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ESTIMATION OF THE MEAN OF SKEWED DISTRIBUTIONS
USING SYSTEMATIC STATISTICS

Ph. D. Thesis Submitted to Iowa State University, August 1970

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ESTIMATION OF THE MEAN OF SKEWED DISTRIBUTIONS USING SYSTEMATIC STATISTICS

Donald Hughes McElhone and Wayne Fuller

ABSTRACT

This paper investigates the use of simple linear combinations of order statistics to estimate the mean of the Weibull distribution. Formulas for calculating the expected values, variances, and covariances of the order statistics of Weibull random variables with integer shape parameter are developed. The asymptotic values of the expectations and covariances are derived for the largest order statistics. These results are used to study estimators of the mean of the form $\hat{W} = (\sum_{i=1}^{n-r} y_{(i)} + ry_{(n-r)})/D$ where the $y_{(i)}$ are the ordered observations of a sample of size n and D is a suitably chosen denominator. This type of estimator replaces each of the r largest observations with the $(r+1)$ -st largest observation. In particular the Winsorized mean is the estimate $\hat{W}$ with $D=n$. Two other forms of D are considered; $D = n-r$ and $D = n-r+1$. For the exponential $D = n-r$ gives an unbiased estimator and $D = n-r+1$ gives the minimum mean square error estimator.

These three estimates are compared using mean square error as the criterion. The asymptotic results are used to show that the mean square error of the once censored estimate is less than the mean square error of $\bar{y}$ for Weibull distributions with shape parameter greater than one. The maximum likelihood estimate for the mean is developed and compared with $\bar{y}$.

Tests of the null hypothesis that the shape parameter of the Weibull is one are investigated. A test suggested by Fuller, a test based on the log-Weibull estimator and the likelihood ratio test are compared. Fuller's test has the advantage of having a known distribution, but the power of Fuller's test is somewhat less than that of the log-Weibull test. The likelihood ratio test was more powerful than either, but the likelihood test studied is a simple versus simple test while the others are simple versus composite tests.

A technique is also presented which allows the calculation of expected values and expected crossproducts of the order statistics of symmetric distributions using the expected values and crossproducts of the order statistics of the related half distribution.

TABLE OF CONTENTS

ABSTRACT	Page iii
CHAPTER I. INTRODUCTION AND LITERATURE REVIEW	1
CHAPTER II. WEIBULL ORDER STATISTICS	22
CHAPTER III. ASYMPTOTIC MOMENTS FOR THE WEIBULL	40
CHAPTER IV. THE EXPONENTIAL DISTRIBUTION	53
CHAPTER V. ESTIMATION AND TESTING IN THE WEIBULL DISTRIBUTION	62
CHAPTER VI. MOMENTS OF SYMMETRIC DISTRIBUTIONS FROM THEIR HALF DISTRIBUTIONS	99
CHAPTER VII. SUMMARY	113
BIBLIOGRAPHY	122
APPENDIX	126

CHAPTER I. INTRODUCTION AND LITERATURE REVIEW

In this thesis we will study testing and estimation for the mean of a class of skewed distributions. The estimators used will be systematic statistics, defined to be linear combinations of the order statistics of the sample. Let x_1 be the observations in a sample of size n . If they are put in increasing order, we shall denote the i -th largest order statistic by $x_{(i)}$. Then $x_{(i)}$ is greater than or equal to $x_{(i-1)}$. A review of the literature is provided later in this chapter. In Chapter II we will develop a procedure for numerical calculation of expected values and covariances (including variances) of Weibull order statistics. In Chapter III, the asymptotic case is considered and asymptotic expected values and covariances for the highest order statistics are developed. Chapter IV considers the special case of the exponential distribution. Chapter V investigates comparative efficiency of several estimates of parameters of the Weibull. In addition we have studied three tests of departure from the exponential where the alternative is the Weibull distribution. In Chapter VI we will develop a combinatorial form which generates the moments of a symmetric distribution from those of the related half distribution. In particular, tables of the expected values and expected cross products of the double Weibull are given.

A. Literature Review

For a discussion of the properties of order statistics, see Wilks (48, 49). The book edited by Sarhan and Greenberg (37) covers the general topic of order statistics by presenting many of the important papers on the subject including general theory and several articles on particular distributions.

The following result in Renyi (32) plays an important role in this thesis. If X is an exponential random variable with parameter b , that is, $f(x) = b^{-1}e^{-x/b}$, let $X_{(1)}$ be the ordered random variables for a sample of n independent observations. Renyi presents a heuristic proof that the random variables

$$Z_k = (n-k+1)(X_{(k)} - X_{(k-1)}) \quad k = 1, 2, \dots, n \quad (1.1)$$

where $X_{(0)} \equiv 0$, are independent, identically distributed as exponential with parameter b . We present an elementary non-heuristic proof of this important result.

It is known that the joint density of the order statistics of a sample of size n is

$$f(x_{(1)}, x_{(2)}, \dots, x_{(n)}) = n! f(x_{(1)})f(x_{(2)}) \dots f(x_{(n)}). \quad (1.2)$$

For the exponential density this becomes

$$f(x_{(1)}, x_{(2)}, \dots, x_{(n)}) = n! b^{-n} \exp(-b \sum_{i=1}^n x_{(i)}). \quad (1.3)$$

Consider the transformation of Equation 1.1.

$$Z_1 = n X_{(1)} \quad (1.4)$$

$$Z_2 = (n-1)(X_{(2)} - X_{(1)})$$

$$\vdots$$

$$Z_k = (n-k+1)(X_{(k)} - X_{(k-1)})$$

$$\vdots$$

$$Z_n = (X_{(n)} - X_{(n-1)}) .$$

The inverse transformation is

$$X_{(1)} = Z_1/n$$

$$X_{(2)} = Z_1/n + Z_2/(n-1)$$

$$\vdots$$

$$X_{(k)} = Z_1/n + Z_2/(n-1) + \dots + Z_k/(n-k+1)$$

$$\vdots$$

$$X_{(n)} = Z_1/n + Z_2/(n-1) + \dots + Z_n .$$

The Jacobian of the transformation is given by

$$\begin{vmatrix} \frac{1}{n} & 0 & 0 & 0 & \dots & 0 \\ \frac{1}{n} & \frac{1}{n-1} & 0 & 0 & \dots & 0 \\ \frac{1}{n} & \frac{1}{n-1} & \frac{1}{n-2} & 0 & \dots & 0 \\ \frac{1}{n} & \frac{1}{n-1} & \frac{1}{n-2} & \frac{1}{n-3} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n} & \frac{1}{n-1} & \frac{1}{n-2} & \frac{1}{n-3} & \dots & 1 \end{vmatrix} = 1/n! \quad (1.5)$$

Therefore the joint density of the Z_k is given by

$$\begin{aligned}
 f(z_1, z_2, \dots, z_n) &= f\left(\frac{z_1}{n}\right)f\left(\frac{z_1}{n} + \frac{z_2}{n-1}\right) \dots f\left(\frac{\sum_{i=1}^n z_i}{(n-1+1)}\right) \\
 &= b^{-n} \exp(-b^{-1} z_1/n) \dots \exp(-b^{-1} \sum_{i=1}^n z_i/(n-1)) \\
 &= b^{-n} \exp(-b^{-1} (z_1 + z_2 + \dots + z_n)) \\
 &= \prod_{i=1}^n (b^{-1} \exp(-b^{-1} z_i)) \quad . \quad (1.6)
 \end{aligned}$$

Therefore the random variables Z_k are independent and identically distributed as exponential with parameter b .

Furthermore since $X_{(j)} = \sum_{i=1}^j Z_i/(n-i+1)$, $X_{(j)}$ is independent of Z_k if $j < k$.

Waloddi Weibull, a Swedish engineer, in 1939 (46), developed a distribution for the study of breaking strengths. This distribution has come to be known by his name. There are many equivalent forms of the distribution, but we shall write it as

$$F(t; a, c) = 1 - \exp(-(t/c)^{1/a}) \quad t > 0, a > 0, c > 0. \quad (1.7)$$

Normally the shape parameter of the Weibull is the reciprocal of the manner in which we will use it. The density function is

$$\begin{aligned}
 f(t; a, c) &= \frac{t^{1/a-1}}{a c^{1/a}} \exp(-(t/c)^{1/a}) \quad t > 0 \\
 &= 0 \quad \text{otherwise} .
 \end{aligned}$$

The parameter c is called the scale parameter or characteristic life and is the $100(1-e^{-1})$ percent point of the

distribution. We call the parameter a the shape parameter. As a decreases, the mode of the density approaches c and, in the limit the density becomes degenerate at c .

An often used function when studying the Weibull is the reliability function $R(t) = 1 - F(t)$. Johns and Lieberman (25) present a technique for constructing confidence intervals for $R(t)$ based on a transformed distribution. Consider a reparameterization of Equation 1.7 by letting $c = e^v$. The distribution function becomes

$$F_T(t; a, v) = 1 - \exp(-\exp((\log t - v)/a)).$$

Let t_0 be the time at which the reliability is to be estimated and let $T' = T/t_0$. Hence

$$\begin{aligned} F_{T'}(t; a, v) &= 1 - \exp(-\exp((\log t - (v - \log t_0))/a))) \\ &= F_T(t; v - \log t_0, a) \end{aligned}$$

so that T' is a Weibull variate. Let $m = v - \log t_0$, then $F_{T'}(t; a, m) = F_T(t; a, m)$ and $R(t_0) = \exp(-\exp(-m/a))$. Thus, finding a lower confidence bound for m/a is equivalent to finding a lower confidence bound for $R(t_0)$. Let $Y = \log T'$, then

$$P(Y < t) = P(T' < e^t) = 1 - \exp(-\exp((t-m)/a)).$$

Let $Y_{(1)} = \log T_{(1)} - \log t_0$ be the order statistics of the sample and consider estimates for m and a

$$Z_a = \sum_{i=1}^r A_i Y_{(i)}$$

$$Z_b = \sum_{i=1}^r B_i Y_{(i)}$$

where $\sum_{i=1}^r A_i = 1$, $\sum_{i=1}^r B_i = 0$, and r is a constant percent amount of censoring. Johns and Lieberman (25) give tables of the A_i and B_i such that the quantities

$$V_a = \frac{1}{a}(Z_a - m)$$

$$V_b = \frac{1}{a} Z_b$$

have joint distribution which is parameter free and asymptotically normal. Define $L(t)$ so that for fixed γ , $0 < \gamma < 1$

$$P(L(t) < tV_b - V_a) = \gamma$$

for all t .

Suppose $P(V_b > 0) = 1$, then $L(t)$ is a monotone increasing function of t and $P(L(Z_a/Z_b) < m/a) = \gamma$. Therefore $L(Z_a/Z_b)$ is a lower confidence bound for m/a having confidence coefficient γ . Hence for the Weibull, the corresponding lower confidence bound for $R(t_0)$ is

$$L^*(Z_a/Z_b) = \exp(-\exp(-L(Z_a/Z_b))).$$

The procedure, then, is to compute Z_a/Z_b from the sample data and given the confidence coefficient γ , use the tabular values of $L^*(Z_a/Z_b)$ to find the lower confidence bound. A point estimate for the reliability is given by $R^*(t_0) =$

$\exp(-\exp(-Z_a/Z_b))$.

Mann (30) has shown that if the first m of n sample values are observable then, for the Weibull, the set of the first m order statistics is minimally sufficient. Furthermore, for m greater than two, no joint complete sufficient statistics exist and, therefore, no unique minimum variance unbiased estimates for the parameters exist under the Lehman-Scheffe theorem. Mann gives estimators based on just the first two order statistics.

If we consider the standardized Weibull by letting $z = t/c$, we have

$$g(z;a) = \frac{z^{\frac{1}{a}-1}}{a} \exp(-z^{1/a}), \quad z > 0, \quad (1.9)$$

$$= 0 \quad \text{otherwise.}$$

The distribution function is given by

$$G(z;a) = 1 - \exp(-z^{1/a}).$$

Splitstone (40) shows that for the above density function, the curve has a maximum at $(1-a)^a$ and inflection points at

$$z_{\text{infl}} = \frac{3(1-a) \pm \sqrt{(5-a)(1-a)^a}}{2}$$

and discusses four cases for the parameter a . We will be primarily interested in the case where the shape parameter a is greater than one. In this case the density function is convex over its entire domain. The second case is for a equal to one which is the usual exponential density. Case three

is for a in the half closed interval $[1/2, 1)$. In the case the density has a mode and is concave below the mode and convex for z greater than the mode. Finally, for a less than one-half the density has a mode and an inflection point on either side of the mode.

The median of the density is at $t = c(\log 2)^2$, where $\log$ will denote the natural logarithm throughout this thesis.

The moments of the distribution are found in the usual manner.

$$E(t^m) = \int_0^{\infty} (t^m t^{\frac{1}{a}-1} \exp(-(t/c)^{1/a}) / ac^{1/a}) dt.$$

Let $z = (t/c)^{1/a}$ and the integral becomes

$$\int_0^{\infty} c^m z^{am} \exp(-z) dz$$

so that

$$E(t^m) = c^m \Gamma(am+1) \quad (1.10)$$

Γ is the gamma function. In particular, for q an integer

$$\Gamma(q+1) = q!.$$

There is a great volume of literature available on the Weibull distribution. We begin the discussion of estimation by considering the maximum likelihood estimators (MLE). Let $b = c^{1/a}$ then we write the density as

$$f(t; a, b) = \frac{t^{\frac{1}{a}-1}}{ab} \exp(-t^{1/a}/b) \quad (1.11)$$

The log-likelihood function is then

$$\log L = -n \log a - n \log b + \sum \left(\frac{1}{a}-1\right) \log t_1 - \frac{1}{b} \sum t_1^{1/a}.$$

Taking the partial derivatives with respect to a and b and setting them equal to zero we have the MLE of a and b as the solution of

$$\hat{b} = \frac{1}{n} \sum t_1^{1/\hat{a}} \quad (1.12)$$

$$-\hat{a} + \frac{1}{n\hat{b}} \sum t_1^{1/\hat{a}} \log t_1 = \frac{1}{n} \sum \log t_1.$$

The estimator for c is then

$$\hat{c} = \left(\frac{1}{n} \sum t_1^{1/\hat{a}} \right)^{\hat{a}}.$$

It has been shown that the maximum likelihood estimators are consistent, asymptotically normal, and asymptotically efficient. Methods for solution are given by Harter and Moore (22) for both the censored and noncensored cases using initial estimates as starting points. Halperin (19) discusses a general procedure for a constant percentage of censoring leading to estimates with the same properties as the MLE. Menon (31) uses the transformation $w = (t/c)^{1/a}$ to get to the standard exponential variate and arrives at the estimate for a

$$\hat{a} = \left(\frac{6}{\pi^2} (\sum \log t_1^2 - (\sum \log t_1)^2/n)/(n-1) \right)^{1/2}$$

which has asymptotic efficiency of about 55 percent. For the case of a known scale parameter he gives an unbiased estimator of α which has asymptotic efficiency of approximately 84 percent. Menon's procedure does not use order statistics and is not applicable to the case of censored samples.

In the area of least square estimation applied to order statistics, Lloyd (27) studied distributions which are dependent on location and scale parameters only. If the expectations and covariances (and variances) of the order statistics are known, then generalized least squares performed on the order statistics give minimum variance, unbiased linear estimates. Let y_i , $i = 1, 2, \dots, n$, be a sample of n independent observations and $y_{(i)}$ be the ordered observations. Let $u_{(i)} = (y_{(i)} - \mu)/\sigma$ be a standardized observation with expectation $E(u_{(i)}) = \alpha_i$ and $\text{Cov}(u_{(i)}u_{(j)}) = w_{ij}$. The α_i and w_{ij} are dependent on the parent distribution, but not on μ or σ . Then in terms of the original observations $E(y_{(i)}) = \mu + \sigma\alpha_i$ and $\text{Cov}(y_{(i)}y_{(j)}) = \sigma^2 w_{ij}$. If we write these equations in matrix notation by letting Y be the vector of ordered observations, then

$$E(Y) = u\mathbf{1} + \sigma\alpha = p\theta$$

where α is the vector of α_i . $\mathbf{1}$ is a vector of ones, $p = (\mathbf{1}, \alpha)$, and hence $\theta' = (\mu, \sigma)$. Furthermore

$$V(Y) = \sigma^2 w$$

where w is an $n \times n$ positive definite, symmetric matrix with typical element w_{ij} . The estimator is given by

$$\hat{\theta} = (p'w^{-1}p)^{-1}p'w^{-1}y$$

and

$$V(\hat{\theta}) = \sigma^2(p'w^{-1}p)^{-1}.$$

Lloyd further shows that the variance of the location parameter, $V(\hat{\mu}) \leq \sigma^2/n$. Sarhan (36) uses Lloyd's procedure to derive the particular estimates for the rectangular, triangular, double, and single exponential distributions.

Blom (2) considers a class of estimates which he calls "nearly best estimates". These are estimates with nearly minimum variance. Let Z be a random variable with a known continuous distribution, $F((z-a)/b)$, and let $Z_{(1)}$ be the ordered random variables from a sample of n . Suppose the expectations and covariances of the ordered random variables are not known. Blom suggests making a transformation such that the transformed variables are approximately distributed as functions of uniform random variables. The expected values and covariances of the order statistics of the transformed variables as well known. He then performs least squares on the transformed variables and uses the results to give estimates for the parameters of the original distribution.

We concern ourselves now with censored estimation. In this thesis we shall say that an estimate is censored if the extreme observations are not used in estimation whether they are available or not. However, the size of the whole sample is always known.

David and Johnson (4) make an expansion to approximate the moments of the ordered observations. Unfortunately, the expansion fails for the highest and the lowest order statistics. Their procedure is satisfactory for percentile censoring although the formulas are rather complex. A best linear systematic statistic is derived by performing least squares on the order statistics of the sample. We denote the best linear systematic statistic by BLSS. Sarhan and Greenberg (38) use the BLSS to study the normal distribution for sample sizes up to ten which are singly and double censored. They give tables of the expectations and covariances for the estimator. For the noncensored case the estimator $\Sigma x_{(i)}/n$, the usual mean, is the BLSS. The estimators are of the form

$$p = \sum_{i=r_1+1}^{n-r_2} a_i y_{(i)}$$

where r_1 and r_2 are the amounts of censoring. Tables of the covariances of the estimates of location and scale parameters are given. Dixon (5), in studying the normal distribution, suggests the estimate

$$\hat{m} = (\alpha x_{(1)} + x_{(2)} + \dots + (r+1)x_{(n-r)})/(n+\alpha-1) \quad (1.13)$$

where α is chosen to make the estimate unbiased and r is number of observations censored. Note that Equation 1.13 is used only for censoring the large observations even though the distribution is symmetric.

The estimator above is similar to those suggested by Epstein and Sobel (14, 15) for life testing situations. A life test is an experiment in which the observations occur in order and the experiment is terminated after the $(n-r)$ -th failure or at some particular time, or some combination of both. Initially they studied the one parameter exponential distribution and gave the estimator for b

$$\hat{E}_r = (\sum_{i=1}^{n-r} x_{(i)} + r x_{(n-r)})/(n-r). \quad (1.14)$$

The probability density function of the first $n-r$ of n ordered observations is

$$f(x_{(1)}, x_{(2)}, \dots, x_{(n-r)}; b) = \frac{n!}{r!} \frac{1}{b^{n-r}} \exp(-(\sum_{i=1}^{n-r} x_{(i)} + r x_{(n-r)})/b) \quad (1.15)$$

and $\hat{E}_r$ maximizes Equation 1.15. Hence $\hat{E}_r$ is the maximum likelihood estimate (MLE) for b . The estimate $\hat{E}_r$ has the same precision as the mean based on a sample of size $(n-r)$. In Chapter IV we shall show that $\hat{E}_r$ is unbiased. Let us compute the Cramér-Rao lower bound $1/E(\partial \log f / \partial \theta)^2$, where f is given by Equation 1.15. We write Equation 1.15 as

$$f(x_{(1)}, x_{(2)}, \dots, x_{(n-r)}; b) = \frac{n!}{r! b^{n-r}} \exp(-(n-r)\hat{E}_r/b)$$

$$\log f = \log(n!) - \log(r!) - (n-r) \log(b) - (n-r)\hat{E}_r/b$$

$$\frac{\partial \log(f)}{\partial b} = -\frac{(n-r)}{b} + \frac{(n-r)\hat{E}_r}{b^2} \quad (1.16)$$

Hence

$$\begin{aligned} E\left(\frac{\partial \log f}{\partial b}\right)^2 &= E\left(-\frac{(n-r)}{b} + \frac{(n-r)\hat{E}_r}{b^2}\right)^2 \\ &= (n-r)^2 E\left(\frac{1}{b^2} + \frac{\hat{E}_r^2}{b^4} - \frac{2\hat{E}_r}{b^3}\right) \\ &= (n-r)/b^2 \end{aligned}$$

Therefore the Cramér-Rao lower bound is $b^2/(n-r)$. But as will be shown in Chapter IV, the variance of the estimator is $b^2/(n-r)$. Hence $\hat{E}_r$ is minimum variance and efficient since any other unbiased estimate based on the first $n-r$ of n order statistics has variance at least $b^2/(n-r)$. Using the random variable Z_1 defined in Equation 1.1 we write the estimate as

$$\hat{E}_r = \sum_{i=1}^{n-r} z_i / (n-r) \quad (1.17)$$

where the observations z_1 are independent exponentials with parameter b . The characteristic function of the Z_1 is

$$g_{Z_1}(t) = (1 - itb)^{-1}$$

hence the quantity $p = 2(n-r)\hat{E}_r/b$ has characteristic function

$$g_p(t) = (1-12t)^{-(n-r)}$$

which is a chi-square variate with $2(n-r)$ degrees of freedom. Epstein and Sobel's second paper (15) discusses a life test to be terminated at the $(n-r)$ -th failure or a preset time, T , where the underlying distribution is exponential. Epstein (11) investigated the two parameter exponential which has distribution function $F(x) = 1 - e^{-(x-A)/b}$, $x > A$, and suggested as an estimate of the location parameter A

$$\hat{A} = x_{(1)} - \hat{E}_r/n$$

and developed an F test for A .

Bartholomew (1) studied the exponential distribution and estimated the parameter by

$$\hat{b} = \frac{1}{k} \sum_{i=1}^n (a_i t_i + (1-a_i) T_i)$$

where the T_i are prechosen failure times for the i -th experiment. The observations t_i are known if and only if they fail before their termination time T_i . Hence $t_i \leq T_i$. Finally $a_i = 1$ if $t_i \leq T_i$ and $a_i = 0$ otherwise, and $k = \sum a_i$. Then the t_i are not order statistics in the strictest sense. The usual case would be for $T_i = T$ for all i . The estimate is asymptotically normally distributed. Searles (39) extended this estimate to more general distributions.

Jackson (24) developed a test to measure departures from the exponential. Letting e_1 be the expected value of the 1-th standard exponential order statistics, he suggests as a test statistic T where

$$T = \sum e_1 x_{(1)} / \sum x_{(1)}. \quad (1.18)$$

It is easily seen that T is independent of the parameter b . He used the results of Renyi to write T as

$$T = \sum c_1 z_1 / \sum z_1 \quad (1.19)$$

where $c_1 = 1$ and $c_1 = 1 + e_{1-1}$ for $1 = 2, 3, \dots, n$. He finds the distribution function of the test as

$$F(x) = \sum_k ((x - c_k)^{n-1} / \prod_{\substack{j=1 \\ j \neq k}}^n (c_j - c_k)) \quad (1.20)$$

which is dependent on the sample size. Furthermore, for sample sizes greater than 30 a normal approximation is satisfactory and T is asymptotically normal with mean two and variance n^{-1} . Jackson presents the results of a Monte Carlo study giving the empirical power of the T test testing $H_0: f(x) = e^{-x}$ versus the alternative $H_1: f(x) = xe^{-x}$. At the five percent level this power is .40 for a sample of size 10 and .56 for a sample of size 20. He suggests that similar results would occur for other alternatives. The test statistic is expanded to include the censored case. He uses the estimate $\hat{E}_r$ and the statistic

$$T_r = \sum_{i=1}^{n-r} e^{ix_{(i)}} / \hat{E}_r. \quad (1.21)$$

We shall discuss a similar test in a later chapter.

Harter and Moore (23) used the estimator $\hat{E}_r$ for estimating the scale parameter of type II extreme value distributions. If T is an exponential random variable with parameter b then $Y = T^{-1/k}$ has type II asymptotic distribution for largest values with scale parameter $v_1 = b^{-1/k}$, shape parameter $k > 0$ and distribution function given by $F_1(y; v_1, k) = \exp(-(y/v_1)^{-k})$. The MLE of v_1 , if the shape parameter is known, is

$$\begin{aligned} \hat{v}_1 &= (\hat{E}_r)^{-1/k} = ((\sum_{i=1}^{n-r} t_{(i)} + r t_{(r)}) / (n-r))^{-1/k} \\ &= ((n-r) / \sum_{i=1}^{n-r} y_{(n-i)}^{-k} + r y_{(r)}^{-k})^{1/k}. \end{aligned}$$

The type II asymptotic distribution for smallest values is derived from the exponential by the transformation $X = -T^{-1/k}$ and has distribution function $F_2(x; v_2, k) = 1 - \exp(-(x/v_2)^{-k})$. The estimator in this case is derived by the symmetry of the distribution of largest values and hence $v_1 = -v_2$. The estimate of v_2 is

$$\hat{v}_2 = -((n-r) / \sum_{i=1}^{n-r} (-x_{(i)})^{-k} + r (x_{(n-r)})^{-k})^{1/k}.$$

The estimators $\hat{v}_1$ and $\hat{v}_2$ are MLE since they are one-to-one functions of the MLE $\hat{E}_r$. Somewhat earlier Harter and

Moore (23) had considered the Weibull distribution with a known shape parameter. From Equation 1.7 let $X = T^{1/a}$, then the random variable X is distributed exponentially with parameter $c^{1/a}$. Hence Epstein's estimator $\hat{E}_r = \frac{1}{n-r} (\sum_{i=1}^{n-r} x_{(i)} + rx_{(n-r)})$, based on the transformed variables estimates $c^{1/a}$. Therefore $(\hat{E}_r)^a$ is the maximum likelihood estimate of c . An unbiased estimate for c based on $\hat{E}_r$ is $\hat{c} = ((n-r)^a \Gamma(n-r) / \Gamma(n-r+a)) \hat{E}_r$.

A similar type estimator developed by Charles Winsor and studied by Tukey with McLaughlin (43) and Dixon (6) is called the Winsorized mean. This estimate censors the s smallest and the r largest observations and replaces them with $x_{(s+1)}$ and $x_{(r+1)}$ respectively. Then

$$\hat{s}_r^w = \frac{1}{n} ((s+1)x_{(s+1)} + x_{(s+2)} + \dots + x_{(n-r-1)} + (r+1)x_{(n-r)}).$$

Tukey, McLaughlin, and Dixon studied $\hat{s}_r^w$ for symmetric distributions, primarily the normal distribution. For symmetric Winsorization a t-type test was developed.

$$t^* = (\hat{s}_r^w - m) / \sqrt{\sum (x_{(i)} - \hat{s}_r^w)^2 / n(n-1)}$$

where m is the hypothesized center of the symmetric distribution. They suggest no criterion for preselecting the amount of censoring, rather they suggest the comparison of several amounts of censoring with the "best" result being chosen. The comparison procedure assumes that all of the

observations are available.

For skewed distributions we shall investigate censoring from only the long tailed side. In particular, for the Weibull, we consider the Winsorized mean $\hat{W}_r$ which we shall write as $\hat{W}_r$. Then

$$\hat{W}_r = \frac{1}{n} \left(\sum_{i=1}^{n-r} x_{(i)} + rx_{(n-r)} \right).$$

Note that $\hat{W}_r$ is related to $\hat{E}_r$ by $\hat{W}_r = \frac{n-r}{n} \hat{E}_r$.

Splitstone's thesis (40) reiterates the graphical techniques for estimation in the Weibull as well as other approaches on an easily understood level. He and many others have investigated a transformation technique in which the logarithms of the observations are taken. If we write the Weibull density as

$$f(t; a, b) = \frac{t^{\frac{1}{a}-1}}{ac^{1/a}} \exp(-(t/c)^{1/a})$$

and let

$$X = \log T$$

$$c^1 = \log c$$

where T is the Weibull random variable then X is distributed as the extreme value distribution with density function

$$h(x; a, c^1) = \frac{1}{a} \exp\left(\frac{x-c^1}{a}\right) \exp\left(-\exp\left(\frac{x-c^1}{a}\right)\right)$$

It is seen that the parameters of h are of the location and scale types. Lieblein and Zelen (26) derive weights for the order statistics which give best linear unbiased estimates of the parameters of the extreme value distribution for both censored and noncensored samples up to size six. White (47) uses Lloyd's least square procedure to derive estimators

$$\hat{a} = \sum_{i=1}^{n-r} a_i x_{(i)}$$

and

$$\hat{c}^1 = \sum_{i=1}^{n-r} b_i x_{(i)}$$

for censored samples with $r=0, \dots, n-1$ and gives the coefficients for samples up to size 20. Tables of the expected values and covariances of these estimators are given. Downton (8) also uses the logarithmic transformation of the Weibull and notes that

$$E(\log T) = \hat{c}^1 - \delta a$$

and

$$V(\log T) = \frac{\pi^2 a^2}{6}$$

where δ is Euler's constant. He further suggests an estimate based on the original data. He defines random variables V and W with realizations v and w such that

$$v = \sum_{i=1}^n t_{(i)}$$

and

$$w = \sum_{i=1}^n t_{(i)}$$

where $t_{(i)}$ are the order statistics of the Weibull. He arrives at unbiased estimates

$$\hat{a} = - \frac{n+1}{n(n-1) \log 2} v + \frac{2}{n(n-1) \log 2} w$$

and

$$\hat{c}_1 = \frac{(n-1) \log 2 - (n+1)}{n(n-1) \log 2} \delta v + \frac{2 \delta}{n(n-1) \log 2} w$$

and computes the covariances of the estimates. Downton was not able to show asymptotic normality of his estimates.

Finally, as a single source, Mann (30) provides a short discourse on most methods of estimation of the parameters of the Weibull and a very complete bibliography.

CHAPTER II. WEIBULL ORDER STATISTICS

A. Expected Moments of Weibull Order Statistics

Consider the Weibull density of the form of Equation

1.11

$$f(y;a,b) = \frac{1}{ab} y^{\frac{1}{a}-1} \exp(-y^{1/a}/b), \quad y > 0, \quad b > 0, \quad a > 0$$

$$= 0 \quad \text{otherwise}.$$
(2.1)

The transformation $X = Y^{1/a}$ is one to one, order preserving, and the random variable X is distributed as an exponential with parameter b . Then if $Y_{(1)}$ are the order statistics drawn from the distribution of Equation 2.1, $X_{(1)} = Y_{(1)}^{1/a}$. We have previously shown in the literature review that

$$Z_k = (n-k+1)(X_{(k)} - X_{(k-1)})$$
(2.2)

$$k = 1, 2, \dots, n$$

are independent and identically distributed as exponential with parameter b . Furthermore, Z_1 is independent of $X_{(j)}$ if i is greater than j . Now the moment generating function for the exponential is given by

$$\text{mgf}_{Z_1}(t) = (1-tb)^{-1}.$$
(2.3)

The m -th moment of the exponential may be found by taking derivatives with respect to t .

$$E(Z_1^m) = \Gamma(m+1)b^m \quad (2.4)$$

where Γ is the Gamma function. Suppose we set b equal to unity and rewrite Equation 2.2 as

$$X_{(k)} = X_{(k-1)} + Z_k/(n-k+1) \quad (2.5)$$

From Equation 2.5 since $X_{(0)} = 0$

$$E(X_{(1)}^m) = E((Z_1/n)^m) \quad (2.6)$$

Now evaluating the right-hand side of Equation 2.6 using Equation 2.4 we have for m a positive integer.

$$E(X_{(1)}^m) = m! \left(\frac{1}{n}\right)^m \quad (2.7)$$

Continuing and using the binomial theorem we have

$$\begin{aligned} E(X_{(2)}^m) &= E((X_{(1)} + Z_2/(n-1))^m) \\ &= E\left(\sum_{i=0}^m \binom{m}{i} X_{(1)}^i (Z_2/(n-1))^{m-i}\right) \end{aligned}$$

but $X_{(1)}$ and Z_2 are independent so we have

$$E(X_{(2)}^m) = \sum_{i=0}^m \binom{m}{i} E(X_{(1)}^i) E((Z_2/(n-1))^{m-i})$$

Now using Equations 2.6 and 2.7 we have

$$E(X_{(2)}^m) = m! \sum_{i=0}^m \left(\frac{1}{n}\right)^i \left(\frac{1}{n-1}\right)^{m-i} \quad (2.8)$$

Assume that

$$E(X_{(k)}^m) = m! \sum_{i_1=0}^m \sum_{i_2=0}^{m-i_1} \dots \sum_{i_{k-1}=0}^{m-i_1-i_2-\dots-i_{k-2}} \left(\frac{1}{n} \right)^{i_1} \left(\frac{1}{n-1} \right)^{i_2} \dots \left(\frac{1}{n-k+2} \right)^{i_{k-1}} \left(\frac{1}{n-k+1} \right)^{m-\sum_{j=1}^{k-1} i_j} \quad (2.9)$$

Let $S(n, k, m)$ be the sum on the right hand side of 2.9. Hence $E(X_{(k)}^m) = m! S(n, k, m)$. Now, using the assumption of 2.9, we find the expected moments of the $(k+1)$ -st order statistic.

$$E(X_{(k+1)}^m) = E\left(X_{(k)} + \frac{Z_{k+1}}{n-k}\right)^m \quad (2.10)$$

$$= \sum_{i=0}^m \binom{m}{i} E\left(\left(\frac{Z_{k+1}}{n-k}\right)^i\right) E(X_{(k)}^{m-i})$$

$$= \sum_{i=0}^m \frac{m!}{i!(m-i)!} \left(\frac{1}{n-k}\right)^i E(X_{(k)}^{m-i})$$

$$E(X_{(k+1)}^m) = m! \sum_{i=0}^m \left(\frac{1}{n-k}\right)^i E(X_{(k)}^{m-i}) / (m-i)! \quad (2.10a)$$

$$= m! \sum_{i=0}^m \frac{1}{(m-i)!} \left(\frac{1}{n-k}\right)^i (m-i)! S(n, k, m-i)$$

$$= m! \sum_{i=0}^m \left(\frac{1}{n-k}\right)^i S(n, k, m-i)$$

$$= m! S(n, k+1, m)$$

which is the desired result. Hence Equation 2.9 is the general formula for computing a particular expected value.

For evaluation of an entire table for $k = 1, 2, \dots, n$,

it is computationally more efficient to use the relationship from Equation 2.10a

$$E(X_{(k)}^m) = m! \sum_{i=0}^m E(X_{(k-1)}^i) \left(\frac{1}{n-k+1} \right)^{m-i} / i! \quad (2.11)$$

Equation 2.11 has been evaluated for $n = 1, 2, \dots, 25$; $k = 1, 2, \dots, n$; and $m = 1, 2, \dots, 10$ by McElhone and Larsen in (28) and a portion of that table appears here as Table 1. The table is used in the following manner. Recall that if Y is a Weibull random variable with shape parameter a ,

$$E(Y_{(k)}^a) = E(X_{(k)}^{aq}) \quad (2.12)$$

where X is an exponential random variable. Hence the value of $E(Y_{(k)}^a)$ is found in the aq -th column of the table. Because of space restrictions, the sixth through the tenth columns appear directly below the first through fifth columns. For example, the value of the second moment of the third order statistic of a Weibull with shape parameter two from a sample of size five is 1.7871. If the shape parameter were five in the above case the value would be $5.6431E+02 = 564.31$.

Relaxing the condition that b be unity, it is seen that for X 's distributed exponentially with parameter b , the moments are given by

$$E(X_{(k)}^m) = m! b^m S(n, k, m). \quad (2.13)$$

Hence the values of the q -th moment of a Weibull with shape

TABLE 1.
Moments of the K-th order statistic of a sample of
size N from the exponential distribution

M E(X _(K))					
N= 2					
K	M=1/6	2/7	3/8	4/9	5/10
1	5.0000E-01	5.0000E-01	7.5000E-01	1.5000E+00	3.7500E+00
	1.1250E+01	3.9375E+01	1.5750E+02	7.0875E+02	3.5438E+03
2	1.5000E+00	3.5000E+00	1.1250E+01	4.6500E+01	2.3625E+02
	1.4288E+03	1.0041E+04	8.0482E+04	7.2505E+05	7.2539E+06
N= 3					
K	M=1/6	2/7	3/8	4/9	5/10
1	3.3333E-01	2.2222E-01	2.2222E-01	2.9630E-01	4.9383E-01
	9.8765E-01	2.3045E+00	6.1454E+00	1.8436E+01	6.1454E+01
2	8.3333E-01	1.0556E+00	1.8055E+00	3.9074E+00	1.0262E+01
	3.1775E+01	1.1352E+02	4.6021E+02	2.0893E+03	1.0506E+04
3	1.8333E+00	4.7222E+00	1.5972E+01	6.7796E+01	3.4924E+02
	2.1272E+03	1.5004E+04	1.2049E+05	1.0865E+06	1.0875E+07
N= 4					
K	M=1/6	2/7	3/8	4/9	5/10
1	2.5000E-01	1.2500E-01	9.3750E-02	9.3750E-02	1.1719E-01
	1.7578E-01	3.0762E-01	6.1523E-01	1.3843E+00	3.4607E+00
2	5.8333E-01	5.1389E-01	6.0764E-01	9.0393E-01	1.6237E+00
	3.4233E+00	8.2952E+00	2.2736E+01	6.9584E+01	2.3532E+02
3	1.0833E+00	1.5972E+00	3.0035E+00	6.9109E+00	1.8901E+01
	6.0126E+01	2.1874E+02	8.9767E+02	4.1088E+03	2.0773E+04
4	2.0833E+00	5.7639E+00	2.0295E+01	8.8091E+01	4.5936E+02
	2.8163E+03	1.9933E+04	1.6036E+05	1.4473E+06	1.4493E+07
N= 5					
K	M=1/6	2/7	3/8	4/9	5/10
1	2.0000E-01	8.0000E-02	4.8000E-02	3.8400E-02	3.8400E-02
	4.6080E-02	6.4512E-02	1.0322E-01	1.8579E-01	3.7159E-01
2	4.5000E-01	3.0500E-01	2.7675E-01	3.1515E-01	4.3234E-01
	6.9458E-01	1.2800E+00	2.6633E+00	6.1772E+00	1.5805E+01
3	7.8333E-01	8.2722E-01	1.1040E+00	1.7871E+00	3.4108E+00
	7.5163E+00	1.8818E+01	5.2844E+01	1.6468E+02	5.6431E+02
4	1.2833E+00	2.1105E+00	4.2698E+00	1.0327E+01	2.9228E+01
	9.5199E+01	3.5201E+02	1.4609E+03	6.7378E+03	3.4238E+04
5	2.2833E+00	6.6772E+00	2.4301E+01	1.0753E+02	5.6689E+02
	3.4965E+03	2.4828E+04	2.0008E+05	1.8075E+06	1.8108E+07
N= 6					
K	M=1/6	2/7	3/8	4/9	5/10
1	1.6667E-01	5.5555E-02	2.7778E-02	1.8518E-02	1.5432E-02
	1.5432E-02	1.8004E-02	2.4005E-02	3.6008E-02	6.0013E-02

TABLE 1. (CONTINUED)

2	3.6667E-01 1.9932E-01	2.0222E-01 2.9705E-01	1.4911E-01 4.9928E-01	1.3781E-01 9.3453E-01	1.5324E-01 1.9275E+00
3	6.1667E-01 1.6851E+00	5.1055E-01 3.2460E+00	5.3203E-01 6.9911E+00	6.6983E-01 1.6660E+01	9.9053E-01 4.3526E+01
4	9.5000E-01 1.3347E+01	1.1439E+00 3.4390E+01	1.6759E+00 9.8696E+01	2.9044E+00 3.1266E+02	5.8312E+00 1.0846E+03
5	1.4500E+00 1.3613E+02	2.5939E+00 5.1082E+02	5.5667E+00 2.1420E+03	1.4038E+01 9.9500E+03	4.0926E+01 5.0806E+04
6	2.4500E+00 4.1686E+03	7.4939E+00 2.9691E+04	2.8048E+01 2.3967E+05	1.2623E+02 2.1669E+06	6.7208E+02 2.1719E+07
N= 7					
K	M=1/6	2/7	3/8	4/9	5/10
1	1.4286E-01 6.1199E-03	4.0816E-02 6.1199E-03	1.7493E-02 6.9941E-03	9.9958E-03 8.9925E-03	7.1399E-03 1.2846E-02
2	3.0952E-01 7.1305E-02	1.4399E-01 8.9309E-02	8.9488E-02 1.2607E-01	6.9654E-02 1.9805E-01	6.5185E-02 3.4259E-01
3	5.0952E-01 5.1935E-01	3.4780E-01 8.1640E-01	2.9817E-01 1.4323E+00	3.0819E-01 2.7752E+00	3.7337E-01 5.8838E+00
4	7.5952E-01 3.2395E+00	7.2756E-01 6.4854E+00	8.4384E-01 1.4403E+01	1.1520E+00 3.5167E+01	1.8134E+00 9.3645E+01
5	1.0929E+00 2.0928E+01	1.4561E+00 5.5318E+01	2.3000E+00 1.6191E+02	4.2186E+00 5.2073E+02	8.8445E+00 1.8269E+03
6	1.5929E+00 1.8220E+02	3.0490E+00 6.9303E+02	6.8734E+00 2.9340E+03	1.7965E+01 1.3721E+04	5.3758E+01 7.0383E+04
7	2.5929E+00 4.8330E+03	8.2347E+00 3.4524E+04	3.1577E+01 2.7913E+05	1.4428E+02 2.5258E+06	7.7514E+02 2.5326E+07
N= 8					
K	M=1/6	2/7	3/8	4/9	5/10
1	1.2500E-01 2.7466E-03	3.1250E-02 2.4033E-03	1.1719E-02 2.4033E-03	5.8594E-03 2.7037E-03	3.6621E-03 3.3796E-03
2	2.6786E-01 2.9733E-02	1.0778E-01 3.2136E-02	5.7910E-02 3.9130E-02	3.8951E-02 5.3000E-02	3.1484E-02 7.9003E-02
3	4.3452E-01 1.9602E-01	2.5262E-01 2.6083E-01	1.8422E-01 3.8689E-01	1.6176E-01 6.3306E-01	1.6629E-01 1.1320E+00
4	6.3452E-01 1.0582E+00	5.0643E-01 1.7423E+00	4.8808E-01 3.1746E+00	5.5223E-01 6.3441E+00	7.1852E-01 1.3790E+01
5	8.8452E-01 5.4207E+00	9.4869E-01 1.1228E+01	1.1996E+00 2.5631E+01	1.7518E+00 6.3981E+01	2.9083E+00 1.7338E+02
6	1.2179E+00 3.0233E+01	1.7606E+00 8.1772E+01	2.9602E+00 2.4368E+02	5.6987E+00 7.9471E+02	1.2406E+01 2.8178E+03
7	1.7179E+00 2.3286E+02	3.4784E+00 8.9678E+02	8.1779E+00 3.8307E+03	2.2054E+01 1.8029E+04	6.7542E+01 9.2888E+04
8	2.7179E+00 5.4902E+03	8.9142E+00 3.9328E+04	3.4920E+01 3.1845E+05	1.6174E+02 2.8840E+06	8.7622E+02 2.8931E+07

TABLE 1. (CONTINUED)

N= 9					
K	M=1/6	2/7	3/8	4/9	5/10
1	1.1111E-01	2.4691E-02	8.2304E-03	3.6580E-03	2.0322E-03
	1.3548E-03	1.0537E-03	9.3665E-04	9.3665E-04	1.0407E-03
2	2.3611E-01	8.3719E-02	3.9625E-02	2.3470E-02	1.6701E-02
	1.3881E-02	1.3199E-02	1.4136E-02	1.6835E-02	2.2056E-02
3	3.7897E-01	1.9200E-01	1.2191E-01	9.3133E-02	8.3224E-02
	8.5216E-02	9.8415E-02	1.2661E-01	1.7953E-01	2.7794E-01
4	5.4563E-01	3.7387E-01	3.0885E-01	2.9903E-01	3.3242E-01
	4.1763E-01	5.8565E-01	9.0744E-01	1.5398E+00	2.8368E+00
5	7.4563E-01	6.7213E-01	7.1212E-01	8.6873E-01	1.2011E+00
	1.8590E+00	3.1882E+00	6.0084E+00	1.2347E+01	2.7455E+01
6	9.9563E-01	1.1699E+00	1.5896E+00	2.4583E+00	4.2740E+00
	8.2700E+00	1.7661E+01	4.1328E+01	1.0527E+02	2.8991E+02
7	1.3290E+00	2.0559E+00	3.6455E+00	7.3190E+00	1.6472E+01
	4.1214E+01	1.1383E+02	3.4486E+02	1.1393E+03	4.0799E+03
8	1.8290E+00	3.8849E+00	9.4728E+00	2.6265E+01	8.2134E+01
	2.8761E+02	1.1205E+03	4.8267E+03	2.2854E+04	1.1824E+05
9	2.8290E+00	9.5428E+00	3.8101E+01	1.7867E+02	9.7548E+02
	6.1405E+03	4.4104E+04	3.5766E+05	3.2417E+06	3.2532E+07
N=10					
K	M=1/6	2/7	3/8	4/9	5/10
1	1.0000E-01	2.0000E-02	6.0000E-03	2.4000E-03	1.2000E-03
	7.2000E-04	5.0400E-04	4.0320E-04	3.6288E-04	3.6288E-04
2	2.1111E-01	6.6913E-02	2.8304E-02	1.4980E-02	9.5221E-03
	7.0680E-03	6.0013E-03	5.7376E-03	6.0987E-03	7.1295E-03
3	3.3611E-01	1.5094E-01	8.4907E-02	5.7433E-02	4.5418E-02
	4.1131E-02	4.1991E-02	4.7727E-02	5.9761E-02	8.1641E-02
4	4.7897E-01	2.8779E-01	2.0825E-01	1.7643E-01	1.7144E-01
	1.8808E-01	2.3007E-01	3.1065E-01	4.5887E-01	7.3499E-01
5	6.4563E-01	5.0300E-01	4.5975E-01	4.8293E-01	5.7388E-01
	7.6196E-01	1.1190E+00	1.8026E+00	3.1604E+00	5.9828E+00
6	8.4563E-01	8.4125E-01	9.6450E-01	1.2545E+00	1.8284E+00
	2.9560E+00	5.2574E+00	1.0214E+01	2.1530E+01	4.8883E+01
7	1.0956E+00	1.3891E+00	2.0063E+00	3.2608E+00	5.9044E+00
	1.1813E+01	2.5929E+01	6.2070E+01	1.6108E+02	4.5028E+02
8	1.4290E+00	2.3417E+00	4.3480E+00	9.0582E+00	2.1001E+01
	5.3815E+01	1.5150E+02	4.6605E+02	1.5584E+03	5.6330E+03
9	1.9290E+00	4.2707E+00	1.0754E+01	3.0566E+01	9.7417E+01
	3.4606E+02	1.3627E+03	5.9168E+03	2.8176E+04	1.4636E+05
10	2.9290E+00	1.0129E+01	4.1140E+01	1.9513E+02	1.0730E+03
	6.7843E+03	4.8853E+04	3.9674E+05	3.5987E+06	3.6129E+07

parameter a and scale parameter b is found by multiplying the associated unit scale tabular value by b^{aq} .

The closed form for the expected values of exponential order statistics is known and is given in Equation 4.5. Our computed values for the exponential are in the first column of Table 1. The tabular values agree to the fifth significant digit with the values calculated from Equation 4.5.

From Equations 2.12 and 2.13

$$E(Y_{(k)}) = a! b^a S(n, k, a) \quad (2.14)$$

$$E(Y_{(k)}^2) = (2a)! b^{2a} S(n, k, 2a). \quad (2.15)$$

The variance is then

$$V(Y_{(k)}) = b^{2a} ((2a)! S(n, k, 2a) - (a!)^2 S^2(n, k, a)). \quad (2.16)$$

In terms of the exponential variates we have

$$V(Y_{(k)}) = E(X_{(k)}^{2a}) - E^2(X_{(k)}^a).$$

Now from Equation 1.10 unordered Weibull variates have expectation

$$E(Y^m) = (am)! b^{am} \quad (2.17)$$

so that

$$E(Y_{(k)}^m) = E(Y^m) S(n, k, am) \quad (2.18)$$

B. Expected Crossproducts of
Weibull Order Statistics

From Equation 2.5 we write, for $j \leq k$

$$\begin{aligned} X_{(k)} &= X_{(k-1)} + \frac{Z_k}{n-k+1} \\ X_{(k)} &= X_{(k-2)} + \frac{Z_{k-1}}{n-k+2} + \frac{Z_k}{n-k+1} \\ &\vdots \\ X_{(k)} &= X_{(k-j)} + \sum_{i=1}^j \frac{Z_{k-i+1}}{n-k+i} . \end{aligned}$$

Let $W_{k,j} = \sum_{i=1}^j Z_{k-i+1} / (n-k+i)$. Then

$$X_{(k)} = X_{(k-j)} + W_{k,j} \quad (2.19)$$

$$W_{k,j} = W_{k,j-1} + \frac{Z_{k-j+1}}{n-k+j} \quad (2.20)$$

$$W_{k,0} = 0 .$$

Taking the expected values of the m -th power of Equation 2.20 using the binomial theorem

$$E(W_{k,j}^m) = m! \sum_{i=0}^m E(W_{k,j-1}^i) \left(\frac{1}{n-k+j} \right)^{m-i} / i! \quad (2.21)$$

We are now able to generate the moments of the crossproducts by using Equation 2.19. In fact the computational formula is (2.22).

$$\begin{aligned}
E((X_{(k-j)}X_{(k)})^m) &= E((X_{(k-j)}(X_{(k-j)} + w_{k,j}))^m) \quad (2.22) \\
&= E((X_{(k-j)}^2 + X_{(k-j)}w_{k,j})^m) \\
&= \sum_{i=0}^m \binom{m}{i} E(X_{(k-j)}^{m+1-i})E(w_{k,j}^{i-1}) .
\end{aligned}$$

The terms in the summation are precalculated from Equations 2.9 and 2.21. For b not unity, multiply the right-hand side of Equation 2.22 by b^m . From the above and Equation 2.9, we have the covariances as

$$\text{Cov}(X_{(k)}^p, X_{(j)}^p) = E((X_{(k)}X_{(j)})^p) - E(X_{(k)}^p)E(X_{(j)}^p) . \quad (2.23)$$

Equation 2.23 has been evaluated for the same values of n and k as the expected value table. The entire table is presented in (28) and a portion of it appears here as Table 2. The covariance of a Weibull variate with shape parameter a is found in column a of the table. For example, if a equals three then $C(Y_{(1)}, Y_{(3)})$ for a sample of size five is 0.14866.

The closed form for the covariances of exponential order statistics is known and is given in Equation 4.6. Our computed values for the exponential are in the first column of Table 2. When compared to the exact results, our results show some small error in the fifth digit. The

TABLE 2.

Covariances of the K-th and J-th order statistics of
a sample of size N from the Weibull distribution
with shape parameter M

$$C(Y_{(K)}, Y_{(J)})$$

			Shape parameter				
N	K	J	M=1	2	3	4	5
2	1	1	2.5000E-01	1.2500E+00	1.0688E+01	1.5525E+02	3.5297E+03
2	1	2	2.5000E-01	2.2500E+00	2.7563E+01	5.0625E+02	1.3514E+04
2	2	2	1.2500E+00	3.4250E+01	1.3022E+03	7.8320E+04	7.1981E+06
3	1	1	1.1111E-01	2.4691E-01	9.3827E-01	6.0576E+00	6.1210E+01
3	1	2	1.1111E-01	3.9506E-01	1.9383E+00	1.4486E+01	1.5975E+02
3	1	3	1.1111E-01	6.9136E-01	5.2716E+00	5.6625E+01	8.4790E+02
3	2	2	3.6111E-01	2.7932E+00	2.8515E+01	4.4494E+02	1.0401E+04
3	2	3	3.6112E-01	4.6450E+00	6.8001E+01	1.3707E+03	3.8428E+04
3	3	3	1.3611E+00	4.5497E+01	1.8721E+03	1.1590E+05	1.0753E+07
4	1	1	6.2500E-02	7.8125E-02	1.6699E-01	6.0644E-01	3.4470E+00
4	1	2	6.2500E-02	1.1979E-01	3.1934E-01	1.3070E+00	7.9466E+00
4	1	3	6.2501E-02	1.8229E-01	6.5332E-01	3.3187E+00	2.3699E+01
4	1	4	6.2501E-02	3.0729E-01	1.7432E+00	1.3436E+01	1.4102E+02
4	2	2	1.7361E-01	6.3985E-01	3.0540E+00	2.1919E+01	2.3268E+02
4	2	3	1.7361E-01	9.4772E-01	5.8455E+00	4.9576E+01	5.8624E+02
4	2	4	1.7361E-01	1.5635E+00	1.4725E+01	1.7987E+02	2.9198E+03
4	3	3	4.2362E-01	4.3598E+00	5.1105E+01	8.4991E+02	2.0416E+04
4	3	4	4.2362E-01	6.9060E+00	1.1536E+02	2.5048E+03	7.3195E+04
4	4	4	1.4236E+00	5.4869E+01	2.4044E+03	1.5260E+05	1.4282E+07
5	1	1	4.0000E-02	3.2000E-02	4.3776E-02	1.0174E-01	3.7011E-01
5	1	2	4.0000E-02	4.8000E-02	8.0496E-02	2.0819E-01	8.0283E-01
5	1	3	4.0000E-02	6.9333E-02	1.4866E-01	4.5524E-01	1.9756E+00
5	1	4	4.0001E-02	1.0133E-01	2.9409E-01	1.1545E+00	6.1599E+00
5	1	5	4.0001E-02	1.6533E-01	7.5777E-01	4.6052E+00	3.7516E+01
5	2	2	1.0250E-01	2.2212E-01	6.1799E-01	2.5639E+00	1.5618E+01
5	2	3	1.0250E-01	3.1512E-01	1.0930E+00	5.2105E+00	3.4631E+01
5	2	4	1.0250E-01	4.5463E-01	2.0914E+00	1.2428E+01	9.8137E+01
5	2	5	1.0250E-01	7.3362E-01	5.2319E+00	4.6828E+01	5.4400E+02
5	3	3	2.1361E-01	1.1028E+00	6.2975E+00	4.9650E+01	5.5267E+02
5	3	4	2.1361E-01	1.5588E+00	1.1427E+01	1.0749E+02	1.3508E+03
5	3	5	2.1361E-01	2.4707E+00	2.7221E+01	3.6801E+02	6.3857E+03
5	4	4	4.6361E-01	5.8723E+00	7.6968E+01	1.3543E+03	3.3384E+04
5	4	5	4.6361E-01	8.9948E+00	1.6670E+02	3.8502E+03	1.1663E+05
5	5	5	1.4636E+00	6.2948E+01	2.9060E+03	1.8852E+05	1.7786E+07
6	1	1	2.7778E-02	1.5432E-02	1.4660E-02	2.3662E-02	5.9775E-02
6	1	2	2.7778E-02	2.2839E-02	2.6327E-02	4.6938E-02	1.2506E-01

TABLE 2. (CONTINUED)

6	1	3	2.7778E-02	3.2099E-02	4.6119E-02	9.5168E-02	2.7978E-01
6	1	4	2.7778E-02	4.4444E-02	8.1767E-02	2.0447E-01	6.9892E-01
6	1	5	2.7778E-02	6.2962E-02	1.5607E-01	5.0669E-01	2.1827E+00
6	1	6	2.7778E-02	9.9999E-02	3.8801E-01	1.9605E+00	1.3123E+01
6	2	2	6.7777E-02	9.6913E-02	1.77C8E-01	4.8029E-01	1.9040E+00
6	2	3	6.7777E-02	1.3439E-01	3.0057E-01	9.2389E-01	3.9578E+00
6	2	4	6.7777E-02	1.8437E-01	5.2065E-01	1.9037E+00	9.2800E+00
6	2	5	6.7777E-02	2.5933E-01	9.7546E-01	4.5555E+00	2.7398E+01
6	2	6	6.7777E-02	4.0926E-01	2.3839E+00	1.7052E+01	1.5592E+02
6	3	3	1.3028E-01	4.0917E-01	1.4021E+00	6.5425E+00	4.2545E+01
6	3	4	1.3028E-01	5.5396E-01	2.3488E+00	1.2699E+01	9.1251E+C1
6	3	5	1.3028E-01	7.7114E-01	4.2815E+00	2.8851E+01	2.4789E+02
6	3	6	1.3028E-01	1.2055E+00	1.0197E+01	1.0268E+02	1.2963E+03
6	4	4	2.4139E-01	1.5959E+00	1.0539E+01	9.0260E+01	1.0506E+03
6	4	5	2.4139E-01	2.1851E+00	1.8378E+01	1.8860E+02	2.5009E+03
6	4	6	2.4139E-01	3.3635E+00	4.1931E+01	6.1616E+02	1.1315E+04
6	5	5	4.9139E-01	7.3096E+00	1.0514E+02	1.9449E+03	4.9131E+04
6	5	6	4.9139E-01	1.0921E+01	2.2039E+02	5.3659E+03	1.6779E+05
6	6	6	1.4914E+00	7.0073E+01	3.3819E+03	2.2374E+05	2.1267E+07
7	1	1	2.0408E-02	8.3299E-03	5.8139E-03	6.8942E-03	1.2795E-02
7	1	2	2.0408E-02	1.2217E-02	1.0276E-02	1.3396E-02	2.6137E-02
7	1	3	2.0408E-02	1.6882E-02	1.7430E-02	2.5951E-02	5.5247E-02
7	1	4	2.0408E-02	2.2713E-02	2.9184E-02	5.1211E-02	1.2346E-01
7	1	5	2.0408E-02	3.0487E-02	4.9854E-02	1.0698E-01	3.0549E-01
7	1	6	2.0408E-02	4.2149E-02	9.2104E-02	2.5746E-01	9.3806E-01
7	1	7	2.0408E-02	6.5472E-02	2.2158E-01	9.6281E-01	5.4904E+00
7	2	2	4.8186E-02	4.8921E-02	6.3297E-02	1.2122E-01	3.3834E-01
7	2	3	4.8186E-02	6.6889E-02	1.0475E-01	2.2547E-01	6.7622E-01
7	2	4	4.8186E-02	8.9349E-02	1.7229E-01	4.3118E-01	1.4422E+00
7	2	5	4.8185E-02	1.1929E-01	2.9032E-01	8.7830E-01	3.4301E+00
7	2	6	4.8185E-02	1.6421E-01	5.3030E-01	2.0687E+00	1.0170E+01
7	2	7	4.8185E-02	2.5405E-01	1.2620E+00	7.5755E+00	5.7488E+01
7	3	3	8.8185E-02	1.8722E-01	4.3045E-01	1.3373E+00	5.7443E+00
7	3	4	8.8185E-02	2.4770E-01	6.9130E-01	2.4521E+00	1.1518E+01
7	3	5	8.8184E-02	3.2834E-01	1.1433E+00	4.8245E+00	2.5950E+C1
7	3	6	8.8184E-02	4.4929E-01	2.0556E+00	1.1026E+01	7.3244E+01
7	3	7	8.8183E-02	6.9120E-01	4.8180E+00	3.9192E+01	3.9388E+02
7	4	4	1.5068E-01	6.2268E-01	2.5274E+00	1.3076E+01	9.0357E+01
7	4	5	1.5068E-01	8.1685E-01	4.0676E+00	2.4463E+01	1.8833E+02
7	4	6	1.5068E-01	1.1081E+00	7.1446E+00	5.3439E+01	4.9372E+02
7	4	7	1.5068E-01	1.6906E+00	1.6365E+01	1.8150E+02	2.4578E+03
7	5	5	2.6179E-01	2.0983E+00	1.5638E+01	1.4412E+02	1.7487E+03
7	5	6	2.6179E-01	2.8070E+00	2.6439E+01	2.9241E+02	4.0679E+03
7	5	7	2.6179E-01	4.2242E+00	5.8272E+01	9.1884E+02	1.7722E+04

TABLE 2. (CONTINUED)

7	6	6	5.1179E-01	8.6692E+00	1.3496E+02	2.6112E+03	6.7493E+04
7	6	7	5.1179E-01	1.2703E+01	2.7546E+02	7.0219E+03	2.2590E+05
7	7	7	1.5118E+00	7.6466E+01	3.8359E+03	2.5831E+05	2.4726E+07
8	1	1	1.5625E-02	4.8828E-03	2.6092E-03	2.3689E-03	3.3662E-03
8	1	2	1.5625E-02	7.1150E-03	4.5599E-03	4.5358E-03	6.7602E-03
8	1	3	1.5625E-02	9.7191E-03	7.5680E-03	8.5228E-03	1.3761E-02
8	1	4	1.5625E-02	1.2844E-02	1.2232E-02	1.5971E-02	2.8760E-02
8	1	5	1.5625E-02	1.6750E-02	1.9711E-02	3.0541E-02	6.3223E-02
8	1	6	1.5625E-02	2.1959E-02	3.2612E-02	6.1924E-02	1.5331E-01
8	1	7	1.5624E-02	2.9771E-02	5.8555E-02	1.4469E-01	4.5961E-01
8	1	8	1.5624E-02	4.5396E-02	1.3681E-01	5.2328E-01	2.6063E+00
8	2	2	3.6034E-02	2.7334E-02	2.6379E-02	3.7613E-02	7.8012E-02
8	2	3	3.6034E-02	3.7014E-02	4.2907E-02	6.8384E-02	1.5184E-01
8	2	4	3.6033E-02	4.8631E-02	6.8366E-02	1.2497E-01	3.0591E-01
8	2	5	3.6033E-02	6.3151E-02	1.0898E-01	2.3434E-01	6.5237E-01
8	2	6	3.6033E-02	8.2511E-02	1.7876E-01	4.6753E-01	1.5417E+00
8	2	7	3.6033E-02	1.1155E-01	3.1858E-01	1.0771E+00	4.5161E+00
8	2	8	3.6033E-02	1.6963E-01	7.3886E-01	3.8403E+00	2.5013E+01
8	3	3	6.3811E-02	9.7947E-02	1.6208E-01	3.6072E-01	1.1043E+00
8	3	4	6.3811E-02	1.2773E-01	2.5354E-01	6.3845E-01	2.1243E+00
8	3	5	6.3811E-02	1.6495E-01	3.9852E-01	1.1668E+00	4.3554E+00
8	3	6	6.3810E-02	2.1459E-01	6.4629E-01	2.2786E+00	9.9471E+00
8	3	7	6.3810E-02	2.8904E-01	1.1405E+00	5.1510E+00	2.8246E+01
8	3	8	6.3809E-02	4.3794E-01	2.6193E+00	1.8018E+01	1.5153E+02
8	4	4	1.0381E-01	2.9576E-01	8.2002E-01	2.8696E+00	1.3274E+01
8	4	5	1.0381E-01	3.7912E-01	1.2645E+00	5.0644E+00	2.5812E+01
8	4	6	1.0381E-01	4.9028E-01	2.0188E+00	9.6015E+00	5.6230E+01
8	4	7	1.0381E-01	6.5702E-01	3.5140E+00	2.1136E+01	1.5281E+02
8	4	8	1.0381E-01	9.9049E-01	7.9596E+00	7.1952E+01	7.8305E+02
8	5	5	1.6631E-01	8.5181E-01	3.9816E+00	2.2562E+01	1.6492E+02
8	5	6	1.6631E-01	1.0921E+00	6.2124E+00	4.0951E+01	3.3530E+02
8	5	7	1.6631E-01	1.4526E+00	1.0595E+01	8.6604E+01	8.5256E+02
8	5	8	1.6631E-01	2.1735E+00	2.3504E+01	2.8280E+02	4.0685E+03
8	6	6	2.7742E-01	2.5990E+00	2.1470E+01	2.1121E+02	2.6639E+03
8	6	7	2.7742E-01	3.4151E+00	3.5402E+01	4.1801E+02	6.0716E+03
8	6	8	2.7742E-01	5.0471E+00	7.5829E+01	1.2708E+03	2.5596E+04
8	7	7	5.2741E-01	9.9549E+00	1.6598E+02	3.3443E+03	8.8326E+04
8	7	8	5.2741E-01	1.4360E+01	3.3131E+02	8.7955E+03	2.9032E+05
8	8	8	1.5274E+00	8.2274E+01	4.2708E+03	2.9230E+05	2.8163E+07
9	1	1	1.2346E-02	3.0483E-03	1.2871E-03	9.2327E-04	1.0366E-03
9	1	2	1.2346E-02	4.4200E-03	2.2301E-03	1.7484E-03	2.0555E-03
9	1	3	1.2346E-02	5.9877E-03	3.6439E-03	3.2143E-03	4.0736E-03
9	1	4	1.2346E-02	7.8167E-03	5.7505E-03	5.8170E-03	8.1413E-03
9	1	5	1.2346E-02	1.0011E-02	8.9368E-03	1.0538E-02	1.6659E-02

TABLE 2. (CONTINUED)

9	1	6	1.2346E-02	1.2755E-02	1.3949E-02	1.9544E-02	3.5801E-02
9	1	7	1.2345E-02	1.6413E-02	2.2460E-02	3.8517E-02	8.4768E-02
9	1	8	1.2345E-02	2.1900E-02	3.9342E-02	8.7529E-02	2.4758E-01
9	1	9	1.2345E-02	3.2873E-02	8.9566E-02	3.0677E-01	1.3591E+00
9	2	2	2.7971E-02	1.6462E-02	1.2310E-02	1.3585E-02	2.1777E-02
9	2	3	2.7971E-02	2.2135E-02	1.9775E-02	2.4295E-02	4.1583E-02
9	2	4	2.7971E-02	2.8755E-02	3.0835E-02	4.3065E-02	8.0647E-02
9	2	5	2.7971E-02	3.6698E-02	4.7496E-02	7.6767E-02	1.6103E-01
9	2	6	2.7971E-02	4.6627E-02	7.3614E-02	1.4054E-01	3.3907E-01
9	2	7	2.7970E-02	5.9866E-02	1.1785E-01	2.7399E-01	7.8886E-01
9	2	8	2.7970E-02	7.9723E-02	2.0537E-01	6.1656E-01	2.2673E+00
9	2	9	2.7970E-02	1.1944E-01	4.6510E-01	2.1391E+00	1.2239E+01
9	3	3	4.8379E-02	5.6270E-02	7.0354E-02	1.1793E-01	2.7101E-01
9	3	4	4.8379E-02	7.2653E-02	1.0808E-01	2.0374E-01	5.0673E-01
9	3	5	4.8379E-02	9.2313E-02	1.6463E-01	3.5591E-01	9.8136E-01
9	3	6	4.8379E-02	1.1689E-01	2.5290E-01	6.4095E-01	2.0136E+00
9	3	7	4.8378E-02	1.4965E-01	4.0189E-01	1.2322E+00	4.5802E+00
9	3	8	4.8378E-02	1.9880E-01	6.9578E-01	2.7381E+00	1.2894E+01
9	3	9	4.8378E-02	2.9709E-01	1.5651E+00	9.3761E+00	6.8102E+01
9	4	4	7.6156E-02	1.5925E-01	3.2224E-01	8.1802E-01	2.7263E+00
9	4	5	7.6156E-02	2.0119E-01	4.8374E-01	1.3919E+00	5.0773E+00
9	4	6	7.6156E-02	2.5361E-01	7.3454E-01	2.4530E+00	1.0071E+01
9	4	7	7.6155E-02	3.2351E-01	1.1560E+00	4.6290E+00	2.2229E+01
9	4	8	7.6154E-02	4.2836E-01	1.9838E+00	1.0114E+01	6.0852E+01
9	4	9	7.6155E-02	6.3804E-01	4.4227E+00	3.4028E+01	3.1205E+02
9	5	5	1.1615E-01	4.1697E-01	1.3519E+00	5.2537E+00	2.6012E+01
9	5	6	1.1615E-01	5.2245E-01	2.0204E+00	8.9871E+00	4.9267E+01
9	5	7	1.1615E-01	6.6309E-01	3.1369E+00	1.6526E+01	1.0428E+02
9	5	8	1.1615E-01	8.7405E-01	5.3184E+00	3.5257E+01	2.7434E+02
9	5	9	1.1615E-01	1.2960E+00	1.1708E+01	1.1569E+02	1.3483E+03
9	6	6	1.7865E-01	1.0895E+00	5.7432E+00	3.5285E+01	2.7164E+02
9	6	7	1.7865E-01	1.3727E+00	8.7413E+00	6.2438E+01	5.4041E+02
9	6	8	1.7865E-01	1.7975E+00	1.4552E+01	1.2848E+02	1.3380E+03
9	6	9	1.7865E-01	2.6469E+00	3.1427E+01	4.0566E+02	6.1541E+03
9	7	7	2.8976E-01	3.0922E+00	2.7925E+01	2.9129E+02	3.8085E+03
9	7	8	2.8976E-01	4.0054E+00	4.5102E+01	5.6432E+02	8.5241E+03
9	7	9	2.8976E-01	5.8319E+00	9.4302E+01	1.6672E+03	3.4906E+04
9	8	8	5.3976E-01	1.1172E+01	1.9788E+02	4.1369E+03	1.1149E+05
9	8	9	5.3976E-01	1.5907E+01	3.8751E+02	1.0669E+04	3.6052E+05
9	9	9	1.5398E+00	8.7604E+01	4.6888E+03	3.2574E+05	3.1580E+07
10	1	1	1.0000E-02	2.0000E-03	6.8400E-04	3.9744E-04	3.6144E-04
10	1	2	1.0000E-02	2.8889E-03	1.1773E-03	7.4620E-04	7.0969E-04
10	1	3	1.0000E-02	3.8889E-03	1.9011E-03	1.3495E-03	1.3786E-03
10	1	4	1.0000E-02	5.0317E-03	2.9486E-03	2.3819E-03	2.6684E-03

TABLE 2. (CONTINUED)

10	1	5	1.0000E-02	6.3651E-03	4.4708E-03	4.1594E-03	5.2036E-03
10	1	6	1.0000E-02	7.9651E-03	6.7293E-03	7.3022E-03	1.0389E-02
10	1	7	1.0000E-02	9.9651E-03	1.0227E-02	1.3168E-02	2.1791E-02
10	1	8	9.9998E-03	1.2632E-02	1.6092E-02	2.5288E-02	5.0350E-02
10	1	9	9.9996E-03	1.6632E-02	2.7588E-02	5.6018E-02	1.4330E-01
10	1	10	9.9996E-03	2.4632E-02	6.1380E-02	1.9072E-01	7.6212E-01
10	2	2	2.2346E-02	1.0502E-02	6.2669E-03	5.5132E-03	7.0388E-03
10	2	3	2.2346E-02	1.4047E-02	9.9716E-03	9.7373E-03	1.3247E-02
10	2	4	2.2346E-02	1.8098E-02	1.5308E-02	1.6885E-02	2.5001E-02
10	2	5	2.2346E-02	2.2824E-02	2.3035E-02	2.9087E-02	4.7765E-02
10	2	6	2.2346E-02	2.8495E-02	3.4468E-02	5.0516E-02	9.3774E-02
10	2	7	2.2347E-02	3.5585E-02	5.2136E-02	9.0289E-02	1.9391E-01
10	2	8	2.2345E-02	4.5037E-02	8.1696E-02	1.7206E-01	4.4248E-01
10	2	9	2.2345E-02	5.9215E-02	1.3954E-01	3.7847E-01	1.2448E+00
10	2	10	2.2345E-02	8.7572E-02	3.0926E-01	1.2789E+00	6.5386E+00
10	3	3	3.7970E-02	3.4650E-02	3.3922E-02	4.4429E-02	7.9578E-02
10	3	4	3.7970E-02	4.4414E-02	5.1433E-02	7.5414E-02	1.4575E-01
10	3	5	3.7970E-02	5.5806E-02	7.6677E-02	1.2777E-01	2.7165E-01
10	3	6	3.7970E-02	6.9476E-02	1.1391E-01	2.1896E-01	5.2239E-01
10	3	7	3.7972E-02	8.6563E-02	1.7128E-01	3.8704E-01	1.0613E+00
10	3	8	3.7970E-02	1.0935E-01	2.6703E-01	7.3053E-01	2.3844E+00
10	3	9	3.7969E-02	1.4352E-01	4.5402E-01	1.5927E+00	6.6109E+00
10	3	10	3.7969E-02	2.1187E-01	1.0013E+00	5.3317E+00	3.4187E+01
10	4	4	5.8379E-02	9.3608E-02	1.4471E-01	2.7952E-01	7.0560E-01
10	4	5	5.8378E-02	1.1708E-01	2.1325E-01	4.6384E-01	1.2755E+00
10	4	6	5.8378E-02	1.4524E-01	3.1390E-01	7.8152E-01	2.3899E+00
10	4	7	5.8381E-02	1.8044E-01	4.6846E-01	1.3620E+00	4.7474E+00
10	4	8	5.8378E-02	2.2738E-01	7.2568E-01	2.5393E+00	1.0454E+01
10	4	9	5.8377E-02	2.9778E-01	1.2265E+00	5.4735E+00	2.8443E+01
10	4	10	5.8377E-02	4.3859E-01	2.6883E+00	1.8101E+01	1.4413E+02
10	5	5	8.6156E-02	2.2992E-01	5.5059E-01	1.5694E+00	5.6534E+00
10	5	6	8.6156E-02	2.8391E-01	8.0083E-01	2.5864E+00	1.0241E+01
10	5	7	8.6160E-02	3.5141E-01	1.1834E+00	4.4239E+00	1.9749E+01
10	5	8	8.6155E-02	4.4141E-01	1.8176E+00	8.1134E+00	4.2340E+01
10	5	9	8.6155E-02	5.7639E-01	3.0480E+00	1.7223E+01	1.1230E+02
10	5	10	8.6155E-02	8.4638E-01	6.6254E+00	5.6032E+01	5.5356E+02
10	6	6	1.2615E-01	5.4682E-01	2.0258E+00	8.6401E+00	4.5540E+01
10	6	7	1.2616E-01	6.7337E-01	2.9531E+00	1.4401E+01	8.4295E+01
10	6	8	1.2615E-01	8.4212E-01	4.4822E+00	2.5810E+01	1.7404E+02
10	6	9	1.2615E-01	1.0952E+00	7.4342E+00	5.3615E+01	4.4513E+02
10	6	10	1.2615E-01	1.6014E+00	1.5972E+01	1.7041E+02	2.1088E+03
10	7	7	1.8866E-01	1.3313E+00	7.7875E+00	5.1438E+01	4.1542E+02
10	7	8	1.8866E-01	1.6542E+00	1.1613E+01	8.9071E+01	8.1071E+02
10	7	9	1.8866E-01	2.1386E+00	1.8946E+01	1.7903E+02	1.9610E+03

TABLE 2. (CONTINUED)

10	7	10	1.8866E-01	3.1074E+00	3.9988E+01	5.4894E+02	8.7301E+03
10	8	8	2.9976E-01	3.5746E+00	3.4910E+01	3.8400E+02	5.1919E+03
10	8	9	2.9976E-01	4.5763E+00	5.5407E+01	7.3021E+02	1.1432E+04
10	8	10	2.9976E-01	6.5800E+00	1.1347E+02	2.1039E+03	4.5614E+04
10	9	9	5.4976E-01	1.2327E+01	2.3042E+02	4.9825E+03	1.3687E+05
10	9	10	5.4976E-01	1.7359E+01	4.4382E+02	1.2628E+04	4.3603E+05
10	10	10	1.5497E+00	9.2537E+01	5.0918E+03	3.5867E+05	3.4978E+07

computation was executed in single precision and there is evidence of round off error.

Consider the summation in Equation 2.9. The form $S(n,n,m)$, for the largest order statistic, is monotone increasing in m . This follows since the difference $S(n,n,k) - S(n,n,m-1)$ is made up of the sum of all of the possible products of $(1/2)^{i_1} (1/3)^{i_2} \dots (1/n)^{i_{n-1}}$ such that $\sum i_j = m$. Each of these products is less than $(1/2)^m$. Now we count the number of these products. We are dealing with $n-1$ fractions. First we count the products made up of a single fraction. There are $\binom{n-1}{1}$ of type $(1/1)^m$. Now there are $\binom{n-1}{2}$ ways to pick the products of pairs of fractions. There are $2! \binom{n-1}{2}$ products of the type $(1/1)^{m-1} (1/j)$ since we must permute the power; that is, either the $(1/1)$ or the $(1/j)$ may have the power $m-1$. There are also $2! \binom{n-1}{2}$ products of the type $(1/1)^{m-2} (1/j)^2$. The highest power can decrease to $\left\lfloor \frac{m}{2} \right\rfloor$, where $\left\lfloor \frac{m}{i} \right\rfloor$ is the largest integer function. Therefore, the number of products of the type $(1/1)^{m-k} (1/j)^k$ is $2! \left\lfloor \frac{m}{2} \right\rfloor \binom{n-1}{2}$. Now looking at the products of three fractions we see that these number less than or equal to $3! \left\lfloor \frac{m}{3} \right\rfloor \binom{n-1}{3}$. This procedure is continued until we count the single product $(1/2)(1/3) \dots (1/n)$. Adding these we see that the total number of products is less than or equal to

$$n + \sum_{i=2}^{n-2} \left\lfloor \frac{m}{i} \right\rfloor i! \binom{n-1}{i}.$$

$$\text{Hence } S(n,n,m) - S(n,n,m-1) \leq \left(\frac{1}{2}\right)^m \left(n + \sum_{i=2}^{n-2} \left[\frac{m}{i}\right] i! \binom{n-1}{i}\right)$$

For a fixed n this difference approaches zero as m increases.

The values of $S(n,k,m)$ were computed in conjunction with Table 1. The values of $S(n,n,m)$ were found to converge rapidly as the moments increased and became stabilized by $m = 10$ for all values of n up to 25. Hence we are able to write the higher moments of the largest exponential order statistic as

$$E(X_{(n)}^m) = m! S(n,n,10) \quad (2.24)$$

for m greater than ten. The values of $S(n,n,10)$ are in Table 3 for sample sizes of five to 25. Hence $E(X_{(5)}^{20})$ in a sample of size five is given by $20! (2.7661)$.

Table 3.
Values of $S(n,n,10)$

n	S	n	S	n	S
5	2.7661	12	3.9668	19	4.6185
6	3.0096	13	4.0796	20	4.6918
7	3.2188	14	4.1843	21	4.7616
8	3.4019	15	4.2820	22	4.8283
9	3.5648	16	4.3737	23	4.8920
10	3.7114	17	4.4599	24	4.9531
11	3.8446	18	4.5413	25	5.0018

CHAPTER III. ASYMPTOTIC MOMENTS FOR THE WEIBULL

Recall the transformation $X = Y^{1/a}$ which takes the Weibull variate, Y , with shape parameter a and scale parameter b into the exponential with scale parameter b . In this chapter we will investigate the asymptotic properties of the exponential order statistics and use the results to obtain the asymptotic moments of the order statistics of the Weibull.

Let the index k be associated with the order statistic $X_{(n-k+1)}$ of a sample of size n drawn from the distribution $F(x)$ which has $f(x)$ as the density function. Gumbel (16) defines u_k to be the k -th characteristic

$$u_k = F^{-1}(1 - \frac{k}{n}) \quad (3.1)$$

that is, u_k is the value such that $P(X < u_k) = 1 - \frac{k}{n}$. Hence u_k is the 100 $(n-k)/n$ percentage point of the distribution. Furthermore call $m(x)$ the intensity function and v_k the extremal intensity where

$$m(x) = f(x)/(1-F(x)) \quad (3.2)$$

and

$$v_k = \frac{n}{k} f(u_k) . \quad (3.3)$$

Finally call the random variable w_k the k -th reduced extreme where

$$w_k = v_k(X_{(n-k+1)} - u_k) \quad (3.4)$$

where the x 's are the order statistics. The reader should note that the first reduced extreme is associated with the largest order statistic. For the exponential distribution with parameter b , $F(x) = 1 - e^{-x/b}$ and $f(x) = b^{-1} e^{-x/b}$. Hence $F^{-1}(y) = -b \log(1-y)$ and

$$u_k = -b \log(1 - (1 - k/n)) \quad (3.5)$$

$$= b \log(n/k)$$

$$m(x) = \frac{b^{-1} e^{-x/b}}{1 - (1 - e^{-x/b})}$$

$$= b^{-1}$$

$$v_k = \frac{n}{k} (b^{-1} e^{-b^{-1}(b \log(n/k))})$$

$$= \frac{n}{k} b^{-1} e^{-\log(n/k)}$$

$$= b^{-1} \frac{n}{k} \frac{k}{n}$$

$$= b^{-1}$$

$$w_k = b^{-1}(X_{(n-k+1)} - b \log(n/k))$$

$$= b^{-1} X_{(n-k+1)} - \log(n/k)$$

where $\log$ is the natural logarithm.

Fisher and Tippett (17) introduced three types of

asymptotic distributions for the largest observation of a sample. Gumbel, in (37), quotes from an article written in French by the Russian probabilist B. V. Gnedenko in which Gnedenko showed that these are the only possible limiting forms and gave necessary and sufficient conditions for memberships in each class based on the initial distribution of the random variable X .

One of these limiting forms is the extreme value distribution

$$F(z) = \exp(-\exp(-z)) \quad (3.6)$$

The condition for this limiting form of the largest observation is that

$$\lim_{n \rightarrow \infty} nP(X > u_1 + x/v_1) = \exp(-x) \quad (3.7)$$

Equation 3.7 holds for all n for the exponential since

$$u_1 = b \log(n)$$

and since $P(X > r) = 1 - F(r)$ we have

$$\begin{aligned} \lim_{n \rightarrow \infty} nP(X > b \log n + bx) \\ &= \lim_{n \rightarrow \infty} n(\exp(-\frac{1}{b}(b \log n + bx))) \\ &= \lim_{n \rightarrow \infty} n(\exp(-\log n) \exp(-x)) \end{aligned}$$

$$\begin{aligned}
&= \text{Limit } n\left(\frac{1}{n} \exp(-x)\right) \\
&= \exp(-x) .
\end{aligned}$$

Fisher and Tippett (17, p. 183) gave another criterion which leads to the limiting form of Equation 3.6.

"Any frequency distribution such that $(1-p)\exp(x/c)$ approaches one, has limiting form $\exp\{-nA\exp(-x/c)\}$ where $p = P(X < x)$ and $1/c = -d(\log(1-p))/dx$ "

For the exponential distribution

$$\begin{aligned}
1/c &= - \frac{d}{dx} (\log (1-(1-e^{-x/b}))) \\
&= - \frac{d}{dx} (\log (e^{-x/b})) \\
&= - (-1/b) \\
&= 1/b .
\end{aligned}$$

Hence $c = b$ and

$$(1-p)\exp(x/b) = \exp(-x/b)\exp(x/b) = 1$$

then the condition is exact for all values of n for the exponential distribution. The limiting form is then $\exp(-n\exp(-x/b))$. A third condition, given by Dodd (7), is that if $F(x)$ is a continuous distribution function such that $F(x)$ is less than one for all finite values of X , has two derivatives, and if $\text{Limit}_{n \rightarrow \infty} \frac{d}{dx} (1/m(x)) = 0$, then $\text{Limit}_{n \rightarrow \infty} P(v_1(X_{(n)} - u_1) - w) = \exp(-\exp(-w))$ for all real numbers w .

The quantities $m(x)$, v_1 , and u_1 are defined by Equations 3.1, 3.2, and 3.3. Again the exponential distribution meets the condition exactly. The asymptotic form we will work with is then

$$F(w_1) = \exp(-\exp(-w_1)) \quad (3.8)$$

where w_1 is the first reduced extreme as defined in Equation 3.4. The asymptotic density is given by

$$f(w_1) = \exp(-w_1)\exp(-\exp(-w_1)). \quad (3.9)$$

We substitute for w_1 using Equation 3.5 to arrive at the asymptotic distribution in terms of the exponential variates, for the largest observation

$$F(x_{(n)}) = \exp(-n \exp(-x/b)) \quad (3.10)$$

and the density function

$$f(x_{(n)}) = nb^{-1} \exp(-x/b)\exp(-n \exp(-x/b)). \quad (3.11)$$

A. Asymptotic Moment Generating Function for the Largest Observation

The moment generating function is derived in the usual manner. In terms of the reduced extreme

$$G_{w_1}(t) = \int_{-\infty}^{\infty} \exp(tw)\exp(-w)\exp(-\exp(-w))dw$$

which, through the transformation $z = \exp(-w)$, leads to

$$\int_0^{\infty} z^{-t} \exp(-z) dz = \int_0^{\infty} z^{(1-t)-1} \exp(-z) dz$$

which is a gamma function. The moment generating function is then

$$G_{w_1}(t) = \Gamma(1-t). \quad (3.12)$$

From Equation 3.5 we make the transformation from the reduced extreme to that of the largest order statistic of the exponential distribution. It is known that if a random variable Z has moment generating function $G_Z(t)$ then the random variable $Y = pZ + q$ has large sample moment generating function $G_Y(t) = \exp(qt)G_Z(pt)$. Hence

$$G_{X(n)}(t) = \exp(bt \log n) \Gamma(1-bt). \quad (3.13)$$

We are able to get the asymptotic moments by the differentiation of Equation 3.13. It is necessary to take the derivatives of the gamma function. It is known that

$$\Gamma^{(m)}(x) = \int_0^{\infty} \exp(-z) z^{x-1} (\log z)^m dz \quad (3.14)$$

where $\Gamma^{(m)}(x)$ is the m -th derivative of the gamma function. For our situation Equation 3.14 becomes

$$\Gamma^{(m)}(1-bt) = (-b)^m \int_0^{\infty} \exp(-z) z^{-bt} (\log z)^m dz. \quad (3.15)$$

This occurs since

$$\Gamma(1-bt) = \int_0^{\infty} \exp(-z) \exp(-bt \log z) dz$$

$$\begin{aligned}\frac{d}{dt} \Gamma(1-bt) &= \int_0^{\infty} \frac{d}{dt} \exp(-z) \exp(-bt \log z) dz \\ &= \int_0^{\infty} \exp(-z) (-b \log z) \exp(-bt \log z) dz .\end{aligned}$$

Hence

$$\Gamma^{(1)}(1-bt) = (-b) \int_0^{\infty} \exp(-z) \exp(-bt \log z) \log z dz .$$

Now taking the second derivative

$$\begin{aligned}\frac{d}{dt} \Gamma^{(1)}(1-bt) &= \\ &(-b) \int_0^{\infty} \exp(-z) (-b \log z) \exp(-bt \log z) \log z dz \\ \Gamma^{(2)}(1-bt) &= (-b)^2 \int_0^{\infty} \exp(-z) \exp(-bt \log z) (\log z)^2 dz\end{aligned}$$

and the result of Equation 3.15 can be shown by mathematical induction. Now Equation 3.15 is, in itself, difficult to evaluate. However, for the moments, the derivative is evaluated at t equal to zero and we have $\Gamma^{(m)}(1)$ where

$$\Gamma^{(m)}(1) = (-b)^m \int_0^{\infty} \exp(-z) (\log z)^m dz \quad (3.16)$$

The integral of Equation 3.16 can be evaluated by using psi functions. From (44) the psi function is $\psi(z) = \Gamma^{(1)}(z)/\Gamma(z)$. Hence $\Gamma^{(1)}(z) = \psi(z) \Gamma(z)$. Taking derivatives and setting z equal to one, we have

$$\Gamma^{(m+1)}(1) = \sum_{i=0}^m \binom{m}{i} \Gamma^{(i)}(1) \psi^{(m-i)}(1)$$

Now $\psi^{(n)}(1) = (-1)^{n+1} n! \zeta(n+1)$, where $\zeta(s) = \sum_{k=1}^{\infty} k^{-s}$ is the Riemann zeta function. The gamma, psi, and zeta functions are all tabulated in (44). A partial table of values of the integral is found in Table 4. In the table the notation E_n represents the n -th power of ten.

Note that $\Gamma^{(0)}(1)$ is one. Furthermore, notice that the sign of $(-b)^m$ is always the same as the sign of the integral so that Equation 3.16 represents a positive quantity for all values of m .

Table 4.
Values of $I(m) = \left| \int_0^{\infty} e^{-z} \log^m z \, dz \right|$

m	$I(m)$	m	$I(m)$
1	.577216E0	6	.715067E3
2	.197811E1	7	.501985E4
3	.544487E1	8	.402436E5
4	.235615E2	9	.362526E6
5	.117839E3	10	.362704E7

B. Generating Functions for Lower Order Statistics

Now recall the relation of Equation 2.2 which we shall write as

$$Z_{n-k+1} = k(X_{(n-k+1)} - X_{(n-k)}) \quad (3.17)$$

$$k = 1, 2, \dots, n$$

Now write Equation 3.17 for k equal to one

$$X_{(n)} = X_{(n-1)} + Z_n . \quad (3.18)$$

Since $X_{(n-1)}$ and Z_n are independent we are able to write the moment generating function of the largest observation as

$$G_{X_{(n)}}(t) = G_{X_{(n-1)}+Z_n}(t) = G_{X_{(n-1)}}(t)G_{Z_n}(t) . \quad (3.19)$$

Since Z_n is an exponential variate it has moment generating function

$$G_{Z_n}(t) = (1-bt)^{-1} \quad (3.20)$$

and Equation 3.19 becomes

$$G_{X_{(n)}}(t) = G_{X_{(n-1)}}(t) (1-bt)^{-1} . \quad (3.21)$$

Multiplying Equation 3.21 by $(1-bt)$ gives us the moment generating function for $X_{(n-1)}$ in terms of the moment generating function of $X_{(n)}$.

$$G_{X_{(n-1)}}(t) = G_{X_{(n)}}(t) (1-bt) . \quad (3.22)$$

From Equation 3.13 for $G_{X_{(n)}}(t)$ we have

$$G_{X_{(n-1)}}(t) = \exp(bt \log n) \Gamma(1-bt)(1-bt) . \quad (3.23)$$

The method above is easily extended, for k small with respect

to n , to obtain the asymptotic moment generating function for $X_{(n-k)}$ as

$$G_{X_{(n-k)}}(t) = \exp(bt \log n) \Gamma(1-bt) \prod_{j=1}^k (1 - \frac{bt}{j}) \quad (3.24)$$

Taking derivatives we have

$$G_{X_{(n-k)}}^{(m)}(t) = \sum_{i=0}^m \binom{m}{i} G_{X_{(n)}}^{(m-i)}(t) h(i, k, t) \quad (3.25)$$

$$k = 1, 2, \dots$$

where

$$G_{X_{(n)}}^{(m-1)}(t) = \frac{d^{m-1}}{dt^{m-1}} G_{X_{(n)}}(t) \quad (3.26)$$

and

$$h(i, k, t) = \frac{d^i}{dt^i} \prod_{j=1}^k (1 - \frac{bt}{j})$$

C. Asymptotic Moments

From Equation 3.13 we find the approximate asymptotic moments of the largest observation as

$$E^*(X_{(n)}^m) = b^m \sum_{i=0}^m \binom{m}{i} (\log n)^{m-i} I(i) \quad (3.27)$$

where $I(i)$ are the absolute values of the integral in Table 4 with $I(0) = 1$. The notation $E^*(X)$ will be used to denote the asymptotic expectation of X .

To find the asymptotic moments of lower order statistics

we use Equation 3.25 evaluated at t equal to zero. Letting b equal one we have

$$E^*(X_{(n-k)}^m) = \sum_{i=0}^m \binom{m}{i} h(i, k, 0) E^*(X_{(n)}^{m-i}). \quad (3.28)$$

There is some difficulty in evaluating $h(i, k, 0)$.

Note that

$$\prod_{j=1}^k (1-t/j) = (1-t)(1-t/2)(1-t/3)\dots(1-t/k)$$

which is a polynomial of degree k in the variable t . Hence

there are at most k nonzero derivatives of $\prod_{j=1}^k (1-t/j)$, and $h(i, k, 0) = 0$ for $i > k$. Individual values may be found

recursively using

$$h(i, k+1, 0) = -\frac{1}{k+1} h(i-1, k, 0) - h(i, k, 0) \quad (3.29)$$

(3.29)

$$i = 1, 2, \dots \quad k = 1, 2, \dots$$

since $\prod_{j=1}^{k+1} (1-t/j) = (1 - \frac{t}{k+1}) \prod_{j=1}^k (1-t/j)$. Several values appear

in Table 5.

If the condition that $b = 1$ is relaxed the right side of Equation 3.28 must be multiplied by b^m .

Table 5.
Values of $h(1,k,0)$

k	1	0	1	2	3	4	5
1	1		-1	0	0	0	0
2	1		-3/2	1	0	0	0
3	1		-11/6	12/6	-1	0	0
4	1		-25/12	35/12	-30/12	1	0

D. Asymptotic Crossproducts

In Chapter II we used the relation $X_{(k-j)} + W_{k,j} = X_{(k)}$, where X is an exponential random variable and $W_{k,j} = W_{k,j-1} + Z_{k-j+1}/(n-k+j)$, to find the moments of the crossproducts of two exponential order statistics. The Z_i are the exponential variates defined in Equation 2.2. The moments of the $W_{k,j}$ are given by the recursive relation

$$E(W_{k,j}^m) = m! \sum_{i=0}^m E(W_{k,j-1}^i) \left(\frac{1}{n-k+j}\right)^{m-i} / i!$$

Hence the asymptotic moments of the crossproducts are

$$\begin{aligned}
 E^*((X_{(k-j)}X_{(k)})^m) &= E^*((X_{(k-j)}(X_{(k-j)} + W_{k,j}))^m) \quad (3.30) \\
 &= E^*((X_{(k-j)}^2 + X_{(k-j)}W_{k,j})^m) \\
 &= \sum_{i=0}^m \binom{m}{i} E^*(X_{(k-j)}^{m+1-i}) E(W_{k,j}^{i-1}).
 \end{aligned}$$

Then the approximate asymptotic covariances are given by

$$C^*(X_{(k)}^m, X_{(j)}^m) = E^*((X_{(k)}X_{(j)})^m) - E^*(X_{(k)}^m)E^*(X_{(j)}^m) \quad (3.31)$$

where C^* denotes the asymptotic covariance.

CHAPTER IV. THE EXPONENTIAL DISTRIBUTION

The exponential distribution is a Weibull distribution with a shape parameter of one. The exponential density is given by

$$\begin{aligned} g(x;b) &= b^{-1} \exp(-x/b) \quad x > 0, b > 0 \\ &= 0 \quad \text{otherwise.} \end{aligned} \quad (4.1)$$

Furthermore

$$E(X) = b, \quad V(X) = b^2. \quad (4.2)$$

From Equation 2.7, the expected value of the k -th order statistic is

$$E(X_{(k)}) = bS(n,k,1) \quad (4.3)$$

and has as its variance

$$V(X_{(k)}) = b^2(S(n,k,2) - S^2(n,k,1)). \quad (4.4)$$

Sarhan (36) has written these relations as

$$E(X_{(k)}) = b \sum_{j=1}^k 1/(n-j+1) \quad (4.5)$$

and

$$V(X_{(k)}) = b^2 \sum_{j=1}^k 1/(n-j+1)^2. \quad (4.6)$$

Furthermore, if k is less than 1, the covariance is

$$C(X_{(k)}, X_{(1)}) = V(X_{(k)}) \quad (4.7)$$

The values for Equations 4.5 and 4.7 may be found in the first columns of Table 1 and Table 2.

Considering the asymptotic case we have from Equation 3.27, setting m equal to one

$$E^*(X_{(n)}) = b(\log n + I(1)) = b(\log n + .5772) \quad (4.8)$$

This is not surprising since it is known that $\int_1^n \frac{dt}{t} = \log n$. Hence $\log n$ approximates the sum of the reciprocals $\sum_{t=1}^n 1/t$. Values of Equation 4.8 for $b = 1$ are compared with the finite form of Equation 4.3 with $k = n$, in Table 6. The asymptotic formula is a close, simple, approximation to the exact value. The error is about 4 percent at $n = 5$ and decreases to about one percent for $n = 100$.

To compute the lower asymptotic expectations, we use Equation 3.28

$$\begin{aligned} E^*(X_{(n-k)}) &= E^*(X_{(n)}) - b \sum_{j=1}^k 1/j \\ &= b(\log n + I(1)) - \sum_{j=1}^k 1/j \end{aligned} \quad (4.9)$$

$$k = 1, 2, \dots, n-1$$

The error of the approximation is a constant and equal to the error in the approximation to the largest order

Table 6.
Comparison of the finite and asymptotic forms of the
expected value of the largest exponential order statistic

n	$E^*(X_{(n)})$	$E^*(X_{(n)}) - E(X_{(n)})$	n	$E^*(X_{(n)})$	$E^*(X_{(n)}) - E(X_{(n)})$
5	2.3800	.0967	55	4.5845	.0087
10	2.9783	.0493	60	4.6715	.0083
15	3.3512	.0330	65	4.7516	.0076
20	3.6225	.0248	70	4.8257	.0071
25	3.8358	.0198	75	4.8947	.0066
30	3.9784	.0165	80	4.9592	.0062
35	4.1275	.0142	85	5.0198	.0058
40	4.2661	.0124	90	5.0770	.0055
45	4.3838	.0110	95	5.1311	.0052
50	4.4892	.0099	100	5.1824	.0049

statistic. From Equation 3.27 and setting b equal to one we have the second asymptotic moment of the largest order statistic of an exponential distribution as

$$\begin{aligned} E^*(X_{(n)}^2) &= \log^2 n + 2I(1) \log n + I(2) \\ &= \log^2 n + 1.1544 \log n + 1.9781. \end{aligned} \quad (4.10)$$

The $I(1)$ are from Table 4. Equations 4.8 and 4.10 give us the asymptotic variance of the largest observation.

$$\begin{aligned} V^*(X_{(n)}) &= E^*(X_{(n)}^2) - E^{*2}(X_{(n)}) \quad (4.11) \\ &= (\log^2 n + 2I(1) \log n + I(2)) - (\log^2 n \\ &\quad + 2I(1) \log n + (I(1))^2) \\ &= I(2) - (I(1))^2. \end{aligned}$$

From (4.4), it is known that $I(2) - (I(1))^2 = \pi^2/6$. Hence Equation 4.11 is independent of sample size. We note that from Equation 4.6

$$V(X_{(k)}) = \sum_{j=1}^k 1/(n-j+1)^2$$

which can be written for $k = n$ as

$$V(X_{(n)}) = \sum_{j=1}^n 1/j^2. \quad (4.12)$$

Now if we let n increase without bound it is known that

$$\lim_{n \rightarrow \infty} \sum_{j=1}^n 1/j^2 = \pi^2/6 \quad (4.13)$$

which is exactly the same as Equation 4.11. If the asymptotic value is compared to the values in Table 2 we see that the error of approximation is only about two percent at n equal to 25.

From Equation 3.28 we have

$$E^*(X_{(n-1)}) = E^*(X_{(n)}) - 1$$

and

$$E^*(X_{(n-1)}^2) = E^*(X_{(n)}^2) - 2E^*(X_{(n)})$$

hence

$$\begin{aligned} V^*(X_{(n-1)}) &= E^*(X_{(n-1)}^2) - E^{*2}(X_{(n)}) \\ &= [E^*(X_{(n)}^2) - 2E^*(X_{(n)})] - [E^*(X_{(n)}) - 1]^2 \\ &= E^*(X_{(n)}^2) - E^{*2}(X_{(n)}) - 1 \\ V^*(X_{(n-1)}) &= V^*(X_{(n)}) - 1. \end{aligned} \quad (4.14)$$

Continuing this procedure we find that

$$\begin{aligned} V^*(X_{(n-k)}) &= V^*(X_{(n)}) - \sum_{j=1}^k 1/j^2 \\ &= \frac{\pi^2}{6} - \sum_{j=1}^k 1/j^2 \end{aligned} \quad (4.15)$$

which is exactly the value we would expect based on Equation

4.6. The use of the asymptotic expression as an approximation for the variance of an exponential order statistic has a constant error over the whole sample and that error is

$$\begin{aligned} V^*(X_{(n)}) - V(X_{(n)}) &= \frac{\pi^2}{6} - \sum_{i=1}^n 1/i^2 \\ &= \sum_{i=n+1}^{\infty} 1/i^2. \end{aligned}$$

Hence the percent error for the largest statistic is

$$(100 \sum_{i=n+1}^{\infty} 1/i^2) / (\pi^2/6). \text{ The asymptotic forms developed}$$

above for expectations and variances give us close approximations for values outside the range of Tables 1 and 2.

These results give us a quick test value based on the larger asymptotic variances. Furthermore, it is interesting that the theory based on the asymptotic properties for the largest order statistic fits so well over the whole sample value range.

A. Censored Estimation of the Mean

We shall next consider a special class of estimators of the mean, b , of an exponential distribution. Consider the random variables Z_r defined in Equation 1.6. Since the Z_r are independently and identically distributed as exponential with parameter b , their characteristic function is given by

$$g_{Z_r}(t) = (1 - itb)^{-1}. \quad (4.16)$$

Consider a sum of the Z_r . It is seen that

$$\sum_{j=1}^{n-r} Z_j = X_{(1)} + X_{(2)} \dots + X_{(n-r-1)} + rX_{(n-r)} \quad (4.17)$$

Let $z_k = (n-k+1)(x_{(k)} - x_{(k-1)})$ be the observed values of Z_k and

$$\hat{w} = \sum_{j=1}^{n-r} z_j / D \quad (4.18)$$

where D is a suitably chosen constant denominator. The characteristic function of $\hat{w}$ is then

$$g_{\hat{w}}(t) = (1 - itb/D)^{-(n-r)} \quad (4.19)$$

so that its expectation is

$$E(\hat{w}) = (n-r)b/D. \quad (4.20)$$

Let $h = \hat{w} - b$. The random variable h has as its characteristic function

$$g_h(t) = \exp(-itb)(1 - itb/D)^{-(n-r)} \quad (4.21)$$

and expectation

$$E(h) = b\left(\frac{(n-r)-D}{D}\right). \quad (4.22)$$

The right-hand side of Equation 4.22 is exactly the bias of the statistic $\hat{w}$ for estimating b . Furthermore, the second derivative of Equation 4.21 gives the mean square (MSE) of $\hat{w}$ for estimating b .

$$\begin{aligned} \text{MSE}(\hat{w}) &= b^2(1-2((n-r)/D) + ((n-r)/D^2) \\ &\quad + ((n-r)/D)^2) \end{aligned} \quad (4.23)$$

Consider the following special cases of the estimator $\hat{w}$.

$$\bar{X} = \sum_{j=1}^n Z_j/n \quad (4.24)$$

$$\hat{E}_r = \sum_{j=1}^{n-r} Z_j/(n-r) \quad (4.25)$$

$$\hat{W}_r = \sum_{j=1}^{n-r} Z_j/n \quad (4.26)$$

$$\hat{M}_r = \sum_{j=1}^{n-r} Z_j/(n-r+1) \quad (4.27)$$

Equation 4.24 is the usual mean estimator. $\hat{E}_r$ is the estimator discussed earlier which was developed by Epstein and $\hat{W}_r$ is the Winsorized mean. The estimator $\hat{M}_r$ is developed by minimizing Equation 4.23 with respect to D and is thus the minimum MSE of this class of estimators for the exponential. The expected values of these estimators are

$$E(\bar{X}) = b \quad (4.28)$$

$$E(\hat{E}_r) = b$$

$$E(\hat{W}_r) = b(1-r/n)$$

$$E(\hat{M}_r) = b(1-1/(n-r+1)) .$$

Note that $\bar{X}$ and $\hat{E}_r$ are unbiased and all of the estimators are consistent for fixed r . The MSE of the estimates are

$$\text{MSE}(\bar{X}) = b^2/n \quad (4.29)$$

$$\text{MSE}(\hat{E}_r) = (b^2/n)(1+r/(n-r))$$

$$\text{MSE}(\hat{W}_r) = (b^2/n)(1+r(r-1)/n)$$

$$\text{MSE}(\hat{M}_r) = (b^2/n)(n-(n-r)/(n-r+1))$$

If we define relative efficiency with respect to mean of the estimators as $\text{MSE}(\bar{X})/\text{MSE}(\hat{W})$ and denote this by REM, then the REM of the three estimators are given by

$$\text{REM}(\hat{E}_r) = (n-r)/n \quad (4.30)$$

$$\text{REM}(\hat{W}_r) = n/(n+r(r-1))$$

$$\text{REM}(\hat{M}_r) = (n-r+1)/n$$

Note that in the case of the exponential, the maximum likelihood estimate for b is $\bar{X}$. The values of REM will be tabulated for the Weibull distribution in the next chapter. We note that $\hat{M}_0$ is the minimum mean square error estimate for the parameter of the exponential and $\text{REM}(\hat{M}_0) = (n+1)/n$ which is greater than one and therefore more efficient than the usual mean.

CHAPTER V. ESTIMATION AND TESTING IN THE WEIBULL DISTRIBUTION

For convenience, we have chosen to write the Weibull distribution in this paper as

$$\begin{aligned} f(y;a,b) &= \frac{1}{ab} y^{\frac{1}{a}-1} \exp(-y^{1/a}/b) \quad y > 0, a > 0, b > 0 \\ &= 0 \quad \text{otherwise} \end{aligned} \quad (5.1)$$

A. Maximum Likelihood Estimation for the Mean

In Chapter I the maximum likelihood estimates (MLE) for the parameters of the Weibull distribution were developed. From Equation 1.12 the MLE are the solutions of

$$\begin{aligned} \hat{b} &= \frac{1}{n} \sum y^{1/\hat{a}} \\ -\hat{a} + \frac{1}{n\hat{b}} \sum y^{1/\hat{a}} \log y &= \frac{1}{n} \sum \log y \end{aligned} \quad (5.2)$$

Furthermore the mean of the Weibull as derived in Equation 1.10 is

$$E(Y) = m = c \Gamma(a+1) \quad (5.3)$$

where $b^a = c$. Hence the maximum likelihood estimate of m is

$$\hat{m} = \hat{c} \Gamma(\hat{a}+1) \quad (5.4)$$

where $\hat{c} = \left(\frac{1}{n} \sum y^{1/\hat{a}} \right)^{\hat{a}}$. By a two parameter Taylor series expansion

$$\hat{m} - m \doteq \Gamma(a+1)(\hat{c} - c) + c \Gamma^{(1)}(a+1)(\hat{a} - a) . \quad (5.5)$$

It is known that the MLE of the Weibull distribution are consistent, asymptotically efficient, and asymptotically normal. Hence

$$E^*(\hat{m} - m) = 0 \quad (5.6)$$

and

$$\begin{aligned} V^*(\hat{m}) &\doteq (\Gamma(a+1))^2 V^*(\hat{c}) + c^2 (\Gamma^{(1)}(a+1))^2 V^*(\hat{a}) \\ &\quad + 2c \Gamma(a+1) \Gamma^{(1)}(a+1) C^*(\hat{a}, \hat{c}) \end{aligned} \quad (5.7)$$

where E^* , V^* , and C^* represent the asymptotic expected values, variances, and covariances. It is known¹ that the limiting covariance matrix of the MLE of the parameters c and a is given by

$$\frac{1}{n} \begin{pmatrix} 1.1087(ca)^2 & -.2570ca^2 \\ -.2570ca^2 & .6079a^2 \end{pmatrix} \quad (5.8)$$

Substituting into Equation 5.7 we have

$$\begin{aligned} n V^*(\hat{m}) &\doteq 1.1087 (\Gamma(a+1)ca)^2 + .6079 (\Gamma^{(1)}(a+1)c a)^2 \\ &\quad - .5140 \Gamma(a+1) \Gamma^{(1)}(a+1) c^2 a^2 . \end{aligned} \quad (5.9)$$

From Equation 1.10 we have

$$E(Y) = c \Gamma(a+1) \quad (5.10)$$

$$E(Y^2) = c^2 \Gamma(2a+1)$$

¹The derivation of this result is given in the Appendix.

and therefore

$$V(Y) = c^2(\Gamma(2a+1) - (\Gamma(a+1))^2) . \quad (5.11)$$

Then from Equations 5.10 and 5.11 the mean square error (MSE) of the estimate of the mean $\bar{y} = \frac{1}{n} \sum y_i$ is given by

$$MSE(\bar{y}) = \frac{c^2}{n} (\Gamma(2a+1) - (\Gamma(a+1))^2) . \quad (5.12)$$

If we compare the approximate asymptotic variance of the MLE with the MSE of $\bar{y}$, we have an asymptotic relative efficiency.

$$\begin{aligned} \frac{V^*(\hat{m})}{MSE(\bar{y})} &= \\ &= \frac{1.1087(\Gamma(a+1)a)^2 + .6079(\Gamma^{(1)}(a+1)a)^2 - .514\Gamma(a+1)\Gamma^{(1)}(a+1)a^2}{\Gamma(2a+1) - (\Gamma(a+1))^2} \\ &= \frac{1.1087a^2 + .6079(\psi(a+1)a)^2 - .514\psi(a+1)a^2}{\frac{\Gamma(2a+1)}{(\Gamma(a+1))^2} - 1} \end{aligned} \quad (5.13)$$

where $\psi(a+1) = \Gamma^{(1)}(a+1)/\Gamma(a+1)$. Note that Equation 5.13 is independent of the scale parameter. In Table 7 we compare this efficiency for several values of the shape parameter. It is seen that as the shape parameter increases beyond one the relative efficiency decreases, that is the MLE becomes more efficient with respect to $\bar{y}$. Study of Equation 5.13 shows why this reduction occurs since $\Gamma(2a+1)$ is clearly the dominate term as the shape parameter increases. Hence, for

Table 7.
Asymptotic relative efficiency $V^*(m)/MSE(\bar{y})$

Shape parameter	.5	1.0	2.0	3.0	4.0	5.0
Efficiency	.99795	1.0000	.92240	.68068	.39884	.17563

larger values of the shape parameter, the maximum likelihood estimator provides large gains in efficiency. The gains increase from about 10 percent for $a = 2$ to over 500 percent at $a = 5$.

B. Censored Estimation

In the previous chapter we studied the class of estimators

$$\hat{w} = (\sum_{i=1}^{n-r} y_{(i)} + ry_{(n-r)})/D \quad (5.14)$$

for the mean of the exponential distribution. We propose to now investigate this class of estimators for estimation of the mean of the Weibull distribution. We have defined in Chapter IV the relative efficiency (REM) of the class of estimators defined by w as being the mean square error of the mean, $\bar{y}$, divided by the mean square error of w . For a given sample size n let

$$A' = (1, 1, \dots, 1, r+1, 0, \dots, 0) \quad (5.15)$$

be a $l \times n$ vector giving the linear combination of the order statistics of the estimator $\hat{w}$. Let Y be the $n \times 1$ vector of ordered observations. Then

$$\hat{w} = A'Y/D \quad (5.16)$$

and the expectation is

$$E(\hat{w}) = b^a A' \epsilon / D \quad (5.17)$$

where ϵ is an $n \times 1$ vector of the expected values of the Weibull variates with scale parameter one. These values were computed in Chapter II for samples up to size ten. A table for samples up to 25 is found in (28). Now

$$V(A'Y/D) = b^{2a} A' \Phi A / D^2 \quad (5.18)$$

where Φ is an $n \times n$ matrix of covariances of the Weibull with unit scale parameter. A partial table of these values is found in Chapter II. Now the squared bias of the estimator is given by

$$\text{Bias}^2(\hat{w}) = b^{2a} (A' \epsilon - D a!)^2 / D^2 \quad (5.19)$$

Hence the mean square error of $\hat{w}$ is

$$\text{MSE}(\hat{w}) = \frac{b^{2a}}{D^2} (A' \Phi A + (A' \epsilon - D a!)^2) \quad (5.20)$$

Using Equation 5.12 the relative efficiency with respect to the mean is

$$\text{REM}(\hat{W}) = \left(\frac{D^2}{n} \right) \frac{((2a)! - (a!)^2)}{A!A + (A!E - Da!)^2} \quad (5.21)$$

Recall from Chapter IV that we have introduced three cases of the estimator $\hat{W}$; $\hat{W}_r$, the Winsorized mean; $\hat{E}_r$, the Epstein mean; and $\hat{M}_r$, the minimum mean square estimator of this class for the exponential. The relative efficiencies for these estimators are given in Table 8 for samples of size two to 25, shape parameters of one through five, and for the censoring indicated by r . For r equal to zero the relative efficiencies are all equal to one.

From the empirical results of Table 8, we see that for all amounts of censoring, the REMs for the exponential are less than those for the corresponding estimator for the Weibulls with shape parameter greater than one. More importantly, as the shape parameter increases, the efficiency increases. Note however, that as censoring increases beyond two, the efficiency decreases for a given shape parameter. The gains in efficiency are quite large for large shape parameters. For the once censored case in a sample of size 25, the gains range from 20 percent for $a = 2$ up to 1000 percent at $a = 5$. The largest gains in efficiency for the once censored samples occur when the sample size is small.

Table 8.
REMs of r of n censored estimators of the mean
of a Weibull with integer shape parameter

Shape parameter n	r	$\hat{W}_r$	$\hat{E}_r^1$	$\hat{M}_r$	$\hat{W}_r$	$\hat{E}_r^2$	$\hat{M}_r$	$\hat{W}_r$	$\hat{E}_r^3$
2	1	1.00	0.50	1.00	2.86	1.67	2.86	8.94	5.43
3	1	1.00	0.67	1.00	2.27	1.69	2.27	6.34	4.81
4	1	1.00	0.75	1.00	1.99	1.63	1.99	5.13	4.25
5	1	1.00	0.80	1.00	1.82	1.57	1.82	4.42	3.84
5	2	0.71	0.60	0.80	1.58	1.61	1.64	4.50	4.54
6	1	1.00	0.83	1.00	1.70	1.52	1.70	3.95	3.54
6	2	0.75	0.67	0.83	1.49	1.55	1.55	3.98	4.07
7	1	1.00	0.86	1.00	1.62	1.48	1.62	3.61	3.30
7	2	0.78	0.71	0.86	1.42	1.50	1.49	3.61	3.72
8	1	1.00	0.88	1.00	1.56	1.44	1.56	3.35	3.11
8	1	0.80	0.75	0.88	1.37	1.46	1.44	3.34	3.45
9	1	1.00	0.89	1.00	1.51	1.41	1.51	3.15	2.96
9	2	0.82	0.78	0.89	1.34	1.43	1.40	3.12	3.24
10	1	1.00	0.90	1.00	1.47	1.39	1.47	2.99	2.83
10	2	0.83	0.80	0.90	1.31	1.40	1.37	2.95	3.07
10	3	0.62	0.70	0.80	1.04	1.26	1.18	2.54	2.78
11	1	1.00	0.91	1.00	1.44	1.37	1.44	2.85	2.72
11	2	0.85	0.82	0.91	1.28	1.38	1.34	2.81	2.93
11	3	0.65	0.73	0.82	1.02	1.24	1.17	2.41	2.64
12	1	1.00	0.92	1.00	1.41	1.35	1.41	2.74	2.63
12	2	0.86	0.83	0.92	1.26	1.35	1.32	2.69	2.81
12	3	0.67	0.75	0.83	1.01	1.22	1.15	2.30	2.52
13	1	1.00	0.92	1.00	1.38	1.33	1.38	2.64	2.54
13	2	0.87	0.85	0.92	1.24	1.34	1.30	2.59	2.70
13	3	0.68	0.77	0.85	1.00	1.21	1.14	2.21	2.42
14	1	1.00	0.93	1.00	1.36	1.32	1.36	2.56	2.47
14	2	0.87	0.86	0.93	1.23	1.32	1.28	2.50	2.61
14	3	0.70	0.79	0.86	0.99	1.20	1.13	2.13	2.34
15	1	1.00	0.93	1.00	1.34	1.30	1.34	2.49	2.41
15	2	0.88	0.87	0.93	1.22	1.30	1.27	2.43	2.54
15	3	0.71	0.80	0.87	0.98	1.18	1.12	2.06	2.26
15	4	0.56	0.73	0.80	0.80	1.04	0.97	1.79	2.00
16	1	1.00	0.94	1.00	1.33	1.29	1.33	2.42	2.35
16	2	0.89	0.87	0.94	1.20	1.29	1.26	2.36	2.47
16	3	0.73	0.81	0.87	0.98	1.17	1.11	2.00	2.20
16	4	0.57	0.75	0.81	0.79	1.03	0.96	1.74	1.94
17	1	1.00	0.94	1.00	1.31	1.28	1.31	2.36	2.30
17	2	0.89	0.88	0.94	1.19	1.28	1.24	2.30	2.40
17	3	0.74	0.82	0.88	0.97	1.16	1.10	1.95	2.14
17	4	0.59	0.76	0.82	0.79	1.03	0.96	1.69	1.89
18	1	1.00	0.94	1.00	1.30	1.27	1.30	2.31	2.25
18	2	0.90	0.89	0.94	1.18	1.27	1.23	2.24	2.35

$\hat{M}_r$	$\hat{W}_r$	$\hat{E}_r^4$	$\hat{M}_r$	$\hat{W}_r$	$\hat{E}_r^5$	$\hat{M}_r$
8.94	30.04	18.71	30.04	106.03	67.50	106.03
6.34	20.12	15.43	20.12	68.97	53.39	68.97
5.13	15.55	12.95	15.55	51.94	43.48	51.94
4.42	12.89	11.23	12.89	42.12	36.78	42.12
4.56	14.67	14.67	14.73	51.27	51.14	51.30
3.95	11.15	9.99	11.15	35.71	32.03	35.71
4.05	12.56	12.64	12.63	43.20	43.23	43.27
3.61	9.91	9.05	9.91	31.19	28.49	31.19
3.68	11.06	11.18	11.14	37.47	37.59	37.56
3.35	8.99	8.32	8.99	27.82	25.75	27.82
3.41	9.94	10.08	10.03	33.19	33.36	33.29
3.15	8.27	7.74	8.27	25.21	23.57	25.21
3.19	9.08	9.23	9.16	29.88	30.07	29.98
2.99	7.69	7.25	7.69	23.12	21.79	23.12
3.02	8.38	8.54	8.47	27.24	27.24	27.34
2.68	7.70	7.98	7.87	26.09	26.44	26.30
2.85	7.21	6.85	7.21	21.41	20.31	21.41
2.87	7.82	7.97	7.90	25.08	25.29	25.19
2.55	7.14	7.41	7.31	23.92	24.27	24.13
2.74	6.81	6.50	6.81	19.99	19.06	19.99
2.75	7.34	7.50	7.42	23.28	23.49	23.39
2.44	6.68	6.94	6.84	22.12	22.46	22.32
2.64	6.47	6.21	6.47	18.78	17.98	18.78
2.65	6.94	7.09	7.02	21.76	21.97	21.87
2.34	6.29	6.54	6.44	20.60	20.93	20.80
2.56	6.17	5.95	6.17	17.74	17.05	17.74
2.56	6.60	6.75	6.67	20.46	20.67	20.56
2.26	5.95	6.20	6.11	19.30	19.62	19.50
2.49	5.92	5.72	5.92	16.84	16.24	16.84
2.48	6.30	6.44	6.37	19.33	19.54	19.43
2.19	5.66	5.90	5.81	18.18	18.49	18.37
1.94	5.23	5.45	5.38	17.48	17.74	17.66
2.42	5.69	5.51	5.69	16.04	15.51	16.04
2.41	6.03	6.18	6.11	18.34	18.55	18.44
2.13	5.41	5.64	5.56	17.20	17.50	17.39
1.88	4.98	5.20	5.13	16.49	16.74	16.67
2.36	5.49	5.33	5.49	15.34	14.87	15.34
2.35	5.80	5.94	5.87	17.47	17.67	17.57
2.07	5.19	5.41	5.33	16.33	16.63	16.52
1.82	4.76	4.97	4.91	15.62	15.87	15.80
2.31	5.31	5.17	5.31	14.71	14.29	14.71
2.30	5.59	5.73	5.66	16.69	16.89	16.79

Table 8. (Continued)

Shape parameter			$\hat{E}_r^1$	$\hat{M}_r$	$\hat{W}_r$	$\hat{E}_r^2$	$\hat{M}_r$	$\hat{W}_r$	$\hat{E}_r^3$
n	r	$\hat{W}_r$							
18	3	0.75	0.83	0.89	0.97	1.16	1.09	1.90	2.09
18	4	0.60	0.78	0.83	0.79	1.02	0.95	1.64	1.84
19	1	1.00	0.95	1.00	1.29	1.26	1.29	2.26	2.21
19	2	0.90	0.89	0.95	1.18	1.26	1.22	2.20	2.30
19	3	0.76	0.84	0.89	0.96	1.15	1.09	1.86	2.04
19	4	0.61	0.79	0.84	0.78	1.01	0.95	1.60	1.79
20	1	1.00	0.95	1.00	1.28	1.25	1.28	2.22	2.17
20	2	0.91	0.90	0.95	1.17	1.25	1.22	2.15	2.25
20	3	0.77	0.85	0.90	0.96	1.14	1.08	1.82	2.00
20	4	0.62	0.80	0.85	0.78	1.01	0.94	1.57	1.75
20	5	0.50	0.75	0.80	0.65	0.89	0.82	1.39	1.57
21	1	1.00	0.95	1.00	1.27	1.24	1.27	2.18	2.14
21	2	0.91	0.90	0.95	1.16	1.24	1.21	2.11	2.21
21	3	0.78	0.86	0.90	0.96	1.14	1.08	1.79	1.96
21	4	0.64	0.81	0.86	0.78	1.01	0.94	1.53	1.72
21	5	0.51	0.76	0.81	0.65	0.88	0.82	1.36	1.53
22	1	1.00	0.95	1.00	1.26	1.24	1.26	2.15	2.11
22	2	0.92	0.91	0.95	1.16	1.24	1.20	2.08	2.18
22	3	0.79	0.86	0.91	0.91	1.04	1.00	1.67	1.78
22	4	0.65	0.82	0.86	0.76	0.95	0.89	1.47	1.62
22	5	0.52	0.77	0.82	0.64	0.85	0.79	1.31	1.47
23	1	1.00	0.96	1.00	1.25	1.23	1.25	2.11	2.07
23	2	0.92	0.91	0.96	1.15	1.23	1.19	2.04	2.13
23	3	0.79	0.87	0.91	0.95	1.12	1.07	1.73	1.89
23	4	0.66	0.83	0.87	0.78	1.00	0.94	1.48	1.66
23	5	0.53	0.78	0.83	0.65	0.88	0.82	1.30	1.47
24	1	1.00	0.96	1.00	1.24	1.22	1.24	2.08	2.05
24	2	0.92	0.92	0.96	1.14	1.22	1.19	2.01	2.10
24	3	0.80	0.87	0.92	0.95	1.12	1.06	1.70	1.86
24	4	0.67	0.83	0.87	0.78	1.00	0.93	1.46	1.63
24	5	0.55	0.79	0.83	0.65	0.88	0.82	1.28	1.45
25	1	1.00	0.96	1.00	1.24	1.22	1.24	2.05	2.02
25	2	0.93	0.92	0.96	1.14	1.21	1.18	1.98	2.07
25	3	0.81	0.88	0.92	0.95	1.11	1.06	1.68	1.83
25	4	0.68	0.84	0.88	0.78	0.99	0.93	1.48	1.60
25	5	0.56	0.80	0.84	0.65	0.88	0.82	1.26	1.42
25	6	0.45	0.76	0.80	0.55	0.77	0.72	1.13	1.29

$\hat{M}_r$	$\hat{W}_r$	$\hat{E}_r$	$\hat{M}_r$	$\hat{W}_r$	$\hat{E}_r$	$\hat{M}_r$
2.02	4.99	5.21	5.13	15.57	15.86	15.75
1.78	4.56	4.77	4.71	14.85	15.10	15.02
2.26	5.14	5.02	5.14	14.15	13.77	14.15
2.25	5.41	5.54	5.47	16.00	16.19	16.09
1.98	4.81	5.02	4.95	14.88	15.16	15.06
1.74	4.39	4.59	4.53	14.16	14.41	14.34
2.22	5.00	4.88	5.00	13.64	13.30	13.64
2.20	5.24	5.37	5.30	15.37	15.56	15.47
1.94	4.65	4.86	4.78	14.26	14.54	14.44
1.70	4.23	4.43	4.37	13.55	13.79	13.72
1.52	3.96	4.14	4.09	13.14	13.33	13.28
2.18	4.86	4.76	4.86	13.18	12.87	13.18
2.16	5.08	5.21	5.15	14.80	14.99	14.90
1.90	4.51	4.71	4.64	13.71	13.98	13.88
1.66	4.09	4.29	4.23	12.99	13.23	13.16
1.49	3.82	3.99	3.95	12.57	12.76	12.72
2.15	4.78	4.68	4.78	12.78	12.47	12.78
2.11	4.39	4.38	4.39	13.15	13.08	13.12
1.74	4.24	4.37	4.32	12.99	13.17	13.11
1.58	3.92	4.08	4.03	12.43	12.64	12.58
1.43	3.67	3.83	3.79	12.05	12.23	12.18
2.11	4.63	4.54	4.63	12.37	12.11	12.37
2.09	4.82	4.93	4.87	13.81	13.99	13.90
1.83	4.26	4.45	4.38	12.74	12.99	12.90
1.61	3.84	4.03	3.98	12.02	12.25	12.19
1.43	3.57	3.74	3.70	11.60	11.79	11.74
2.08	4.52	4.44	4.52	12.01	11.77	12.01
2.06	4.70	4.81	4.75	13.38	13.55	13.47
1.81	4.15	4.33	4.26	12.32	12.56	12.48
1.58	3.74	3.92	3.87	11.60	11.83	11.76
1.41	3.47	3.63	3.59	11.17	11.36	11.32
2.05	4.42	4.35	4.42	11.68	11.46	11.68
2.03	4.59	4.70	4.64	12.98	13.15	13.06
1.78	4.04	4.22	4.16	11.98	12.17	12.08
1.55	3.64	3.82	3.77	11.21	11.43	11.37
1.38	3.37	3.53	3.49	10.78	10.97	10.92
1.25	3.19	3.33	3.30	10.52	10.67	10.64

For a given shape parameter, if the values in Table 8 are multiplied by the corresponding values from Table 7, we have an indication of relative efficiency of the censored estimates with respect to the maximum likelihood estimate.

That is

$$\frac{V^*(\hat{\eta})}{MSE(\bar{x})} \frac{MSE(\bar{x})}{MSE(\hat{\eta})} = \frac{V^*(\hat{\eta})}{MSE(\hat{\eta})} .$$

For the once censored case with a sample size of 25 these values are all greater than or equal to one. That is, once censored estimators of this type have a lower mean square error than does the maximum likelihood estimate. The gains in efficiency for the once censored case range from about 15 percent for $a = 2$ to over 300 percent for $a = 5$. From the results of Splitstone (40) on the variances of the maximum likelihood estimators of the parameters of the Weibull for finite samples we know that the asymptotic variance differs from the finite result by less than six percent for a sample size of 25. For example Splitstone's estimate of the variance of the estimated shape parameter of the Weibull based on a sample of size 25 is .102 while the asymptotic result is .096 . Of course the difference decreases as the sample size increases. This result suggests that for moderate sized samples, since the asymptotic variance of the shape parameter approximates that of the true variance, the use of the asymptotic form in place of the true value provides a good indication of relative efficiency.

In Chapter IV we considered the estimate $\hat{M}_0$, the minimum mean square error estimate of the mean of the exponential distribution

$$\text{MSE}(M_0) = \left(\frac{c}{n+1}\right)^2 (n\Gamma(2a+1) - (n-1)(\Gamma(a+1))^2).$$

Recalling that $\text{MSE}(\bar{X}) = \frac{1}{n}c^2(\Gamma(2a+1) - (\Gamma(a+1))^2)$, note that as the sample size increases we see that $\text{REM}(\hat{M}_0)$ approaches unity quite rapidly. Therefore the use of the REM instead of the relative efficiency based on $\hat{M}_0$ does not have a significant effect on the values of Table 8.

C. Estimation of the Sum of the Largest Order Statistics

We now consider the estimation of the expected value of the sum of the largest two order statistics, $E(Y_{(n-1)} + Y_{(n)})$. We shall compare the two estimators $Y_{(n-1)} + Y_{(n)}$ and $2Y_{(n-1)}$. This study will allow us to make some specific statements about the once censored estimators for the mean based on a sample of n observations which are discussed above. We must first develop some preliminary results.

We will show that for Weibull order statistics, $Y_{(i)}$ with integer shape parameter, a , larger than one that $C(Y_{(n-j)}, Y_{(n)}) > C(Y_{(n-j)}, Y_{(n-1)})$. We will do this by considering the exponential random variable X such that $X_{(i)}^a = Y_{(i)}$. In Chapter IV we have stated that for the exponential

distribution $C(X_{(n-j)}, X_{(n)}) = C(X_{(n-j)}, X_{(n-1)})$ since $C(X_{(n-j)}, X_{(n-k)}) = V(X_{(n-j)})$ if $n-j < n-k$. We now show for $a > 1$ that $C(X_{(n-j)}^a, X_{(n)}^a) > C(X_{(n-j)}^a, X_{(n-1)}^a)$. From Equation 2.19 we have

$$X_{(n)} = X_{(n-j)} + W_{n,j} \quad (5.22)$$

$$X_{(n-1)} = X_{(n-j)} + W_{n-1,j-1}$$

and from Equation 2.22

$$\begin{aligned} E((X_{(n-j)} X_{(n)})^a) &= E(X_{(n-j)}^a \sum_{i=0}^a \binom{a}{i} X_{(n-j)}^i W_{n,j}^{a-i}) \\ &= \sum_{i=0}^a \binom{a}{i} E(X_{(n-j)}^{a+i}) E(W_{n,j}^{a-i}) \end{aligned} \quad (5.23)$$

$$E((X_{(n-j)} X_{(n-1)})^a) = \sum_{i=0}^a \binom{a}{i} E(X_{(n-j)}^{a+i}) E(W_{n-1,j-1}^{a-i}).$$

Hence

$$\begin{aligned} C(X_{(n-j)}^a, X_{(n)}^a) &= \sum_{i=0}^a \binom{a}{i} E(X_{(n-j)}^{a+i}) E(W_{n,j}^{a-i}) - E(X_{(n-j)}^a) E(X_{(n)}^a) \\ &= \sum_{i=0}^a \binom{a}{i} E(X_{(n-j)}^{a+i}) E(W_{n,j}^{a-i}) - E(X_{(n-j)}^a) \sum_{i=0}^a \binom{a}{i} E(X_{(n-j)}^i) E(W_{n,j}^{a-i}) \\ &= \sum_{i=0}^a \binom{a}{i} (E(X_{(n-j)}^{a+i}) - E(X_{(n-j)}^a) E(X_{(n-j)}^i)) E(W_{n,j}^{a-i}). \end{aligned} \quad (5.24)$$

Similarly

$$\begin{aligned} C(X_{(n-j)}^a, X_{(n-1)}^a) &= \sum_{i=0}^a \binom{a}{i} (E(X_{(n-j)}^{a+i}) \\ &\quad - E(X_{(n-j)}^a) E(X_{(n-j)}^i)) E(W_{n-1,j-1}^{a-i}). \end{aligned} \quad (5.25)$$

Hence $C(Y_{(n-j)}, Y_{(n)}) > C(Y_{(n-j)}, Y_{(n-1)})$ if $E(W_{n,j}^{a-1}) -$

$E(W_{n-1,j-1}^{a-1}) > 0$, but

$$W_{n,j} = X_{(n)} - X_{(n-j)} > X_{(n-1)} - X_{(n-j)} = W_{n-1,j-1} \quad (5.26)$$

since $X_{(n)} > X_{(n-1)}$. Therefore we have the desired result.

Lemma: For Y a Weibull random variable with integer shape parameter $a > 1$

$$C(Y_{(n-j)}, Y_{(n)}) > C(Y_{(n-j)}, Y_{(n-1)}) \quad (5.27)$$

for $j \geq 1$.

From Equation 5.27 we write

$$\begin{aligned} C(Y_{(n-j)}, Y_{(n)}) - C(Y_{(n-j)}, Y_{(n-1)}) &> 0 \\ E(Y_{(n-j)}(Y_{(n)} - Y_{(n-1)})) - E(Y_{(n-j)})E(Y_{(n)} - Y_{(n-1)}) &> 0 \\ C(Y_{(n-j)}, Y_{(n)} - Y_{(n-1)}) &> 0 \quad j \geq 1. \end{aligned} \quad (5.28)$$

That is, the covariance of any lower Weibull order statistic with the difference $Y_{(n)} - Y_{(n-1)}$ is positive.

We now investigate the variance of certain linear combinations of order statistics. To facilitate the discussion we introduce the $n \times 1$ vectors

$$A'_1 = (1, \dots, 1, 1, 1) \quad (5.29)$$

$$A'_2 = (1, \dots, 1, 2, 0)$$

$$A_3' = (0, \dots, 0, -1, 1)$$

$$B_1' = (0, \dots, 0, 1, 1)$$

$$B_2' = (0, \dots, 0, 2, 0)$$

$$D' = (1, \dots, 1, 0, 0).$$

Then $A_1 = A_2 + A_3$, $B_1 = B_2 + A_3$, $A_1 = D + B$, and $A_2 = D + B_2$. Let Y be the $n \times 1$ vector of Weibull order statistics and recall $V(Y) = \Sigma$ and $E(Y) = \epsilon$. The $n \times n$ matrix Σ and the $n \times 1$ vector ϵ are known from the results of Chapter II. Hence

$$A_1'Y = \sum_{i=1}^{n-1} Y_{(i)} + Y_{(n)} \quad (5.30)$$

$$A_2'Y = \sum_{i=1}^{n-1} Y_{(i)} + Y_{(n-1)}$$

$$B_1'Y = Y_{(n-1)} + Y_{(n)}$$

$$B_2'Y = 2Y_{(n-1)}$$

$$D'Y = \sum_{i=1}^{n-2} Y_{(i)}.$$

It can be easily shown by using the matrices of Equation 5.29 and the result in Equation 5.28 that for Weibulls with $a > 1$

$$\begin{aligned} 0 < V(Y_{(n-1)} + Y_{(n)}) - V(2Y_{(n-1)}) &< V\left(\sum_{i=1}^{n-1} Y_{(i)} + Y_{(n)}\right) \\ &- V\left(\sum_{i=1}^{n-1} Y_{(i)} + Y_{(n-1)}\right). \end{aligned} \quad (5.31)$$

We will estimate the mean of the Weibull by estimating the

sum $E(\sum_{i=1}^n Y_{(i)})$ with $Y_{(1)} + \dots + 2Y_{(n-1)}$. The sum $Y_{(1)} + \dots + Y_{(n)}$ of observations is unbiased for $E(\sum_{i=1}^n Y_{(i)})$. We will compare the mean square errors of the two estimates. Now

$$\begin{aligned} \text{MSE}(\sum_{i=1}^{n-1} Y_{(i)} + Y_{(n)}) - \text{MSE}(\sum_{i=1}^{n-1} Y_{(i)} + Y_{(n-1)}) &= \\ A_1' A_1 - A_2' A_2 - ((A_2 - A_1)' \epsilon)^2 & \quad (5.32) \\ = (A_2 + A_3)' (A_2 + A_3) - A_2' A_2 - (A_3' \epsilon)^2 \\ = A_3' A_3 + 2A_2' A_3 - (A_3' \epsilon)^2. \end{aligned}$$

We estimate the sum $E(Y_{(n-1)} + Y_{(n)})$ with $2Y_{(n-1)}$. The sum $Y_{(n-1)} + Y_{(n)}$ of observations is unbiased for $E(Y_{(n-1)} + Y_{(n)})$. We will compare the mean square errors of the two estimates.

$$\begin{aligned} \text{MSE}(Y_{(n-1)} + Y_{(n)}) - \text{MSE}(2Y_{(n-1)}) &= \\ B_1' B_1 - B_2' B_2 - ((B_2 - B_1)' \epsilon)^2 & \quad (5.33) \\ = (B_2 + A_3)' (B_2 + A_3) - B_2' B_2 - (A_3' \epsilon)^2 \\ = A_3' A_3 + 2B_2' A_3 - (A_3' \epsilon)^2. \end{aligned}$$

Hence the difference of the left-hand sides of Equations 5.32 and 5.33 is $2(A_2 - B_2)' A_3$. But

$$\begin{aligned} 2(A_2 - B_2)' A_3 &= 2D' A_3 \\ &= 2C(\sum_{i=1}^{n-2} Y_{(i)}, Y_{(n)} - Y_{(n-1)}) . \end{aligned}$$

From Equation 5.28 this covariance is positive. Then we have

$$\begin{aligned} \text{MSE}(Y_{(n-1)} + Y_{(n)}) - \text{MSE}(2Y_{(n-1)}) &< \text{MSE}\left(\sum_{i=1}^{n-1} Y_{(i)} + Y_{(n)}\right) \\ &- \text{MSE}\left(\sum_{i=1}^{n-1} Y_{(i)} + Y_{(n-1)}\right). \end{aligned} \quad (5.34)$$

We would wish to show that the difference in Equation 5.33 is positive. Unfortunately this result is extremely difficult to prove. The covariance $2B_2'A_3$ is positive from Equation 5.28 for $a > 1$. If $A_3'A_3 > (A_3'\epsilon)^2$ we would have the desired result. That is if $V(Y_{(n)} - Y_{(n-1)}) > (E(Y_{(n)} - Y_{(n-1)}))^2$ we would have the difference in 5.33 being positive. Under this assumption we would have, taking the square root of both sides $\sqrt{V(Y_{(n)} - Y_{(n-1)})}/E(Y_{(n)} - Y_{(n-1)}) > 1$. But this is exactly the coefficient of variation of $Y_{(n)} - Y_{(n-1)}$. Hence if the coefficient of variation were greater than one the right-hand side of Equation 5.33 would be positive. For the exponential distribution it is known that the coefficient of variation is one. Now for the Weibull with shape parameter $a > 1$ it is heuristically clear that the difference $X_{(n)}^a - X_{(n-1)}^a$ would be more variable than $X_{(n)} - X_{(n-1)}$ since the distribution is more skewed and hence the coefficient of variation would be greater than one. This is verified for small samples.

Now we are able to show that the left-hand side of Equation 5.34 is positive for the larger sample sizes. We shall do this using the asymptotic theory developed in

Chapter III. In particular we use a special case of Equation 3.28, namely

$$E^*(X_{(n-1)}^m) = E^*(X_{(n)}^m) - mE^*(X_{(n)}^{m-1}). \quad (5.35)$$

Now we have already stated that $Y_{(n-1)} + Y_{(n)}$ estimates $E(Y_{(n-1)} + Y_{(n)})$ unbiasedly. However $2Y_{(n-1)}$ is biased for $E(Y_{(n-1)} + Y_{(n)})$ and the bias is

$$\begin{aligned} E^*(X_{(n-1)}^a - X_{(n)}^a) &= E^*(X_{(n-1)}^a) - E^*(X_{(n)}^a) \\ &= E^*(X_{(n)}^a) - aE^*(X_{(n)}^{a-1}) - E^*(X_{(n)}^a) \\ &= -aE^*(X_{(n)}^{a-1}) \end{aligned} \quad (5.36)$$

$$(\text{Bias } (2Y_{(n-1)}))^2 = a^2 E^{*2}(X_{(n)}^{a-1}). \quad (5.37)$$

Next we find the asymptotic variance of the random variable $2Y_{(n-1)}$

$$\begin{aligned} V^*(2Y_{(n-1)}) &= V^*(2X_{(n-1)}^a) = 4(E^*(X_{(n-1)}^{2a}) - E^{*2}(X_{(n-1)}^a)) \quad (5.38) \\ &= 4(E^*(X_{(n)}^{2a}) - 2aE^*(X_{(n)}^{2a-1}) - (E^*(X_{(n)}^a) - aE^*(X_{(n)}^{a-1}))^2) \\ &= 4E^*(X_{(n)}^{2a}) - 8aE^*(X_{(n)}^{2a-1}) - 4E^{*2}(X_{(n)}^a) \\ &\quad - 4a^2E^{*2}(X_{(n)}^{a-1}) + 8aE^*(X_{(n)}^a)E^*(X_{(n)}^{a-1}). \end{aligned}$$

Hence the asymptotic mean square error of $2Y_{(n-1)}$ is

$$\begin{aligned} \text{MSE}^*(2Y_{(n-1)}) &= 4E^*(X_{(n)}^{2a}) - 8aE^*(X_{(n)}^{2a-1}) - 4E^{*2}(X_{(n)}^a) \\ &\quad - 3a^2E^{*2}(X_{(n)}^{a-1}) + 8aE^*(X_{(n)}^a)E^*(X_{(n)}^{a-1}). \end{aligned} \quad (5.39)$$

Now the approximate asymptotic variance and thus the mean square error of $Y_{(n-1)} + Y_{(n)}$ is

$$\text{MSE}^*(Y_{(n-1)} + Y_{(n)}) = V^*(X_{(n-1)}^a) + V^*(X_{(n)}^a) + 2C^*(X_{(n-1)}^a, X_{(n)}^a) \quad (5.40)$$

$$\begin{aligned} &= E^*(X_{(n-1)}^{2a}) - E^{*2}(X_{(n-1)}^a) + E^*(X_{(n)}^{2a}) - E^{*2}(X_{(n)}^a) \\ &\quad + 2(E^*(X_{(n-1)}^a X_{(n)}^a) - E^*(X_{(n)}^a) E^*(X_{(n-1)}^a)) \\ &= 2E^*(X_{(n)}^{2a}) - 2aE^*(X_{(n)}^{2a-1}) - 4E^{*2}(X_{(n)}^a) - a^2 E^{*2}(X_{(n)}^{a-1}) \\ &\quad + 4aE^*(X_{(n)}^a) E^*(X_{(n)}^{a-1}) + 2E^*(X_{(n-1)}^a X_{(n)}^a). \end{aligned}$$

From Equation 3.30 and using the fact that $E(W_{n,1}^{a-1}) = (a-1)!$ we have

$$\begin{aligned} E^*(X_{(n-1)}^a X_{(n)}^a) &= \sum_{i=1}^a \frac{a!}{i!} E^*(X_{(n-1)}^{a+1}) \\ &= \sum_{i=1}^a \frac{a!}{i!} (E^*(X_{(n)}^{a+1}) - (a+1)E^*(X_{(n)}^{a+1-1})). \end{aligned} \quad (5.41)$$

Now the highest two indices of 5.41 contain the highest moments of $X_{(n)}$. Substituting 5.41 into Equation 5.40 we write

$$\begin{aligned} \text{MSE}^*(Y_{(n-1)} + Y_{(n)}) &= 4E^*(X_{(n)}^{2a}) - 4aE^*(X_{(n)}^{2a-1}) \\ &\quad - 2(2a-1)E^*(X_{(n)}^{2a-2}) - 4E^{*2}(X_{(n)}^a) - a^2 E^*(X_{(n)}^{a-1}) \\ &\quad + 4aE^*(X_{(n)}^a) E^*(X_{(n)}^{a-1}) \\ &\quad + 2a! \sum_{i=1}^{a-2} (E^*(X_{(n)}^{a+1}) - (a+1)E^*(X_{(n)}^{a+1-1}))/i! \end{aligned} \quad (5.42)$$

Now taking the difference of Equations 5.41 and 5.42

$$\text{MSE}^*(Y_{(n-1)} + Y_{(n)}) - \text{MSE}^*(2Y_{(n-1)}) \quad (5.43)$$

$$\begin{aligned} &= 4aE^*(X_{(n)}^{2a-1}) - (4a-2)E^*(X_{(n)}^{2a-2}) + 2a^2E^*(X_{(n)}^{a-1}) \\ &\quad - 4aE^*(X_{(n)}^a)E^*(X_{(n)}^{a-1}) + 2a! \sum_{i=0}^{a-2} (E^*(X_{(n)}^{a+1-i}) \\ &\quad - (a+1)E^*(X_{(n)}^{a+1-i-1}))/i! \end{aligned}$$

The difference has, as we would expect, a value of zero for a equal to one. Substituting Equation 3.27 into Equation 5.43 we see that the difference $\text{MSE}^*(Y_{(n-1)} + Y_{(n)}) - \text{MSE}^*(2Y_{(n-1)})$ is expressible as a polynomial in $\log n$ since $E^*(X_{(n)}^m) = \sum_{i=0}^m \binom{m}{i} (\log n)^{m-i} I(i)$. Study of Equation 5.43 shows us that the highest possible degree of this polynomial is $2a-1$. However if we separate the two terms of this degree, letting $R(i)$ be a polynomial of degree i in $\log n$, we have

$$\begin{aligned} 4a(E^*(X_{(n)}^{2a-1}) - E^*(X_{(n)}^a)E^*(X_{(n)}^{a-1})) &= 4a((\log n)^{2a-1} \\ &\quad + (2a-1)I(1)(\log n)^{2a-2} + R(2a-3) - ((\log n)^a \\ &\quad + aI(1)(\log n)^{a-1} + R(a-2))((\log n)^{a-1} \\ &\quad + (a-1)I(1)(\log n)^{a-2} + R(a-3)) \\ &= 4a((\log n)^{2a-1} + (2a-1)I(1)(\log n)^{2a-2} - \log n)^{2a-1} \\ &\quad - aI(1)(\log n)^{2a-2} - (a-1)I(1)(\log n)^{2a-2} + R(2a-3)) \end{aligned}$$

$$= 4a(R(2a-3))$$

where $R(2a-3) = 0$ if $2a-3 < 0$.

Hence the degree of the polynomial is $2a-2$. Collecting the terms in 5.43 which have degree $2a-2$, we have

$$-(4a-2)E^*(X_{(n)}^{2a-2}) + 2a^2E^*(X_{(n)}^{a-1}) + 2a(a-1)E^*(X_{(n)}^{2a-2}) = \\ (4a^2 - 6a + 2)(\log n)^{2a-2} + R(2a-3).$$

Then we are able to write Equation 5.43 as

$$MSE^*(Y_{(n-1)} + Y_{(n)}) - MSE^*(2Y_{(n-1)}) = \\ (4a^2 - 6a + 2)(\log n)^{2a-2} + R(2a-3) \quad (5.44)$$

where Y is a Weibull with shape parameter a . Now due to the binomial form of $E(X_{(n)}^m)$ we know that the terms of $R(2a-3)$ are of the form $c_1 (\log n)^{2a-2-1}$, $1 = 1, 2, \dots$, $2a-2$ where c_1 are polynomials in a and the constants $I(j)$. The values of $I(j)$ are in Table 4. Hence the right-hand side of Equation 5.44 is a continuous function in a . The derivative of this function is

$$(8a-6)(\log n)^{2a-2} + 2(4a^2-6a+2)(\log(\log n))(\log n)^{2a-2} \\ + \sum_{i=1}^{2a-2} \frac{d}{da} c_1 (\log n)^{2a-2-1} \quad (5.45)$$

Since the leading coefficient of the right-hand side of Equation 5.43 is positive for $a > 1$ and the leading

coefficient of the derivative is positive for $a > 1$ we conclude that for large sample sizes, the difference in Equation 5.43 is an increasing function in a and is positive for $a > 1$. We are only able to check integer values of a , for example for $a = 2$

$$\begin{aligned} \text{MSE}^*(Y_{(n-1)} + Y_{(n)}) - \text{MSE}^*(2Y_{(n-1)}) &= 8E^*(X_{(n)}^3) \\ &\quad - 2E^*(X_{(n)}^2) + 8(E^*(X_{(n)}^2) - E^*(X_{(n)}^2) - E^*(X_{(n)})) \\ &= 6(\log n)^2 + 25.27 \log n + 32.8 > 0. \end{aligned}$$

Now from Equations 5.44 and 5.45 we conclude that

$$\text{MSE}^*(Y_{(n-1)} + Y_{(n)}) - \text{MSE}^*(2Y_{(n-1)}) > 0 \quad (5.46)$$

for $a > 1$. Therefore from Equation 5.34

$$\text{MSE}(\bar{Y}_{(1)}^{n-1} + Y_{(n-1)}) < \text{MSE}(\bar{Y}_{(1)}^{n-1} + Y_{(n)})$$

and

$$\text{MSE}(\hat{W}_1) < \text{MSE}(\bar{Y}) \quad (5.47)$$

for large n . We have already shown that 5.47 is true for samples of size up to 25. Hence we conclude that 5.47 is true for all values of n .

Then we have the relative efficiency with respect to the mean

$$\text{REM}(\hat{W}_1) = \frac{\text{MSE}(\bar{Y})}{\text{MSE}(\hat{W}_1)} = 1 + \frac{\text{MSE}(\bar{Y}) - \text{MSE}(\hat{W}_1)}{\text{MSE}(\hat{W}_1)} \geq 1 \quad (5.48)$$

since the $\text{MSE}(\hat{W}_1)$ is positive and expressible using Equations 3.27 and 3.28 as a polynomial in $\log n$ of degree $2a$. Hence the quotient on the right-hand side of Equation 5.48 approaches zero as n increases without bound since the highest power of $\log n$ in the numerator is $2a-2$. We have already noted that equality also occurs if a is equal to one.

D. Tests for Departure from the Exponential

It is known that the distribution of the k -th order statistic in a sample of size n from a distribution with distribution function F and density function f is

$$f_{X_{(k)}}(x) = k \binom{n}{k} F(x)^{k-1} (1-F(x))^{n-k} f(x) \quad (5.49)$$

Hence the distribution of the k -th order statistic of a standard exponential is

$$f_{X_{(k)}}(x) = k \binom{n}{k} (1-e^{-x})^{k-1} e^{-x(n-k)} e^{-x} \quad (5.50)$$

Using the binomial theorem on Equation 5.50 gives us

$$f_{X_{(k)}}(x) = k \binom{n}{k} \sum_{i=0}^{k-1} \binom{k-1}{i} (-1)^i e^{(1+n-k+1)x} \quad (5.51)$$

Now from Fuller (18) if Δ is a small number

$$\int x^{(1+\Delta)} e^{-ax} dx = \int (a^{-1}y)^{1+\Delta} e^{-y} a^{-1} dy$$

where $y = ax$. Hence

$$\int_0^\infty x^{1+\Delta} e^{-ax} dx = a^{-2-\Delta} \Gamma(2+\Delta) . \quad (5.52)$$

Furthermore if we expand the result of Equation 5.52 as a linear function of Δ we have

$$a^{-2-\Delta} \Gamma(2+\Delta) \doteq a^{-2} \Gamma(2) - a^{-2} (\log a) \Gamma(2) \Delta + a^{-2} \Gamma^{(1)}(2) \Delta^2 \quad (5.53)$$

where Γ is the gamma function. Hence $\Gamma(2) = 1$ and $\Gamma^{(1)}(2) = .4227$. From Equation 5.52 the expected value of the k -th standard exponential order statistic to the power $1+\Delta$ is

$$\begin{aligned} E(X_{(k)}^{1+\Delta}) &= \int_0^\infty x^{1+\Delta} f_{X_{(k)}}(x) dx \\ &= k \binom{n}{k} \sum_{i=0}^{k-1} \binom{k-1}{i} (-1)^i (1+n-k+1)^{-2-\Delta} \Gamma(2+\Delta) . \end{aligned} \quad (5.54)$$

Using the approximation of Equation 5.53 we write Equation 5.54 as

$$\begin{aligned} E(X_{(k)}^{1+\Delta}) &\doteq k \binom{n}{k} \sum_{i=0}^{k-1} \binom{k-1}{i} (-1)^i ((1+n-k+1)^{-2} \\ &\quad (1 + (-\log(1+n-k+1) + .4227)\Delta)) . \end{aligned} \quad (5.55)$$

Letting

$$A_k = \sum_{i=0}^{k-1} 1/n-1$$

and

$$B_k = k \binom{n}{k} \sum_{i=0}^{k-1} \binom{k-1}{i} (-1)^i (1+n-k+1)^{-2} \log(1+n-k+1)$$

so we write Equation 5.55 as

$$E(X_{(k)}^{1+\Delta}) = A_k + (.4227A_k - B_k)\Delta \quad (5.56)$$

Recall Equation 1.1 and consider the random variables

$$Z_k = (n-k+1)(X_{(n-k+1)}^{1+\Delta} - X_{(n-k)}^{1+\Delta}) \quad (5.57)$$

to be approximately uncorrelated. Then

$$E(Z_k) \doteq 1 + (.4227 + D_k)\Delta \quad (5.58)$$

where

$$D_k = -(n-k+1)(B_k - B_{k-1})$$

$$D_k = -(n-k+1) \binom{n}{k-1} \sum_{i=0}^{k-1} \binom{k-1}{i} (-1)^i \frac{\log(1+n-k+1)}{(1+n-k+1)} \quad (5.59)$$

The D_k are increasing with k and sum to zero. The values for D_k for sample sizes from 2 to 30 are found in Table 9. Suppose there is an initial estimate for a say a_0 . Write the variables Z_k as

$$Z_k = (n-k+1)(Y_{(k)}^{1/a_0} - Y_{(k-1)}^{1/a_0}) \quad (5.60)$$

where $Y_{(0)}$ is understood to be zero. Now let $1/a_0 = (1/a)(1+\Delta)$ and hence from Equation 5.58

$$E(Z_k) \doteq c^{1/a_0} + (.4227 + D_k)c^{1/a_0} \Delta \quad (5.61)$$

Therefore it is possible to estimate $c^{1/a}$ and $c^{1/a} \Delta$ by the

TABLE 9.

Coefficients for the T statistic for a sample of size N					
N	VALUES 1,2,...,N				
2	-6.9315E-01	6.9315E-01			
3	-1.0986E+00	1.1778E-01	9.8083E-01		
4	-1.3863E+00	-2.3557E-01	4.7113E-01	1.1507E+00	
5	-1.6094E+00	-4.9372E-01	1.5167E-01	6.8411E-01	1.2674E+00
6	-1.7918E+00	-6.9783E-01	-8.5500E-02	3.8883E-01	8.3175E-01
	1.3545E+00				
7	-1.9459E+00	-8.6686E-01	-2.7527E-01	1.6752E-01	5.5481E-01
	9.4253E-01	1.4232E+00			
8	-2.0794E+00	-1.0112E+00	-4.3385E-01	-1.0961E-02	3.4601E-01
	6.8009E-01	1.0300E+00	1.4793E+00		
9	-2.1972E+00	-1.1372E+00	-5.7024E-01	-1.6108E-01	1.7608E-01
	4.8146E-01	7.7941E-01	1.1016E+00	1.5266E+00	
10	-2.3026E+00	-1.2490E+00	-6.8997E-01	-2.9087E-01	3.3603E-02
	3.1977E-01	5.8926E-01	8.6090E-01	1.1618E+00	1.5671E+00
11	-2.3979E+00	-1.3495E+00	-7.9671E-01	-4.0530E-01	-9.0599E-02
	1.8265E-01	4.3403E-01	6.7796E-01	9.2951E-01	1.2134E+00
	1.6025E+00				
12	-2.4849E+00	-1.4408E+00	-8.9305E-01	-5.0771E-01	-2.0049E-01
	6.3248E-02	3.0205E-01	5.2831E-01	7.5279E-01	9.8842E-01
	1.2584E+00	1.6337E+00			
13	-2.5649E+00	-1.5244E+00	-9.8084E-01	-6.0042E-01	-2.9912E-01
	-4.2682E-02	1.8683E-01	4.0080E-01	6.0800E-01	8.1713E-01
	1.0398E+00	1.2981E+00	1.6617E+00		
14	-2.6391E+00	-1.6015E+00	-1.0615E+00	-6.8513E-01	-3.8864E-01
	-1.3799E-01	8.4390E-02	2.8928E-01	4.8444E-01	6.7665E-01
	8.7333E-01	1.0852E+00	1.3336E+00	1.6869E+00	
15	-2.7081E+00	-1.6732E+00	-1.1361E+00	-7.6314E-01	-4.7062E-01

TABLE 9. (CONTINUED)

	-2.2467E-01	-7.9598E-03	1.8993E-01	3.7620E-01	5.5660E-01
	7.3667E-01	9.2302E-01	1.1257E+00	1.3656E+00	1.7099E+00
16	-2.7726E+00	-1.7400E+00	-1.2055E+00	-8.3543E-01	-5.4626E-01
	-3.0421E-01	-9.2107E-02	1.0023E-01	2.7964E-01	4.5130E-01
	6.1978E-01	7.8981E-01	9.6743E-01	1.1623E+00	1.3946E+00
	1.7309E+00				
17	-2.8332E+00	-1.8026E+00	-1.2703E+00	-9.0280E-01	-6.1649E-01
	-3.7772E-01	-1.6944E-01	1.8371E-02	1.9232E-01	3.5725E-01
	5.1714E-01	6.7577E-01	8.3732E-01	1.0075E+00	1.1955E+00
	1.4212E+00	1.7503E+00			
18	-2.8904E+00	-1.8615E+00	-1.3312E+00	-9.6587E-01	-6.8203E-01
	-4.4607E-01	-2.4102E-01	-5.6964E-02	1.1254E-01	2.7210E-01
	4.2537E-01	5.7554E-01	7.2588E-01	8.8019E-01	1.0438E+00
	1.2258E+00	1.4456E+00	1.7682E+00		
19	-2.9444E+00	-1.9172E+00	-1.3886E+00	-1.0252E+00	-7.4349E-01
	-5.0995E-01	-3.0766E-01	-1.2678E-01	3.9029E-02	1.9422E-01
	3.4220E-01	4.8585E-01	6.2786E-01	7.7111E-01	9.1914E-01
	1.0771E+00	1.2537E+00	1.4682E+00	1.7849E+00	
20	-2.9957E+00	-1.9699E+00	-1.4428E+00	-1.0811E+00	-8.0134E-01
	-5.6992E-01	-3.7002E-01	-1.9185E-01	-2.9168E-02	1.2238E-01
	2.6605E-01	4.0450E-01	5.4008E-01	6.7513E-01	8.1225E-01
	9.5478E-01	1.1076E+00	1.2794E+00	1.4892E+00	1.8004E+00
21	-3.0445E+00	-2.0199E+00	-1.4943E+00	-1.1341E+00	-8.5599E-01
	-6.2644E-01	-4.2862E-01	-2.5281E-01	-9.2798E-02	5.5673E-02
	1.9576E-01	3.2996E-01	4.6041E-01	5.8911E-01	7.1814E-01
	8.4989E-01	9.8755E-01	1.1359E+00	1.3034E+00	1.5087E+00
	1.8150E+00				
22	-3.0910E+00	-2.0676E+00	-1.5432E+00	-1.1844E+00	-9.0779E-01
	-6.7989E-01	-4.8391E-01	-3.1015E-01	-1.5246E-01	-6.6233E-02
	1.3043E-01	2.6109E-01	3.8735E-01	5.1099E-01	6.3375E-01
	7.5752E-01	8.8454E-01	1.0179E+00	1.1621E+00	1.3257E+00
	1.5270E+00	1.8287E+00			
23	-3.1355E+00	-2.1131E+00	-1.5898E+00	-1.2323E+00	-9.5701E-01
	-7.3059E-01	-5.3624E-01	-3.6429E-01	-2.0863E-01	-6.5080E-02
	6.9370E-02	1.9704E-01	3.1981E-01	4.3931E-01	5.5707E-01

TABLE 9. (CONTINUED)

	6.7465E-01	7.9377E-01	9.1657E-01	1.0460E+00	1.1866E+00
	1.3465E+00	1.5442E+00	1.8417E+00		
24	-3.1781E+00	-2.1566E+00	-1.6344E+00	-1.2780E+00	-1.0039E+00
	-7.7882E-01	-5.8593E-01	-4.1559E-01	-2.6171E-01	-1.2016E-01
	1.2034E-02	1.3713E-01	2.5694E-01	3.7299E-01	4.8667E-01
	5.9931E-01	7.1232E-01	8.2731E-01	9.4632E-01	1.0722E+00
	1.2094E+00	1.3661E+00	1.5604E+00	1.8539E+00	
25	-3.2189E+00	-2.1983E+00	-1.6770E+00	-1.3216E+00	-1.0487E+00
	-8.2480E-01	-6.3322E-01	-4.6432E-01	-3.1203E-01	-1.7225E-01
	-4.2030E-02	8.0841E-02	1.9811E-01	3.1125E-01	4.2152E-01
	5.3011E-01	6.3824E-01	7.4717E-01	8.5847E-01	9.7406E-01
	1.0968E+00	1.2309E+00	1.3846E+00	1.5757E+00	1.8655E+00
26	-3.2581E+00	-2.2384E+00	-1.7179E+00	-1.3635E+00	-1.0915E+00
	-8.6873E-01	-6.7834E-01	-5.1074E-01	-3.5987E-01	-2.2166E-01
	-9.3187E-02	2.7730E-02	1.4280E-01	2.5342E-01	3.6082E-01
	4.6603E-01	5.7015E-01	6.7428E-01	7.7956E-01	8.8754E-01
	1.0000E+00	1.1198E+00	1.2511E+00	1.4020E+00	1.5902E+00
	1.8765E+00				
27	-3.2959E+00	-2.2768E+00	-1.7572E+00	-1.4036E+00	-1.1326E+00
	-9.1080E-01	-7.2149E-01	-5.5506E-01	-4.0548E-01	-2.6807E-01
	-1.4175E-01	-2.2554E-02	9.0594E-02	1.9905E-01	3.0393E-01
	4.0628E-01	5.0714E-01	6.0715E-01	7.0781E-01	8.0984E-01
	9.1480E-01	1.0244E+00	1.1415E+00	1.2702E+00	1.4184E+00
	1.6039E+00	1.8870E+00			
28	-3.3317E+00	-2.3139E+00	-1.7950E+00	-1.4422E+00	-1.1720E+00
	-9.5116E-01	-7.6283E-01	-5.9747E-01	-4.4904E-01	-3.1350E-01
	-1.8797E-01	-7.0316E-02	4.1171E-02	1.4765E-01	2.5042E-01
	3.5028E-01	4.4823E-01	5.4504E-01	6.4165E-01	7.3910E-01
	8.3800E-01	9.4016E-01	1.0473E+00	1.1619E+00	1.2883E+00
	1.4341E+00	1.6170E+00	1.8970E+00		
29	-3.3671E+00	-2.3496E+00	-1.8315E+00	-1.4794E+00	-1.2100E+00
	-9.8994E-01	-8.0251E-01	-6.3812E-01	-4.9075E-01	-3.5636E-01
	-2.3208E-01	-1.1580E-01	-5.8674E-02	9.8940E-02	1.9965E-01
	2.9750E-01	3.9317E-01	4.8709E-01	5.8045E-01	6.7385E-01
	7.6862E-01	8.6443E-01	9.6473E-01	1.0690E+00	1.1813E+00
	1.3054E+00	1.4489E+00	1.6294E+00	1.9065E+00	

TABLE 9. (CONTINUED)

30	-3.4014E+00	-2.3842E+00	-1.8666E+00	-1.5152E+00	-1.2465E+00
	-1.0273E+00	-8.4066E-01	-6.7716E-01	-5.3076E-01	-3.9740E-01
	-2.7426E-01	-1.5922E-01	-5.0710E-02	5.2772E-02	1.5210E-01
	2.4746E-01	3.4441E-01	4.3187E-01	5.2202E-01	6.1428E-01
	7.0391E-01	7.9448E-01	8.8987E-01	9.8710E-01	1.0900E+00
	1.1994E+00	1.3216E+00	1.4631E+00	1.6413E+00	1.9157E+00

use of ordinary regression of Z_k on D_k . If these estimates are d_0 and d_1 then $\hat{\Lambda} = d_1/d_0$ and

$$\hat{a} = a_0(1 + \hat{\Lambda}) \quad (5.62)$$

Equation 5.62 can be iterated for a better estimate. If a_0 is the true value of the parameter then Λ is a ratio of linear combinations of independent exponential random variables.

Define the statistic

$$T = \frac{\sum_{k=1}^n D_k Z_k}{\sum_{k=1}^n Z_k} \quad (5.63)$$

The statistic is a monotone function of Λ and is clearly independent of the scale parameter. Furthermore, the T statistic is appealing for testing the hypothesis $H_0: a = a_0$ versus $H_A: a \neq a_0$. The distribution of the T statistic under the null hypothesis is given by

$$P(T > r) = \sum_{i=1}^m \frac{(D_{n-i+1} - r)^{n-1}}{\pi(D_{n-i+1} - D_j)} \quad (5.64)$$

$j=1$
 $j \neq n-i+1$

where $D_{n-m} < r < D_{n-m+1}$. Note that Equation 5.64 is identical, except for the coefficients, to the distribution of the test suggested by Jackson (24) in Equation 1.20. The critical values for Equation 5.64 which we shall call Fuller's T , are given in Table 10. Fuller's T may be written in terms of the original variates as

TABLE 10.

Critical values of the T statistic based on a sample of size N
 $P(T > \text{Value}) = C$

N	C=.99	.975	.95	.90	.10	.05	.025	.01
2	-.67928	-.65848	-.62383	-.55452	.55452	.62838	.65849	.67928
3	-.93957	-.84714	-.74928	-.59568	.55719	.68127	.76901	.84686
4	-1.0078	-.87258	-.73906	-.57083	.53013	.65816	.75928	.86267
5	-1.0094	-.85491	-.71215	-.54238	.50743	.63015	.73156	.84126
6	-.98776	-.82605	-.68245	-.51729	.48500	.60402	.70282	.81212
7	-.95844	-.79551	-.65453	-.49485	.46461	.57956	.67572	.78305
8	-.92741	-.76624	-.62892	-.47466	.44627	.55734	.65067	.75559
9	-.89705	-.73896	-.60556	-.45653	.42978	.53724	.62783	.73013
10	-.86825	-.71378	-.58444	-.44021	.41492	.51905	.60702	.70670
11					.40146	.50253	.58806	.68519
12					.38922	.48746	.57071	.66542
13					.37803	.47367	.55479	.64721
14					.36775	.46099	.54013	.63038
15					.35828	.44929	.52658	.61480
16					.34951	.43844	.51401	.60032
17					.34137	.42836	.50231	.58682
18					.33377	.41895	.49139	.57241
19					.32655	.41014	.48114	.56236
20	-.66712	-.54599	-.44638	-.33662	.32000	.40188	.47155	.55126
21					.31374	.39411	.46252	.54081
22					.30784	.38679	.45400	.53094
23					.30227	.37987	.44596	.52160
24					.29699	.37331	.43831	.51275
25					.29182	.36692	.43090	.50416
26					.28723	.36119	.42419	.49635
27					.28270	.35556	.41764	.48874
28					.27838	.35018	.41137	.48145
29					.27427	.34507	.40541	.47454
30	-.55661	-.45669	-.37401	-.28287	.27032	.34016	.39969	.46789

$$T = \frac{\sum G_k Y_{(k)}^{1/a_0}}{\sum Y_{(k)}^{1/a_0}} \quad (5.65)$$

where $G_k = (n-k+1)D_k - (n-k)D_{k+1}$. Now from Equation 5.63, assuming the null hypothesis is true

$$E\left(\sum_{k=1}^n D_k Z_k\right) = \sum_{k=1}^n D_k E(Z_k) = 0 \quad (5.66)$$

since $E(Z_k)$ is the same for all k and the sum of the D_k is zero.

It has been suggested by Fuller (18) that the T statistic is approximately normal with variance $\sum D_k^2/n^2$. For $n = 30$, the sum of squares is 44.16 and the square root is 6.65. It is known that if X is distributed $N(0, a^2)$ then $\frac{1}{a} X$ is distributed as $N(0, 1)$. Hence the critical values of Fuller's T for $n = 30$ multiplied by $\sqrt{n^2/\sum D_k^2} = 4.51$, should be approximately equal to the corresponding critical values of the normal distribution. The results in Table 11 suggest that the use of a normal approximation for small sample sizes would be somewhat inaccurate. Jackson (24) suggests that for his similar test, given in Equation 1.21, the approach to normality is rather slow, it appears that the normal approximation to Fuller's T would be satisfactory for large samples, especially for two tailed tests.

White (47) and others suggest transformed estimators based on the extreme value distribution as discussed in Chapter I. The transformation is the natural logarithm

Table 11.
Comparison of T to normal
for $n = 30$

Level	.99	.975	.95	.90	.10	.05	.025	.01
Normal	-2.33	-1.96	-1.64	-1.28	1.28	1.64	1.96	2.33
T	-2.51	-2.06	-1.68	-1.28	1.22	1.54	1.80	2.12

of the observations. That is if X is a Weibull random variable then if $Y = \log X$ the distribution of Y is

$$\begin{aligned}
 F(y) &= P(Y < y) = P(\log X < y) & (5.67) \\
 &= P(X < e^y) \\
 &= 1 - \exp(-(e^y/c)^{1/a}) \\
 &= 1 - \exp(-\exp(\frac{1}{a}(y - \log c))).
 \end{aligned}$$

Let $A = \log c$ and $B = a$, then the reduced random variable $U = (Y-A)/B$ has distribution

$$F(u) = P(Y < A + Bu) = 1 - e^{-e^x}. \quad (5.68)$$

The estimators are given by

$$\hat{A} = \sum A_1 Y_{(1)}, \quad \hat{B} = \sum B_1 Y_{(1)} \quad (5.69)$$

where the coefficients A_1 and B_1 are generated using the least square procedure of Lloyd (27) on the reduced variables. Hence the estimates are unbiased and have minimum variance in the class of unbiased estimates. White then gives tables of the expected values, variances, and covariances of the reduced log Weibull variate and develops and tabulates the coefficients. Finally White computes the variances and covariances of the estimates.

As a test of the hypothesis $H_0: a = a_0$ versus $H_A: a \neq a_0$, we can use the statistic

$$W = \frac{\hat{B} - a_0}{\sqrt{V(\hat{B})}}. \quad (5.70)$$

Unfortunately little is known about the distribution of the estimators and Monte Carlo techniques must be used to get the critical points of the test.

Consider the likelihood ratio test of $H_0: a = a_0$ versus $H_A: a = a_1$. Recall the density functions

$$f(y; a, c) = \frac{c^{-1/a}}{a} y^{\frac{1}{a}-1} \exp(-(y/c)^{1/a}).$$

The likelihood function is

$$L = \left(\frac{1}{a}\right)^n c^{-\frac{n}{a}} \prod_{i=1}^n y_i^{\frac{1}{a}-1} \exp\left(-\frac{1}{c^{1/a}} \sum y_i^{1/a}\right). \quad (5.71)$$

Hence the likelihood ratio is

$$\frac{L_1}{L_0} = \frac{\left(\frac{1}{a_1}\right)^n c^{-\frac{n}{a_1}} \prod_{i=1}^n y_i^{\frac{1}{a_1}-1} \exp\left(-\frac{1}{c^{1/a_1}} \sum y_i^{1/a_1}\right)}{\left(\frac{1}{a_0}\right)^n c^{-\frac{n}{a_0}} \prod_{i=1}^n y_i^{\frac{1}{a_0}-1} \exp\left(-\frac{1}{c^{1/a_0}} \sum y_i^{1/a_0}\right)}. \quad (5.72)$$

From Equation 1.12 the MLE estimate for the scale parameter is

$$\hat{c} = \left(\frac{1}{n} \sum y_i^{1/\hat{a}}\right)^{\hat{a}}.$$

Substituting the above into Equation 5.43 the likelihood ratio is

$$\frac{L_1}{L_0} = \left(\frac{a_0}{a_1} \right)^n \left(\frac{\frac{1}{n} \sum y_i^{1/a_1}}{\frac{1}{n} \sum y_i^{1/a_0}} \right)^{-n} \frac{\prod y_i^{1/a_1 - 1/a_0}}{e^{-1}} \cdot (5.73)$$

Therefore the test is to accept H_0 if

$$R = \left(\frac{\sum y_i^{1/a_0}}{\sum y_i^{1/a_1}} \right)^n (\prod y_i)^{1/a_1 - 1/a_0} > k \quad (5.74)$$

The distribution of the test is not known so that Monte Carlo methods are used to determine the critical points.

The three tests given in Equation 5.63, 5.70, and 5.74 were compared using Monte Carlo methods. The null hypothesis was that the shape parameter, a , was one, that is, that the underlying distribution was exponential. The alternatives investigated were Weibull with shape parameters .8, 1.2, 2.0, and 3.0. The Monte Carlo study consisted in generating 6,000 samples of size 5, 10, and 20. The three null test statistics were computed for each sample using the exponential as the underlying distribution and then ordered in increasing magnitude. Then the critical values under the null hypothesis were found by identifying the desired percentage points. The test was then calculated using as the underlying distribution a Weibull with shape parameter equal to that of the alternative hypothesis. These statistics were then placed in increasing order. The fraction of the sample exceeding the previously determined critical value gave the empirical

power of the tests. For the alternative shape parameter larger than one the critical point for a one sided test with alternative hypothesis shape parameter greater than one was used. Conversely the power of the one sided test for shape parameter less than one was compared for the Weibull with shape parameter of .8. The results appear in Table 12. The likelihood test proved to be the most powerful since the likelihood ratio test is always a most powerful test. Furthermore since the underlying distribution of the alternative was always that of the alternative hypothesis the simple versus simple likelihood ratio was in its most ideal situation. Had we used a test statistic based on an underlying distribution with some other shape parameter than that given in the alternative hypothesis we would expect the power to be reduced. The power of the log Weibull based test and Fuller's T were approximately the same. As the underlying distribution departs from the exponential the power of the three tests becomes more uniform. The power also increases rapidly as sample size increases.

Table 12.

Powers of tests based on a sample of size N of the hypothesis that the distribution is exponential where the alternative distribution is Weibull with shape parameter a

Size of test .10				.05			.025			.01		
	T	W	R	T	W	R	T	W	R	T	W	R
N=5												
a=.8	.193	.197	.208	.115	.119	.157	.060	.065	.089	.020	.025	.051
1.2	.218	.226	.272	.128	.142	.164	.076	.083	.090	.039	.046	.047
2.0	.610	.678	.682	.492	.524	.554	.372	.405	.420	.290	.274	.336
3.0	.822	.893	.897	.740	.749	.783	.643	.684	.690	.504	.531	.535
N=10												
a=.8	.282	.282	.299	.176	.180	.214	.084	.093	.151	.031	.033	.061
1.2	.282	.288	.308	.173	.183	.207	.104	.108	.140	.061	.062	.086
2.0	.845	.884	.904	.753	.813	.821	.659	.726	.751	.556	.632	.641
3.0	.976	.981	.984	.959	.962	.972	.921	.952	.961	.876	.911	.926
N=20												
a=.8	.421	.441	.512	.281	.286	.321	.204	.205	.241	.094	.090	.140
1.2	.407	.412	.548	.275	.285	.338	.185	.187	.231	.112	.114	.160
2.0	.980	.987	.994	.961	.977	.981	.934	.943	.960	.890	.921	.930
3.0	1.00	1.00	1.00	.999	1.00	1.00	.998	1.00	1.00	.996	.999	.999

¹T is Fuller's T statistic, W is White's log Weibull statistic, and R is the likelihood ratio test. R is simple versus simple where the alternative hypothesis is the alternative used to get power.

CHAPTER VI. MOMENTS OF SYMMETRIC DISTRIBUTIONS
FROM THEIR HALF DISTRIBUTIONS

Suppose Y is a random variable with density function $f(y) \geq 0$ for $Y > 0$ and $f(y) = 0$ otherwise. Let Z be the random variable with density function $g(z) = (1/2)f(z)$ if $z > 0$ and $g(z) = (1/2)f(-z)$ if $z < 0$. Consider the conditional density of z given that $z > 0$. We know that

$$\begin{aligned} g(z|z > 0) &= g(z, z > 0)/P(z > 0) & (6.1) \\ &= g(z, z > 0)/(1/2) \\ &= 2g(z, z > 0) \\ &= 2(1/2)f(z) \end{aligned}$$

$$g(z|z > 0) = f(z). \quad (6.2)$$

Suppose a sample of size n is drawn from $g(z)$. Let J be the random variable equal to the number of observations greater than zero. Obviously J is a binomial variate with parameter $p = 0.5$. Then the conditional distribution of the $J = j$ positive values of a sample of size n from the symmetric density $g(z)$ given that $J = j$ is precisely that of a sample of size j drawn from the half distribution $f(z)$. Similarly, the conditional distribution of the $n-j$ negative observations given that $J = j$ is precisely that of a sample

of size $n-j$ drawn from the half distribution $f(-z)$. If the $Z_{(i)}$ are the ordered random variables, we are able to write for fixed k , $0 < k < j$

$$g(z_{(n-k)} | J = j) = f_{X_{(j-k)}}(z_{(j-k)}) \quad (6.3)$$

where $f(z_{(j-k)})$ is the distribution of the $(j-k)$ -th order statistic from a sample of j observations. Since there are $n-j$ observations less than zero we use the same argument for $k \geq j$ to write

$$g(z_{(n-k)} | J = j) = f_{X_{(n-j+1)}}(-z_{(k-j+1)}) \quad (6.4)$$

where $f(-z_{(k-j+1)})$ is the distribution of the $(k-j+1)$ -st order statistic from a sample of size $n-j$. Then for fixed k

$$\begin{aligned} E(Z_{(n-k)} | J = j) &= E(Y_{(j-k)} \text{ of } j) \text{ if } 0 < k < j \quad (6.5) \\ &= E(-Y_{(k-j+1)} \text{ of } n-j) \text{ if } k \geq j \end{aligned}$$

where the expected values on the right-hand side are based on samples of size j and $n-j$ respectively. Since J is a random $E(Z_{(n-k)}) = E_j(E(Z_{(n-k)} | J = j))$. The expected value, E_j , is the sum over all the possible values of J multiplied by $P(J = j)$. Hence for any k , $0 \leq k \leq n-1$

$$\begin{aligned} E(Z_{(n-k)}) &= \sum_{j=0}^k P(n, j) E(-Y_{(k-j+1)} \text{ of } n-j) \quad (6.6) \\ &+ \sum_{j=k+1}^n P(n, j) E(Y_{(j-k)} \text{ of } j) \end{aligned}$$

where $P(n, j) = \binom{n}{j} (1/2)^n$.

Through a similar argument we are able to find the expected values of the crossproducts of order statistics of the symmetric distribution as

$$\begin{aligned} E(Z_{(1)} Z_{(k)}) &= \sum_{j=0}^{i-1} P(n, j) E(Y_{(1-j)} Y_{(k-j)} \text{ of } n-j) \quad (6.7) \\ &+ \sum_{j=1}^{k-1} P(n, j) E(-Y_{(j-1+1)} \text{ of } j) E(Y_{(k-j)} \text{ of } n-j) \\ &+ \sum_{j=k}^n P(n, j) E((-Y_{(j-k+1)}) (-Y_{(j-1+1)}) \text{ of } j) \end{aligned}$$

where $i < k$.

A. Moments of the Double Weibull

In Chapter II the moments for the order statistics of the Weibull distribution with integer shape parameter were derived from the moments of a transformed random variable X where $X = Y^{1/a}$ and Y is the Weibull variate. The recursive relation defining the moments is given by

$$E(X_{(k)}^m) = m! \sum_{i=0}^m E(X_{(k-1)}^i) \left(\frac{1}{n-k+1} \right)^{m-1}. \quad (6.8)$$

From the transformation we have the q -th moment for a Weibull with scale parameter c and shape parameter a ,

$$E(Y_{(k)}^q) = c^q E(X_{(k)}^{aq}). \quad (6.9)$$

The only restriction is that aq be a positive integer. If

we write the density function of Y as

$$\begin{aligned} f(y;a,b) &= \frac{1}{ab} y^{\frac{1}{a}-1} \exp(-y^{1/a}/b) \quad y > 0, a > 0, b > 0 \\ &= 0 \quad \text{otherwise} \end{aligned} \quad (6.10)$$

then the symmetric density related to the above density is given by

$$\begin{aligned} f(z;a,b) &= \frac{1}{2ab} |z|^{\frac{1}{a}-1} \exp(-|z|^{1/a}/b) \quad -\infty < z < \infty \\ a > 0, b > 0 \end{aligned} \quad (6.11)$$

We seek the moments of Z as a function of the moments of Y .

Hence from Equations 6.6 and 6.9 we have

$$\begin{aligned} E(Z_{(n-k)}^m) &= \sum_{j=0}^k P(n,j) E((-X_{(k-j+1)})^m \text{ of } n-j) \\ &+ \sum_{j=k+1}^n P(n,j) E(X_{(j-k)}^m \text{ of } j) . \end{aligned} \quad (6.12)$$

Z^m is the q -th moment of a double Weibull with shape parameter a such that $m = qa$. The values derived from Equation 6.12 may be multiplied by b^{aq} to generate the moments of Weibulls with nonunit scale parameters. Figure 1, presented here, from (29) is for the double Weibull and we find the value for the q -th moment of a double Weibull with shape parameter a in the aq -th column of the table. For example, if $Y_{(4)}$ is the fourth order statistic in a sample of size five with shape parameter two then $E(Y_{(4)}) = 0.830695$ and $E(Y_{(4)}^2) =$

MOMENTS OF THE KTH ORDER STATISTIC
FROM THE DOUBLE EXPONENTIAL
DISTRIBUTION
 $E((X(K))^{**M})$

		MOMENT									
N	K	1	2	3	4	5	6	7	8	9	10
2	2	7.50000E-01	2.00000E 00	5.62500E 00	2.40000E 01	1.18125E 02	7.20000E 02	5.02030E 03	4.03199E 04	3.62524E 05	3.62676E 06
3	3	1.12500E-01	2.68055E 00	8.43750E 00	3.49491E 01	1.77187E 02	1.07181E 03	7.53046E 03	6.03632E 04	5.43785E 05	5.44044E 06
3	2	0.00000E-01	6.33690E-01	0.00000E-01	2.10185E 00	0.00000E-01	1.63812E 01	0.00000E-01	2.33176E 02	0.00000E-01	5.20426E 03
4	4	1.38542E 00	3.36111E 00	1.09744E 01	4.58991E 01	2.34607E 02	1.42367E 03	1.00226E 04	8.04064E 04	7.24679E 05	7.25210E 06
4	3	3.43750E-01	6.38890E-01	8.26821E-01	2.10185E 00	4.92019E 00	1.63811E 01	5.57210E 01	2.33176E 02	1.03594E 03	5.26382E 03
5	5	1.58354E 00	3.99371E 00	1.33735E 01	5.65934E 01	2.91205E 02	1.77305E 03	1.25043E 04	1.00413E 05	9.05439E 05	9.05287E 06
5	4	5.72916E-01	6.30695E-01	1.37803E 00	3.11604E 00	8.21365E 00	2.50640E 01	9.28683E 01	3.79126E 02	1.72652E 03	8.70723E 03
5	3	0.00000E-01	3.51180E-01	0.00000E-01	5.79359E-01	0.00000E-01	2.15682E 00	0.00000E-01	1.42435E 01	0.00000E-01	1.47214E 02
6	6	1.75624E 00	4.57036E 00	1.56633E 01	6.70350E 01	3.47063E 02	2.12012E 03	1.49778E 04	1.20354E 05	1.08601E 06	1.06727E 07
6	5	7.49998E-01	1.07045E 00	1.92440E 00	4.38558E 00	1.18893E 01	3.77176E 01	1.36741E 02	5.61564E 02	2.56911E 03	1.29861E 04
6	4	2.18750E-01	3.51180E-01	2.35212E-01	5.79358E-01	8.62289E-01	2.15682E 00	4.72307E 00	1.42435E 01	4.12302E 01	1.47189E 02
7	7	1.89947E 00	5.12664E 00	1.78561E 01	7.72395E 01	4.02239E 02	2.46490E 03	1.74427E 04	1.40318E 05	1.26641E 06	1.26917E 07
7	6	8.90873E-01	1.32350E 00	2.49444E 00	5.80605E 00	1.60414E 01	5.14625E 01	1.88411E 02	7.76875E 02	3.56781E 03	1.80700E 04
7	5	3.82812E-01	4.37830E-01	4.99296E-01	8.29421E-01	1.50901E 00	3.35521E 00	8.26536E 00	2.32847E 01	7.21493E 01	2.47527E 02
7	4	0.00000E-01	2.35648E-01	0.00000E-01	2.45942E-01	0.00000E-01	5.58966E-01	0.00000E-01	2.20004E 00	0.00000E-01	1.33023E 01
8	8	2.02441E 00	5.52692E 00	1.99682E 01	8.72236E 01	4.56754E 02	2.80746E 03	1.90992E 04	1.60217E 05	1.44665E 06	1.44895E 07
8	7	1.02350E 00	1.57623E 00	3.05710E 00	7.35039E 00	2.06365E 01	6.69390E 01	2.46983E 02	1.02385E 03	4.71942E 03	2.39747E 04
8	6	5.16992E-01	5.59139E-01	7.16441E-01	1.17951E 00	2.25640E 00	5.03295E 00	1.26925E 01	3.59353E 01	1.12645E 02	3.27974E 02
9	9	1.59130E-01	2.35547E-01	1.37307E-01	2.45941E-01	2.63351E-01	5.53966E-01	8.86761E-01	2.20003E 00	4.64649E 00	1.92977E 01
9	8	2.13578E 00	6.10151E 00	2.20021E 01	9.70023E 01	5.10644E 02	3.14739E 03	2.23476E 04	1.80082E 05	1.62673E 06	1.62971E 07
9	7	1.13529E 00	1.83018E 00	3.69741E 00	8.99400E 00	2.56335E 01	8.40277E 01	3.12337E 02	1.30143E 03	6.02061E 03	3.06096E 04
9	6	6.32226E-01	6.96653E-01	9.51013E-01	1.60002E 00	3.14758E 00	7.12850E 00	1.82406E 01	5.21432E 01	1.64770E 02	5.70753E 02
9	5	2.86523E-01	2.84111E-01	2.47296E-01	3.33483E-01	4.74032E-01	8.41674E-01	1.59617E 00	3.51913E 00	8.36311E 00	2.21397E 01
9	4	0.00000E-01	1.75068E-01	0.00000E-01	1.50265E-01	0.00000E-01	2.05581E-01	0.00000E-01	5.51119E-01	0.00000E-01	2.20475E 00
10	10	2.23300E 00	6.54867E 00	2.39666E 01	1.06509E 02	5.63938E 02	3.40625E 03	2.47830E 04	1.99912E 05	1.80664E 06	1.81033E 07
10	9	1.23599E 00	2.07715E 00	4.32064E 00	1.07222E 01	3.09944E 01	1.02624E 02	3.54174E 02	1.60882E 03	7.46818E 03	3.81195E 04
10	8	7.34124E-01	3.42302E-01	1.20450E 00	2.00119E 00	4.13876E 00	9.64122E 00	2.49902E 01	7.21340E 01	2.29672E 02	8.00017E 02
10	7	3.94466E-01	3.56306E-01	3.59537E-01	4.77295E-01	7.18183E-01	1.26573E 00	2.49170E 00	5.49779E 00	1.33333E 01	3.53976E 01
10	6	1.24609E-01	1.75068E-01	7.89341E-02	1.30265E-01	1.07303E-01	2.05581E-01	2.52871E-01	5.51116E-01	9.06445E-01	2.23300E 00
11	11	2.32672E 00	6.97170E 00	2.53680E 01	1.15995E 02	6.16664E 02	3.82262E 03	2.72205E 04	2.19709E 05	1.90639E 06	1.99090E 07
11	10	1.32662E 00	2.31829E 00	4.95293E 00	1.25232E 01	3.66858E 01	1.22635E 02	4.62214E 02	1.94497E 03	9.05907E 03	4.63462E 04
11	9	8.25921E-01	9.91999E-01	1.47533E 00	2.61747E 00	5.37368E 00	1.25749E 01	3.29925E 01	9.61275E 01	3.03233E 02	1.07971E 03
11	8	4.89332E-01	4.43138E-01	4.82294E-01	6.51097E-01	1.01563E 00	1.81810E 00	3.65061E 00	8.15009E 00	2.00013E 01	5.35276E 01
11	7	2.28450E-01	2.05724E-01	1.44713E-01	1.73141E-01	1.97640E-01	2.99093E-01	4.62597E-01	8.56186E-01	1.66168E 00	3.63739E 00

Figure 1. Moments of the K-th order statistic from the double exponential distribution.

11	6	0.00000E-01	1.38280E-01	0.00000E-01	7.88133E-02	0.00000E-01	9.23660E-02	0.00000E-01	1.85023E-01	0.00000E-01	5.47458E-01
12	12	2.41006E-00	7.37338E-00	2.77113E-01	1.25233E-02	6.68844E-02	4.15702E-03	2.96455E-04	2.39472E-05	2.16593E-06	2.17137E-07
12	11	1.41001E-00	2.55327E-00	5.59126E-00	1.43270E-01	4.26814E-01	1.43976E-02	5.46200E-02	2.30000E-03	1.07903E-04	5.53275E-04
12	10	9.09675E-01	1.14340E-00	1.76128E-00	3.20450E-00	6.71393E-00	1.59318E-01	4.22360E-01	1.24717E-02	4.01470E-02	1.41351E-03
12	9	5.74659E-01	5.37753E-01	6.17463E-01	8.56387E-01	1.37291E-00	2.50413E-00	5.11180E-00	1.15573E-01	2.06640E-01	7.73436E-01
12	8	2.18678E-01	2.53898E-01	2.11956E-01	2.40518E-01	3.01091E-01	4.46041E-01	7.20235E-01	1.35552E-00	2.87171E-00	5.34205E-00
12	7	1.02132E-01	1.38260E-01	5.05721E-02	7.88133E-02	5.20073E-02	9.33653E-02	9.30972E-02	1.85023E-01	2.47208E-01	5.47194E-01
12	6	2.48690E-00	7.75509E-00	2.95012E-01	1.34310E-02	7.20501E-02	4.40393E-03	3.20629E-04	2.58203E-05	2.34542E-06	2.39173E-07
12	5	1.40656E-00	2.78203E-00	6.23327E-00	1.62045E-01	4.89525E-01	1.66569E-02	6.35601E-02	2.70040E-03	1.26592E-04	6.50328E-04
12	4	9.05799E-01	1.29512E-00	2.06020E-00	3.83839E-00	3.19025E-00	1.97119E-01	5.29001E-01	1.56370E-02	5.10007E-02	1.80406E-03
12	3	6.52594E-01	6.37637E-01	7.64901E-01	1.09155E-00	1.79287E-00	3.33145E-00	6.90570E-00	1.58067E-01	3.93908E-01	1.07705E-02
12	2	3.99305E-01	3.13027E-01	2.35728E-01	3.27263E-01	4.27979E-01	6.42572E-01	1.07532E-00	1.99578E-00	4.05414E-00	8.94680E-00
12	1	1.39674E-01	1.58293E-01	9.39193E-02	1.01725E-01	9.80706E-02	1.31590E-01	1.72800E-01	2.73935E-01	4.58244E-01	8.47848E-01
12	0	0.00000E-01	1.13764E-01	0.00000E-01	5.20835E-02	0.00000E-01	4.87714E-02	0.00000E-01	7.54552E-02	0.00000E-01	1.70436E-01
13	14	2.55341E-00	8.12143E-00	3.12415E-01	1.43236E-02	7.71357E-02	4.32028E-03	3.44731E-04	2.70923E-05	2.52471E-06	2.53200E-07
13	13	1.55840E-00	3.00465E-00	6.87712E-00	1.82698E-01	5.54774E-01	1.90345E-02	7.31063E-02	3.11014E-03	1.46630E-04	7.55115E-04
13	12	1.05832E-00	1.44628E-00	2.37013E-00	4.51556E-00	9.80296E-00	2.39131E-01	6.46567E-01	1.93929E-02	6.24526E-02	2.25692E-03
13	11	7.24539E-01	7.40906E-01	9.23786E-01	1.35542E-00	2.27700E-00	4.30731E-00	9.05945E-00	2.09022E-01	5.30772E-01	1.45403E-02
13	10	4.72732E-01	3.79466E-01	3.67639E-01	4.31879E-01	5.32561E-01	8.91943E-01	1.52150E-00	2.06509E-00	5.09467E-00	1.31400E-01
13	9	2.67136E-01	1.93438E-01	1.30199E-01	1.33955E-01	1.49732E-01	1.93703E-01	2.72057E-01	4.31542E-01	7.40236E-01	1.33094E-00
13	8	8.63906E-02	1.13766E-01	3.40815E-02	5.20335E-02	2.91093E-02	4.27714E-02	4.06071E-02	7.54547E-02	3.46095E-02	1.72317E-01
13	7	2.62508E-00	8.47149E-00	3.29358E-01	1.52019E-02	8.22330E-02	5.14921E-03	3.63760E-04	2.98369E-05	2.70333E-06	2.71227E-07
13	6	1.62507E-00	3.22132E-00	7.52138E-00	2.02755E-01	6.22359E-01	2.15240E-02	8.31507E-02	3.56150E-03	1.67991E-04	8.66942E-04
13	5	1.12504E-00	1.59626E-00	2.60939E-00	5.23273E-00	1.15472E-01	2.85920E-01	7.81715E-01	2.35619E-02	7.75868E-02	2.77264E-03
13	4	7.91469E-01	8.46325E-01	1.09310E-00	1.64688E-00	2.32593E-00	5.43760E-00	1.15970E-01	2.71679E-01	6.93633E-01	1.91545E-02
13	3	5.40479E-01	4.51002E-01	4.58169E-01	5.53087E-01	7.67309E-01	1.19877E-00	2.30109E-00	3.97405E-00	8.27323E-00	1.86337E-01
13	2	3.37238E-01	2.36394E-01	1.86729E-01	1.87863E-01	2.13066E-01	2.78265E-01	4.02566E-01	6.45513E-01	1.12301E-00	2.13600E-00
13	1	1.61983E-01	1.29005E-01	6.54027E-02	6.55946E-02	5.47300E-02	6.60307E-02	7.62800E-02	1.10172E-01	1.58403E-01	2.66151E-01
13	0	0.00000E-01	9.63493E-02	0.00000E-01	3.66422E-02	0.00000E-01	2.81320E-02	0.00000E-01	3.53203E-02	0.00000E-01	4.40353E-02
14	15	2.68758E-00	8.00744E-00	3.45272E-01	1.60666E-02	8.72338E-02	5.47641E-03	3.92720E-04	3.18200E-05	2.88201E-06	2.09240E-07
14	14	1.68758E-00	3.43227E-00	3.16493E-00	2.23167E-01	6.92099E-01	2.41193E-02	9.37029E-02	4.03003E-03	1.90651E-04	9.25906E-04
14	13	1.18756E-00	1.74470E-00	3.01653E-00	5.93686E-00	1.34101E-01	3.35637E-01	9.28556E-01	2.82043E-02	9.34599E-02	3.35404E-03
14	12	8.54104E-01	9.53037E-01	1.27181E-00	1.96403E-00	3.43999E-00	6.72767E-00	1.45404E-01	3.44376E-01	8.87157E-01	2.46760E-02
14	11	6.03554E-01	5.26190E-01	5.56953E-01	6.93057E-01	9.83932E-01	1.56771E-00	2.76690E-00	5.35023E-00	1.12831E-00	2.56546E-01
14	10	4.01694E-01	2.85508E-01	2.40844E-01	2.47712E-01	2.90730E-01	3.87097E-01	5.72116E-01	9.31364E-01	1.65153E-00	3.16311E-00
14	9	2.29012E-01	1.54404E-01	7.65274E-02	3.01130E-02	8.36129E-02	5.69295E-02	1.19909E-01	1.69079E-01	2.55215E-01	4.28474E-01
14	8	7.47737E-02	6.63492E-02	2.53724E-02	3.66421E-02	1.75947E-02	2.81320E-02	2.01013E-02	3.53200E-02	3.38943E-02	6.40498E-02
14	7	2.74640E-00	9.13054E-00	3.61904E-01	1.59103E-02	9.22298E-02	5.80192E-03	4.16610E-04	3.37610E-05	3.06164E-06	3.07245E-07
14	6	1.74640E-00	3.63773E-00	8.80689E-00	2.43889E-01	7.63001E-01	2.68151E-02	1.04744E-03	4.52273E-03	2.14506E-04	1.11183E-05
14	5	1.24639E-00	1.89133E-00	3.35029E-00	6.77517E-00	1.54100E-01	3.90023E-01	1.00916E-02	3.33295E-02	1.11071E-03	4.00602E-03
14	4	9.12997E-01	1.06042E-00	1.45396E-00	2.30011E-00	4.11834E-00	8.10130E-00	1.79091E-01	4.20449E-01	1.11385E-02	3.11901E-02
14	3	6.62703E-01	6.04028E-01	6.63602E-01	8.49171E-01	1.23370E-00	2.00312E-00	3.59180E-00	7.04036E-00	1.50153E-01	3.44414E-01
14	2	4.61629E-01	3.39370E-01	3.03976E-01	3.10304E-01	3.64480E-01	5.22722E-01	7.87414E-01	1.30103E-00	2.30294E-00	4.33190E-00
14	1	2.01013E-01	1.86974E-01	1.30566E-01	1.18148E-01	1.18376E-01	1.38453E-01	1.77403E-01	2.52164E-01	3.89013E-01	6.40400E-01
14	0	1.41239E-01	1.07874E-01	4.79257E-02	4.52059E-02	3.32344E-02	3.76111E-02	3.79691E-02	5.04330E-02	6.40105E-02	9.76487E-02
15	15	0.00000E-01	8.33836E-02	0.00000E-01	2.73000E-02	0.00000E-01	1.74680E-02	0.00000E-01	1.82166E-02	0.00000E-01	2.73000E-02
15	14	2.80196E-00	9.44187E-00	3.77721E-01	1.75577E-02	9.71625E-02	6.12579E-03	4.44042E-04	3.57594E-05	3.24031E-06	2.28242E-07
15	13	1.80196E-00	3.83795E-00	9.44654E-00	2.64031E-01	8.37409E-01	1.16258E-02	5.03966E-03	2.38773E-04	1.24848E-05	5.55555E-06
15	12	1.30195E-00	2.03599E-00	3.60962E-00	7.59500E-00	1.75205E-01	4.49430E-01	1.26954E-02	3.86649E-02	1.30527E-03	4.72040E-03
15	11	9.68588E-01	1.16803E-00	1.65363E-00	2.67558E-00	4.84206E-00	9.80230E-00	2.17210E-01	5.25180E-01	1.37915E-02	3.00072E-02

Figure 1. (Continued)

18 14	7.18429E-01	5.83790E-01	7.77592E-01	1.02196E-00	1.51758E-00	2.50898E-00	4.56751E-00	9.07821E-00	1.95485E-01	4.52411E-01
18 13	5.17815E-01	3.94640E-01	3.67227E-01	3.99925E-01	4.95609E-01	6.87897E-01	1.05494E-00	1.77064E-00	3.22411E-00	6.31238E-00
18 12	3.49257E-01	2.24342E-01	1.68536E-01	1.55301E-01	1.62221E-01	1.92370E-01	2.92346E-01	3.64185E-01	5.70990E-01	9.64647E-01
18 11	2.01543E-01	1.27467E-01	7.08979E-02	5.97642E-02	5.07671E-02	5.37256E-02	5.96353E-02	7.61269E-02	1.02951E-01	1.53379E-01
18 10	6.53929E-02	3.33036E-02	1.92102E-02	2.70080E-02	1.13164E-02	1.74680E-02	1.08863E-02	1.83165E-02	1.53323E-02	2.78624E-02
18 9	2.35459E-02	9.74235E-03	3.95103E-03	1.85833E-03	1.02053E-03	6.44506E-03	4.54130E-03	3.76929E-03	3.41884E-03	3.43232E-03
18 8	1.35459E-02	4.03317E-03	1.06834E-03	2.86106E-03	9.12701E-04	3.24837E-04	1.22227E-03	5.57954E-04	2.66188E-04	1.36472E-04
18 7	1.35459E-02	2.17052E-03	4.03361E-03	8.44427E-04	1.97427E-04	5.10775E-04	1.45172E-04	4.50571E-04	1.51851E-04	5.52483E-04
18 6	1.02123E-02	1.27552E-03	1.65503E-03	3.06311E-03	5.66393E-04	1.15922E-04	2.59713E-04	6.34611E-04	1.67641E-04	4.75955E-04
18 5	7.71152E-03	7.64933E-04	8.98382E-04	1.21100E-04	1.83629E-04	3.08336E-04	5.70550E-04	1.14803E-04	2.49783E-04	5.83100E-04
18 4	5.70003E-03	4.56592E-04	4.39378E-04	4.92397E-04	6.25203E-04	8.65319E-04	1.30112E-03	2.35211E-03	4.33669E-03	8.50024E-03
18 3	4.03003E-03	2.66761E-04	2.10893E-04	1.99568E-04	2.14622E-04	2.60150E-04	3.48235E-04	5.10759E-04	8.12618E-04	1.20950E-03
18 2	2.57114E-03	1.52980E-04	9.59151E-05	7.94157E-05	7.20434E-05	7.61754E-05	8.79632E-05	1.12909E-04	1.56621E-04	2.34909E-04
18 1	1.25132E-03	9.23858E-05	3.64994E-05	3.27431E-05	2.15050E-05	2.28568E-05	2.06840E-05	2.55503E-05	2.91294E-05	4.09710E-05
19 10	0.00000E-01	7.33809E-02	0.00000E-01	2.06356E-02	0.00000E-01	1.14804E-02	0.00000E-01	1.02787E-02	0.00000E-01	1.32644E-02
20 20	2.90459E-00	1.00328E-01	4.08155E-01	1.94016E-02	1.06904E-03	6.76878E-03	4.87870E-04	3.96444E-05	3.59722E-06	3.61212E-07
20 19	1.90459E-00	4.22362E-00	1.07169E-01	3.07543E-01	9.80587E-01	3.54574E-02	1.40637E-03	6.14207E-03	2.95013E-04	1.53132E-05
20 18	1.40459E-00	2.31903E-00	4.38147E-00	9.32056E-00	2.20728E-01	5.76993E-01	1.65367E-02	5.16713E-02	1.75031E-03	6.39680E-03
20 17	1.07124E-00	1.39264E-00	2.06243E-00	3.47860E-00	6.53850E-00	1.35538E-01	3.07355E-01	7.57543E-01	2.01688E-02	5.76157E-02
20 16	8.21202E-01	8.47037E-01	1.02544E-00	1.41617E-00	2.19033E-00	3.74595E-00	7.01657E-00	1.42867E-01	3.13992E-01	7.39106E-01
20 15	6.21003E-01	5.16621E-01	5.17199E-01	5.95324E-01	7.74165E-01	1.11756E-00	1.77266E-00	3.06055E-00	5.71203E-00	1.14167E-01
20 14	4.53668E-01	3.11858E-01	2.57796E-01	2.51067E-01	2.77623E-01	3.43414E-01	4.68445E-01	6.93107E-01	1.12629E-00	1.94868E-00
20 13	3.03923E-01	1.63010E-01	1.23802E-01	1.03927E-01	5.81913E-02	1.05516E-01	1.24987E-01	1.62817E-01	2.29356E-01	3.49033E-01
20 12	1.79377E-01	1.03735E-01	5.40344E-02	4.26491E-02	3.28339E-02	3.24272E-02	3.80450E-02	4.67285E-02	6.36646E-02	8.67285E-02
20 11	5.80096E-02	7.33808E-02	1.50044E-02	2.06356E-02	7.65867E-03	1.14804E-02	6.33112E-03	1.02786E-02	7.61266E-03	1.32560E-02
21 21	2.95221E-00	1.03140E-01	4.22837E-01	2.02071E-02	1.11715E-03	7.03796E-03	5.11526E-04	4.15930E-05	3.77546E-05	3.79135E-07
21 20	1.95221E-00	4.40955E-00	1.13466E-01	3.29156E-01	1.06796E-02	3.85087E-02	1.53474E-03	6.72671E-03	3.22627E-04	1.68464E-05
21 19	1.45221E-00	2.45734E-00	4.73252E-00	1.02220E-01	2.45066E-01	6.47011E-01	1.86934E-02	5.67921E-02	2.00253E-03	7.34645E-03
21 18	1.11587E-00	1.49920E-00	2.27517E-00	3.91197E-00	7.47000E-00	1.56831E-01	3.59648E-01	8.94540E-01	2.39976E-02	6.89786E-02
21 17	3.68847E-01	9.29776E-01	1.15227E-00	1.63679E-00	2.53005E-00	4.48311E-00	8.51093E-00	1.75286E-01	3.89002E-01	9.23111E-01
21 16	6.68735E-01	5.82273E-01	6.00394E-01	7.10180E-01	9.43257E-01	1.33706E-00	2.23451E-00	3.91215E-00	7.38596E-00	1.49023E-01
21 15	5.01674E-01	3.59491E-01	3.09211E-01	3.09937E-01	3.51433E-01	4.43814E-01	6.16387E-01	9.32895E-01	1.52545E-00	2.66963E-00
21 14	3.57655E-01	2.16592E-01	1.54966E-01	1.33527E-01	1.30006E-01	1.42614E-01	1.72559E-01	2.28520E-01	3.27659E-01	5.03369E-01
21 13	2.29740E-01	1.28441E-01	7.31611E-02	5.61510E-02	4.64934E-02	4.52320E-02	4.76825E-02	5.60479E-02	7.08675E-02	9.69411E-02
21 12	1.12273E-01	0.05955E-02	2.86486E-02	2.46466E-02	1.46211E-02	1.47417E-02	1.20867E-02	1.40406E-02	1.45317E-02	1.90736E-02
21 11	0.00000E-01	6.54445E-02	0.00000E-01	1.62236E-02	0.00000E-01	7.89296E-03	0.00000E-01	6.14036E-03	0.00000E-01	6.84256E-03
22 22	2.99766E-00	1.05865E-01	4.27223E-01	2.10022E-02	1.16408E-03	7.40565E-03	5.35069E-04	4.35336E-05	3.95356E-06	3.97150E-07
22 21	1.99766E-00	4.59115E-00	1.19729E-01	3.50925E-01	1.14771E-02	4.16388E-02	1.66722E-03	7.33294E-03	3.52608E-04	1.84456E-05
22 20	1.49766E-00	2.59349E-00	5.68617E-00	1.11467E-01	2.70399E-01	7.20756E-01	2.09067E-02	6.64230E-02	2.27393E-03	8.37563E-03
22 19	1.16432E-00	1.59505E-00	2.49260E-00	4.36518E-00	8.46208E-00	1.79958E-01	4.16907E-01	1.04613E-02	2.82713E-02	8.17531E-02
22 18	9.14316E-01	1.01289E-00	1.29639E-00	1.87249E-00	3.00561E-00	5.30281E-00	1.01982E-01	2.12367E-01	4.75742E-01	1.13784E-02
22 17	7.14233E-01	6.47135E-01	6.88652E-01	8.35387E-01	1.13311E-01	1.69609E-00	2.77427E-00	4.92037E-00	9.39562E-00	1.91463E-01
22 16	5.47355E-01	4.09172E-01	3.65037E-01	3.76294E-01	4.36960E-01	5.62981E-01	7.95518E-01	1.22214E-00	2.02454E-00	3.58253E-00
22 15	4.03766E-01	2.53031E-01	1.89505E-01	1.67742E-01	1.68155E-01	1.88456E-01	2.32535E-01	3.13060E-01	4.55513E-01	7.09253E-01
22 14	2.76925E-01	1.52822E-01	9.43340E-02	7.31019E-02	6.32365E-02	6.23895E-02	6.76007E-02	8.05712E-02	1.03824E-01	1.43714E-01
22 13	1.61584E-01	9.32215E-02	4.25058E-02	3.16653E-02	2.23088E-02	2.04490E-02	1.89112E-02	2.06240E-02	2.32468E-02	2.92496E-02
22 12	5.30990E-02	6.54445E-02	1.20197E-02	1.62236E-02	5.39592E-03	7.89296E-03	3.89720E-03	6.14031E-03	4.07036E-03	6.83664E-03
23 23	3.04114E-00	1.08509E-01	4.51476E-01	2.17874E-02	1.21224E-03	7.72189E-03	5.58570E-04	4.54815E-05	4.13151E-06	4.15100E-07
23 22	2.04114E-00	4.76864E-00	1.25949E-01	3.72829E-01	1.22376E-02	4.48442E-02	1.60370E-03	7.96029E-03	3.83739E-04	2.01102E-05
23 21	1.54114E-00	2.72750E-00	5.44193E-00	1.20932E-01	2.96689E-01	7.98153E-01	2.34158E-02	7.45671E-02	2.56545E-03	9.48602E-03

Figure 1. (Continued)

23 20	1.20780E-00	1.70008E-00	2.71443E-00	4.83726E-00	9.51366E-00	2.04776E-01	4.79220E-01	1.21280E-02	3.30103E-02	9.60165E-02
23 19	9.57501E-01	1.09617E-00	1.43937E-00	2.12282E-00	3.46709E-00	6.20727E-00	1.20873E-01	2.54405E-01	5.75127E-01	1.30605E-02
23 18	7.57766E-01	7.13068E-01	7.81664E-01	9.71327E-01	1.34427E-00	2.04676E-00	3.56719E-00	6.10238E-00	1.17792E-01	2.47260E-01
23 17	5.90665E-01	4.60518E-01	4.25119E-01	4.50222E-01	5.34837E-01	7.02501E-01	1.00932E-00	1.57317E-00	2.63898E-00	4.72021E-00
23 16	4.47675E-01	2.91810E-01	2.27706E-01	2.07317E-01	2.13242E-01	2.44076E-01	3.06823E-01	4.19766E-01	6.19466E-01	9.76274E-01
23 15	3.21495E-01	1.60322E-01	1.18107E-01	9.35399E-02	8.36295E-02	8.41634E-02	9.32432E-02	1.12980E-01	1.47070E-01	2.07408E-01
23 14	2.07593E-01	1.10047E-01	5.74009E-02	4.13097E-02	3.15139E-02	2.65108E-02	2.77124E-02	3.01560E-02	3.51241E-02	4.40007E-02
23 13	1.01774E-01	7.13487E-02	2.30382E-02	1.91287E-02	1.03422E-02	9.06856E-03	7.46964E-03	8.23144E-03	7.00070E-03	9.64513E-03
23 12	0.00000E-01	5.90036E-02	0.00000E-01	1.20543E-02	0.00000E-01	5.62865E-03	0.00000E-01	3.85899E-03	0.00000E-01	2.74805E-03
24 24	3.00200E-00	1.11078E-01	4.65361E-01	2.25630E-02	1.25925E-03	8.03669E-03	5.82011E-04	4.74215E-05	4.30932E-06	4.33057E-07
24 23	2.08280E-00	4.94221E-00	1.32127E-01	3.94850E-01	1.31102E-02	4.81213E-02	1.94406E-03	8.60820E-03	4.16000E-04	2.10396E-05
24 22	1.50280E-00	2.85940E-00	5.79936E-00	1.30597E-01	3.23866E-01	8.79127E-01	2.59798E-02	8.32264E-02	2.07720E-03	1.06792E-04
24 21	1.24947E-00	1.80420E-00	2.93995E-00	5.32725E-00	1.06235E-01	2.31335E-01	5.46702E-01	1.39502E-02	3.82338E-02	1.11344E-03
24 20	9.99472E-01	1.17946E-00	1.50600E-00	2.30729E-00	3.96444E-00	7.19337E-00	1.41868E-01	3.01690E-01	6.80064E-01	1.67054E-02
24 19	7.99452E-01	7.79684E-01	8.79126E-01	1.11784E-00	1.57715E-00	2.44105E-00	4.10916E-00	7.47194E-00	1.45759E-01	3.02460E-01
24 18	6.32708E-01	5.13220E-01	4.59279E-01	5.31765E-01	6.45622E-01	8.63905E-01	1.26129E-00	1.99555E-00	3.30510E-00	6.11811E-00
24 17	4.89589E-01	3.32525E-01	2.69302E-01	2.52187E-01	2.65787E-01	3.10519E-01	3.97392E-01	5.52213E-01	8.26139E-01	1.31730E-00
24 16	3.63847E-01	2.10379E-01	1.44513E-01	1.17575E-01	1.08151E-01	1.11190E-01	1.25624E-01	1.54862E-01	2.05937E-01	2.92526E-01
24 15	2.50908E-01	1.30227E-01	7.40982E-02	5.34319E-02	4.27605E-02	3.91322E-02	3.91756E-02	4.31744E-02	5.13150E-02	6.54661E-02
24 14	1.46951E-01	8.17944E-02	3.42166E-02	2.42637E-02	1.57686E-02	1.36408E-02	1.16640E-02	1.19312E-02	1.24438E-02	1.46090E-02
24 13	4.83823E-02	5.90035E-02	9.82729E-03	1.30543E-02	3.92919E-03	5.62864E-03	2.51269E-03	3.85897E-03	2.31173E-03	3.73369E-03
25 25	3.12280E-00	1.13577E-01	4.78990E-01	2.33294E-02	1.30591E-03	8.35011E-03	6.05391E-04	4.93587E-05	4.48699E-06	4.50090E-07
25 24	2.12281E-00	5.11203E-00	1.38261E-01	4.16972E-01	1.39442E-02	5.14684E-02	2.03717E-03	9.27645E-03	4.48374E-04	2.76328E-05
25 23	1.62280E-00	2.99923E-00	6.15807E-00	1.40450E-01	3.51986E-01	9.63604E-01	2.06770E-02	9.24026E-02	3.20946E-03	1.19560E-04
25 22	1.28747E-00	1.90736E-00	3.16084E-00	5.03426E-00	1.17903E-01	2.59602E-01	6.19390E-01	1.59321E-02	4.39604E-02	1.29310E-03
25 21	1.03947E-00	1.26262E-00	1.73832E-00	2.66542E-00	4.49753E-00	8.27777E-00	1.65045E-01	3.54498E-01	6.15453E-01	1.98410E-02
25 20	8.39463E-01	8.46838E-01	9.30747E-01	1.27476E-00	1.93211E-00	2.88076E-00	4.91576E-00	9.04480E-00	1.70255E-01	3.73149E-01
25 19	6.72752E-01	5.67028E-01	5.57326E-01	6.20936E-01	7.69810E-01	1.04865E-00	1.55491E-00	2.49106E-00	4.28004E-00	7.81409E-00
25 18	5.29738E-01	3.74858E-01	3.14301E-01	3.02469E-01	3.26283E-01	3.88825E-01	5.06263E-01	7.14193E-01	1.08271E-01	1.74634E-01
25 17	4.04274E-01	2.42569E-01	1.73679E-01	1.45338E-01	1.37231E-01	1.44118E-01	1.66041E-01	2.07986E-01	2.80661E-01	4.03722E-01
25 16	2.91976E-01	1.53152E-01	9.26622E-02	6.82175E-02	5.64534E-02	5.26508E-02	5.39380E-02	6.04176E-02	7.30370E-02	9.43996E-02
25 15	1.89307E-01	9.58390E-02	4.62521E-02	3.15702E-02	2.22212E-02	1.80543E-02	1.70319E-02	1.73089E-02	1.87260E-02	2.19708E-02
25 14	9.30429E-02	6.39197E-02	1.88986E-02	1.52202E-02	7.55616E-03	7.00539E-03	4.83203E-03	5.08654E-03	4.44515E-03	5.21673E-03
25 13	0.00000E-01	5.36777E-02	0.00000E-01	1.07079E-02	0.00000E-01	4.13715E-03	0.00000E-01	2.52903E-03	0.00000E-01	2.19130E-03

Figure 1. (Continued)

3.11684. For values not in the table use the relation

$$E(Y_{(j)}^m) = -E(Y_{(n-j)}^m).$$

From Equations 6.7 and 6.9 the moments of the cross-products are

$$\begin{aligned} E((Z_{(1)}Z_{(k)})^m) &= \sum_{j=0}^{i=1} P(n, j) E((X_{(1-j)}X_{(k-j)})^m \text{ of } n-j) \\ &+ \sum_{j=1}^{k-1} P(n, j) E((-X_{(j-i+1)})^m \text{ of } j) E((X_{(j-k)}^m) \text{ of } n-j) \\ &+ \sum_{j=k}^n P(n, j) E(((X_{(j-k+1)})^m (-X_{(j-i+1)})^m \text{ of } j) \quad (6.13) \end{aligned}$$

where $1 < j$. For the double Weibull with shape parameter a , the q -th moment of the crossproducts is found in the aq -th column of Table 13. For example $E(Z_{(1)}Z_{(2)})$ of a double Weibull with shape parameter of three in a sample of five is 77.428.

Our results were compared to those of Sarhan (36) for the double exponential of samples up to size five and agreed on the fifth digit.

The combinatorial forms of Equations 6.6 and 6.7 are not dependent on the distribution. Thus we are able to generate the moments of the order statistics of any symmetric distribution from the moments of the related half distribution. For example, we are able to get the moments of the isocoles triangular from the right triangular distribution, the uniform $(-a, a)$ from the uniform $(0, a)$, and the normal from the

TABLE 13.

Expected crossproducts of the 1-th and j-th order statistics
of the double Weibull with shape parameter a based on a
sample of size n

n	1	j	a	1	2	3	4	5
2	1	2		0.0000E+00	4.0000E+00	0.0000E+00	5.7600E+02	1.5625E-02
3	1	2		4.7222E-01	3.1574E+00	2.4210E+01	4.3591E+02	1.0503E+04
3	1	3		-9.4444E-01	5.6852E+00	-4.8420E+01	8.5618E+02	-2.1007E+04
3	2	3		4.7222E-01	3.1574E+00	2.4210E+01	4.3591E+02	1.0503E+04
4	1	2		9.4444E-01	5.0370E+00	4.8420E+01	8.2137E+02	2.1006E+04
4	1	3		-2.0139E-01	1.9259E+00	-1.9176E+00	8.5245E+01	-1.5428E+02
4	1	4		-1.6875E+00	9.4444E+00	-9.4922E+01	1.6271E+03	-4.1860E+04
4	2	3		2.0139E-01	6.2963E-01	1.9176E+00	1.5643E+01	1.5425E+02
4	2	4		-2.0139E-01	1.9259E+00	-1.9176E+00	8.5245E+01	-1.5428E+02
4	3	4		9.4444E-01	5.0370E+00	4.8420E+01	8.2137E+02	2.1006E+04
5	1	2		1.3927E+00	7.5396E+00	7.7428E+01	1.3330E+03	3.4582E+04
5	1	3		2.6403E-01	1.9088E+00	7.3287E+00	8.9736E+01	1.0848E+03
5	1	4		-7.1062E-01	2.6471E+00	-1.1948E+01	1.2044E+02	-1.4660E+03
5	1	5		-2.3387E+00	1.3976E+01	-1.5024E+02	2.6316E+03	-6.8790E+04
5	2	3		2.8017E-01	6.5757E-01	2.4844E+00	1.8083E+01	1.9505E+02
5	2	4		-1.1375E-01	5.1787E-01	-3.4963E-01	5.8829E+00	-9.0309E+00
5	2	5		-7.1062E-01	2.6471E+00	-1.1948E+01	1.2044E+02	-1.4660E+03
5	3	4		2.8017E-01	6.5757E-01	2.4844E+00	1.8083E+01	1.9505E+02
5	3	5		2.6403E-01	1.9088E+00	7.3287E+00	8.9736E+01	1.0848E+03
5	4	5		1.3927E+00	7.5396E+00	7.7428E+01	1.3330E+03	3.4582E+04
6	1	2		1.8169E+00	1.0377E+01	1.1124E+02	1.9526E+03	5.1232E+04
6	1	3		6.5102E-01	2.7013E+00	1.5153E+01	1.5440E+02	2.1846E+03
6	1	4		-1.9828E-01	1.4855E+00	-9.6233E-01	3.5731E+01	-3.1863E+01
6	1	5		-1.1569E+00	4.0380E+00	-2.2613E+01	2.1367E+02	-2.8895E+03
6	1	6		-2.9297E+00	1.8945E+01	-2.1405E+02	3.8405E+03	-1.0174E+05
6	2	3		4.3738E-01	1.0267E+00	4.4733E+00	3.3432E+01	3.7493E+02
6	2	4		2.7656E-02	3.7795E-01	4.3789E-01	3.7624E+00	1.8311E+01
6	2	5		-3.9656E-01	7.9769E-01	-1.9247E+00	1.0124E+01	-6.3717E+01
6	2	6		-1.1569E+00	4.0380E+00	-2.2613E+01	2.1367E+02	-2.8895E+03
6	3	4		1.7062E-01	2.4365E-01	5.2445E-01	2.2200E+00	1.3540E+01
6	3	5		2.7656E-02	3.7795E-01	4.3789E-01	3.7624E+00	1.8311E+01
6	3	6		-1.9828E-01	1.4855E+00	-9.6233E-01	3.5731E+01	-3.1863E+01
6	4	5		4.3738E-01	1.0267E+00	4.4733E+00	3.3432E+01	3.7493E+02
6	4	6		6.5102E-01	2.7013E+00	1.5153E+01	1.5440E+02	2.1846E+03
6	5	6		1.8169E+00	1.0377E+01	1.1124E+02	1.9526E+03	5.1232E+04
7	1	2		2.2191E+00	1.3445E+01	1.4940E+02	2.6725E+03	7.0861E+04

TABLE 13. (CONTINUED)

7	1	3	1.0019E+00	3.8727E+00	2.4427E+01	2.4999E+02	3.6688E+03
7	1	4	1.8242E-01	1.4635E+00	3.5740E+00	3.7153E+01	2.8577E+02
7	1	5	-5.8033E-01	1.8624E+00	-5.6878E+00	4.5311E+01	-3.5833E+02
7	1	6	-1.5678E+00	5.8102E+00	-3.5737E+01	3.4628E+02	-4.8420E+03
7	1	7	-3.4744E+00	2.4199E+01	-2.8537E+02	5.2382E+03	-1.4050E+05
7	2	3	6.2082E-01	1.5456E+00	7.2274E+00	5.5638E+01	6.3704E+02
7	2	4	1.8483E-01	4.9114E-01	1.2148E+00	6.5164E+00	4.5085E+01
7	2	5	-1.9424E-01	4.2070E-01	-3.9476E-01	2.7276E+00	-5.3675E+00
7	2	6	-6.6632E-01	1.3003E+00	-3.9645E+00	1.9986E+01	-1.4152E+02
7	2	7	-1.5678E+00	5.8102E+00	-3.5737E+01	3.4628E+02	-4.8420E+03
7	3	4	1.9676E-01	2.5669E-01	5.9432E-01	2.4790E+00	1.5490E+01
7	3	5	1.3785E-02	1.6543E-01	1.0518E-01	6.6606E-01	1.8225E+00
7	3	6	-1.9424E-01	4.2070E-01	-3.9476E-01	2.7276E+00	-5.3675E+00
7	3	7	-5.8033E-01	1.8624E+00	-5.6878E+00	4.5311E+01	-3.5833E+02
7	4	5	1.9676E-01	2.5669E-01	5.9432E-01	2.4790E+00	1.5490E+01
7	4	6	1.8483E-01	4.9114E-01	1.2148E+00	6.5164E+00	4.5085E+01
7	4	7	1.8242E-01	1.4635E+00	3.5740E+00	3.7152E+01	2.8577E+02
7	5	6	6.2082E-01	1.5456E+00	7.2274E+00	5.5638E+01	6.3704E+02
7	5	7	1.0019E+00	3.8727E+00	2.4427E+01	2.4999E+02	3.6688E+03
7	6	7	2.2191E+00	1.3445E+01	1.4940E+02	2.6725E+03	7.0861E+04
8	1	2	2.6013E+00	1.6682E+01	1.9148E+02	3.4867E+03	9.3377E+04
8	1	3	1.3307E+00	5.2905E+00	3.5619E+01	3.7421E+02	5.6214E+03
8	1	4	5.0656E-01	1.9008E+00	7.5526E+00	5.8647E+01	5.7877E+02
8	1	5	-1.8180E-01	1.2493E+00	-6.1836E-01	2.0125E+01	-1.1986E+01
8	1	6	-9.2315E-01	2.6302E+00	-1.0613E+01	7.4376E+01	-7.0250E+02
8	1	7	-1.9536E+00	7.8501E+00	-5.1588E+01	5.1722E+02	-7.4138E+03
8	1	8	-3.9814E+00	2.9649E+01	-3.6330E+02	6.8118E+03	-1.8486E+05
8	2	3	8.1437E-01	2.1753E+00	1.0722E+01	8.4957E+01	9.9065E+02
8	2	4	3.4825E-01	7.2731E-01	2.2180E+00	1.1562E+01	8.5291E+01
8	2	5	-2.1475E-02	3.5669E-01	2.3637E-01	2.2564E+00	7.0029E+00
8	2	6	-4.0464E-01	6.3053E-01	-1.1958E+00	4.6105E+00	-2.0689E+01
8	2	7	-9.2801E-01	1.9702E+00	-6.7333E+00	3.5361E+01	-2.6235E+02
8	2	8	-1.9536E+00	7.8501E+00	-5.1588E+01	5.1722E+02	-7.4138E+03
8	3	4	2.7323E-01	3.8060E-01	9.9559E-01	4.3791E+00	2.8613E+01
8	3	5	6.4308E-02	1.5324E-01	1.8687E-01	6.8480E-01	2.7434E+00
8	3	6	-1.3778E-01	2.0197E-01	-1.3990E-01	6.0984E-01	-9.4102E-01
8	3	7	-4.0464E-01	6.3053E-01	-1.1958E+00	4.6105E+00	-2.0689E+01
8	3	8	-9.2315E-01	2.6302E+00	-1.0613E+01	7.4376E+01	-7.0250E+02
8	4	5	1.3897E-01	1.2594E-01	1.9511E-01	5.4364E-01	2.2339E+00
8	4	6	6.4308E-02	1.5324E-01	1.8687E-01	6.8480E-01	2.7434E+00
8	4	7	-2.1475E-02	3.5669E-01	2.3637E-01	2.2564E+00	7.0029E+00
8	4	8	-1.8180E-01	1.2493E+00	-6.1836E-01	2.0125E+01	-1.1986E+01
8	5	6	2.7323E-01	3.8060E-01	9.9559E-01	4.3791E+00	2.8613E+01
8	5	7	3.4825E-01	7.2731E-01	2.2180E+00	1.1562E+01	8.5291E+01

TABLE 13. (CONTINUED)

8	5	8	5.0656E-01	1.9008E+00	7.5526E+00	5.8647E+01	5.7877E+02
8	6	7	8.1437E-01	2.1753E+00	1.0722E+01	8.4957E+01	9.9065E+02
8	6	8	1.3307E+00	5.2905E+00	3.5619E+01	3.7421E+02	5.6214E+03
8	7	8	2.6013E+00	1.6682E+01	1.9148E+02	3.4867E+03	9.3377E+04
9	1	2	2.9653E+00	2.0049E+01	2.3708E+02	4.3902E+03	1.1869E+05
9	1	3	1.6434E+00	6.8931E+00	4.8788E+01	5.2734E+02	8.0846E+03
9	1	4	8.0071E-01	2.6015E+00	1.2152E+01	9.2473E+01	9.6983E+02
9	1	5	1.3877E-01	1.2233E+00	2.1492E+00	2.0653E+01	1.1403E+02
9	1	6	-4.9567E-01	1.4875E+00	-3.4613E+00	2.4161E+01	-1.4032E+02
9	1	7	-1.2424E+00	3.6499E+00	-1.6578E+01	1.1770E+02	-1.1718E+03
9	1	8	-2.3191E+00	1.0096E+01	-7.0134E+01	7.2711E+02	-1.0657E+04
9	1	9	-4.4563E+00	3.5235E+01	-4.4707E+02	8.5503E+03	-2.3463E+05
9	2	3	1.0114E+00	2.8994E+00	1.4963E+01	1.2185E+02	1.4455E+03
9	2	4	5.1451E-01	1.0530E+00	3.5379E+00	1.8922E+01	1.4422E+02
9	2	5	1.3920E-01	4.2995E-01	7.6825E-01	3.4967E+00	1.7611E+01
9	2	6	-2.0860E-01	4.0049E-01	-3.1667E-01	1.8738E+00	-2.7083E+00
9	2	7	-6.0944E-01	9.5984E-01	-2.2410E+00	8.4032E+00	-4.2756E+01
9	2	8	-1.1829E+00	2.7784E+00	-1.0327E+01	5.6928E+01	-4.3802E+02
9	2	9	-2.3191E+00	1.0096E+01	-7.0134E+01	7.2711E+02	-1.0657E+04
9	3	4	3.7309E-01	5.6420E-01	1.5919E+00	7.2839E+00	4.8902E+01
9	3	5	1.4165E-01	2.1045E-01	3.6789E-01	1.2271E+00	5.6160E+00
9	3	6	-6.2128E-02	1.4446E-01	2.1887E-03	3.4969E-01	3.5229E-01
9	3	7	-2.8909E-01	3.1698E-01	-4.2408E-01	1.1302E+00	-3.5277E+00
9	3	8	-6.0944E-01	9.5984E-01	-2.2410E+00	8.4032E+00	-4.2756E+01
9	3	9	-1.2424E+00	3.6499E+00	-1.6578E+01	1.1770E+02	-1.1718E+03
9	4	5	1.5065E-01	1.3133E-01	2.1092E-01	5.8845E-01	2.4493E+00
9	4	6	4.5478E-02	8.2825E-02	6.8626E-02	1.8519E-01	5.0633E-01
9	4	7	-6.2128E-02	1.4446E-01	2.1888E-03	3.4969E-01	3.5229E-01
9	4	8	-2.0860E-01	4.0049E-01	-3.1667E-01	1.8738E+00	-2.7083E+00
9	4	9	-4.9567E-01	1.4875E+00	-3.4613E+00	2.4161E+01	-1.4032E+02
9	5	6	1.5065E-01	1.3133E-01	2.1092E-01	5.8845E-01	2.4493E+00
9	5	7	1.4165E-01	2.1045E-01	3.6789E-01	1.2271E+00	5.6160E+00
9	5	8	1.3920E-01	4.2995E-01	7.6825E-01	3.4967E+00	1.7611E+01
9	5	9	1.3877E-01	1.2233E+00	2.1492E+00	2.0653E+01	1.1403E+02
9	6	7	3.7309E-01	5.6420E-01	1.5919E+00	7.2839E+00	4.8902E+01
9	6	8	5.1451E-01	1.0530E+00	3.5379E+00	1.8922E+01	1.4422E+02
9	6	9	8.0071E-01	2.6015E+00	1.2152E+01	9.2473E+01	9.6983E+02
9	7	8	1.0114E+00	2.8994E+00	1.4963E+01	1.2185E+02	1.4455E+03
9	7	9	1.6434E+00	6.8931E+00	4.8788E+01	5.2734E+02	8.0846E+03
9	8	9	2.9653E+00	2.0049E+01	2.3708E+02	4.3902E+03	1.1869E+05
10	1	2	3.3127E+00	2.3518E+01	2.8586E+02	5.3782E+03	1.4672E+05
10	1	3	1.9428E+00	8.6434E+00	6.3892E+01	7.0981E+02	1.1090E+04
10	1	4	1.0762E+00	3.4863E+00	1.7706E+01	1.3793E+02	1.4948E+03

TABLE 13. (CONTINUED)

10	1	5	4.1853E-01	1.5021E+00	4.6283E+00	3.0444E+01	2.3219E+02
10	1	6	-1.6594E-01	1.0937E+00	-4.4953E-01	1.3176E+01	-6.0467E+00
10	1	7	-7.8129E-01	1.9890E+00	-6.3906E+00	3.7209E+01	-2.7382E+02
10	1	8	-1.5449E+00	4.8574E+00	-2.3829E+01	1.7519E+02	-1.8033E+03
10	1	9	-2.6672E+00	1.2505E+01	-9.1271E+01	9.7648E+02	-1.4612E+04
10	1	10	-4.9036E+00	4.0918E+01	-5.3601E+02	1.0444E+04	-2.8964E+05
10	2	3	1.2088E+00	3.7069E+00	1.9949E+01	1.6673E+02	2.0103E+03
10	2	4	6.8190E-01	1.4544E+00	5.2195E+00	2.8874E+01	2.2606E+02
10	2	5	2.9408E-01	5.9085E-01	1.3972E+00	5.8435E+00	3.2566E+01
10	2	6	-4.1122E-02	3.4779E-01	1.4877E-01	1.6068E+00	3.5023E+00
10	2	7	-3.8656E-01	5.5514E-01	-8.6209E-01	2.8611E+00	-9.9085E+00
10	2	8	-8.1045E-01	1.3876E+00	-3.6347E+00	1.4220E+01	-7.6477E+01
10	2	9	-1.4311E+00	3.7056E+00	-1.4789E+01	8.5400E+01	-6.7905E+02
10	2	10	-2.6672E+00	1.2505E+01	-9.1271E+01	9.7648E+02	-1.4612E+04
10	3	4	4.8543E-01	7.9569E-01	2.3809E+00	1.1286E+01	7.7601E+01
10	3	5	2.3278E-01	3.1096E-01	6.3382E-01	2.1461E+00	1.0308E+01
10	3	6	2.2482E-02	1.5093E-01	1.2517E-01	4.6495E-01	1.3816E+00
10	3	7	-1.8766E-01	1.9661E-01	-1.5673E-01	4.3341E-01	-7.2931E-01
10	3	8	-4.4124E-01	4.9754E-01	-8.2512E-01	2.1753E+00	-7.7254E+00
10	3	9	-8.1045E-01	1.3876E+00	-3.6347E+00	1.4220E+01	-7.6477E+01
10	3	10	-1.5449E+00	4.8574E+00	-2.3829E+01	1.7519E+02	-1.8033E+03
10	4	5	1.9516E-01	1.8562E-01	3.3327E-01	9.9141E-01	4.3401E+00
10	4	6	6.9200E-02	8.2608E-02	8.6499E-02	2.0342E-01	6.1750E-01
10	4	7	-4.9409E-02	8.3693E-02	-2.8670E-03	1.1227E-01	6.1272E-02
10	4	8	-1.8766E-01	1.9661E-01	-1.5673E-01	4.3341E-01	-7.2931E-01
10	4	9	-3.8656E-01	5.5514E-01	-8.6209E-01	2.8611E+00	-9.9085E+00
10	4	10	-7.8129E-01	1.9890E+00	-6.3906E+00	3.7209E+01	-2.7382E+02
10	5	6	1.1538E-01	7.5654E-02	8.9094E-02	1.8100E-01	5.4232E-01
10	5	7	6.9200E-02	8.2608E-02	8.6499E-02	2.0342E-01	6.1750E-01
10	5	8	2.2482E-02	1.5093E-01	1.2517E-01	4.6495E-01	1.3816E+00
10	5	9	-4.1122E-02	3.4779E-01	1.4877E-01	1.6068E+00	3.5023E+00
10	5	10	-1.6594E-01	1.0937E+00	-4.4953E-01	1.3176E+01	-6.0467E+00
10	6	7	1.9516E-01	1.8562E-01	3.3327E-01	9.9141E-01	4.3401E+00
10	6	8	2.3278E-01	3.1096E-01	6.3382E-01	2.1461E+00	1.0308E+01
10	6	9	2.9408E-01	5.9085E-01	1.3972E+00	5.8435E+00	3.2566E+01
10	6	10	4.1853E-01	1.5021E+00	4.6283E+00	3.0444E+01	2.3219E+02
10	7	8	4.8543E-01	7.9569E-01	2.3809E+00	1.1286E+01	7.7601E+01
10	7	9	6.8190E-01	1.4544E+00	5.2195E+00	2.8874E+01	2.2606E+02
10	7	10	1.0762E+00	3.4863E+00	1.7706E+01	1.3793E+02	1.4948E+03
10	8	9	1.2088E+00	3.7069E+00	1.9949E+01	1.6673E+02	2.0103E+03
10	8	10	1.9428E+00	8.6434E+00	6.3892E+01	7.0981E+02	1.1090E+04
10	9	10	3.3127E+00	2.3518E+01	2.8586E+02	5.3782E+03	1.4672E+05

half normal. This procedure is much more efficient than the usual direct integration technique used, for example, by Sarhan (36) to generate the moments of the double exponential. Sarhan gives the equation

$$E(X_{(r)}^m) = \int_{-\infty}^0 x_{(r)}^m f_1(x_{(r)}) dx + \int_0^{\infty} x_{(r)}^m f_2(x_{(r)}) dx$$

where

$$f_1(x_{(r)}) = \frac{n!}{(r-1)!(n-r)!} \left(\frac{1}{2}e^{x(r)}\right)\left(1 - \frac{1}{2}e^{x(r)}\right)^{n-r}$$

and

$$f_2(x_{(r)}) = \frac{n!}{(r-1)!(n-r)!} \left(1 - \frac{1}{2}e^{-x(r)}\right)\left(\frac{1}{2}e^{-x(r)}\right)^{n-r+1}$$

The functions f_1 and f_2 above must be expanded using the binomial theorem and then integrated by parts. The procedure as followed by Sarhan becomes intolerable as sample size and moments increase.

CHAPTER VII. SUMMARY

Let $X_{(i)}$ be the ordered random variables for a sample of n observations selected from an exponential with parameter b . Then it is known that

$$Z_k = (n-k+1)(X_{(k)} - X_{(k-1)}) \quad k = 1, 2, \dots, n \quad (7.1)$$

where $X_{(0)} = 0$ are identically and independently distributed as exponential with parameter b . Furthermore we showed that $X_{(i)}$ is independent of Z_j if $i < j$.

If Y is a Weibull random variable with density

$$f(y, a, b) = \frac{1}{ab} (y)^{\frac{1}{a}-1} \exp(-(y^{1/a}/b)) \quad y > 0, a > 0,$$

$$b > 0 \quad (7.2)$$

the mean and variance are given by

$$E(y) = b^a \Gamma(a+1)$$

$$V(y) = b^{2a} (\Gamma(2a+1) - (\Gamma(a+1))^2).$$

The transformation $X = Y^{1/a}$ is one to one, order preserving and the random variable X is distributed as an exponential with parameter b . Now from 7.1

$$X_{(k)} = X_{(k-1)} + Z_k / (n-k+1) \quad (7.3)$$

and

$$E(Y_{(k)}) = E(X_{(k)}^a) = E((X_{(k-1)} + Z_k / (n-k+1))^a). \quad (7.4)$$

If a is an integer we use the binomial theorem to obtain

$$E(Y_{(k)}) = a!b^a \sum_{i=0}^a E(X_{(k-1)}^i) \left(\frac{1}{n-k+1}\right)^{a-i} / i! \quad (7.5)$$

We may rewrite Equation 7.3 as

$$X_{(k)} = X_{(k-j)} + W_{k,j} \quad (7.6)$$

where $W_{k,j}$, defined in Equation 2.20, is a weighted sum of Z 's. Hence by the binomial theorem

$$\begin{aligned} E(Y_{(k-j)} Y_{(k)}) &= E((X_{(k-j)} (X_{(k-j)} + W_{k,j}))^a) \quad (7.7) \\ &= E((X_{(k-j)}^2 + X_{(k-j)} W_{k,j})^a) \\ &= b^{2a} \sum_{i=0}^a \binom{a}{i} E(X_{(k-j)}^{a+i}) E(W_{k,j}^{a-i}). \end{aligned}$$

Both of the expected values contained in the expression on the right hand side of Equation 7.7 are known. Hence Equations 7.5 and 7.7 may be used to compute the variances and covariances of the Weibull with integer shape parameter.

The asymptotic expectations and asymptotic covariances of Weibull order statistics were developed by deriving the asymptotic moment generating functions. Asymptotically the largest exponential order statistic has an extreme value distribution. Hence the mgf of the largest order statistic of an exponential distribution is

$$G_{X_{(n)}}(t) = \exp(bt \log n) \Gamma(1-bt) \quad (7.8)$$

Using the independence of Z_k and X_1 we have the mgf of the lower order statistics,

$$G_{X_{(n-k)}}(t) = \exp(bt \log n) \prod_{j=1}^k (1 - \frac{bt}{j}). \quad (7.9)$$

The asymptotic expectations are

$$E^*(Y_{(n)}) = E^*(X_{(n)}^a) = b^a \sum_{i=0}^a \binom{a}{i} (\log n)^{a-i} I(i) \quad (7.10)$$

$$E^*(Y_{(n-k)}) = E^*(X_{(n-k)}^a) = b^a \sum_{i=0}^a \binom{a}{i} h(i, k, 0) E^*(X_{(n)}^{a-i})$$

$$k = 1, 2, \dots \quad (7.11)$$

where $I(i) = \left| \int_0^\infty e^{-z} (\log z)^i dz \right|$ and $h(i, k, 0) = \frac{d^i}{dt^i} \prod_{j=1}^k (1 - t/j)$ evaluated at $t = 0$. Using Equation 7.5 the asymptotic expectations of the crossproducts are given by

$$E^*(Y_{(k-j)} Y_{(k)}) = E^*(X_{(k-j)}^a X_{(k)}^a)$$

$$= \sum_{i=0}^a \binom{a}{i} E^*(X_{(k-j)}^{a+i}) E(W_{k,j}^{a-i}) \quad (7.12)$$

and asymptotic variances and covariances may be calculated in the usual manner.

The expected values and covariances of the order statistics of the exponential are well known.

$$E(X_{(k)}) = b \sum_{j=1}^k 1/(n-j+1) \quad (7.13)$$

$$V(X_{(k)}) = b^2 \sum_{j=1}^k 1/(n-j+1)^2$$

$$C(X_{(j)} X_{(k)}) = V(X_{(j)}) \text{ if } j < k.$$

The asymptotic approximations are given by

$$E^*(X_{(n-k)}) = \log n + I(1) - \sum_{j=1}^k 1/j \quad (7.14)$$

$$V^*(X_{(n-k)}) = \frac{\pi^2}{6} - \sum_{j=1}^k 1/j^2 \quad k = 1, 2, \dots$$

Letting $a = 1$ in Equation 7.10 we have $E^*(X_{(n)}) = b(\log n + I(1))$. The error incurred by using the asymptotic formula ranges from about four percent at $n = 5$ to 0.1 percent at $n = 100$. Also $V^*(X_{(n)}) = \pi^2/6$ which is independent of sample size and differs from $V(X_{(n)})$ by about two percent at $n = 25$.

The class of estimators of the mean

$$\hat{w} = \left(\sum_{i=0}^{n-r} X_{(i)} + rX_{(n-r)} \right) / D \quad (7.15)$$

is introduced. In terms of the random variables Z_i , Equation 7.15 may be written

$$\hat{w} = \sum_{i=0}^{n-r} Z_i / D.$$

Three members of this class are studied as estimators for the mean of the Weibull distribution. These are i) the Winsorized mean where $D = n$, ii) the estimate due to Epstein where $D = n-r$, and iii) the member of the class derived to minimize the mean square error of $\hat{w}$ for the exponential distribution where $D = n-r+1$. The MSE of these estimates is compared to the MSE of $\bar{Y}$. There is a large gain in efficiency with respect to the mean in using $\hat{w}$ instead of $\bar{Y}$ as the shape parameter increases. An argument is

presented to show that in large samples from a Weibull with shape parameter greater than one $MSE(\hat{W}_1) < MSE(\bar{Y})$ where $\hat{W}_1$ is the once censored Winsorized mean.

The maximum likelihood estimator of the mean of the Weibull is given by

$$\hat{m} = \hat{c} \Gamma(\hat{a}+1) \quad (7.16)$$

where c and a are the scale and shape parameters respectively. Employing a Taylor series expansion the asymptotic variance of $\hat{m}$ is expressed as a function of the asymptotic variances and covariances of $\hat{c}$ and $\hat{a}$. As the shape parameter increases above one large gains in efficiency are made by using the MLE instead of $\bar{Y}$ as an estimator of the mean.

Three tests for the shape parameter of the Weibull are investigated in a Monte Carlo study. The first test statistic may be written

$$T = \frac{\sum_{k=1}^n D_k Z_k}{\sum_{k=1}^n Z_k} \quad (7.17)$$

where the D_k are constants and

$$Z_k = (n-k+1) \left(Y_{(k)}^{1/a_0} - Y_{(k-1)}^{1/a_0} \right)$$

The $Y_{(k)}$ are the Weibull order statistics and a_0 is the hypothesized value of the shape parameter. If the null hypothesis is true $E(T) = 0$. If the true value of the shape parameter differs from a_0 then $E(T)$ differs from

zero. Hence T is intuitively appealing for testing $H_0: a = a_0$ versus $H_A: a \neq a_0$. The distribution of T is known.

The second test is derived from the extreme value distribution. We know that if Y is a Weibull variate then $\log Y$ has the extreme value distribution. The expected values and covariances of the order statistics of the extreme value distribution are known. The parameters of this distribution are of the location and scale type so that least squares may be used to obtain linear estimators of the parameters. A test statistic analogous to Student's t is

$$W = \frac{\hat{B} - a_0}{\sqrt{V(\hat{B})}} \quad (7.18)$$

where $\hat{B}$ is the least squares estimate of a . Little is known of the distribution of W .

The third test investigated is the likelihood ratio test for $H_0: a = a_0$ versus $H_A: a = a_1$. The test statistic is

$$R = \left(\frac{y_1^{1/a_0}}{y_1^{1/a_1}} \right)^{n \left(\prod y_i \right)^{1/a_1 - 1/a_0}}$$

where H_0 is accepted if $R > k$. The distribution of R is not known for this test so that Monte Carlo methods were used to establish critical values.

When the three tests were compared using Monte Carlo methods the T and W statistics were found to have approximately the same power with W generally more powerful. The likelihood ratio test was the most powerful when the underlying distribution used for the alternative was Weibull with $a = a_1$. The likelihood ratio test would be expected to have less power if the alternative were not that used in constructing the test.

A combinatorial form is developed expressing the expected values and expected crossproducts of the order statistics of a symmetric distribution as a function of the expected values and expected crossproducts of the order statistics of the related half distribution. If Z is the symmetric random variable then $g(z) = \frac{1}{2} f(z)$ if $z > 0$ and $g(z) = \frac{1}{2} f(-z)$ if $z < 0$ where $f(y)$ is the density of the half distribution. It is argued that the conditional density of the j largest observations of a sample of size n from the symmetric variate given that $J = j$ of them are greater than zero is precisely that of a sample of size j from the density of the positive half distribution. The number of observations greater than zero is a binomial random variable with parameter $p = \frac{1}{2}$. The expected values of the order statistics of the symmetric random variables are then given by the summation over the random variable J . The form for the crossproducts is developed in a similar manner. In particular the expected values

119-121

and expected crossproducts of the double Weibull distribution are tabulated.

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APPENDIX

Presented here is the derivation of the limiting covariance matrix of the maximum likelihood estimates of parameters c and a of the Weibull distribution. The maximum likelihood estimates $\hat{c}$ and $\hat{a}$ for the parameters of the density $f(y;a,c)$ based on a sample of size n are approximately distributed, for large n , as a multivariate normal with means c and a and covariance matrix given by

$$\hat{\Sigma} = -\frac{1}{n} \begin{pmatrix} E\left(\frac{\partial^2}{\partial c^2} \log f(y;a,c)\right) & E\left(\frac{\partial^2}{\partial c \partial a} \log f(y;a,c)\right) \\ \text{sym.} & E\left(\frac{\partial^2}{\partial a^2} \log f(y;a,c)\right) \end{pmatrix}^{-1}$$

The density of the Weibull with scale parameter c and shape parameter a is

$$f(y;a,c) = \frac{1}{ac} \left(\frac{y}{c}\right)^{\frac{1}{a}-1} \exp\left(-\left(\frac{y}{c}\right)^{\frac{1}{a}}\right) \quad y > 0.$$

Hence

$$\log f(y;a,c) = -\log a - \log c + \left(\frac{1}{a} - 1\right) \log(y/c) - \left(\frac{y}{c}\right)^{\frac{1}{a}}.$$

Taking the partial derivatives we have

$$\frac{\partial^2}{\partial c^2} \log f(y;a,c) = \frac{1}{ac^2} - \frac{1}{ac^2} \left(\frac{y}{c}\right)^{1/a} - \frac{1}{a^2 c^2} \left(\frac{y}{c}\right)^{1/a}$$

$$\frac{\partial^2}{\partial c \partial a} \log f(y;a,c) = \frac{1}{a^2 c} - \frac{1}{a^2 c} \left(\frac{y}{c}\right)^{1/a} - \frac{1}{a^3 c} \left(\frac{y}{c}\right)^{1/a} \log(y/c)$$

$$\frac{\partial^2}{\partial a^2} \log f(y;a,c) = \frac{1}{a^2} + \frac{2}{a^2} \log(y/c) - \frac{2}{a^3} \left(\frac{y}{c}\right)^{1/a} \log(y/c)$$

$$-\frac{1}{a^4} (y/c)^{1/a} (\log (y/c))^2 .$$

Furthermore using the transformation $z = (y/c)^{1/a}$,
we find the expected values

$$E((y/c)^{1/a}) = \Gamma(2)$$

$$E(\log (y/c)) = \Gamma^{(1)}(1)a$$

$$E((y/c)^{1/a} \log (y/c)) = \Gamma^{(1)}(2)a$$

$$E((y/c)^{1/a} (\log (y/c))^2) = \Gamma^{(2)}(2)a^2 .$$

From (44) we know that $\Gamma(2) = 1$, $\Gamma^{(1)}(2) = 1-.5772$,

$\Gamma^{(2)}(2) = (\frac{\pi^2}{6} - 1) + (1 - .5772)^2$, and $\Gamma^{(1)}(1) = -.5772$.

Hence

$$\begin{aligned} \frac{1}{4} &= \frac{1}{n} \left(\begin{array}{cc} \frac{1}{(ac)^2} & \frac{.4228}{ca^2} \\ \text{sym.} & \frac{1}{a^2} \left(\frac{\pi^2}{6} + (.4228)^2 \right) \end{array} \right)^{-1} \\ &= \frac{1}{n} \left(\begin{array}{cc} 1.1087(ac)^2 & -.2570ca^2 \\ -.2570ca^2 & .6079a^2 \end{array} \right) \end{aligned}$$

which is exactly Equation 5.8 .