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**The Residual Tune Splitting due to Linear Coupling –
Theory and Correction**

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1. Introduction

The tune splitting due to random a_1 has been found to be large in RHIC. $|\nu_1 - \nu_2| \sim 200 \times 10^{-3}$ was found for the $\beta^* = 2$ lattice for the worse error distribution out of 10 error distributions tried. Most of this tune splitting can be corrected with a 2 family a_1 correction system,¹ but a residual tune splitting remains that is still appreciable. The residual tune splitting, $|\nu_1 - \nu_2|$, that remained in RHIC after correction with the 2 family correction system was found to be about 18×10^{-3} .

The residual $|\nu_1 - \nu_2|$ appears to be due to higher order effects of the random a_1 . The theory of these effects indicates that the harmonics of a_1 near $\nu_x + \nu_y$ are the important harmonics. However, the harmonics closest to $\nu_x + \nu_y$ do not contribute much to the residual $|\nu_1 - \nu_2|$. A correction system has been proposed and simulated that appears able to reduce the residual $|\nu_1 - \nu_2|$. The lack of a dominant harmonic complicates the correction.

2. Residual Tune Splitting in RHIC – Definition and Properties

Magnitude of Tune Splitting in RHIC

Table 1: Results for the correction of the tune splitting for a $\beta^* = 2$ RHIC lattice using a 2 family tune splitting correction system set to make $\Delta\nu = 0$.

Error Field Dist.	—Uncorrected—			—Corrected—		
	ν_1	ν_2	$ \nu_1 - \nu_2 $ / 10^{-3}	ν_1	ν_2	$ \nu_1 - \nu_2 $ / 10^{-3}
1	0.796	0.854	59	0.828	0.823	6
2	0.707	0.935	228	0.838	0.819	19
3	0.869	0.783	86	0.825	0.829	4
4	0.772	0.883	111	0.831	0.823	7
5	0.779	0.872	93	0.836	0.820	16
6	0.848	0.805	43	0.832	0.821	11
7	0.840	0.847	7	0.852	0.834	18
8	0.742	0.895	153	0.838	0.818	20
9	0.785	0.866	81	0.828	0.823	6
10	0.749	0.891	142	0.823	0.827	5

¹ G. Parzen, AD/RHIC-AP-72 (1988).

Before Correction

Largest $|\nu_1 - \nu_2| = 228 \times 10^{-3}$. Note $\nu = 28.935$, for error field #2, is almost in $\nu = 29$ stopband. $\beta_1 \simeq 140$ shows this. Stopband danger can be avoided by going to $\beta^* = 6$, and correcting at $\beta^* = 6$.

Residual $|\nu_1 - \nu_2|$ - After Correction

Correction makes $\Delta\nu = 0$.

$$\Delta\nu = \frac{1}{4\pi\rho} \int ds \, a_1 (\beta_x \beta_y)^{1/2} \exp[i\bar{\nu}(\theta_x - \theta_y)],$$

$$\theta_x = \psi_x/\nu_x, \quad \theta_y = \psi_y/\nu_y, \quad \bar{\nu} = (\nu_x + \nu_y)/2.$$

Largest residual $|\nu_1 - \nu_2| = 20 \times 10^{-3}$. Note $\nu_x = 28.826$, $\nu_y = 28.821$, $|\nu_x - \nu_y| = 5 \times 10^{-3}$.

Residual tune splitting is $|\nu_1 - \nu_2|$ after correction, $\Delta\nu = 0$.

We will see that after correction

$$\text{Residual } |\nu_1 - \nu_2| \sim a_1^2$$

whereas before correction

$$|\nu_1 - \nu_2| \sim a_1$$

An additional reason for having a residual tune splitting correction system, is to correct the β -functions, β_1, β_2 . One finds $\beta_1 \sim 100$ m after correction with the 2 family system for $\beta^* = 2$. Correction of the residual tune splitting also corrects β_1, β_2 to a considerable extent.

Lowest Order Analytical Result for $\nu_1 - \nu_2$

$$\begin{aligned} \nu_{1,2} &= \bar{\nu} \pm \left[\left(\frac{\nu_x - \nu_y}{2} \right)^2 + |\Delta\nu|^2 \right]^{\frac{1}{2}}, \\ \Delta\nu &= \frac{1}{4\pi\rho} \int ds \, a_1 (\beta_x \beta_y)^{\frac{1}{2}} \exp[i\bar{\nu}(\theta_x - \theta_y)], \\ \bar{\nu} &= \frac{1}{2}(\nu_x + \nu_y), \quad \theta_x = \psi_x/\nu_x, \quad \theta_y = \psi_y/\nu_y. \end{aligned} \tag{1}$$

ν_1 is the normal mode that goes to ν_x when $a_1 \rightarrow 0$, and ν_2 is the corresponding mode for ν_y . For the $\pm$ sign, the $+$ sign is used when $\nu_x > \nu_y$ for ν_1 and the opposite sign for ν_2 . Close enough to the $\nu_x = \nu_y$ resonance, $\nu_1 - \nu_x \sim |\Delta\nu|$ and is linear in a_1 . Far enough from the $\nu_x = \nu_y$ resonance, $|\nu_1 - \nu_x| \simeq |\Delta\nu|^2$ and is quadratic in a_1 .

$$\begin{aligned} |\nu_1 - \nu_2| &= 2 \left[\left(\frac{\nu_x - \nu_y}{2} \right)^2 + |\Delta\nu|^2 \right]^{\frac{1}{2}} \\ \frac{\nu_1 + \nu_2}{2} &= \frac{\nu_x + \nu_y}{2} \end{aligned} \tag{2}$$

One cannot reduce $|\nu_1 - \nu_2|$ below $2|\Delta\nu|$ using ν_x, ν_y . One can control $(\nu_1 + \nu_2)/2$ using ν_x, ν_y .

Equation (1) is useful only near the $\nu_x = \nu_y$ resonance line, as it neglects some a_1^2 terms.

Comments on Figure 1

Figure (1a) shows how well Eq. (1) describes the tune shifts due to a_1 . ν_1 and ν_2 have been computed for random a_1 distribution #2, and ν_1 and ν_2 are plotted against a_1 . $a_1 = 0.8$ means that the a at each element has been reduced by the same factor of 0.8. As ν_x, ν_y is on the resonance lines $\nu_x = \nu_y$, Eq. (1) predicts straight lines for ν_1 and ν_2 versus a_1 , and the tune splitting agrees well with $|\nu_1 - \nu_2| = 2\Delta\nu$.

Figure (1b) shows $|\nu_1 - \nu_2|$ versus a_1 after a 2 family correction that makes $\Delta\nu = 0$. One sees that the residual $|\nu_1 - \nu_2|$ after correction is quadratic in a_1 . The crosses indicate a a_1^2 curve normalized at $a_1 = 1$.

Figure (1c) shows an error distribution, #7, where the higher order a_1 terms in the ν -shift dominates even before correction.

3. Analytical Results for Residual $|\nu_1 - \nu_2|$

For the case when the tune splitting has been corrected by making $\Delta\nu = 0$ then

$$\begin{aligned}
 \nu_1 &= \nu_x + \frac{1}{2\nu_x} \Delta_x \\
 \nu_2 &= \nu_y + \frac{1}{2\nu_y} \Delta_y \\
 \Delta_x &= 4\nu_x \nu_y \sum_{n \neq 0} \frac{|c_n|^2}{n(n - \nu_x - \nu_y)} \\
 \Delta_y &= 4\nu_x \nu_y \sum_{n \neq 0} \frac{|b_n|^2}{n(n - \nu_x - \nu_y)} \\
 c_n &= \frac{1}{4\pi\rho} \int ds a_1 (\beta_x \beta_y)^{\frac{1}{2}} \exp[i(\nu_x \theta_x + (n - \nu_x) \theta_y)] \\
 b_n &= \frac{1}{4\pi\rho} \int ds a_1 (\beta_x \beta_y)^{\frac{1}{2}} \exp[i((n - \nu_y) \theta_x + \nu_y \theta_y)] \\
 \theta_x &= \psi_x / \nu_x, \quad \theta_y = \psi_y / \nu_y
 \end{aligned} \tag{3}$$

Above holds when ν_x, ν_y are close to the resonance line, $\nu_x = \nu_y$. These results will be derived in a separate paper. A numerical check of these results is given below.

Comments on Eq. (3) for ν_1, ν_2

1. In 1-dimensional case, $\nu_1 = \nu_x + 2\nu_x \sum_n |b_n|^2 / (n(n - 2\nu_x))$, when $b_0 = 0$; $b_n = (1/4\pi\rho) \int ds b_1 \beta \exp(in\theta)$. A similar result to Eq. (3).
2. Importance of $n \sim \nu_x + \nu_y$ harmonic because of the resonance denominator, $1/(n - \nu_x - \nu_y)$. $n \simeq 58$, for RHIC, is the harmonic closest to $\nu_x + \nu_y$.
3. b_n, c_n are similar to sum resonance stopband integrals corresponding to different ν_x, ν_y on the resonance line, $\nu_x + \nu_y = n$.

$$c_n \rightarrow \nu_x, n - \nu_x$$

$$b_n \rightarrow n - \nu_y, \nu_y$$

4. For n close to $\nu_x + \nu_y$, b_n and c_n are nearly equal. Thus $n = 58$ for RHIC causes the largest change in ν_1 and ν_2 , but does not contribute to the residual tune splitting, $|\nu_1 - \nu_2|$ when $\Delta\nu = 0$.
5. Note that $\Delta x, \Delta y$ can be written in the integral form $\Delta x \sim \int ds ds' a_1(s) (\beta_x(s) \beta_y(s))^{\frac{1}{2}} a_1(s') (\beta_x(s') \beta_y(s'))^{\frac{1}{2}} g(s - s')$. This form may be useful in designing a correction system for the residual $|\nu_1 - \nu_2|$.

Numerical Test of Analytical Result for the Residual $|\nu_1 - \nu_2|$

The following table shows the results of a numerical check of the results in Eq. (3). In this check, the a_1 around the ring was excited according to the formula

$$a_1 = A \cos[n(\theta_x + \theta_y)],$$

$$\theta_x = \psi_x / \nu_x, \quad \theta_y = \psi_y / \nu_y.$$

Only 12 high- β quadrupoles in the insertions were excited in this way. The b_n and c_n in Eq. (3) were computed and the theoretical results from Eq. (3) were compared with computed results for ν_1, ν_2 .

n	Computed		Theory	
	$(\nu_1 - \nu_x)$ / 10^{-3}	$(\nu_2 - \nu_y)$ / 10^{-3}	$(\nu_1 - \nu_x)$ / 10^{-3}	$(\nu_2 - \nu_y)$ / 10^{-3}
59	9	7	9	9
58	32	30	33	33
57	-3	-1	-1	-1
56	25	27	26	26
55	6	7	6	6

4. Correction of the Residual Tune Splitting²

Optimize the 2 Family Tune Splitting Correction

Instead of the setting $\Delta\nu = 0$, set corrections to minimize $|\nu_1 - \nu_2|$. These results are shown in Table 2.

Table 2: Residual tune splitting resulting from two different methods of tune splitting correction using the 2-family tune splitting correction system.

Error Field Distribution	$ \nu_1 - \nu_2 /10^{-3}$ $\Delta\nu = 0$ Correction	$ \nu_1 - \nu_2 /10^{-3}$ minimize $ \nu_1 - \nu_2 $ Correction
	$\beta^* = 2$	
1	6	5
2	19	18
3	4	3
4	7	7
5	16	8
6	11	11
7	18	7
8	20	16
9	6	6
10	5	4

The Enlarged a_1 Correction System

For the enlarged a_1 correction system, there are 12 a_1 correctors, one at each high- β quad, Q2 or Q3. In the correction scheme that was simulated, these 12 a_1 correctors were excited according to the equation

$$a_1 = A \cos(\psi_x + \psi_y) + B \sin(\psi_x + \psi_y)$$

² G. Parzen, AD/RHIC-82 (1990).

Table 3: The reduction in the residual tune splitting obtained using the enlarged a_1 correction system, compared with the results obtained with the 2-family correction system, minimizing $|\nu_1 - \nu_2|$.

Error Field Distribution	$ \nu_1 - \nu_2 /10^{-3}$ 2 Family Correction System	$ \nu_1 - \nu_2 /10^{-3}$ enlarged a_1 Correction System
$\beta^* = 2$		
1	5	5
2	16	2.8
3	3	3
4	5	5
5	5	5
6	9	3
7	14	1.5
8	10	2
9	6	3
10	1	3

The total correction field is given by

$$a_1 = A \cos(\psi_x + \psi_y) + B \sin(\psi_x + \psi_y) + C (a_1 \text{ due to Family 1}) + D (a_1 \text{ due to Family 2})$$

These 4 parameters A, B, C, D are varied to minimize $|\nu_1 - \nu_2|$. The results using the enlarged a_1 correction system are shown in Table 3.

Varying the Harmonic Number of the a_1 Correction

A further improvement in $|\nu_1 - \nu_2|$ and in the correction of β_1, β_2 is obtained by exciting the 12 correctors according to

$$a_1 = A \cos n(\theta_x + \theta_y) + B \sin n(\theta_x + \theta_y)$$

One now has 5 parameters to vary, A, B, C, D, n .

The effect of varying n is shown on the next figure, Fig. 2. This additional parameter is sometimes useful in correcting β_1, β_2 as well as ν_1, ν_2 .

Conclusions

To correct the tune splitting, one has to reduce $\Delta\nu, \Delta_x, \Delta_y$. Thus a correction system with a minimum of 4 adjustable parameters is required, depending on the location of the a_1 correctors. The correction system described here is appealing because it emphasizes the important harmonics, the $|\nu_x - \nu_y|$ harmonic (0 for RHIC) and the $\nu_x + \nu_y$ harmonic (58

for RHIC). Probably, other arrangements of the a_1 correctors, with 4 or more parameters, may also work well.

The setting of the 4 or more parameters required may be difficult, partly because of the lack of a dominant harmonic. The setting of the parameters may be made easier by providing observation stations that measure β_1, β_2 or the 4 parameters of the R matrix that defines the normal mode coordinates. At the moment, this may seem like too much effort for this effect as far as RHIC is concerned.

$\beta^* = 2$, Seed = 2

$b_1 = 0$, $\gamma_x, \gamma_y = .824, .824$

un corrected, $\Delta V = .11$

(1a)

Corrected

Uncorrected, $\Delta V = .011$

$\beta^* = 2$, Seed = 7

$\gamma_x, \gamma_y = .826, .821$

(1b)

Fig. 1.

$$a_1 = A \cos n(\theta_x + \theta_y) + B \sinh n(\theta_x + \theta_y)$$

$$B^* = 2, \text{ Seed} = 2$$

Fig. 2