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Informal Report

FIELDS AND FORCES FOR RECTANGULAR BEAMS IN AN AGS

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ABSTRACT

The direct space-charge fields and forces that are present in a proton beam which has a rectangular cross section and a uniform charge density are considered. We calculate the horizontal and vertical betatron tune shifts, corresponding to these forces, which occur within such a beam. In addition, a direct extension of this rectangular model allows us to obtain analytical expressions for the tune changes characteristic of a beam profile which has symmetrical or unsymmetrical linear variations of charge density at the horizontal edges of the beam.

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I. INTRODUCTION

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In proton storage rings the beam is, in general, larger in the median plane than in the normal plane. Thus, the particles of different energies execute horizontal betatron oscillations about a spread of equilibrium orbits determined by the dispersion of the ring, while their motion in the vertical plane is a betatron oscillation about an equilibrium orbit in the median plane. To a first approximation, then, the beam can be pictured as having a rectangular cross section and a charge density which is constant in the center and drops off linearly at the horizontal edges. It is our purpose in this paper to study in some detail the fields and forces due to the space charge for such a rectangular model of the beam. The knowledge of these self-forces allows us to calculate the changes in the horizontal and vertical betatron tunes for particles within the beam. We shall not consider, in this report, the effect of the forces arising from the beam images. An initial estimate of the total change in betatron tune can be determined by adding the effect of the image fields on the central particle, originally calculated by Laslett,¹ and analyzed further by Zotter,² and Month and Gluckstern,³ to that due to the direct space-charge fields within the beam, initially derived by Teng.⁴ This procedure has recently been followed by Zotter⁵ who has evaluated the betatron tune shifts for a stacked beam made up of elliptical beams each with a Gaussian distribution.

The rectangular model, which we develop here, has the virtue of yielding relatively simple analytical expressions for the fields, forces, and betatron tune changes inside the beam. In addition, a direct extension of this model permits us to generalize our result to the case of a beam profile having either symmetrical or unsymmetrical horizontal regions at the edges of the beam, in which the charge density decreases linearly to zero.

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1. L.J. Laslett, Proc. 1963 Summer Study on Storage Rings, Accelerators, and Experimentation at Super-High Energies, BNL 7534, p. 324 (1963).
 2. B.W. Zotter, CERN Rept., CERN-ISR-TH/72-8 (1972).
 3. M. Month and R.L. Gluckstern, Particle Accelerators **6**, 19 (1974).
 4. L.C. Teng, Argonne National Lab. Rept., ANLAD-59 (1960).
 5. B.W. Zotter, CERN Rept., CERN-ISR-TH/75-5 (1975).

II. FIELDS AND POTENTIALS DUE TO A MOVING LINE CHARGE

A moving line charge, that is, a continuous linear distribution of charges each moving with a uniform velocity (u), produces an electric field and a magnetic flux density (expressed in cylindrical coordinates) given by

$$E_r = \frac{\lambda}{2\pi\epsilon_0 r} \quad , \quad (1)$$

and

$$B_\theta = \frac{\mu_0 \lambda u}{2\pi r} \quad . \quad (2)$$

We shall use SI units in all equations. For example, λ is the linear charge density in Coulomb per meter and r is the perpendicular distance from the line charge to the field point (r, θ) in meters. Corresponding to these fields one has a vector (A_k) and scalar (φ) potential with the components

$$A_r = 0 \quad (3)$$

$$A_\theta = 0 \quad (4)$$

$$A_z = - \frac{\mu_0 \lambda u}{2\pi} \ln r \quad (5)$$

and

$$\varphi = - \frac{\lambda}{2\pi\epsilon_0} \ln r \quad (6)$$

III. FIELDS DUE TO A RECTANGULAR BEAM OF UNIFORM DENSITY

We now consider the potentials produced by a multitude of linear current elements distributed uniformly over a rectangular cross section which has a horizontal width of $2a$ and a vertical height of $2b$. The uniform line density is

$$\lambda = \frac{\lambda_L}{4ab} d\xi d\eta \quad , \quad (7)$$

and the potentials are

$$A_z(x, y) = - \frac{\mu_0 \lambda_L u}{16\pi ab} \int_{-a}^a d\xi \int_{-b}^b d\eta \ln[(x - \xi)^2 + (y - \eta)^2] \quad (8)$$

and

$$\varphi(x,y) = - \frac{\lambda_L}{16\pi\epsilon_0 ab} \int_{-a}^a d\xi \int_{-b}^b d\eta \ln[(x - \xi)^2 + (y - \eta)^2] . \quad (9)$$

To take advantage of the symmetry of the problem we have changed to rectangular coordinates, expressing the field point by (x,y) and the source point by (ξ,η) . The electric and magnetic fields are now respectively equal to minus the gradient of the scalar potential and the curl of the vector potential. Thus we find

$$E_x(x,y) = \frac{\lambda_L}{16\pi\epsilon_0 ab} M(x,y) , \quad (10)$$

$$E_y(x,y) = \frac{\lambda_L}{16\pi\epsilon_0 ab} N(x,y) , \quad (11)$$

$$B_x(x,y) = - \frac{\mu_0 \lambda_L u}{16\pi ab} N(x,y) , \quad (12)$$

and

$$B_y(x,y) = \frac{\mu_0 \lambda_L u}{16\pi ab} M(x,y) , \quad (13)$$

where the functions, M and N , are integrals over the beam cross section

$$M(x,y) = \int_{-a}^a d\xi \int_{-b}^b d\eta \frac{2(x - \xi)}{(x - \xi)^2 + (y - \eta)^2} \quad (14)$$

and

$$N(x,y) = \int_{-a}^a d\xi \int_{-b}^b d\eta \frac{2(y - \eta)}{(x - \xi)^2 + (y - \eta)^2} \quad (15)$$

These integrals can be evaluated in terms of elementary functions with the result that

$$\begin{aligned} M(x,y) = & (y + b) \ln \left[\frac{(y+b)^2 + (x+a)^2}{(y+b)^2 + (x-a)^2} \right] - (y - b) \ln \left[\frac{(y-b)^2 + (x+a)^2}{(y-b)^2 + (x-a)^2} \right] \\ & + 2(x + a) \left[\arctan \left(\frac{y+b}{x+a} \right) - \arctan \left(\frac{y-b}{x+a} \right) \right] \\ & - 2(x - a) \left[\arctan \left(\frac{y+b}{x-a} \right) - \arctan \left(\frac{y-b}{x-a} \right) \right] ; \end{aligned} \quad (16)$$

while N is equal to M with x interchanged with y and a interchanged with b .

IV. FORCE ON A MOVING PROTON

Since each proton in the beam is moving with a velocity equal to u , the total force exerted on it by the electric and magnetic fields due to all the other protons is given by the Lorentz force expression

$$\vec{F} = e(\vec{E} + \vec{u} \times \vec{B}) \quad (17)$$

When the space-charge fields, Eqs. (10-13), are inserted into this formula, one obtains the well-known relationships,

$$F_x = \frac{e\lambda_L}{16\pi\epsilon_0 ab\gamma^2} M(x,y) = \frac{E_x}{\gamma^2} \quad , \quad (18)$$

and

$$F_y = \frac{e\lambda_L}{16\pi\epsilon_0 ab\gamma^2} N(x,y) = \frac{E_y}{\gamma^2} \quad . \quad (19)$$

We see that the total force on a proton in the beam equals the electric field divided by the square of the particle relativistic factor, γ , the total energy of the proton divided by its rest mass energy. This result can be immediately extended to the case when there are electrons or negative ions trapped in the beam of protons. Since these negative charges are essentially stationary, they only produce an electric field which acts on the moving protons. Defining the degree of neutralization, η , as the ratio of number of trapped negative charges to that of the moving protons and assuming that these trapped charges are distributed over the cross section with a density proportional to the proton density, we find that the net force becomes

$$F_x = \frac{E_x}{\gamma^2} (1 - \eta\gamma^2) \quad , \quad (20)$$

and

$$F_y = \frac{E_y}{\gamma^2} (1 - \eta\gamma^2) \quad . \quad (21)$$

V. FIELDS IN A RECTANGULAR BEAM

As shown in the previous section, the total force acting on a proton in a rectangular beam is expressible in terms of the electric field due to the charges within the beam. In addition, it is the electric field which acts on the relatively stationary trapped electrons and ions. In

this section we shall, therefore, write the electric fields over the cross section of the beam in a convenient manner. When we introduce the normalized variables, $\rho = a/b$, $\alpha = x/a$, $\beta = y/b$, into Eqs. (10), (11), and (16), we derive

$$E_x(\alpha, \beta) = \frac{\lambda_L}{2\pi\epsilon_0 a} G(\alpha, \beta, \rho) \alpha, \quad (22)$$

and

$$E_y(\alpha, \beta) = \frac{\lambda_L}{2\pi\epsilon_0 a} \rho G(\beta, \alpha, 1/\rho) \beta, \quad (23)$$

in which the mean gradient function for the horizontal (x) component of field is

$$\begin{aligned} G(\alpha, \beta, \rho) = & \frac{1}{4\alpha} \left\{ \frac{1+\beta}{2} \ln \left[\frac{(1+\beta)^2 + \rho^2(1+\alpha)^2}{(1+\beta)^2 + \rho^2(1-\alpha)^2} \right] + \frac{1-\beta}{2} \ln \left[\frac{(1-\beta)^2 + \rho^2(1+\alpha)^2}{(1-\beta)^2 + \rho^2(1-\alpha)^2} \right] \right. \\ & + \rho(1+\alpha) \left[\arctan \left(\frac{1+\beta}{\rho(1+\alpha)} \right) + \arctan \left(\frac{1-\beta}{\rho(1+\alpha)} \right) \right] \\ & \left. - \rho(1-\alpha) \left[\arctan \left(\frac{1+\beta}{\rho(1-\alpha)} \right) + \arctan \left(\frac{1-\beta}{\rho(1-\alpha)} \right) \right] \right\}. \quad (24) \end{aligned}$$

We note that the analogous function, which determines the vertical (y) component of the electric field, Eq. (23), is given by the product $\rho G(\beta, \alpha, 1/\rho)$.

In Figs. 1-4 we have plotted the variation of these mean gradient functions for different aspect ratios, ρ , and for characteristic directions within the rectangular beam. For example, Figs. 1 and 2 exhibit the mean gradient function variation along the principal horizontal and vertical axes of the beam, while Fig. 3 shows the change of the horizontal mean gradient function along the vertical axis of the beam. In these figures we have also drawn, for the sake of comparison, the constant field gradient line typical of a beam with circular cross section and uniform density. It is important to emphasize that, throughout this discussion of fields and their variations, we have kept the total charge per unit length, λ_L , unchanged.

Fig. 1. Mean gradient function [see Eq. (22)] along the x axis, $G(\alpha, 0, \rho)$, vs normalized distance, α , from center of rectangular beam.

Fig. 2. Mean gradient function [see Eq. (23)] along the y axis, $\rho G(\beta, 0, 1/\rho)$ vs normalized distance, β , from center of rectangular beam.

Fig. 3. Mean gradient function [see Eq. (22)] normal to y axis, $G(0, \beta, \rho)$, vs normalized distance, β , along the y axis.

Fig. 4. Mean gradient function [see Eq. (22)] along top of beam, $G(\alpha, 1, \rho)$, vs normalized distance, α , along top [$\beta = 1$] of beam.

VI. FIELD AND FORCE GRADIENTS

A glance at Fig. 1 shows that the gradient of the electric field within a rectangular beam is not a constant. In order to calculate the horizontal and vertical gradients at any point in the median plane, we, therefore, differentiate Eqs. (22) and (23) and find that

$$\left. \frac{\partial E_x}{\partial x} \right|_{y=0} = \frac{\lambda_L}{4\pi\epsilon_0 ab} X(\rho, \alpha) \quad , \quad (25)$$

and

$$\left. \frac{\partial E_y}{\partial y} \right|_{y=0} = \frac{\lambda_L}{4\pi\epsilon_0 ab} Y(\rho, \alpha) \quad . \quad (26)$$

Here the X and Y functions are

$$X(\rho, \alpha) = \arctan \left[\frac{1}{\rho(1 + \alpha)} \right] + \arctan \left[\frac{1}{\rho(1 - \alpha)} \right] \quad , \quad (27)$$

and

$$Y(\rho, \alpha) = \arctan [\rho(1 + \alpha)] + \arctan [\rho(1 - \alpha)] \quad . \quad (28)$$

The corresponding Lorentz force gradients are then according to Eqs. (20) and (21)

$$\left. \frac{\partial F_x}{\partial x} \right|_{y=0} = \frac{\lambda_L e}{4\pi\epsilon_0 ab\gamma^2} (1 - \eta_V^2) X(\rho, \alpha) \quad , \quad (29)$$

and

$$\left. \frac{\partial F_y}{\partial y} \right|_{y=0} = \frac{\lambda_L e}{4\pi\epsilon_0 ab\gamma^2} (1 - \eta_V^2) Y(\rho, \alpha) \quad . \quad (30)$$

VII. BETATRON TUNE SHIFTS

The knowledge of the self-force gradients now permits us to readily determine the associated betatron tune changes. We apply first order perturbation theory⁶ to describe the betatron motion of a proton, thereby expressing the betatron tune shifts ($\Delta\nu$) in terms of the radius, R, of the synchrotron and the perturbing gradient forcing function, k. Thus, we have

$$\Delta\nu = \frac{R^2}{2\nu} k \quad . \quad (31)$$

6. E.D. Courant and H.S. Snyder, Ann. Phys. 3, 1 (1958).

The horizontal and vertical gradient forcing functions, which each equal the change in slope per unit particle path length for a unit transverse displacement of the particle position, are related to the Lorentz force gradient according to

$$k_x = - \left(\frac{\partial F_x}{\partial x} \right)_{y=0} \frac{1}{m_p c^2 \gamma \beta^2} , \quad (32)$$

and

$$k_y = - \left(\frac{\partial F_y}{\partial y} \right)_{y=0} \frac{1}{m_p c^2 \gamma \beta^2} . \quad (33)$$

We have written m_p for the proton rest mass, c for the velocity of light, and γ and β for the standard relativistic factors. Combining the last three equations with Eqs. (29) and (30), we derive the tune shift, $\Delta\nu$, due to the direct space-charge forces acting on a proton executing betatron motion about an equilibrium orbit displaced from the geometric center of a rectangular beam by a distance $\alpha(x/a)$:

$$\Delta\nu_x = \frac{-R^2}{2\nu_x} \frac{\lambda_L}{e} \frac{r_p}{ab\gamma^3\beta^2} (1 - \eta\gamma^2) X(\rho, \alpha) . \quad (34)$$

and

$$\Delta\nu_y = \frac{-R^2}{2\nu_y} \frac{\lambda_L}{e} \frac{r_p}{ab\gamma^3\beta^2} (1 - \eta\gamma^2) Y(\rho, \alpha) , \quad (35)$$

wherein the classical proton radius is

$$r_p = \frac{e^2}{4\pi\epsilon_0 m_p c^2} ; \quad (36)$$

and the functions X and Y are those of Eqs. (27) and (28). In Figs. 5 and 6 we present graphs of these betatron tune-change functions for various ratios (ρ) of width to height of the rectangular beam.

VIII. LINEAR DENSITY DECREASE AT HORIZONTAL EDGES

Up to this point in our discussion, we have confined ourselves to a rectangular beam with a uniform charge density and sharp boundaries. Though this model is satisfactory as far as the evaluation of the space-charge forces and fields within the beam is concerned, it does introduce an element of ambiguity in the value of the space-charge depression of the horizontal betatron tune at the very boundary of the beam. Considering

Fig. 5. Horizontal betatron tune change function, [see Eqs.(27) and (34)], $X(\rho, \alpha)$, for a rectangular beam vs normalized distance from center of beam, α .

Fig. 6. Vertical betatron tune change function [see Eqs. (28) and (35)] $Y(\rho, \alpha)$, for a rectangular beam vs normalized distance from center of beam, α .

Fig. 5 for a moment, we see that the horizontal betatron tune shift function rises to a peak as the horizontal edge of the beam ($\alpha = 1$) is approached. Right outside of the beam, the horizontal Lorentz force gradient [see Eqs. (29) and (27)] reverses sign. A proton just outside of the beam would therefore experience a betatron tune change of opposite sign to that just inside of the beam. To remedy this discontinuous behavior and also to provide us with a better understanding of the variation of the horizontal betatron tune at the boundary of the beam, we modify our rectangular model to one which has a uniform density over a width equal to $2a$ and a linearly decreasing density at both horizontal edges over a width w .

Perhaps the best way of determining the field gradient for such a density distribution is to employ a method analogous to that used by Zotter.⁵ We therefore visualize the total beam as made up of a superposition of equal density beams each having a height of $2b$ and a width which increases successively for the set of beams from a value of $2a$ to a final value of $(2a + 2w)$. We then express the total gradient at a field point (x) as that due to a summation of contributions from each component beam [see Eqs. (25) and (26)]. The result is:

$$\left. \frac{\partial E_x}{\partial x} \right|_{y=0} = \frac{\lambda_L}{4\pi\epsilon_0 ab(1+w/2a)w} \int_0^w dz \left[\arctan \frac{b}{x+a+z} + \arctan \frac{b}{a+z-x} \right], \quad (37)$$

and

$$\left. \frac{\partial E_y}{\partial y} \right|_{y=0} = \frac{\lambda_L}{4\pi\epsilon_0 ab(1+w/2a)w} \int_0^w dz \left[\arctan \frac{x+a+z}{b} + \arctan \frac{a+z-x}{b} \right]. \quad (38)$$

The integrals in these equations can be evaluated in terms of elementary functions, and the gradients assume the forms

$$\left. \frac{\partial E_x}{\partial x} \right|_{y=0} = \frac{\lambda_L}{4\pi\epsilon_0 ab} \bar{X}(\rho, \delta, \alpha), \quad (39)$$

and

$$\left. \frac{\partial E_y}{\partial y} \right|_{y=0} = \frac{\lambda_L}{4\pi\epsilon_0 ab} \bar{Y}(\rho, \delta, \alpha). \quad (40)$$

The $\bar{X}$ and $\bar{Y}$ are dependent on three variables including the normalized width, $\delta = w/a$, at each horizontal edge. These functions are explicitly written as:

$$\begin{aligned} \bar{X}(\rho, \delta, \alpha) = & \frac{1}{1+\delta/2} \left\{ \arctan \frac{1}{\rho(1+\delta+\alpha)} + \arctan \frac{1}{\rho(1+\delta-\alpha)} \right. \\ & + \frac{1+\alpha}{\delta} \left[\arctan \frac{1}{\rho(1+\delta+\alpha)} - \arctan \frac{1}{\rho(1+\alpha)} \right] \\ & + \frac{1-\alpha}{\delta} \left[\arctan \frac{1}{\rho(1+\delta-\alpha)} - \arctan \frac{1}{\rho(1-\alpha)} \right] \\ & \left. + \frac{1}{2\rho\delta} \left[\ln \left(\frac{\rho^2(1+\delta+\alpha)^2+1}{\rho^2(1+\alpha)^2+1} \right) + \ln \left(\frac{\rho^2(1+\delta-\alpha)^2+1}{\rho^2(1-\alpha)^2+1} \right) \right] \right\}, \quad (41) \end{aligned}$$

and

$$\begin{aligned} \bar{Y}(\rho, \delta, \alpha) = & \frac{1}{1+\delta/2} \left\{ \arctan \rho(1+\delta+\alpha) + \arctan \rho(1+\delta-\alpha) \right. \\ & + \frac{1+\alpha}{\delta} \left[\arctan \rho(1+\delta+\alpha) - \arctan \rho(1+\alpha) \right] \\ & + \frac{1-\alpha}{\delta} \left[\arctan \rho(1+\delta-\alpha) - \arctan \rho(1-\alpha) \right] \\ & \left. - \frac{1}{2\rho\delta} \left[\ln \left(\frac{\rho^2(1+\delta+\alpha)^2+1}{\rho^2(1+\alpha)^2+1} \right) + \ln \left(\frac{\rho^2(1+\delta-\alpha)^2+1}{\rho^2(1-\alpha)^2+1} \right) \right] \right\}. \quad (42) \end{aligned}$$

The following sequence of expressions then gives the associated change in betatron tunes over the median plane of the beam:

$$\Delta v_x(\alpha) = \Delta v_x(0) P(\rho, \delta, \alpha), \quad (43)$$

$$\Delta v_x(0) = - \frac{R^2}{2v_x} \frac{\lambda_L}{e} \frac{r_p}{ab\gamma^3\beta^2} (1 - \eta_Y^2) \bar{X}(\rho, \delta, 0), \quad (44)$$

$$P(\rho, \delta, \alpha) = \bar{X}(\rho, \delta, \alpha) / \bar{X}(\rho, \delta, 0), \quad (45)$$

$$\Delta v_y(\alpha) = \Delta v_y(0) Q(\rho, \delta, \alpha), \quad (46)$$

$$\Delta v_y(0) = - \frac{R^2}{2v_y} \frac{\lambda_L}{e} \frac{r_p}{ab\gamma^3\beta^2} (1 - \eta_Y^2) \bar{Y}(\rho, \delta, 0), \quad (47)$$

and

$$Q(\rho, \delta, \alpha) = \bar{Y}(\rho, \delta, \alpha) / \bar{Y}(\rho, \delta, 0). \quad (48)$$

It is well to note that the above P and Q functions relate the betatron tune depression at a chosen position in the median plane to that at the center, and they might consequently be called tune spread functions. In Fig. 7 we show, as an example, these functions for the case of beams having a central width to height ratio of $\rho = 2$ and two different width of edges, $\delta = \frac{1}{4}$ and $\frac{1}{2}$. For the sake of comparison, the curves for an abrupt edge, that is $\delta = 0$, are also depicted. It is interesting that protons executing betatron oscillations within the finite edges can experience a horizontal tune depression significantly different from that of the particles in the central region of the beam.

IX. UNEQUAL LINEAR DENSITY AT THE BEAM EDGES

In this last section, we will present, in the interest of completeness, the equations for a beam having a central region (2a) of uniform density, and two horizontal edges of unequal widths, w_1 and w_2 , in which the particle density decreases to zero linearly. Since the width of the central region can vanish ($a = 0$) (the beam then has a triangular charge density distribution) it is desirable in this instance to use the beam half height (b) for normalization. We therefore write the normalized variables as $\rho = a/b$, $\bar{\alpha} = x/b$, $\bar{\delta}_1 = w_1/b$, and $\bar{\delta}_2 = w_2/b$, where w_1 is the edge width associated with positive values of x . Because the method of deriving the tune changes for this charge density distribution is essentially a direct extension of that presented in the previous part of this report, we shall simply list the results. Thus,

$$\Delta v_x(\bar{\alpha}) = \Delta v_x(0) P(\rho, \bar{\delta}_1, \bar{\delta}_2, \bar{\alpha}) \quad , \quad (49)$$

$$\Delta v_x(0) = \frac{-R^2}{2v_x} \frac{\lambda_L}{e} \frac{r_p}{b^2 \gamma^3 \beta^2} (1 - \eta_V^2) X(\rho, \bar{\delta}_1, \bar{\delta}_2, 0) \quad , \quad (50)$$

$$P(\rho, \bar{\delta}_1, \bar{\delta}_2, \bar{\alpha}) = X(\rho, \bar{\delta}_1, \bar{\delta}_2, \bar{\alpha}) / X(\rho, \bar{\delta}_1, \bar{\delta}_2, 0) \quad , \quad (51)$$

$$\Delta v_y(\bar{\alpha}) = \Delta v_y(0) Q(\rho, \bar{\delta}_1, \bar{\delta}_2, \bar{\alpha}) \quad , \quad (52)$$

$$\Delta v_y(0) = \frac{-R^2}{2v_y} \frac{\lambda_L}{e} \frac{r_p}{b^2 \gamma^3 \beta^2} (1 - \eta_V^2) Y(\rho, \bar{\delta}_1, \bar{\delta}_2, 0) \quad , \quad (53)$$

and

$$Q(\rho, \bar{\delta}_1, \bar{\delta}_2, \bar{\alpha}) = Y(\rho, \bar{\delta}_1, \bar{\delta}_2, \bar{\alpha}) / Y(\rho, \bar{\delta}_1, \bar{\delta}_2, 0) \quad . \quad (54)$$

Fig. 7. Horizontal and vertical tune spread functions [see Eqs. (41-48)], $P(\rho,\delta,\alpha)$ and $Q(\rho,\delta,\alpha)$, for a rectangular beam with a linear decreasing density at the horizontal edges [$\rho = a/b$, $\delta = w/a$] vs normalized distance from center of beam, $\alpha = x/a$.

The betatron tune change functions which now depend on four variables are

$$\begin{aligned}
 X(\rho, \delta_1, \delta_2, \alpha) = & \frac{1}{\rho + \delta_1/4 + \delta_2/4} \left\{ \arctan \frac{1}{\rho + \delta_1 - \alpha} + \arctan \frac{1}{\rho + \delta_2 + \alpha} \right. \\
 & + \frac{\rho - \alpha}{\delta_1} \left[\arctan \frac{1}{\rho + \delta_1 - \alpha} - \arctan \frac{1}{\rho - \alpha} \right] \\
 & + \frac{\rho + \alpha}{\delta_2} \left[\arctan \frac{1}{\rho + \delta_2 + \alpha} - \arctan \frac{1}{\rho + \alpha} \right] \\
 & \left. + \frac{1}{2\delta_1} \ln \left[\frac{(\rho + \delta_1 - \alpha)^2 + 1}{(\rho - \alpha)^2 + 1} \right] + \frac{1}{2\delta_2} \ln \left[\frac{(\rho + \delta_2 + \alpha)^2 + 1}{(\rho + \alpha)^2 + 1} \right] \right\} , \quad (55)
 \end{aligned}$$

and

$$\begin{aligned}
 Y(\rho, \delta_1, \delta_2, \alpha) = & \frac{1}{\rho + \delta_1/4 + \delta_2/4} \left\{ \arctan (\rho + \delta_1 - \alpha) + \arctan (\rho + \delta_2 + \alpha) \right. \\
 & + \frac{\rho - \alpha}{\delta_1} \left[\arctan (\rho + \delta_1 - \alpha) - \arctan (\rho - \alpha) \right] \\
 & + \frac{\rho + \alpha}{\delta_2} \left[\arctan (\rho + \delta_2 + \alpha) - \arctan (\rho + \alpha) \right] \\
 & \left. - \frac{1}{2\delta_1} \ln \left[\frac{(\rho + \delta_1 - \alpha)^2 + 1}{(\rho - \alpha)^2 + 1} \right] - \frac{1}{2\delta_2} \ln \left[\frac{(\rho + \delta_2 + \alpha)^2 + 1}{(\rho + \alpha)^2 + 1} \right] \right\} . \quad (56)
 \end{aligned}$$

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