1857

K-1798
Part 4

MASTER

INTERNAL RAREFIED GAS FLOWS WITH BACKSCATTERING
PART 4: PLANE POISEUILLE FLOW
WITH BACKSCATTERING

AUTHOR:

A. S. Berman

UNION CARBIDE CORPORATION
NUCLEAR DIVISION
OAK RIDGE GASEOUS DIFFUSION PLANT

operated for the **ATOMIC ENERGY COMMISSION** *under* **U. S. GOVERNMENT Contract W-7405 eng 26**

**UNION
CARBIDE**

OAK RIDGE GASEOUS DIFFUSION PLANT
P. O. Box P
Oak Ridge, Tennessee 37830

DISTRIBUTION OF THIS DOCUMENT IS UNLIMITED

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency Thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

DISCLAIMER

Portions of this document may be illegible in electronic image products. Images are produced from the best available original document.

Printed in the United States of America. Available from
National Technical Information Service
U.S. Department of Commerce
5285 Port Royal Road, Springfield, Virginia 22151
Price: Printed Copy \$3.00; Microfiche \$0.95

This report was prepared as an account of work sponsored by the United States Government. Neither the United States nor the United States Atomic Energy Commission, nor any of their employees, nor any of their contractors, subcontractors, or their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness or usefulness of any information, apparatus, product or process disclosed, or represents that its use would not infringe privately owned rights.

Date of Issue: June 16, 1971

Report Number: K-1798, Part 4

Subject Category: PHYSICS

INTERNAL RAREFIED GAS FLOWS WITH BACKSCATTERING
PART 4: PLANE POISEUILLE FLOW WITH BACKSCATTERING

A. S. Berman
University of Minnesota

This report was prepared as an account of work sponsored by the United States Government. Neither the United States nor the United States Atomic Energy Commission, nor any of their employees, nor any of their contractors, subcontractors, or their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness or usefulness of any information, apparatus, product or process disclosed, or represents that its use would not infringe privately owned rights.

UNION CARBIDE CORPORATION
NUCLEAR DIVISION
Oak Ridge Gaseous Diffusion Plant
Oak Ridge, Tennessee

DISTRIBUTION OF THIS DOCUMENT IS UNLIMITED

lad

THIS PAGE
WAS INTENTIONALLY
LEFT BLANK

Report Number: K-1798, Part 4

Subject Category: PHYSICS

Title: INTERNAL RAREFIED GAS
FLOWS WITH BACKSCATTERING
PART 4

Author: A. S. Berman

ABSTRACT

Plane Poiseuille flow at arbitrary Knudsen number is computed using the BGK model equation and a boundary condition which permits a fraction $(1 - \alpha)$ of molecules incident on the walls to be scattered back along their incident path while a fraction α are diffusely scattered.

The inclusion of backscattering reduces the transport in the free molecule and near free molecule regions while the transport in the continuum region is relatively unaffected.

With a value of $\alpha = 0.94$, obtained in Part 2 of this sequence of reports, comparison of predicted transport with the transport of hydrogen between glass plates observed by Gaede is made. It is found that the inclusion of backscattering significantly improves the agreement between observed and predicted flow.

THIS PAGE
WAS INTENTIONALLY
LEFT BLANK

INTERNAL RAREFIED GAS FLOWS WITH BACKSCATTERING
PART 4: PLANE POISEUILLE FLOW WITH BACKSCATTERING

INTRODUCTION

Previous parts of this report have dealt with the effects of backscattering on free molecule and near continuum flows. In this report we extend our study to an internal flow and obtain a solution valid for all Knudsen numbers and for a sufficiently small fraction of backscattering. Again, we use the BGK model equation which has proved to be a useful tool for the analysis of internal rarefied gas flows.

The variational principle described in Part 2 is used to obtain an estimate of the transport with backscattering included. For 6% backscattering ($\alpha = 0.94$), the value obtained in Part 2, we compare the predicted transport with the experimental results of Gaede [1] for the flow of hydrogen between glass plates.

PLANE POISEUILLE FLOW

We examine the flow between two infinite, parallel flat plates located at $y = \pm(d/2)$. The z axis is taken to be the direction of the flow which is driven by a small pressure gradient and we write

$$p(z) = p_0(1 - kz), \quad (1)$$

with $k = -(1/p_0)(dp/dz)$ and (dp/dz) constant. The temperature of the plates is assumed constant and equal to T_0 .

For this configuration we expect $f(\vec{c}, \vec{r}) = f(\vec{c}, y, z)$ so that the BGK model equation is

$$c_y \frac{\partial f}{\partial y} + c_z \frac{\partial f}{\partial z} + \nu f = \nu f_{eq} \quad (2)$$

where ν is the collision frequency (assumed dependent of $\vec{c}$),

$$f_{eq} = \frac{n}{(2\pi RT)^{3/2}} e^{-(\vec{c} - \vec{q})^2/2RT} \quad (3)$$

and n , $\vec{q}$, and T are the local gas density, velocity, and temperature, respectively. The boundary condition at the plates with a fraction $(1 - \alpha)$ backscattered and a fraction α diffusely scattered is

$$f^+(\vec{c}, \bar{z}d/2, z) = \frac{\alpha A(z)}{(2\pi RT)^{3/2}} e^{-c^2/2RT} + (1 - \alpha)f^-(-\vec{c}, \bar{z}d/2, z) ; c_y \geq 0 . \quad (4)$$

As in the Kramer problem, we introduce the dimensionless quantities

$$Y = v_0 y / \sqrt{2RT_0} ; Z = v_0 z / \sqrt{2RT_0} ; \beta = v_0 d / \sqrt{2RT_0} ;$$

$$\vec{\xi} = \vec{c} / \sqrt{2RT_0} ; w = q_z / \sqrt{2RT_0}$$

and obtain for (2)

$$\xi_y \frac{\partial f}{\partial Y} + \xi_z \frac{\partial f}{\partial Z} + \frac{v}{v_0} f = \frac{v}{v_0} f_{eq} . \quad (5)$$

We assume that a small pressure gradient leads to sufficiently small flows so that we take $w \ll 1$ and linearize about some global equilibrium distribution function

$$f_0 = \frac{n_0}{(2\pi RT_0)^{3/2}} e^{-\xi^2} . \quad (6)$$

The procedure is the same as that used in Part 3 and leads to the linearized BGK equation

$$\xi_y \frac{\partial h}{\partial Y} + \xi_z \frac{\partial h}{\partial Z} + h = \eta + 2w\xi_z + (\xi^2 - \frac{3}{2}t) \quad (7)$$

where h , η , t are the dimensionless perturbations defined by

$$f = f_0(1 + h) \quad (8)$$

$$n = n_0(1 + \eta) \quad (9)$$

$$T = T_0(1 + t) . \quad (10)$$

In terms of dimensionless variables, the gas properties are given by

$$\eta = \pi^{-3/2} \int_{\vec{\xi}} h e^{-\xi^2} d\vec{\xi} \quad (11)$$

$$w = \pi^{-3/2} \int_{\vec{\xi}} \xi_z h e^{-\xi^2} d\vec{\xi} \quad (12)$$

$$t = (2\pi^{-3/2}/3) \int_{\vec{\xi}} \xi^2 e^{-\xi^2} h d\vec{\xi} - \eta . \quad (13)$$

From the gas law we have

$$p = \rho RT = mnRT = p_0(1 + \eta + t) \quad (14)$$

so that with equation (1), in terms of dimensionless variables, we require

$$\eta + t = -KZ \quad (15)$$

where

$$K = k\sqrt{2RT_0}/v_0 . \quad (16)$$

We will seek a solution for which the temperature is constant, i.e.,

$t = 0$, so that from (15) we take

$$\eta = -KZ . \quad (17)$$

This suggests we write the perturbation to the distribution function as

$$h(\vec{\xi}, Y, Z) = -KZ + \xi_z \phi(\xi_x, \xi_y, Y, Z) \quad (18)$$

since this form of h , with the use of (11), (12), and (13) leads to (17)

and to $t = 0$, but still permits a nonzero velocity w . With this choice of h , the BGK equation (7) becomes

$$\xi_y \frac{\partial \phi}{\partial Y} + \xi_z \frac{\partial \phi}{\partial Z} + \phi = K + 2w . \quad (19)$$

Linearization of the boundary condition (4), and use of (18), leads to the condition

$$\xi_z \phi^+(\xi_x, \xi_y, \frac{\beta}{2}, Z) = \alpha \left[\frac{A(Z)}{n_0} - 1 + KZ \right] \quad (20)$$

$$-(1 - \alpha) \xi_z \phi^-(\xi_x, \xi_y, \frac{\beta}{2}, Z) ; \xi_y \geq 0 .$$

A choice of $A(Z)$ which will satisfy (20) for all values of ξ_z and Z is

$$A(Z) = n_0(1 - KZ) \quad (21)$$

with which the boundary condition on ϕ becomes

$$\phi^+(\xi_x, \xi_y, \frac{\beta}{2}, Z) = -(1 - \alpha)\phi^-(-\xi_x, -\xi_y, \frac{\beta}{2}, Z) ; \xi_y \geq 0 . \quad (22)$$

We note that Z does not appear explicitly in the differential equation (19) or in the boundary condition (22) and hence we seek a solution of the form $\phi \neq \phi(Z)$. The differential equation and boundary condition for ϕ are

$$\xi_y \frac{\partial \phi}{\partial Y} + \phi = K + 2w(Y) \quad (23)$$

$$\phi^+(\xi_x, \xi_y, \frac{\beta}{2}) = -(1 - \alpha)\phi^-(-\xi_x, -\xi_y, \frac{\beta}{2}) ; \xi_y \geq 0 . \quad (24)$$

Formal integration of (23) along lines of constant $\vec{\xi}$ leads to

$$\begin{aligned} \phi(\xi_x, \xi_y, Y) &= e^{-[(\beta/2) + Y]/\xi_y} \phi^+(\xi_x, \xi_y, \frac{\beta}{2}) \\ &+ \int_{-\beta/2}^Y \frac{1}{\xi_y} (K + 2w) e^{+(Y' - Y)/\xi_y} dY' ; \xi_y > 0 \end{aligned} \quad (25)$$

and

$$\begin{aligned} \phi(\xi_x, \xi_y, Y) &= e^{[(\beta/2) - Y]/\xi_y} \phi^+(\xi_x, \xi_y, \frac{\beta}{2}) \\ &- \int_Y^{\beta/2} \frac{1}{\xi_y} (K + 2w) e^{(Y' - Y)/\xi_y} dY' ; \xi_y < 0 . \end{aligned} \quad (26)$$

In order to progress to an integral equation for the gas velocity w , we need to eliminate the boundary terms ϕ^+ from equations (25) and (26).

The details are given in Appendix A to this report. The result is

$$\begin{aligned} \phi(\vec{\xi}, Y) &= (1 - \alpha) \int_{-\beta/2}^{\beta/2} \frac{1}{\xi_y} (K + 2w) e^{(\beta - Y' - Y)/\xi_y} dY' \\ &- \int_Y^{\beta/2} \frac{1}{\xi_y} (K + 2w) e^{(Y' - Y)/\xi_y} dY' ; \xi_y < 0 \end{aligned} \quad (27)$$

$$\begin{aligned} \phi(\vec{\xi}, Y) = & -(1 - \alpha) \int_{-\beta/2}^{\beta/2} \frac{1}{\xi_y} (K + 2w) e^{-(Y' + Y + \beta)/\xi_y} dY' \\ & + \int_{-\beta/2}^Y \frac{1}{\xi_y} (K + 2w) e^{-(Y' - Y)/\xi_y} dY' ; \xi_y > 0 . \end{aligned} \quad (28)$$

In obtaining these equations, as shown in Appendix A, we have assumed that the amount of backscattering is sufficiently small and have ignored any terms of order $(1 - \alpha)^2$ or higher.

From equations (12) and (18) one obtains for the velocity

$$\pi^{3/2} w(Y) = \int_{\xi_y < 0} \xi_z^2 \phi e^{-\xi^2} d\vec{\xi} + \int_{\xi_y > 0} \xi_z^2 \phi e^{-\xi^2} d\vec{\xi} \quad (29)$$

and, since ϕ depends only on the y component of molecular velocity ξ_y , the integrations over ξ_x and ξ_z can be carried out. The result is

$$\pi^{1/2} w(Y) = \int_{-\infty}^0 (\phi/2) e^{-\xi_y^2} d\xi_y + \int_0^{\infty} (\phi/2) e^{-\xi_y^2} d\xi_y . \quad (30)$$

The integral equation for w follows from (30) by using equations (27) and (28). Thus,

$$\begin{aligned} \pi^{1/2} w(Y) = & \int_{-\infty}^0 e^{-\xi_y^2} \left[(1 - \alpha) \int_{-\beta/2}^{\beta/2} \frac{1}{\xi_y} \left(\frac{K}{2} + w \right) e^{(\beta - Y' - Y)/\xi_y} dY' \right. \\ & \left. - \int_Y^{\beta/2} \frac{1}{\xi_y} \left(\frac{K}{2} + w \right) e^{(Y' - Y)/\xi_y} dY' \right] d\xi_y \\ & + \int_0^{\infty} e^{-\xi_y^2} \left[-(1 - \alpha) \int_{-\beta/2}^{\beta/2} \frac{1}{\xi_y} \left(\frac{K}{2} + w \right) e^{-(Y' + Y + \beta)/\xi_y} dY' \right. \\ & \left. + \int_{-\beta/2}^Y \frac{1}{\xi_y} \left(\frac{K}{2} + w \right) e^{-(Y - Y')/\xi_y} dY' \right] d\xi_y . \end{aligned}$$

If the variable of integration in the first integral is changed from ξ_y to $(-\xi_y)$, then terms can be grouped to write, with interchange of the order of integration,

$$\begin{aligned} \pi^{1/2} w(Y) = & -(1 - \alpha) \int_{-\beta/2}^{\beta/2} \left(\frac{K}{2} + w \right) \left[\int_0^{\infty} \frac{1}{\xi_y} e^{-\xi_y^2 - (\beta - Y' - Y)/\xi_y} d\xi_y \right. \\ & \left. + \int_0^{\infty} \frac{1}{\xi_y} e^{-\xi_y^2 - (Y' + Y + \beta)/\xi_y} d\xi_y \right] dY' \\ & + \int_{-\beta/2}^{\beta/2} \left(\frac{K}{2} + w \right) \int_0^{\infty} \frac{1}{\xi_y} e^{-\xi_y^2 - (|Y' - Y|)/\xi_y} d\xi_y dY' . \end{aligned}$$

Introducing the Abramowitz functions*

$$J_n(x) = \int_0^{\infty} t^n e^{-t^2 - (x/t)} dt \quad (31)$$

we obtain, finally, the integral equation

$$\begin{aligned} \pi^{1/2} w(Y) = & \int_{-\beta/2}^{\beta/2} \left[\frac{K}{2} + w(Y') \right] J_{-1}(|Y' - Y|) dY' \\ & - (1 - \alpha) \int_{-\beta/2}^{\beta/2} \left[\frac{K}{2} + w(Y') \right] [J_{-1}(\beta - Y' - Y) + J_{-1}(\beta + Y' + Y)] dY' . \end{aligned} \quad (32)$$

It is convenient to transform the dependent variable in (32) by the relation

$$w(Y) = -(K/2)[1 - \psi(Y)] . \quad (33)$$

The integral equation for ψ is

$$\begin{aligned} \pi^{1/2} \psi(Y) = & \pi^{1/2} + \int_{-\beta/2}^{\beta/2} \psi(Y') J_{-1}(|Y - Y'|) dY' \\ & - (1 - \alpha) \int_{-\beta/2}^{\beta/2} \psi(Y') [J_{-1}(\beta - Y' - Y) + J_{-1}(Y' + Y + \beta)] dY' . \end{aligned} \quad (34)$$

*Useful properties of these functions are given in the Appendix of Part 2.

Equation (33) reduces to that given by Cercignani and Daneri [2] for pure diffuse reflection if one sets $\alpha = 1$. In equation (33), β is related to the Knudsen number (λ/d) where λ is the mean free path defined by the relation for gas viscosity $\mu = (1/2)\rho_0\lambda(8RT_0/\pi)^{1/2}$. Thus, choosing $v_0 = p_0/\mu$ one finds from the definition $\beta = v_0 d/\sqrt{2RT_0}$ that

$$\beta = (\sqrt{\pi}/2)(\lambda/d)^{-1}. \quad (35)$$

If V is the volume flow per unit width of channel, then

$$V = \frac{2RT_0}{v_0} \int_{-\beta/2}^{\beta/2} w dY. \quad (36)$$

A dimensionless flow parameter, in common use at Oak Ridge Gaseous Diffusion Plant, is defined by

$$W = \frac{4p_0 V}{d\Delta p \sqrt{8RT_0/\pi}}. \quad (37)$$

In terms of the variable $\psi(Y)$ the flow rate W is given by*

$$W = \frac{\sqrt{\pi}d}{\ell} \left[\frac{1}{\beta} - \frac{1}{\beta^2} \int_{-\beta/2}^{\beta/2} \psi(Y) dY \right] \quad (38)$$

where ℓ is the channel length. Thus, calculation of W could proceed by solving the integral equation (34) for $\psi(Y)$ and performing the quadrature indicated in (38). Alternatively, if we are interested only in the flow rate W , and not in the detailed variation of flow velocity in the cross section, we can apply the variational technique discussed in Part 2 of this report.

For the integral equation (34), the variational method provides the bound

$$\max F\{g(Y)\} \leq \int_{-\beta/2}^{\beta/2} \psi(Y) dY$$

*The dimensionless flow W is simply related to the flow variable $Q(\beta)$ used by Cercignani and Daneri [2] by the equation $W = (\sqrt{\pi}d/\ell)Q$.

so that the bound on the flow is

$$W \geq (\sqrt{\pi}d/\ell) \left[-\frac{1}{\beta} + \frac{1}{\beta^2} \max F\{g(Y)\} \right]. \quad (39)$$

For the trial function $g(Y)$ we select

$$g(Y) = AY^2 + B, \quad (40)$$

form the functional

$$\begin{aligned} F\{g(Y)\} = & 2 \int_{-\beta/2}^{\beta/2} g(Y) dY + \frac{1}{\sqrt{\pi}} \int_{-\beta/2}^{\beta/2} \int_{-\beta/2}^{\beta/2} g(Y)g(X) [J_{-1}(|X - Y|) \\ & - (1 - \alpha)J_{-1}(\beta - X - Y) - (1 - \alpha)J_{-1}(\beta + X + Y)] dXdY - \int_{-\beta/2}^{\beta/2} g^2(Y) dY, \end{aligned} \quad (41)$$

and maximize $F\{g(Y)\}$ with respect to the parameters A and B in the trial function.

The result (details in Appendix B) is

$$\begin{aligned} \max F\{AY^2 + B\} = & (A^2/2)(I_{44} - I_4) + (B^2/2)(I_{00} - I_0) \\ & + AB(I_{22} - I_2) + AI_2 + BI_0 \end{aligned} \quad (42)$$

with

$$A = (I_0 I_{22} - I_2 I_{00})/\Delta \quad (43)$$

$$B = [I_0(I_4 - I_{44}) + I_2(I_{22} - I_2)]/\Delta \quad (44)$$

$$\Delta = (I_{00} - I_0)(I_{44} - I_4) - (I_{22} - I_2)^2 \quad (45)$$

$$I_0 = 2\beta \quad (46)$$

$$I_2 = \frac{\beta^3}{6} \quad (47)$$

$$I_4 = \frac{\beta^5}{40} \quad (48)$$

$$I_{00} = 2\beta - \frac{2(2 - \alpha)}{\sqrt{\pi}} + \frac{4(3 - 2\alpha)}{\sqrt{\pi}} J_1(\beta) - \frac{4(1 - \alpha)}{\sqrt{\pi}} J_1(2\beta) \quad (49)$$

$$I_{22} = \frac{\beta^3}{6} + [1 + (1 - \alpha)] \left[\beta - \frac{\beta^2}{2\sqrt{\pi}} - \frac{4}{\sqrt{\pi}} \right] + \frac{2}{\sqrt{\pi}} \left\{ 2\beta[1 + 2(1 - \alpha)]J_0(\beta) - 4\beta(1 - \alpha)J_0(2\beta) + \left[\frac{\beta^2}{2} + 4 + (\beta^2 + 8)(1 - \alpha) \right]J_1(\beta) - \left[\left(\frac{\beta^2}{2} + 4 \right)(1 - \alpha) \right]J_1(2\beta) + 2\beta J_2(\beta) - 2\beta(1 - \alpha)J_2(2\beta) \right\}, \quad (50)$$

$$I_{44} = \frac{\beta^5}{40} + \frac{\beta^3}{6} - \frac{\beta^4}{8\sqrt{\pi}} - \frac{16}{\sqrt{\pi}} + (1 - \alpha) \left[6\beta + \frac{\beta^3}{2} - \frac{4\beta^2}{\sqrt{\pi}} - \frac{16}{\sqrt{\pi}} - \frac{\beta^4}{8\sqrt{4}} \right] + \frac{2}{\sqrt{\pi}} \left\{ 2\beta[\beta^2 + 4 + 8\beta(1 - \alpha)]J_0(\beta) - 4\beta(1 - \alpha)(\beta^2 + 4)J_0(2\beta) + [16 + 8\beta^2 + \frac{\beta^4}{8} + (1 - \alpha)(\frac{\beta^4}{4} + 32)]J_1(\beta) - (1 - \alpha)(16 + \frac{\beta^4}{8} + 12\beta^2)J_1(2\beta) + [\beta^3 + 16\beta + 8\beta(1 - \alpha)]J_2(\beta) - (1 - \alpha)(\beta^3 + 20\beta)J_2(2\beta) \right\}, \quad (51)$$

and the J_n are the Abramowitz functions defined previously.

Equations (42) and (39) furnish a bound to the flow W as a function of β .

COMPARISON WITH EXPERIMENT

Gaede [1] measured the flow of hydrogen between flat parallel glass plates as a function of pressure. The channel dimensions were: length $\ell = 0.12$ cm, width $b = 3.4$ cm, and nominal spacing $d = 4 \times 10^{-4}$ cm. All the flow measurements were made at temperatures ranging from 16° to 18°C .

It was recognized by Gaede that the nominal spacing of 4×10^{-4} cm was not known with sufficient accuracy for use in the calculation of predicted flow rates and that the data could furnish a better estimate of the spacing. To obtain such an estimate of the spacing d , we proceeded as follows.

We assumed that the flows measured by Gaede at the three highest pressures (707, 652, and 484 mm Hg) could be described adequately by continuum plane Poiseuille flow with a slip correction. For a slip coefficient we used the values obtained in Part 3 of this report, i.e., $\zeta = 1.137\lambda$ for $\alpha = 1$,

and $\zeta = 1.015\lambda$ for $\alpha = 0.94$. In this way we obtained the equations

$$\bar{p}d^3 = (12\mu\ell/b)G - 1.137(3\pi/2)\mu \sqrt{8RT/\pi} d^2 \quad (52)$$

for $\alpha = 1$, and

$$\bar{p}d^3 = (12\mu\ell/b)G - 1.015(3\pi/2)\mu \sqrt{8RT/\pi} d^2 \quad (53)$$

for $\alpha = 0.94$, where G is the observed volume flow rate in cc/sec measured at the average pressure $\bar{p}$. In the calculation of d , a temperature of 17°C was used along with a viscosity $\mu = 87.4 \times 10^{-6}$ poise for hydrogen at this temperature.

Solution of (52) and (53) for the flows measured at the three highest pressures gave three values of d for each value of α . These were averaged for each α leading to the estimates

$$d = 2.880 \times 10^{-4} \quad \text{for } \alpha = 1$$

and

$$d = 2.909 \times 10^{-4} \quad \text{for } \alpha = 0.94 .$$

With these values of d , equation (39) was used to predict the pressure dependence of the flow and the results are given in Table C-2 in Appendix C and are shown in Figure 1 where predicted flows (G , cc/sec) are plotted versus average pressure ($\bar{p}$, mm Hg).*

Also shown in the figure and in Table C-1 are the results of Gaede's measurements. The theoretical prediction of the transport is clearly in better agreement with observations for 6% backscattering ($\alpha = 0.94$) than for pure diffuse scattering ($\alpha = 1$).

*The relationship between G and W is

$$G = [(bd \sqrt{8RT/\pi})/4] W$$

and the average pressure $\bar{p}$ in millimeters of mercury is related to β by

$$\bar{p}_{\text{mm}} = \frac{(760)\sqrt{\pi} \mu \sqrt{8RT/\pi}}{(2)(1.013)(10^6)d} \beta .$$

Figure 1
BGK SOLUTION FOR PLANE POISEUILLE FLOW

SUMMARY AND CONCLUSIONS

The effect of backscattering on the flow of a rarefied gas between two infinite parallel flat plates has been examined. Predictions of the transport produced by sufficiently small pressure gradients have been made for arbitrary average pressure and for the cases of purely diffuse and partially backscattered reflections at the plates.

In the case of the backscattering boundary condition, the calculations were confined to 6% backscattering, a value obtained in Part 2 of this report.

The inclusion of backscattering reduces the transport in the near free molecule and transition regions below that obtained for a purely diffuse scattering boundary condition. Further, backscattering has little effect on the predicted transport in the near continuum and continuum regimes other than its influence on the slip coefficient as described in Part 3.

A comparison with Gaede's data for the flow of hydrogen between glass plates indicates that the predicted transports agree significantly better with the measured flows when backscattering is included in the theoretical model.

We tentatively conclude that the inclusion of backscattering (in place of pure diffuse scattering) provides an improved model for the prediction of internal rarefied gas flows in systems whose surfaces have undergone no special preparation.

Part 5 of this report will provide further evidence to substantiate this conjecture by examining the transport through capillaries and annuli at arbitrary average pressure.

REFERENCES

1. Gaede, W., Ann. Physik, 41, 289 (1913).
2. Cercignani, C., and Daneri, A., J. Appl. Phys., 34, 3509 (1963).

/jg

APPENDIX A

ELIMINATION OF THE BOUNDARY TERMS
FROM THE INTEGRATED BGK EQUATION.

We have the integrated BGK equations (25) and (26)

$$\phi(\vec{\xi}, Y) = f_1(\xi_y, Y) \phi^+(\vec{\xi}, -\beta/2) + f_2(\xi_y, Y) ; \xi_y > 0 \quad (A-1)$$

$$\phi(\vec{\xi}, Y) = g_1(\xi_y, Y) \phi^+(\vec{\xi}, \beta/2) + g_2(\xi_y, Y) ; \xi_y < 0 \quad (A-2)$$

where for convenience in what follows we have written

$$e^{-[(\beta/2) + Y]/\xi_y} = f_1(\xi_y, Y) \quad (A-3)$$

$$\int_{-\beta/2}^Y \frac{1}{\xi_y} (K + 2w) e^{+(Y' - Y)/\xi_y} dY' = f_2(\xi_y, Y) \quad (A-4)$$

$$e^{-[Y - (\beta/2)]/\xi_y} = g_1(\xi_y, Y) \quad (A-5)$$

$$-\int_Y^{\beta/2} \frac{1}{\xi_y} (K + 2w) e^{(Y' - Y)/\xi_y} dY' = g_2(\xi_y, Y) . \quad (A-6)$$

The boundary condition equation (22) provides the relations

$$\phi^+(\vec{\xi}, -\beta/2) = -(1 - \alpha) \phi^-(\vec{\xi}, -\beta/2) ; \xi_y > 0 \quad (A-7)$$

$$\phi^+(\vec{\xi}, \beta/2) = -(1 - \alpha) \phi^-(\vec{\xi}, \beta/2) ; \xi_y < 0 . \quad (A-8)$$

From (A-1) and (A-2) we obtain at $Y = \pm\beta/2$

$$\phi^-(\vec{\xi}, \beta/2) = f_1(\xi_y, \beta/2) \phi^+(\vec{\xi}, -\beta/2) + f_2(\xi_y, \beta/2) ; \xi_y > 0 \quad (A-9)$$

$$\phi^-(\vec{\xi}, -\beta/2) = g_1(\xi_y, -\beta/2) \phi^+(\vec{\xi}, \beta/2) + g_2(\xi_y, -\beta/2) ; \xi_y < 0 , \quad (A-10)$$

or by replacing $\vec{\xi}$ with $(-\vec{\xi})$

$$\phi^{-}(\vec{\xi}, \beta/2) = f_1(-\xi_y, \beta/2) \phi^{+}(-\vec{\xi}, -\beta/2) + f_2(-\xi_y, \beta/2) ; \xi_y < 0 \quad (\text{A-11})$$

$$\phi^{-}(-\vec{\xi}, -\beta/2) = g_1(-\xi_y, -\beta/2) \phi^{+}(-\vec{\xi}, \beta/2) + g_2(-\xi_y, -\beta/2) ; \xi_y > 0 \quad (\text{A-12})$$

From (A-1) and (A-7) one obtains

$$\phi(\vec{\xi}, Y) = -(1 - \alpha) f_1(\xi_y, Y) \phi^{-}(-\vec{\xi}, -\beta/2) + f_2(\xi_y, Y) ; \xi_y > 0 \quad (\text{A-13})$$

and using (A-12), this becomes

$$\begin{aligned} \phi(\vec{\xi}, Y) = & -(1 - \alpha) f_1(\xi_y, Y) [g_1(-\xi_y, -\beta/2) \phi^{+}(-\vec{\xi}, \beta/2) \\ & + g_2(-\xi_y, -\beta/2)] + f_2(\xi_y, Y) . \end{aligned} \quad (\text{A-14})$$

With $\phi^{+}(-\vec{\xi}, \beta/2)$ from the boundary condition (A-8) we obtain

$$\begin{aligned} \phi(\vec{\xi}, Y) = & -(1 - \alpha) f_1(\xi_y, Y) [-g_1(-\xi_y, -\beta/2) (1 - \alpha) \phi^{-}(\vec{\xi}, \beta/2) \\ & + g_2(-\xi_y, -\beta/2)] + f_2(\xi_y, Y) ; \xi_y > 0 . \end{aligned} \quad (\text{A-15})$$

At $Y = \beta/2$ this becomes

$$\begin{aligned} \phi^{-}(\vec{\xi}, \beta/2) = & -(1 - \alpha) f_1(\xi_y, \beta/2) [-g_1(-\xi_y, -\beta/2) (1 - \alpha) \phi^{-}(\vec{\xi}, \beta/2) \\ & + g_2(-\xi_y, -\beta/2)] + f_2(\xi_y, \beta/2) ; \xi_y > 0 \end{aligned} \quad (\text{A-16})$$

or

$$\begin{aligned} & \phi^{-}(\vec{\xi}, \beta/2) [1 - (1 - \alpha)^2 f_1(\xi_y, \beta/2) g_1(-\xi_y, -\beta/2)] \\ & = -(1 - \alpha) f_1(\xi_y, \beta/2) g_2(-\xi_y, -\beta/2) + f_2(\xi_y, \beta/2) ; \xi_y > 0 . \end{aligned} \quad (\text{A-17})$$

In (A-17), we change the sign of $\vec{\xi}$, multiply by $(1 - \alpha)$ and use the condition (A-8) to obtain finally for $\phi^{+}(\vec{\xi}, \beta/2)$ the equation

$$\phi^+(\vec{\xi}, \beta/2) = \left[\frac{(1 - \alpha)}{1 - (1 - \alpha)^2 f_1(-\xi_y, \beta/2) g_1(\xi_y, \beta/2)} \right] \quad (A-18)$$

$$\cdot [(1 - \alpha) f_1(-\xi_y, \beta/2) g_2(\xi_y, -\beta/2) - f_2(-\xi_y, \beta/2)] \quad ; \quad \xi_y < 0 .$$

The use of (A-18) in the integrated BGK equation leads to an integral equation for the velocity which is much too difficult to deal with. We will content ourselves with restricting our consideration to the case of a sufficiently small amount of backscattering and linearize (A-18) with respect to $(1 - \alpha)$, the fraction of backscattering.

For sufficiently small fractions of backscattering, this linearization produces little error in the computed transport. Higher powers of $(1 - \alpha)$ are associated with multiple backscattering events between two wall points. The maximum error thus occurs in the free molecule limit since, at higher pressures, the contribution of such multiple backscattered molecules to the gas properties at any point is decreased rapidly due to collisions in the gas phase.

In the limit of free molecule flow, we know from the long capillary result that backscattering reduces the transport by the exact factor $\alpha/(2 - \alpha)$. The approximation of this, to first order in $(1 - \alpha)$, is α^2 . Thus for 6% backscattering the exact expression gives 0.887, while the linearized expression yields 0.884, an error of about 0.3%. The result in terms of our original notation when substituted into the integrated BGK equation (26) leads to equation (27) of this report.

In a similar fashion one obtains equation (28) of this report by elimination of the boundary term $\phi^+(\xi_x, \xi_y, -\beta/2)$ from equation (25).

APPENDIX B

MAXIMIZATION OF $F\{g(Y)\}$

With $g(Y) = AY^2 + B$ we must maximize, with respect to A and B, the functional

$$F\{g(Y)\} = 2 \int g(Y) dY + \frac{1}{\sqrt{\pi}} \iint g(Y)g(X)K(J)dYdX - \int g^2(Y)dY \quad (B-1)$$

where the limits on all integrals are understood to be $(-\beta/2, \beta/2)$ and where we have written

$$K(J) = J_{-1}(|X - Y|) - (1 - \alpha)[J_{-1}(\beta - X - Y) + J_{-1}(\beta + X + Y)] . \quad (B-2)$$

Differentiation of (B-1) with respect to A and B, and setting these partial derivatives equal to zero, provides simultaneous algebraic equations for those values of A and B which lead to an extremum in $F\{g(Y)\}$. These values are easily found to be

$$A = (I_0 I_{22} - I_2 I_{00}) / \Delta \quad (B-3)$$

$$B = [I_0(I_4 - I_{44}) + I_2(I_{22} - I_2)] / \Delta \quad (B-4)$$

with

$$\Delta = (I_{44} - I_4)(I_{00} - I_0) - I_{22} - I_2^2 \quad (B-5)$$

which are equations (43), (44), and (45) of the text. In the above, the I integrals are the following

$$I_0 = 2 \int_{-\beta/2}^{\beta/2} dY = 2\beta \quad (B-6)$$

$$I_2 = 2 \int_{-\beta/2}^{\beta/2} Y^2 dY = \frac{\beta^3}{6} \quad (B-7)$$

$$I_4 = 2 \int_{-\beta/2}^{\beta/2} Y^4 dY = \frac{\beta^5}{40} \quad (B-8)$$

$$I_{00} = \frac{2}{\sqrt{\pi}} \int_{-\beta/2}^{\beta/2} \int_{-\beta/2}^{\beta/2} K(J) dX dY \quad (B-9)$$

$$I_{22} = \frac{1}{\sqrt{\pi}} \int_{-\beta/2}^{\beta/2} \int_{-\beta/2}^{\beta/2} (X^2 + Y^2) K(J) dX dY \quad (B-10)$$

$$I_{44} = \frac{2}{\sqrt{\pi}} \int_{-\beta/2}^{\beta/2} \int_{-\beta/2}^{\beta/2} X^2 Y^2 K(J) dX dY . \quad (B-11)$$

The evaluation of these integrals is straightforward but time consuming and tedious. The following results, based on the properties of the Abramowitz functions given in the Appendix of Part 3 of this report, prove very helpful in the evaluation of the integrals.

Let $J_n[h(\pm Y)]$ indicate a J_n function in which the variable of integration appears as $\pm Y$ in the argument of the J_n function. Then

$$\int_a^b J_n[h(\pm Y)] dY = \mp J_{n+1}[h(\pm b)] \pm J_{n+1}[h(\pm a)] . \quad (B-12)$$

Two integrations by parts yield the result

$$\begin{aligned} \int_a^b Y^2 J_n[h(\pm Y)] dY &= \mp b^2 J_{n+1}[h(\pm b)] \pm a^2 J_{n+1}[h(\pm a)] \\ &\quad - 2b J_{n+2}[h(\pm b)] + 2a J_{n+2}[h(\pm a)] \\ &\quad \mp 2J_{n+3}[h(\pm b)] \pm 2J_{n+3}[h(\pm a)] . \end{aligned} \quad (B-13)$$

The recursion formulae

$$J_n(Y) = \frac{n-1}{2} J_{n-2}(Y) + \frac{Y}{2} J_{n-3}(Y) \quad (B-14)$$

$$J_n(0) = \frac{n-1}{2} J_{n-2}(0)$$

with

$$J_0(0) = \sqrt{\pi}/2 \quad (B-15)$$

$$J_1(0) = 1/2 \quad (B-16)$$

are also required to reduce all the I integrals to expressions containing only the three Abramowitz functions J_0 , J_1 , and J_2 . The results of these integrations are given by equations (46) through (51) in the body of this report.

APPENDIX C

COMPARISON OF CALCULATED AND MEASURED FLOWS

Table C-1 gives Gaede's [1] measured values of the volume flow of hydrogen between glass plates (at a nominal temperature of 17°C) and at various pressures.

In Table C-2, the volume flow rates G_1 and $G_{0.94}$ (calculated as described in this report) are given for the two cases, pure diffuse scattering ($\alpha = 1$), and partial backscattering ($\alpha = 0.94$).

As indicated in the text of this report, two slightly different estimates of plate spacing are obtained for the cases $\alpha = 1$ and $\alpha = 0.94$. This is reflected in Table C-2 by the two values of average pressure $\bar{p}_1$ and $\bar{p}_{0.94}$ which result from the fact that the relationship between the theoretical parameter β and the average pressure involves the plate spacing.

TABLE C-1

GAEDE'S [1] FLOW DATA; H₂-GLASS AT 17°C

<u>$\bar{p}$, mm Hg</u>	<u>G, cc/sec</u>	<u>$\bar{p}$, mm Hg</u>	<u>G, cc/sec</u>
0.019	0.540	484	0.609
0.031	0.5335	652	0.741
0.039	0.5233	707	0.779
0.065	0.498		
0.079	0.486		
0.131	0.465		
0.265	0.4408		
0.32	0.431		
0.67	0.4105		
1.08	0.3745		
1.33	0.3588		
2.2	0.332		
2.7	0.334		
4.44	0.3042		
5.45	0.305		
9.3	0.2809		
16.1	0.2726		
23.2	0.2552		
59	0.2639		
114	0.2949		
228	0.3806		
307	0.4555		

TABLE C-2

CALCULATED FLOWS; BGK SOLUTION

$\bar{P}_1$, mm Hg	$\bar{P}_{0.94}$, mm Hg	G_1 , cc/sec	$G_{0.94}$, cc/sec	$\bar{P}_1$, mm Hg	$\bar{P}_{0.94}$, mm Hg	G_1 , cc/sec	$G_{0.94}$, cc/sec
0.01987	0.01967	--	0.696	31.7	31.4	0.280	--
0.3525	0.349	0.554	--	35.2	34.9	0.280	--
0.497	0.492	--	0.486	38.8	38.4	0.280	--
0.705	0.698	0.493	--	39.8	39.4	--	0.259
1.057	1.047	0.459	--	42.6	41.8	0.280	--
1.410	1.397	0.436	--	52.8	52.4	0.283	--
1.763	1.745	0.419	--	70.5	69.8	0.290	--
1.988	1.967	--	0.382	88.1	87.3	0.300	--
2.468	2.442	0.394	--	99.4	98.5	--	0.292
2.82	2.79	0.385	--	105.7	104.7	0.312	--
3.17	3.14	0.376	--	141.0	139.7	0.336	--
3.52	3.49	0.370	--	176.3	174.5	0.362	--
7.05	6.98	0.329	--	211.7	209.4	0.389	--
9.95	9.85	--	0.293	282	279	0.445	--
10.57	10.47	0.310	--	352	349	0.503	--
14.10	13.97	0.298	--	370	366	0.518	--
17.63	17.45	0.292	--	597	591	--	0.701
19.88	19.70	--	0.270	995	985	--	1.046
21.17	20.94	0.287	--				
24.68	24.42	0.284	--				
28.20	27.92	0.282	--				
29.82	29.52	--	0.262				