{E_s} [\sigma_f(\theta) - \nu \sigma_m(\theta)] \quad (13a)$$

Fig. 3. Core displacements.

Fig. 4. Support shell displacements.

and

$$u_{ys}(\theta) = \frac{p_r \max d}{E_c} + \frac{R}{E_s} \int_0^\theta \left[\sigma_m(\phi) - \nu \sigma_c(\phi) \right] \sin \phi d\phi. \quad (14a)$$

The compatibility requirements of displacement are

$$u_{xc}(\theta) = u_{xs}(\theta), \quad (15)$$

$$u_{yc}(\theta) = u_{ys}(\theta). \quad (16)$$

From substituting Eqs. (9), (10), (13a), and (14a) in (15) and (16), the compatibility requirements become

$$\frac{\tau(\theta)d \cos \theta}{G_c} - \frac{p(\theta)d \sin \theta}{E_c} = \frac{R \sin \theta}{E_s} \left[\sigma_f(\theta) - \nu \sigma_m(\theta) \right]; \quad (15a)$$

$$\frac{\tau(\theta)d \sin \theta}{G_c} + \frac{p(\theta)d \cos \theta}{E_c} = \frac{p(\theta=0)d}{E_c} + \frac{R}{E_s} \int_0^\theta \left[\sigma_m(\phi) - \nu \sigma_f(\theta) \right] \sin \phi d\phi. \quad (16a)$$

In Eq. (15a), eliminating first σ_m and σ_f using Eqs. (7a) and (8), then $\tau(\theta)$ and $p(\theta)$ using Eqs. (5) and (6), and grouping terms results in

$$\begin{aligned} & \left\{ \left[\frac{(1+\nu)R^2}{tE_s} \right] \frac{\sin \theta}{2} - \left[\frac{d}{E_c} + \frac{R^2}{E_s t} \right] \sin \theta \right\} a_0 \\ & + \left\{ - \left[\frac{(1+\nu)R^2}{tE_s} \right] \frac{1}{3} \frac{(\cos^3 \theta - 1)}{\sin \theta} - \left[\frac{d}{E_c} + \frac{R^2}{E_s t} \right] \frac{\sin 2\theta}{2} \right\} a_1 \\ & + \left\{ \left[\frac{(1+\nu)R^2}{tE_s} \right] \frac{\sin \theta \cos^2 \theta}{2} - \left[\frac{d}{E_c} + \frac{R^2}{E_s t} \right] \sin \theta \cos 2\theta \right\} a_2 \\ & + \sum_{3,4,5,\dots} \left\{ - \left[\frac{(1+\nu)R^2}{tE_s} \right] \frac{F_a(\theta, n)}{\sin \theta} - \left[\frac{d}{E_c} + \frac{R^2}{E_s t} \right] \sin \theta \cos n\theta \right\} a_n \\ & + \left\{ \left[\frac{(1+\nu)R^2}{tE_s} \right] \frac{G_c}{E_c} \frac{(\cos^3 \theta + 2 - 3 \cos \theta)}{3(\sin \theta)} + \left[\frac{d}{2E_c} \sin 2\theta \right] \right\} b_1 \\ & + \left\{ \left[\frac{(1+\nu)R^2}{tE_s} \right] \frac{G_c}{E_c} \frac{\sin^3 \theta}{2} + \frac{d \cos \theta \sin 2\theta}{E_c} \right\} b_2 \\ & + \sum_{3,4,5,\dots} \left\{ \left[\frac{(1+\nu)R^2}{tE_s} \right] \frac{G_c}{E_c} \frac{F_b(\theta, n)}{\sin \theta} + \left[\frac{d \cos \theta \sin n\theta}{E_c} \right] \right\} b_n = 0. \end{aligned} \quad (15b)$$

In Eq. (16a) we first evaluate the integral designated I_3 :

$$I_3 = \int_0^\theta [\sigma_m(\phi) - \nu \sigma_f(\phi)] \sin \phi \, d\phi.$$

Using Eqs. (7a) and (8) to eliminate σ_f and σ_m gives

$$I_3 = \frac{(1+\nu)R}{t} \int_0^\theta \left[\frac{I_1(\phi) + I_2(\phi)}{\sin \phi} \right] d\phi - \frac{\nu R}{t} \int_0^\theta p(\phi) \sin \phi \, d\phi.$$

Carrying out the integration,

$$I_3 = \frac{(1+\nu)R}{t} (I_4 + I_5) - \frac{\nu R p_r \max}{t} \sum_{n=0,2,3,\dots} a_n M_a(\theta, n) - \frac{\nu R p_r \max}{2t} a_1 \sin^2 \theta$$

where

$$I_y = p_r \max \left[\frac{a_0}{2} (1 - \cos \theta) + \frac{a_1}{3} \left(\frac{\sin^2 \theta}{2} - 2 \ln \cos \frac{\theta}{2} \right) + \frac{a_2}{6} (1 - \cos^3 \theta) + \sum_{n=3,4,5,\dots} a_n L_a(\theta, n) \right]$$

and

$$I_5 = p_r \max \frac{G_c}{E_c} \left[-\frac{b_1}{3} \left(4 \ln \cos \frac{\theta}{2} + \frac{\sin^2 \theta}{2} \right) + \frac{b_2}{2} \left(\frac{2}{3} - \cos \theta + \frac{\cos^3 \theta}{3} \right) + \sum_{n=3,4,5,\dots} b_n L_b(\theta, n) \right],$$

in which

$$M_a(\theta, n) = \left\{ -\frac{1}{2} \left[\frac{\cos(1-n)\theta}{(1-n)} + \frac{\cos(1+n)\theta}{(1+n)} \right] + \frac{1}{(1-n^2)} \right\},$$

$$L_a(\theta, n) = \left\{ \frac{1}{(n^2-4)} \left[I_6 + \frac{(n^2+2)}{(n^2-1)} \right] - \frac{1}{2} \left[\frac{\cos(n-1)\theta}{(n-2)(n-1)} + \frac{\cos(n+1)\theta}{(n+2)(n+1)} \right] \right\},$$

$$L_b(\theta, n) = \left\{ \frac{2}{n(n^2-4)} I_6 + \frac{1}{2} \left[\frac{(\cos(n+1)\theta - 1)}{(n+2)(n+1)} - \frac{\cos(n-1)\theta - 1}{(n-2)(n-1)} \right] \right\},$$

where

$$I_6 = \frac{2}{(n-1)} \left[\cos(n-1)\theta - 1 \right] + \frac{2}{(n-3)} \left[\cos(n-3)\theta - 1 \right] + \dots$$

$$+ \frac{2}{(n-p)} \left[\cos(n-p)\theta - 1 \right] \dots + (\cos 2\theta - 1) + 2 \ln \cos \frac{\theta}{2}, \text{ for } n \text{ odd,}$$

$$p \leq n-2,$$

and

$$I_6 = \frac{2}{(n-1)} \left[\cos(n-1)\theta - 1 \right] + \frac{2}{(n-3)} \left[\cos(n-3)\theta - 1 \right] + \dots$$

$$+ \frac{2}{n-p} \left[\cos(n-p)\theta - 1 \right] \dots + 2(\cos \theta - 1), \text{ for } n \text{ even, } p \leq n-1.$$

Now, substituting this result for t_3 and Eqs. (5) and (6) back in Eq. (16a) and grouping terms, we have

$$\left\{ \left[\frac{d}{E_c} + \frac{G_c}{E_c} \frac{(1+\nu)R^2}{6E_s t} \right] \sin^2 \theta + \frac{4}{3} \frac{(1+\nu)R^2}{E_s t} \frac{G_c}{E_c} \ln \cos \frac{\theta}{2} \right\} b_1$$

$$+ \left\{ \left[\frac{2d}{E_c} + \frac{(1+\nu)R^2}{2E_s t} \frac{G_c}{E_c} \right] \cos \theta - \left[\frac{2d}{E_c} + \frac{(1+\nu)R^2}{6E_s t} \frac{G_c}{E_c} \right] \cos^3 \theta - \frac{G_c}{E_c} \frac{(1+\nu)R^2}{3E_s t} \right\} b_2$$

$$+ \sum_{3,4,5,\dots} \left[\frac{d \sin \theta \sin n\theta}{E_c} - \frac{(1+\nu)R^2}{E_s t} \frac{G_c}{E_c} L_b(\theta, n) \right] b_n$$

$$+ \left[\frac{R^2}{2E_s t} (\nu-1) - \frac{d}{E_c} \right] (1 - \cos \theta) a_0$$

$$+ \left\{ \left[-\frac{d}{E_c} + \frac{\nu R^2}{2E_s t} - \frac{(1+\nu)R^2}{6E_s t} \right] \sin^2 \theta + \frac{2}{3} \frac{(1+\nu)R^2}{E_s t} \ln \cos \frac{\theta}{2} \right\} a_1$$

$$+ \left\{ \left[\frac{\nu R^2}{E_s t} - \frac{d}{E_c} \right] \cos \theta + \left[\frac{2d}{E_c} - \frac{2}{3} \frac{\nu R^2}{E_s t} + \frac{1}{6} \frac{(1+\nu)R^2}{E_s t} \right] \cos^3 \theta - \frac{d}{E_c} - \frac{1}{3} \frac{\nu R^2}{E_s t} \right.$$

$$\left. - \frac{1}{6} \frac{(1+\nu)R^2}{E_s t} \right\} a_2$$

$$+ \sum_{3,4,5,\dots} \left[\frac{d(\cos \theta \cos n\theta - 1)}{E_c} + \frac{\nu R^2}{E_s t} M_a(\theta, n) - \frac{(1+\nu)R^2}{E_s t} L_a(\theta, n) \right] a_n = 0.$$

(16b)

Further simplification can be made by grouping parameters in terms of dimensionless ratios. Let

$$G = \frac{E_c R^2}{E_s t d}, \quad M = \frac{G_c}{E_c},$$

then Eq. (15b) becomes

$$\begin{aligned} & a_0 \left[(1 + \nu) G \frac{\sin \theta}{2} - (1 + G) \sin \theta \right] + a_1 \left[- (1 + \nu) G \frac{(\cos^3 \theta - 1)}{3 \sin \theta} - (1 + G) \frac{\sin 2\theta}{2} \right] \\ & + a_2 \left[(1 + \nu) G \frac{(\sin \theta \cos^2 \theta)}{2} - (1 + G) \sin \theta \cos 2\theta \right] \\ & + \sum_{n=3,4,5,\dots} a_n \left[- (1 + \nu) G \frac{F_a(\theta, n)}{\sin \theta} - (1 + G) \sin \theta \cos n\theta \right] \\ & + b_1 \left[(1 + \nu) GM \frac{(\cos^3 \theta + 2 - 3 \cos \theta)}{3 \sin \theta} + \frac{\sin 2\theta}{2} \right] \\ & + b_2 \left[(1 + \nu) GM \frac{(\sin^3 \theta)}{2} + (\cos \theta \sin 2\theta) \right] \\ & + \sum_{n=3,4,5,\dots} b_n \left[(1 + \nu) GM \frac{F_b(\theta, n)}{\sin \theta} + (\cos \theta \sin n\theta) \right] = 0. \quad (15c) \end{aligned}$$

In terms of G and M, Eq. (16b) becomes

$$\begin{aligned}
 a_0 & \left\{ (1 - \cos \theta) \left[\frac{G(\nu - 1)}{2} - 1 \right] \right\} \\
 & + a_1 \left\{ \left[\frac{G\nu}{2} - 1 - \frac{G(1 + \nu)}{6} \right] \sin^2 \theta + \frac{2}{3} (1 + \nu) G \ln \cos \frac{\theta}{2} \right\} \\
 & + a_2 \left\{ (\nu G - 1) \cos \theta + \left[2 - \frac{2}{3} \nu G + \frac{G(1 + \nu)}{6} \right] \cos^3 \theta - 1 - \frac{\nu G}{3} - \frac{G(1 + \nu)}{6} \right\} \\
 & + \sum_{n=3,4,5,\dots} a_n \left[\cos \theta \cos n\theta - 1 + \nu G M_a(\theta, n) - (1 + \nu) G L_a(\theta, n) \right] \\
 & + b_1 \left\{ \left[1 + \frac{(1 + \nu)}{6} GM \right] \sin^2 \theta + \frac{4}{3} (1 + \nu) GM \ln \cos \frac{\theta}{2} \right\} \\
 & + b_2 \left\{ \left[2 + \frac{(1 + \nu)}{2} GM \right] \cos \theta - \left[2 + \frac{(1 + \nu)}{6} GM \right] \cos^3 \theta - \frac{(1 + \nu)}{3} GM \right\} \\
 & + \sum_{n=3,4,5,\dots} b_n \left[\sin \theta \sin n\theta - (1 + \nu) GM L_b(\theta, n) \right] = 0. \quad (16c)
 \end{aligned}$$

3. Boundary Conditions

Referring to Fig. 5, from a force balance in the y direction, the applied load F in terms of σ_m at $\theta = \pi/2$ is found to be

$$F = 2\pi R t \sigma_m(\pi/2). \quad (17)$$

From Eq. (7a), $\sigma_m(\pi/2)$ is found to be

$$\begin{aligned}
 \sigma_m(\pi/2) = \frac{R p_r \max}{t} & \left\{ \frac{1}{2} \left(a_0 + \frac{G_c}{E_c} b_2 \right) + \sum_{n=1,3,4,5,\dots} \left[\left(a_n + \frac{G_c}{E_c} \frac{n b_n}{2} \right) \left(\frac{1 + \cos \frac{n\pi}{2}}{4 - n^2} \right) \right. \right. \\
 & \left. \left. + \frac{G_c}{E_c} b_n \frac{(1 - \cos \frac{n\pi}{2})}{2n} \right] \right\}.
 \end{aligned}$$

Fig. 5. Boundary conditions.

Substituting this value for $\sigma_m(\pi/2)$, using Eq. (3), and the dimensionless parameters G and M, Eq. (17) becomes

$$\frac{1}{3} [1 + 2M] = \left\{ \frac{1}{2} (a_0 + Mb_2) + \frac{1}{3} (a_1 + 2Mb_1) + \sum_{n=3,4,5,\dots} \left[\left(a_n + \frac{Mnb_n}{2} \right) \left(\frac{1 + \cos \frac{n\pi}{2}}{4 - n^2} \right) + \frac{Mb_n (1 - \cos \frac{n\pi}{2})}{2n} \right] \right\} \quad (17a)$$

4. Evaluation of Coefficients

By evaluating Eqs. (15c) and (16c) at i number of meridional angular positions θ around the support shell and using Eq. (17a), we obtain the matrix equation

$$\begin{bmatrix} C_{11} & C_{12} & \dots & C_{1,2n+1} \\ C_{21} & \dots & & \\ \dots & \dots & \dots & \dots \\ C_{2i+1,1} & \dots & C_{2i+1,2n+1} \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ \dots \\ a_n \\ b_1 \\ \dots \\ b_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \dots \\ \dots \\ 0 \\ \frac{1}{3} [1 + 2M] \end{Bmatrix}$$

where the first i rows are Eq. (15c) evaluated at successive angular positions θ_i , the next i rows are Eq. (16c) evaluated at successive angular positions θ_i , and the last row is Eq. (17a). The coefficients C_{ij} represent the actual coefficients of the equations.

This results in a rectangular $2i + 1$ by $2n + 1$ matrix where i is much larger than n, thus giving many more equations than unknowns. A minimum least square residuals routine on the digital computer is used to find the unknown coefficients a_n and b_n . This method of solving for a_n and b_n results in much more accurate values than would be possible to obtain by evaluating

the equations at only n positions θ , that is, by using exactly as many equations as unknowns and solving the more conventional square matrix. This is because the value of the determinant of the square C matrix decreases rapidly as n is increased, and since only a finite number of significant figures can be carried on the computer, a singular matrix results. Introducing extra rows into the matrix greatly increases the value of the resulting equivalent determinant for any n , because those rows that are nearly identical, thus causing singularity of the equivalent determinant, are rejected in the least-squares routine. In this routine, to insure that the boundary condition (17a) would be fulfilled with good accuracy, it was given a weighting of eight as compared to a weight of one for each of Eqs. (15c) or (16c) at θ_i .

The accuracy of the computed coefficients a_n and b_n determined for any value of n and θ_i was determined by rewriting Eqs. (15) and (16) so the displacement of the core and shell in the x and y direction could be computed. Carrying out similar operations as was done for Eqs. (15b) and (16b) gave the following dimensionless equations.

In the x direction at the interface, we have

$$x \text{ displacement of shell} = x \text{ displacement of core,}$$

or

$$\begin{aligned} & -a_0 \left\{ G \left[\frac{\sin \theta}{2} (1 + \nu) - \sin \theta \right] \right\} - a_1 \left\{ G \left[\frac{(1 + \nu)(1 - \cos^3 \theta)}{3 \sin \theta} - \frac{\sin 2\theta}{2} \right] \right\} \\ & - a_2 \left\{ G \left[(1 + \nu) \frac{(\sin \theta \cos^2 \theta)}{2} - \sin \theta \cos 2\theta \right] \right\} \\ & - \sum_{n=3,4,5,\dots} a_n \left\{ G \left[-(1 + \nu) \frac{F_a(\theta, n)}{\sin \theta} - \sin \theta \cos n\theta \right] \right\} \\ & - b_1 \left[GM(1 + \nu) \frac{(\cos^3 \theta + 2 - 3 \cos \theta)}{3 \sin \theta} \right] - b_2 \left[GM(1 + \nu) \frac{(\sin^3 \theta)}{2} \right] \\ & - \sum_{n=3,4,5,\dots} b_n \left[GM(1 + \nu) \frac{F_b(\theta, n)}{\sin \theta} \right] \\ & = -a_0 \sin \theta - a_1 \frac{\sin 2\theta}{2} - a_2 \sin \theta \cos 2\theta \end{aligned}$$

(continued)

$$\begin{aligned}
 & - \sum_{3,4,5,\dots} a_n \sin \theta \cos n\theta + b_1 \frac{\sin 2\theta}{2} + b_2 \cos \theta \sin 2\theta \\
 & + \sum_{3,4,5,\dots} b_n \cos \theta \sin n\theta.
 \end{aligned} \tag{15d}$$

Also, in the y direction at the interface, we have

y displacement of shell = y displacement of core.

$$\begin{aligned}
 & - a_0 \left[(1 - \cos \theta) \frac{G(\nu - 1)}{2} - 1 \right] - a_1 \left\{ \left[\frac{G\nu}{2} - \frac{G(1 + \nu)}{6} \right] \sin^2 \theta - 1 + \frac{2}{3} (1 + \nu) G \ln \cos \frac{\theta}{2} \right\} \\
 & - a_2 \left\{ \nu G \cos \theta - 1 + \left[-\frac{2}{3} \nu G + \frac{G(1 + \nu)}{6} \right] \cos^3 \theta - \frac{\nu G}{3} - \frac{G(1 + \nu)}{6} \right\} \\
 & - \sum_{3,4,5,\dots} a_n \left[-1 + \nu G M_a(\theta, n) - (1 + \nu) G L_a(\theta, n) \right] \\
 & - b_1 \left[\frac{(1 + \nu)}{6} GM \sin^2 \theta + \frac{4}{3} (1 + \nu) GM \ln \cos \frac{\theta}{2} \right] \\
 & - b_2 \left[- (1 + \nu) GM \left(\frac{1}{3} - \frac{\cos \theta}{2} + \frac{\cos^3 \theta}{6} \right) \right] \\
 & - \sum_{3,4,5,\dots} b_n \left[- (1 + \nu) GM L_b(\theta, n) \right] \\
 & = + a_0 \cos \theta + a_1 \cos^2 \theta + a_2 \cos \theta \cos 2\theta \\
 & + \sum_{3,4,5,\dots} a_n \cos \theta \cos n\theta + b_1 \sin^2 \theta + b_2 \sin \theta \sin 2\theta \\
 & + \sum_{3,4,5,\dots} b_n \sin \theta \sin n\theta.
 \end{aligned} \tag{16d}$$

To obtain the actual displacement in either x or y direction, Eqs. (15c) and (16c) must be multiplied by the displacement factor $p_r \max d/E_c$.

By computing the magnitude of each side of Eqs. (15c) and (16c) at any position θ , the dimensionless displacement of the core and shell at the interface at that θ can be obtained. The difference of these values ideally should be zero. A measure of the solution error is obtained by comparing the

maximum difference to the maximum displacement at any position. This error is caused by using a finite number of terms in the trigonometric series and finite number of positions θ_1 .

5. Theoretical Limitations

It is to be recognized that the formulation of the problem does not account for the variety of boundary conditions that may occur in design applications at the core boundary; i. e., $\theta = \pi/2$. The solution as formulated requires a core shear stress at this position equal in magnitude to the interface shear stress obtained from C_τ , as shown in Fig. 2. In some applications, this boundary shear stress may be zero, in which case $\tau(\pi/2)$ must also be zero instead of a maximum as the design curves show. A redistribution of all other stress must necessarily also occur at this position. However, according to Saint-Venant's principle, the effect of these boundary stresses will be very local, extending into the core no more than the core width d ; hence, the design curves can be considered to apply, regardless of core boundary conditions, for the large majority of applications in which d is small compared with the core radius R , say $d/R \leq 0.25$. In all cases, use of good engineering judgment in this region should give usable results.

RESULTS

1. Convergence

To arrive at the required magnitude of n and θ_1 , a series of test runs were made on the computer. These consisted of computing an error indication over all θ from the displacement difference computed from Eqs. (15c) and (16c) for several values of n and θ_1 and for a wide range of feasible combinations of G , M , and ν . It is to be expected that no error at all should occur at those values of θ for which the equations were solved, with maximum error occurring at some in-between positions.

The graph in Fig. 6 shows the dimensionless displacement of the core and shell in the x and y direction plotted against θ for $n = 7$ and $i = 7$ with G , M , and ν as shown. This choice results in a 15×15 square matrix, hence no least-squares solution is necessary. It is seen that good agreement results for small values of θ , and of course, exact agreement is obtained at

those points θ for which the equations were solved. Large deviations are seen to occur at larger θ values. A dimensionless displacement error of 0.33 exists compared with a maximum displacement of 2.16 or 15.5% error. The value of the determinant of the matrix C for this case was 2.34×10^{-8} .

The same plot is shown in Fig. 7 with i increased to 50, giving a matrix size of 15×101 . A weighting factor of 8 was applied to the boundary equation. The least-squares solution must be used now since more equations than unknowns are available. In this case a maximum dimensionless displacement difference of 0.0034 compared with a maximum dimensionless displacement of 2.17 exists, or 1.57% error, a considerable improvement. Also, the value of the equivalent determinant was increased to 0.0163, or about 7×10^5 times as large as that of the square matrix.

This combination gave error less than 2% for G values to 100. For larger G values, to retain this accuracy, it was necessary to increase n to 10, with a weighting factor of 10 applied to the boundary condition.

2. Displacements

The graphs of Figs. 8 to 10 show the dimensionless displacements D_x and D_y given by Eqs. (15c) and (16c) plotted against meridional position angle θ for values of G from 0 (rigid shell) to 10 (flexible shell). Figures 11 to 13 show D_y for G values from 10 to 1000 (extremely flexible shell). A set of curves is given for M values of 0.1, 0.2, and 0.4 with Poisson's ratio set at 0.3. In most cases the results are nearly independent of Poisson's ratio. To show the effect of Poisson's ratio, some curves have been included which represent a value of zero for Poisson's ratio. To obtain actual displacement from these curves the factor D_x or D_y must be multiplied by the quantity $p_r \max d/E_c$ where $p_r \max$ is as given by Eq. (3); i. e.,

$$x \text{ displacement} = D_x p_r \max d/E_c \quad (15d)$$

$$y \text{ displacement} = D_y p_r \max d/E_c \quad (16d)$$

3. Stresses

Introducing the dimensionless core stress ratios C_p (normal) and C (shear) obtained from Eqs. (5) and (6) by dividing through by $p_r \max$ and $\tau_r \max$ gives

$$C_p = \frac{p(\theta)}{p_r \max} = \sum_{0, 1, 2, 3, \dots} a_n \cos n\theta, \quad (5a)$$

$$C_\tau = \frac{\tau(\theta)}{\tau_r \max} = \frac{\tau(\theta)}{p_r \max} \frac{E_c}{G_c} = \sum_{n=1, 2, 3, \dots} b_n \sin n\theta. \quad (6a)$$

Similarly, introducing the dimensionless flexible shell stress ratios S_m (meridional) and S_f (circumferential) obtained from Eqs. (7a) and (8) by dividing through by $\sigma_{mr}(0)$ and $\sigma_{fr}(0)$ gives

$$S_m = \frac{\sigma_m(\theta)}{\sigma_{mr}(0)} = \frac{R}{\sigma_{mr}(0)t \sin^2 \theta} [I_1(\theta) + I_2(\theta)], \quad (7b)$$

$$S_f = \frac{\sigma_f(\theta)}{\sigma_{fr}(0)} = \frac{p(\theta)R}{\sigma_{fr}(0)t} - \frac{\sigma_m(\theta)}{\sigma_{fr}(0)}. \quad (8a)$$

Stresses $\sigma_{mr}(0)$ and $\sigma_{fr}(0)$, the meridional and circumferential rigid support shell stress at the pole, are also the maximum stress if $G_c/E_c < 0.25$. They are identical and equal to

$$\sigma_{mr}(0) = \sigma_{fr}(0) = \frac{3}{4} \frac{F}{\pi R t \left[1 + 2 \frac{G_c}{E_c} \right]}, \quad (18)$$

which is derived in the Appendix. Substituting Eq. (18) and the values of I_1 and I_2 given for Eq. (7) in Eqs. (7a) and (8a) results in

$$S_m = \frac{2}{\sin^2 \theta} \left[\frac{a_0}{2} \sin^2 \theta + \frac{a_1}{3} (1 - \cos^3 \theta) + \frac{a_2}{8} \sin^2 2\theta + \frac{b_1 M}{3} (\cos^3 \theta + 2 - 3 \cos \theta) \right. \\ \left. + \frac{b_2 M}{2} \sin^4 \theta + \sum_{n=3, 4, 5, \dots} b_n M F_b(\theta, n) - a_n F_a(\theta, n) \right], \quad (7b)$$

$$S_f = 2C_p - S_m. \quad (8b)$$

Another useful parameter for the case of a support shell fabricated from a ductile material is derived from the Hencky-von Mises failure criterion.

Since the normal stress $p(\theta)$ is small compared to $\sigma_m(\theta)$ and $\sigma_f(\theta)$, it can be neglected, giving

$$\sigma_d \equiv \frac{1}{\sqrt{2}} \left\{ [\sigma_m(\theta) - \sigma_c(\theta)]^2 + \sigma_m^2(\theta) + \sigma_f^2(\theta) \right\}^{1/2}. \quad (19)$$

The failure criterion states that to prevent yielding of the support shell, the yield stress obtained from a simple tensile test must be lower than σ_d .

Arbitrarily dividing (19) through by $\sigma_{mr}(0)$ yields

$$N = \frac{\sigma_d}{\sigma_{mr}(0)} = \frac{1}{\sqrt{2}} \left[(S_m - S_c)^2 + S_m^2 + S_c^2 \right]^{1/2}. \quad (19a)$$

In Figs. 14 to 31 are shown the stress parameters C_p , C_τ , S_m , S_f , and N plotted versus θ for a range of G values from 0 (rigid shell) to 1000 (very flexible shell). A set of curves is given for values of M of 0.1, 0.2, and 0.4. Poisson's ratio (ν) is taken as 0.3. The dotted curves shown are for ν of 0, showing the small effect of this parameter in most cases.

In Figs. 32 and 33 are a set of graphs of greater design utility, showing the maximum values attained by C_p , C_τ , D_y , and N plotted against G . Again, curves are given for values of M of 0.1, 0.2, and 0.4. These curves are plotted for the more common cases of G ranges from 1 to 10 and $\nu = 0.3$.

APPLICATION OF METHOD

1. General

For the general ductile shell problem, the method of solution for the stresses will usually be a repeated trial process using the plotted results. Generally, the procedure would be to first assume a rigid shell and compute maximum core stresses from Eqs. (3) and (4), repeated below, using given or estimated material and geometrical parameters.

$$p_r \max = \frac{3}{2} \frac{F}{\pi R^2 [1 + 2 M]}, \quad (3)$$

$$\tau_r \max = M p_r \max. \quad (4)$$

Then from Fig. 32 determine a G value such that the maximum core stresses $p(\theta)$ and $\tau(\theta)$ for a flexible support shell are below the yield stress

for the core material chosen. Equations (5a) and (6a) are used for this calculation.

$$p(\theta)\max = (C_p\max) p_r\max, \quad (5a)$$

$$\tau(\theta)\max = (C_\tau\max) \tau_r\max. \quad (6a)$$

The stresses and yield criterion for the support shell are next determined. From the selected G value $E_c R^2/E_s t d$ and other parameters E_c , R , E_s , d , the resultant shell thickness is determined. This shell thickness is used to compute maximum rigid shell in-plane stresses using Eq. (18).

$$\sigma_{mr} = \frac{3}{4} \frac{E_c}{\pi R t [1 + 2M]}. \quad (18)$$

The known G value determines N from Fig. 33, which when multiplied by σ_{mr} gives the critical von Mises yield criterion for a ductile shell. On the other hand, if the shell will fail according to the maximum stress theory, one finds the maximum in-plane stress σ_f or σ_m by multiplying S_f or S_m from Figs. 23 to 28 by σ_{mr} . It may be found that the thickness given by the initial G value is inadequate, making some repetition necessary.

The displacement of the center of the supported component with respect to the support shell of the supported waist can be computed using D_y from Fig. 32 as found from the selected G value. The dimensionless parameter D_y must be multiplied by $p_r\max d/E_c$ to obtain the actual displacement.

From Figs. 8 through 28, at the appropriate values of G , M , and ν , the normal and shear stresses in the core, meridional and circumferential stresses in the shell, and displacements in the axial and transverse directions of the interface, as a function of θ , can be found. The value of these parameters for a rigid shell corresponds to $G = 0$, hence these graphs give a convenient comparison between a rigid and flexible shell.

2. Example Problem

As an example of the optimization possible using this technique, consider the following.

A spherical instrument package is to be supported by a lightweight cellular core as shown in Fig. 1. It will be assumed that at $\theta = \pi/2$ the core

has a boundary shear stress equal to $\tau(\pi/2)$, hence this problem fits the assumptions of the basic theory exactly. The following design parameters are given:

Sphere radius $R = 12$ in.,
 Sphere weight $W = 1000$ lb,
 Maximum package acceleration $= 100$ g's,
 Core thickness $d = 4$ in.

It is desired to use a core material of resin-impregnated paper honeycomb weighing 3.50 lb/cu ft. Its properties are

Compressive yield stress $= 180$ psi,
 Shear yield stress* $= 85$ psi,
 Modulus of elasticity $E_c = 55,000$ psi,
 Shear modulus of elasticity* $G_c = 11,000$ psi.

Then, from these,

$$F = Wg = 100,000 \text{ lb},$$

$$M = G_c/E_c = 0.20.$$

The support shell will be constructed from aluminum, so

$$E_s = 10 \times 10^6 \text{ psi},$$

$$\nu = 0.30,$$

$$\text{Tensile yield stress} = 40,000 \text{ psi}.$$

It is desired to decrease the weight of the support structure by reducing the thickness of the support shell to the point that yielding will occur in either the core or the shell.

*It is recommended that the method employing a shear frame as described in University of California Institute of Engineering Research Report No. SESM-63-1 and titled "Effective Shear Modulus of Honeycomb Cellular Structure", by Joseph Penzien and Theodor Didriksson, be used to determine the effective shear modulus G_c .

As previously explained, rigid support shell values must first be found. These are

$$\begin{aligned}
 p_{r \max} &= \frac{3}{2} \frac{F}{\pi R^2 [1 + 2M]} \\
 &= \frac{3}{2} \frac{(100,000)}{\pi (12)^2 (1 + 0.4)} \\
 &= 237 \text{ psi.}
 \end{aligned} \tag{3}$$

$$\begin{aligned}
 \tau_{r \max} &= M p_{r \max} \\
 &= (0.2)(237) \\
 &= 47.3.
 \end{aligned} \tag{4}$$

The above calculations show that for a rigid shell the core would fail in compression at the pole, while its maximum shear load is only 55% of the yield stress.

Applying Eq. (6a), the maximum value of C_τ is found.

$$\begin{aligned}
 C_{\tau \max} &= \frac{\text{Shear yield stress of core}}{\tau_{r \max}} \\
 &= \frac{85}{47.3} \\
 &= 1.80.
 \end{aligned}$$

From Fig. 32 at the point $C_\tau = 1.8$, $\nu = 0.3$, $M = 0.2$, the following values were obtained.

$$\begin{aligned}
 G &= 3.25, \\
 C_p \max &= 0.73, \\
 D_y \max &= 1.80.
 \end{aligned}$$

Using the above value of C_p in Eq. (5a) gives the maximum flexible shell core stress.

$$\begin{aligned}
 p(\theta) \max &= (C_p \max) p_{r \max} \\
 &= (0.715)(237) \\
 &= 173 \text{ psi}
 \end{aligned}$$

which is below the yield stress of the material.

Similarly, the maximum displacement of the instrument maximum is found using D_y max in Eq. (16d), giving

$$\begin{aligned}\text{Maximum axial movement} &= \frac{(1.8)(237)(4)}{55,000} \\ &= 0.031 \text{ in.}\end{aligned}$$

This is 1.8 times the displacement that would occur with a rigid support shell. From the selected value of G, the resultant shell thickness is

$$\begin{aligned}t &= \frac{E_c R^2}{E_s G d} \\ &= \frac{(55,000)(12^2)}{(10 \times 10^6)(3.25)(4)} \\ &= 0.061 \text{ in.}\end{aligned}$$

Using this shell thickness, the rigid shell meridional stress is found from Eq. (18) to be

$$\begin{aligned}\sigma_{mr} &= \frac{3}{4} \frac{F}{\pi R t [1 + 2M]} \\ &= \frac{3}{4} \frac{120,000}{(\pi)(12)(0.061)(1.4)} \\ &= 28,000 \text{ psi.}\end{aligned}$$

Applying the von Mises yield criterion, from Fig. 33 at $G = 3.25$, $M = 0.2$, $\nu = 0.3$, N is found to be 0.93. Then, from Eq. (19a),

$$\begin{aligned}\sigma_d &= N \sigma_{mr} \\ &= (0.93)(28,000) \\ &= 26,000 \text{ psi.}\end{aligned}$$

The value of σ_d is considerably below the yield stress of 40,000 psi, showing the shell design is satisfactory.

This does not mean, however, that it is safe to reduce t further. This would allow additional stretch of the shell causing higher shear stresses to occur in the core, which is not permissible with the material specified.

If σ_d was found to be greater than the shell yield stress, the support shell would become the controlling part. It would then be necessary to increase t , thus decreasing G . The core stresses should then be recomputed based on this new value of G .

In this example, from Fig. 15 it is seen that compared to a rigid support shell ($G = 0$) the normal cone stress is decreased at $\theta = 0$ and increased at $\theta = 90^\circ$, resulting in a more uniform stress distribution. As would be expected, the shear stress remains at 0 at $\theta = 0$ and increases to a higher value at $\theta = 90^\circ$ than it does for a rigid shell (see Fig. 18).

Figure 9 shows the x and y displacement parameters as a function of θ . Interpolating to $G = 3.25$, it is seen that the shell has no displacement in the transverse direction at $\theta = 68^\circ$. Below 68° , the shell stretches to a larger circumferential diameter; above 68° the shell compresses to a smaller circumferential diameter. For a rigid support shell, there is, naturally, no change in this diameter. The plot of axial shell displacement (D_y) shows that at the pole the flexible shell moves only 75% as far as the rigid shell, while at the waist it moves 180% as far as the rigid shell.

The distributions of the orthogonal shell stresses σ_f and σ_m are shown in Figs. 24 and 27. These stresses are equal at the pole and reduced to about 75% of the value of rigid shell stress. At the waist the meridional stress is seen to be independent of shell stiffness as is required by the boundary condition. Also, the circumferential stress is nearly zero, while in the rigid shell case it is seen to be compressive and of the same magnitude as the circumferential stress.

The graph of N , the von Mises yield criterion, in Fig. 30 shows that at $G = 3.25$ a fairly uniform yielding condition exists throughout the support shell, with a somewhat higher safety factor around $\theta = 45^\circ$ than at the pole or waist.

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APPENDIX

Core Stresses for the Case of a Rigid Support Shell

For the case of a rigid support shell, any displacement will be constant over the entire shell. Hence, referring to Fig. 34, we see a displacement $u(\pi/2)$ in the y direction of the shell results in normal and shear stresses of

$$p_r(\theta) = \frac{E_c u(\pi/2) \cos \theta}{d}, \quad (a)$$

$$\tau_r(\theta) = \frac{G_c u(\pi/2) \sin \theta}{d}. \quad (b)$$

From the force diagram, a summation of forces in the y direction gives

$$F = \frac{2\pi R^2}{d} u\left(\frac{\pi}{2}\right) E_c \left[\int_0^{\pi/2} \sin \theta \cos^2 \theta d\theta + M \int_0^{\pi/2} \sin^3 \theta d\theta \right]$$

where $M = G_c/E_c$.

Carrying out the integration and solving for $u(\pi/2)$ we have

$$u\left(\frac{\pi}{2}\right) = \frac{3}{2} \frac{Fd}{\pi R^2 E_c (1 + 2M)}$$

Substituting this in Eqs. (a) and (b) gives the desired solution.

$$p_r(\theta) = \frac{3}{2} \frac{F \cos \theta}{\pi R^2 (1 + 2M)}, \quad (c)$$

$$\tau_r(\theta) = \frac{3}{2} \frac{F \sin \theta}{\pi R^2 (2 + 1/M)}. \quad (d)$$

Maximum Rigid Shell Support Stresses

Referring to Fig. 2, a force balance in the y direction shows the shell meridional stress to be

$$\sigma_m(\theta) = \frac{R}{t \sin^2 \theta} \left[\int_0^\theta p(\phi) \cos \phi \sin \phi d\phi + \int_0^\theta \tau(\phi) \sin^2 \phi d\phi \right]$$

Substituting Eqs. (c) and (d) and integrating gives

$$\sigma_{mr}(\theta) = \frac{F}{2\pi Rt[1+2M]} \frac{1}{\sin^2 \theta} \left[(M-1) \cos^3 \theta - 3M \cos \theta + 2M + 1 \right]. \quad (e)$$

Taking the partial derivative of $\sigma_m(\theta)$ with respect to θ and setting it to zero will give an extremum

$$\frac{\partial \sigma_m(\theta)}{\partial \theta} = \cos^4 \theta (M-1) + 3 \cos^2 \theta - 2 \cos \theta (2M+1) + 3M = 0.$$

The position of $\theta = 0$ is obviously one solution, others will depend on values of the variable M .

At $\theta = 0$, Eq. (e) gives, after application of L'Hospital's rule,

$$\sigma_{mr}(0) = \frac{3F}{4\pi Rt[1+2M]}. \quad (f)$$

From the boundary conditions it is known that

$$\sigma_m\left(\frac{\pi}{2}\right) = \frac{F}{2\pi Rt}. \quad (g)$$

Comparing (f) and (g) it is apparent that $\sigma_m(0)$ is the larger when $M < 0.25$, hence this is a maximum value. Note that Eq. (f) was used to compute S_c , S_m , and F regardless of the value of M .

From membrane theory of shells, the shell circumferential stress is known to be

$$\sigma_f(\theta) = \frac{p(t)R}{t} - \sigma_m(\theta). \quad (h)$$

Substituting Eqs. (c) at $\theta = 0$ and (f), in (h) we have at the pole

$$\sigma_f(0) = \frac{3F}{4\pi Rt[1+2M]}. \quad (i)$$

GLL-6310-2582

Fig. 6. Shell-core interface displacement parameter vs meridional position for $M = 0.2$, $\nu = 0.3$, $G = 5.0$, matrix size 15×15 .

GLL-6310-2583

Fig. 7. Shell-core interface displacement parameter vs meridional position for $M = 0.2$, $\nu = 0.3$, $G = 5.0$, matrix size 15×101 .

Fig. 8. Shell-core interface displacement parameter vs meridional position for $M = 0.1$, $\nu = 0.3$ (solid curves), $\nu = 0$ (dashed curves).

GIL-6310-2585

Fig. 9. Shell-core interface displacement parameter vs meridional position for $M = 0.2$, $\nu = 0.3$ (solid curves), $\nu = 0$ (dashed curves).

LLL-6310-2586

Fig. 10. Shell-core interface displacement parameter vs meridional position for $M = 0.4$, $\nu = 0.3$ (solid curves), $\nu = 0$ (dashed curves).

Fig. 11. Shell-core interface displacement parameter vs meridional position for $M = 0.1$, $\nu = 0.3$.

Fig. 12. Shell-core interface displacement parameter vs meridional position for $M = 0.2$, $\nu = 0.3$.

Fig. 13. Shell-core interface displacement parameter vs meridional position for $M = 0.4$, $\nu = 0.3$.

GLL-6310-2590

Fig. 14. Core normal stress parameter vs meridional position for $M = 0.1$, $\nu = 0.3$ (solid curves), $\nu = 0$ (dashed curves).

GIL-6310-2591

Fig. 15. Core normal stress parameter vs meridional position for $M = 0.2$, $\nu = 0.3$ (solid curves), $\nu = 0$ (dashed curves).

Fig. 16. Core normal stress parameter vs meridional position for $M = 0.4$, $\nu = 0.3$ (solid curves), $\nu = 0$ (dashed curves).

GLL-6310-2593

Fig. 17. Core shear stress parameter vs meridional position for $M = 0.1$, $\nu = 0.3$ (solid curves), $\nu = 0$ (dashed curves).

Fig. 18. Core shear stress parameter vs meridional position for $M = 0.2$, $\nu = 0.3$ (solid curves), $\nu = 0$ (dashed curves).

Fig. 19. Core shear stress parameter vs meridional position for $M = 0.4$, $\nu = 0.3$ (solid curves), $\nu = 0$ (dashed curves).

Fig. 20. Core shear stress parameter vs meridional position for $M = 0.1$, $\nu = 0.3$.

Fig. 21. Core shear stress parameter vs meridional position for $M = 0.2$, $\nu = 0.3$.

Fig. 22. Core shear stress parameter vs meridional position for $M = 0.4$, $\nu = 0.3$.

Fig. 23. Shell meridional stress parameter vs meridional position for $M = 0.1$, $\nu = 0.3$ (solid curves), $\nu = 0$ (dashed curves).

GLL-6310-2599

GLL-6310-2600

Fig. 24. Shell meridional stress parameter vs meridional position for $M = 0.2$, $\nu = 0.3$ (solid curves), $\nu = 0$ (dashed curves).

Fig. 25. Shell meridional stress parameter vs meridional position for $M = 0.4$, $\nu = 0.3$ (solid curves), $\nu = 0$ (dashed curves).

Fig. 26. Shell circumferential stress parameter vs meridional position for $M = 0.1$, $\nu = 0.3$ (solid curves), $\nu = 0$ (dashed curves).

GLL-6310-2602

GILL-6310-2603

Fig. 27. Shell circumferential stress parameter vs meridional position for $M = 0.2$, $\nu = 0.3$ (solid curves), $\nu = 0$ (dashed curves).

Fig. 28. Shell circumferential stress parameter vs meridional position for $M = 0.4$, $\nu = 0.3$ (solid curves), $\nu = 0$ (dashed curves).

Fig. 29. Shell von Mises yield parameter vs meridional position for $M = 0.1$, $\nu = 0.3$ (solid curves), $\nu = 0$ (dashed curves).

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Fig. 30. Shell von Mises yield parameter vs meridional position for $M = 0.2$, $\nu = 0.3$ (solid curves), $\nu = 0$ (dashed curves).

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Fig. 31. Shell von Mises yield parameter vs meridional position for $M = 0.4$, $\nu = 0.3$ (solid curves), $\nu = 0$ (dashed curves).

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Fig. 32. Maximum core stress parameters C_τ and C_p and maximum displacement parameter D_y vs G for Poisson's ratio of 0.3 and various values of M .

Fig. 33. Maximum von Mises yield parameter N of support shell vs G for Poisson's ratio of 0.3 and various values of M .

A. NOMENCLATURE

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B. FORCE COMPONENTS

Fig. 34. Rigid support shell displacement and forces.

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