Written: January 1970

Distributed: June 1970

LA-4294
UC-80, REACTOR
TECHNOLOGY
TID-4500

LOS ALAMOS SCIENTIFIC LABORATORY
of the
University of California
LOS ALAMOS • NEW MEXICO

**Hydraulic Effects in
Circulating Fuel Reactors**

by

Felix T. Adler

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TABLE OF CONTENTS

LIST OF SYMBOLS	iv
I. INTRODUCTION	1
II. THE VELOCITY DISTRIBUTION IN TURBULENT FLOW NEAR A WALL	3
A. Equations of Motion	3
B. The Turbulent Mixing Coefficient in Reichardt's Approach	5
C. Reichardt's Representation of the Turbulent Velocity Distribution	6
D. The Law of the Wall of van Driest	8
III. REPRESENTATIVE HYDRAULIC DATA FOR CIRCU- LATING FUEL REACTORS OF THE KING TYPE	10
A. Energy Deposition during Core Transit	10
B. Coefficient of Resistance	11
C. Data for Viscosity μ and Kinematic Viscosity ν	11
D. Hydraulic Parameters for the Fuel Solution	12
E. Representative Fuel Solution Exposures	14
F. The Critical Region: Residence Times and Dimensions	19
IV. IMPLICATIONS FOR THE KING REACTOR EXPERI- MENT AND FULL-SCALE DEVICES	21
APPENDIXES:	
A. Equations of Motion for Turbulent Flow	24
B. Dimensional Analysis of the Law of the Wall	25
C. The Mixing Length Theory of Prandtl	26
D. Relation of Reichardt's and van Driest's Methods to Other Semiempir- ical Methods	28
REFERENCES	30

LIST OF SYMBOLS

A	: Damping constant in van Driest's law of the wall	u, v, w	: Components of fluid velocity
A_*	: $(u_* / \nu) A$	$\bar{u}, \bar{v}, \bar{w}$	: Local time-averaged components of fluid velocity
A_t	: Turbulent mixing coefficient	u', v', w'	: Fluctuating components of fluid velocity
$E(y)$	: Energy deposition per unit volume in a volume element ΔV passing through core at a wall distance y	$u_* = \sqrt{t_0 / \rho}$	: Friction velocity
E_B	: Average energy deposition per unit volume for fluid traveling with mean bulk velocity v_B through core	$(d\bar{u}/dy)_0$	: Value of velocity gradient at pipe wall
H	: Effective core height	y	: Distance from pipe wall
L	: Integral of correlation function	$\eta = (u_* \nu) y$	: Dimensionless wall distance
R_{ij}	: Double correlation function	$\eta_a = (u_* \nu) a$	: Dimensionless pipe radius
$R = \bar{u}d/\nu$	: Reynolds number	κ	: Proportionality constant in mixing length expression
T	: Duration of growth of boundary layer in Einstein-Li model	λ	: Dimensionless coefficient of resistance in pipe flow
$T_0 = v_B/H$	: Average core residence time of a fuel solution volume element	μ	: Viscosity
T_B	: Core residence time for fuel solution volume element in boundary layer	$\nu = \mu/\rho$	: Kinematic viscosity
T_C	: Fuel solution recycle time interval	ρ	: Density of fuel solution
T_m	: Fuel mixing time after core passage	$\bar{\rho}$	: Reference density of fuel solution
T_R	: Residence time of specified fuel solution volume	ρ_f	: Local fuel density
U_m	: Maximum value of $\bar{u}$ at center of pipe	ρ'	: Assumed density of fuel solution in parametric studies
ΔV	: Volume element of fuel solution	$\tau = \tau_t + \tau_v$	: Total shearing stress
a	: Pipe radius	τ_t	: Turbulent shearing stress
d	: Pipe diameter	τ_v	: Viscous shearing stress
l	: Mixing length	τ_0	: Value of viscous shearing stress at pipe wall
r	: Radial distance from center of pipe	$\varphi(y, z)$	: Thermal neutron flux at wall distance y and elevation z in core
r'	: Distance between two points in correlation function R_{ij}	Σ_f^0	: Macroscopic fission cross section at reference fuel solution density
$t_1(y)$	: Transit time through core for volume element ΔV at wall distance y		
u_B	: Mean flow velocity for bulk flow		
$\bar{u}(y)$	: Average velocity in pipe direction at wall distance y		

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ABSTRACT

The well-known results for the boundary layer flow in a pipe are reviewed with the aim of ascertaining the effects of the radial velocity distribution on the fuel exposure in a circulating fuel reactor. The laminar sublayer and the turbulent-laminar transition layer are of particular relevance for the fuel exposure because of the increased residence time of the fuel solution in these regions during core passage. The semi-empirical approaches for the velocity distributions in these regions are reviewed in detail and are compared with available experimental data. The radial distribution of energy deposition in aqueous solutions of ^{235}U corresponding to those of interest for the Kinetic Intense Neutron Generator (KING) reactor and possible full-scale devices are considered and compared with existing data on the limits for safe exposure not leading to fuel decomposition. Although exposures in the KING dynamic experiments appear to be acceptable, full-scale devices may have excessive energy deposition rates in the boundary region. Conjectures are discussed as to methods to avoid any excess local energy deposition in full-scale devices.

I. INTRODUCTION

In the design for the critical experiment for the Kinetic Intense Neutron Generator (KING) reactor, as well as in the conceptual designs for full-scale devices,¹ the bulk flow velocity of the solution carrying the fissile material through the core region is chosen so that the concentration of radiolytic gas developed be kept at a safe level, and so that the solubility limit of H_2O_2 in the solution is not exceeded to avoid precipitation of UO_4 . Both conditions are satisfied for the proposed average flow velocities. Local variations in the flow regime will not affect the total radiolytic gas development, which depends only on the average flow velocity and on the power level of the reactor. Fuel decomposition may, however, occur in any

part of the fuel solution in which the local energy deposition by fission has exceeded a critical value. In a series of static measurements, Thamer² found a value of 70 joule/cm³ for this critical energy deposition in aqueous solutions of ^{235}U . Earlier, in 1962, during the KEWB experiments³ at Atomic International, dynamic measurements involving intense fast bursts of short duration, typically in the millisecond range, led to a much higher critical value of about 500 joule/cm³. The effect of this uncertainty is considered in detail in Secs. III and IV.

The energy deposition in a volume ΔV of the fuel solution is given approximately by the product of the average reactor power and the actual residence time, T_R , of ΔV in the core. T_R depends, however, on

the actual velocity of ΔV in its passage through the core, and not only on the average bulk flow velocity. In contrast to the determination of the safe limit for total gas development, it is therefore necessary to consider the velocity distribution in the flow to ascertain that fuel decomposition with associated corrosion effects cannot occur during the dynamic critical experiments for the KING reactor or in the full-scale devices. Thus, a combined hydraulic-neutronic analysis of the proposed flowing core reactor concepts becomes necessary. As a basis for such studies and associated safety analyses, some of the relevant literature on hydrodynamic models and data for pipe flow are summarized. In particular, results dealing with the laminar sublayer and the turbulent laminar

transition layer of the boundary layer are stressed because the residence time of the fuel solution in these layers may differ considerably from its average value.

It is apparent from the design and hydraulic parameters for the KING reactor critical experiment, summarized in Table I, that for flow speeds of experimental interest the Reynolds number, R , will be in the range from 10^5 to 5×10^7 ; the flow in the core pipe will therefore be turbulent. The problem of turbulent pipe flow has been investigated extensively.⁴⁻¹² The so-called universal law of the wall for the average velocity, $U(y)$, in the pipe direction at a distance y from the wall has been derived from a variety of physical models: Reynolds' principle of similarity, Prandtl's "mixing length" approach, Taylor's theory

TABLE I

REFERENCE DATA FOR KING REACTOR DYNAMIC CRITICAL EXPERIMENT

Reactor Configuration

Inside diameter of pipe	5.032 in.	(12.78 cm)
Core height	28 in.	(71 cm)
Fuel solution density	1.146 g/cm ³	
Maximum fuel velocity	~ 18 ft/sec	(550 cm/sec)
Pressure head	58 ft	

Kinematic Viscosity of Water (Ref. 4)

Temp, °C	0	20	50	100
$V = \mu/\rho$, cm ² /sec	0.0178	0.01	0.0056	0.003

Reynolds' Number $R = V d/\nu$ for pipe diameter $d = 25.6$ cm

V = average flow speed

ν = kinematic viscosity

V (cm/sec)	50	100	200	300	400	500	550
T °C							
0	0.72×10^5	1.45×10^5	2.9×10^5	4.3×10^5	5.8×10^5	7.2×10^5	7.9×10^5
20	1.3×10^5	2.6×10^5	5.2×10^5	7.8×10^5	1.0×10^6	1.3×10^6	1.4×10^6
50	2.3×10^5	4.6×10^5	9.2×10^5	1.4×10^6	1.8×10^6	2.3×10^6	2.5×10^6
100	4.25×10^5	8.5×10^5	1.7×10^6	2.55×10^6	3.4×10^6	4.3×10^6	4.7×10^6

of vorticity transport;¹³ and von Kármán's similarity hypothesis have been used as starting points in this context. Various assumptions for the mixing length have also been introduced to obtain better agreement between predictions for the radial variations of $\bar{v}(y)$ with the experimental data. Szablewski,¹⁴ for example, introduced an expression for the mixing length that includes the pipe radius and the viscosity as parameters. These physical models are used in many applications for the description of turbulent boundary layer phenomena.⁶ The models differ markedly in their physical assumptions, but for pipe flow they all lead ultimately to very similar results for $\bar{u}(y)$ in the turbulent boundary layer. On the other hand, the laminar sublayer and the transition layer, that is, the regions of particular interest for the analysis of the KING reactor experiment, are relatively poorly described by these models. It is therefore necessary to consider in detail alternative methods, developed more recently by Reichardt¹⁵ and van Driest¹⁶ that provide a better description of the mean velocity and of the Reynolds stresses in the region of interest for the applications considered here. According to a recent review by Rotta,¹⁷ these models are still considered as the most satisfactory representations of the velocity profiles near the wall in pipe or channel flow.

In Sec. II, the results of Reichardt and van Driest are summarized, together with the experimental data^{15, 18, 19} supporting the conclusions obtained from these models. In Sec. III, hydraulic data for water²⁰ and uranyl sulfate²¹ are summarized, and preliminary estimates for the energy deposition in various regions of the boundary layer are obtained and compared with available experimental data.^{3, 22} It must be emphasized that these estimates are preliminary, and that no detailed analysis of cross flow effects is made, nor are the statistical theories of turbulence considered. Only the very remarkable model for

the sublayer flow of Einstein and Li^{23, 24} is briefly discussed, in which the flow in the entire boundary layer is treated as an unsteady process. Section IV summarizes the preliminary conclusions from this study for the KING reactor concept.

The mathematical background for the derivations used is reviewed in the Appendixes. Appendix A summarizes the derivation of the basic equation of motion from the Stokes-Navier equation. A derivation of the law of the wall using only dimensional arguments is given in Appendix B; Appendix C summarizes the results of the mixing length approach. Finally, in Appendix D, the results obtained from Reichardt's approach are compared with those obtained from the von Kármán and Prandtl models.

More detailed hydrodynamic and coupled hydraulic-neutronic studies seem indicated before reaching a definitive conclusion on the energy deposition rates that are critical for design considerations. The density variations created by energy deposition during core passage of the fuel solution are treated here only by parametric studies. An investigation based on the methods presented recently by Herring²⁵ may become desirable at a time when detailed design studies for a full-scale device are undertaken. An experimental investigation of the effects of the gas bubbles on the kinematic viscosity of the fuel solutions would be of particular interest. Even recent summary reports²⁶ on cocurrent flow do not provide information on this topic.

The recent publications of Delhay^{27, 28} and the Proceedings of the 1968 AFOSR-IFP Stanford Conference on computations for turbulent boundary layers²⁹ became available only after completion of this work, and are not considered in this report.

II. THE VELOCITY DISTRIBUTION IN TURBULENT FLOW NEAR A WALL

A. Equations of Motion

For laminar flow in a pipe, the frictional shearing stress, τ_v , is related

to the fluid velocity gradient du/dy by

$$\tau_v = \mu \frac{du}{dy}, \quad (1)$$

where y denotes the perpendicular distance from the pipe wall, μ is the viscosity of the fluid, and u is the velocity of the fluid in laminar flow.

In turbulent flow, the fluid velocity components u , v , w , are expressed in the form

$$\begin{aligned} u &= \bar{u} + u' \\ v &= \bar{v} + v' \\ w &= \bar{w} + w' \end{aligned} \quad (2)$$

where $\bar{u}$, $\bar{v}$, and $\bar{w}$ denote the local, time-averaged velocity components, and u' , v' , and w' are the fluctuating components of the fluid velocity. The apparent or virtual stresses, often called the Reynolds stresses, occurring in turbulent flow are expressed as the averages $\overline{u'^2}$, $\overline{u'v'}$, etc., of the fluctuating components. As discussed in Appendix A, the turbulent shearing stress in pipe flow can be expressed in the form

$$\tau_t = A_t \frac{d\bar{u}}{dy}, \quad (3)$$

which is analogous to the equation for laminar flow. Here, however, $d\bar{u}/dy$ is the radial component of the gradient of the local average velocity in the pipe direction, and the proportionality factor, A_t , is the turbulent mixing coefficient describing the exchange of momentum between neighboring fluid regions.

The turbulent mixing coefficient corresponds to the viscosity, μ , in laminar flow, and is often called the apparent or eddy viscosity. In contrast to μ , the coefficient A_t depends on the flow velocity which in turn varies with the radial distance from the center line of the pipe. Therefore, Eq. (3) cannot be used to calculate the turbulent velocity distribution in pipe flow directly; Eq. (3) must be

complemented not only by the equations describing the boundary conditions at the wall, but also by an expression relating A_t to the mean and turbulent velocity components. Unless one were to consider also the statistical theory of turbulence, this relation can only be obtained empirically. Its formulation for a specific flow regime, for example through the Prandtl theory of the mixing length,⁵ provides the basis of all necessarily semiempirical treatments of turbulent flow.

The boundary conditions to be satisfied by the mean velocity components are assumed to be the same as in ordinary laminar flow; that is, the mean velocity components must vanish at a solid wall (no-slip condition). Moreover, all turbulent velocity components must vanish at the walls; consequently, these components will be very small near the wall. Therefore, as can be seen from Eq. (A-5), all the components of the Reynolds stresses must vanish at a wall, and the only stresses that act near the wall are the viscous stresses of laminar flow.

As a further consequence, in the immediate vicinity of a wall, the turbulent (Reynolds) stresses will be small compared to the viscous stresses, and in every flow there exists a very thin layer next to the wall that behaves like a region of laminar flow. This layer is known as the laminar sublayer; in this layer the viscous forces dominate, as compared to the inertial forces, and no turbulence can exist in it. Proceeding from the wall into the fluid, a transitional layer adjoins the laminar sublayer.⁵ In the transitional layer, the velocity fluctuations are large enough to give rise to turbulent shearing stresses that are comparable in magnitude to the viscous stresses. At still larger distances from the wall, the turbulent stresses dominate the viscous stresses, and a turbulent boundary layer is formed.

The existence of these layers and the average flow characteristics in them

determine the actual residence time of the different elements of the fuel solution in the core region.

B. The Turbulent Mixing Coefficient in Reichardt's Approach

Various hypothetical models have been presented in the literature^{4-8, 14, 17} for the determination of the turbulent mixing coefficient A_t . Reichardt has developed a direct approach, choosing for A_t a simple mathematical function, compatible with the boundary conditions for the velocity components, and containing parameters which can be determined directly from the available experimental data. Furthermore, the whole boundary layer, including the laminar sublayer, is treated in a unified manner.

The basic equation for the total (viscous plus turbulent) stress, applicable throughout the whole boundary layer, is then

$$\tau = \tau_v + \tau_t = \mu \frac{d\bar{u}}{dy} + A_t \frac{d\bar{u}}{dy} \quad (4)$$

since $\bar{u} = u$ in the laminar sublayer. To obtain a dimensionless form for this equation, the friction velocity u_* , defined by

$$u_*^2 = \tau_0 / \rho, \quad (5)$$

where τ_0 is the value of the viscous shearing stress at the pipe wall, and a dimensionless length, η , defined in terms of the kinematic viscosity ν and u_* ,

$$\eta = (u_* / \nu) y \quad (6)$$

are then introduced, so that

$$\frac{\tau}{\tau_0} = \frac{\mu + A_t}{\rho u_*^2} \frac{d\bar{u}}{dy} = \left(1 + \frac{A_t}{\mu}\right) \frac{d(\bar{u}/u_*)}{d\eta} \quad (7)$$

since

$$\nu = \mu / \rho \quad (8)$$

Equation (8) shows that the dimensionless velocity, $\bar{u}/u_*$, depends only on the two parameters, τ/τ_0 and A_t/μ .

From the extensive sets of experiments by Nikuradse³⁰ and Reichardt¹⁸ one deduces that

$$(\tau/\tau_0) = 1 - (y/a), \quad (9)$$

where a is the pipe radius; for A/μ in the turbulent boundary layer, that is, not too close to the wall,

$$A/\mu = \kappa \eta = 0.4 \eta. \quad (10)$$

As the wall distance increases, Eq. (10) becomes more inaccurate. On the other hand, Eq. (10) is also not valid in the laminar sublayer where $A_t = 0$, because the flow is fully laminar.

A better approximation for small values of y is obtained by considering the velocity components of fluctuation. The velocity component $u = \bar{u} + u'$ must vanish at the wall, so that u' , like $\bar{u}$, increases at least linearly with y near the wall. As shown in Appendix D, one obtains from the Reynolds stress tensor, using also the boundary conditions and the equation of continuity, the stronger condition that near the wall A_t/μ is proportional to y^2 , or, under certain conditions, to y^4 .

Reichardt therefore postulates that A_t/μ be represented by a function $f(\eta)$ which, together with its first and second derivatives, vanishes at the wall ($y = 0$), and which asymptotically (far from the wall) approaches $\kappa \eta$, to be in agreement with the data of Nikuradse.³⁰ To represent A_t/μ , Reichardt chooses the function

$$A_t/\mu = f(\eta) = \kappa \left(\eta - \eta_1, \tanh \frac{\eta}{\eta_1} \right), \quad (11)$$

where κ and η_1 are determined so as to fit the experimental data of Nikuradse³⁰ and Reichardt.^{18, 19} The choice of

$$\kappa = 0.4, \quad \eta_1 = 11 \quad (12)$$

provides good agreement between the calculated and measured values of $\bar{u}/u_*$.

C. Reichardt's Representation of the Turbulent Velocity Distribution

From Eqs. (7), (11), and (12), one obtains the final expression for the radial distribution of the velocity component $\bar{u}(y)$ in the form

$$\begin{aligned} \bar{u}/u_* &= \int_0^{\eta'} d\eta' \frac{\tau/\tau_0}{1 + (A_t/\mu)} , \\ &= \int_0^{\eta} d\eta' \frac{1 - (\eta'/\eta_a)}{1 + 0.4[\eta' - 11 \tanh(\eta'/11)]} , \end{aligned} \quad (13)$$

where

$$\eta_a = (u_*/\nu)a \quad (14)$$

is the dimensionless value for the pipe radius.

Equation (13) has to be numerically integrated for given values of a . Reichardt has shown that one can obtain an approximate expression for the boundary layer region in the form

$$\begin{aligned} \bar{u}/u_* &= 2.5 \ln(1 + 0.4\eta) \\ &+ 7.8 \left[1 - e^{-\eta/11} - \frac{\eta}{11} e^{-0.33\eta} \right] . \end{aligned} \quad (15)$$

This "law of the wall" is shown in Fig. 1 in terms of the dimensionless variables $\bar{u}/u_*$ and η for the range of η values relevant for the applications considered here.

Figure 2 illustrates the transition of $\bar{u}/u_*$ occurring near $\eta = 7$ from laminar flow to turbulent flow and the approach of A_t/μ to its asymptotic value. For $\eta \approx 80$, $\bar{u}/u_*$ determined from Eq. (13) would approach the purely logarithmic form of the von Kármán law of the wall, discussed in Appendix D.

Fig. 1. Reichardt's law of the wall (Ref. 15).

The comparison of the semiempirical law of the wall of Reichardt with the experimental data of Nikuradse³⁰ and Reichardt¹⁸ is shown in Fig. 3.

The data of Reichardt¹⁸ were obtained by measurements in a rectangular channel 25 cm high, 100 cm wide, and 16 m long at a maximum flow speed of 80 cm/sec. The data apply also to flow in a pipe of circular cross section since $\bar{u}/u_*$ near the wall is independent of the channel geometry. As von Kármán formulated it, "the fluid near the wall does not know that another boundary exists;" the universal law of the wall, as shown in Fig. 1, is

Fig. 2. $\bar{u}/u_*$ and A_t/u in the transition region (after Reichardt, Ref. 15).

Fig. 3. Comparison of Reichardt's law of the wall with the measurements of Reichardt (Ref. 15) and Nikuradse (Ref. 30).

Curve 1: von Kármán's law of the wall

Curve 2: $\bar{u}/u_*$ for viscous flow

Curve 3: Reichardt's law of the wall

independent of a , as can be verified readily by comparing numerically the complete expression for $\bar{u}/u_*$ given in Eq. (13) with the approximate expression for $\bar{u}/u_*$, Eq. (15), which does not contain a .

Values of $A_t/(\mu\eta_r)$ outside the boundary layer have also been measured by Reichardt,¹⁰ who indicates that his measurements in the region of fully turbulent flow (near the center of the pipe) can be represented by

$$A_t/\mu\eta_r = \frac{\kappa}{3} \left[\frac{1}{2} + \left(\frac{r}{a} \right)^2 \right] \left[1 - \left(\frac{r}{a} \right)^2 \right], \quad (16)$$

where r is the radial distance from the center line of the pipe.

Introducing Eq. (16) into Eq. (13), and noting that

$$\eta = (u_* / \nu)(a - r),$$

one obtains for the central region of the pipe

$$\bar{u}/u_* = \frac{U_m}{u_*} - \frac{u_*}{\nu} \int_0^r dr' \frac{(r'/a)}{1 + \frac{\kappa}{3} \left[\frac{1}{2} + \left(\frac{r'}{a} \right)^2 \right] \left[1 - \left(\frac{r'}{a} \right)^2 \right] \eta_a}, \quad (17)$$

where U_m is the value $\bar{u}$ at the center of the pipe. Neglecting the first term in the denominator, which would give a correction of only a fraction of a percent, one obtains finally

$$\begin{aligned} \bar{u}/u_* = & \frac{U_m}{u_*} - \frac{1}{\kappa} \ln \frac{\frac{1}{2} + \left(\frac{r}{a} \right)^2}{1 - \left(\frac{r}{a} \right)^2} \\ & + \frac{1}{\kappa} \ln \frac{1}{2} \end{aligned} \quad (18)$$

for any center line distance r , except the region of boundary layer near the wall.

The semilogarithmic plot of Fig. 4 shows the good agreement of Reichardt's data with Eq. (18) up to values of $r = 0.95a$.

Since Eq. (17) approaches the general expression Eq. (13) in the turbulent boundary layer region, it is possible to formulate an equation for the velocity distribution which is valid for the whole pipe cross section in the form

Fig. 4. Comparison of Reichardt's data for the central region in pipe flow with $\bar{u}/u_*$ calculated from Eq. (18).

$$\bar{u}/u_* = \frac{1}{\kappa} \ln \left\{ (1 + \kappa\eta) \frac{1.5 \left(1 + \frac{\kappa}{a}\right)}{1 + 2 \left(\frac{\kappa}{a}\right)^2} \right\} + 7.8 \left\{ 1 - e^{-\eta/11} - \frac{\eta}{11} e^{-0.33\eta} \right\}, \quad (19)$$

where, from experiment, $\kappa = 0.4$, η is the dimensionless wall distance, r is the distance from the center, and a is the pipe radius. For $r \leq 0.9a$, Eq. (19) reduces to Eq. (17).

Equation (19) can be used to obtain a total velocity profile. In Fig. 5, the velocity profile obtained from Eq. (19) is shown for parameters representative for the KING reactor experiment. It should be noted that the sharp velocity drop starts at a distance of about $0.97a$ from the center. Thus, only in about 6% of the total cross-sectional area the flow rate is noticeably smaller than the average flow rate. The consequence of this fact for the exposure of the fuel solution is discussed in Secs. III and IV.

D. The Law of the Wall of van Driest

The model proposed by van Driest¹⁶ is basically different from Reichardt's approach. Van Driest starts with the expression for the turbulent stress, see Eq. (C-5) in Appendix C, obtained from the Prandtl mixing length theory, and postulates for the total shear stress near the wall

$$\tau = \tau_v + \tau_t = \mu \frac{d\bar{u}}{dy} + \rho l^2 \left(\frac{\partial \bar{u}}{\partial y} \right)^2. \quad (20)$$

Fig. 5. Velocity profile for representative KING reactor configuration.

The novel feature introduced by van Driest is his assumption for the mixing length, which he defines as

$$l = \kappa y \left[1 - e^{-y/A} \right], \quad (21)$$

where A and κ are constants determined from experimental data. The damping factor $1 - \exp(-y/A)$ is introduced to account for the viscous damping of the turbulent mixing near the wall.

This hypothesis has the advantage that for small y the mixing length, because of the presence of the damping factor, becomes proportional to y^2 , so that the turbulent stress near the wall becomes proportional to y^4 . As discussed above for the Reichardt approach, and in more detail in Appendix D, this provides for a better agreement with the experimental data in the laminar sublayer and in the transition layer than the Prandtl choice (κ proportional to y). In the turbulent boundary layer, however, the damping factor reduces approximately to unity so that

$$l \sim \kappa y,$$

in agreement with Prandtl's model, and, again, in good agreement with the experimental data,³⁰ if the value of 0.4 is chosen for κ .

Van Driest obtains the specific functional form of l by considering the analogy between turbulent mixing and the fluid motion near an infinite flat plate undergoing simple harmonic motion parallel to the plate in an infinite fluid. In the latter case, it was shown by Stokes³¹ that the amplitude of the fluid motion diminishes with increasing distance from the plate as $\exp(-y/A)$ when A is a constant depending on the frequency of oscillation of the plate and the kinematic viscosity of the fluid. If now the plate is considered as fixed and the fluid is assumed to oscillate relative to the plate (due to the turbulent fluctuations), the fluid oscillations are damped by a factor $[1 - \exp(-y/A)]$

due to the presence of the fixed plate, i.e., the wall. This assumption gives the correct exponential decrease of the damping effect as the distance from the plate increases.

Introducing the dimensionless variables η and $\bar{u}/u_*$, one obtains from Eqs. (20) and (21) the relation

$$\frac{\tau}{\tau_0} = \frac{d}{d\eta} \frac{\bar{u}}{u_*} + \kappa^2 \eta^2 \left[1 - e^{-\eta/A_*} \right]^2 \left(\frac{d}{d\eta} \frac{\bar{u}}{u_*} \right)^2, \quad (22)$$

where

$$A_* = \frac{u_*}{\nu} A. \quad (23)$$

From Eq. (22) one obtains immediately

$$\frac{d(\bar{u}/u_*)}{d\eta} = \frac{-1 + \sqrt{1 + \frac{4\tau}{\tau_0} \kappa^2 \eta^2 [1 - \exp(-\eta/A_*)]^2}}{2\kappa^2 \eta^2 [1 - \exp(-\eta/A_*)]^2}, \quad (24)$$

so that

$$\frac{\bar{u}}{u_*} = \int_0^\eta d\eta \, 2(\tau/\tau_0) \left\{ 1 + \sqrt{1 + 4(\tau/\tau_0) \kappa^2 \eta^2 [1 - \exp(-\eta/A_*)]^2} \right\}^{-1}. \quad (25)$$

Van Driest assumes, moreover, that $\tau = \tau_0$; this approximation can easily be replaced by Eq. (9) to obtain results also valid far from the wall.

With the choice of $\kappa = 0.4$ made, Eq. (25) contains only the one parameter, A_* .

Van Driest has compared this law of the wall with the experimental data of Laufer.¹⁹ For the choice of

$$A_* = 26, \quad (26)$$

the agreement between theory and experiment obtained by van Driest¹⁶ appears to be excellent, as shown in Fig. 6.

The data of Laufer were obtained by measurements on air flow at Reynolds numbers of 50,000 and 500,000. The test section was a 16-ft-long brass tube of inside diameter of 9.72 in.

Fig. 6. Semilogarithmic plot of velocity profiles for turbulent flow near smooth and rough walls. Comparison of theory and experiment for a smooth pipe. (Reproduced from van Driest (Ref. 16), *Journal of the Aeronautical Sciences* 23, 1008 (1956), by permission of AIAA).

To attain agreement with the results of Nikuradse,³⁰ the slightly different choice of $A_* = 27$ has to be made. Comparisons of the resulting distribution for $\bar{u}/u_*$ are shown in Fig. 7.

It is apparent that the variation with A_* is small. This also shows the

Reichardt velocity distribution, obtained by the exact integration of Eq. (13). It is apparent that the agreement with the result of van Driest, obtained by entirely different assumptions, is excellent.

Very recently, Faure, Dumas, and Verollet³² have reported measurements of velocity and average temperature profiles in a wind tunnel with heated walls. The test section appears to have a length of 3.75 m and a width of 80 cm. The wall temperature is 25.8°C below the temperature of the air in the tunnel. Mean velocities were measured with a total pressure tube of external diameter of 0.8 mm, and the mean temperature was measured with thermocouples of 0.1 mm diameter. The results for the appropriately scaled temperature and velocity measurements are shown in Fig. 8. The agreement with either van Driest's or Reichardt's model up to $\eta = 300$

Fig. 7. Comparison of the Reichardt and van Driest velocity distributions.
Curve 1: van Driest's law, with $A_* = 26$
Curve 2: van Driest's law, with $A_* = 27$
Curve 3: Reichardt's law

Fig. 8. Mean local velocity and average temperature profiles (after Faure et al., Ref. 32).

is excellent as far as $\bar{u}/u_*$ is concerned.

As discussed in Appendix D, the asymptotic relations valid for $\bar{u}/u_*$ in the laminar sublayer and in the fully turbulent region are readily obtained from Eq. (25), in full agreement with the results of Reichardt and others.

III. REPRESENTATIVE HYDRAULIC DATA FOR CIRCULATING FUEL REACTORS OF THE KING TYPE

A. Energy Deposition during Core Transit

The energy deposition $E(y)$ per unit volume in a volume element ΔV of the fuel solution during a transit through the reactor core region is proportional to the transit time $t_1(y)$ of a ΔV element through the core. For a wall distance y of ΔV and the average axial velocity $\bar{u}(y)$, one has in a first approximation

$$t_1(y) = H/\bar{u}(y) \quad , \quad (27)$$

where H is the active core height.

Therefore,

$$E(y) = \epsilon \int_0^H \Sigma_f^0 \frac{\rho(y,z)}{\rho_0} \varphi(y,z) \frac{dz}{\bar{u}(y)} \quad , \quad (28)$$

where $\epsilon = 3.2 \times 10^{-11}$ (joule/fission) is the energy deposition per fission, $\varphi(y,z)$ is the thermal flux, $\rho_f(y,z)$ is the local fuel density, and Σ_f^0 is the macroscopic fission cross section at density ρ_0 .

To estimate $E(y)$, it is necessary to use the appropriate values of $\bar{u}(y,z)$ that are given by the velocity distributions resulting from the law of the wall discussed in the previous sections. As illustrated in Figs. 6 and 7, the velocity $\bar{u}(y)$ depends only on the dimensionless wall distance η and the friction velocity u_* that can be expressed as

$$u_* = u_B \sqrt{\lambda/8} \quad , \quad (29)$$

where u_B is the mean flow velocity characterizing the bulk flow and λ is the dimensionless coefficient of resistance which relates the shear at the wall τ_0 to u_B :

$$\tau_0 = \frac{1}{8} \lambda \rho u_B^2 \quad (30)$$

For pipe flow it is well established⁶ that λ is a universal function of the Reynolds number κ

$$R = u_B d / \nu \quad (31)$$

where $\nu = \mu / \rho$ is the kinematic viscosity and d is the pipe diameter.

In a first approximation, the value of $E(y)$ will therefore depend, on the one hand, only on the design parameters d and u_B and the desired power level characterizing the neutron flux ϕ , and, on the other, on one fluid property, the kinematic viscosity ν , which in turn is a function of the fluid temperature T and density ρ . Parametric studies of $E(y)$ can therefore readily be performed, once the data for $\lambda(\kappa)$ and ν are available.

B. Coefficient of Resistance

The experimental data on λ have been summarized, for example, by Schlichting,⁵ and are shown in Fig. 9. For the application considered here, extremely high values of the Reynolds number are of interest. Prandtl^{3,3} derived a universal law of friction for flow in smooth pipes from the

Fig. 9. Experimental data and theoretical relations for the frictional resistance in smooth pipes (after Schlichting, Ref. 5).
Curve 1: Poiseuille law for laminar flow
Curve 2: Blasius law
Curve 3: λ from Prandtl's law of friction, Eq. (32).

asymptotically valid law of the wall of Eq. (D-6) in the form

$$\frac{1}{\sqrt{\lambda}} = 2.035 \log_{10} (\kappa \sqrt{\lambda}) - 0.91 \quad (32)$$

The values of λ obtained from this law are also shown as curve 3 in Fig. 9. The agreement between the theoretical values with the experimental data is excellent. It is also apparent that for large Reynolds numbers the experimental data diverge markedly from the often-used Blasius law

$$\lambda_{\text{Blasius}} = 0.3164 / \kappa^{1/4} \quad (33)$$

shown in curve 2 of Fig. 9.

For the applications considered here, the relevant Reynolds numbers are far larger than those for which experimental information is readily available.

Figure 10 and Table II show λ vs κ for the region of interest here, as calculated from Eq. (32).

C. Data for Viscosity μ and Kinematical Viscosity ν

The viscosity μ and density ρ are well known as functions of temperature for water and uranyl sulfate. The relevant data are summarized in Tables III and IV and Fig. 11. Figure 12 shows that the values of ν for water and uranyl sulfate, as well as their temperature dependency, are similar and that extrapolations from water to uranyl sulfate solutions should not create too much uncertainty. On the other hand, no information is available on the effect of the additive components (H_2SO_4 , $CuSO_4$) in the fuel solution on its viscosity. More important, no data are available to estimate the effect of density fluctuations on the kinematic viscosity. In the following, a parametric approach is used, in which it is assumed that μ is not affected by the bubble formation in the fuel, and that ν can be calculated from the approximation

Fig. 10. Dimensionless coefficient of resistance λ vs $\log_{10} R$ for large values of R .

TABLE II
PRANDTL'S LAW OF FRICTION

$\log_{10} R$	λ
5.000	0.01799
5.301	0.01564
5.477	0.01447
5.602	0.01371
5.699	0.01316
5.778	0.01274
5.845	0.01239
5.903	0.01210
5.954	0.01186
6.000	0.01165
6.301	0.01037
6.477	0.009722
6.602	0.009295
6.699	0.008982
6.778	0.008738
6.845	0.008539
6.903	0.008372
6.954	0.008229
7.000	0.008104
7.176	0.007647
7.301	0.007345
7.398	0.007123
7.477	0.006949
7.544	0.006806
7.602	0.006686
7.653	0.006583
7.699	0.006493

R = Reynold's number

λ = Dimensionless coefficient of resistance

TABLE III
SOME PROPERTIES OF WATER AT SATURATION PRESSURES

Temperature °C	°F	Specific Weight (lb/ft ³)	Absolute Viscosity (cP)	Kinematic Viscosity (cm ² /sec)
0	32	62.418	1.792	0.01793
	35	62.422	1.692	0.01693
4	39.2	62.427	1.567	0.01567
	40	62.426	1.546	0.01546
5		62.425	1.519	0.01519
	45	62.420	1.419	0.01419
10	50	62.40	1.308	0.01308
	55	62.38	1.210	0.01211
15		62.37	1.140	0.01141
	60	62.35	1.124	0.01130
	65	62.34	1.047	0.01050
20		62.30	1.005	0.01010
20.2	68.4	62.30	1.000	0.01000
	70	62.29	0.9785	0.009985
	75	62.25	0.9166	0.009182
25		62.24	0.8937	0.008959
	80	62.20	0.8609	0.008638
	85	62.16	0.8062	0.008065
30		62.15	0.8007	0.008018
	90	62.11	0.7644	0.007677
35		62.05	0.7225	0.007267
	100	62.00	0.6843	0.006884
40		61.94	0.6560	0.006608
	105	61.92	0.6439	0.006487
	110	61.84	0.6160	0.006208
45		61.82	0.5988	0.006041
	115	61.80	0.5871	0.005920
	120	61.73	0.5599	0.005660
50		61.68	0.5494	0.005558
	125	61.65	0.5345	0.005409
	130	61.54	0.5155	0.005223
55		61.53	0.5064	0.005130
	135	61.46	0.4890	0.004963
60		61.38	0.4688	0.004768
	145	61.27	0.4498	0.00458
65		61.22	0.4355	0.00444
	150	61.20	0.4300	0.00438
	155	61.09	0.4255	0.00435
70		61.04	0.4061	0.00415
	160	61.01	0.4000	0.00409
	165	60.90	0.3855	0.00396
75		60.86	0.3799	0.00390
	170	60.79	0.3713	0.00382
	175	60.68	0.3690	0.00368
80		60.66	0.3545	0.00366
	180	60.57	0.3469	0.00357
85		60.46	0.3355	0.00346
	185	60.35	0.3247	0.00337
	190	60.31	0.3165	0.00328
90		60.24	0.3146	0.00326
	195	60.13	0.3049	0.00317
	200	60.00	0.2955	0.00308
100		59.83	0.2838	0.00296
105		59.77	0.270	0.00282
110		59.38	0.250	0.00263
120		58.86	0.232	0.00246
130		58.34	0.212	0.00227
140		57.80	0.196	0.00216
150		57.24	0.184	0.00201
160		56.66	0.174	0.00191

λ (cP) = 0.01 dyne/cm²

Source: Ref. 20.

$$\nu(T_1, \rho) = \frac{\mu(T)}{\rho} \left(\frac{\bar{\rho}}{\rho} \right), \quad (34)$$

i.e., the viscosity μ , as measured at the reference density $\bar{\rho}$ can be used, and that ν is changed in the ratio $(\bar{\rho}/\rho)$, where ρ is the actual fuel fluid density due to the presence of the bubbles.

D. Hydraulic Parameters for the Fuel Solution

The data summarized in the previous two sections have been used to calculate

TABLE IV
DENSITY AND VISCOSITY OF LIGHT WATER SOLUTIONS OF URANYL SULFATE

Temp. (°C)	Density			Viscosity				
	g U/liter of Solution			g U/liter of Solution				
	60.6 (g/cm ³)	101.0 (g/cm ³)	20.2 (cP)	59.7 (cP)	100.4 (cP)	201.4 (cP)	301.1 (cP)	400.7 (cP)
30	--	1.1340	0.857	0.946	1.04	1.40	1.97	3.04
45	1.0715	1.1275	0.629	0.701	0.773	1.05	1.39	1.98
60	1.0649	1.1199	0.492	0.536	0.605	0.820	1.06	1.41
75	1.0562	1.1111	0.403	0.436	0.483	0.638	0.821	1.10
90	1.0468	1.1008	0.329	0.365	0.399	0.517	0.675	0.884
100	1.0395	1.0952	0.292	0.327	0.351	0.454	0.589	0.767
125	1.0203	1.0742	0.238	0.264	0.273	0.347	0.442	0.543
150	0.9984	1.0528	0.194	0.213	0.223	0.275	0.335	0.437
175	0.9735	1.0293	0.164	0.178	0.188	0.226	0.269	0.360
200	0.9460	1.0030	0.142	0.153	0.158	0.192	0.230	0.298
225	0.9156	0.9745	--	0.136	0.141	--	0.209	0.264
250	--	0.9440	--	0.125	0.130	--	0.190	0.238
325	0.7725*	0.8450*	0.086*	0.091*	0.096*	0.109*	0.139*	0.174*

*Extrapolated values

Source: Ref. 21.

the friction velocity u_* and the length scaling factor ν/u_* as functions of temperature $T(^{\circ}\text{C})$ and density. Table V shows the values of u_* and ν/u_* under the assumption

Fig. 11. Viscosity of light water solutions of uranyl sulfate (after Barnett et al., Ref. 21).

that ν is calculated as if the solution had the viscosity and density of water. To account for the bubble formation, the parametric change described by Eq. (34) has been made. The most striking result is that the effect of temperature changes appears to be more important than that of a density change. Furthermore, for a temperature change from 20 to 100°C, u_* varies but little (at most by 10%), and the length scaling factor ν/u_* changes by a factor of three. A parametric density change of 20%, on the other hand, practically does not affect the value of u_* , and the length scaling factor varies only by about 12%. Thus,

Fig. 12. Comparison of kinematic viscosity of water and uranyl sulfate.

TABLE V
HYDRAULIC PARAMETERS AS FUNCTIONS OF TEMPERATURE AND DENSITY

T°C	μ (cP)	ρ (g/cm ³)	ν (cm ² /sec)	R	λ	$u_0 = U_p/\sqrt{\lambda/8}$	$\tau_0 = \mu u_0$	ν/u_0 (cm)
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
0	1.793	1.0	1.793×10^{-3}	0.39×10^6	0.0137	22.76	518.0	7.88×10^{-4}
20	1.005	0.952	1.010×10^{-3}	0.69×10^6	0.0124	21.65	468.0	4.67×10^{-4}
50	0.5494	0.988	0.556×10^{-3}	1.26×10^6	0.0108	20.21	403.5	2.75×10^{-4}
80	0.3565	0.972	0.366×10^{-3}	1.91×10^6	0.0104	19.83	382.0	1.85×10^{-4}
100	0.2838	0.958	0.296×10^{-3}	2.36×10^6	0.0102	19.64	369.5	1.51×10^{-4}
Density and Kinematic Viscosity Modified by Reduction of Density to $\rho' = 0.97$								
20		0.898	0.909×10^{-3}	0.77×10^6	0.0122	21.48	414.0	4.23×10^{-4}
50		0.889	0.500×10^{-3}	1.40×10^6	0.0107	20.11	359.0	2.49×10^{-4}
80		0.875	0.329×10^{-3}	2.13×10^6	0.0103	19.73	340.5	1.67×10^{-4}
100		0.862	0.266×10^{-3}	2.63×10^6	0.0099	19.35	323.0	1.37×10^{-4}
Density and Kinematic Viscosity Modified by Reduction of Density to $\rho' = 0.85$								
20		0.798	0.808×10^{-3}	0.87×10^6	0.0119	21.21	359.0	3.81×10^{-4}
50		0.790	0.445×10^{-3}	1.57×10^6	0.0106	20.02	316.5	2.22×10^{-4}
80		0.778	0.293×10^{-3}	2.29×10^6	0.0101	19.54	297.0	1.50×10^{-4}
100		0.766	0.237×10^{-3}	2.95×10^6	0.0098	19.25	284.0	1.23×10^{-4}

1 (cP) = 0.01 dyne sec/cm²

Cols. (1) through (4) after V. L. Streeter, Ref. 20.

μ : absolute viscosity in centipoise
(or 0.01 dyne/sec/cm²)
 ρ : density of fuel solution
 $\nu = \mu/\rho$: kinematic viscosity, $\frac{\text{cm}^2}{\text{sec}}$
R: Reynolds number = $\frac{2r u_0 \rho}{\mu}$
a: pipe radius
 U_0 : 550 cm/sec average bulk velocity of fluid
 u_0 : friction velocity
 τ_0 : shear stress at wall
 ν/u_0 : scale factor from dimensionless units η to true wall distance

it appears that the temperature change affecting the viscosity during the core transit will influence the flow characteristics more than the density change, provided that one may trust the simple model described by Eq. (34).

Table VI shows the corresponding results for a uranyl sulfate solution containing about 60 g of uranium per liter of solution. Between 30 and 150°C, the change in u_0 is of the order of 12% and the length scaling factor ν/u_0 changes by a factor 3.6 to 3.7. For a parametric change of density of 20%, the change in u_0 is again small, and the change of u_0/ν is of the order of 20%; that is, larger than water, but still small compared to the effect of temperature variations.

E. Representative Fuel Solution Experiments

Results from interrelated neutronic-hydrodynamic calculations are needed to obtain the energy deposition $E(y)$ from Eq. (28): the local flux $\varphi(y,z)$ depends on

the density $\rho(y,z)$, the form of which in turn depends on $\varphi(y,z)$ and $\bar{u}(y)$. To obtain preliminary estimates for $E(y)$, it appears suitable to average over the axial variable so that

$$E(y) \approx \epsilon \Sigma_f^0 \frac{\bar{\rho}(y)}{\rho_0} \bar{\varphi}(y) \frac{H}{\bar{u}(y)}, \quad (35)$$

where $\bar{\rho}(y)$ and $\bar{\varphi}(y)$ represent suitable axial average values.

For a parametric survey, it seems appropriate to choose the extreme values $\bar{\rho}(y) = 0.8$, valid near core exit, and $\bar{\rho}(y) = 1.0$, which applies at the core entrance of the flow.

For the mean flow condition, i.e., average energy deposition, assuming bulk flow parameters, Eq. (35) reduces to

$$E_B = \epsilon \Sigma_f^0 \bar{\varphi} \frac{\bar{\rho}}{\rho_0} \frac{H}{u_B}, \quad (36)$$

where E_B is the energy deposition for mean flow conditions, $\bar{\rho}$ is the average density in the core, $\bar{\varphi}$ the average design flux,

TABLE VI
HYDRAULIC PARAMETERS AS FUNCTIONS OF TEMPERATURE AND DENSITY FOR URANYL SULFATE SOLUTIONS

T°C	μ (cP)	ρ (g/cm ³)	ν (cm ² /sec)	κ	λ	$u_* = \bar{u}_B \sqrt{\lambda/8}$	$\tau_o = \rho u_*^2$	ν/u_* (cm)
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
30	0.946	1.0845*	0.00872	8.03×10^6	121×10^{-4}	21.39	496.2	4.08×10^{-4}
45	0.701	1.0715	0.00654	1.07×10^6	114×10^{-4}	20.76	461.9	3.15×10^{-4}
60	0.536	1.0649	0.00503	1.39×10^6	110×10^{-4}	20.39	442.9	2.47×10^{-4}
75	0.436	1.0562	0.00413	1.70×10^6	107×10^{-4}	20.11	427.3	2.05×10^{-4}
90	0.365	1.0468	0.00349	2.01×10^6	104×10^{-4}	19.83	411.7	1.76×10^{-4}
100	0.327	1.0395	0.00315	2.22×10^6	103×10^{-4}	19.73	404.9	1.60×10^{-4}
125	0.264	1.0203	0.00259	2.70×10^6	100×10^{-4}	19.45	385.8	1.33×10^{-4}
150	0.213	0.9984	0.00213	3.29×10^6	96×10^{-4}	19.05	362.4	1.12×10^{-4}

Density and Kinematic Viscosity Modified by Reduction of Density to $\rho' = 0.8\rho$

30	0.868	0.0109	6.42×10^6	123×10^{-4}	21.47	403.7	5.05×10^{-4}
45	0.857	0.00818	8.56×10^6	119×10^{-4}	21.21	385.6	3.86×10^{-4}
60	0.852	0.00629	1.11×10^6	114×10^{-4}	20.76	367.3	3.03×10^{-4}
75	0.845	0.00516	1.36×10^6	111×10^{-4}	20.49	354.7	2.52×10^{-4}
90	0.837	0.00436	1.61×10^6	108×10^{-4}	20.21	341.8	2.16×10^{-4}
100	0.832	0.00394	1.78×10^6	106×10^{-4}	20.02	333.5	1.97×10^{-4}
125	0.816	0.00324	2.16×10^6	103×10^{-4}	19.73	317.8	1.64×10^{-4}
150	0.799	0.00266	2.63×10^6	100×10^{-4}	19.45	302.1	1.37×10^{-4}

Cols. (1) through (4) after Barnett et al., Ref. 21.

μ : absolute viscosity in centipoise
(or 0.01 dyne/sec/cm²)
 ρ : density of fuel solution
 $\nu = \mu/\rho$: kinematic viscosity
 R : Reynolds number = $\frac{2r\bar{u}_B}{\nu}$
 a : pipe radius
 $\bar{u}_B$: 550 cm/sec average bulk velocity of fluid
 u_* : friction velocity
 τ_o : shear stress at wall
 ν/u_* : scale factor from dimensionless units η to true wall distance

and u_B the design velocity for bulk flow. For E_B , one can choose representative design values; e.g.,

$$E_B = 30 \text{ joule/cm}^3 \quad \text{or} \quad E_B = 60 \text{ joule/cm}^3.$$

Combining Eqs. (35) and (36), one obtains

$$\frac{E(y)}{E_B} = \left(\frac{\tilde{\rho}(y)}{\bar{\rho}} \right) \left(\frac{\tilde{\Phi}(y)}{\bar{\Phi}} \right) \left(\frac{u_B}{\bar{u}(y)} \right) \quad (37)$$

This simple relation expresses the local energy deposition at wall distance y in terms of the radial density variation ($\tilde{\rho}(y)/\bar{\rho}$), the radial flux enhancement factor ($f = \tilde{\Phi}(y)/\bar{\Phi}$), and the ratio of the bulk flow velocity u_B to the local mean velocity in the axial direction $\bar{u}(y)$ at wall distance y . The local energy deposition depends on the fluid temperature only implicitly; that is, through the parametric dependency of $\bar{u}(y)$ on the kinematic viscosity which in turn is a function of the

temperature. $\bar{u}(y)$ also depends on the density, as shown in Tables V and VI.

Figures 13a through 13d illustrate the behavior of the energy deposition $E(y)$ as a function of temperature for various values of flux enhancement. The left ordinate scale in these figures applies for a bulk energy deposition of $E_B = 30 \text{ J/cm}^3$, whereas the right-hand scale applies for $E_B = 60 \text{ J/cm}^3$. For other values of E_B , one obtains $E(y)$ by simple rescaling of the ordinate. The dependence on temperature in the boundary layer region is apparent from Figs. 13a through 13c. The energy deposition becomes very large for very small values of y . Figure 13d shows that the influence of the temperature variation decreases with increasing distance from the wall. It is interesting to note that an increase of 25 to 30% over the bulk energy deposition rate persists up to 0.25 cm into the fluid; i.e., over about 8% of the total core

Fig. 13. Energy deposition $E(y)$ as function of position and temperature

volume. At very small distances from the wall, $E(y)$ may be 5 to 10 times larger than E_B .

Figures 14a through 14c illustrate the case of reduced density ($\rho = 0.8$). The density variation enters $E(y)$ not only through the factor $\tilde{\rho}/\rho$, but also through the density dependence of $\bar{u}(y)$, shown in Tables V and VI. Comparison of Figs. 13 and 14 shows that this effect, although present, does not affect the qualitative aspects of the results for $E(y)$.

In a realistic situation, the fluid temperature will increase during the core transit. At least for the KING experiment, and presumably also for a large-scale

device, this temperature rise is modest (10 to 20°C) so that an appropriate average value and the corresponding curve can be chosen to indicate the order of magnitude of the energy deposition.

In calculating $E(y)$, it has been assumed that a fluid volume ΔV moves in the axial direction with the speed $\bar{u}(y)$. Obviously, the volume ΔV will also move in the radial and tangential directions due to the turbulence. Thus the effective value of $\bar{u}(y)$ will change during the core transit of ΔV . No direct experimental evidence appears to be available to estimate the magnitude of this effect.

Some evidence is, however, available

Fig. 14. Energy deposition $E(y)$ as function of position and temperature at reduced density.

from the experiments of Laufer,¹⁵ the analysis of Schubauer,^{3,4} and, indirectly, from the statistical theory of turbulence.

Laufer has measured the intensity of turbulent fluctuations, defined by

$$\hat{u} \equiv \sqrt{u'^2} \quad (\text{axial direction}) ,$$

$$\hat{v} \equiv \sqrt{v'^2} \quad (\text{radial direction}) ,$$

$$\hat{w} \equiv \sqrt{w'^2} \quad (\text{tangential direction}) .$$

The results of Laufer¹⁹ are shown in Figs. 15a through 15c.

From Tables V and VI one obtains

$$u_{*}/u_{\text{Bulk}} \approx 0.035 \text{ to } 0.04$$

so that the values of $\hat{u}$ will be at most of the order of 60 cm/sec. Since the sign of u' also fluctuates, this sizable fluctuation of velocity in the axial direction may not affect the results shown in Figs. 13 and 14. On the other hand, rms values $\hat{v}$ of the radial velocity fluctuations are much smaller:

$$\hat{v} < 1.2 u_{*} \sim 24 \text{ cm/sec.}$$

In the region of interest (wall distance $y < .03a$), $\hat{v}$ approaches zero rapidly, and $\hat{v} \lesssim 10 \text{ cm/sec.}$

The value of $\hat{v}$ alone, however, gives no indication of the radial range covered by a fluid volume ΔV during the core transit. To obtain such an estimate, data on the average size of the turbulent eddies are needed.

Data on the spatial structure of turbulence can be obtained by the simultaneous observation of the velocity fluctuations at two neighboring points $\vec{x}_1$ and $\vec{x}_2$. The (double) correlation function R defined by

$$R_{ij} = \frac{\overline{u'_i(\vec{x}_1)u'_j(\vec{x}_1 + \vec{r}')}}{\hat{u}_i(\vec{x}_1)\hat{u}_j(\vec{x}_1 + \vec{r}')} \quad (38)$$

(a) $\hat{u}$ distribution

(b) $\hat{u}$ distribution near wall

(c) $\hat{v}$ and $\hat{w}$ distributions

Fig. 15. Intensity of fluctuations as function of radius (after Laufer, Ref. 19).

was introduced by Taylor.³⁶ As r' is increased, the correlation function decreases very rapidly, and it even assumes negative values, as shown in Fig. 16. The integral of the correlation function

$$L = \int_0^r R(r') dr' \quad (39)$$

defines a characteristic length for the structure of turbulence in the flow. This length, also known as the scale of

Fig. 16. Correlation between velocity at center and velocity at radius r . (Reproduced from The Scientific Papers of Sir Geoffrey Ingram Taylor, T. K. Batchelor (Ed.), Vol. II, page 376, by permission of Cambridge University Press.)

turbulence,⁵ gives a measure of the extent of the mass which moves as a unit, and thus gives an indication of the average size of the turbulent eddies, or of the "lumps of liquid" in Prandtl's theory of the mixing length.⁴ The correlation function R for the turbulent fluctuations of the longitudinal (axial) velocity component u' , measured at the center of the pipe³⁶ and at a radial distance r' from the center, is shown in Fig. 16. From $R(r)$ one obtains for the characteristic length L defined in Eq. (39) the value

$$L \approx 0.14 \left(\frac{1}{2} d \right), \quad (40)$$

where d is the pipe diameter.

Thus, for a pipe radius of 6.35 cm, one finds

$$L \sim 0.45 \text{ cm}.$$

Clearly, the fluid lumps near the wall will be smaller, as can be seen from Prandtl's assumption that near the wall the mixing length λ is approximately

$$\lambda \sim 0.4y \quad (y \text{ wall distance}).$$

Furthermore, Schubauer³⁴ shows that the

analysis of the data of Laufer¹⁸ and Klebanoff³⁷ indicates small spatial scales of turbulence in the y and z directions (i.e., normal to the direction of bulk flow), and this conveys the picture of u (bulk) currents running fiberlike in the x (flow) direction.¹⁹

Thus it appears that in a first approximation the values for E(y) of Figs. 13 and 14 should provide an estimate for the actual fuel exposures.

A more detailed study would have to include an analysis of statistical turbulence theory predictions, which would go far beyond the intent of this review, which is focused on the phenomenological approach and on the more immediate extrapolation from available experimental data.

A review of the statistical theory and of the relevant experimental techniques (hot-wire anemometry) has recently been given by Hinze,⁹ and in particular, at the 1961 conference on turbulence at Marseille.¹⁰ More recent measurements by Faure and his collaborators³⁸ of the space-time correlation of velocity fluctuations could also be useful for more detailed studies of the fluid behavior near the wall.

F. The Critical Region: Residence Times and Dimensions

The graphs representing E(y) of Figs. 13 and 14 show that within a very thin layer near the wall the energy deposition may become quite large. These energy deposition values may be compared with known threshold values for fuel solution decomposition. From static measurements, Thamer² deduces a safe limit of exposure of 70 joule/cm³, while on the other hand, the KEWB experiments³ indicate that single bursts of about 2 msec length do not lead to fuel decomposition, even though the energy decomposition during the bursts is of the order of 580 joule/cm³.

Obviously, two entirely different time intervals are relevant here:

1. The actual exposure time, determining the energy deposition.

2. The available "healing time" for the solution between exposures.

It is interesting to note that Thamer² apparently indicates a safe recycle time of 8 h for complete reconstitution of the fuel solution for the KING reactor dynamic critical experiment. On the other hand, the report on the KEWB experiments³ does not indicate the repetition rate of the burst experiments. Thus, the problem of the required healing time is open at present. Moreover, for the KING reactor experiment, and even more so for full-scale devices, the effects of turbulent mixing after the fuel passes the core become important. The actual configuration of deflectionors and baffles may ensure an intimate mixing of highly exposed fuel to the main bulk flow encountering a much lower, and safer, exposure. In such a situation the healing time for the total solution may well be reduced very much, as compared to that found from static experiments with uniformly exposed small samples.

The actual exposure time of elements of the fuel solution is of independent interest. The design value for the average exposure time in one core transit corresponding to the mean bulk flow at 550 cm/sec is between 0.11 and 0.13 sec for the KING reactor dynamic critical experiment. It is thus about 60 times larger than the exposure during a KEWB burst; on the other hand, the design value for the average fuel energy deposition of 34 to 47 joule/cm³ is far below the safe limit of 70 joule/cm³ obtained from static measurements.⁹

The estimated residence times at the limit between the viscous and the viscous-turbulent boundary layer are, however, larger by about a factor 6 than the average core residence times, ranging from 0.64 to 0.75 sec; that is, about 300 to 400 times those encountered in the KEWB experiments. It may therefore be necessary to consider the static measurement data on permissible energy deposition as representative for safe operation for those fuel elements which are

within the boundary layer.

Figures 13 and 14 show that under any condition of temperature or average flow velocity, even at $u_B \approx 5500$ cm/sec, an extremely thin layer of fluid will encounter an energy deposition which is larger than 70 joule/cm²; i.e., that a situation can arise which will lead to corrosion of the core pipe in a full-scale device. From these figures it is also apparent that under favorable circumstances the relevant fluid layer may have a thickness of only 0.01 cm.

On the other hand, some fluid may be trapped for even longer times and statistical fluctuations may occur, so that parts of a much larger fluid volume may be exposed to large energy depositions.

These open questions are closely related to the problems of the detailed structure of the wall flow. Kline and Runstadler³⁰ have investigated the wall experimentally by dye injection experiments. As described by Rotta,¹⁷ these experiments show that when dye is injected very near to the wall into a turbulent boundary layer, the pattern of flow appears to consist of an array of "islands of hesitation" and longitudinal vortices, interspersed with areas of faster moving fluid. The islands of hesitation appear as long filaments stretched in the direction of the flow which move downstream more slowly than the surrounding fluid. The vortices apparently originate as a breakdown along the edges of the islands of hesitation. The primary orientation of the vortex elements is in the flow direction, but at a slight angle to the wall, so that its distance from the wall increases as it moves downstream. After the vortex element reaches a certain critical distance from the wall, it breaks up into turbulent "hash." Figure 17 shows schematically this behavior of the flow. It is interesting to note that this qualitative picture is in agreement with the deductions of Schubauer³⁴ who started from a quite different basis.

Fig. 17. Diagrammatic sketch of flow model in wall layers of fully developed turbulent boundary layer. (Reproduced from Transactions of the ASME, Series E, Journal of Applied Mechanics, 26, 167 (1959), Kline and Runstadler.)

No quantitative data on this effect appear to be available, but the islands of hesitation could conceivably increase the estimated energy depositions obtained from the average radial velocity distribution, based on the approaches of Reichardt or van Driest. In this context, the model of Einstein and Li³³ may provide some guidance. In their approach, the flow, even in the laminar sublayer, is considered as an inherently unsteady process, in which the laminar sublayer alternately builds up and is destroyed. The model assumes that an unsteady laminar (or nearly laminar) flow exists, bounded on one side by the wall and on the other by the turbulent flow. The nature of this boundary is not specified. One assumes that near the wall a viscosity-controlled sublayer builds up in thickness; after a certain growth the layer separates, moves into the transition region, and fluid of high velocity is transported from the turbulent region to the wall where the shear stresses slow it down. This cycle would repeat itself, over and over again. In this model, the boundary layer equation for the sublayer reduces to

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}, \quad (41)$$

where u is the instantaneous velocity and y the wall distance. Equation (41) can be integrated immediately, yielding

$$u(y,t) = \frac{2u_0}{\sqrt{\pi}} \int_0^{y/\delta\sqrt{\nu t}} \exp(-\alpha^2) d\alpha, \quad (42)$$

where u_0 is the velocity at $t = 0$.

Einstein and Li^{23,24} assume that the process of growth of the boundary layer continues over a time interval $t = 0$ to $t = T$, and that the duration of the decay is negligible due to the effectiveness of turbulent mixing. With these assumptions, the mean velocity distribution as a function of y , the shear stress at the wall, and the root mean square of the axial velocity fluctuations can be calculated by integrating the instantaneous values obtained from Eq. (41) over time. It is interesting to note that the results depend on only one parameter, namely (Tu_0^2/ν) .

Figures 18 and 19 show the result of Einstein and Li for the dimensionless average velocity distribution $\bar{u}/u_*$. The agreement of approach with the data of Laufer¹⁹ and with the results of van Driest¹⁸ and

Fig. 18. Mean-velocity distribution near wall (after Einstein and Li, Ref. 23).

Fig. 19. Root mean square velocity fluctuation in direction of flow near wall (after Einstein and Li, Ref. 23).

Reichardt,¹⁸ as shown in Fig. 7, is striking.

The agreement for velocity fluctuation distribution with the data of Laufer appears to be good for $\eta < 15$, but poor for $\eta > 15$. This can be understood as indicating that the model of Einstein and Li ceases to be valid outside the immediate neighborhood of the wall.

For the time T necessary to accomplish a cycle of growth and decay of the laminar layer, Einstein and Li, assuming that the theoretical velocity v_0 at the limit of the laminar region is

$$v_0 = 11.6 u_*, \quad (43)$$

obtain the value

$$T = \frac{\nu}{u_*^2} \left(\frac{v_0}{u_*} \right)^2 = \frac{11.6^2 \nu}{u_*^2}, \quad (44)$$

where v_0 is found as the point where the laminar flow velocity distribution and the extrapolated logarithmic law of the wall velocity would intersect.

Einstein and Li^{23,24} report measurements on flow of oil in a pipe performed in a 4.37-cm-diam Lucite tube with water flowing at a bulk speed of 226 cm/sec.

The kinematic viscosity quoted is 0.0995, corresponding to a Reynolds number of 9.9×10^4 . With these data, Einstein and Li report a cycle time of $T = 0.011$ sec and a thickness of the viscous sublayer of 0.01 cm, which agree closely with the estimates obtained previously from Reichardt's model for the KING reactor dynamic experiment.

It appears, therefore, that even under the assumption that the laminar sublayer is inherently unsteady, one would obtain regions, similar to those shown in Figs. 13 and 14, where the energy deposition may exceed safe limits.

IV. IMPLICATIONS FOR THE KING REACTOR EXPERIMENT AND FULL-SCALE DEVICES

It is apparent from Figs. 13a through c and Figs. 14a through c that there will

always exist a region where the energy deposition will locally exceed a permissible limit. To evaluate this effect as well as others pertinent to the reactor performance, one has to account for the roles of the various characteristic times for the system:

a. The core residence time $T_0 = v_B/H$ for fuel traveling with or near the bulk flow speed.

b. The core residence time T_B for fuel moving axially through the boundary layer region.

c. The recycle time T_c ; i.e., the time before a given fuel solution element reenters the core.

d. The mixing time T_m for thorough mixing of the fuel solution after core passage.

The mixing time will depend on the mechanical design of conduits and curved pipes leading from the core to the fuel pump and of baffles inserted to accelerate mixing. For rapid and thorough mixing, the healing time for the fuel solution may be considerably reduced compared to the values obtained from static experiments² with small, uniformly exposed, fuel samples. Quantitative predictions of this effect are not possible because T_m depends on design features and because no experimental data are available for the healing time of a heavily exposed fuel sample which, immediately after exposure, is mixed intimately with a large quantity of fuel solution that had only an exposure compatible with the static experiment limit.

In the KING reactor experiment, the fuel solution will have a relatively long resting time between short-duration runs (perhaps 10 to 100 sec) at full reactor power. One should therefore anticipate no problem of fuel decomposition. For future applications to full-scale devices, it would be very desirable to make measurements of exposed fuel samples, mixed to normal solution, after a power run to further investigate healing times after

mixture is accomplished. After completion of the dynamic tests, it may also be desirable to examine the inner surface of the zirconium core tube of the KING reactor for any indications of traces of corrosion or microscopic deposition of decomposed fuel.

In a full-scale device, the problem of boundary layer effects can conceivably be avoided by one of the following approaches:

a. The core tube surfaces exposed to fuel solution could be artificially roughened to break up the boundary layer, or to reduce the laminar and transition layer to still smaller dimensions. Early measurements by Nikuradse^{4,5} show that the velocity gradient near a rough wall is less steep than that near a smooth wall. Although extensive experimental results for the turbulent boundary layer are available, the transition layer and laminar sublayer have been less fully investigated because the effects encountered there depend on the details of the machining or preparation of the wall. Nikuradse, for example, used circular pipes covered on the inside as tightly as possible with sand of definite grain size glued to the wall. For roughened metal surfaces, extrapolation of available results may be more difficult; more important, artificial roughening of the core tube in a full-scale device may introduce additional corrosion or deposition problems.

b. The problem presented by the laminar sublayer and the transition layer could conceivably be avoided altogether by using an orificed jet, as indicated schematically in Fig. 20, so that the fuel solution is not in contact with the core tube during fuel solution transit through the core. Thus the entire fuel solution would be exposed effectively at a rate corresponding to the bulk flow velocity.

In view of the density change of the fuel solution during core transit, the orifice should have a smaller diameter than the core tube. Assuming that the entire

Fig. 20. Schematic of fuel solution flow in orificed core pipe

expansion of the fuel solution is radial, and not also axial, an air gap of about 10% of the entrance radius of the fuel jet appears indicated. For a full-scale device, the spreading of a jet after an orifice will have to be investigated in detail. Based on the mixing length model, the behavior of turbulent jets has been investigated in detail^{41,42} and experimental data on the velocity distribution have also been obtained.^{43,44} An inductive theory based on experimental information has been developed by Reichardt.^{44,45} The novel feature in need of investigation for the application considered here is introduced by the prescribed density variations of the fuel solution jet during core transit. No experimental or theoretical work on the spreading of a jet under such conditions appears to exist. Moreover, the fact that the jet is free and bounded by a medium (air) of entirely different density and kinematic viscosity may introduce further problems for the theoretical analysis of this novel situation. The estimate of the air gap given above is therefore entirely conjectural, based only on the density change and

excluding the effects of the spreading of the jet; on the other hand, the phenomenon of the vena contracta may by itself reduce the required size for the air gap.

The orifice could be designed so that there is full separation of the fuel solution from the boundary, as was shown in a classic paper by Prandtl,⁴⁶ and illustrated in Fig. 21.

In a full-scale device, the xenon poisoning may become of importance in limiting maximum power attainable, because for economic reasons the amount of fuel in circulation will have to be limited, and a recycle time short compared to the ^{135}Xe lifetime will be necessary. Unless ^{135}Xe or its precursors are removed before the fuel is recirculated, as anticipated in the HRE and LMFR designs, the xenon poisoning may limit the steady-state thermal flux to the neighborhood of 10^{16} neutrons/cm²-sec, as indicated for thermal reactors, for example, by Isbin.⁴⁷

The use of an orificed jet may conceivably also permit attainment of very high pulsed fluxes in a full-scale device, for example, by using rapidly rotating control drums in the beryllium reflector. If the pulse length were short compared to the core passage time, one could surmise that the total fuel exposure would remain within permissible limits, because the outer part of the jet may have an overexposure of about 20% more than the bulk flow exposure, corresponding to the estimated increase in fission density near the reflector, as shown by preliminary calculations.

The orificing could, of course, also be used in connection with a central irradiation island by having an orifice that produces a hollow core jet; that is, a fuel solution shaped as a cylindrical shell.

An alternative approach indicated by King⁴⁸ to overcome the boundary layer effect is based on the injection of water from recombination of radiolytic gas along the core tube walls to form a protective layer not containing fissile material.

Fig. 21. Flow through orifice with separation of flow from wall. (Reproduced from Prandtl, *Gesammelte Abhandlungen*, Vol. II, page 769, by permission of Springer-Verlag.)

APPENDIX A

EQUATIONS OF MOTION FOR TURBULENT FLOW

In the theory of turbulent flow,^{3,5,7} it is assumed that the turbulent flow motion can be separated into a mean motion with velocity components $\bar{u}$, $\bar{v}$, $\bar{w}$ and into a fluctuation or eddying motion with velocity components u' , v' , w' relative to the mean motion. The velocity components, the pressure, and the density in a compressible turbulent flow are therefore expressed as

$$\begin{aligned} u &= \bar{u} + u' ; v = \bar{v} + v' ; w = \bar{w} + w' ; \\ \rho &= \bar{\rho} + \rho' ; T = \bar{T} + T' , \end{aligned} \quad (A-1)$$

where the barred quantities are defined as time averages; e.g.,

$$\bar{u} = \frac{1}{t_1} \int_{t_0}^{t_0 + t_1} u \, dt . \quad (A-2)$$

In this approach it is implied that the averaging time, t_1 , is chosen so that the mean values are independent of time, or, if the flow characteristics change, so that the averages vary only slowly, and independently from the eddying motion. Therefore, by definition, the time averages of all the quantities describing the fluctuations must vanish:

$$\bar{u'} = \bar{v'} = \bar{w'} = \bar{\rho'} = \bar{p'} = \bar{T'} = 0 . \quad (A-3)$$

On the other hand

$$\begin{aligned} \overline{u^2} &= \frac{1}{t_1} \int_{t_0}^{t_0 + t_1} dt (\bar{u} + u')^2 \\ &= (\bar{u})^2 + \frac{1}{t_1} \int_{t_0}^{t_0 + t_1} dt [2\bar{u} u' + u'^2] \\ &= (\bar{u})^2 + \overline{u'^2} , \end{aligned} \quad (A-4)$$

and

$$\begin{aligned} \overline{u v} &= \frac{1}{t_1} \int_{t_0}^{t_0 + t_1} dt (\bar{u} + u')(\bar{v} + v') \\ &= \bar{u} \bar{v} + \overline{u' v'} . \end{aligned} \quad (A-5)$$

The last product will in general not vanish; it is related to the transport of momentum perpendicular to the direction of flow, as can be seen from qualitative considerations.

In turbulent flow, the apparent or virtual stresses (often called the Reynolds

stresses) are expressed as the averages of the fluctuating components u' , v' , w' of the fluid velocity. For incompressible flow, the stress tensor due to the turbulent velocity components is given by⁵

$$\begin{pmatrix} \sigma_x' & \tau_{xy}' & \tau_{xz}' \\ \tau_{xy}' & \sigma_y' & \tau_{yz}' \\ \tau_{xz}' & \tau_{yz}' & \sigma_z' \end{pmatrix}$$

$$= - \begin{pmatrix} \overline{\rho u'^2} & \overline{\rho u'v'} & \overline{\rho u'w'} \\ \overline{\rho u'v'} & \overline{\rho v'^2} & \overline{\rho v'w'} \\ \overline{\rho u'w'} & \overline{\rho v'w'} & \overline{\rho w'^2} \end{pmatrix}.$$

To obtain a consistent picture of pipe flow near the wall, it is necessary to consider the total stresses; i.e., the sum of the viscous stresses and the Reynolds stresses. The tensor of the total stresses is defined as

$$\begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{pmatrix} = \begin{pmatrix} -\bar{p} + 2\mu \frac{\partial \bar{u}}{\partial x} - \overline{\rho u'^2} & \mu \left(\frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x} \right) - \overline{\rho u'v'} & \mu \left(\frac{\partial \bar{u}}{\partial z} + \frac{\partial \bar{w}}{\partial x} \right) - \overline{\rho u'w'} \\ -\left(\frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x} \right) - \overline{\rho u'v'} & -\bar{p} + 2\mu \frac{\partial \bar{v}}{\partial y} - \overline{\rho v'^2} & \mu \left(\frac{\partial \bar{v}}{\partial z} + \frac{\partial \bar{w}}{\partial y} \right) - \overline{\rho v'w'} \\ -\left(\frac{\partial \bar{u}}{\partial z} + \frac{\partial \bar{w}}{\partial x} \right) - \overline{\rho u'w'} & \mu \left(\frac{\partial \bar{v}}{\partial z} + \frac{\partial \bar{w}}{\partial y} \right) - \overline{\rho v'w'} & -\bar{p} + 2\mu \frac{\partial \bar{w}}{\partial z} - \overline{\rho w'^2} \end{pmatrix},$$

(A-7)

where $\bar{p}$ indicates average pressure.

For the case of pipe flow (x is flow direction, y is distance perpendicular to the wall), the relevant component τ_{xy} of the stress tensor becomes

$$\tau_{xy} = \mu \frac{\partial \bar{u}}{\partial y} - \overline{\rho u'v'} \quad (\text{A-8})$$

since $\frac{\partial \bar{u}}{\partial x} \approx 0$ is assumed.

Assuming further that $\bar{u} = \bar{u}(y)$, and introducing the concept of the turbulent mixing coefficient, A_t , to describe the effect of the Reynolds stresses, one obtains

$$\tau_t = -\overline{\rho u'v'} = A_t \frac{d\bar{u}}{dy} \quad (\text{A-9})$$

and, substituting in Eq. (A-8),

$$\tau_{xy} = \tau = \tau_t + \tau_v = \mu \frac{\partial \bar{u}}{\partial y} + A_t \frac{d\bar{u}}{dy}, \quad (\text{A-10})$$

which is the starting point for the analysis by Reichardt.

APPENDIX B

DIMENSIONAL ANALYSIS OF THE LAW OF THE WALL

The analytical form of the velocity distribution for flow near a wall, but outside the laminar sublayer and the transitional layer, can be determined by simple dimensional considerations, as shown by Squire.^{4,9} Let τ_0 be the Reynolds shear stress τ_t in the turbulent boundary layer, which is assumed to be constant over a narrow radial region. Then Eq. (3) is written

in the form

$$\frac{\tau_t}{\rho} = \frac{\tau_0}{\rho} = u_*^2 = \frac{A_t}{\rho} \frac{d\bar{u}}{dy} \quad (\text{B-1})$$

The quantity $\sqrt{\frac{\tau_0}{\rho}}$ has the dimension (force/area)^{1/2}/(mass/volume); i.e., of a velocity. For an incompressible fluid in a region where the Reynolds stresses dominate over the viscous stresses, the turbulent mixing coefficient, A_t , can depend only on τ_0 and the distance y from the wall:

$$A_t = A_t(y, \tau_c) \quad (B-2)$$

A_t/ρ has, however, the dimension of (length)²/(time); therefore, it can be expressed most simply in the form

$$\epsilon = Ku_* y \quad (B-3)$$

where K is a constant.

Combining Eqs. (B-1) and (B-3) one obtains

$$u_*^2 = Ku_* y \frac{d\bar{u}}{dy} \quad (B-4)$$

or by integration

$$\bar{u} = \frac{u_*}{K} \ln y + C \quad (B-5)$$

C being a constant of integration to be determined by comparison with experimental data.

The velocity distribution law, Eq. (B-5), is limited to distances from the wall which are large in comparison to the thickness of the laminar sublayer. On the other hand, the pipe radius does not enter, so Eq. (B-5) cannot describe the velocity distribution near the center of the pipe, where Eq. (B-5) would lead to a radial velocity gradient at the center of the pipe.

The approximation Eq. (B-5) can be improved in two ways: the turbulent shear stress τ_t used above can be replaced by the total shear stress given in Eq. (4), which is again assumed to be constant, so that

$$\tau = \tau_v + \tau_t = \bar{\tau} = (\mu + A_t) \frac{d\bar{u}}{dy} \quad (B-6)$$

Second, the expression for A_t/ρ can be modified to account for the existence of a laminar sublayer:

$$\epsilon = Ku_*(y - y_0) \quad (B-7)$$

Equation (B-7) accounts for the vanishing of the turbulent mixing coefficient

in the laminar sublayer, and y_0 is a measure of its thickness.

Introducing again the friction velocity u_* , one obtains, combining Eqs. (B-6) and (B-7),

$$\frac{d\bar{u}}{dy} = \frac{u_*^2}{\nu + Ku_*(y - y_0)} \quad (B-8)$$

where $\nu = \mu/\rho$ is the kinematic viscosity; therefore,

$$\frac{\bar{u}}{u_*} = \frac{1}{K} \ln \left[\frac{u_*(y - y_0)}{\nu} + \frac{1}{K} \right] + C' \quad (B-9)$$

where C' is an integration constant. Utilizing experimental results (e.g., the classical measurements of Nikuradse³⁰), K and C' can be determined directly, and the so-called universal law of the wall is obtained in the form

$$\frac{\bar{u}}{u_*} = 2.50 \ln \left(\frac{u_* y}{\nu} - 5.3 \right) + 5.5 \quad (B-10)$$

In this form the law of the wall may hold even in parts of the transition layer.

APPENDIX C

THE MIXING LENGTH THEORY OF PRANDTL

In his classic paper¹² describing the phenomena of turbulent pipe flow, Prandtl postulated the following mechanism for the motion: fluid particles coalesce into "lumps" that move bodily and stay together for a given distance traversed, both in the longitudinal direction (parallel to the pipe axis) and in the transverse (radial) direction, as the fluid moves along the pipe wall in turbulent motion. In this picture, the concept of the mixing length λ plays a central role; λ is defined as that distance in the transverse (radial) direction which must be covered by a fluid lump traveling with its original mean velocity in order to make the difference between the velocity of the lump and the velocity in the newly reached region equal to

the mean transverse fluctuation in turbulent flow.

The concept of mixing length is analogous to the concept of the mean free path in the kinetic theory of gases. Prandtl, in fact, considers it as expressing the properties of "molar," in contrast to "molecular," motion.

In terms of this model, Prandtl derived an expression for the exchange coefficient A_t by analyzing the motion of representative fluid lumps. Consider a lump coming from a radial wall distance $y_1 - \ell$, as indicated in Fig. (C-1), and moving to a radial wall distance y_1 ; at y_1 , the lump will move slower than with the velocity prevailing at y_1 :

$$\bar{u}(y_1) - \bar{u}(y_1 - \ell) \approx \ell \left. \frac{d\bar{u}}{dy} \right|_{y_1} \equiv \Delta u_1 \quad (C-1)$$

Similarly, a lump coming from $y_1 + \ell$ would have at y_1 an excess velocity given by

$$\bar{u}(y_1 + \ell) - \bar{u}(y_1) \approx \ell \left. \frac{d\bar{u}}{dy} \right|_{y_1} \equiv \Delta u_2 \quad (C-2)$$

Prandtl next considers the velocity differences caused by the transverse motion as making up the turbulent or eddy velocity components u' at y_1 :

$$|u'| = \frac{1}{2} (|\Delta u_1| + |\Delta u_2|) = \ell \left| \frac{d\bar{u}}{dy} \right|_{y_1} \quad (C-3)$$

Fig. C-1. Schematic of mixing length concept.

In the mixing length theory, one furthermore assumes that the average of a product of the fluctuation velocity components may be replaced by the product of the averages of the magnitudes of the components:

$$\overline{u'v'} = -c \overline{|u'|} \cdot \overline{|v'|} \quad (C-4)$$

where c is an arbitrary coefficient: $0 < c \leq 1$.

With this assumption one obtains from Eq. (C-3) and the corresponding equation for v' for the xy -component of the Reynolds stresses discussed in Appendix A the expression

$$\tau_t = \tau_{xy} = -\overline{\rho u'v'} = \rho c \left(\ell \left. \frac{d\bar{u}}{dy} \right| \right)^2 \quad (C-5)$$

The turbulent shearing stress must change sign with $d\bar{u}/dy$; therefore, absorbing the constant c into ℓ^2 , one sets

$$\tau_t = \rho \ell^2 \left| \frac{d\bar{u}}{dy} \right| \frac{d\bar{u}}{dy} \quad (C-6)$$

From this relation and Eq. (3) one obtains immediately that

$$A_t = \rho \ell^2 \left| \frac{d\bar{u}}{dy} \right| \quad (C-7)$$

This physical approach provides for an intuitive appreciation of the effects of turbulent mixing; it does not provide, however, a tool for calculating the velocity distribution, unless further assumptions are made as to the functional dependence of ℓ on the wall distance y .

Prandtl assumed that near the wall

$$\ell(y) = \kappa y \quad (C-8)$$

where κ is a dimensionless constant, the value of which is deduced from experiment³⁰ as 0.4.

More complicated expressions for $\lambda(y)$ have been introduced to obtain better agreement between the predicted radial variations of $\bar{u}$ with the experimental data. Szablewski,¹⁴ for example, introduced an expression for the mixing length that includes the pipe radius, a , and the viscosity, μ , as parameters:

$$\lambda/r_0 = \kappa(y/r_0) + \mu(y/r_0)^2 \quad (C-9)$$

From Eqs. (C-6) and (C-9), Szablewski¹⁴ obtained the velocity distribution function

$$\frac{\bar{u}}{u_*} = \frac{1}{\kappa^2} \frac{1 - \sqrt{1 + 4\kappa^2 \left[\frac{u_*}{\nu} (y - \delta_0) \right]^2}}{2 \frac{u_*}{\nu} (y - \delta)} + \frac{1}{\kappa} \ln \left\{ 2\kappa \frac{u_*}{\nu} (y - \delta_0) + \sqrt{1 + 4\kappa^2 \left[\frac{u_*}{\nu} (y - \delta_0) \right]^2} \right\} + \frac{u_* \delta_0}{\nu} \quad (C-10)$$

valid for

$$0 \leq \frac{u_*}{\nu} (y - \delta_0) \ll \kappa \sqrt{\frac{\lambda}{32}} \quad (C-11)$$

where δ_0 is the thickness of the laminar sublayer and λ is the dimensionless coefficient of resistance in pipe flow. Equation (C-11) does not give as good agreement for $\bar{u}/u_*$ with the experimental data as the distribution functions obtained by Reichardt and van Driest in the laminar sublayer and the transition layer. Moreover it appears more cumbersome to use because of its more complicated form.

APPENDIX D

RELATION OF REICHARDT'S AND VAN DRIEST'S METHODS TO OTHER SEMIEMPIRICAL METHODS

To compare Eq. (18) with other expressions for the velocity distribution, it is useful to write

$$\ln \frac{1 + 2\left(\frac{y}{r}\right)^2}{1 - \left(\frac{y}{r}\right)^2} = - \ln \frac{\eta}{\eta_r} + \ln \frac{1 + 2\left(\frac{y}{r}\right)^2}{1 + \left(\frac{y}{r}\right)^2} \quad (D-1)$$

so that

$$\frac{\bar{u}}{u_*} = \frac{u_m}{u_*} + \frac{1}{\kappa} \ln (\eta/\eta_r) - \frac{1}{\kappa} \ln \frac{1 + 2\left(\frac{y}{r}\right)^2}{1 + \frac{y}{r}} \quad (D-2)$$

This equation shows clearly that the logarithmic law of the wall on von Kármán and Prandtl is contained in the results of Reichardt; moreover, it becomes apparent why the logarithmic law of the wall holds approximately through a large part of the pipe cross section: the additive term

$$- \frac{1}{\kappa} \ln \frac{1 + 2\left(\frac{y}{r}\right)^2}{1 + \frac{y}{r}} \quad (D-3)$$

varies relatively slowly and is dominated by the term $\ln(\eta/\eta_r)$.

It is interesting to note that these results provide a qualitative explanation for the validity of the logarithmic profile throughout much of the pipe cross section, although the law of the wall is based on assumptions which are valid only near the wall. In the derivations of Prandtl and von Kármán, it is assumed that $\frac{A}{\mu\eta_r} \sim \kappa \frac{y}{r}$, and that $\tau \approx \tau_0$; thus,

$$\frac{\tau/\tau_0}{A/(\mu\eta_r)} \sim \frac{1}{\kappa y/r}$$

Actually, however, $\frac{\tau}{\tau_0} = 1 - \frac{y}{r}$ and $A/(\mu\eta_r)$ deviates soon from its approximate value $\kappa(y/a)$; thus approximately,

$$\frac{\tau/\tau_0}{A/(\mu\eta_r)} = \frac{1 - \frac{y}{r}}{\kappa \frac{y}{r} (1 - \frac{y}{r})} \approx \frac{1}{\kappa \left(\frac{y}{r}\right)} \quad (D-4)$$

Thus, the errors in the expressions for τ/τ_0 and $A/(\mu\eta_T)$ cancel to the first order, and a more exact treatment leads in first order to a logarithmic dependency of $\bar{u}/u_*$ on η .

By the appropriate series expansions one obtains immediately from either Eq. (19) or Eq. (25) that in the laminar sub-layer

$$\bar{u}/u_* = \eta \quad (\text{valid for } \eta \lesssim 5) \quad (\text{D-5})$$

and that in the turbulent boundary layer

$$\begin{aligned} \bar{u}/u_* &= 2.5 \ln(\eta - 5.3) + 5.5 \\ &(\text{valid for } \eta \gtrsim 70) ; \end{aligned} \quad (\text{D-6})$$

i.e., the customary law of the wall.⁵

The boundary condition of Reichardt used in Eq. (11) can be justified as follows: at the wall, both $\bar{u}$ and u' must vanish. One may therefore postulate that near the wall

$$u' = \alpha(x, t) \bar{u} = \alpha(x, t) \left(\frac{d\bar{u}}{dy} \right)_0 y, \quad (\text{D-7})$$

where α is a function of time and the coordinate x (parallel to wall).

From the equation of continuity (v' being the fluctuation velocity normal to the wall)

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0 \quad (\text{D-8})$$

one obtains

$$\begin{aligned} v' &= - \int_0^y dy' \frac{\partial u'}{\partial x} \\ &= - \frac{1}{2} \left(\frac{d\bar{u}}{dy} \right)_0 y^2 \frac{\partial \alpha}{\partial x}. \end{aligned} \quad (\text{D-9})$$

Hence one obtains from the definition of τ_t

$$\begin{aligned} \tau_t &= - \rho \overline{u'v'} = \frac{1}{2} \left(\frac{d\bar{u}}{dy} \right)_0^2 y^3 \alpha \frac{\partial \alpha}{\partial x} \\ &= \frac{1}{4} \rho \frac{\partial \alpha^2}{\partial x} \left(\frac{d\bar{u}}{dy} \right)_0^2 y^3. \end{aligned} \quad (\text{D-10})$$

Instead of Eq. (10), one now obtains for the exchange coefficient

$$\begin{aligned} \frac{A}{\mu} &= \frac{\tau_t}{\tau_v} = \frac{1}{4} \rho \frac{\partial \alpha^2}{\partial x} \left(\frac{d\bar{u}}{dy} \right)_0^2 y^3 / \mu \left(\frac{d\bar{u}}{dy} \right)_0 \\ &= \frac{1}{4} \frac{\partial \alpha^2}{\partial \left(\frac{yu_*}{\nu} \right)} \left(\frac{yu_*}{\nu} \right)^3. \end{aligned} \quad (\text{D-11})$$

If only two fluctuation velocity components are considered, it thus follows from the use of the continuity equation that A/μ increases as y^3 near the wall. This conclusion is, however, open to question. Van Driest¹⁶ has developed an alternative approach which leads to the result that A/μ increases as y^4 near the wall. This formulation allows for an excellent fit of the classic data by Laufer,¹⁹ but it has the disadvantage that the continuity equation is apparently not satisfied near the wall. In Reichardt's¹⁶ derivation, the third velocity fluctuation component is, however, neglected. If the third component is also considered, one obtains

$$\frac{A}{\mu} = \frac{1}{2\nu^2} \frac{\partial u'}{\partial \eta} \frac{\partial}{\partial z} \left(\frac{\partial w'}{\partial \eta} \right)_0 y^3 \quad (\text{D-12})$$

If $\frac{\partial u'}{\partial \eta}$ and $\frac{\partial}{\partial z} \left(\frac{\partial w'}{\partial \eta} \right)_0$ are correlated, then the y^3 dependence of A/μ remains valid; if there is no correlation, then one finds a y^4 rise of A/μ , in agreement with van Driest.

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