325
12/2/62

COMPUTER GENERATION AND TESTING
OF
RANDOM NUMBERS

MASTER

L. J. Gannon and L. A. Schmittroth

**PHILLIPS
PETROLEUM
COMPANY**

ATOMIC ENERGY DIVISION

**NATIONAL REACTOR TESTING STATION
US ATOMIC ENERGY COMMISSION**

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency Thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

DISCLAIMER

Portions of this document may be illegible in electronic image products. Images are produced from the best available original document.

PRICE \$1.00

Available from the
Office of Technical Services
U. S. Department of Commerce
Washington 25, D. C.

LEGAL NOTICE

This report was prepared as an account of Government sponsored work. Neither the United States, nor the Commission, nor any person acting on behalf of the Commission:

A. Makes any warranty or representation, express or implied, with respect to the accuracy, completeness, or usefulness of the information contained in this report, or that the use of any information, apparatus, method, or process disclosed in this report may not infringe privately owned rights; or

B. Assumes any liabilities with respect to the use of, or for damages resulting from the use of any information, apparatus, method, or process disclosed in this report.

As used in the above, "person acting on behalf of the Commission" includes any employee or contractor of the Commission, or employee of such contractor, to the extent that such employee or contractor of the Commission, or employee of such contractor prepares, disseminates, or provides access to, any information pursuant to his employment or contract with the Commission, or his employment with such contractor.

Printed in USA

IDO-16921
AEC Research and Development Report
Mathematics and Computers
TID-4500 (22nd Ed.)
Issued: August 20, 1963

COMPUTER GENERATION AND TESTING
OF
RANDOM NUMBERS

L. J. Gannon
L. A. Schmittroth

PHILLIPS
PETROLEUM
COMPANY

Atomic Energy Division
Contract AT(10-1)-205
Idaho Operations Office
U. S. ATOMIC ENERGY COMMISSION

THIS PAGE
WAS INTENTIONALLY
LEFT BLANK

COMPUTER GENERATION AND TESTING OF RANDOM NUMBERS

Abstract

This report discusses the need for random numbers particularly in regard to Monte Carlo applications and the two methods most commonly used to obtain random numbers. These methods are the multiplicative congruential method and the mixed congruential method, and both methods involve somewhat arbitrary parameters. Proofs for the lengths of period for both methods and a derivation of an expression for the serial correlation coefficient in terms of the parameters of the mixed congruential method are presented. The results of a series of statistical tests made to determine satisfactory parameters are listed. The length of run test was found to be the most powerful test in that it rejected the largest number of parameters as unsatisfactory. The multiplicative method (for the parameters tested) was found superior to the mixed congruential method. The multiplicative method multiplier $\lambda = 2^{18} + 3$ satisfactorily passed all tests and has a short multiply time.

TABLE OF CONTENTS

	Page
ABSTRACT	iii
LIST OF TABLES	v
INTRODUCTION	1
NEED FOR RANDOM NUMBERS	1
Monte Carlo Method	1
Example of the Monte Carlo Method	2
Traditional Sources of Random Numbers not Adequate	3
Definition of Random Numbers	4
DESCRIPTION OF TWO RANDOM NUMBER GENERATORS	4
Mid Square Method	4
Multiplicative Congruential Method	5
Mixed Congruential Method	5
PROOFS AND DERIVATION	6
Length of Period for the Multiplicative Congruential Methods	6
Length of Period for the Mixed Congruential Method	9
Correlation Coefficient for the Mixed Congruential Method	12
STATISTICAL TESTS	19
General Comments and Discussion of the Computer Program	19
Frequency Test	21
Matrix Test	22
Auto-correlation Coefficient	23
Frequency of Lengths of Runs	24
Frequencies in Subsequences	27
Conclusions	28
LITERATURE CITED	37

LIST OF TABLES

Table	Page
1. Comparison of Chi-square Values of Frequencies in 102^4 Subintervals with Generator Parameters	30
2. Comparison of Chi-square Values of Frequencies in 32 by 32 Matrix with Generator Parameters	31
3. Auto-correlation Coefficients	32
4. Lengths of Runs	34
5. Chi-square Values of Frequencies in 102^4 Subintervals, for 8 Subsequences of the Sequence Generated by Using $\lambda = 2^{18} + 3$ and $\mu = 0$	36

INTRODUCTION

Random numbers have been used for years in various statistical sampling techniques and have been obtained from different sources. With the development of high speed digital computers and the Monte Carlo method, interest has increased in random numbers, means of obtaining them, and various tests for randomness.

This paper discusses the need for random numbers (particularly in regard to Monte Carlo applications) and the two methods most commonly used to obtain random numbers. These methods involve somewhat arbitrary parameters and this paper presents the results of statistical tests made to determine satisfactory parameters.

NEED FOR RANDOM NUMBERS

Monte Carlo Method

Random numbers drawn from a uniform distribution on the unit interval are required in a wide variety of both scientific and commercial applications. The colorful term Monte Carlo is commonly applied to any computation method that uses random numbers, and was first used by von Neumann and Ulam at Los Alamos in 1946. Nuclear Physicists have pioneered the use of Monte Carlo methods in the study of nuclear phenomena and in reactor calculations.

Extensive use of the Monte Carlo method is now made in a wide variety of situations. In addition to the nuclear field some of the applications include the studies of cell growth, traffic flow, scheduling, business, and

social-economic systems. These types of problems are of a probabilistic nature and in some cases the Monte Carlo method has been the best method or the only method of attacking the problem. There is another class of problems that does not fall in this category, yet surprisingly the Monte Carlo method has been used with varying degrees of success in computing multiple integrals, solving differential and integral equations, finding eigenvalues, and inverting matrices.

Two basic requirements for the successful use of the Monte Carlo method are a high speed digital computer and a source of a large number of random numbers. The electronic computer has been in common use for about 10 or 15 years, and most of the investigation of random numbers has been in this period.

Example of the Monte Carlo Method

To illustrate how random numbers are used in the Monte Carlo method, we consider a nuclear reactor calculation. Here we simplify the problem to tracking a neutron as it collides and moves about in a region of the reactor. Let us suppose that the region is composed of a homogeneous mixture of hydrogen, oxygen, aluminium, and uranium. Given that a neutron with a certain energy suffers a collision with an isotope, the probability that the collision will be with a particular isotope of the mixture can be computed from the concentrations and the cross sections. In our example let P_H , P_O , P_{Al} , and P_U be these probabilities. Then $P_H + P_O + P_{Al} + P_U = 1$. We plot these in the unit interval as shown in Figure 1.

Fig. 1.--Plot of collision probabilities

Then it is clear that drawing a random number from the unit interval and locating it in the proper position selects an isotope with the proper frequency. In other words if $0 \leq r \leq P_H$, where r is a random number the collision is with H . If $P_H < r \leq P_H + P_O$, the collision is with O , etc. In a similar manner a random number is used to determine the type of event, i.e., scattering, fission, or absorption, by comparing the random number to the probabilities of these events for the isotope involved in the collision. Random numbers are also used to determine the scattering angle, new direction cosines, and the distance to the next collision. Thus we see that in tracking a neutron many random numbers are used.

Traditional Sources of Random Numbers not Adequate

Conceptually random numbers can be obtained by tossing a coin, rolling dice, drawing balls from urns, and similar actions which give rise to random events. However, these methods are much too slow to produce random numbers in the volume necessary for large scale calculations. Other physical sources of random numbers (mechanical equipment, pulses from electronic circuits, radioactive decay and other phenomena which give rise to random events) have not proved practical. There is a question if the source always produces "truly random" numbers, and it is impossible to exactly reproduce a given calculation. This requirement of being able to exactly reproduce a given calculation is extremely important and the comparison between solving the problem on the computer and working it out with pencil and paper can be made. One must be able to go back and check and recheck the intermediate steps, and always get the same answer to a given problem. If one gets two different answers to a problem, he can't tell which one is correct, or if either is correct. Thus algorithms have been developed by means of which sequences of "pseudo random" numbers can be produced and reproduced within the computer.

Definition of Random Numbers

The concept of "pseudo-random numbers or sequence" has been defined by D. H. Lehmer [7] to be a "vague notion embodying the idea of a sequence in which each term is unpredictable to the uninitiated and whose digits pass a certain number of tests traditional with statisticians and depending somewhat on the use to which the sequence is to be put."

H. Cramer [2] has stated, "It does not seem possible to give a precise definition of what is meant by the word 'random'." In the remainder of this paper the term random numbers will apply to both random numbers and pseudo-random numbers.

DESCRIPTION OF TWO RANDOM NUMBER GENERATORS

The rather stringent requirements for a large number of random numbers, uniformly distributed, obtainable in a minimum of time, and exactly reproducible have been more or less satisfied by several algorithms or random number generators.

Mid Square Method

One of the first algorithms for producing random numbers was proposed by von Neumann and Metropolis. This is the mid square method in which each number in the sequence is defined as the middle set of digits of the square of the previous number. Consider as an example the first number in the sequence to be 1357 (assume we are working with four digit numbers). Then this number squared is 01841449 and the second number in the sequence is 8414, the middle four digits of the squared number.

D. Lehmer [7] points out difficulties that have caused this method to be abandoned. Numbers of the type OOX Y or XY00 soon develop and are either

self perpetuating or quickly degenerate to zero. We will not consider this method in any more detail.

Multiplicative Congruential Method

D. H. Lehmer [7] proposed the multiplicative congruential method.

His specific generator was

$$x_{n+1} \equiv 25x_n \pmod{10^8 + 1}$$

for a decimal machine with an eight digit word. (The word size of a computer is analogous to the keyboard capacity of a calculator.) That is, the random number x_{n+1} is obtained from the random number x_n by a multiplication. The multiplicative congruential method as most commonly now used is

$$x_{n+1} \equiv \lambda x_n \pmod{P}$$

Here $P = 10^b$ or 2^b and b is the number of digits or bits in the computer word. This modulus is chosen so that no division is necessary to form the congruence. Two numbers of b bits (or b digits) each are multiplied to form a product of $2b$ bits (or digits). The desired number consists of the b least significant bits (or digits). To avoid making the distinction between the binary and decimal computers, and bits and digits, only the binary computer will be considered in detail in what follows.

Mixed Congruential Method

A. Rotenberg [8] proposed the mixed congruential method

$$x_{n+1} \equiv \lambda x_n + \mu \pmod{P}$$

This is similar to the multiplicative method, only now a constant μ is added.

Since the word length of the computer is finite, only a finite number of numbers can be represented. Also it is obvious that once a number in the

sequence is repeated, a cycle is set up and the sequence from there on repeats itself. It is clear that we want a sequence with a long period (the number of numbers that are generated before a number is repeated).

As will be shown in the next section, the multiplicative congruential method has a maximum period of 2^{b-2} and this is obtained when the initial random number is odd and the multiplier has a value $\lambda \equiv \pm 3 \pmod{8}$. The mixed congruential method has a maximum period 2^b and is obtained with a multiplier $\lambda \equiv 2^a + 1$, $a > 2$, and the additive constant μ is odd.

The above calculations are performed in the computer with integer arithmetic (as distinguished from floating point) and the fixed binary (decimal) point is assumed to be to the right of the right most bit during the calculation. Now since we want numbers on the unit interval, we merely reassign the binary point to the left of the left most bit and this gives the desired values $0 < x_n < 1$.

PROOFS AND DERIVATION

Length of Period for the Multiplicative Method

In this section we want to determine a multiplier λ and an initial number x_0 such that the multiplicative method

$$x_{i+1} \equiv \lambda x_i \pmod{2^b}$$

generates a sequence with the maximum period. We can make two preliminary observations. The first is that if the multiplier λ is even, i.e., $\lambda = 2k$, the period is very short. In fact

$$x_{i+b} \equiv (2k)^b x_i \pmod{2^b}$$

means that $x_{i+b} = x_i$ and that the sequence can not be longer than b . Hence, λ must be odd in order to obtain a period of any useful length. Likewise, if the initial number x_0 is even, the common factor 2 appears both in x_0 and the modulus 2^b , hence, the modulus is effectively reduced, which reduces the period. Thus x_0 must also be odd.

To continue further we will need to recall several concepts and definitions from number theory. We do this and proceed as in [5].

Residue Class: A class of all integers which are mutually congruent for a given modulus. There are m distinct residue classes mod m , corresponding to the terms of a complete residue system mod m ; collectively they comprise the class of all integers.

Complete Residue System: For a given modulus m , a set of m numbers congruent in some order to the residues $0, 1, \dots, m-1$.

Reduced Residue System: A subset of a complete residue system, containing all terms which are relatively prime to m .

Euler's phi-function $\phi(m)$ denotes the number of positive integers less than m and prime to m , so a reduced residue system contains $\phi(m)$ terms. If the prime factorization of m is $m = \prod p_i^{e_i}$ (here $\prod$ represents the product notation, p_i the prime factors of m , and e_i the number of times p_i appears as a factor of m), then $\phi(m) = \prod (p_i - 1) p_i^{e_i - 1}$.

Power Residues: The residues of the successive powers of a number: $x^n \pmod{m}$ for $n = 1, 2, 3, \dots$.

Order of $x \pmod{m}$: When x and m are relatively prime, the least positive exponent h with $x^h \equiv 1 \pmod{m}$. Euler showed that $x^{\phi(m)} \equiv 1 \pmod{m}$, from which it follows that h is a divisor of $\phi(m)$.

Primitive Root of m : A number x whose order $h = \phi(m)$.

The powers of a primitive root x generate a reduced system of residues mod m (the largest possible set), and all primitive roots occur in this set as powers whose exponents are relatively prime to $\phi(m)$. Therefore, if m has a primitive root, it has $\phi(\phi(m))$ primitive roots. As an example consider $m = 7$ and $x = 3$, here the sequence is 3, 2, 6, 4, 5, 1 and then repeats. Note that all 6 positive integers less than 7 appear, showing that 3 is a primitive root of the prime modulus 7. With the exception of $m = 4$, only numbers of the form p^n or $2p^n$, for p an odd prime, have primitive roots.

With this information we are now in a position to determine the maximum period and the corresponding requirement for the multiplier. We recall that the order h of any number $\lambda \pmod{2^b}$ is a divisor of $\phi(2^b) = 2^{b-1}$, (the number of odd positive integers less than 2^b) and so is of the form 2^e for $e \leq b - 1$. If $m = 2^b$ had primitive roots e would equal $b - 1$, however, m has no primitive roots (m is even and not 4) therefore, $e < b - 1$. From our earlier observation λ has to be odd, and we can represent all odd numbers in the form $8t \pm 1$ or $8t \pm 3$. We observe that

$$(8t \pm 1)^{2^{b-3}} = 1 \pm 2^b t + \dots \equiv 1 \pmod{2^b}.$$

So that the order of

$$\lambda \equiv (8t \pm 1) \pmod{2^b}$$

is a divisor of 2^{b-3} , and any multiplier $\lambda \equiv 1 \pmod{8}$ will generate a sequence whose period is not greater than 2^{b-3} . We next show that a multiplier $\lambda \equiv 3 \pmod{8}$ leads to a sequence with a period greater than 2^{b-3} .

$$(\pm 3 + 8t)^{2^{b-3}} = 3^{2^{b-3}} \pm 2^{b-3} \cdot 3^{2^{b-3}-1} \cdot 8t + \dots \equiv 3^{2^{b-3}} \pmod{2^b}$$

$$\equiv (4-1)^{2^{b-3}} \equiv 1 - 2^{b-1} \not\equiv 1 \pmod{2^b}.$$

This shows that the order of $\lambda \equiv 3 \pmod{8}$ is greater than 2^{b-3} . We then try 2^{b-2} .

$$\begin{aligned} (\pm 3 + 8t)^{2^{b-2}} &= 3^{2^{b-2}} \pm 2^{b-2} \cdot 3^{2^{b-2}-1} \cdot 8t + \dots \equiv 3^{2^{b-2}} \pmod{2^b} \\ &\equiv (-1 + 4)^{2^{b-2}} \equiv 1 \pmod{2^b} \end{aligned}$$

So the order of $x \equiv 3 \pmod{8}$ is $h = 2^{b-2}$. In our earlier discussion we saw that the order h must divide $\phi(m) = 2^{b-1}$ and must be less than 2^{b-1} . Hence, $h = 2^{b-2}$ is the maximum order and corresponds to the maximum period of the multiplicative congruential method with a choice of a multiplier $\lambda \equiv 3 \pmod{8}$ and the initial number odd.

A final remark must be made concerning the multiplicative method. If random bits (or digits) are required, they should be taken from the left most (high-order) end, because the right most (low-order) bits are far from random.

Length of Period for the Mixed Congruential Methods

Recall that the mixed congruential method

$$x_{n+1} \equiv \lambda x_n + \mu \pmod{2^b}$$

is similar to the multiplicative method, only now a constant μ is added. As in the previous section, we now determine the maximum period for the mixed method and the requirements on λ and μ to obtain the maximum period.

The following lemma is used in the proof of the maximum period of the mixed congruential method.

Lemma:

$$x_{i+n} \equiv \lambda^n x_i + \mu \frac{\lambda^n - 1}{\lambda - 1} \pmod{M} \quad (1.1)$$

We prove this by induction. It is by definition true for $n = 1$.

$$x_{i+1} \equiv \lambda x_i + \mu \pmod{M}$$

We assume it is true for case n ,

$$x_{i+n} \equiv \lambda^n x_i + \mu \frac{\lambda^n - 1}{\lambda - 1} \pmod{M}.$$

Then prove it for case $n + 1$.

$$\begin{aligned} x_{i+n+1} &\equiv \lambda x_{i+n} + \mu \pmod{M} \\ &\equiv \lambda \left(\lambda^n x_i + \mu \frac{\lambda^n - 1}{\lambda - 1} \right) + \mu \pmod{M} \\ &\equiv \lambda^{n+1} x_i + \mu \frac{\lambda^{n+1} - \lambda}{\lambda - 1} + \mu \frac{\lambda - 1}{\lambda - 1} \pmod{M} \\ &\equiv \lambda^{n+1} x_i + \mu \frac{\lambda^{n+1} - 1}{\lambda - 1} \pmod{M} \end{aligned}$$

Q.E.D.

This lemma is used in the proof of the maximum period which follows and can be used to extend the lag in the serial correlation coefficient which is discussed later.

To continue the proof of the maximum period, it is clear that the sequence repeats when $x_1 = x_{i+n}$. Thus the problem is to find the minimum value of n for which the sequence repeats itself and this is the maximum period.

Thus

$$x_1 = x_{i+n} \equiv \lambda^n x_i + \mu \frac{\lambda^n - 1}{\lambda - 1} \pmod{M}$$

follows from the lemma. Subtract x_i from each side of the congruence and factor

$$0 \equiv (\lambda^n - 1) x_i + \mu \frac{\lambda^n - 1}{\lambda - 1} \pmod{M}.$$

Factor and bring to a common denominator

$$0 \equiv \left(\frac{\lambda^n - 1}{\lambda - 1} \right) (x_i (\lambda - 1) + \mu) \pmod{M}.$$

Recalling that $\lambda = 2^a + 1$, $a \geq 2$, and that μ is odd we obtain

$$0 \equiv \left(\frac{(2^a + 1)^n - 1}{2^a} \right) (2^a x_i + \mu) \pmod{2^b}.$$

The second factor is odd, since μ is odd, and the first factor must be divisible by 2^b to satisfy the congruence. This is equivalent to

$$(2^a+1)^n \equiv 1 \pmod{2^{b+a}}.$$

Writing the congruence as an equation we obtain

$$(2^a+1)^n - 1 = m 2^{b+a}$$

where m is a positive integer. Expanding the first term, we have

$$\sum_{k=2}^n \binom{n}{k} 2^{ak} + n \cdot 2^a + 1 - 1 = m 2^{b+a}$$

where $\binom{n}{k}$ are the binomial coefficients $\frac{n!}{(n-k)! k!}$. Dividing by 2^{b+a} yields

$$\sum_{k=2}^n \binom{n}{k} 2^{a(k-1)-b} + n \cdot 2^{-b} = m \quad (1.2)$$

We let $n = r 2^s$, and $k = t 2^i$, where t and r are odd, and then use these substitutions to express the binomial coefficient in the form

$$\binom{n}{k} = \frac{n}{k} \binom{n-1}{k-1} = \frac{r}{t} \binom{n-1}{k-1} \cdot \frac{2^s}{2^i} = \text{an integer (a binomial coefficient)}.$$

We use this expression for the binomial coefficient and the substitutions to rewrite (1.2) in the following form.

$$m = \sum_{k=2}^n \frac{r}{t} \binom{n-1}{k-1} \cdot \frac{2^s}{2^i} \cdot 2^{a(t2^i-1)-b} + r \cdot 2^s \cdot 2^{-b}.$$

Now factor a 2^{s-b} to obtain

$$m = \left[\sum_{k=2}^n \frac{r}{t} \binom{n-1}{k-1} \frac{2^{a(t2^i-1)}}{2^i} + r \right] \cdot 2^{s-b}. \quad (1.3)$$

Since $\binom{n}{k}$ is an integer, the first term is an integer except possibly for a factor of 2. However,

$$\frac{2^{a(t2^i-1)}}{2^i} = 2^{a(t2^i-1)-i} > 1$$

since $a > 1$, t is a positive integer, i a non negative integer, and $t > 2$ if $i = 0$, so that the exponent of 2 is a positive integer. Hence, the first term in the brackets of (1.3) is even, and since r is odd, the quantity in the brackets is odd. Hence, $s \geq b$ for m to be an integer, hence, $n = 2^b$ = the period for the mixed congruential method (recall that b is the number of bits in the computer word). Thus we see that the mixed methods have a larger period than the multiplicative method. This is of little practical concern, however, as it would take the IBM 709 computer over three months of steady computing to generate the maximum sequence for the multiplicative method.

Correlation Coefficient for the Mixed Congruential Method

The correlation coefficient ρ serves as a measure of a property of a distribution that might be called the degree of linear dependence between variables. For the mixed congruential method Coveyou [1] first derived an approximate formula for the correlation coefficient in terms of the parameters λ and μ . This represents the most significant step in relating the statistical properties of the generated sequence to the parameters of the generator. Coveyou's approximation was improved by Greenberger [4], and we present his derivation in what follows.

The correlation coefficient is defined to be

$$\rho = \frac{\alpha - m_1 \cdot m_2}{\sigma_1 \sigma_2} \quad (2.1)$$

We will compute $\rho(x_n, x_{n+1})$ for the full sequence of random numbers (x_n) generated by the method $x_{n+1} \equiv \lambda x_n + \mu \pmod{P}$, where λ and μ are chosen so that the period is $P = 2^b$.

$$m_1 = m_2 = E(x_n) = E(x_{n+1})$$

is the expected or mean value of x_n , and

$$\sigma_1 = \sigma_2 = \sqrt{E(x_n^2) - (E(x_n))^2}$$

is the standard deviation.

$$\alpha = E(x_n, x_{n+1})$$

is the expected or mean value of $x_n x_{n+1}$. Equation (2.1) now becomes

$$\rho = \frac{E(x_n, x_{n+1}) - (E(x_n))^2}{E(x_n^2) - (E(x_n))^2} \quad (2.2)$$

Values for $E(x_n)$ and $E(x_n^2)$ are readily obtained.

$$E(x_n) = \sum_{x=0}^{P-1} x = \frac{P-1}{2} \quad (2.3)$$

$$E(x_n^2) = \sum_{x=0}^{P-1} x^2 = \frac{(P-1)(2P-1)}{6} \quad (2.4)$$

To obtain an expression for $E(x_n, x_{n+1})$, equate x_{n+1} with the remainder r_n in the following equation.

$$\lambda x_n + \mu = q_n P + r_n \quad (n = 0, 1, \dots) \quad (2.5)$$

where q_n and $r_n < P$, and are non-negative integers, uniquely determined for each x_n by the Euclidean algorithm. Omitting the subscript in (2.5) for convenience we obtain

$$\begin{aligned} E(x_n, x_{n+1}) &= \frac{1}{P} \sum_{x_n=0}^{P-1} x_n x_{n+1} = \frac{1}{P} \sum_{x=0}^{P-1} x(\lambda x + \mu - qP) \\ &= \lambda E(x^2) + \mu E(x) - \sum_{x=0}^{P-1} xq. \end{aligned} \quad (2.6)$$

The first two terms are easily evaluated, while the evaluation of $\sum xq$ presents a problem. We now work out an expression for the value of $\sum xq$.

Let x_n in (2.5) take on consecutive integral values starting with zero, and define

$$\lambda \bar{x}_q + \mu = qP + \bar{r}_q. \quad (2.7)$$

Here $\bar{x}_q$ is the initial (and smallest) x and $\bar{r}_q$ is the smallest r for a given value of q . By virtue of its being the smallest of its set, $\bar{r}_q < \lambda$, except when $q = 0$, and then $\bar{r}_0 = \mu$. For $q = 1, 2, \dots, \lambda$, we note that $\bar{r}_q$ is the unique non-negative integer less than λ defined by

$$\bar{r}_q \equiv \mu - qP \pmod{\lambda}. \quad (2.8)$$

It follows that the integers, $\bar{r}_q$, are distinct and $(\bar{r}_1, \bar{r}_2, \dots, \bar{r}_\lambda)$ are a permutation of the integers $(0, 1, 2, \dots, \lambda-1)$. Later we will use the following inequalities involving $(\bar{r}_1, \bar{r}_2, \dots, \bar{r}_\lambda)$

$$\sum_{q=1}^{\lambda} q(\lambda-q) \leq \sum_{q=1}^{\lambda} \bar{r}_q q \leq \sum_{q=1}^{\lambda} q(q-1) \quad (2.9)$$

or

$$\frac{\lambda^3}{6} - \frac{\lambda}{6} \leq \sum_{q=1}^{\lambda} \bar{r}_q q \leq \frac{\lambda^3}{3} - \frac{\lambda}{3}. \quad (2.10)$$

These relationships result from repeated application of a basic inequality which we state in terms of integers $k_1 \leq k_2$, and $k_3 \leq k_4$.

$$k_1 k_4 + k_2 k_3 \leq k_1 k_3 + k_2 k_4 \quad (2.11)$$

which follows from

$$(k_2 - k_1)(k_4 - k_3) \geq 0. \quad (2.12)$$

To demonstrate the application of (2.11) in obtaining (2.9), assume the contrary, i.e.,

$$\sum_{q=1}^{\lambda} \bar{r}_q q > \sum_{q=1}^{\lambda} q(q-1) \dots \quad (2.13)$$

Expand this summation and write

$$\bar{r}_1 \cdot 1 + \dots + \bar{r}_j j + \dots + \bar{r}_{\lambda} \lambda > 0 \cdot 1 + 1 \cdot 2 + \dots + (j-1)j + \dots + (\lambda-1)\lambda \quad (2.14)$$

Now assume $\bar{r}_{\lambda} \neq \lambda - 1$ (otherwise we could cancel the last term on each side)

and that $\bar{r}_j = \lambda - 1$. We then have $\bar{r}_{\lambda} < \bar{r}_j$ and can use (2.11) to write

$$\bar{r}_{\lambda} \lambda + \bar{r}_j j \geq \bar{r}_{\lambda} j + \bar{r}_j \lambda \quad (2.15)$$

Interchanging $\bar{r}_j$ and $\bar{r}_{\lambda}$ will strengthen the assumed inequality (2.13). Then the last term in each sequence, $(\lambda-1)\lambda$, can be cancelled and process repeated until all except the first terms are cancelled which state that zero is greater than a positive integer, a contradiction. Hence, our assumption (2.13) was false. In a similar manner it follows that

$$\sum_{q=1}^{\lambda} q(\lambda-q) \leq \sum_{q=1}^{\lambda} \bar{r}_q q \quad (2.16)$$

We now return to the evaluation of $\sum xq$ in (2.6). We can picture q as a function of x as a bar graph where the abscissa represents the x values which run from zero to $P-1$, and the ordinate represents the q values which run from zero to λ for the case $\mu \geq \lambda$ (and to $\lambda-1$ for the case $\mu < \lambda$ which we will consider later). For $\bar{x}_q \leq x < \bar{x}_{q+1}$ the function has the value q . So

$$\sum_{x=0}^{P-1} xq = \sum_{q=1}^{\lambda} q \sum_{x=\bar{x}_q}^{\bar{x}_{q+1}-1} x \quad (2.17)$$

Graphically this is summing over the vertical bars of the bar graph. An equivalent and more convenient form, which can be considered as summing over

the horizontal bars of the graph, is

$$\begin{aligned}
 \sum_{x=0}^{P-1} xq &= \sum_{q=1}^{\lambda} \sum_{x=x_q}^{P-1} x \\
 &= \frac{1}{2} \sum_{q=1}^{\lambda} P(P-1) - \bar{x}_q(\bar{x}_q-1) \\
 &= \frac{1}{2} \lambda P^2 - \frac{\lambda P}{2} - \frac{1}{2} \sum_{q=1}^{\lambda} \bar{x}_q(\bar{x}_q-1). \tag{2.18}
 \end{aligned}$$

Recalling from (2.7) that

$$\bar{x}_q = \frac{1}{\lambda} (Pq + \bar{r}_q - \mu) \tag{2.19}$$

and substituting in (2.18) we obtain

$$\begin{aligned}
 \sum_{x=0}^{P-1} xq &= \frac{\lambda P^2}{2} - \frac{\lambda P}{2} - \frac{\mu^2}{2\lambda} - \frac{\mu}{2} - \frac{P^2}{2\lambda^2} \sum_{q=1}^{\lambda} q^2 + P \frac{\lambda+2\mu}{2\lambda^2} \sum_{q=1}^{\lambda} q - \frac{1}{2\lambda^2} \sum_{q=1}^{\lambda} \bar{r}_q^2 \\
 &\quad + \frac{\lambda+2\mu}{2\lambda^2} \sum_{q=1}^{\lambda} \bar{r}_q - \frac{P}{\lambda^2} \sum_{q=1}^{\lambda} \bar{r}_q q. \tag{2.20}
 \end{aligned}$$

From the definition of $\bar{r}_q$ in (2.19) it follows that

$$\sum_{q=1}^{\lambda} \bar{r}_q^2 = \sum_{q=1}^{\lambda-1} q^2 \tag{2.21}$$

and that

$$\sum_{q=1}^{\lambda} \bar{r}_q = \sum_{q=1}^{\lambda-1} q. \tag{2.22}$$

Evaluating all the summations of (2.20) except the last, and combining like terms yields

$$\sum_{x=0}^{P-1} xq = \frac{\lambda P^2}{3} - \frac{\lambda P}{4} - \frac{\mu^2}{2\lambda} + \frac{P}{4} - \frac{\mu}{2\lambda} + \frac{\mu P}{2\lambda} + \frac{\lambda}{12} + \frac{\mu P}{2} - \frac{1}{12\lambda} - \frac{P^2}{4} - \frac{P^2}{12\lambda} - \frac{PS}{\lambda^2} \quad (2.23)$$

where

$$S = \sum_{q=1}^{\lambda} \bar{r}_q q. \quad (2.24)$$

Rewrite (2.2), using (2.3), (2.4), and (2.6).

$$\begin{aligned} \rho &= \frac{E(x_n, x_{n+1}) - (E(x_n))^2}{E(x_n^2) - (E(x_n))^2} \\ &= \frac{\lambda E(x_n^2) + \mu E(x) - (E(x_n))^2 - \sum_{x=0}^{P-1} xq}{E(x_n^2) - (E(x_n))^2} \\ &= \frac{12 \left(\frac{\lambda}{6}(P-1)(2P-1) + \frac{\mu}{2}(P-1) - \frac{1}{4}(P-1)(P-1) - \sum_{x=0}^{P-1} xq \right)}{2(P-1)(2P-1) - 3(P-1)(P-1)} \quad (2.25) \end{aligned}$$

Carry out the division by $P^2 - 1$, the denominator in (2.25), using the value for $\sum xq$ in (2.23) and neglect terms whose order of magnitude is $1/P$ or less. This yields the correlation coefficient which we set out to determine

$$\rho(x_n, x_{n+1}) \doteq \frac{1}{\lambda} - \frac{6\mu}{\lambda P} \left(1 - \frac{\mu}{P}\right) + \frac{12}{P} \left(\frac{S}{\lambda^2} - \frac{\lambda}{4}\right). \quad (2.26)$$

P is usually of the order of 10^{10} so that the approximation ($\doteq$) is an equality for all practical purposes. Recalling that

$$S = \sum_{q=0}^{\lambda} \bar{r}_q q$$

and (2.10) we can set limits on S and the last term of (2.26).

$$\frac{\lambda}{6} (\lambda^2 - 1) \leq S \leq \frac{\lambda}{3} (\lambda^2 - 1) \quad (2.27)$$

Multiply equation (2.27) by $12/\lambda^2$, add 3λ , and divide by P to obtain

$$\frac{-\lambda}{P} \leq \frac{12}{P} \left(\frac{S}{\lambda^2} - \frac{\lambda}{4} \right) \leq \frac{\lambda}{P} \quad (2.28)$$

where terms of magnitude less than $1/P$ are omitted.

We return now to the case $\mu < \lambda$. Using the same approach and recalling the comment prior to (2.17), we again obtain (2.23), only now with a slightly different value for S . Because $\mu < \lambda$ and of the magnitude of P , (2.26) then reduces to

$$\rho(x_n, x_{n+1}) \doteq \frac{1}{\lambda} + \frac{12}{P} \left(\frac{S}{\lambda^2} - \frac{\lambda}{4} \right) \quad (2.29)$$

where

$$S = \sum_{q=1}^{\lambda-1} \bar{r}_q q \quad (2.30)$$

and the inequality (2.28) again holds. The values of S as given by (2.24) and (2.30) corresponding to $\mu \geq \lambda$ and $\mu < \lambda$ differ in definition by the upper limit on the summation (λ and $\lambda-1$) and hence only by a negligible amount.

We conclude our discussion and derivation of the serial correlation coefficient by mentioning several additional items. The first is that the results obtained above can be readily used for computing the correlation coefficient for any order n . This can be done by using the results of lemma (1.1) in (2.26) where λ is replaced by λ^n and μ is replaced by $\mu \frac{\lambda^n - 1}{\lambda - 1}$.

Another point is that a small correlation coefficient is not a sufficient condition for randomness. An example will be given shortly. Also, the above derivation assumes that the whole sequence is generated, something

that is never done in practice. Certain subsequences may be far from random. This is demonstrated by the using $\lambda = 2^{18} + 1$, $\mu = 1$, $x_0 = 1$, and $P = 2^{35}$. Substituting these values for λ and μ in (2.29) yields a value for ρ of less than 10^{-5} . Yet the first several hundred numbers in the sequence are less than $P/2$ which is far from random [3].

Thus (2.29) can be used for picking possible values for λ and μ , yet the sequence produced must still be carefully checked. The sequence should be subjected to certain standard tests and also checked in the light of Lehmer's definition "... depending somewhat on the use to which the sequence is to be put."

STATISTICAL TESTS

General Comments and Discussion of the Computer Program

There are many possible tests for randomness. In keeping with the spirit of Lehmer's definition, we have selected the tests described in the following sections chiefly because of their acceptance as standard and because of Monte Carlo applications. In most Monte Carlo applications one is more interested in random numbers than in random digits. That is, the random number appears as a multiplier or additive constant and only a few digits of the result are significant, or a few leading bits or digits are used as an index to pick a quantity from a table. Thus we are not considering the usual tests for randomness of digits.

A computer program was written in the FAP language for the IBM 709/7090 to test random number generators. The test is actually a series of tests which are performed on the sequence of numbers as they are generated. The input for the computer program consists of the sample size, the initial random number, the multiplier λ , the constant μ (these last three numbers are

specified in octal), and the number of samples. The sample size is the number of numbers that are generated by a set of parameters (λ and μ) and subjected to the series of tests. A sample size of 10,240 was used in all but one test.

The initial random number is the number used to obtain the first number in the sequence. The number 123456227651_8 (the subscript 8 indicates an octal number) was used to start the first sequence for each set of parameters, with one exception which is noted later. The last number in any sequence or sample was used as the initial number for the following sequence generated by the same parameters.

Our main concern is to determine suitable parameters λ and μ . For the multiplicative method $\mu = 0$ and hence the tests are only for the multiplier λ . Three multipliers for the multiplicative method were tested. Various computing installations use these multipliers and one would expect them to be satisfactory, as indeed the following tests demonstrate.

For the mixed methods multipliers of the type $\lambda = 2^a + 1$ were tested. Here we were looking for a multiplier with only a few bits and consequently a fast multiplier. To illustrate this, the computer can multiply 1637_8 by 1001_8 faster than it can multiply 1637_8 by 5714_8 . To test a range of multipliers we held $\mu = 1$ for all but one multiplier. The different μ value had an appreciable effect on the resulting statistics. However, there are $2^{34} - 1$ possible values of μ that could be used with a λ value and it seemed more profitable to vary λ than μ .

One generator $\lambda = 2^{34} + 1$ and $\mu = 1$, has been mentioned [4] as unsatisfactory and was included to see how this would show up in the tests. One can notice in the following sections that it is definitely unsatisfactory.

About one minute of 709 computer time is required for a sample of 10,240 numbers.

The last item for input to the computer program is the number of samples. Three samples of 10,240 were used to test all the generators considered, and the results of the test on these samples are presented in tables in the following sections. Nine additional samples of 10,240 were tested, for each of two generators. For some simplification of discussion in the following sections we shall refer to these tests on the additional samples as the "extended tests on M1 and M2." We define M1 to be $\lambda = 2^{18} + 3$ with $\mu = 0$ and M2 to be $\lambda = 2^9 + 1$ with $\mu = 1$.

Frequency Test

A good random number generator is expected to distribute the numbers uniformly on the unit interval. To test this distribution we divide the unit interval into K subintervals (in our test $K = 1024$); then generate N numbers (this corresponds to our sample size of 10,240) and count or tally the number of times a number appears in each subinterval. In the computer program the 10 leading bits of a number are used as an index to determine the proper subinterval.

The expected number in each subinterval is $E = N/K = 10,240/1024 = 10$. To test the deviation from the expected value we compute the chi-square value

$$\chi^2 = \sum_{i=1}^K \frac{(f_i - E_i)^2}{E_i} = \sum_{i=1}^K \frac{(f_i - 10)^2}{10} = \frac{1}{10} \sum_{i=1}^{1024} (f_i - 10)^2$$

where f_i is the observed frequency in subinterval i . We then compare this chi-square value with the chi-square distribution with 1023 degrees of freedom which has a 95 per cent confidence interval of $936.3 \leq \chi^2 \leq 1113.5$. Table 1 lists these computed chi-square values as a function of the generator parameters. We make the following observations on the contents of Table 1:

Three of the generators have one value that is slightly low, one set

of values is consistently low, and two sets are way off scale. We had anticipated excessive values with $\lambda = 2^{34} + 1$ and $\mu = 1$. However, the chi-square value of zero for $\lambda = 2^{12} + 1$ and $\mu = 1$ is rather surprising. This means that this generator placed the expected number in each subinterval; something that is "too random" and unacceptable in most cases. If one had an application where extreme uniformity was essential and some other characteristics of randomness were of minor importance, the generator $\lambda = 2^{12} + 1$ and $\mu = 1$ would merit consideration.

The frequency chi-square values for M1 and M2 obtained in the extended tests were all within the 95 per cent confidence interval.

Matrix Test

The question arises if there is undue correlation between certain numbers in the sequence. M. L. Juncosa [6] proposed the test whereby one forms a K by K matrix whose elements K_{ij} represent the number of times a number in the i'th interval is followed by a number in the j'th interval. Equal values are expected for all matrix elements, and a chi-square value is computed and compared with the expected chi-square distribution with $K^2 - 1$ degrees of freedom.

For our test we selected a 32 by 32 matrix. This matrix has 1024 elements and thus the test is somewhat similar to our frequency test. Table 2 shows the results of this test on the sequences formed by our selected parameters. The results are similar to those in Table 1, except that two additional generators fail to pass this test, namely $\lambda = 2^{16} + 1$ with $\mu = 1$, and $\lambda = 2^{17} + 1$ with $\mu = 1$. It is interesting to note that changing the μ value makes $\lambda = 2^{17} + 1$ an acceptable multiplier as far as this test is concerned.

The corresponding chi-square values from the extended tests on M1 and M2 were all within the 95 per cent confidence interval, with the exception of

one value of 1119.0 for M1 which barely exceeds the upper limit of 1113.5 (one even expects this number to be exceeded 2.5 per cent of the time).

Auto-correlation Coefficient

The auto-correlation coefficient [5] is defined by

$$C_h = \frac{1}{N} \sum_{i=1}^N x_i \cdot x_{i+h}$$

and likewise serves as a measure of the independence of the numbers x_n of the sequence. For a truly random sample the values of C_h are approximately normally distributed about the mean = $1/4$ for $h > 0$, and mean = $1/3$ for $h = 0$, with a standard deviation = $.3/\sqrt{N}$ in both cases.

With choices of λ with the form $2^a + 1$, one may be concerned about every 2nd, 4th, 8th . . . etc. numbers (and others for that matter) being correlated. This concern is considered in more detail in a later test, but we will mention here that in some Monte Carlo application every j 'th number in a sequence is associated with a particular variable.

In our test on the auto-correlation coefficient we let $h = 0, 1, \dots, 9$, and these coefficients are listed in Table 3. We observe that the generator, $\lambda = 2^{15} + 1$ with $\mu = 1$, had passed the two previous tests, yet here shows the property that we were concerned about, namely that 2nd, 4th, and 8th numbers are unduly correlated. As in the previous test changing the μ value for $\lambda = 2^{17} + 1$ is beneficial and brings the coefficients into line. Most of the coefficients for $\lambda = 2^{12} + 1$ and $\mu = 1$ are within the limits, and this is somewhat surprising considering the results of the previous tests.

The extended tests on M1 and M2 showed all values for M1 within the expected limits and all but 2 for M2, and these were below the lower limit by less than .001.

Frequency of Lengths of Runs

An occurrence of k consecutive numbers forming a monotonic increasing (or decreasing) sequence which is broken by the $k + 1$ st number is called a run of length k . Comparing the observed frequencies of lengths of runs with expected values has been recommended as a good test for randomness [5] and we present the derivation of the expected values as follows.

We first consider an equivalent definition of a sequence of length k . For N distinct numbers $x_1, \dots, x_N$ we write an $N-1$ bit binary sequence S whose n 'th term is a 0 if $x_n < x_{n+1}$ and a 1 if $x_n > x_{n+1}$. A subsequence of k zeroes (or ones) bracketed by ones (or zeroes) forms a sequence of length k . To determine the expected number of runs of length k in a sequence of N numbers we observe that if the x_i are arranged in increasing sequence $x_{i1}, x_{i2}, \dots, x_{iN}$, then all $N!$ permutations of the subscripts are equally likely, since $P(x_r < x_s) = P(x_r > x_s) = 1/2$. Therefore, an equivalent problem is to consider the $N!$ permutations of the first N integers, form the $N - 1$ bit binary sequence S as before for each permutation, and count the number of occurrences of a run of length k in the whole set of S 's. The process is lengthy but straight-forward. For example consider $k=1$: in this instance four consecutive values of x_i are needed for an "inside run" of length 1 and three consecutive values of x_i are needed for an "end" run. Examples of this are $2 < 3 > 1 < 4$ gives $..010..$ for S , and $1 < 3 > 2$ which gives $01...$ for S . The total number of "end" runs of length 1 is $2 \cdot 2 \cdot 2 \binom{N}{3} \cdot (N-3)!$. Here the first 2 indicates two ends to choose, the second 2 corresponds to choice of up or down, and the third 2 indicates the two ways to make such a pattern with three numbers ($1 < 3 > 2$ or $2 < 3 > 1$). The $\binom{N}{3}$ is the number of ways to choose three numbers from N to form the run, and $(N-3)!$ is the number of permutations of $N - 3$. In like manner there are $2 \cdot 5 \cdot \binom{N}{4} (N-4)! (N-3)$ "inside" runs. Here the 2 corresponds

to choice of runs up or down, and the 5 indicates five ways to make such a pattern from four numbers.

$$1. \quad 1 < 3 > 2 < 4$$

$$2. \quad 1 < 4 > 2 < 3$$

$$3. \quad 2 < 3 > 1 < 4$$

$$4. \quad 2 < 4 > 1 < 3$$

$$5. \quad 3 < 4 > 1 < 2$$

The $\binom{N}{4}$ is the number of ways to choose four numbers from N , the $(N-4)!$ is the number of permutations of $N - 4$, and the $(N-3)$ indicates the number of places the pattern could appear in the sequence S . Adding the total "end" runs and the total "inside" runs and dividing by $N!$, the total number of sequences S , gives the expected number of runs of length 1. We define A_N^1 to be the expected number of runs i in a sequence of length N . Then

$$A_N^1 = \frac{\left(2 \cdot 2 \cdot 2 \cdot \binom{N}{3} (N-3)! + 2 \cdot 5 \binom{N}{4} (N-4)! (N-3) \right)}{N!} \quad (3.1)$$

$$\text{Recalling that } \binom{N}{3} = \frac{N(N-1)(N-2)}{3!} = \frac{N!}{3!(N-3)!}$$

and substituting for $\binom{N}{3}$ and $\binom{N}{4}$ in (3.1), and cancelling the $N!$, the $(N-3)!$, and the $(N-4)!$ we obtain

$$A_N^1 = \frac{8}{3!} + \frac{10(N-3)}{4!} = \frac{32 + 10N - 30}{24} = \frac{5N + 1}{12}$$

as the expected number of runs of length 1 in a sequence of N random numbers.

The formula for a run of length k is

$$A_N^k = \frac{2 \cdot 2(k+1) \binom{N}{k+2} (N-k-2)! + 2(k + (k+1)^2) \binom{N}{k+2} (N-k-3)! (N-k-2)}{N!}$$

Here the $(k+1)$ in the first term corresponds to the number of ways $k + 1$ integers can be arranged in the prescribed pattern for an "end" run, and the $(k + (k+1)^2)$ in the second term is the number of ways $k + 3$ integers can be combined to form the prescribed pattern for an "inside" run. The remaining factors have the same significance as for the case $k = 1$. Carrying out the necessary algebra we find the simplified formula for the expected number of runs of length k in a sequence of length N where $k < N - 1$ to be

$$A_N^k = \frac{2 \left((k^2 + 3k + 1) N - (k^3 + 3k^2 - k - 4) \right)}{(k+3)!}.$$

Table 4 compares the observed lengths of sequences formed by the various generators with the expected values. The chi-square value is computed from the frequencies in five cells, the first four cells correspond to the lengths of run 1, 2, 3, and 4, while the fifth cell corresponds to all lengths of run greater than four. Thus we compare the computed chi-square values with a chi-square distribution with four degrees of freedom.

This length of run test is by far the most discriminating test, in the sense that this test rejects over half of the generators tested as unacceptable. While some of the generators would produce statistics in other tests that were questionable the statistics for these same generators are not even close to the expected values in the length of run test.

In keeping with our previous comments on $\lambda = 2^{17} + 1$, we note that the larger μ value reduces the chi-square value by more than a factor of 300, yet it is still not acceptable as far as this length of run test is concerned.

It was chiefly on the basis of this test that $\lambda = 2^{18} + 3$ with $\mu = 0$ (called M1 in our discussion) and $\lambda = 2^9 + 1$ with $\mu = 1$ (called M2 in our discussion) were selected for more extensive testing. All of the nine additional

chi-square values for M1 were in the 95 per cent confidence interval, while two of the values for M2 exceeded 15. This is significant enough to reject M2 as a satisfactory generator.

Frequencies in Subsequences

In our discussion on the auto-correlation coefficient we were concerned about the correlation between every j 'th number. We now discuss the subject further and another test for randomness.

Consider the following example. Suppose that one desires the coordinates (x_i, y_i, z_i) of a large number of points uniformly distributed throughout the unit cube. Then one merely uses one of the random number generators to generate a sequence of random numbers r_i ($0 < r_i < 1$) and assigns the desired coordinates these values, i.e., $x_1 = r_1$, $y_1 = r_2$, $z_1 = r_3$ and $x_2 = r_4$, $y_2 = r_5$, $z_2 = r_6$, etc. This is the type of application we mentioned earlier where every j 'th number (in our example $j = 3$) is associated with a given variable. Thus one sees that difficulties are encountered if the subsequences of a given sequence are not random. (In our example the points would not be uniformly distributed.)

To test for this randomness of subsequences, one sets up j tables, each with K subintervals, and tallies every j 'th number of the sequence in the j 'th table as we did in the frequency test. Table 5 lists the chi-square values of the frequencies in the subintervals for each of the j tables where $j = 8$. That is, the 1st, 9th, 17th, ... numbers are tallied in the table or cell corresponding to the 1st subsequence, and the 2nd, 10th, 18th, ... numbers are tallied in the cell for the 2nd subsequence, etc. For this test the initial random number was 24232366751₈ and the sample size was 81,920. These are the two exceptions that were mentioned in the discussion regarding the computer program. However,

with the sample size of 81,920, $j = 8$ tables, and $K = 1024$ subintervals we have the same expected values as in the regular frequency test.

This is a rather time consuming test and thus we only used it for $j = 8$ and the generator $\lambda = 2^{18} + 3$ with $\mu = 0$. It would be advisable to repeat this test for a different value of j if one had an application similar to the one above where every j 'th number is associated with a particular parameter.

Notice that all of the chi-square values are within the 95 per cent confidence interval.

This concludes our series of tests.

Conclusions

The length of run test is the most powerful of the foregoing tests. It rejected the most generator parameters as unsatisfactory, and rejected all the parameters that were rejected by any of the other tests.

On the basis of these tests, the multiplicative method is superior to the mixed method, and the multiplier $\lambda = 2^{18} + 3$ forms a satisfactory generator for the multiplicative method

$$x_{i+1} \equiv \lambda x_i \pmod{2^{35}}.$$

The following subroutine coded in the FAP computer language can be used on the IBM 709, 7090, 7040 computers to obtain random numbers from the above sequence.

1	RAND	LDQ	=01000003
2		MFY	RN
3		STQ	RN
4		CLA	RN
5		TRA	1,4
6	RN	0CT	123456227651

Instruction number 4 can be omitted if the random number is not needed in the accumulator, and instruction number 5 can be modified for various types of

return linkage. To obtain a floating point number replace instruction number 4 with the following three.

CLA =0200

LLS 27

FAD =02000000000000

TABLE 1.--Comparison of Chi-square values of frequencies in 1024 subintervals with generator parameters^a

Generator Parameters		Chi-square of Frequencies	Generator Parameters		Chi-square of Frequencies
λ	μ		λ	μ	
$2^7 + 1$	1	1,036.6 956.0 1,004.2	$2^{16} + 1$	1	1,073.2 990.4 1,045.6
$2^8 + 1$	1	1,002.4 941.2 1,040.4	$2^{17} + 1$	1	1,014.0 976.2 1,020.0
$2^9 + 1$	1	1,025.2 1,036.8 1,040.4	$2^{17} + 1$	b	954.6 1,009.6 1,070.2
$2^{10} + 1$	1	965.4 1,040.8 953.8	$2^{18} + 1$	1	928.0 954.6 962.8
$2^{11} + 1$	1	1,032.2 1,104.0 962.6	$2^{19} + 1$	1	841.0 856.2 883.8
$2^{12} + 1$	1	0.0 0.0 0.0	$2^{34} + 1$	1	5,232,640.0 5,232,640.0 5,232,640.0
$2^{13} + 1$	1	931.0 973.8 1,051.2	$2^{18} + 3$	0	1,106.0 1,005.2 962.8
$2^{14} + 1$	1	1,044.4 1,107.0 1,072.4	11060471625 ₈	0	1,003.2 999.6 962.6
$2^{15} + 1$	1	911.2 1,030.4 956.2	343277244615 ₈	0	1,022.4 990.6 1,025.4

^aData is for three samples of 10,240 for each set of parameters. The 95 per cent confidence interval for the Chi-square distribution with 1023 degrees of freedom is $936.3 \leq \chi^2 \leq 1113.5$.

^b $\mu = 1230452101_8$

TABLE 2.--Comparision of Chi-square values of frequencies in 32 X 32 matrix
with generator parameters^a

Generator Parameters		Chi-square of Frequencies	Generator Parameters		Chi-square of Frequencies
λ	μ		λ	μ	
$2^7 + 1$	1	1,058.0 968.0 1,104.8	$2^{16} + 1$	1	20,191.2 13,830.6 14,239.4
$2^8 + 1$	1	985.8 1,029.4 1,054.2	$2^{17} + 1$	1	75,676.0 71,160.6 71,274.8
$2^9 + 1$	1	1,008.8 1,031.4 1,073.0	$2^{17} + 1$	b	1,004.2 1,055.8 992.0
$2^{10} + 1$	1	942.0 972.8 1,019.0	$2^{18} + 1$	1	97,742.8 97,151.8 97,978.4
$2^{11} + 1$	1	1,029.6 1,074.8 977.8	$2^{19} + 1$	1	49,790.6 49,572.8 49,991.8
$2^{12} + 1$	1	0.0 0.0 0.0	$2^{34} + 1$	1	2,610,688.0 2,611,200.0 2,611,200.0
$2^{13} + 1$	1	1,015.8 914.0 1,048.2	$2^{18} + 3$	0	1,023.2 1,040.8 980.2
$2^{14} + 1$	1	1,045.8 955.2 907.2	11060471625 ₈	0	964.8 1,074.4 1,010.8
$2^{15} + 1$	1	1,004.4 939.4 944.8	343277244615 ₈	0	1,017.4 1,051.2 1,101.2

^aData is for three samples of 10,240 for each set of parameters. The 95 per cent confidence interval for the Chi-square distribution with 1023 degrees of freedom is $936.3 \leq \chi^2 \leq 1113.5$.

^b $\mu = 1230452101_8$

TABLE 3.--Auto-correlation coefficients^a

$$C_h = \frac{1}{N} \sum_{i=1}^N x_i \cdot x_{i+h}$$

Generator Parameters		h values									
λ	μ	0	1	2	3	4	5	6	7	8	9
$2^7 + 1$	1	3358	2548	2540	2538	2546	2528	2537	2538	2536	2544
		3304	2474	2452	2464	2484	2470	2466	2477	2476	2470
		3337	2509	2501	2504	2494	2492	2503	2502	2509	2492
$2^8 + 1$	1	3317	2481	2495	2480	2482	2478	2475	2475	2500	2482
		3318	2470	2482	2483	2487	2474	2476	2477	2491	2485
		3322	2471	2470	2489	2485	2471	2480	2478	2464	2490
$2^9 + 1$	1	3305	2462	2461	2450	2464	2463	2469	2469	2454	2480
		3326	2507	2498	2500	2493	2488	2489	2480	2490	2484
		3343	2532	2513	2523	2532	2518	2516	2525	2531	2527
$2^{10} + 1$	1	3354	2513	2517	2507	2528	2522	2523	2531	2530	2527
		3364	2543	2556	2545	2551	2537	2537	2550	2530	2538
		3384	2553	2547	2566	2556	2557	2549	2560	2552	2550
$2^{11} + 1$	1	3357	2528	2537	2543	2541	2538	2526	2529	2536	2535
		3273	2430	2449	2441	2444	2424	2428	2433	2424	2429
		3384	2553	2547	2545	2541	2535	2543	2557	2530	2550
$2^{12} + 1$	1	3327	2480	2486	2494	2463	2487	2483	2487	2480	2486
		3420	2571	2567	2554	2534	2567	2575	2564	2566	2560
		3334	2526	2500	2508	2498	2498	2521	2506	2491	2511
$2^{13} + 1$	1	3352	2511	2555	2517	2521	2516	2548	2525	2452	2520
		3331	2497	2507	2487	2497	2493	2517	2498	2447	2498
		3337	2488	2512	2484	2492	2482	2530	2482	2429	2493
$2^{14} + 1$	1	3396	2578	2606	2562	2514	2567	2555	2581	2496	2570
		3315	2475	2515	2478	2437	2488	2481	2491	2374	2583
		3340	2503	2541	2509	2460	2511	2532	2502	2373	2517
$2^{15} + 1$	1	3324	2502	2668	2495	2671	2490	2458	2485	2267	2486
		3339	2501	2669	2509	2736	2509	2427	2513	2270	2512
		3335	2505	2665	2510	2743	2501	2403	2508	2314	2499

TABLE 3.--Continued

Generator Parameters		h values									
λ	μ	0	1	2	3	4	5	6	7	8	9
$2^{16} + 1$	1	3340	2499	2267	2514	2796	2495	2429	2522	2414	2499
		3337	2501	2272	2502	2828	2506	2342	2493	2554	2497
		3366	2518	2323	2529	2763	2539	2300	2534	2588	2530
$2^{17} + 1$	1	3339	2420	2473	2597	2377	2564	2529	2444	2134	2500
		3340	2463	2414	2575	2132	2418	2497	2574	2713	2473
		3349	2529	2421	2426	2138	2599	2500	2456	2740	2558
$2^{17} + 1$	b	3307	2505	2493	2483	2464	2488	2478	2491	2492	2481
		3323	2494	2488	2492	2472	2492	2482	2504	2492	2494
		3364	2527	2530	2542	2542	2523	2530	2520	2521	2544
$2^{18} + 1$	1	3337	2177	2374	2891	2131	2710	2599	2239	2709	2460
		3337	2311	2129	2792	2709	2164	2463	2776	2415	2351
		3327	2495	2117	2190	2717	2876	2434	2203	2425	2635
$2^{19} + 1$	1	3331	2369	2126	2593	2703	2464	2467	2473	2433	2580
		3319	2110	2691	2444	2397	2582	2437	2452	2539	2452
		3369	2158	2759	2475	2466	2627	2491	2500	2603	2487
$2^{34} + 1$	1	3950	3324	2699	3324	3950	3324	2699	3324	3950	3324
		3950	3324	2699	3324	3950	3324	2699	3324	3950	3324
		3950	3324	2699	3324	3950	3324	2699	3324	3950	3324
$2^{18} + 3$	0	3344	2513	2521	2519	2517	2523	2521	2521	2512	2510
		3353	2530	2517	2519	2515	2525	2520	2515	2525	2523
		3319	2492	2504	2477	2486	2487	2487	2478	2499	2492
c	0	3374	2533	2524	2517	2515	2530	2518	2535	2530	2545
		3379	2563	2567	2552	2549	2562	2561	2548	2564	2545
		3359	2518	2522	2504	2523	2526	2518	2536	2521	2511
d	0	3372	2552	2548	2544	2536	2543	2543	2553	2562	2540
		3331	2501	2512	2506	2507	2504	2503	2518	2498	2511
		3362	2541	2545	2550	2554	2531	2535	2545	2548	2525

^aAll values of C_h are times 10^{-4} . The expected value of C_0 is $.333 \pm .006$ and of C_h is $.250 \pm .006$ for $h > 0$. The .006 represents two standard deviations and $N = 10,231$.

^b $\mu = 1230452011_8$

^c $\lambda = 11060471625_8$

^d $\lambda = 343277244615_8$

TABLE 4.-- Lengths of runs^a

Generator Parameters		χ^2 Chi-square	Lengths of Runs						
λ	μ		1	2	3	4	5	6	≥ 7
		Expected Values							
		$.48 \leq \chi^2 \leq 11.14$ b	4266.8	1877.1	540.31	117.8	20.8	3.1	.4
		Observed Values							
$2^7 + 1$	1	6.73	4234	1836	539	137	27	5	0
		10.81	4281	1893	559	112	8	1	0
		1.18	4251	1852	548	126	21	5	0
$2^8 + 1$	1	6.03	4345	1831	519	135	23	2	1
		2.59	4246	1918	540	108	18	2	0
		4.87	4234	1927	516	111	29	1	1
$2^9 + 1$	1	5.07	4302	1918	533	102	15	3	0
		5.50	4283	1841	571	102	23	5	1
		7.66	4163	1861	583	125	14	3	2
$2^{10} + 1$	1	8.35	4363	1787	553	131	20	3	0
		14.20	4261	1951	549	88	15	0	0
		2.40	4212	1920	529	113	19	6	2
$2^{11} + 1$	1	8.09	4266	1938	540	92	19	2	0
		9.83	4340	1855	526	105	33	3	1
		10.78	4265	1926	534	89	31	0	1
$2^{12} + 1$	1	105.70	4283	2040	424	66	30	14	13
		109.31	4355	1977	423	71	34	13	15
		88.27	4114	2005	432	112	32	12	16
$2^{13} + 1$	1	136.70	4444	2013	367	69	26	5	26
		95.90	4414	2011	378	81	22	1	24
		78.50	4411	1972	422	67	18	5	24
$2^{14} + 1$	1	179.25	4425	1966	373	74	35	14	25
		225.81	4481	1976	351	68	44	16	19
		168.51	4461	1979	351	84	34	13	21
$2^{15} + 1$	1	349.30	5054	1864	253	174	0	0	0
		266.22	4920	1881	289	172	0	0	0
		281.09	4899	1989	244	147	8	0	0
$2^{16} + 1$	1	1,022.00	2765	2798	528	73	0	0	0
		1,021.60	2822	2790	569	32	0	0	0
		884.77	2987	2733	571	18	0	0	0

TABLE 4.--Continued

Generator Parameters		χ^2	Lengths of Runs						
λ	μ	Chi-square	1	2	3	4	5	6	≥ 7
Expected Values									
		$.48 \leq \chi^2 \leq 11.14$ ^b	4266.8	1877.1	540.31	117.8	20.8	3.1	.4
Observed Values									
$2^{17} + 1$	1	1,682.70	3802	3215	2	0	0	0	0
		1,074.10	3276	2965	344	0	0	0	0
		881.20	3018	2166	962	0	0	0	0
$2^{17} + 1$	c	204.29	3819	2262	578	40	0	0	0
		221.81	3808	2255	600	30	0	0	0
		198.96	3887	2226	597	27	0	0	0
$2^{18} + 1$	1	1,107.90	4830	2698	4	0	0	0	0
		1,752.10	2967	1627	1339	0	0	0	0
		12,951.00	2168	0	840	1080	246	0	0
$2^{19} + 1$	1	2,510.50	2714	1103	1093	471	31	0	0
		2,446.30	6984	1627	0	0	0	0	0
		2,602.50	7092	1573	0	0	0	0	0
$2^{34} + 1$	1	10,243.00	2561	0	2559	0	0	0	0
		10,243.00	2561	0	2559	0	0	0	0
		10,243.00	2561	0	2559	0	0	0	0
$2^{18} + 3$	0	1.89	4263	1875	525	122	22	5	3
		4.96	4203	1889	542	114	30	4	0
		7.49	4377	1834	561	101	20	1	0
d	0	6.67	4180	1868	551	140	18	3	0
		5.34	4237	1841	538	137	25	3	2
		3.06	4314	1835	559	109	22	4	1
e	0	2.19	4218	1878	551	114	26	4	0
		2.16	4296	1868	556	107	17	3	1
		4.71	4287	1913	532	113	13	2	0

^aData is for three samples of 10,240 for each set of parameters.

^bThis represents the 95 percent confidence interval for a Chi-square distribution with four degrees of freedom.

^c $\mu = 1230452101_8$

^d $\lambda = 11060471625_8$

^e $\lambda = 343277244615_8$

TABLE 5.--Chi-square values of frequencies in 1024 subintervals, for 8 subsequences of the sequence generated by using $\lambda = 2^{18} + 3$ and $\mu = 0^a$

Subsequence Number	Chi-square of Frequencies		
	Sample 1	Sample 2	Sample 3
1	984.4	1,069.0	1,062.8
2	1,016.4	942.4	925.6
3	1,000.6	1,023.2	1,022.2
4	989.6	1,017.0	1,055.8
5	936.2	1,010.0	982.6
6	964.2	990.6	1,084.4
7	966.6	1,054.0	1,033.2
8	1,006.8	999.2	1,024.0

^aEach sample size was 81,920 or 10,240 in each subsequence. The 95 per cent confidence interval for a Chi-square distribution with 1023 degrees of freedom is $936.6 \leq x^2 \leq 1113.5$.

LITERATURE CITED

- [1] Coveyou, R. R. Serial correlation in the generation of pseudo-random numbers. J. Assoc. Comput. Mach. 7 (1960) 72-74.
- [2] Cramer, Herald. Mathematical methods of statistics. Princeton University Press, 1946, p. 138
- [3] Greenberger, Martin. Notes on a new pseudo-random number generator. J. Assoc. Comput. Mach. 8 (1961) 163-167.
- [4] Greenberger, Martin. An a priori determination of serial correlation in computer generated random numbers. Math. Comp. 15 (1961) 383-389. Also private communication to the author.
- [5] International Business Machines Corp. Random number generation and testing reference manual C20-8011. New York: 1959.
- [6] Juncosa, M. L. Random number generation on the BRL high-speed computing machines. Ballistic Research Laboratories, Report number 855 (Aberdeen Proving Ground, 1953).
- [7] Lehmer, D. H. Mathematical methods in large-scale computing units. Annals Comp. Laboratory Harvard Univ. 26 (1951) 141-146.
- [8] Rotenberg, A. A new pseudo-random number generator. J. Assoc. Comput. Mach. 7 (1960) 75-77.

**PHILLIPS
PETROLEUM
COMPANY**

ATOMIC ENERGY DIVISION